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The Decomposition of Rademacher-Walsh Spectra

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1 Introduction

Rademacher-Walsh transforms have been used by a number of authors in the area of digital design, in particular Hurst and Edwards (see [5], [4], [2] and other work of these authors). For an n-variable switching function these transforms are square matrices of order 2^n which when multiplied by 2^n entries in the table specifying the function result in a spectral vector containing 2^n spectral coefficients. Since these transforms are orthogonal it is a simple matter to multiply the spectral vector by the transform again to recover the table specifying the function (except for a constant scaling factor). However several of the design techniques involve a manipulation of the spectral vector transforming it to the spectral vector of a new function. Clearly we could generate the table specifying the function by use of the full transform again, but this is a rather extravagent representation of the function and our motivation below is to try to develop a technique which will express the function into a much simpler sum of products expression (an "on-array" for the function). This expression will not in general be minimal but will hopefully be reasonably close to this.

For our purpose we develop in sections 3 and 4 a general theorem concerning the relation between the spectrum of a given function and the spectra of all the subfunctions involved in a general Shannon decomposition of the original function. This connection itself turns out, as might be expected, to be a Rademacher-Walsh transform of the correct order .

In section 5 we develop two cost parameters for evaluating a given spectrum to decide which variables the decomposition should center upon.

The technique adopted is to successively decompose the function two variables at a time, the hope being that if the correct variables are chosen several of the resulting subfunctions will be trivial. The final section looks in detail at three examples to compare the results we obtain with minimal expressions and with the minterm expressions.

2. The Transform Background

This is not the place to go into great detail of the theory of Rademacher-Walsh transforms and their applications to logic design (see, for example [1], [3] , [5]). However we will give enough of the background material for the later sections to be intelligible.

Consider a function of n variables, $f(x_1, \ldots, x_n)$ which assumes values in 0, 1 . For this report we shall assume that f is completely specified although the techniques described below apply directly to partially specified functions as well. One way of defining f is by a table giving the value assumed by f for all combinations of values of the inputs.

For example $f(x_1, x_2, x_3)$ is defined by the table below. For a particular function $f(x_1, \ldots, x_n)$ we shall denote the 2^n entry column vector defining the function for all combinations of values of the variables by $\underset{\sim}{F}$, which will be called the specification vector for the function.

x_1	x_2	x_3	f
0	0	0	0
1	0	0	1
0	1	0	1
1	1	0	0
0	0	1	1
1	0	1	0
0	1	1	0
1	1	1	1

The orthogonal Rademacher-Walsh transforms which we are considering are defined below. Our definition is based on the Hadamard ordering, since this is the most convenient for our purpose ([10]). There are several different possible orderings for the rows of the transforms which lead to the various alternate names, though, of course, the information content is the same in all cases (see [5]).

Definition

$$\Delta_n = \begin{bmatrix} \Delta_{n-1} & \Delta_{n-1} \\ \Delta_{n-1} & -\Delta_{n-1} \end{bmatrix} \quad \text{for each } n \geq 1$$

$$\Delta_0 = [1]$$

Hence Δ_n is a $2^n \times 2^n$ matrix. The first three transforms are listed below. For convenience we shall adopt the convention of writing negative numbers as $\bar{z}$ rather than $-z$ so -1 will appear as $\bar{1}$

$$\Delta_1 = \begin{bmatrix} 1 & 1 \\ 1 & \bar{1} \end{bmatrix}$$

$$\Delta_2 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \bar{1} & 1 & \bar{1} \\ 1 & 1 & \bar{1} & \bar{1} \\ 1 & \bar{1} & \bar{1} & 1 \end{bmatrix}$$

$$\Delta_3 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \bar{1} & 1 & \bar{1} & 1 & \bar{1} & 1 & \bar{1} \\ 1 & 1 & \bar{1} & \bar{1} & 1 & 1 & \bar{1} & \bar{1} \\ 1 & \bar{1} & \bar{1} & 1 & 1 & \bar{1} & \bar{1} & 1 \\ 1 & 1 & 1 & 1 & \bar{1} & \bar{1} & \bar{1} & \bar{1} \\ 1 & \bar{1} & 1 & \bar{1} & \bar{1} & 1 & \bar{1} & 1 \\ 1 & 1 & \bar{1} & \bar{1} & \bar{1} & \bar{1} & 1 & 1 \\ 1 & \bar{1} & \bar{1} & 1 & \bar{1} & 1 & 1 & \bar{1} \end{bmatrix}$$

The nth order transform is applied to the specification vector giving a 2^n entry column vector of spectral coefficients (the "spectrum" of the function), viz

$$\underset{\sim}{R} = \Delta_n \underset{\sim}{F}$$

For convenience it is customary to code $\underset{\sim}{F}$ by replacing 0 entries by $\bar{1}$, leaving 1 entries as 1. This makes it much easier to handle "don't cares" later on.

The entries in $\underset{\sim}{R}$ measure the correlation of $\underset{\sim}{F}$ with particular variables and exclusive or functions of these variables. The entries in $\underset{\sim}{R}$ are labelled accordingly. For example in the 3-variable case

$$\underset{\sim}{R} = \begin{bmatrix} R_0 \\ R_1 \\ R_2 \\ R_{12} \\ R_3 \\ R_{13} \\ R_{23} \\ R_{123} \end{bmatrix}$$

where (i) R_0 is a measure of the number of 1's in $\underset{\sim}{F}$.

(ii) R_1 measures the correlation of the function with x_1.

(iii) R_{12} measures the correlation with $x_1 \oplus x_2$.

(iv) R_{123} measures the correlation with $x_1 \oplus x_2 \oplus x_3$.

We are presently not interested in the meaning of the spectral coefficients but in their manipulation of the information content. For more details of the above see [1], [5], [6], or [7].

Since $\Delta_n^{\,2} = 2^n \underset{\sim}{I}$ we can easily recover $\underset{\sim}{F}$ from $\underset{\sim}{R}$ by

$$\underset{\sim}{F} = 1/2^n \quad \Delta_n \underset{\sim}{R}$$

However this recovers a table specification for the function which is often a rather extravagent way to represent it. It is the aim of this report to present a method of moving directly from $\underset{\sim}{R}$ to a sum of products (on-array) form for $f(x_1, \ldots, x_n)$ without ever recovering $\underset{\sim}{F}$. The relevance of such a technique is that many of the spectral design methods involve the manipulation of $\underset{\sim}{R}$ to deduce the spectrum of a new function. From this spectrum it is necessary to calculate a specification for the function.

3. The Basic Single Variable Detachment

Consider a Shannon decomposition of a function of n variables $f(x_1, \ldots, x_n)$. We have $f(x_1, \ldots, x_n) = \bar{x}_n f(x_1, \ldots, x_{n-1}, 0) + x_n f(x_1, \ldots, x_{n-1}, 1)$

If $f(x_1, \ldots, x_n)$, $f(x_1, \ldots, x_{n-1}, 0)$, and $f(x_1, \ldots, x_{n-1}, 1)$ have specification vectors $\underset{\sim}{F}$, $\underset{\sim}{F}_0$, and $\underset{\sim}{F}_1$ respectively then clearly

$$\underset{\sim}{F} = \begin{bmatrix} \underset{\sim}{F}_0 \\ \underset{\sim}{F}_1 \end{bmatrix}$$

Now if $\underset{\sim}{R}$ is the spectral coefficient vector for $\underset{\sim}{F}$, i.e. $\underset{\sim}{R} = \Delta_n \underset{\sim}{F}$ then

$$\underset{\sim}{R} = \Delta_n \underset{\sim}{F}$$

$$= \begin{bmatrix} \Delta_{n-1} & \Delta_{n-1} \\ \Delta_{n-1} & -\Delta_{n-1} \end{bmatrix} \begin{bmatrix} \underset{\sim}{F}_0 \\ \underset{\sim}{F}_1 \end{bmatrix}$$

$$= \left[\begin{array}{c} \Delta_{n-1} \; \underset{\sim}{F}_0 \; + \; \Delta_{n-1} \; \underset{\sim}{F}_1 \\ \\ \Delta_{n-1} \; \underset{\sim}{F}_0 \; - \; \Delta_{n-1} \; \underset{\sim}{F}_1 \end{array} \right]$$

Let $\underset{\sim}{R}$ be partitioned into two equal halves each of 2^{n-1} entries, denoted $\underset{\sim}{R}_0$ and $\underset{\sim}{R}_1$ so that

$$\underset{\sim}{R} = \begin{bmatrix} \underset{\sim}{R}_0 \\ \underset{\sim}{R}_1 \end{bmatrix}$$

then

$$\left\{ \begin{array}{lcl} \underset{\sim}{R}_0 & = & \Delta_{n-1} \; \underset{\sim}{F}_0 \; + \; \Delta_{n-1} \; \underset{\sim}{F}_1 \\ \underset{\sim}{R}_1 & = & \Delta_{n-1} \; \underset{\sim}{F}_0 \; + \; \Delta_{n-1} \; \underset{\sim}{F}_1 \end{array} \right.$$

or

$$\left\{ \begin{array}{lcl} \Delta_{n-1} \; \underset{\sim}{F}_0 & = & 1/2 \; (\underset{\sim}{R}_0 + \underset{\sim}{R}_1) \\ \Delta_{n-1} \; \underset{\sim}{F}_1 & = & 1/2 \; (\underset{\sim}{R}_0 - \underset{\sim}{R}_1) \end{array} \right.$$

The two left-hand sides here are just the spectral coefficient vectors for $\underset{\sim}{F}_0$ and $\underset{\sim}{F}_1$ which will be denoted $\underset{\sim}{R}^0$ and $\underset{\sim}{R}^1$ respectively, i.e.

$$\left\{ \begin{array}{lcl} \underset{\sim}{R}^0 & = & 1/2 \; (\underset{\sim}{R}_0 + \underset{\sim}{R}_1) \\ \underset{\sim}{R}^1 & = & 1/2 \; (\underset{\sim}{R}_0 - \underset{\sim}{R}_1) \end{array} \right.$$

Let us look a little more at the nature of the arrangement of the entries in $\underset{\sim}{R}$. The purpose of using the Hadamard ordering is that it is easy to identify the R entry in any particular row from the row number, (assuming the rows are numbered from 0 to $2^n - 1$).

Given row number i we express it in binary $a_0\, a_1 \dots a_{n-1}$ where

$$i = \sum_{j=0}^{n-1} a_j \; 2^j$$

then the entry in row number i is $R_{(\alpha)}$ where (α) is a string of up to n digits subject to the constraint that (α) includes j if and only if $a_j = 1$.

For example the entry in row 5 will always be R_{13} , the entry in row 7 will be R_{123} .

In particular if the entry in row i ($i \leq 2^{n-1} - 1$) is $R_{(\alpha)}$ then the entry in row $i + 2^{n-1}$ will be $R_{(\alpha)n}$.

Hence we have

$$R^0_{(\alpha)} = 1/2 \; (R_{(\alpha)} + R_{(\alpha)n})$$

$$R^1_{(\alpha)} = 1/2 \; (R_{(\alpha)} - R_{(\alpha)n})$$

This relationship may be clarified if we give the 3-variable situation in detail. For convenience we shall often write a column vector as a row vector , the entries being separated by commas.

For the three variable function

$f(x_1, x_2, x_3) = \bar{x}_3 \, f(x_1, x_2, 0) + x_3 \, f(x_1, x_2, 1)$ with corresponding specification vectors $\underset{\sim}{F}$, $\underset{\sim}{F}_0$, and $\underset{\sim}{F}_1$ we have the three spectral coefficient vectors $\underset{\sim}{R}$, $\underset{\sim}{R}^0$, and $\underset{\sim}{R}^1$ respectively where

$$\underset{\approx}{R} = [R_0, R_1, R_2, R_{12}, R_3, R_{13}, R_{23}, R_{123}]$$

$$\underset{\approx}{R}^0 = [R_0^0, R_1^0, R_2^0, R_{12}^0]$$

$$\underset{\approx}{R}^1 = [R_0^1, R_1^1, R_2^1, R_{12}^1]$$

and the result above gives us the following:

$$R_0^0 = 1/2\ (R_0 + R_3) \qquad R_0^1 = 1/2\ (R_0 - R_3)$$

$$R_1^0 = 1/2\ (R_1 + R_{13}) \qquad R_1^1 = 1/2\ (R_1 - R_{13})$$

$$R_2^0 = 1/2\ (R_2 + R_{23}) \qquad R_2^1 = 1/2\ (R_2 - R_{23})$$

$$R_{12}^0 = 1/2\ (R_{12} + R_{123}) \qquad R_{12}^1 = 1/2\ (R_{12} - R_{123})$$

We may write the result rather more succinctly in the following form:

$$\begin{bmatrix} \underset{\approx}{R}^0 \\ \underset{\approx}{R}^1 \end{bmatrix} = \begin{bmatrix} \Delta_{n-1}\ \underset{\sim}{F}_0 \\ \Delta_{n-1}\ \underset{\sim}{F}_1 \end{bmatrix} = 1/2 \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \underset{\sim}{R}_0 \\ \underset{\sim}{R}_1 \end{bmatrix}$$

$$\text{i.e.} \quad \begin{bmatrix} \underset{\approx}{R}^0 \\ \underset{\approx}{R}^1 \end{bmatrix} = 1/2 \quad \Delta_1 \ \underset{\sim}{R}$$

as long as we remember to partition R into the correct number of equal-sized vectors.

This is an example of the more general theorem which we deduce in the next section.

4. The General Theorem

Before giving the result a little more notation is required. As before we consider a function of n variables $f(x_1, \ldots, x_n)$ with specification vector $\underset{\sim}{F}$ and spectral coefficient vector $\underset{\sim}{R}$, i.e. $\underset{\sim}{R} = \Delta_n \underset{\sim}{F}$.

We shall consider the effect of detaching m variables from $f(x_1, \ldots, x_n)$ giving a result which gives the spectral coefficient vectors for each of the 2^m functions resulting from the Shannon decomposition.

$$\begin{aligned} f(x_1, \ldots, x_n) = {} & \bar{x}_n \ldots \bar{x}_{n-m+1} f(x_1, \ldots, x_{n-m}, 0, \ldots, 0) + \\ & + \bar{x}_n \ldots \bar{x}_{n-m+2}\, x_{n-m+1} f(x_1, \ldots, x_{n-m}, 1, 0, \ldots, 0) + \cdots + \\ & + x_n \ldots x_{n-m+1} f(x_1, \ldots, x_{n-m}, 1, \ldots, 1). \end{aligned}$$

Let $M = 2^m$. $\underset{\sim}{F}$ may be partitioned into M equal column vectors, $\underset{\sim}{F} = \underset{\sim}{F}_0, \ldots, \underset{\sim}{F}_{M-1}$ where $\underset{\sim}{F}_i$ is the specification vector for $f(x_1, \ldots, x_{n-m}, a_0, a_1, \ldots, a_{m-1})$ with $i = \sum_{j=0}^{m-1} a_j \; 2^j$.

In the same fashion the spectral coefficient vector can also be partitioned into M equal column vectors, $\underset{\sim}{R} = \left[\underset{\sim}{R}_0, \ldots, \underset{\sim}{R}_{M-1}\right]$.

Finally the M functions of $n-m$ variables represented by the $\underset{\sim}{F}_i$ will each have a spectral coefficient vector denoted by $\underset{\sim}{R}^i$, viz

$$\underset{\sim}{R}^i = \Delta_{n-m} \underset{\sim}{F}_i \quad (\text{each } i, \; 0 \le i \le M-1)$$

<u>Theorem</u>

$$\begin{bmatrix} \underset{\sim}{R}^0 \\ \vdots \\ \underset{\sim}{R}^{M-1} \end{bmatrix} = 1/M \;\; \Delta_m \underset{\sim}{R}$$

where $\underset{\sim}{R}$ is partitioned into M equal-sized column vectors.

The proof of the theorem is by induction on m. The previous section has established the basis for $m = 1$. Assume the result for some positive integral value of m. We shall prove the result for $m + 1$.

Consider a function which has already had m variables detached. If we detach one further variable we shall be considering 2 M functions.

For example

$$f(x_1, \ldots, x_{n-m}, a_o, \ldots, a_{m-1})$$

$$\bar{x}_{n-m} \; f(x_1, \ldots, x_{n-m-1}, 0, a_o, \ldots, a_{m-1})$$
$$+ x_{n-m} \; f(x_1, \ldots, x_{n-m-1}, 1, a_o, \ldots, a_{m-1})$$

If $i = \sum_{j=0}^{m-1} a_j 2^j$ then we partition

$$\underset{\sim}{F}_i = \begin{bmatrix} \underset{\sim}{G}_{2i} \\ \underset{\sim}{G}_{2i+1} \end{bmatrix}$$

so that $\underset{\sim}{G}_k$ is the specification vector for

$$f(x_1, \ldots, x_{n-m-1}, b, a_0, \ldots, a_{m-1}) \text{ where}$$

$$k = 2 \sum_{j=0}^{m-1} a_j 2^j + b$$

Further let $\underset{\sim}{S}^k = \Delta_{n-m-1} \underset{\sim}{G}_k$ so that $\underset{\sim}{S}^k$ is the spectral coefficient vector for $\underset{\sim}{G}_k$.

Now consider the two functions $f(x_1, \ldots, x_{n-1}, 0)$ and $f(x_1, \ldots, x_{n-1}, 1)$ with specification vectors $\underset{\sim}{F}_0{}^*$ and $\underset{\sim}{F}_1{}^*$.

Let $\underset{\sim}{R}_0{}^* = \Delta_{n-1} \underset{\sim}{F}_0{}^*$

and $\underset{\sim}{R}_1{}^* = \Delta_{n-1} \underset{\sim}{F}_1{}^*$

Now by the induction hypothesis

$$\begin{bmatrix} \underset{\approx}{S}^0 \\ \vdots \\ \underset{\approx}{S}^{M-1} \end{bmatrix} = \frac{1}{M} \Delta_m \underset{\sim}{R}_0{}^* \quad \text{and} \quad \begin{bmatrix} \underset{\approx}{S}^M \\ \vdots \\ \underset{\approx}{S}^{2M-1} \end{bmatrix} = \frac{1}{M} \Delta_m \underset{\sim}{R}_1{}^*$$

However, by the result of the previous section

$$\begin{bmatrix} \underset{\sim}{R}_0{}^* \\ \underset{\sim}{R}_1{}^* \end{bmatrix} = \frac{1}{2} \Delta_1 \underset{\sim}{R}$$

Consequently

$$\begin{bmatrix} \underset{\sim}{S}^0 \\ \vdots \\ \underset{\sim}{S}^{2M-1} \end{bmatrix} = \frac{1}{M} \Delta_m \begin{bmatrix} \underset{\sim}{R}_0^* \\ \underset{\sim}{R}_1^* \end{bmatrix} = \frac{1}{2M} \Delta_m \begin{bmatrix} 1 & 1 \\ 1 & \bar{1} \end{bmatrix} \underset{\sim}{R}$$

$$= \frac{1}{2M} \begin{bmatrix} \Delta_m & \Delta_m \\ \Delta_m & -\Delta_m \end{bmatrix} \underset{\sim}{R}$$

$$= \frac{1}{2M} \begin{bmatrix} \Delta_{m+1} \end{bmatrix} \underset{\sim}{R}$$

which establishes the result.

5 The Decomposition Method

The result of the previous section has direct relevance to the reexpression of some function defined solely by its spectral coefficient vector $\underset{\sim}{R}$ into sum of products form.

Consider a 4-variable function $f(x_l, x_k, x_j, x_i)$

If $f(x_\ell, x_k, x_j, x_i) = \bar{x}_i \bar{x}_j f_o + \bar{x}_i x_j f_1 + x_i \bar{x}_j f_2 + x_i x_j f_3$

and f_o, f_1, f_2, f_3 have specification vectors $\underset{\sim}{F}_0, \underset{\sim}{F}_1, \underset{\sim}{F}_2, \underset{\sim}{F}_3$ then we have, using our usual notation that

$$\begin{bmatrix} \underset{\sim}{R}^0 \\ \underset{\sim}{R}^1 \\ \underset{\sim}{R}^2 \\ \underset{\sim}{R}^3 \end{bmatrix} = 1/4 \quad \Delta_2 \quad \underset{\sim}{R}$$

$$= 1/4 \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \bar{1} & 1 & \bar{1} \\ 1 & 1 & \bar{1} & \bar{1} \\ 1 & \bar{1} & \bar{1} & 1 \end{bmatrix} \begin{bmatrix} \underset{\sim}{R}_0 \\ \underset{\sim}{R}_1 \\ \underset{\sim}{R}_2 \\ \underset{\sim}{R}_3 \end{bmatrix}$$

$$= 1/4 \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \bar{1} & 1 & \bar{1} \\ 1 & 1 & \bar{1} & \bar{1} \\ 1 & \bar{1} & \bar{1} & 1 \end{bmatrix} \begin{bmatrix} R_o & R_\ell & R_k & R_{k\ell} \\ R_j & R_{j\ell} & R_{jk} & R_{jk\ell} \\ R_i & R_{i\ell} & R_{ik} & R_{ik\ell} \\ R_{ij} & R_{ij\ell} & R_{ijk} & R_{ijk\ell} \end{bmatrix}$$

Ideally in this situation we wish to detach two variables x_i, x_j to make the resulting functions $\underset{\sim}{F}_0, \ldots, \underset{\sim}{F}_4$ as simple as possible. This choice must be based upon the entries in $\underset{\sim}{R}$. We describe below the two heuristic cost parameters that have been used satisfactorily although this area is still the subject of current investigation. The above matrix given for a 4-variable problem is just as applicable for larger numbers of variables viz

$$\begin{bmatrix} \underset{\sim}{R}^0 \\ \underset{\sim}{R}^1 \\ \underset{\sim}{R}^2 \\ \underset{\sim}{R}^3 \end{bmatrix} = 1/4 \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \bar{1} & 1 & \bar{1} \\ 1 & 1 & \bar{1} & \bar{1} \\ 1 & \bar{1} & \bar{1} & 1 \end{bmatrix} \begin{bmatrix} \underset{\sim}{R}_0 \\ \underset{\sim}{R}_1 \\ \underset{\sim}{R}_2 \\ \underset{\sim}{R}_3 \end{bmatrix}$$

where $\underset{\sim}{R}_0$, $\underset{\sim}{R}_1$, $\underset{\sim}{R}_2$, $\underset{\sim}{R}_3$ will be arranged in the Hadomard ordering of entries which will then be the resulting order of entries in $\underset{\sim}{R}^0$, $\underset{\sim}{R}^1$, $\underset{\sim}{R}^2$ and $\underset{\sim}{R}^3$. This is made clear in example 3 in §6 below which is a 6 variable example.

Initially we define the cost parameters in terms of the 4-variable function $f(x_\ell, x_k, x_j, x_i)$ used above, following this with the general definition.

Cost Parameters

(a) For $f(x_\ell, x_k, x_j, x_i)$.

1. Choose x_i, x_j to maximize C_{ij} where $C_{ij} = |R_{ij}| + |R_{ij\ell}| + |R_{ijk}| + |R_{ijk\ell}|$

2. Choose x_i, x_j to maximize C_{ij}* where

$$C_{ij}^* = (|R_o| + |R_\ell| + |R_k| + |R_{k\ell}|) + 2(|R_j| + |R_{j\ell}| + |R_{jk}| + |R_{jk\ell}| + |R_i| + |R_{i\ell}| + |R_{ik}| + |R_{ik\ell}|) + 4\,C_{ij}$$

(b) For $f(x_1, .., x_n)$, to detach x_i and x_j.

1. Choose x_i, x_j to maximize C_{ij} where

$$C_{ij} = \sum_{\substack{\text{all } \alpha \\ k=0, .., n-2}} R_{ij\,(\alpha)}$$

distinct

where α any string of k digits from $\{1, .., n\}$ not including i or j.

2. Choose x_i, x_j to maximize C_{ij}^* where

$$C_{ij}^* = \sum_{\substack{\text{all } \alpha \\ k=0, .., n-2}} R_{(\alpha)} + 2 \sum_{\substack{\text{all } \beta \\ k=0, .., n-2}} (R_{i(\beta)} + R_{j(\beta)})$$

$$+ 4 \sum_{\substack{\text{all } \gamma \\ k=0, .., n-2}} R_{ij(\gamma)}$$

where α, β, γ, are strings of k distinct digits from $\{1, .., n\}$ not including i or j.

These cost parameters do have some intuitive base behind them in that high values for both C_{ij} and C_{ij}^* indicate correlation with functions of type $x_i\bar{x}_j + \bar{x}_ix_j$ or $x_ix_j + \bar{x}_i\bar{x}_j$ so it is likely that these product terms will have to appear explicitly in any final sum of products expression. Consequently we would hope to not be paying too high a penalty in imposing the partition based on detaching x_i and x_j .

The definition given above for C_{ij}^* can in practice be modified to one which is considerably easier to calculate. Since the spectral coefficients are fixed so is the sum of the absolute values, viz.

Let $S = \sum_{\substack{\text{all } \alpha \\ k=0,\ ..,\ n}} R_{(\alpha)}$

where α is a string of k distinct digits from $\{1, .., n\}$. S is fixed for any particular function.

For a particular i, j $(1 \le i, j \le n ; i \ne j)$ let

$$A = \sum_{\substack{\text{all } \alpha \\ k=0,\ ..,\ n-2}} R_{(\alpha)}$$

$$B_{ij} = \sum_{\substack{\text{all } \beta \\ k=0,\ ..,\ n-2}} R_{i(\beta)} + R_{j(\beta)}$$

where α, β are strings of k distinct digits from $\{1, .., n\}$ not including i or j.

Then

$$C^*_{ij} = A_{ij} + 2B_{ij} + 4C_{ij}$$

However $A_{ij} + B_{ij} + C_{ij} = S$

Hence $C^*_{ij} = 2S + 2C_{ij} - A_{ij}$

Since S is fixed for the particular function we can use $2C_{ij} - A_{ij}$ as our C^*_{ij} cost parameter;

<u>Definition</u>

$$C^*_{ij} = 2 \sum_{\substack{\text{all } \alpha \\ k=0,\ ..,\ n-2}} R_{ij\ (\alpha)} - \sum_{\substack{\text{all } \beta \\ k=0,\ ..,\ n-2}} R_{(\beta)}$$

where α, β strings of k distinct digits from $\{1, .., n\}$ not including i or j.

This simplifies the calculation of C^*_{ij} since it now involves only half the spectral coefficients whereas formerly it involved all of them.

The method we are using always detaches two variables at a time though a similar approach could be used to detach a higher number of variables. From the examples we have investigated this does not appear to be such a good general approach, it being preferable to detach the variables two at a time.

The resulting method is consequently to detach two variables determined by the cost parameter and evaluate the spectral coefficient vectors for the resulting $\underset{\sim}{F}_i$ functions. The method is then repeated for each of the $\underset{\sim}{F}_i$ functions to successively detach all but two of the variables. In practice we would hope that a number of the $\underset{\sim}{F}_i$ functions are either 0 or 1. A number of examples are considered in the next section.

§ 6. Examples

Two examples will be considered in detail, both being 4-variable examples. For each of them we shall look at all 6 possible pairs of variables to detach and compare the results obtained with our cost parameters and with both the minimal sum of products form and the minterm form. Since we are only considering sum of product forms we shall not allow XOR functions even when these display obvious advantages. Further the sum of products cost we use will be the direct one without any further simplification of $\underset{\sim}{F}_1$, $\underset{\sim}{F}_2$, $\underset{\sim}{F}_3$, or $\underset{\sim}{F}_4$ even though on some occasions this is trivially obvious. The cost function used to compare the results will be one of the common switching theory cost functions namely the number of terms plus the number of literals.

Example 6.1

$$f(x_1, x_2, x_3, x_4) = x_3x_4 + \bar{x}_2x_4 + x_2\bar{x}_3\bar{x}_4 + \bar{x}_1x_2x_3$$

which is a minimal form for the function, with a cost of 14.

The function is illustrated on the map below. Its minterm cost is 45.

We shall assume that the variables being detached in each case below are x_i and x_j in the form

$$f(x_\ell, x_k, x_j, x_i) = \bar{x}_i\bar{x}_jf_o + \bar{x}_ix_jf_1 + x_i\bar{x}_jf_2 + x_ix_jf_3$$

with corresponding specification vectors $\underset{\sim}{F}_0$, $\underset{\sim}{F}_1$, $\underset{\sim}{F}_2$, $\underset{\sim}{F}_3$ and corresponding spectral coefficient vectors $\underset{\sim}{R}^0$, $\underset{\sim}{R}^1$, $\underset{\sim}{R}^2$, $\underset{\sim}{R}^3$ respectively.

$x_1 x_2 \backslash x_3 x_4$	00	01	11	10
00	0	1	1	0
01	1	0	1	1
11	1	0	1	0
10	0	1	1	0

We have

$\underset{\sim}{F} = \left[\bar{1}, 1, \bar{1}, 1, 1, \bar{1}, 1, 1, \bar{1}, 1, \bar{1}, 1, 1, \bar{1}, \bar{1}, 1\right]$ recalling that 0 is coded as $\bar{1}$.

The spectral coefficient vector

$$\underset{\sim}{R} = \left[R_0, R_1, R_2, R_{12}, R_3, R_{13}, R_{23}, R_{123}, R_4, R_{14}, R_{24}, R_{124}, R_{34}, R_{134}, R_{234}, R_{1234}\right]$$

$$= \left[2, 2, \bar{2}, \bar{2}, \bar{2}, \bar{2}, 2, 2, \bar{6}, 2, \overline{10}, \bar{2}, 6, \bar{2}, \bar{6}, 2\right]$$

a) If $i = 1$, $j = 2$.

$$\begin{bmatrix} \underset{\sim}{R}^0 \\ \underset{\sim}{R}^1 \\ \underset{\sim}{R}^2 \\ \underset{\sim}{R}^3 \end{bmatrix} = 1/4 \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \bar{1} & 1 & \bar{1} \\ 1 & 1 & \bar{1} & \bar{1} \\ 1 & \bar{1} & \bar{1} & 1 \end{bmatrix} \begin{bmatrix} R_0 & R_3 & R_4 & R_{34} \\ R_2 & R_{23} & R_{24} & R_{234} \\ R_1 & R_{13} & R_{14} & R_{134} \\ R_{12} & R_{123} & R_{124} & R_{1234} \end{bmatrix}$$

$$= 1/4 \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \bar{1} & 1 & \bar{1} \\ 1 & 1 & \bar{1} & \bar{1} \\ 1 & \bar{1} & \bar{1} & 1 \end{bmatrix} \begin{bmatrix} 2 & \bar{2} & \bar{6} & 6 \\ \bar{2} & 2 & \overline{10} & \bar{6} \\ 2 & \bar{2} & 2 & \bar{2} \\ \bar{2} & 2 & \bar{2} & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & \bar{4} & 0 \\ 2 & \bar{2} & 2 & 2 \\ 0 & 0 & \bar{4} & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

so that
$$\underset{\sim}{R}^0 = (0, 0, \bar{4}, 0)$$
$$\underset{\sim}{R}^1 = (2, \bar{2}, 2, 2)$$
$$\underset{\sim}{R}^2 = (0, 0, \bar{4}, 0)$$
$$\underset{\sim}{R}^3 = (0, 0, 0, 4)$$

Most of the two-variable spectra are immediately recognizable though to evaluate the functions in detail we use

$$\underset{\sim}{F}_0 = 1/4 \ \Delta_2 \quad \underset{\sim}{R}^0 = (\bar{1}, \bar{1}, 1, 1)$$

$$\text{so} \quad f_0 = x_4$$

similarly $f_1 = x_3 + \bar{x}_4$

$$f_2 = x_4$$

$$f_3 = x_3 x_4 + \bar{x}_3 \bar{x}_4$$

so that

$$f(x_1, x_2, x_3, x_4) = \bar{x}_1 \bar{x}_2 x_4 + \bar{x}_1 x_2 (x_3 + \bar{x}_4) + x_1 \bar{x}_2 x_4 + x_1 x_2 (x_3 x_4 + \bar{x}_3 \bar{x}_4)$$

which has a cost of 26.

b) $i = 1, j = 3$

$$\begin{bmatrix} \underset{\sim}{R}^0 \\ \underset{\sim}{R}^1 \\ \underset{\sim}{R}^2 \\ \underset{\sim}{R}^3 \end{bmatrix} = 1/4 \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \bar{1} & 1 & \bar{1} \\ 1 & 1 & \bar{1} & \bar{1} \\ 1 & \bar{1} & \bar{1} & 1 \end{bmatrix} \begin{bmatrix} 2 & \bar{2} & \bar{6} & \overline{10} \\ \bar{2} & 2 & 6 & \bar{6} \\ 2 & \bar{2} & 2 & \bar{2} \\ \bar{2} & 2 & \bar{2} & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & \bar{4} \\ 2 & \bar{2} & \bar{2} & \bar{2} \\ 0 & 0 & 0 & \bar{4} \\ 0 & 0 & \bar{4} & 0 \end{bmatrix}$$

Hence
$$f_0 = x_2 \bar{x}_4 + \bar{x}_2 x_4$$
$$f_1 = x_2 + x_4$$
$$f_2 = x_2 \bar{x}_4 + \bar{x}_2 x_4$$
$$f_3 = x_4$$

so $f = \bar{x}_1 \bar{x}_3(x_2 \bar{x}_4 + \bar{x}_2 x_4) + \bar{x}_1 x_3 (x_2 + x_4) + x_1 \bar{x}_3 (x_2 \bar{x}_4 + \bar{x}_2 x_4) + x_1 x_3 x_4$

which has a cost of 32, though in practice $f_0 = f_2$ would be noticed leading to an obvious simplification.

c) $i = 1,\ j = 4$

$$\begin{bmatrix} \underset{\sim}{R}^0 \\ \underset{\sim}{R}^1 \\ \underset{\sim}{R}^2 \\ \underset{\sim}{R}^3 \end{bmatrix} = 1/4 \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \bar{1} & 1 & \bar{1} \\ 1 & 1 & \bar{1} & \bar{1} \\ 1 & \bar{1} & \bar{1} & 1 \end{bmatrix} \begin{bmatrix} 2 & \bar{2} & \bar{2} & 2 \\ \bar{6} & \overline{10} & 6 & \bar{6} \\ 2 & \bar{2} & \bar{2} & 2 \\ 2 & \bar{2} & \bar{2} & 2 \end{bmatrix} = \begin{bmatrix} 0 & \bar{4} & 0 & 0 \\ 2 & 2 & \bar{2} & 2 \\ \bar{2} & \bar{2} & 2 & \bar{2} \\ 2 & 2 & \bar{2} & 2 \end{bmatrix}$$

Hence

$$f_0 = x_2$$

$$f_1 = \bar{x}_2 + x_3$$

$$f_2 = x_2\,\bar{x}_3$$

$$f_3 = \bar{x}_2 + x_3$$

so $f = \bar{x}_1\,\bar{x}_4\,x_2 + \bar{x}_1\,x_4\,(\bar{x}_2 + x_3) + x_1\,\bar{x}_4\,x_2\,\bar{x}_3 + x_1\,x_4\,(\bar{x}_2 + x_3)$

which has a cost of 25.

d) $i = 2,\ j = 3$

$$\begin{bmatrix} \underset{\sim}{R}^0 \\ \underset{\sim}{R}^1 \\ \underset{\sim}{R}^2 \\ \underset{\sim}{R}^3 \end{bmatrix} = 1/4 \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \bar{1} & 1 & \bar{1} \\ 1 & 1 & \bar{1} & \bar{1} \\ 1 & \bar{1} & \bar{1} & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 & \bar{6} & 2 \\ \bar{2} & \bar{2} & 6 & \bar{2} \\ \bar{2} & \bar{2} & \overline{10} & \bar{2} \\ 2 & 2 & \bar{6} & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & \bar{4} & 0 \\ 0 & 0 & \bar{4} & 0 \\ 0 & 0 & 4 & 0 \\ 2 & 2 & \bar{2} & 2 \end{bmatrix}$$

Hence

$$f_0 = x_4$$

$$f_1 = x_4$$

$$f_2 = \bar{x}_4$$

$$f_3 = \bar{x}_1 + x_4$$

so $f = \bar{x}_2\,\bar{x}_3\,x_4 + \bar{x}_2\,x_3\,x_4 + x_2\,\bar{x}_3\,\bar{x}_4 + x_2\,x_3\,(\bar{x}_1 + x_4)$

which has a cost of 20.

e) $i = 2,\ j = 4$

$$\begin{bmatrix} \underset{\sim}{R}^0 \\ \underset{\sim}{R}^1 \\ \underset{\sim}{R}^2 \\ \underset{\sim}{R}^3 \end{bmatrix} = 1/4 \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \bar{1} & 1 & \bar{1} \\ 1 & 1 & \bar{1} & \bar{1} \\ 1 & \bar{1} & \bar{1} & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 & \bar{2} & \bar{2} \\ \bar{6} & 2 & 6 & \bar{2} \\ \bar{2} & \bar{2} & 2 & 2 \\ \overline{10} & \bar{2} & \bar{6} & 2 \end{bmatrix} = \begin{bmatrix} \bar{4} & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 \\ 2 & 2 & 2 & \bar{2} \\ 0 & 0 & \bar{4} & 0 \end{bmatrix}$$

Hence

$$f_0 = 0$$

$$f_1 = 1$$

$$f_2 = \bar{x}_1 + \bar{x}_3$$

$$f_3 = x_3$$

so $f = \bar{x}_2\, x_4 + x_2\, \bar{x}_4\, (\bar{x}_1 + \bar{x}_3) + x_2\, x_4\, x_3$

which has a cost of 15.

f) $i = 3,\ j = 4$

$$\begin{bmatrix} \underset{\sim}{R}^0 \\ \underset{\sim}{R}^1 \\ \underset{\sim}{R}^2 \\ \underset{\sim}{R}^3 \end{bmatrix} = 1/4 \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \bar{1} & 1 & \bar{1} \\ 1 & 1 & \bar{1} & \bar{1} \\ 1 & \bar{1} & \bar{1} & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 & \bar{2} & \bar{2} \\ \bar{6} & 2 & \overline{10} & \bar{2} \\ \bar{2} & \bar{2} & 2 & 2 \\ 6 & \bar{2} & \bar{6} & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & \bar{4} & 0 \\ 0 & 0 & 4 & 0 \\ \bar{2} & 2 & \bar{2} & \bar{2} \\ 4 & 0 & 0 & 0 \end{bmatrix}$$

Hence

$$f_0 = x_2$$

$$f_1 = \bar{x}_2$$

$$f_2 = \bar{x}_1\, x_2$$

$$f_3 = 1$$

so $f = \bar{x}_3\, \bar{x}_4\, x_2 + \bar{x}_3\, x_4\, \bar{x}_2 + x_3\, \bar{x}_4\, \bar{x}_1\, x_2 + x_3\, x_4$

which has a cost of 16.

Let us compare the results obtained for the six possible choices of variables detached.

variables detached	C_{ij}	C_{ij}^*	cost of expression
x_1, x_2	8	0	26
x_1, x_3	8	4	32
x_1, x_4	8	8	25
x_2, x_3	12	12	20
x_2, x_4	20	32	15
x_3, x_4	16	24	16
minimal expression cost			14
minterm expression cost			45

The table shows that, at least for this one example the cost parameters used are a very reasonable reflection of the cost of the resulting expression, with C_{ij}^* being rather more discriminating. Further we have expressions in both (e) and (f) that are reasonably close to the minimal sum of products expression for the function.

Example 6.2

This example does not have quite so much simplification available as example 6.1.

$f(x_1, x_2, x_3, x_4) = x_3 x_4 + \bar{x}_1 \bar{x}_2 x_4 + \bar{x}_1 x_2 \bar{x}_4 + x_1 x_2 x_4 +$

$x_1 \bar{x}_2 \bar{x}_3 \bar{x}_4$ is a minimal form for the function with a cost of 20. The minterm cost for the expression is 45. The function is illustrated in in the map below.

$x_1 x_2$ \ $x_3 x_4$	00	01	11	10
00	0	1	1	0
01	1	0	1	1
11	0	1	1	0
10	1	0	1	0

For this function $F = [\,\bar{1}, 1, \bar{1}, 1, 1, \bar{1}, 1, 1, 1, \bar{1}, \bar{1}, 1, \bar{1}, 1, \bar{1}, 1\,]$

and $\underset{\sim}{R} = [R_0, R_1, R_2, R_{12}, R_3, R_{13}, R_{23}, R_{123}, R_4, R_{14}, R_{24}, R_{124}, R_{34}, R_{134},$

$R_{234}, R_{1234}]$

$= [2, 2, \bar{2}, \bar{2}, \bar{2}, \bar{2}, 2, 2, \bar{6}, 2, \bar{2}, \overline{10}, 6, \bar{2}, 2, \bar{6}]$

The table of our cost parameters is given below

variables detached	C_{ij}	C^*_{ij}
x_1, x_2	20	24
x_1, x_3	12	4
x_1, x_4	20	32
x_2, x_3	12	12
x_2, x_4	20	32
x_3, x_4	16	24

In this case C_{ij} gives no discrimination between three possible pairs while C^*_{ij} still does not discriminate C^*_{14} from C^*_{24}. This is to be expected since an interchange of $\bar{x}_1$ and x_2 in the minimal form leaves the function unchanged.

As in example 1 we shall consider all 6 possible detachments in order to compare the results.

(a) $i = 1,\ j = 2$

$$\begin{bmatrix} \underset{\sim}{R}^0 \\ \underset{\sim}{R}^1 \\ \underset{\sim}{R}^2 \\ \underset{\sim}{R}^3 \end{bmatrix} = 1/4 \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \bar{1} & 1 & \bar{1} \\ 1 & 1 & \bar{1} & \bar{1} \\ 1 & \bar{1} & \bar{1} & 1 \end{bmatrix} \begin{bmatrix} 2 & \bar{2} & \bar{6} & 6 \\ \bar{2} & 2 & \bar{2} & 2 \\ 2 & \bar{2} & 2 & \bar{2} \\ \bar{2} & 2 & \overline{10} & \bar{6} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \bar{4} & 0 \\ 2 & \bar{2} & 2 & 2 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & \bar{4} & 0 \end{bmatrix}$$

so

$$f_0 = x_4$$
$$f_1 = x_3 + \bar{x}_4$$
$$f_2 = x_3\, x_4 + \bar{x}_3\, \bar{x}_4$$
$$f_3 = x_4$$

so $$f = \bar{x}_1\, \bar{x}_2\, x_4 + \bar{x}_1\, x_2(x_3 + \bar{x}_4) + x_1\, \bar{x}_2(x_3\, x_4 + \bar{x}_3\, \bar{x}_4) + x_1\, x_2\, x_4$$

which has a cost of 26.

(b) $i = 1,\ j = 3$

$$\begin{bmatrix} \underset{\sim}{R}^0 \\ \underset{\sim}{R}^1 \\ \underset{\sim}{R}^2 \\ \underset{\sim}{R}^3 \end{bmatrix} = 1/4 \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \bar{1} & 1 & \bar{1} \\ 1 & 1 & \bar{1} & \bar{1} \\ 1 & \bar{1} & \bar{1} & 1 \end{bmatrix} \begin{bmatrix} 2 & \bar{2} & \bar{6} & \bar{2} \\ \bar{2} & 2 & 6 & 2 \\ 2 & \bar{2} & 2 & \overline{10} \\ \bar{2} & 2 & \bar{2} & \bar{6} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & \bar{4} \\ 2 & \bar{2} & \bar{2} & \bar{2} \\ 0 & 0 & 0 & 4 \\ 0 & 0 & \bar{4} & 0 \end{bmatrix}$$

$$f_0 = x_2\,\bar{x}_4 + \bar{x}_2\,x_4$$

$$f_1 = x_2 + x_4$$

so

$$f_2 = x_2\,x_4 + \bar{x}_2\,\bar{x}_4$$

$$f_3 = x_4$$

so $f = \bar{x}_1\,\bar{x}_3\,(x_2\,\bar{x}_4 + \bar{x}_2\,x_4) + \bar{x}_1\,x_3\,(x_2 + x_4) + x_1\,\bar{x}_3(x_2\,x_4 + \bar{x}_2\,\bar{x}_4) + x_1\,x_3\,x_4$ which has a cost of 32.

(c) $i = 1,\ j = 4$

$$\begin{bmatrix} \underset{\sim}{R}^0 \\ \underset{\sim}{R}^1 \\ \underset{\sim}{R}^2 \\ \underset{\sim}{R}^3 \end{bmatrix} = 1/4 \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \bar{1} & 1 & \bar{1} \\ 1 & 1 & \bar{1} & \bar{1} \\ 1 & \bar{1} & \bar{1} & 1 \end{bmatrix} \begin{bmatrix} 2 & \bar{2} & \bar{2} & 2 \\ \bar{6} & \bar{2} & 6 & 2 \\ 2 & \bar{2} & \bar{2} & 2 \\ 2 & \overline{10} & \bar{2} & \bar{6} \end{bmatrix} = \begin{bmatrix} 0 & \bar{4} & 0 & 0 \\ 2 & 2 & \bar{2} & 2 \\ \bar{2} & 2 & 2 & 2 \\ 2 & \bar{2} & \bar{2} & \bar{2} \end{bmatrix}$$

$$f_0 = x_2$$

$$f_1 = \bar{x}_2 + x_3$$

so

$$f_2 = \bar{x}_2\,\bar{x}_3$$

$$f_3 = x_2 + x_3$$

so $f = \bar{x}_1 \bar{x}_4 x_2 + \bar{x}_1 x_4(\bar{x}_2 + x_3) + x_1 \bar{x}_4 \bar{x}_2 \bar{x}_3 + x_1 x_4(x_2 + x_3)$

which has a cost of 25.

(d) $i = 2, j = 3$

$$\begin{bmatrix} \underset{\sim}{R}^0 \\ \underset{\sim}{R}^1 \\ \underset{\sim}{R}^2 \\ \underset{\sim}{R}^3 \end{bmatrix} = 1/4 \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \bar{1} & 1 & \bar{1} \\ 1 & 1 & \bar{1} & \bar{1} \\ 1 & \bar{1} & \bar{1} & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 & \bar{6} & 2 \\ \bar{2} & \bar{2} & 6 & \bar{2} \\ \bar{2} & \bar{2} & \bar{2} & \overline{10} \\ 2 & 2 & 2 & \bar{6} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & \bar{4} \\ 0 & 0 & \bar{4} & 0 \\ 0 & 0 & 0 & 4 \\ 2 & 2 & \bar{2} & 2 \end{bmatrix}$$

so

$$f_0 = \bar{x}_1 x_4 + x_1 \bar{x}_4$$

$$f_1 = x_4$$

$$f_2 = x_1 x_4 + \bar{x}_1 \bar{x}_4$$

$$f_3 = \bar{x}_1 + x_4$$

so $f = \bar{x}_2 \bar{x}_3 (\bar{x}_1 x_4 + x_1 \bar{x}_4) + \bar{x}_2 x_3 x_4 + x_2 \bar{x}_3(x_1 x_4 + \bar{x}_1 \bar{x}_4) +$
$x_2 x_3(\bar{x}_1 + x_4)$ which has a cost of 32.

(e) $i = 2,\ j = 4$

$$\begin{bmatrix} \underset{\sim}{R}^0 \\ \underset{\sim}{R}^1 \\ \underset{\sim}{R}^2 \\ \underset{\sim}{R}^3 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \bar{1} & 1 & \bar{1} \\ 1 & 1 & \bar{1} & \bar{1} \\ 1 & \bar{1} & \bar{1} & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 & \bar{2} & \bar{2} \\ \bar{6} & 2 & 6 & \bar{2} \\ \bar{2} & \bar{2} & 2 & 2 \\ \bar{2} & \overline{10} & 2 & \bar{6} \end{bmatrix} = \begin{bmatrix} \bar{2} & \bar{2} & 2 & \bar{2} \\ 2 & 2 & \bar{2} & 2 \\ 0 & 4 & 0 & 0 \\ 2 & \bar{2} & \bar{2} & \bar{2} \end{bmatrix}$$

so

$$f_0 = x_1 \bar{x}_3$$
$$f_1 = \bar{x}_1 + x_3$$
$$f_2 = \bar{x}_1$$
$$f_3 = x_1 + x_3$$

so $\quad f = \bar{x}_2 \bar{x}_4 x_1 \bar{x}_3 + \bar{x}_2 x_4 (\bar{x}_1 + x_3) + x_2 \bar{x}_4 \bar{x}_1 + x_2 x_4 (x_1 + x_3)$

which has a cost of 25.

(f) $i = 3,\ j = 4$

$$\begin{bmatrix} \underset{\sim}{R}^0 \\ \underset{\sim}{R}^1 \\ \underset{\sim}{R}^2 \\ \underset{\sim}{R}^3 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \bar{1} & 1 & \bar{1} \\ 1 & 1 & \bar{1} & \bar{1} \\ 1 & \bar{1} & \bar{1} & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 & \bar{2} & \bar{2} \\ \bar{6} & 2 & \bar{2} & \overline{10} \\ \bar{2} & \bar{2} & 2 & 2 \\ \bar{6} & \bar{2} & 2 & \bar{6} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & \bar{4} \\ 0 & 0 & 0 & 4 \\ \bar{2} & 2 & \bar{2} & \bar{2} \\ 4 & 0 & 0 & 0 \end{bmatrix}$$

so

$$f_0 = x_1 \bar{x}_2 + \bar{x}_1 x_2$$
$$f_1 = x_1 x_2 + \bar{x}_1 \bar{x}_2$$
$$f_2 = \bar{x}_1 x_2$$
$$f_3 = 1$$

so $\quad f = \bar{x}_3 \bar{x}_4 (x_1 \bar{x}_2 + \bar{x}_1 x_2) + \bar{x}_3 x_4 (x_1 x_2 + \bar{x}_1 \bar{x}_2) + x_3 \bar{x}_4 \bar{x}_1 x_2 +$

$x_3 x_4$ which has a cost of 28.

The results in the six cases are given in the table below, the cost parameters we have used again appearing reasonable although we have not approached as close to the minimal sum of products expression. In practice a method would identify equal and closely related subfunctions, for example in (f) above $f_0 = \bar{f}_1$ which is obvious from its spectrum.

variables detached	C_{ij}	C^*_{ij}	cost of expansion
x_1, x_2	20	24	26
x_1, x_3	12	4	32
x_1, x_4	20	32	25
x_2, x_3	12	12	32
x_2, x_4	20	32	25
x_3, x_4	16	24	28
minimal expression cost			20
minterm expression cost			45

Example 6.3

This is a larger six-variable example and we shall only consider the solution suggested by our method and compare it with both the minterm realization and a minimal realization.

The function $f(x_1, x_2, x_3, x_4, x_5, x_6)$ has a spectrum containing 64 coefficients which are listed below. The method is very tedious when described in the detail given here, but is of course perfectly amenable to a computer treatment which is the normal method for such problems.

Spectral Coefficients:

$R_{0} = 2$
$R_{1} = 2$
$R_{2} = \overline{10}$
$R_{3} = \overline{2}$
$R_{4} = \overline{2}$
$R_{5} = 2$
$R_{6} = \overline{2}$

$R_{12} = \overline{2}$
$R_{13} = \overline{10}$
$R_{14} = \overline{2}$
$R_{15} = 2$
$R_{16} = 6$
$R_{23} = 10$
$R_{24} = \overline{6}$
$R_{25} = \overline{2}$
$R_{26} = 2$
$R_{34} = 2$
$R_{35} = \overline{2}$
$R_{36} = \overline{6}$
$R_{45} = \overline{18}$
$R_{46} = \overline{30}$
$R_{56} = \overline{10}$

$R_{123} = 10$
$R_{124} = \overline{14}$
$R_{125} = 6$
$R_{126} = 2$
$R_{134} = 10$
$R_{135} = \overline{10}$
$R_{136} = \overline{6}$
$R_{145} = 14$
$R_{146} = \overline{6}$
$R_{156} = \overline{2}$
$R_{234} = 6$
$R_{235} = 2$
$R_{236} = 6$
$R_{245} = 2$
$R_{246} = \overline{2}$
$R_{256} = 2$
$R_{345} = 18$
$R_{346} = \overline{26}$
$R_{356} = 2$
$R_{456} = 10$

$R_{123456} = 2$
$R_{12345} = \overline{2}$
$R_{12346} = 2$
$R_{12356} = \overline{2}$
$R_{12456} = \overline{2}$
$R_{13456} = \overline{2}$
$R_{23456} = \overline{6}$

$R_{1234} = 6$
$R_{1235} = 2$
$R_{1236} = \overline{2}$
$R_{1245} = \overline{6}$
$R_{1246} = \overline{2}$
$R_{1256} = 2$
$R_{1345} = \overline{6}$
$R_{1346} = 6$
$R_{1356} = 2$
$R_{1456} = 2$
$R_{2345} = \overline{2}$
$R_{2346} = \overline{6}$
$R_{2356} = 6$
$R_{2456} = \overline{2}$
$R_{3456} = \overline{2}$

From this spectrum we can calculate the following cost parameters for pairs of variables which could be detached.

variables detached	C_{ij}	C_{ij}^{*}
x_1, x_2	60	$\overline{16}$
x_1, x_3	80	56
x_1, x_4	84	100
x_1, x_5	64	8
x_1, x_6	48	8
x_2, x_3	72	32
x_2, x_4	68	68
x_2, x_5	48	$\overline{24}$
x_2, x_6	48	$\overline{8}$
x_3, x_4	104	152
x_3, x_5	70	48
x_3, x_6	84	76
x_4, x_5	96	112
x_4, x_6	108	140
x_5, x_6	56	16

Based on these cost parameters we shall detach x_3 and x_4, putting $i = 3$ and $j = 4$ in the result given previously, viz if $f(x_1, x_2, x_3, x_4, x_5, x_6) = \bar{x}_3 \bar{x}_4 f_0 + \bar{x}_3 x_4 f_1 + x_3 \bar{x}_4 f_2 + x_3 x_4 f_3$ then we have:

$$\begin{bmatrix} \underset{\sim}{R}^0 \\ \underset{\sim}{R}^1 \\ \underset{\sim}{R}^2 \\ \underset{\sim}{R}^3 \end{bmatrix} = 1/4 \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \bar{1} & 1 & \bar{1} \\ 1 & 1 & \bar{1} & \bar{1} \\ 1 & \bar{1} & \bar{1} & 1 \end{bmatrix} \begin{bmatrix} \underset{\sim}{R}_0 \\ \underset{\sim}{R}_1 \\ \underset{\sim}{R}_2 \\ \underset{\sim}{R}_3 \end{bmatrix}$$

where

$$\begin{bmatrix} \underset{\sim}{R}_0 \\ \underset{\sim}{R}_1 \\ \underset{\sim}{R}_2 \\ \underset{\sim}{R}_3 \end{bmatrix} = \begin{bmatrix} R_0 & R_1 & R_2 & R_{12} & R_5 & R_{15} & R_{25} & R_{125} \\ R_4 & R_{14} & R_{24} & R_{124} & R_{45} & R_{145} & R_{245} & R_{1245} \\ R_3 & R_{13} & R_{23} & R_{123} & R_{35} & R_{135} & R_{235} & R_{1235} \\ R_{34} & R_{134} & R_{234} & R_{1234} & R_{345} & R_{1345} & R_{2345} & R_{12345} \end{bmatrix.$$

$$\begin{matrix} R_6 & R_{16} & R_{26} & R_{126} & R_{56} & R_{156} & R_{256} & R_{1256} \\ R_{46} & R_{146} & R_{246} & R_{1246} & R_{456} & R_{1456} & R_{2456} & R_{12456} \\ R_{36} & R_{136} & R_{236} & R_{1236} & R_{356} & R_{1356} & R_{2356} & R_{12356} \\ R_{346} & R_{1346} & R_{2346} & R_{12346} & R_{3456} & R_{13456} & R_{23456} & R_{123456} \end{matrix} \Bigg]$$

$$= \begin{bmatrix} 2 & 2 & \bar{10} & \bar{2} & 2 & 2 & \bar{2} & 6 & \bar{2} & 6 & 2 & 2 & \bar{10} & \bar{2} & 2 & 2 \\ \bar{2} & \bar{2} & \bar{6} & \bar{14} & \bar{18} & 14 & 2 & \bar{6} & \bar{30} & \bar{6} & \bar{2} & \bar{2} & 10 & 2 & \bar{2} & \bar{2} \\ \bar{2} & \bar{10} & 10 & 10 & \bar{2} & \bar{10} & 2 & 2 & \bar{6} & \bar{6} & 6 & \bar{2} & 2 & 2 & 6 & \bar{2} \\ 2 & 10 & 6 & 6 & 18 & \bar{6} & \bar{2} & \bar{2} & \bar{26} & 6 & \bar{6} & 2 & \bar{2} & \bar{2} & \bar{6} & 2 \end{bmatrix}$$

Hence

$$\begin{bmatrix} \underset{\sim}{R}^0 \\ \underset{\sim}{R}^1 \\ \underset{\sim}{R}^2 \\ \underset{\sim}{R}^3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bar{16} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \bar{4} & 0 & 4 & 0 & \bar{4} & 0 & 4 & 12 & 0 & 4 & 0 & \bar{4} & 0 & 4 & 0 \\ 0 & 0 & \bar{8} & \bar{8} & \bar{8} & 8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 6 & \bar{2} & 2 & 10 & \bar{2} & \bar{2} & 2 & 2 & 6 & \bar{2} & 2 & \bar{6} & \bar{2} & \bar{2} & 2 \end{bmatrix}$$

the coefficients being in the order listed above in $\underset{\sim}{R}_0$; x_3 and x_4 of course no longer appear since these are the variables which were detached. Since R^0 has $R_6 = 16$ and all the remaining coefficients are 0 it follows immediately that this function is x_6.

For the remaining three functions (f_1, f_2, and f_3) we shall repeat the procedure to detach 2 variables. The cost functions are given in the table below.

variables detached	f_1		f_2		f_3	
	C_{ij}	C_{ij}^*	C_{ij}	C_{ij}^*	C_{ij}	C_{ij}^*
x_1, x_2	8	6	8	8	8	$\overline{4}$
x_1, x_5	8	6	8	8	8	8
x_1, x_6	0	0	0	$\overline{16}$	12	8
x_2, x_5	8	6	0	0	8	0
x_2, x_6	8	8	0	$\overline{16}$	8	$\overline{4}$
x_5, x_6	8	8	0	$\overline{16}$	12	12

Based on these parameters we shall detach x_2, x_6 for f_1 ; x_1, x_5 for f_2 ; and x_5, x_6 for f_3. A detail investigation of the alternatives reveals that detaching x_5 and x_6 for f_1 leads to an identically sized solution to that resulting form detaching x_2 and x_6. The same is true if x_1, x_2 are detached from f_2 instead of x_1, x_5. For f_3 detaching x_1, x_6 leads to a solution that has a cost one greater than that resulting form detaching x_5, x_6.

Using the method detailed in examples 1 and 2 and assuming that

$$f_1 = \bar{x}_2\ \bar{x}_6\ f_{10} + \bar{x}_2\ x_6\ f_{11} + x_2\ \bar{x}_6\ f_{12} + x_2\ x_6\ f_{13}$$

with similar notation for the other f_2 and f_3 then we have

$$\begin{bmatrix} \underset{\sim}{R}^{10} \\ \underset{\sim}{R}^{11} \\ \underset{\sim}{R}^{12} \\ \underset{\sim}{R}^{13} \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \bar{1} & 1 & \bar{1} \\ 1 & 1 & \bar{1} & \bar{1} \\ 1 & \bar{1} & \bar{1} & 1 \end{bmatrix} \begin{bmatrix} 0 & \bar{4} & 0 & \bar{4} \\ 12 & 0 & \bar{4} & 0 \\ 0 & 4 & 0 & 4 \\ 4 & 0 & 4 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 & 0 \\ \bar{4} & 0 & 0 & 0 \\ 2 & \bar{2} & \bar{2} & \bar{2} \\ \bar{2} & \bar{2} & 2 & \bar{2} \end{bmatrix} \text{ so } \quad \begin{aligned} f_{10} &= 1 \\ f_{11} &= 0 \\ f_{12} &= x_1 + x_5 \\ f_{13} &= x_1\ \bar{x}_5 \end{aligned}$$

$$\begin{bmatrix} \underset{\sim}{R}^{20} \\ \underset{\sim}{R}^{21} \\ \underset{\sim}{R}^{22} \\ \underset{\sim}{R}^{23} \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \bar{1} & 1 & \bar{1} \\ 1 & \bar{1} & 1 & \bar{1} \\ 1 & \bar{1} & \bar{1} & 1 \end{bmatrix} \begin{bmatrix} 0 & \bar{8} & 0 & 0 \\ \bar{8} & 0 & 0 & 0 \\ 0 & \bar{8} & 0 & 0 \\ 8 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \bar{4} & 0 & 0 \\ 0 & \bar{4} & 0 & 0 \\ \bar{4} & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 \end{bmatrix} \text{ so } \quad \begin{aligned} f_{20} &= x_2 \\ f_{21} &= x_2 \\ f_{22} &= 0 \\ f_{23} &= 1 \end{aligned}$$

$$\begin{bmatrix} \underset{\sim}{R}^{30} \\ \underset{\sim}{R}^{31} \\ \underset{\sim}{R}^{32} \\ \underset{\sim}{R}^{33} \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \bar{1} & 1 & \bar{1} \\ 1 & 1 & \bar{1} & \bar{1} \\ 1 & \bar{1} & \bar{1} & 1 \end{bmatrix} \begin{bmatrix} 2 & 6 & \bar{2} & 2 \\ 2 & 6 & \bar{2} & 2 \\ 10 & \bar{2} & \bar{2} & 2 \\ \bar{6} & \bar{2} & \bar{2} & 2 \end{bmatrix} = \begin{bmatrix} 2 & 2 & \bar{2} & 2 \\ 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ \bar{4} & 0 & 0 & 0 \end{bmatrix} \quad \begin{aligned} f_{30} &= \bar{x}_1 + x_2 \\ f_{31} &= 1 \\ f_{32} &= \bar{x}_1 \\ f_{33} &= 0 \end{aligned}$$

Hence the final expression for the function is

$$
\begin{aligned}
f\,(x_1, x_2, x_3, x_4, x_5, x_6) \;=\; & \bar{x}_3\bar{x}_4x_6 \;+ \\
& + \bar{x}_3x_4 \;\{\bar{x}_2\bar{x}_6 + x_2\bar{x}_6\,(x_1 + x_5) + x_2x_6x_1\bar{x}_5\,\} \\
& + x_3\bar{x}_4 \;\{\bar{x}_1\bar{x}_5x_2 + \bar{x}_1x_5x_2 + x_1x_5\,\} \\
& + x_3x_4 \;\{\bar{x}_5\bar{x}_6\,(\bar{x}_1 + x_2) + \bar{x}_5x_6 + x_5\bar{x}_6\bar{x}_1\,\}
\end{aligned}
$$

In any practical implementation there are a number of obvious simplifications which would be incorporated. For example $f_{20} = f_{21} = x_2$ should be identified since then f_{20} and f_{21} can be combined into a single term. A slightly more complex identification might notice, for example, that both f_{30} and f_{32} include $\bar{x}_1$ with a resulting simplification.

If we write f as a sum of products incorporating these two simplifications we have

$$
\begin{aligned}
f\,(x_1, x_2, x_3, x_4, x_5, x_6) = {} & \bar{x}_3\bar{x}_4x_6 + \bar{x}_2\bar{x}_3x_4\bar{x}_6 + x_1x_2\bar{x}_3x_4\bar{x}_6 \;+ \\
& + x_2\bar{x}_3x_4x_5\bar{x}_6 + x_1x_2\bar{x}_3x_4\bar{x}_5x_6 + \bar{x}_1x_2x_3\bar{x}_4 + x_1x_3\bar{x}_4x_5 \;+ \\
& + \bar{x}_1x_3x_4x_5 + x_2x_3x_4\bar{x}_5\bar{x}_6 + x_3x_4\bar{x}_5x_6.
\end{aligned}
$$

Let us consider the cost of the solutions. The cost of the above solution is 54, while the cost prior to the obvious simplification was 68. A detail consideration of the function, whose map is given below, leads to the conclusion that a minimal sum of products form has a cost of 44 while the direct sum of minterm form has a cost of 231.

As a matter of interest when we initially choose x_3, x_4 to detach it was reasonable to have alternatively chosen x_4, x_6 which had almost identical cost parameters. If we had done so the resulting expression for f would be as below, which has a cost of 83 as given which reduces to 69 if some obvious simplifications are included.

$x_1\ x_3\ x_4$ \ $x_2\ x_5\ x_6$	000	001	011	010	100	101	111	110
0 0 0	0	1	1	0	0	1	1	0
0 0 1	1	0	0	1	0	0	0	1
0 1 1	1	1	0	1	1	1	0	1
0 1 0	0	0	0	0	1	1	1	1
1 0 0	0	1	1	0	0	1	1	0
1 0 1	1	0	0	1	1	1	0	1
1 1 1	0	1	0	0	1	1	0	0
1 1 0	0	0	1	1	0	0	1	1

$$\begin{aligned} f\,(x_1, x_2, x_3, x_4, x_5, x_6) = {} & \bar{x}_4\bar{x}_6\ \{\bar{x}_1x_2x_3 + x_1x_2x_3x_5 + x_1\bar{x}_2x_3x_5\ \} \\ & + \bar{x}_4x_6\ \{\ \bar{x}_1\bar{x}_3 + \bar{x}_1x_3x_2 + x_1\bar{x}_3 + x_1x_3x_5\ \} \\ & + x_4\bar{x}_6\ \{\bar{x}_1\bar{x}_3\ (\bar{x}_2 + x_5) + \bar{x}_1x_3 + x_1\bar{x}_3 + x_1x_3x_2\bar{x}_5\ \} \\ & + x_4x_6\ \{x_3\bar{x}_5 + \bar{x}_3\bar{x}_5x_4x_6\ \}. \end{aligned}$$

This function was randomly generated subject to the constraint of having 33 1's on its map.

References

[1] K.G. Beauchamp, "Walsh Functions and their Applications," Academic Press, London, 1975.

[2] C.R. Edwards, "The application of the Rademacher - Walsh transform to Boolean function classification and threshold-logic synthesis," Trans. IEEE, vol. C - 24, 1975, pp. 48-62.

[3] H.F. Harmuth, "Transmission of Information by Orthogonal Functions," Springer, New York, 1969.

[4] S.L.Hurst, "Application of Chow parameters and Rademacher - Walsh matrices in the synthesis of binary functions," Computer J., Vol. 16, 1973, pp. 165-173.

[5] S.L. Hurst, "Logical Processing of Digital Signals," Crane Russak, New York (to be published shortly).

[6] R.J. Lechner, "A transform approach to logic design," Trans. IEEE, Vol. C-19, 1970, pp. 627-640.

[7] R.J. Lechner, "Harmonic analysis of switching functions," in "Recent Developments in Switching Theory (ed. A. Mukhopadhyay), Academic Press, New York, 1971.

[8] R.E.A.C. Paley, "On orthogonal matrices," J. Math, Physics, Vol. 12, 1933, pp. 311-320.

[9] H. Rademacher, "Einige sätze über reihen von allgemeinen orthogonalfunktionen," Math. Ann., Vol. 87, 1922, pp. 112-138.

[10] L.J. Ulman, "Computation of the Hadamard transform and the R transform in ordered form," Trans. IEEE, Vol EC-19, 1970, pp. 359-360.

[11] J.L. Walsh, "A closed set of orthogonal functions," Amer. J. Math., Vol. 45, 1923, pp. 5-24.