THERMAL STRESSES IN A FINITE SOLID-PROPELLANT GRAIN

by

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TABLE OF CONTENTS

LIST OF FIGURES

LIST OF SYMBOLS

a internal radius of hollow grain configuration; radius of solid grain configuration

b external radius of hollow propellant grain

c external radius of casing

L semi-length of cylinder

r radial cylindrical coordinate

T temperature

T_o zero thermal-stress temperature

T_S surface temperature

ΔT temperature field, $(T - T_o)$

t time

u radial displacement

v angular displacement

w axial displacement

z axial cylindrical coordinate

α_T coefficient of thermal dilatation

γ material property, $(3\lambda + 2\mu)\ \alpha_T/(\lambda + 2\mu)$

δ_{ij} Kronecker delta

ϵ_{ij} strain component

θ angular cylindrical coordinate

k thermal diffusivity

λ Lame constant

μ viscosity of material

σ_{ij} stress component

Φ potential of thermoelastic displacement defined by Eq. (6)

Subscripts

i,j,k summation indices, 1, 2, 3

C casing

P propellant

I. INTRODUCTION

The thermal stresses in an infinite cylinder are readily obtained on the assumption of plane strain which is nonrestrictive only for very long cylinders. For a short cylinder, however, where the length is of the same order of magnitude as the diameter, the thermal stresses are influenced by the cylinder end conditions; and, the plane strain solution is no longer a good approximation unless the special boundary condition, dictated by the plane strain conditions, is maintained at the ends of the cylinder.

An analysis of the unsteady thermoelastic stress problem, for a short circular cylinder with a concentric hole, is of practical importance since it has direct application to the stress analysis of solid-propellant grain configurations. In reference 1 the pressure, temperature cycling and vertical storage problems, in elastic or viscoelastic finite cylinders, have been treated for a time-dependent temperature distribution throughout the cylinder.

This thesis deals with the determination of the unsteady thermal stresses in a finite length, hollow, circular cylinder with an axially symmetrical temperature field. The equations of equilibrium are reduced to a single governing differential equation which is solved by the method of separation of variables, yielding the thermal stresses in terms of the prescribed temperature distribution. The exact solution for the temperature distribution for the finite, hollow, circular cylinder, however, is not known. Therefore, the method of solution employed for the determination of the thermal stresses is illustrated for the case of the finite, solid cylinder for which the unsteady temperature distribution is known[2].

II. THE UNSTEADY THERMOELASTIC PROBLEM FOR THE FINITE CIRCULAR CYLINDER

For an elastic solid with a nonsteady temperature field $\Delta T(x_r,t)$, the stresses, σ_{ij}, and the strains, ϵ_{ij}, produced, are connected by the thermoelastic stress-strain relations[3]

$$\sigma_{ij} = \lambda\, \delta_{ij}\, \epsilon_{kk} + 2\mu\, \epsilon_{ij} - (3\lambda + 2\mu)\, \delta_{ij}\, \alpha_T\, \Delta T \quad i,j = 1,2,3 \tag{1}$$

where λ, μ are the Lame constants, α_T is the coefficient of thermal dilatation, and δ_{ij} is Kronecker's delta. In terms of the cylindrical coordinates (r, θ, z), the strain-displacement relations are[3]

$$\epsilon_{rr} = \frac{\partial u}{\partial r},\ \epsilon_{\theta\theta} = \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta},\ \epsilon_{zz} = \frac{\partial w}{\partial z}$$

$$\epsilon_{r\theta} = \epsilon_{\theta r} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r}\right),\ \epsilon_{rz} = \epsilon_{zr} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right)$$

$$\epsilon_{\theta z} = \epsilon_{z\theta} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{1}{r}\frac{\partial w}{\partial \theta}\right) \tag{2}$$

where u, v, and w represent the components of the displacement vector in the r, θ, and z directions, respectively. With axial symmetry prevailing, i.e, $\frac{\partial}{\partial \theta} = 0$ and $v = 0$, the thermoelastic stress-strain relations, for the circular cylinder, reduce to

$$\sigma_{rr} = \lambda \left(\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} \right) + 2\mu \frac{\partial u}{\partial r} - (3\lambda + 2\mu)\, \alpha_T\, \Delta T$$

$$\sigma_{\theta\theta} = \lambda \left(\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} \right) + 2\mu \frac{u}{r} - (3\lambda + 2\mu)\, \alpha_T\, \Delta T$$

$$\sigma_{zz} = \lambda \left(\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} \right) + 2\mu \frac{\partial w}{\partial z} - (3\lambda + 2\mu)\, \alpha_T\, \Delta T \qquad (3)$$

$$\sigma_{rz} = \sigma_{zr} = \mu \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \right)$$

$$\sigma_{r\theta} = \sigma_{\theta r} = \sigma_{\theta z} = \sigma_{z\theta} = 0$$

or

$$\sigma_{rr} = (\lambda + 2\mu)\, u_r + \lambda \left(w_z + \frac{u}{r}\right) - (3\lambda + 2\mu)\, \alpha_T\, \Delta T \qquad (3.1)$$

$$\sigma_{\theta\theta} = (\lambda + 2\mu) \frac{u}{r} + \lambda (u_r + w_z) - (3\lambda + 2\mu)\, \alpha_T\, \Delta T \qquad (3.2)$$

$$\sigma_{zz} = (\lambda + 2\mu)\, w_z + \lambda \left(u_r + \frac{u}{r}\right) - (3\lambda + 2\mu)\, \alpha_T\, \Delta T \qquad (3.3)$$

$$\sigma_{rz} = \sigma_{zr} = \mu(u_z + w_r) \qquad (3.4)$$

where the subscripts on the displacements indicate partial differentiation. Assuming that there are no body-forces within the cylinder and that axial symmetry prevails, the equations of equilibrium[3],

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} + R = 0$$

$$\frac{\partial \sigma_{rz}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\theta z}}{\partial \theta} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\sigma_{rz}}{r} + Z = 0 \qquad (4)$$

$$\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \sigma_{\theta z}}{\partial z} + 2 \frac{\sigma_{r\theta}}{r} + \Theta = 0$$

where the body-force components in the r, θ, and z directions are denoted by R, Θ, and Z, respectively, reduce to

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0 \tag{4.1}$$

$$\frac{\partial \sigma_{rz}}{\partial r} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\sigma_{rz}}{r} = 0 \tag{4.2}$$

By means of the thermoelastic stress-strain relations, i.e., Eqs. (3.1) through (3.4), Eqs. (4.1) and (4.2) become, respectively,

$$(\lambda + 2\mu)\ (u_{rr} + \frac{u_r}{r} - \frac{u}{r^2}) + (\lambda + \mu)\ w_{zr} + \mu\ u_{zz} - m\ \ \Delta T_r = 0 \tag{5.1}$$

$$(\lambda + 2\mu)\ w_{zz} + (\lambda + \mu)\ (u_{zr} + \frac{u_z}{r}) + \mu\ (w_{rr} + \frac{w_r}{r}) - m\ \ \Delta T_z = 0 \tag{5.2}$$

where $\quad m = (3\lambda + 2\mu)\ \alpha_T$.

Now defining a "potential of thermoelastic displacement", Φ, as

$$u = \Phi_r\ , \quad w = \Phi_z \tag{6}$$

where the subscripts on Φ imply partial differentiation as before, then Eqs. (5.1) and (5.2) become, respectively,

$$(\lambda + 2\mu)\ (\Phi_{rrr} + \frac{\Phi_{rr}}{r} - \frac{\Phi_r}{r^2} + \Phi_{zzr}) - m\ \ \Delta T_r = 0 \tag{7.1}$$

and

$$(\lambda + 2\mu)\ (\Phi_{zzz} + \Phi_{zrr} + \frac{\Phi_{zr}}{r}) - m\ \Delta T_z = 0 \qquad (7.2)$$

Integrating Eq. (7.1) with respect to r gives

$$(\lambda + 2\mu)\ (\Phi_{rr} + \frac{\Phi_r}{r} + \Phi_{zz}) - m\ \Delta T = f(z)\ , \qquad (8.1)$$

and integrating Eq. (7.2) with respect to z gives

$$(\lambda + 2\mu)\ (\Phi_{rr} + \frac{\Phi_r}{r} + \Phi_{zz}) - m\ \Delta T = g(r) \qquad (8.2)$$

so that from Eqs. (8.1) and (8.2)

$$f(z) = g(r) = 0 \qquad (9)$$

and

$$\nabla^2 \Phi = \gamma\ \Delta T \qquad (10)$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\ \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}$$

and

$$\gamma = \frac{m}{\lambda + 2\mu} = \frac{(3\lambda + 2\mu)}{(\lambda + 2\mu)}\ \alpha_T$$

The thermoelastic stress-strain relations, i.e., Eqs. (3), may be written in terms of the displacement potential Φ, defined by Eqs. (6), as

$$\sigma_{rr} = \lambda \nabla^2 \Phi + 2\mu \, \Phi_{rr} - (3\lambda + 2\mu) \, \alpha_T \, \Delta T$$

$$\sigma_{\theta\theta} = \lambda \nabla^2 \Phi + \frac{2\mu}{r} \Phi_r - (3\lambda + 2\mu) \, \alpha_T \, \Delta T$$

$$\sigma_{zz} = \lambda \nabla^2 \Phi + 2\mu \, \Phi_{zz} - (3\lambda + 2\mu) \, \alpha_T \, \Delta T \qquad (11)$$

$$\sigma_{rz} = \sigma_{zr} = 2\mu \, \Phi_{rz}$$

$$\sigma_{r\theta} = \sigma_{\theta r} = \sigma_{\theta z} = \sigma_{z\theta} = 0$$

which, by means of Eq. (10) reduce, for the case of an unsteady temperature distribution, to

$$\sigma_{rr} = 2\mu \, (\Phi_{rr} - \gamma \, \Delta T) \qquad (12.1)$$

$$\sigma_{\theta\theta} = 2\mu \, \left(\frac{1}{r} \Phi_r - \gamma \, \Delta T\right) \qquad (12.2)$$

$$\sigma_{zz} = 2\mu \, (\Phi_{zz} - \gamma \, \Delta T) \qquad (12.3)$$

$$\sigma_{rz} = \sigma_{zr} = 2\mu \, \Phi_{rz} \qquad (12.4)$$

and

$$\sigma_{r\theta} = \sigma_{\theta r} = \sigma_{\theta z} = \sigma_{z\theta} = 0 \qquad (12.5)$$

III. SOLUTION OF THE GOVERNING DIFFERENTIAL EQUATION

The solution to the governing differential equation

$$\nabla^2 \Phi = \gamma \Delta T \tag{10}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}$$

is composed of two parts: (1) a general solution, Φ_h, i.e. the solution to Laplace's equation, $\nabla^2 \Phi_h = 0$; and (2), a particular solution, Φ_p, i.e. a solution of Poisson's equation $\nabla^2 \Phi_p = \gamma \Delta T$.

Assuming the general solution Φ_h to be of the form

$$\Phi_h = R(r)\ Z(z) \tag{13.1}$$

then Laplace's equation $\nabla^2 \Phi_h = 0$, in cylindrical coordinates with axial symmetry, becomes, by means of Eq. (13.1),

$$\nabla^2 \Phi_n = R''(r)\ Z(z) + \frac{1}{r} R'(r)\ Z(z) + R(r)\ Z''(z) = 0 \tag{13.2}$$

where the prime denotes differentiation with respect to the argument. By separating the variables in Eq. (13.2) there follows

$$\frac{1}{R} (R'' + \frac{1}{r} R') = - \frac{Z''}{Z} = \beta^2 \tag{13.3}$$

where β^2 is the separation constant.

Equation (13.3) implies the two ordinary differential equations

$$r^2 R'' + r R' - \beta^2 r^2 R = 0 \qquad (13.4)$$

and

$$Z'' + \beta^2 Z = 0 \qquad (13.5)$$

The sign of the separation constant was chosen in such a way that trigometric functions, rather than exponential functions, will be introduced in the z-direction for reasons that will be apparent later.

Equation (13.4) is the modified Bessel's equation with the solution

$$R(r) = A\ I_o(\beta r) + B\ K_o(\beta r) \qquad (13.6)$$

and Eq. (13.5) has the solution

$$Z(z) = C \sin(\beta z) + D \cos(\beta z) \qquad (13.7)$$

so that the general solution, Φ_h, given by Eq. (13.1), may be written as

$$\Phi_h = \sum_n \left[A_n\ I_o(\beta_n r) + B_n\ K_o(\beta_n r)\right] \left[C_n \sin(\beta_n z) + D_n \cos(\beta_n z)\right] \qquad (13.8)$$

where A_n, B_n, C_n, and D_n are "constants" of integration that may be functions of time, and the β_n are the characteristic eigen values for this boundary-value problem.

A particular solution of Eq. (10) may be written as[3]

$$\Phi_p(P) = -\frac{\gamma}{4\pi} \int_V \frac{\Delta T(Q)}{r(P,Q)}\ dV_Q \qquad (14)$$

where $\Phi_p(P)$ is the displacement potential at a point P in the cylinder, $\Delta T(Q)$ is the difference between the temperature at some point Q in the cylinder and the temperature for zero-stress conditions; $r(P,Q)$ is the distance between points P and Q, and dV_Q is a differential volume containing the point Q. The volume integral, Eq. (14), is usually quite difficult to handle, so another method of obtaining the particular solution, Φ_p, will be used.

The temperature distribution, ΔT, satisfies the Fourier heat conduction equation

$$k \nabla^2 (\Delta T) = \frac{\partial}{\partial t} (\Delta T) \tag{15}$$

Considering the functions Φ and ΔT, in Eq. (10), as functions of both the space and time, i.e.

$$\Phi = \Phi(r, z, t) \tag{16}$$

and

$$\Delta T = \Delta T(r, z, t) \tag{17}$$

then, the differentiation of Eq. (10), with respect to the time (t), gives

$$\nabla^2 \left(\frac{\partial \Phi}{\partial t} \right) = \gamma \frac{\partial \Delta T}{\partial t} \tag{18}$$

which becomes, by means of Eq. (15),

$$\nabla^2 \left(\frac{\partial \Phi}{\partial t} \right) = \gamma\, k \nabla^2 (\Delta T) \tag{19}$$

or

$$\nabla^2 \left(\frac{\partial \Phi}{\partial t} - \gamma k \; \Delta T \right) = 0 \tag{20}$$

A particular solution of Eq. (20) may be obtained by setting

$$\frac{\partial\Phi}{\partial t} = \gamma k \ \Delta T \tag{21}$$

so that, upon integrating with respect to time,

$$\Phi(r,z,t) = \gamma k \int_{t_1}^{t} \Delta T(r,z,\tau) \, d\tau + \psi(r,z) \tag{22}$$

where t_1 is some reference time and ψ is, as yet, an undetermined function of the space coordinates. The substitution of Eq. (22) into Eq. (10) yields

$$\gamma k \ \nabla^2 \int_{t_1}^{t} \Delta T(r,z,\tau) \, d\tau + \nabla^2 \psi \ (r,z) = \gamma \ \Delta T(r,z,t) \tag{23}$$

which, by means of the Fourier heat conduction equation, reduces to

$$\nabla^2 \psi \ (r,z) = \gamma \ \Delta T(r,z,t_1) \tag{24}$$

At the time of the cast, say $t = 0$, the thermal stresses are zero; therefore, if t_1 is chosen as zero one concludes that $\psi = 0$ and

$$\Phi_p(r,z,t) = \gamma k \int_0^t \Delta T(r,z,\tau) \, d\tau \quad . \tag{25}$$

Since linear stress-strain relations are used in this analysis, the problem is essentially linear, and the general solution to Eq. (10) can be written as the sum of the general solution, Φ_h, and the particular solution, Φ_p, i.e.

$$\Phi(r,z,t) = \gamma k \int_0^t \Delta T(r,z,\tau)\, d\tau + \sum_n \left[A_n I_o(\beta_n r) + B_n K_o(\beta_n r) \right] \left[C_n \sin(\beta_n z) + D_n \cos(\beta_n z) \right] \qquad (26)$$

IV. SOLUTION OF THE THERMAL STRESS PROBLEM IN A SIMPLY SUPPORTED HOLLOW GRAIN

Consider a finite length, hollow, circular cylinder, $(-L < z < L, a < r < b)$ as shown in Figure 1, encased by an elastic shell at $r = b$, and simply supported with end plates that constrain the ends $(z = L, z = -L)$ from experiencing a radial displacement. The following boundary conditions will then apply:

$$\text{at } r = a, \qquad \sigma_{rz_P}(a,z,t) = 0 \tag{27}$$

$$\text{at } r = b, \qquad u_P(b,z,t) = u_C(b,z,t) \tag{28}$$

$$w_P(b,z,t) = w_C(b,z,t) \tag{29}$$

$$\text{at } r = c, \qquad \sigma_{rz_C}(c,z,t) = 0 \tag{30}$$

$$\text{and at the ends,} \qquad u_P(r,L,t) = u_P(r,-L,t) = 0 \tag{31}$$

$$u_C(r,L,t) = u_C(r,-L,t) = 0 \tag{32}$$

where the subscript "P" refers to the propellant and "C" to the casing. Furthermore, the net force in the axial direction must be zero, or

$$\int_{\bar{S}} \bar{\bar{\sigma}}_{ij} \cdot d\bar{S} = 0 \quad . \tag{33}$$

Considering a vector pointing outward from the surface $\bar{S}$ as positive, then the boundary condition, Eq. (33), using Eqs. (27) and (30), may be written as

$$2\pi \int_a^c r\left[\sigma_{zz}(r,L,t) - \sigma_{zz}(r,-L,t)\right] dr = 0 \tag{34}$$

Since, from Eq. (26),

$$u(r,z,t) = \Phi_r(r,z,t) = \gamma k \int_0^t \frac{\partial}{\partial r}\left[\Delta T\ (r,z,\tau)\right] d\tau$$

$$+ \beta\left[AI_1\ (\beta r) - B\ \ K_1\ (\beta r)\right]\left[C\ \sin\ (\beta z) + D\ \cos\ (\beta z)\right] \tag{35.1}$$

$$w(r,z,t) = \Phi_z\ (r,z,t) = \gamma k \int_0^t \frac{\partial}{\partial z}\left[\Delta T(r,z,\tau)\right] d\tau$$

$$+ \beta\left[A\ I_o\ (\beta r) + B\ K_o\ (\beta r)\right]\left[C\ \cos\ (\beta z) - D\ \sin\ (\beta z)\right] \tag{35.2}$$

$$\Phi_{rz}(r,z,t) = \gamma k \int_0^t \frac{\partial^2}{\partial r \partial z}\left[\Delta T(r,z,\tau)\right] d\tau$$

$$+ \beta^2\left[A\ I_1\ (\beta r) - B\ K_1\ (\beta r)\right]\left[C\ \cos\ (\beta z) - D\ \sin\ (\beta z)\right] \tag{35.3}$$

$$\Phi_{zz}(r,z,t) = \gamma k \int_0^t \frac{\partial^2}{\partial z^2}\left[\Delta T(r,z,\tau)\right] d\tau$$

$$- \beta^2\left[A\ I_o\ (\beta r) + B\ K_o\ (\beta r)\right]\left[C\ \sin\ (\beta z) + D\ \cos\ (\beta z)\right] \tag{35.4}$$

and

$$\Phi_{rr}(r,z,t) = \gamma k \int_0^t \frac{\partial^2}{\partial r^2}\left[\Delta T(r,z,\tau)\right] d\tau$$

$$+ \beta\left\{ \beta\left[A\, I_o\, (\beta r) + B\, K_o\, (\beta r)\right] - \frac{1}{r}\left[A\, I_1\, (\beta r) - B\, K_1\, (\beta r)\right]\right\}$$

$$\left[C \sin(\beta z) + D \cos(\beta z)\right] \qquad (35.5)$$

then, the boundary condition

$$u_p\,(r,L,t) = u_p\,(r, -L, t) = 0 \qquad (28)$$

gives, from Eq. (35.1)

$$\gamma_P\; k_P \int_0^t \frac{\partial}{\partial r}\left[\Delta T_p\,(r,L,\tau)\right] d\tau + \beta\left[A_p\, I_1\,(\beta r) - B_p\, K_1\,(\beta r)\right]$$

$$\left[C_p \sin(\beta L) + D_p \cos(\beta L)\right] = 0 \qquad (36.1)$$

and

$$\gamma_P\; k_P \int_0^t \frac{\partial}{\partial r}\left[\Delta T_p\,(r, -L, \tau)\right] d\tau + \beta\left[A_p\, I_1\,(\beta r) - B_p\, K_1\,(\beta r)\right]$$

$$\left[- C_p \sin(\beta L) + D_p \cos(\beta L)\right] = 0 \quad . \qquad (36.2)$$

Too, the boundary condition

$$u_C\,(r,L,t) = u_C\,(r, -L, t) = 0$$

gives

$$\gamma_C k_C \int_0^t \frac{\partial}{\partial r}\left[\Delta T_C(r,L,\tau)\right] d\tau + \beta\left[A_C I_1(\beta r) - B_C K_1(\beta r)\right]$$

$$\left[C_C \sin(\beta L) + D_C \cos(\beta L)\right] = 0 \qquad (37.1)$$

and

$$\gamma_C k_C \int_0^t \frac{\partial}{\partial r}\left[\Delta T_C(r,-L,\tau)\right] d\tau + \beta\left[A_C I_1(\beta r) - B_C K_1(\beta r)\right]$$

$$\left[- C_C \sin(\beta L) + D_C \cos(\beta L)\right] = 0 \qquad (37.2)$$

The condition that the net force in the axial direction must be zero gives, from Eqs. (12.3), (34) and (35.4)

$$\mu_P \int_a^b \left\{ r \gamma_P k_P \int_0^t \frac{\partial^2}{\partial z^2}\left[\Delta T_P(r,z,\tau)\right]_{z=L} d\tau - \right.$$

$$r \gamma_P k_P \int_0^t \frac{\partial^2}{\partial z^2}\left[\Delta T_P(r,z,\tau)\right]_{z=-L} d\tau - 2\beta^2 r\left[A_P I_o(\beta r) + \right.$$

$$\left. B_P K_o(\beta r)\right] C_P \sin(\beta L) - r\gamma_P\left[\Delta T_P(r,L,t) - \Delta T_P(r,-L,t)\right]\Big\} dr +$$

$$\mu_C \int_b^c \left\{ r \gamma_C k_C \int_0^t \frac{\partial^2}{\partial z^2}\left[\Delta T_C(r,z,\tau)\right]_{z=L} d\tau - \right.$$

$$r \gamma_C k_C \int_0^t \frac{\partial^2}{\partial z^2}\left[\Delta T_C(r,z,\tau)\right]_{z=-L} d\tau -$$

$$2\beta^2 r\left[A_C I_o(\beta r) + B_C K_o(\beta r)\right] C_C \sin(\beta L) -$$

$$\left. r \gamma_C\left[\Delta T_C(r,L,t) - \Delta T_C(r,-L,t)\right]\right\} dr = 0 \qquad (38)$$

Now, if the temperature distribution throughout the propellant and the casing is of the form

$$\Delta T\ (r,z,t) = A + \sum_{n=1}^{\infty} f(t)\ g(r)\ \cos\ (\beta_n z) \tag{39}$$

where the β_n are roots of the equation $\cos\ (\beta L) = 0$, i.e. $\beta = \beta_n = \frac{(2n-1)\pi}{2L}$ for $n = 1, 2, \ldots$, and A is a constant, then

$$\Delta T(r,L,t) = \Delta T\ (r,-L,t) = A = \text{constant} \tag{40}$$

$$\frac{\partial}{\partial r}\left[\Delta T\ (r,L,t)\right] = \frac{\partial}{\partial r}\left[\Delta T\ (r,-L,t)\right] = 0 \tag{41}$$

and

$$\frac{\partial^2}{\partial z^2}\left[\Delta T\ (r,z,\tau)\right]_{z=L} = \frac{\partial^2}{\partial z^2}\left[\Delta T\ (r,z,\tau)\right]_{z=-L} = 0\ . \tag{42}$$

If the temperature distribution throughout the propellant and casing is of the form

$$\Delta T\ (r,z,t) = B + \sum_{n=1}^{\infty} f_1(t)\ g_1(r)\ \sin\ (\beta_n z) \tag{43}$$

where the β_n are roots of the equation $\sin\ (\beta L) = 0$, i.e. $\beta = \beta_n = \frac{n\pi}{L}$ for $n = 1, 2, \ldots$, and B is a constant, then

$$\Delta T\ (r,L,t) = \Delta T\ (r,-L,t) = B = \text{constant} \tag{44}$$

$$\frac{\partial}{\partial r}\left[\Delta T\ (r,L,t)\right] = \frac{\partial}{\partial r}\left[\Delta T\ (r,-L,t)\right] = 0 \tag{45}$$

and

$$\frac{\partial^2}{\partial z^2}\left[\Delta T\ (r,z,\tau)\right]_{z=L} = \frac{\partial^2}{\partial z^2}\left[\Delta T\ (r,z,\tau)\right]_{z=-L} = 0\ . \tag{46}$$

Now, by means of Eqs. (40) through (42), Eqs. (36.1) through (38), respectively, reduce to

$$\beta\left[A_P\, I_1\ (\beta r) - B_P\, K_1\ (\beta r)\right]\left[C_P\ \sin\ (\beta L) + D_P\cos\ (\beta L)\right] = 0 \tag{47.1}$$

$$\beta\left[A_P\, I_1\ (\beta r) - B_P\, K_1\ (\beta r)\right]\left[-\, C_P\sin\ (\beta L) + D_P\cos\ (\beta L)\right] = 0 \tag{47.2}$$

$$\beta\left[A_C\, I_1\ (\beta r) - B_C\, K_1\ (\beta r)\right]\left[C_C\sin\ (\beta L) + D_C\cos\ (\beta L)\right] = 0 \tag{47.3}$$

$$\beta\left[A_C\, I_1\ (\beta r) - B_C\, K_1\ (\beta r)\right]\left[-\, C_C\sin\ (\beta L) + D_C\cos\ (\beta L)\right] = 0 \tag{47.4}$$

and

$$\beta^2\sin\ (\beta L)\left\{\mu_P\, C_P\int_a^b\left[A_P\, r\, I_o\ (\beta r) + B_P\, r\, K_o\ (\beta r)\right]dr\right.$$

$$\left. +\, \mu_C\, C_C\int_b^c\left[A_C\, r\, I_o\ (\beta r) + B_C\, r\, K_o\ (\beta r)\right]dr\right\} = 0 \tag{47.5}$$

Equations (44) through (46) yield the same set of equations, Eqs. (47.1) through (47.5), from Eqs. (36.1) through (38).

For a temperature distribution of the form given by Eq. (39), Eq. (47.5) yields

$$C_P = C_C = 0 \tag{48}$$

and Eqs. (47.1) through (47.4) are automatically satisfied since

$$\cos(\beta L) = 0 \quad \text{or} \quad \beta = \beta_n = \frac{(2n-1)\pi}{2L}, \text{ for } n = 1, 2, \ldots \tag{49}$$

For a temperature distribution of the form given by Eq. (43), Eq. (47.5) is automatically satisfied since

$$\sin(\beta L) = 0 \quad \text{or} \quad \beta = \beta_n = \frac{n\pi}{L}, \text{ for } n = 1, 2, \ldots \tag{50}$$

and Eqs. (47.1) through (47.4) yield

$$D_P = D_C = 0 \tag{51}$$

Thus, two cases arise which will be considered individually:

Case I: $\Delta T\,(r,z,t)$ is as given by Eq. (39) ,

and Case II: $\Delta T\,(r,z,t)$ is as given by Eq. (43) .

A. The Case of an "Even" Temperature Distribution in the Propellant and the Casing

The "potential of thermoelastic displacement" $\Phi\ (r,z,t)$ is, from Eqs. (26), (48), and (49)

$$\Phi\ (r,z,t) = \gamma k \int_0^t \Delta T\ (r,z,\tau)\ d\tau + \sum_{n=1}^{\infty} \Big[a_n\ I_o\ (\beta_n r) + b_n\ K_o\ (\beta_n r) \Big] \cos\ (\beta_n z) \qquad (52)$$

where $\Delta T\ (r,z,t)$ is given by Eq. (39), $\beta_n = \frac{(2n-1)\pi}{2L}$, for n = 1,2,.., and a_n and b_n are constants to be evaluated from the boundary conditions.

Then, from Eq. (52),

$$u\ (r,z,t) = \Phi_r\ (r,z,t) = \gamma k \int_0^t \frac{\partial}{\partial r} \Big[\Delta T\ (r,z,\tau) \Big]\ d\tau + \sum_{n=1}^{\infty} \beta_n \Big[a_n\ I_1\ (\beta_n r) - b_n\ K_1\ (\beta_n r) \Big] \cos\ (\beta_n z) \qquad (53.1)$$

$$w\ (r,z,t) = \Phi_z\ (r,z,t) = \gamma k \int_0^t \frac{\partial}{\partial z} \Big[\Delta T\ (r,z,\tau) \Big]\ d\tau - \sum_{n=1}^{\infty} \beta_n \Big[a_n\ I_o\ (\beta_n r) + b_n\ K_o\ (\beta_n r) \Big] \sin\ (\beta_n z) \qquad (53.2)$$

$$\Phi_{rz}\ (r,z,t) = \gamma k \int_0^t \frac{\partial^2}{\partial r \partial z}\left[\Delta T\ (r,z,\tau)\right] d\tau$$

$$- \sum_{n=1}^{\infty} \beta_n^{\ 2}\left[a_n\ I_1\ (\beta_n r) - b_n\ K_1\ (\beta_n r)\right] \sin\ (\beta_n z) \tag{53.3}$$

$$\Phi_{zz}\ (r,z,t) = \gamma k \int_0^t \frac{\partial^2}{\partial z^2}\left[\Delta T\ (r,z,\tau)\right] d\tau$$

$$- \sum_{n=1}^{\infty} \beta_n^{\ 2}\left[a_n\ I_o\ (\beta_n r) + b_n\ K_o\ (\beta_n r)\right] \cos\ (\beta_n z) \tag{53.4}$$

and

$$\Phi_{rr}\ (r,z,t) = \gamma k \int_0^t \frac{\partial^2}{\partial r^2}\left[\Delta T\ (r,z,\tau)\right] d\tau$$

$$+ \sum_{n=1}^{\infty} \beta_n \left\{\beta_n \left[a_n\ I_o\ (\beta_n r) + b_n\ K_o\ (\beta_n r)\right] -\right.$$

$$\left.\frac{1}{r}\left[a_n\ I_1\ (\beta_n r) - b_n\ K_1\ (\beta_n r)\right]\right\} \cos\ (\beta_n z) \tag{53.5}$$

and the boundary conditions (27), (30), (28) and (29), respectively, give

$$\gamma_p\ k_p \int_0^t \frac{\partial^2}{\partial r \partial z}\left[\Delta T_p\ (r,z,\tau)\right]_{r=a} d\tau =$$

$$\sum_{n=1}^{\infty} \beta_n^{\ 2}\left[a_{n_p}\ I_1\ (\beta_n a) - b_{n_p}\ K_1\ (\beta_n a)\right] \sin\ (\beta_n z) \tag{54}$$

$$\gamma_C k_C \int_0^t \frac{\partial^2}{\partial r \partial z}\left[\Delta T_C(r,z,\tau)\right]_{r=c} d\tau =$$

$$\sum_{n=1}^{\infty} \beta_n^2 \left[a_{n_C} I_1 (\beta_n c) - b_{n_C} K_1 (\beta_n c)\right] \sin (\beta_n z) \tag{55}$$

$$\gamma_C k_C \int_0^t \frac{\partial}{\partial r}\left[\Delta T_C (r,z,\tau)\right]_{r=b} d\tau -$$

$$\gamma_P k_P \int_0^t \frac{\partial}{\partial r}\left[\Delta T_P (r,z,\tau)\right]_{r=b} d\tau =$$

$$\sum_{n=1}^{\infty} \beta_n \left[a_{n_P} I_1 (\beta_n b) - b_{n_P} K_1 (\beta_n b) -\right.$$

$$\left. a_{n_C} I_1 (\beta_n b) + b_{n_C} K_1 (\beta_n b)\right] \cos (\beta_n z) \tag{56}$$

and

$$\gamma_C k_C \int_0^t \frac{\partial}{\partial z}\left[\Delta T_C(r,z,\tau)\right]_{r=b} d\tau -$$

$$\gamma_P k_P \int_0^t \frac{\partial}{\partial z}\left[\Delta T_P(r,z,\tau)\right]_{r=b} d\tau =$$

$$\sum_{n=1}^{\infty} \beta_n\left[a_{n_C} I_o (\beta_n b) + b_{n_C} K_o (\beta_n b) -\right.$$

$$\left. a_{n_P} I_o (\beta_n b) - b_{n_P} K_o (\beta_n b)\right] \sin (\beta_n z) \tag{57}$$

where

$$\beta_n = \frac{(2n-1)\pi}{2L}, \quad \text{for } n = 1, 2, \ldots \quad . \tag{49}$$

Now, according to the theory of Fourier series, for a half-range sine expansion, one obtains from Eqs. (54), (55) and (57), respectively

$$\beta_n^{\,2}\left[a_{n_p} I_1\ (\beta_n a) - b_n\ K_1\ (\beta_n a)\right] =$$

$$\frac{2}{L}\int_0^L \sin\ (\beta_n z)\ \gamma_P\ \ k_P \int_0^t \frac{\partial^2}{\partial r \partial z}\left[\Delta T_P(r,z,\tau)\right]_{r=a} d\tau\ dz \tag{58}$$

$$\beta_n^{\,2}\left[a_{n_C}\ I_1\ (\beta_n c)\ = b_{n_C}\ K_1\ (\beta_n c)\right] =$$

$$\frac{2}{L}\int_0^L \sin\ (\beta_n z)\ \gamma_C\ \ k_C \int_0^t \frac{\partial^2}{\partial r \partial z}\left[\Delta T_C(r,z,\tau)\right]_{r=c} d\tau\ dz \tag{59}$$

and

$$\beta_n\left[a_{n_C}\ I_o\ (\beta_n b) + b_{n_C}\ K_o\ (\beta_n b) - a_{n_p} I_o\ (\beta_n b) - b_{n_p}\ K_o\ (\beta_n b)\right] =$$

$$\frac{2}{L}\int_0^L \sin\ (\beta_n z)\Big\{\gamma_C\ \ k_C \int_0^t \frac{\partial}{\partial z}\left[\Delta T_C(b,z,\tau)\right] d\tau\ -$$

$$\gamma_P\ \ k_P \int_0^t \frac{\partial}{\partial z}\left[\Delta T_P\ (b,z,\tau)\right] d\tau \Big\}\ dz \tag{60}$$

Too, according to the theory of Fourier series, for a half-range cosine expansion, one obtains from Eq. (56)

$$\beta_n \left[a_{n_P} I_1 (\beta_n b) - b_{n_P} K_1 (\beta_n b) - a_{n_C} I_1 (\beta_n b) + b_{n_C} K_1 (\beta_n b) \right] =$$

$$\frac{2}{L} \int_0^L \cos (\beta_n z) \left\{ \gamma_C k_C \int_0^t \frac{\partial}{\partial r} \left[\Delta T_C (r,z,\tau) \right]_{r=b} d\tau - \right.$$

$$\left. \gamma_P k_P \int_0^t \frac{\partial}{\partial r} \left[\Delta T_P (r,z,\tau) \right]_{r=b} d\tau \right\} dz \quad . \qquad (61)$$

The simultaneous solution of Eqs. (58) through (61), for the four "constants" a_{n_P}, b_{n_P}, a_{n_C} and b_{n_C}, yields

$$a_{n_P} = \frac{2\gamma_C\, k_C}{\beta_n^{\,2}\, L} \cdot \frac{K_1\,(\beta_n a)}{U_1\left[\beta_n c, \beta_n a\right]} \int_0^L \sin\,(\beta_n z) \int_0^t \frac{\partial^2}{\partial r \partial z}\left[\Delta T_C(r,z,\tau)\right]_{r=c} d\tau\; dz\; -$$

$$\frac{2\gamma_P\, k_P}{\beta_n^{\,2}\, L} \cdot \frac{K_1\,(\beta_n c)}{U_1\left[\beta_n c, \beta_n a\right]} \int_0^L \sin\,(\beta_n z) \int_0^t \frac{\partial^2}{\partial r \partial z}\left[\Delta T_P(r,z,\tau)\right]_{r=a} d\tau\; dz\; +$$

$$\frac{2K_1(\beta_n a)}{(L/b)} \; \frac{U_o\left[\beta_n b, \beta_n c\right]}{U_1\left[\beta_n c, \beta_n a\right]} \left\{\gamma_C\, k_C \int_0^L \cos(\beta_n z) \int_0^t \frac{\partial}{\partial r}\left[\Delta T_C(r,z,\tau)\right]_{r=b} d\tau\; dz\; -\right.$$

$$\left.\gamma_P\, k_P \int_0^L \cos\,(\beta_n z) \int_0^t \frac{\partial}{\partial r}\left[\Delta T_P(r,z,\tau)\right]_{r=b} d\tau\; dz\right\}\; -$$

$$\frac{2K_1(\beta_n a)}{(L/b)} \; \frac{U_1\left[\beta_n c, \beta_n b\right]}{U_1\left[\beta_n c, \beta_n a\right]} \left\{\gamma_C\, k_C \int_0^L \sin\,(\beta_n z) \int_0^t \frac{\partial}{\partial z}\left[\Delta T_C(b,z,\tau)\right] d\tau\; dz\; -\right.$$

$$\left.\gamma_P\, k_P \int_0^L \sin\,(\beta_n z) \int_0^t \frac{\partial}{\partial z}\left[\Delta T_P(b,z,\tau)\right] d\tau\; dz\right\}, \tag{62}$$

$$b_{n_p} = \frac{2\gamma_C\, k_C}{\beta_n^{\,2}\, L}\ \frac{I_1(\beta_n a)}{U_1\left[\beta_n c, \beta_n a\right]} \int_0^L \sin\,(\beta_n z) \int_0^t \frac{\partial^2}{\partial r \partial z}\left[\Delta T_C(r,z,\tau)\right]_{r=c} d\tau\ dz\ -$$

$$\frac{2\gamma_P\, k_P}{\beta_n^{\,2}\, L}\ \frac{I_1(\beta_n c)}{U_1\left[\beta_n c, \beta_n a\right]} \int_0^L \sin\,(\beta_n z) \int_0^t \frac{\partial^2}{\partial r \partial z}\left[\Delta T_P(r,z,\tau)\right]_{r=a} d\tau\ dz\ +$$

$$\frac{2I_1(\beta_n a)}{(L/b)}\ \frac{U_o\left[\beta_n b, \beta_n c\right]}{U_1\left[\beta_n c, \beta_n a\right]}\left\{\gamma_C\, k_C \int_0^L \cos\,(\beta_n z) \int_0^t \frac{\partial}{\partial r}\left[\Delta T_C(r,z,\tau)\right]_{r=b} d\tau\ dz\ -\right.$$

$$\left.\gamma_P\, k_P \int_0^L \cos\,(\beta_n z) \int_0^t \frac{\partial}{\partial r}\left[\Delta T(r,z,\tau)\right]_{r=b} d\tau\ dz\right\} -$$

$$\frac{2I_1(\beta_n a)}{(L/b)}\ \frac{U_1\left[\beta_n c, \beta_n b\right]}{U_1\left[\beta_n c, \beta_n a\right]}\left\{\gamma_C\, k_C \int_0^L \sin\,(\beta_n z) \int_0^t \frac{\partial}{\partial z}\left[\Delta T_C(b,z,\tau)\right] d\tau\ dz\ -\right.$$

$$\left.\gamma_P\, k_P \int_0^L \sin\,(\beta_n z) \int_0^t \frac{\partial}{\partial z}\left[\Delta T_P(b,z,\tau)\right] d\tau\ dz\right\},$$

(63)

$$a_{n_C} = \frac{2\gamma_C\ k_C}{\beta_n^{\ 2}\ L}\ \frac{K_1(\beta_n a)}{U_1\left[\beta_n c, \beta_n a\right]} \int_0^L \sin\ (\beta_n z) \int_0^t \frac{\partial^2}{\partial r \partial z}\left[\Delta T_C(r,z,\tau)\right]_{r=c} d\tau\ dz\ -$$

$$\frac{2\gamma_P\ k_P}{\beta_n^{\ 2} L}\ \frac{K_1(\beta_n c)}{U_1\left[\beta_n c, \beta_n a\right]} \int_0^L \sin\ (\beta_n z) \int_0^t \frac{\partial^2}{\partial r \partial z}\left[\Delta T_P(r,z,\tau)\right]_{r=a} d\tau\ dz\ +$$

$$\frac{2K_1(\beta_n c)}{(L/b)}\ \frac{U_o\left[\beta_n b, \beta_n a\right]}{U_1\left[\beta_n c, \beta_n a\right]}\left\{\gamma_C\ k_C \int_0^L \cos\ (\beta_n z) \int_0^t \frac{\partial}{\partial r}\left[\Delta T_C(r,z,\tau)\right]_{r=b} d\tau\ dz\ -\right.$$

$$\left.\gamma_P\ k_P \int_0^L \cos\ (\beta_n z) \int_0^t \frac{\partial}{\partial r}\left[\Delta T_P(r,z,\tau)\right]_{r=b} d\tau\ dz\right\} -$$

$$\frac{2K_1(\beta_n c)}{(L/b)}\ \frac{U_1\left[\beta_n a, \beta_n b\right]}{U_1\left[\beta_n c, \beta_n a\right]}\left\{\gamma_C\ k_C \int_0^L \sin\ (\beta_n z) \int_0^t \frac{\partial}{\partial z}\left[\Delta T_C(b,z,\tau)\right] d\tau\ dz\ -\right.$$

$$\left.\gamma_P\ k_P \int_0^L \sin\ (\beta_n z) \int_0^t \frac{\partial}{\partial z}\left[\Delta T_P(b,z,\tau)\right] d\tau\ dz\right\} \qquad (64)$$

and

$$b_{n_C} = \frac{2\gamma_C\, k_C}{\beta_n^{\,2}\, L}\; \frac{I_1(\beta_n a)}{U_1\left[\beta_n c, \beta_n a\right]} \int_0^L \sin\,(\beta_n z) \int_0^t \frac{\partial^2}{\partial r \partial z}\left[\Delta T_C(r,z,\tau)\right]_{r=c} d\tau\; dz\; -$$

$$\frac{2\gamma_P\, k_P}{\beta_n^{\,2}\, L}\; \frac{I_1(\beta_n c)}{U_1\left[\beta_n c, \beta_n a\right]} \int_0^L \sin\,(\beta_n z) \int_0^t \frac{\partial^2}{\partial r \partial z}\left[\Delta T_P(r,z,\tau)\right]_{r=a} d\tau\; dz\; +$$

$$\frac{2I_1(\beta_n c)}{(L/b)}\; \frac{U_o\left[\beta_n b, \beta_n a\right]}{U_1\left[\beta_n c, \beta_n a\right]} \left\{\gamma_C\, k_C \int_0^L \cos\,(\beta_n z) \int_0^t \frac{\partial}{\partial r}\left[\Delta T_C(r,z,\tau)\right]_{r=b} d\tau\; dz\; -\right.$$

$$\left.\gamma_P\, k_P \int_0^L \cos\,(\beta_n z) \int_0^t \frac{\partial}{\partial r}\left[\Delta T_P(r,z,\tau)\right]_{r=b} d\tau\; dz\right\} -$$

$$\frac{2I_1(\beta_n c)}{(L/b)}\; \frac{U_1\left[\beta_n a, \beta_n b\right]}{U_1\left[\beta_n c, \beta_n a\right]} \left\{\gamma_C\, k_C \int_0^L \sin\,(\beta_n z) \int_0^t \frac{\partial}{\partial z}\left[\Delta T_C(b,z,\tau)\right] d\tau\; dz\; -\right.$$

$$\left.\gamma_P\, k_P \int_0^L \sin\,(\beta_n z) \int_0^t \frac{\partial}{\partial z}\left[\Delta T_P(b,z,\tau)\right] d\tau\; dz\right\} \qquad (65)$$

where

$$U_o\left[\beta_n r_1, \beta_n r_2\right] = I_o(\beta_n r_1)\, K_1(\beta_n r_2) + I_1(\beta_n r_2)\, K_o(\beta_n r_1) \quad (66)$$

and

$$U_1\left[\beta_n r_1, \beta_n r_2\right] = I_1(\beta_n r_1)\, K_1(\beta_n r_2) - I_1(\beta_n r_2)\, K_1(\beta_n r_1). \quad (67)$$

The thermal stresses in the propellant and the casing are then given by Eqs. (12.1) through (12.5) with $\Delta T(r,z,t)$ defined by Eq. (39); the respective partial derivatives of the displacement potential, $\Phi(r,z,t)$, given by Eqs. (53.1) through (53.5); and the constants a_{n_p}, b_{n_p}, a_{n_C}, and b_{n_C} given, respectively, by Eqs. (62) through (65).

B. "Odd" Temperature Distribution

The "potential of thermoelastic displacement" $\Phi(r,z,t)$ is, from Eqs. (26), (50) and (51)

$$\Phi(r,z,t) = \gamma k \int_0^t \Delta T(r,z,\tau)\, d\tau + \sum_{n=1}^{\infty} \left[c_n I_o(\beta_n r) + d_n K_o(\beta_n r)\right] \sin(\beta_n z) \qquad (68)$$

where $\Delta T(r,z,t)$ is given by Eq. (43), $\beta_n = \frac{n\pi}{L}$, $n = 1, 2, \ldots$, and c_n and d_n are constants to be evaluated from the boundary conditions.

Now, from Eq. (68)

$$u(r,z,t) = \Phi_r(r,z,t) = \gamma k \int_0^t \frac{\partial}{\partial r}\left[\Delta T(r,z,\tau)\right] d\tau + \sum_{n=1}^{\infty} \beta_n \left[c_n I_1(\beta_n r) - d_n K_1(\beta_n r)\right] \sin(\beta_n z) \quad , \qquad (69)$$

$$w(r,z,t) = \Phi_z(r,z,t) = \gamma k \int_0^t \frac{\partial}{\partial r}\left[\Delta T(r,z,\tau)\right] d\tau + \sum_{n=1}^{\infty} \beta_n \left[c_n I_o(\beta_n r) + d_n K_o(\beta_n r)\right] \cos(\beta_n z), \qquad (70)$$

$$\Phi_{rz}(r,z,t) = \gamma k \int_0^t \frac{\partial^2}{\partial r \partial z} \left[\Delta T(r,z,\tau) \right] d\tau \; +$$

$$\sum_{n=1}^{\infty} \beta_n^{\,2} \left[c_n \, I_1(\beta_n r) - d_n \, K_1(\beta_n r) \right] \cos\,(\beta_n z) \qquad (71)$$

$$\Phi_{zz}(r,z,t) = \gamma k \int_0^t \frac{\partial^2}{\partial z^2} \left[\Delta T(r,z,\tau) \right] d\tau \; -$$

$$\sum_{n=1}^{\infty} \beta_n^{\,2} \left[c_n \, I_o(\beta_n r) + d_n \, K_o(\beta_n r) \right] \sin\,(\beta_n z) \; , \qquad (72)$$

and

$$\Phi_{rr}(r,z,t) = \gamma k \int_0^t \frac{\partial^2}{\partial r^2} \left[\Delta T(r,z,\tau) \right] d\tau \; +$$

$$\sum_{n=1}^{\infty} \beta_n \left\{ \beta_n \left[c_n \, I_o(\beta_n r) + d_n \, K_o(\beta_n r) \right] - \right.$$

$$\left. \frac{1}{r} \left[c_n \, I_1(\beta_n r) - d_n \, K_1(\beta_n r) \right] \right\} \sin\,(\beta_n z) \qquad (73)$$

The boundary conditions (27), (30), (28) and (29), respectively, become

$$- \gamma_p \, k_p \int_0^t \frac{\partial^2}{\partial r \partial z} \left[\Delta T_p(r,z,\tau) \right]_{r=a} d\tau =$$

$$\sum_{n=1}^{\infty} \beta_n^{\,2} \left[c_{n_p} I_1(\beta_n a) - d_{n_p} K_1(\beta_n a) \right] \cos\,(\beta_n z) \; , \qquad (74)$$

$$- \gamma_C k_C \int_0^t \frac{\partial^2}{\partial r \partial z} \left[\Delta T_C(r,z,\tau) \right]_{r=c} d\tau =$$

$$\sum_{n=1}^{\infty} \beta_n^2 \left[c_{n_C} I_1(\beta_n c) - d_{n_C} K_1(\beta_n c) \right] \cos (\beta_n z) \ , \qquad (75)$$

$$\gamma_C k_C \int_0^t \frac{\partial}{\partial r} \left[\Delta T_C(r,z,\tau) \right]_{r=b} d\tau - \gamma_P k_P \int_0^t \frac{\partial}{\partial r} \left[\Delta T_P(r,z,\tau) \right]_{r=b} d\tau =$$

$$\sum_{n=1}^{\infty} \beta_n \left[c_{n_P} I_1(\beta_n b) - d_{n_P} K_1(\beta_n b) - c_{n_C} I_1(\beta_n b) + d_{n_C} K_1(\beta_n b) \right] \sin (\beta_n z) \ , \qquad (76)$$

and

$$\gamma_C k_C \int_0^t \frac{\partial}{\partial z} \left[\Delta T_C(b,z,\tau) \right] d\tau - \gamma_P k_P \int_0^t \frac{\partial}{\partial z} \left[\Delta T_P(b,z,\tau) \right] d\tau =$$

$$\sum_{n=1}^{\infty} \beta_n \left[c_{n_P} I_o(\beta_n b) + d_{n_P} K_o(\beta_n b) - c_{n_C} I_o(\beta_n b) - d_{n_C} K_o(\beta_n b) \right] \cos (\beta_n z) \qquad (77)$$

where

$$\beta_n = \frac{n\pi}{L} \ , \text{ for } n = 1, 2, \ldots \qquad (50)$$

According to the theory of Fourier series for a cosine expansion in z, with $-L < z < L$, one obtains from Eqs. (74), (75) and (77), respectively

$$\beta_n^2 \left[c_{n_P} I_1(\beta_n a) - d_{n_P} K_1(\beta_n a) \right] =$$

$$- \frac{1}{L} \int_{-L}^{L} \cos (\beta_n z) \ \gamma_P k_P \int_0^t \frac{\partial^2}{\partial r \partial z} \left[\Delta T_P(r,z,\tau) \right]_{r=a} d\tau dz \ , \qquad (78)$$

$$\beta_n^{\,2}\left[c_{n_C} I_1(\beta_n c) - d_{n_C} K_1(\beta_n c)\right] =$$

$$-\frac{1}{L}\int_{-L}^{L} \cos\ (\beta_n z)\ \gamma_C\ k_C \int_0^t \frac{\partial^2}{\partial r \partial z}\left[\Delta T_C(r,z,\tau)\right]_{r=c} d\tau\ dz\ , \tag{79}$$

and

$$\beta_n\left[c_{n_P}\ I_o(\beta_n b) + d_{n_P}\ K_o(\beta_n b) - c_{n_C}\ I_o(\beta_n b) - d_{n_P}\ K_o(\beta_n b)\right] =$$

$$\frac{1}{L}\int_{-L}^{L} \cos\ (\beta_n z)\left\{\gamma_C\ k_C \int_0^t \frac{\partial}{\partial z}\left[\Delta T_C(b,z,\tau)\right] d\tau\ - \right.$$

$$\left.\gamma_P\ k_P \int_0^t \frac{\partial}{\partial z}\left[\Delta T_P(b,z,\tau)\right] d\tau\right\} dz\ . \tag{80}$$

Now, according to the theory of Fourier series for a sine expansion in z, with $-L < z < L$, one obtains from Eq. (76)

$$\beta_n\left[c_{n_P}\ I_1(\beta_n b) - d_{n_P}\ K_1(\beta_n b) - c_{n_C}\ I_1(\beta_n b) + d_{n_C}\ K_1(\beta_n b)\right] =$$

$$\frac{1}{L}\int_{-L}^{L} \sin\ (\beta_n z)\left\{\gamma_C\ k_C \int_0^t \frac{\partial}{\partial r}\left[\Delta T_C(r,z,\tau)\right]_{r=b} d\tau\ - \right.$$

$$\left.\gamma_P\ k_P \int_0^t \frac{\partial}{\partial r}\left[\Delta T_P(r,z,\tau)\right]_{r=b} d\tau\right\} dz\ . \tag{81}$$

The simultaneous solution of Eqs. (78) through (81) for the four "constants" c_{n_P}, d_{n_P}, c_{n_C} and d_{n_C}, yields

$$c_{n_P} = \frac{\gamma_P\, k_P}{\beta_n^{\,2}\, L}\; \frac{K_1(\beta_n c)}{U_1\left[\beta_n c, \beta_n a\right]} \int_{-L}^{L} \cos(\beta_n z) \int_0^t \frac{\partial^2}{\partial r \partial z}\left[\Delta T_P(r,z,\tau)\right]_{r=a} d\tau\, dz\; -$$

$$\frac{\gamma_C\, k_C}{\beta_n^{\,2}\, L}\; \frac{K_1(\beta_n a)}{U_1\left[\beta_n c, \beta_n a\right]} \int_{-L}^{L} \cos(\beta_n z) \int_0^t \frac{\partial^2}{\partial r \partial z}\left[\Delta T_C(r,z,\tau)\right]_{r=c} d\tau\, dz\; +$$

$$\frac{K_1(\beta_n a)}{(L/b)}\; \frac{U_o\left[\beta_n c, \beta_n b\right]}{U_1\left[\beta_n c, \beta_n a\right]} \Big\{ \gamma_C\, k_C \int_{-L}^{L} \sin(\beta_n z) \int_0^t \frac{\partial}{\partial r}\left[\Delta T_C(r,z,\tau)\right]_{r=b} d\tau\, dz\; -$$

$$\gamma_P\, k_P \int_{-L}^{L} \sin\,(\beta_n z) \int_0^t \frac{\partial}{\partial r}\left[\Delta T_P(r,z,\tau)\right]_{r=b} d\tau\, dz \Big\}\; +$$

$$\frac{K_1(\beta_n a)}{(L/b)}\; \frac{U_1\left[\beta_n c, \beta_n b\right]}{U_1\left[\beta_n c, \beta_n a\right]} \Big\{ \gamma_C\, k_C \int_{-L}^{L} \cos(\beta_n z) \int_0^t \frac{\partial}{\partial z}\left[\Delta T_C(b,z,\tau)\right] d\tau\, dz\; -$$

$$\gamma_P\, k_P \int_{-L}^{L} \cos(\beta_n z) \int_0^t \frac{\partial}{\partial z}\left[\Delta T_P(b,z,\tau)\right] d\tau\, dz \Big\}\,, \tag{82}$$

$$d_{n_P} = \frac{\gamma_P\, k_P}{\beta_n^{\,2}\, L}\, \frac{I_1(\beta_n c)}{U_1\left[\beta_n c, \beta_n a\right]} \int_{-L}^{L} \cos(\beta_n z) \int_0^t \frac{\partial^2}{\partial r \partial z} \left[\Delta T_P(r,z,\tau)\right]_{r=a} d\tau\; dz -$$

$$\frac{\gamma_C\, k_C}{\beta_n^{\,2}\, L}\, \frac{I_1(\beta_n a)}{U_1\left[\beta_n c, \beta_n a\right]} \int_{-L}^{L} \cos(\beta_n z) \int_0^t \frac{\partial^2}{\partial r \partial z} \left[\Delta T_C(r,z,\tau)\right]_{r=c} d\tau\; dz +$$

$$\frac{I_1(\beta_n a)}{(L/b)}\, \frac{U_o\left[\beta_n c, \beta_n b\right]}{U_1\left[\beta_n c, \beta_n a\right]} \left\{ \gamma_C\, k_C \int_{-L}^{L} \sin\,(\beta_n z) \int_0^t \frac{\partial}{\partial r} \left[\Delta T_C(r,z,\tau)\right]_{r=b} d\tau\; dz - \right.$$

$$\left. \gamma_P\, k_P \int_{-L}^{L} \sin(\beta_n z) \int_0^t \frac{\partial}{\partial r} \left[\Delta T_P(r,z,\tau)\right]_{r=b} d\tau\; dz \right\} +$$

$$\frac{I_1(\beta_n a)}{(L/b)}\, \frac{U_1\left[\beta_n c, \beta_n b\right]}{U_1\left[\beta_n c, \beta_n a\right]} \left\{ \gamma_C\, k_C \int_{-L}^{L} \cos(\beta_n z) \int_0^t \frac{\partial}{\partial z} \left[\Delta T_C(b,z,\tau)\right] d\tau\; dz - \right.$$

$$\left. \gamma_P\, k_P \int_{-L}^{L} \cos(\beta_n z) \int_0^t \frac{\partial}{\partial z} \left[\Delta T_P(b,z,\tau)\right] d\tau\; dz \right\}, \qquad (83)$$

$$c_{n_C} = \frac{\gamma_P\, k_P}{\beta_n^{\,2}\, L} \; \frac{K_1(\beta_n c)}{U_1\left[\beta_n c, \beta_n a\right]} \int_{-L}^{L} \cos(\beta_n z) \int_0^t \frac{\partial^2}{\partial r \partial z} \left[\Delta T_P(r,z,\tau)\right]_{r=a} d\tau \; dz \; -$$

$$\frac{\gamma_C\, k_C}{\beta_n^{\,2}\, L} \; \frac{K_1(\beta_n a)}{U_1\left[\beta_n c, \beta_n a\right]} \int_{-L}^{L} \cos(\beta_n z) \int_0^t \frac{\partial^2}{\partial r \partial z} \left[\Delta T_C(r,z,\tau)\right]_{r=c} d\tau \; dz \; +$$

$$\frac{K_1(\beta_n c)}{(L/b)} \; \frac{U_o\left[\beta_n b, \beta_n a\right]}{U_1\left[\beta_n c, \beta_n a\right]} \left\{ \gamma_C\, k_C \int_{-L}^{L} \sin(\beta_n z) \int_0^t \frac{\partial}{\partial r} \left[\Delta T_C(r,z,\tau)\right]_{r=b} d\tau \; dz \; - \right.$$

$$\left. \gamma_P\, k_P \int_{-L}^{L} \sin\,(\beta_n z) \int_0^t \frac{\partial}{\partial r} \left[\Delta T_P(r,z,\tau)\right]_{r=b} d\tau \; dz \right\} +$$

$$\frac{K_1(\beta_n c)}{(L/b)} \; \frac{U_1\left[\beta_n a, \beta_n b\right]}{U_1\left[\beta_n c, \beta_n a\right]} \left\{ \gamma_C\, k_C \int_{-L}^{L} \cos(\beta_n z) \int_0^t \frac{\partial}{\partial z} \left[\Delta T_C(b,z,\tau)\right] d\tau \; dz \; - \right.$$

$$\left. \gamma_P\, k_P \int_{-L}^{L} \cos(\beta_n z) \int_0^t \frac{\partial}{\partial z} \left[\Delta T_P(b,z,\tau)\right] d\tau \; dz \right\} \tag{84}$$

and

$$d_{n_C} = \frac{\gamma_P\ k_P}{\beta_n^{\ 2}\ L}\ \frac{I_1(\beta_n c)}{U_1\left[\beta_n c, \beta_n a\right]} \int_{-L}^{L} \cos(\beta_n z) \int_0^t \frac{\partial^2}{\partial r \partial z} \left[\Delta T_P(r,z,\tau)\right]_{r=a} d\tau\ dz\ -$$

$$\frac{\gamma_C\ k_C}{\beta_n^{\ 2}\ L}\ \frac{I_1(\beta_n a)}{U_1\left[\beta_n c, \beta_n a\right]} \int_{-L}^{L} \cos(\beta_n z) \int_0^t \frac{\partial^2}{\partial r \partial z} \left[\Delta T_C(r,z,\tau)\right]_{r=c} d\tau\ dz\ +$$

$$\frac{I_1(\beta_n c)}{(L/b)}\ \frac{U_o\left[\beta_n b, \beta_n a\right]}{U_1\left[\beta_n c, \beta_n a\right]} \left\{\gamma_C\ k_C \int_{-L}^{L} \sin(\beta_n z) \int_0^t \frac{\partial}{\partial r} \left[\Delta T_C(r,z,\tau)\right]_{r=b} d\tau\ dz\ -\right.$$

$$\left.\gamma_P\ k_P \int_{-L}^{L} \sin\ (\beta_n z) \int_0^t \frac{\partial}{\partial r} \left[\Delta T_P(r,z,\tau)\right]_{r=b} d\tau\ dz\right\} +$$

$$\frac{I_1(\beta_n c)}{(L/b)}\ \frac{U_1\left[\beta_n a, \beta_n b\right]}{U_1\left[\beta_n c, \beta_n a\right]} \left\{\gamma_C\ k_C \int_{-L}^{L} \cos(\beta_n z) \int_0^t \frac{\partial}{\partial z} \left[\Delta T_C(b,z,\tau)\right] d\tau\ dz\ -\right.$$

$$\left.\gamma_P\ k_P \int_{-L}^{L} \cos(\beta_n z) \int_0^t \frac{\partial}{\partial z} \left[\Delta T_P(b,z,\tau)\right] d\tau\ dz\right\} \tag{85}$$

where the functions $U_o\left[\beta_n r_1, \beta_n r_2\right]$ and $U_1\left[\beta_n r_1, \beta_n r_2\right]$ are defined by Eqs. (66) and (67), respectively.

The thermal stresses in the propellant and the casing are then given by Eqs. (12.1) through (12.5), with $\Delta T(r,z,t)$ defined by Eq. (43); the respective partial derivatives of the displacement potential $\Phi(r,z,t)$ are given by Eqs. (69) through (73); and the constants c_{n_P}, d_{n_P}, c_{n_C}, and d_{n_C} are obtained, respectively, by Eqs. (82) through (85).

V. THE FINITE SOLID CIRCULAR CYLINDER

Consider the solid circular cylinder ($-L < z < L$, $0 \leq r < a$) as shown in Figure 2. A solution to the governing differential equation, $\nabla^2 \Phi = \gamma \ \Delta T$, is, from Eq. (26),

$$\Phi(r,z,t) = \gamma k \int_0^t \Delta T(r,z,\tau) \, d\tau + \left[A \, I_o \, (\beta r) + B \, K_o \, (\beta r)\right] \left[C \sin (\beta z) + D \cos (\beta z)\right] \quad (26a)$$

However, the displacement potential, Φ, is finite at $r = 0$, thus the constant of integration B must be equal to zero, i.e.

$$B = 0 \quad , \quad (86)$$

and Eq. (26a) becomes

$$\Phi(r,z,t) = \gamma k \int_0^t \Delta T(r,z,\tau) \, d\tau + I_o(\beta r) \left[E \sin (\beta z) + F \cos (\beta z)\right] \quad (87)$$

where E and F are constants. The boundary condition at the ends of the cylinder is

$$u(r,L,t) = u(r,-L,t) = 0 \quad , \quad (31)$$

and the condition that the net force in the axial direction is zero, requires

$$2\pi \int_0^a r \left[\sigma_{zz}(r,L,t) - \sigma_{zz}(r,-L,t)\right] dr = 0 \quad (34)$$

From Eqs. (87) and (12.3) it is found that

$$u(r,L,t) = \Phi_r(r,L,t) = \gamma k \int_0^t \frac{\partial}{\partial r}\left[\Delta T\ (r,L,\tau)\right] d\tau$$

$$+ \beta\ I_1\ (\beta r)\left[E \sin\ (\beta L) + F \cos\ (\beta L)\right] = 0 \qquad (88)$$

$$u(r,-L,t) = \Phi_r(r,-L,t) = \gamma k \int_0^t \frac{\partial}{\partial r}\left[\Delta T\ (r,-L,\tau)\right] d\tau$$

$$+ \beta\ I_1\ (\beta r)\left[-E \sin\ (\beta L) + F \cos\ (\beta L)\right] = 0 \qquad (89)$$

and

$$\int_0^a r \left\{\gamma k \int_0^t \frac{\partial^2}{\partial z^2}\left[\Delta T(r,z,\tau)\right]_{z=L} d\tau - \gamma k \int_0^t \frac{\partial^2}{\partial z^2}\left[\Delta T(r,z,\tau)\right]_{z=-L} d\tau\right.$$

$$\left. - 2E\ \beta^2 \sin\ (\beta L)\ I_o\ (\beta r) - \gamma\left[\Delta T(r,L,t) - \Delta T(r,-L,t)\right]\right\} dr = 0 \qquad (90)$$

Now, for a finite solid circular cylinder ($-L < z < L,\ o \le r < a$), with zero initial temperature, the non-steady-state temperature distribution is given, for a constant surface temperature T_S, by[2]

$$\Delta T(r,z,t) = T - T_o = T_S -$$

$$\frac{8T_S}{\pi a} \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \frac{(-1)^n\ J_o\ (r\alpha_m)}{(2n+1)\ \alpha_m\ J_1(a\alpha_m)} \cos\left[\frac{(2n+1)\pi z}{2L}\right]$$

$$e^{-kt\left[\alpha_m^2 + (2n+1)^2\ \pi^2/4L^2\right]} \qquad (91)$$

where T is the temperature in the cylinder at a point (r,z) at time t;

T_o is the zero thermal stress temperature of the cylinder; T_S is the surface temperature of the cylinder, and the α_m are the roots of the equation $J_o(a\alpha) = 0$.

Equation (91) may be written as

$$\Delta T(r,z,t) = T_S + \frac{4T_S}{aL} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{(-1)^n J_o(r\alpha_m)}{\beta_n \alpha_m J_1(a\alpha_m)} \cos(\beta_n z)\, e^{-kt(\alpha_m^2+\beta_n^2)} \tag{92}$$

where $\beta_n = \frac{(2n-1)\pi}{2L}$, for $n = 1, 2, \ldots$

Thus, from Eq. (92)

$$\Delta T(r,L,t) = \Delta T(r,-L,t) = T_S = \text{constant} \tag{93}$$

and

$$\frac{\partial^2}{\partial z^2}\Big[\Delta T(r,z,t)\Big]_{z=L} = \frac{\partial^2}{\partial z^2}\Big[\Delta T(r,\ z,t)\Big]_{z=-L} = 0 \tag{94}$$

Equations (88), (89) and (90) now become, by means of Eqs. (93) and (94),

$$\beta\, I_1(\beta r)\Big[E \sin(\beta L) + F \cos(\beta L)\Big] = 0 \tag{95}$$

$$\beta\, I_1(\beta r)\Big[-E \sin(\beta L) + F \cos(\beta L)\Big] = 0 \tag{96}$$

and

$$E\, \beta^2 \sin(\beta L) \int_0^a r\, I_o(\beta r)\, dr = 0 \ . \tag{97}$$

The temperature distribution, given by Eq. (92), is of Case I of the form

$$\Delta T\ (r,z,t) = A + \sum_{n=1}^{\infty} f(t)\ g(r)\ \cos\ (\beta_n z) \tag{39}$$

where

$$\cos\ (\beta L) = 0 \quad \text{or} \quad \beta = \beta_n = \frac{(2n-1)\pi}{2L}\ , \ \text{for}\ n = 1,\ 2,\ \ldots \tag{49}$$

and Eqs. (95) through (97) are satisfied, when

$$E = 0\ \ . \tag{98}$$

The "potential of thermoelastic displacement" $\Phi(r,z,t)$ then becomes, from Eqs. (87), (49) and (98)

$$\Phi(r,z,t) = \gamma k \int_0^t \Delta T(r,z,\tau)\ d\tau + \sum_{n=1}^{\infty} a_n\ I_o\ (\beta_n r)\ \cos\ (\beta_n z) \tag{99}$$

where $\Delta T(r,z,t)$ is given by Eq. (92), a_n is a constant to be determined from the remaining boundary conditions, and β_n is given by Eq. (49).

The boundary condition that $\sigma_{rz}(a,z,t) = 0$, which is the only additional boundary condition other than the end conditions for the particular problem under consideration here, requires that

$$\gamma k \int_0^t \frac{\partial^2}{\partial r \partial z} \Big[\Delta T(r,z,\tau)\Big]_{r=a} d\tau = \sum_{n=1}^{\infty} \beta_n^{\ 2}\ I_1\ (\beta_n a)\ a_n\ \sin\ (\beta_n z) \tag{100}$$

so that, according to the theory of Fourier series for a half-range sine expansion, one obtains

$$\beta_n^{\ 2}\ I_1\ (\beta_n a)\ a_n = \frac{2}{L} \int_0^L \sin\ (\beta_n z)\ \gamma k \int_0^t \frac{\partial^2}{\partial r \partial z} \Big[\Delta T(r,z,\tau)\Big]_{r=a} d\tau\ dz$$

or

$$a_n = \frac{2\gamma\ k}{\beta_n^{\ 2}\ L\ I_1\ (\beta_n a)} \int_0^L \sin\ (\beta_n z) \int_0^t \frac{\partial^2}{\partial r \partial z} \Big[\Delta T(r,z,\tau)\Big]_{r=a} d\tau\ dz \tag{101}$$

Now, since from Eq. (92)

$$\frac{\partial^2}{\partial r \partial z} \Big[\Delta T(r,z,\tau)\Big]_{r=a} = \frac{4T_S}{aL} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (-1)^n \sin\ (\beta_n z)\ e^{-k\tau(\alpha_m^{\ 2}+\beta_n^{\ 2})} \tag{102}$$

then

$$\int_0^L \sin\ (\beta_n z) \int_0^t \frac{\partial^2}{\partial r \partial z} \Big[\Delta T(r,z,\tau)\Big]_{r=a} d\tau\ dz$$

$$= \frac{4T_S}{aL} \int_0^L \sin\ (\beta_n z) \int_0^t \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (-1)^n \sin(\beta_n z)\ e^{-k\tau(\alpha_m^{\ 2}+\beta_n^{\ 2})} d\tau dz \tag{103}$$

And since

$$\beta_n = \frac{(2n-1)\pi}{2L}\ , \text{ for } n = 1,\ 2,\ \ldots \tag{49}$$

then the right hand side of Eq. (103), upon integration with respect to τ, becomes

$$\frac{4T_S}{aL} \int_0^L \sin\left[\frac{(2n-1)\pi z}{2L}\right] \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{(-1)^{n-1}\left[e^{-kt(\alpha_m^{\ 2}+\beta_n^{\ 2})} - 1\right]}{(\alpha_m^{\ 2} + \beta_n^{\ 2})}$$

$$\sin\left[\frac{(2n-1)\pi z}{2L}\right] dz\ . \tag{104}$$

Letting

$$x = \frac{\pi z}{2L} \tag{105}$$

then Eq. (101) becomes, by means of (104) and (105),

$$a_n = \frac{16\,\gamma\,T_S}{\beta_n^{\,2}\,aL\pi\,I_1\,(\beta_n a)}$$

$$\int_0^{\pi/2} \sin\left[(2n-1)\,x\right] \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{(-1)^{n-1}\left[e^{-kt(\alpha_m^{\,2}+\beta_n^{\,2})} - 1\right]}{(\alpha_m^{\,2}+\beta_n^{\,2})}$$

$$\sin\left[(2n-1)\,x\right] dx \tag{106}$$

However, since

$$\int_0^{\pi/2} \sin\,(mx)\,\sin\,(kx)\,dx = 0 \quad \text{for } m \neq k,\ m \text{ and } k \text{ integers} \tag{107}$$

and

$$\int_0^{\pi/2} \sin^2\left[(2n-1)\,x\right] dx = \frac{\pi}{4} \qquad \text{for } n = 1, 2, \ldots \tag{108}$$

then Eq. (106) gives

$$a_n = \frac{4\gamma T_S}{aL}\,\frac{(-1)^{n-1}}{I_1(a\,\beta_n)} \sum_{m=1}^{\infty} \frac{e^{-kt(\alpha_m^{\,2}+\beta_n^{\,2})} - 1}{(\alpha_m^{\,2}+\beta_n^{\,2})} \tag{109}$$

where the α_m are the roots of $J_o(a\alpha) = 0$, and

$\beta_n = \frac{(2n-1)\pi}{2L}$, for $n = 1, 2, \ldots$.

From Eq. (99), upon partial differentiation, one obtains

$$u(r,z,t) = \Phi_r(r,z,t) = \gamma k \int_0^t \frac{\partial}{\partial r}\left[\Delta T(r,z,\tau)\right] d\tau +$$

$$\sum_{n=1}^{\infty} \beta_n\,a_n\,I_1\,(\beta_n r)\,\cos\,(\beta_n z) \tag{110.1}$$

$$w(r,z,t) = \Phi_z(r,z,t) = \gamma k \int_0^t \frac{\partial}{\partial z}\left[\Delta T(r,z,\tau)\right] d\tau -$$

$$\sum_{n=1}^{\infty} \beta_n a_n I_o(\beta_n r) \sin(\beta_n z) \qquad (110.2)$$

$$\sigma_{rr}(r,z,t) = 2\mu(\Phi_{rr} - \gamma\Delta T) = 2\mu\left\{\gamma k \int_0^t \frac{\partial^2}{\partial r^2}\left[\Delta T(r,z,\tau)\right] d\tau -\right.$$

$$\left.\gamma \Delta T(r,z,t) + \sum_{n=1}^{\infty} \beta_n a_n \left[\beta_n I_o(\beta_n r) - \frac{1}{r} I_1(\beta_n r)\right] \cos(\beta_n z)\right\}$$

(110.3)

$$\sigma_{\theta\theta}(r,z,t) = 2\mu\left(\frac{1}{r}\Phi_r - \gamma\Delta T\right) =$$

$$2\mu\left\{\frac{\gamma k}{r} \int_0^t \frac{\partial}{\partial r}\left[\Delta T(r,z,\tau)\right] d\tau - \gamma \Delta T(r,z,t) +\right.$$

$$\left. + \frac{1}{r} \sum_{n=1}^{\infty} \beta_n a_n I_1(\beta_n r) \cos(\beta_n z)\right\} \qquad (110.4)$$

$$\sigma_{zz}(r,z,t) = 2\mu(\Phi_{zz} - \gamma \Delta T) =$$

$$2\mu\left\{\gamma k \int_0^t \frac{\partial^2}{\partial z^2}\left[\Delta T(r,z,\tau)\right] d\tau - \gamma \Delta T(r,z,t) -\right.$$

$$\left.\sum_{n=1}^{\infty} \beta_n^2 a_n I_o(\beta_n r) \cos(\beta_n z)\right\} \qquad (110.5)$$

$$\sigma_{rz}(r,z,t) = 2\mu\ \Phi_{rz} = 2\mu \left\{ \gamma k \int_0^t \frac{\partial^2}{\partial r \partial z} \left[\Delta T(r,z,\tau) \right] d\tau \right.$$

$$\left. - \sum_{n=1}^{\infty} \beta_n^{\ 2}\ a_n\ I_1\ (\beta_n r)\ \sin\ (\beta_n z) \right\} . \qquad (110.6)$$

Furthermore, from Eq. (92)

$$\gamma k \int_0^t \frac{\partial}{\partial r} \left[\Delta T(r,z,\tau) \right] d\tau = \frac{-4\gamma\ T_S}{aL} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{(-1)^{n-1}}{(\alpha_m^{\ 2} + \beta_n^{\ 2})}$$

$$\frac{J_1\ (r\alpha_m)}{\beta_n\ J_1(a\alpha_m)} \cos\ (\beta_n z) \left[e^{-kt\ (\alpha_m^{\ 2} + \beta_n^{\ 2})} - 1 \right] , \qquad (111.1)$$

$$\gamma k \int_0^t \frac{\partial}{\partial z} \left[\Delta T(r,z,\tau) \right] d\tau = \frac{-4\gamma\ T_S}{aL} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{(-1)^{n-1}}{(\alpha_m^{\ 2} + \beta_n^{\ 2})}$$

$$\frac{J_o(r\alpha_m)}{\alpha_m\ J_1(a\alpha_m)}\ \sin\ (\beta_n z) \left[e^{-kt\ (\alpha_m^{\ 2} + \beta_n^{\ 2})} - 1 \right] , \qquad (111.2)$$

$$\gamma k \int_0^t \frac{\partial^2}{\partial r \partial z} \left[\Delta T(r,z,\tau) \right] d\tau = \frac{4\gamma\ T_S}{aL} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{(-1)^{n-1}}{(\alpha_m^{\ 2} + \beta_n^{\ 2})}$$

$$\frac{J_1(r\alpha_m)}{J_1(a\alpha_m)}\ \sin\ (\beta_n z) \left[e^{-kt(\alpha_m^{\ 2} + \beta_n^{\ 2})} - 1 \right] \qquad (111.3)$$

$$\gamma k \int_0^t \frac{\partial^2}{\partial r^2}\left[\Delta T(r,z,\tau)\right] d\tau = \frac{-4\gamma\, T_S}{aL} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{(-1)^{n-1}}{(\alpha_m^{\ 2} + \beta_n^{\ 2})}$$

$$\frac{\left[\alpha_m J_o(r\alpha_m) - \frac{1}{r} J_1(r\alpha_m)\right]}{\beta_n J_1(a\alpha_m)} \cos(\beta_n z) \left[e^{-kt\,(\alpha_m^{\ 2}+\beta_n^{\ 2})} - 1\right], \tag{111.4}$$

and

$$\gamma k \int_0^t \frac{\partial^2}{\partial z^2}\left[\Delta T(r,z,\tau)\right] d\tau = \frac{-4\gamma\, T_S}{aL} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{(-1)^{n-1}}{(\alpha_m^{\ 2} + \beta_n^{\ 2})}$$

$$\frac{\beta_n J_o(r\alpha_m)}{\alpha_m J_1(a\alpha_m)} \cos(\beta_n z) \left[e^{-kt\,(\alpha_m^{\ 2}+\beta_n^{\ 2})} - 1\right]. \tag{111.5}$$

Now, from Eqs. (110.1) through (110.6), by means of Eqs. (111.1) through (111.5), Eq. (109), and Eq. (92), the thermal displacements and stresses are given by

$$u(r,z,t) = \frac{4\gamma\, T_S}{aL} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{(-1)^n}{\beta_n(\alpha_m^{\ 2} + \beta_n^{\ 2})}$$

$$\left[\frac{J_1(r\alpha_m)}{J_1(a\alpha_m)} - \frac{I_1(r\beta_n)}{I_1(a\beta_n)}\right] \cos(\beta_n z) \left[e^{-kt\,(\alpha_m^{\ 2}+\beta_n^{\ 2})} - 1\right] \tag{112}$$

$$w(r,z,t) = \frac{4\gamma\, T_S}{aL} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{(-1)^n}{(\alpha_m^{\ 2} + \beta_n^{\ 2})}$$

$$\left[\frac{J_o(r\alpha_m)}{\alpha_m\, J_1(a\alpha_m)} + \frac{I_o(r\beta_n)}{\beta_n\, I_1(a\beta_n)} \right] \sin\,(\beta_n z) \left[e^{-kt\,(\alpha_m^{\ 2} + \beta_n^{\ 2})} - 1 \right] \tag{113}$$

$$\sigma_{rr}(r,z,t) = 2\mu\ \gamma\ T_S \left\{ \frac{4}{aL} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{(-1)^{n-1}}{\beta_n\ \alpha_m(\alpha_m^{\ 2} + \beta_n^{\ 2})} \right.$$

$$\left[\frac{\frac{\alpha_m}{r} J_1(r\alpha_m) + \beta_n^{\ 2}\ J_o(r\alpha_m)}{J_1(a\alpha_m)} + \frac{\alpha_m\ \beta_n\ I_o(r\beta_n) - \frac{\alpha_m}{r} I_1(r\beta_n)}{I_1(a\beta_n)} \right]$$

$$\cos\,(\beta_n z)\ e^{-kt\,(\alpha_m^{\ 2} + \beta_n^{\ 2})} - \frac{4}{aL} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{(-1)^n}{\beta_n(\alpha_m^{\ 2} + \beta_n^{\ 2})}$$

$$\left. \left[\frac{\alpha_m\ J_o(r\alpha_m) - \frac{1}{r} J_1(r\alpha_m)}{J_1(a\alpha_m)} - \frac{\beta_n\ I_o(r\beta_n) - \frac{1}{r} I_1(r\beta_n)}{I_1(a\beta_n)} \right] \cos\,(\beta_n z) - 1 \right\} \tag{114}$$

$$\sigma_{\theta\theta}(r,z,t) = 2\mu\ \gamma\ T_S \Bigg\{ \frac{4}{aLr} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{(-1)^n}{\beta_n\ \alpha_m(\alpha_m^{\ 2} + \beta_n^{\ 2})}$$

$$\left[\frac{\alpha_m\ J_1(r\alpha_m) - (\alpha_m^{\ 2} + \beta_n^{\ 2})\ r\ J_o(r\alpha_m)}{J_1(a\alpha_m)} + \frac{\alpha_m\ I_1(r\beta_n)}{I_1(a\beta_n)} \right]$$

$$\cos\ (\beta_n z)\ e^{-kt\ (\alpha_m^{\ 2} + \beta_n^{\ 2})} - \frac{4}{aLr} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{(-1)^n}{\beta_n(\alpha_m^{\ 2} + \beta_n^{\ 2})}$$

$$\left[\frac{J_1(r\alpha_m)}{J_1(a\alpha_m)} + \frac{I_1(r\beta_n)}{I_1(a\beta_n)} \right] \cos\ (\beta_n z) - 1 \Bigg\} \tag{115}$$

$$\sigma_{zz}(r,z,t) = 2\mu\ \gamma\ T_S \Bigg\{ \frac{4}{aL} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{(-1)^{n-1}}{\beta_n(\alpha_m^{\ 2} + \beta_n^{\ 2})}$$

$$\left[\frac{\alpha_m\ J_o(r\alpha_m)}{J_1(a\alpha_m)} - \frac{\beta_n\ I_o(r\beta_n)}{I_1(a\beta_n)} \right] \cos\ (\beta_n z)\ e^{-kt(\alpha_m^{\ 2} + \beta_n^{\ 2})} -$$

$$\frac{4}{aL} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{(-1)^n}{(\alpha_m^{\ 2} + \beta_n^{\ 2})} \left[\frac{\beta_n\ J_o(r\alpha_m)}{\alpha_m\ J_1(a\alpha_m)} + \frac{I_o(r\beta_n)}{I_1(a\beta_n)} \right] \cos\ (\beta_n z) - 1 \Bigg\} \tag{116}$$

and

$$\sigma_{r\theta} = \sigma_{\theta r} = \sigma_{\theta z} = \sigma_{z\theta} = 0 \tag{117}$$

where T_S is the surface temperature, a constant; the α_m are the roots of the equation $J_o(a\alpha) = 0$; the β_n are the roots of $\cos(\beta L) = 0$, i.e. $\beta = \beta_n = \frac{(2n-1)\pi}{2L}$, for $n = 1, 2, \ldots$; and $\gamma = (3\lambda+2\mu)\alpha_T/(\lambda+2\mu)$.

VI. DISCUSSION

In order to gain a fundamental understanding of actual solid-propellant thermal stress problems, the geometry of the solid propellant has been idealized as a short, circular cylinder with flat ends. It is felt that the consideration of actual curved ends would only unduly have complicated the analysis.

The method of solution for the thermal stresses in the finite cylinder, that has been presented in this thesis, utilizes an arbitrarily selected set of cylinder end-conditions. Therefore, different end conditions than the ones employed here might have been considered just as easily.

The fundamental difficulties encountered in the thermoelastic analysis of short cylinders are that firstly, the problem is at least two-dimensional and secondly, it has mixed boundary conditions since displacements and/or stresses are specified along at least four distinct boundaries. It is relatively simple to solve the governing differential equation by the method of separation of variables. The greatest difficulties are encountered in satisfying the various boundary conditions. As a matter of fact the method of solution for the thermal stresses that has been presented in this thesis is applicable only when the temperature distribution throughout the propellant and casing exhibits a particular variation in the axial direction, as shown by

Eqs. (39) and (43). With such temperature fields, however, the elastic analytic solutions that have been presented are significant since the simultaneous linear algebraic equations, for the arbitrary constants, are easily solved. It is true that, in principle, an infinite number of these arbitrary constants must be determined. From a practical point of view, however, the arbitrary constants can always be reduced to a finite number by truncating the obtained series solutions for the thermal displacements and stresses.

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IX. VITA

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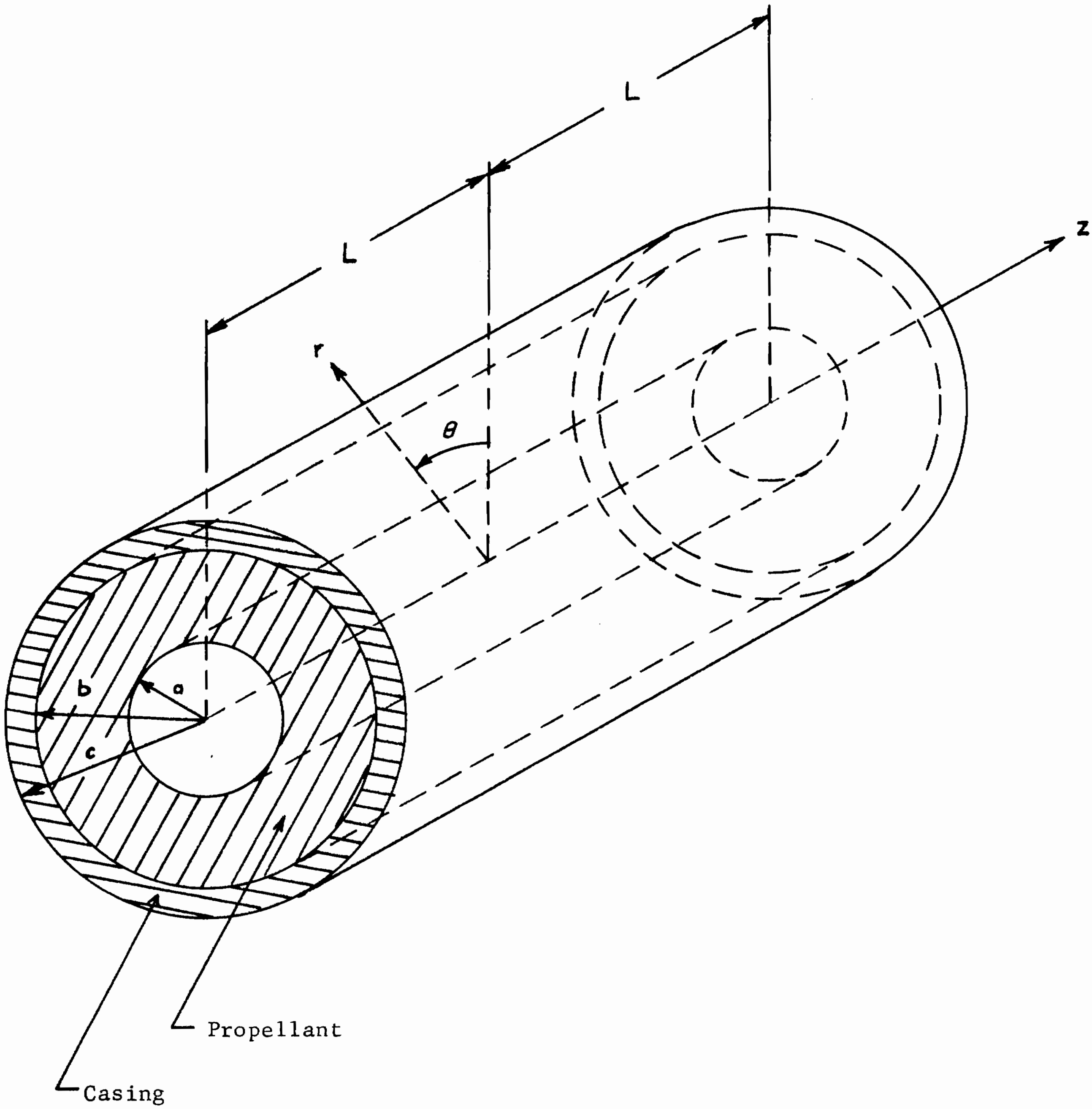
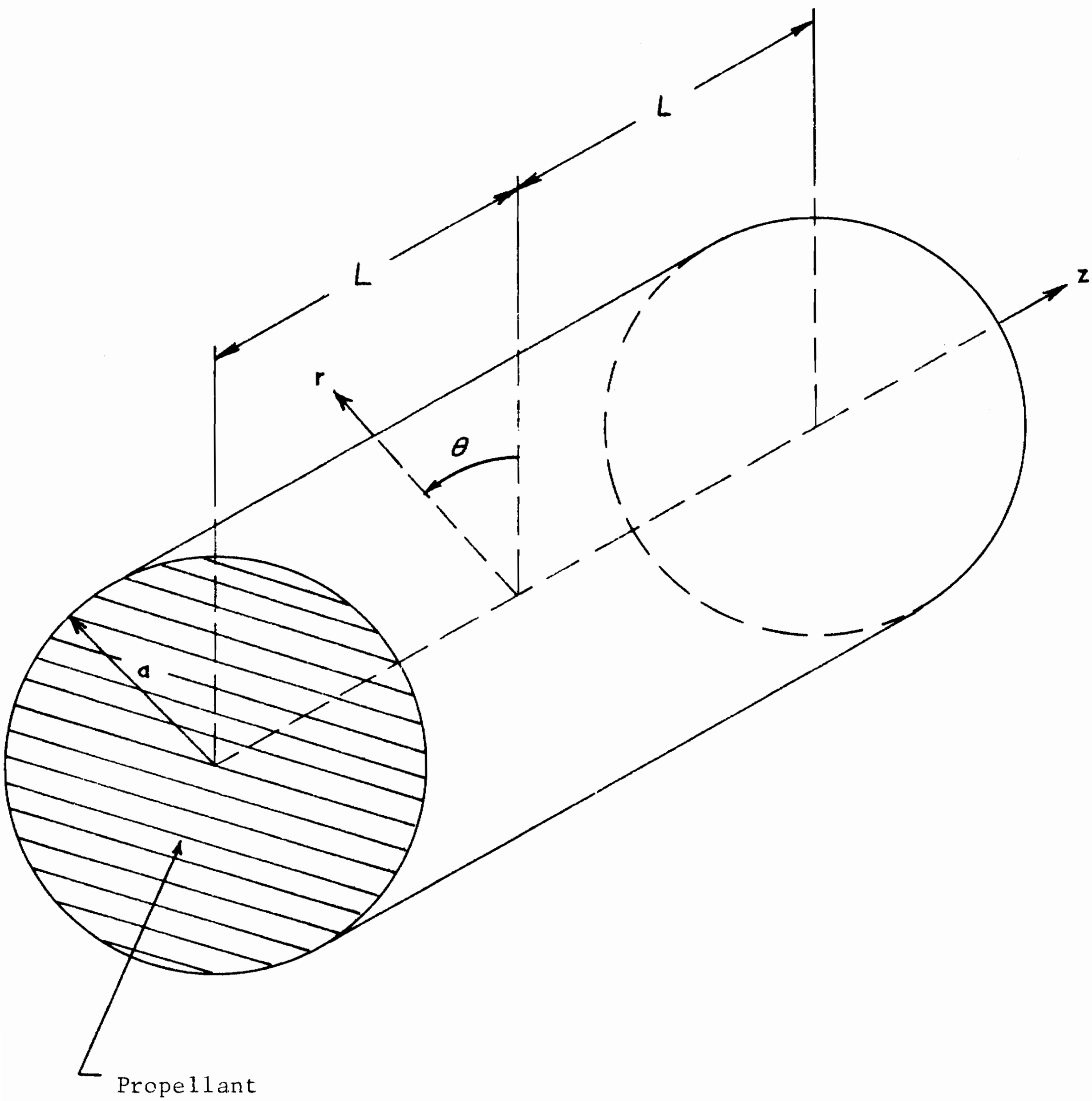
Jürgen P. Fröhlich



Finite, Hollow, Circular Cylinder

FIGURE 1



Finite, Solid, Circular Cylinder

FIGURE 2

Abstract

THERMAL STRESSES IN A FINITE SOLID-PROPELLANT GRAIN

by

Jurgen Paul Frohlich

Aerospace Engineering

This thesis considers an investigation of the thermal stresses in a case-bonded, finite, solid-propellant grain due to an axially symmetrical, unsteady temperature field. The analysis was simplified by idealizing the grain configuration to a short, circular cylinder with a concentric hole.

By means of a "potential of thermoelastic displacement", the equations of equilibrium were reduced to a single governing differential equation of the Poisson type. An arbitrary set of end conditions was chosen, and the corresponding solution for the displacement potential obtained by the method of separation of variables. Thermal displacements were directly obtained from the displacement potential. Thermal stresses were determined from the thermoelastic stress-strain relations expressed in terms of the displacement potential.