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Dilaton shifts, probability measures, and decomposition

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Abstract
In this paper we discuss dilaton shifts (Euler counterterms) arising in decomposition of two-dimensional quantum field theories with higher-form symmetries. Relative shifts between universes are fixed by locality and take a universal form, reflecting underlying (noninvertible, quantum) symmetries. The first part of this paper constructs a general formula for such dilaton shifts, and discusses related computations. In the second part of this paper, we comment on the relation between decomposition and ensembles.

Keywords: decomposition, higher-form symmetries, dilaton shifts, quantum field theory

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1. Introduction

Put simply, for a local quantum field theory to decompose means that it is equivalent to a disjoint union of other local quantum field theories (known in this context as 'universes'). A quantum field theory that one thought one knew, might secretly be a union of several independent quantum field theories masquerading as a unit. Decomposition was first observed in examples in [1], where it was used to resolve apparent inconsistencies in string compactifications on certain stacks known as gerbes [2–4], which are fiber bundles of higher-form symmetry groups, realized physically as gauge theories with trivially-acting subgroups.

Decomposition has been discussed and applied by now extensively to a variety of examples, including Gromov-Witten theory (see e.g. [5–10]), gauged linear sigma models (see e.g. [11–22]), elliptic genera and IR limits of pure supersymmetric gauge theories (see e.g. [23]), adjoint QCD_2 [24], anomaly resolution [25–27], lattice computations [28], and even quivers [29], in not only two-dimensional theories but also three-dimensional (see e.g. [30–32]) as well as four-dimensional gauge theories (see e.g. [33, 34]). It is often associated with the existence of higher-form symmetries—specifically, a (possibly noninvertible [35–38]) $(d-1)$-form symmetry in a d-dimensional quantum field theory [33, 34]. In such theories, decomposition often has the effect of restricting allowed instantons, through a 'multiverse interference effect' created by the superposition of multiple quantum field theories (the universes). See e.g. [39–42] for reviews and additional references.

As first observed in [1], the contributions to a partition function from different universes often have different Euler number counterterms, universal counterterms multiplying world-sheet Ricci curvatures, which in a string theory would correspond to constant dilaton shifts. For this reason, we refer to these counterterms as 'dilaton shifts,' as was the usage in [1]. Although one can add local counterterms to change the dilaton shifts / Euler counterterms in any one Universe or across the entire theory, the differences between Euler counterterms (ratios of dilaton factors) between different universes is well-defined, fixed by locality.

The purpose of this paper is to systematically study these ratios of dilaton shift factors arising between the different universes in decomposition in two dimensions. These factors have a canonical and universal form, reflecting symmetries of the decomposing theory, which we will discuss.

These dilaton shifts are also sometimes interpreted in the literature in the form of probability densities arising in entropy computations. Now, a decomposition is not the same as an ensemble, as we shall discuss, but at least on a connected spacetime, they do appear closely related, which is one motivation to understand dilaton shift factors more systematically.

We begin in section 2 with a brief review of decomposition in two-dimensional theories. In section 3 we give general results for dilaton shifts in orbifolds, gauge theories, and topological field theories in two dimensions, which we check in numerous examples. Briefly, the dilaton shifts are universally proportional to $(\dim R)^2$, where for orbifolds and gauge theories, R is an irreducible representation of some gauge group indexing the universes, and in topological field theories, a ratio of $\dim R$'s is the quantum dimension of an interface between universes. As an aside, in section 3.5.4, we discuss the implications of decomposition for volumes of moduli spaces of flat connections.

In section 4 we discuss how this form is expected on symmetry grounds, related to the presence and properties of interfaces linking the different universes, and also relate these shifts to asymptotic densities of states as in e.g. [43]. In section 5 we outline how the form of this result for dilaton shifts can also be understood by reinterpreting at least some of these theories in the language of coupling to a topological field theory.

In section 6 we compare and contrast these dilaton shifts with probability measures appearing in various discussions. In particular, a decomposition is not the same as an ensemble, as we discuss, though in the special case of a connected spacetime, they are at least naively closely related. We compare structures in mirrors to decompositions to stochastic variables appearing in the SYK construction, and compute entanglement entropy in decomposing theories.

In appendix A we briefly review the notion of invertible field theories, which arise in numerous locations in this paper. In appendix B we collect some finite group representation theory identities, which are used in a number of places. In appendix C we give a presentation of two-dimensional Dijkgraaf-Witten theory utilizing triangulations.

2. Brief review of decomposition

Briefly, decomposition is a property of local quantum field theories, in which they are equivalent to ('decompose into') a disjoint union of other local quantum field theories, known in this context as 'universes.' Formally, decomposition is expected to arise in any d-dimensional quantum field theory with a global $(d-1)$-form symmetry (see e.g. [42] for a recent review).

One of the reasons for interest in decomposition is the fact that it provides counterexamples to old lore in the community, which suggests that local descriptions of disjoint unions should not exist. We can understand the issue as follows. Suppose we have two different theories on the same (connected) spacetime, with partition functions Z_i and local actions S_i, schematically:

$$Z_i = \int [D\phi] \exp(-S_i). \tag{2.1}$$

The partition function of the disjoint union, on a connected spacetime, is the sum of the partition functions of the two separate theories:

$$Z_{1\coprod 2} = Z_1 + Z_2 = \int [D\phi] \exp(-S_1) + \int [D\phi] \exp(-S_2). \tag{2.2}$$

(This is different from a product of QFTs, for which the partition function is a product, and the actions merely add. A free field theory of two scalars is a product of the QFTs for each scalar separately, for example.) However, for this to be a local quantum field theory, we would need to write this as a path integral of a single action $S_{1\coprod 2}$:

$$Z_{1\coprod 2} = \int [D\phi] \exp\left(-S_{1\coprod 2}\right). \tag{2.3}$$

The issue, in a nutshell, is that it is not clear how to construct such an action $S_{1\coprod 2}$ in general. It certainly is not the sum $S_1 + S_2$ (unlike a product of QFTs), for example. The paper [1], and others since, provided examples.

A signature of decomposition in a unitary quantum field theory is the existence of topological projection operators in the spectrum of local operators, a set of operators Π_i that commute with all other operators,

$$[\Pi_i, \mathcal{O}] = 0, \tag{2.4}$$

and which behave like projectors in the sense that

$$\Pi_i \Pi_j = \delta_{ij} \Pi_i, \quad \sum_i \Pi_i = 1. \tag{2.5}$$

As the projectors Π_i commute with local operators, they are simultaneously diagonalizable, with eigenspaces corresponding to the states associated to the different universes, onto which they project. Another signature of decomposition is that partition functions on connected worldsheets can be written as sums, schematically

$$Z = \sum_i Z_i, \tag{2.6}$$

a consequence of the fact that the state space breaks up into eigenspaces of the projection operators. The projection operators are also the conserved defect operators of a $(d-1)$-form symmetry.

Simple examples of decomposition in two-dimensional theories are gauge theories with trivially-acting subgroups [1–4]. Suppose, for example, we have a Γ gauge theory (where Γ might be either finite—an orbifold—or continuous—an ordinary gauge theory) in two dimensions, in which a subgroup $K \subset \Gamma$ acts trivially. For simplicity, let us assume that K lies within the center of Γ. (Decomposition is understood in more general cases, but for our purposes here, this will suffice.) Then, schematically,

$$\mathrm{QFT}\left(\Gamma - \text{gauge theory}\right) = \coprod_{\theta \in \hat{K}} \mathrm{QFT}\left((\Gamma/K) - \text{gauge theory}\right)_{\omega(\theta)}, \tag{2.7}$$

where the $\omega(\theta)$'s indicate discrete theta angles, determined by the image of the extension class $[\omega] \in H^2(\Gamma/K, K)$ corresponding to

$$1 \longrightarrow K \longrightarrow \Gamma \longrightarrow \Gamma/K \longrightarrow 1 \tag{2.8}$$

under the map

$$H^2\left(\Gamma/K, K\right) \longrightarrow H^2\left(\Gamma/K, U(1)\right) \tag{2.9}$$

induced by $\theta : K \to U(1)$. (If Γ is finite, then the 'discrete theta angles' are choices of discrete torsion.)

Now, to be clear, the expression (2.7) has glossed over the 'dilaton shifts' (Euler counterterms) that will be the focus of this paper. Indeed, as these dilaton shifts are merely counterterms, which can be added or subtracted at will, the expression (2.7) is still correct. However, if we pull them out explicitly into (Universe-dependent) factors $\rho(\theta)$, then expression (2.7) can be rewritten (on a connected worldsheet Σ) as

$$\mathrm{QFT}\left(\Gamma - \text{gauge theory}\right) = \coprod_{\theta \in \hat{K}} \rho\left(\theta\right)^{\chi(\Sigma)} \mathrm{QFT}\left((\Gamma/K) - \text{gauge theory}\right)_{\omega(\theta)}, \tag{2.10}$$

which is the starting point for the analysis of this paper.

Decomposition is not restricted to gauge theories, but in fact exists in more general examples. Important special cases include two-dimensional unitary topological field theories, with semisimple local operator algebras.

It is a standard result that two-dimensional unitary topological field theories with semisimple local operator algebras are equivalent to disjoint unions of 'trivial' (invertible field theories, see e.g. [44], [45, section 3.1], which in e.g. [38, section 3], [24, appendix C.1] was observed to be a decomposition. In broad brushstrokes, such topological field theories have a decomposition of the form

$$\coprod_{R} \mathrm{Inv}\left(0, \ln\left(\dim R\right)\right), \tag{2.11}$$

up to an overall dilaton counterterm, where $\mathrm{Inv}(\lambda_1, \lambda_2)$ labels a familiy of invertible field theories, and the decomposition is over irreducible representations. (See appendix A for a review of invertible field theories and our labelling conventions.) For Dijkgraaf-Witten, BF theory, and pure Yang-Mills, these will all be ordinary or projective representations; for the G/G model, these will be integrable representations, and $\dim R$ the quantum dimension. (For completeness, some excellent references on the two-dimensional topological gauge theories appearing in this paper are [46–49].)

Before going on, let us compare cohomological topological field theories to the Schwarz-type theories we study in this paper. Such theories, such as for example the A and B models in two dimensions, have position-independent (hence topological) operators, and hence their topological subsectors decompose. However, no decomposition of the underlying quantum field theory is expected, unless of course the target space has multiple disconnected components or is a gerbe. One difference is that constructions of projection operators from dimension-zero operators rest on a physical assumption of unitarity, which does not typically hold after topological twisting. For this reason, when we speak of decomposition and dimension-zero operators, we refer to untwisted theories, and so exclude the topological A and B models, topological gravity, and topological Kazama-Suzuki cosets. (Put another way, the topological subsector might decompose abstractly as a Frobenius algebra, but one does not expect the quantum field theory in which the topological sector is embedded will decompose.) There can still exist untwisted topological field theories, and these include Dijkgraaf-Witten theory, *BF* theories, and G/G models.

3. Dilaton shifts in decomposition in two dimensions

A subtlety of decomposition in two dimensions is that in computing partition functions on surfaces of genus $g \neq 1$, there can be a dilaton[1] shift (Euler counterterm) appearing on the universes—a counterterm in the action proportional to the worldsheet curvature, that generates a genus-dependent factor multiplying the partition function.

3.1. Prediction

In this section, we conjecture a universal form for such dilaton shifts in orbifolds, gauge theories, and topological field theories, namely that they are proportional to $(\dim R)^2$. In orbifolds and gauge theories, R is an irreducible representation of some group indexing the universes; in topological field theories, $\dim R$'s is instead interpreted as the quantum dimension of an an operator. (Other factors vary slightly due to differences in normalization conventions, so a single result valid for all cases is not possible, a statement of proportionality is the most one can hope to expect.)

Now, in any one universe, we can of course just add a counterterm; however, locality constrains the counterterms between different universes. Adding counterterms to the original (decomposing) theory adds the same counterterm to each Universe, so only differences of counterterms are invariant. If one adds different counterterms to separate universes, then the result in general is not expected to be a local quantum field theory.

In the remaining subsections of this paper, we give a more refined prediction for two-dimensional orbifolds, gauge theories, and unitary topological field theories, which we elaborate in examples.

In section 3.2 we discuss two-dimensional orbifolds with trivially-acting subgroups and no discrete torsion. We formulate a precise conjecture for the dilaton shift factors, which we check in a number of examples. In section 3.3 we repeat for orbifolds with discrete torsion, making a conjecture which is checked in numerous examples. In section 3.4 we consider orbifolds with quantum symmetries, again formulating a conjecture which is checked in examples.

[1] To be clear, these theories are need not be coupled to worldsheet gravity. We refer to this as a 'dilaton shift' solely because of the universal form of the counterterm—proportional to the worldsheet curvature. Such counterterms can appear in any two-dimensional theory, regardless of any worldsheet gravity coupling or lack thereof.

In section 3.5 we perform the same analysis in two-dimensional gauge theories in which a subgroup of the gauge group acts trivially. We also make some general observations about relations between volumes of moduli spaces of flat connections implied by decomposition, in section 3.5.4. Finally, in section 3.6 we discuss dilaton shifts in examples of unitary two-dimensional topological field theories.

Later we will discuss general arguments for these dilaton shift factors, and compare to probability densities.

3.2. Examples in orbifolds without discrete torsion

3.2.1. Conjecture . In an orbifold $[X/\Gamma]$ or gauge theory, for $K \subset \Gamma$ acting trivially, decomposition predicts that [1]

$$\mathrm{QFT}\left([X/\Gamma]\right) = \mathrm{QFT}\left(\left[\frac{X \times \hat{K}}{\Gamma/K}\right]_{\hat{\omega}}\right) \tag{3.1}$$

where $\hat{\omega}$ denotes discrete torsion, and the right-hand side in general is a sum of multiple disconnected components (details depending upon the action of Γ/K on the set of irreducible representations $\hat{K}$ of K). (For example, if K is central in Γ, then Γ/K acts trivially on $\hat{K}$, and there are as many different components, as many universes, as elements of $\hat{K}$ (irreducible representations of K)). The expression above omits dilaton shift (Euler counterterm) factors. The reader should note that if Γ/K exchanges several elements of $\hat{K}$, then the irreducible representations appearing in that orbit all have the same dimension.

We conjecture that the genus g partition function can be written in the following form, which makes dilaton shifts explicit:

$$Z_g\left([X/\Gamma]\right) = \sum_U \left(\frac{\dim R_U}{|K|}\right)^{2-2g} Z_g\left(X_U\right), \tag{3.2}$$

generalizing equation (3.1) above, where U denotes universes, $\dim R_U$ is the dimension of a representative[2] irreducible representation appearing in the orbit of $G = \Gamma/K$, and where X_U is the theory corresponding to Universe U.

In the expression above, we define the orbifold partition function without a dilaton shift by the normalization

$$Z_g\left([X/\Gamma]\right) = \frac{1}{|\Gamma|}\sum_{a_i,b_j} Z(a_1,\ldots,b_g). \tag{3.3}$$

In some of the papers we cite, partition functions are normalized differently, with a factor of $1/|\Gamma|^g$ instead of $1/|\Gamma|$. We shall denote such partition functions by $\tilde{Z}_g$, as

$$\tilde{Z}_g\left([X/\Gamma]\right) = \frac{1}{|\Gamma|^g}\sum_{a_i,b_j} Z(a_1,\ldots,b_g), \tag{3.4}$$

to distinguish them from our conventions above.

[2] As opposed to e.g. the sum of all representations in the orbit.

The two normalizations above are related by a dilaton shift (a factor of $1/|\Gamma|^{g-1}$). More generally, we can always add an Euler counterterm to the ambient theory, so that there is not 'one' partition function so much as a one-parameter family.

Taking into account that ambiguity in shifting the ambient theory by Euler counterterms, a more invariant statement of the conjecture is that the relative dilaton shift between the partition function contribution from a single Universe U and that of the ambient theory is

$$\left(\frac{\dim R_U}{|K|}\right)^{2-2g}. \tag{3.5}$$

Later in section 4 we will discuss how the factors of $\dim R$ seem to be determined by symmetries.

We shall see that this matches dilaton shifts in orbifolds in which K is both abelian and nonabelian. Later in section 3.3 we will generalize this expression to orbifolds with discrete torsion. We will give an expression for dilaton shift factors which is of an identical form; however, the interpretation of the R_U will differ slightly, as we elaborate there.

Our conjecture above generalizes the results in [43] for matrix ensemble eigenvalue densities, see for example [43, equation (43)]. We shall return to this in section 4.

In passing, these dilaton shift factors have also been studied in connection with Frobenius-Schur indicators, see e.g. [50].

3.2.2. Examples with trivially-acting central subgroups . In this section we look at orbifolds in which the trivially-acting subgroup lies in the center, beginning with $[X/D_4]$ with trivially-acting central $\mathbb{Z}_2$.

From[3] [1, section 5.2], on a genus g Riemann surface, it was shown that

$$\tilde{Z}_g([X/D_4]) = 2^{g-1}\left(\tilde{Z}_g([X/\mathbb{Z}_2\times\mathbb{Z}_2]) + \tilde{Z}_g\left([X/\mathbb{Z}_2\times\mathbb{Z}_2]_{\mathrm{dt}}\right)\right), \tag{3.6}$$

in conventions in which the partition function is normalized by $1/|\Gamma|^g$. Converting to our conventions with

$$\tilde{Z}_g([X/D_4]) = \frac{1}{|D_4|^{g-1}}Z_g([X/D_4]),\ \ \tilde{Z}_g([X/\mathbb{Z}_2\times\mathbb{Z}_2]) = \frac{1}{|\mathbb{Z}_2\times\mathbb{Z}_2|^{g-1}}Z_g([X/\mathbb{Z}_2\times\mathbb{Z}_2]), \tag{3.7}$$

we find

$$Z_g([X/D_4]) = 2^{2g-2}\left(Z_g([X/\mathbb{Z}_2\times\mathbb{Z}_2]) + Z_g\left([X/\mathbb{Z}_2\times\mathbb{Z}_2]_{\mathrm{dt}}\right)\right). \tag{3.8}$$

Each of the universes (each a $\mathbb{Z}_2\times\mathbb{Z}_2$ orbifold) is associated with a one-dimensional irreducible representation of $K=\mathbb{Z}_2$, and as $|K|=2$, we see that the result above matches the general prediction (3.2).

More generally, for

$$1 \longrightarrow K \stackrel{\iota}{\longrightarrow} \Gamma \stackrel{\pi}{\longrightarrow} G \longrightarrow 1, \tag{3.9}$$

[3] The reader should note that the formula for the g loop partition function of this orbifold in [1, section 5.2] has a typo: the $|D_4|^2$ should instead be $|D_4|^g$.

for K finite, central, and trivially-acting, the same analysis as [1, section 5] yields

$$Z_g([X/\Gamma]) = \frac{|G|}{|\Gamma|}|K|^{2g}\left(|K|^{-1}\sum_{\rho\in\hat{K}} Z_g\left([X/G]_{\hat{\omega}(\rho)}\right)\right). \tag{3.10}$$

Since

$$\frac{|G|}{|\Gamma|}|K|^{2g-1} = |K|^{2g-2}, \tag{3.11}$$

and K is central, the representation R_U of K associated to each Universe is one-dimensional. Thus, the partition function above can be written in the form

$$Z_g([X/\Gamma]) = |K|^{2g-2}\sum_{\rho\in\hat{K}}(1)^{2-2g} Z_g\left([X/G]_{\hat{\omega}(\rho)}\right), \tag{3.12}$$

and so matches the prediction (3.2).

We will discuss the effect of adding discrete torsion to orbifolds of this form in section 3.3.6.

Next, we will consider several examples of orbifolds in which the trivially-acting subgroup K is abelian, but not necessarily central in Γ.

3.2.3. Noncentral abelian example: $[X/\mathbb{H}]$, $K = \mathbb{Z}_4$. Beginning in this section we will look at examples of orbifolds in which the trivially-acting subgroup K is not in the center (though is still abelian). (Technically, these are 'nonbanded' abelian gerbes.) In these cases, the different universes need not have the same form as one another, and the associated orbits of $\Gamma/K = G$ may contain several representations.

Consider the eight-element group of unit quaternions $\mathbb{H}$, with trivially-acting $K = \langle i\rangle \cong \mathbb{Z}_4 \subset \mathbb{H}$. Decomposition predicts [1, section 5.4]

$$\mathrm{CFT}([X/\mathbb{H}]) = \mathrm{CFT}\left(X \coprod [X/\mathbb{Z}_2] \coprod [X/\mathbb{Z}_2]\right), \tag{3.13}$$

using the fact that $\mathbb{Z}_2 = \mathbb{H}/\langle i\rangle$ acts on $K = \langle i\rangle$ by conjugation, which leaves two irreps of K invariant, and exchanges two others.

From [1, section 5.4],

$$Z_g([X/\mathbb{H}]) = \frac{4^{2g-1}}{|\mathbb{H}|}\left(2\,(\text{untwisted sector}) + 2\,(\text{all sectors})\right), \tag{3.14}$$

$$= \frac{4^{2g-1}}{(2)(4)}\left(2Z_g(X) + 2|\mathbb{Z}_2| Z_g([X/\mathbb{Z}_2])\right), \tag{3.15}$$

$$= 4^{2g-2} Z_g(X) + 4^{2g-2}(2) Z_g([X/\mathbb{Z}_2]), \tag{3.16}$$

$$= 4^{2g-2}\left(Z_g(X) + 2Z_g([X/\mathbb{Z}_2])\right). \tag{3.17}$$

In this case, since $K = \mathbb{Z}_4$ is abelian, all its irreducible representations are one-dimensional. The two $[X/\mathbb{Z}_2]$ universes are associated to one-dimensional representations of K which are invariant under the action of $\mathbb{H}/K = \mathbb{Z}_2$, whereas the Universe X is associated to a pair of one-dimensional representations that are interchanged by $\mathbb{H}/K = \mathbb{Z}_2$. In all three cases, representatives of the orbit of $G = \mathbb{Z}_2$ on the space of irreducible representations are one-dimensional, so in terms of our conjecture (3.2), $\dim R_U = 1$ for every Universe U. In particular, the result above for the partition function matches our prediction (3.2).

3.2.4. Noncentral abelian example: $[X/A_4]$, $K = \mathbb{Z}_2 \times \mathbb{Z}_2$. Consider the group of alternating permutations A_4 on four elements. This has a normal subgroup $K = \mathbb{Z}_2 \times \mathbb{Z}_2$, which we take to act trivially, and $A_4/\mathbb{Z}_2 \times \mathbb{Z}_2 = \mathbb{Z}_3$.

Decomposition predicts [1, section 5.5]

$$\mathrm{CFT}\left([X/A_4]\right) = \mathrm{CFT}\left([X/\mathbb{Z}_3] \coprod X\right), \tag{3.18}$$

using the fact that $\mathbb{Z}_3 = A_4/\mathbb{Z}_2 \times \mathbb{Z}_2$ leaves invariant the trivial representation of $K = \mathbb{Z}_2 \times \mathbb{Z}_2$ and permutes the other three.

Analyzing this example at genus g in the same fashion as [1, section 5.4], we find the untwisted sector of the $\mathbb{Z}_3$ orbifold appears with multiplicity $|\mathbb{Z}_2 \times \mathbb{Z}_2|^{2g}$ (from multiplying any edge by any element of $\mathbb{Z}_2 \times \mathbb{Z}_2$), whereas all twisted sectors appear with multiplicity $|\mathbb{Z}_2 \times \mathbb{Z}_2|^{2g-1}$.

Thus, we find

$$Z_g\left([X/A_4]\right) = \frac{4^{2g-1}}{|A_4|}\left(|\mathbb{Z}_3| Z_g\left([X/\mathbb{Z}_3]\right) + (4-1)\, Z_g(X)\right), \tag{3.19}$$

$$= 4^{2g-2}\left(Z_g\left([X/\mathbb{Z}_3]\right) + Z_g(X)\right). \tag{3.20}$$

In the special case $g = 1$, this reduces to

$$Z_1\left([X/A_4]\right) = Z_1\left([X/\mathbb{Z}_3]\right) + Z_1(X), \tag{3.21}$$

as expected.

The universe $[X/\mathbb{Z}_3]$ is associated to a one-dimensional representation of K (which is invariant under the action of $A_4/K = \mathbb{Z}_3$), and the Universe X is associated to an orbit of A_4/K consisting of three one-dimensional representations of K. In both cases, representatives of the orbit are one-dimensional, so for both universes, $\dim R_U = 1$. In particular, since $|K| = 4$, we see that the result above matches the prediction (3.2).

3.2.5. Noncentral abelian example: $[X/D_n]$, $K = \mathbb{Z}_n$. Consider the $2n$-element dihedral group D_n. Let its normal subgroup $K = \mathbb{Z}_n$ act trivially, and recall $D_n/\mathbb{Z}_n = \mathbb{Z}_2$.

In this case, decomposition predicts [1, section 5.6]

$$\mathrm{CFT}\left([X/D_n]\right) = \mathrm{CFT}\left(\coprod_{\alpha} [X/\mathbb{Z}_2] \coprod_{(n-\alpha)/2} X\right), \tag{3.22}$$

where

$$\alpha = \left\{ \begin{array}{ll} 1 & n \text{ odd}, \\ 2 & n \text{ even}, \end{array} \right. \tag{3.23}$$

using the fact that $\mathbb{Z}_2 = D_n/\mathbb{Z}_n$ leaves α irreps of $K = \mathbb{Z}_n$ invariant and exchanges the rest in pairs.

Proceeding in the same fashion as [1, section 5.4], we compute the partition function at genus g. Here, the untwisted sector appears with multiplicity n^{2g} (arising from multiplying any edge by an element of $\mathbb{Z}_n$, and all twisted sectors appear with multiplicity αn^{2g-1}. The partition function then takes the form

$$Z_g([X/D_n]) = \frac{n^{2g-1}}{|D_n|}\left(\alpha|\mathbb{Z}_2|Z_g([X/\mathbb{Z}_2]) + (n-\alpha)Z_g(X)\right), \tag{3.24}$$

$$= n^{2g-2}\left[\alpha Z_g([X/\mathbb{Z}_2]) + \frac{(n-\alpha)}{2}Z_g(X)\right]. \tag{3.25}$$

In the special case $g = 1$, this reduces to

$$Z_1([X/D_n]) = \alpha Z_1([X/\mathbb{Z}_2]) + \frac{n-\alpha}{2}Z_1(X), \tag{3.26}$$

matching the prediction of decomposition.

The universes $[X/\mathbb{Z}_2]$ are each associated to one-dimensional representations of K (invariant under the action of $D_n/K = \mathbb{Z}_2$), and the universes X are each associated to orbits of $D_n/Z_k = \mathbb{Z}_2$ exchanging pairs of one-dimensional representations of K. As $|K| = n$, we see that the partition functions above match the predictions (3.2).

3.2.6. Nonabelian example: $[X/D_6]$, $K = D_3$. Next, consider the orbifold $[X/D_6]$ by the twelve-element dihedral group D_6, with trivially-acting normal subgroup $K = D_3 \subset D_6$, and recall $D_6/D_3 = \mathbb{Z}_2$. In other examples considered so far, K was abelian; here, since K is nonabelian, it has representations of dimension greater than one, so the $\dim R_U$ factors in (3.2) will be nontrivial in the dilaton shifts in higher-genus partition functions here.

Explicitly, D_6 can be presented as generated by a, b such that

$$a^2 = 1 = b^6, \ aba = b^5. \tag{3.27}$$

It is straightforward to check that $z \equiv b^3$ generates the $\mathbb{Z}_2$ center of D_6, and the normal subgroup D_3 is generated by $\{a, b^2\}$.

Let ξ denote the nontrivial coset in $D_6/D_3 = \mathbb{Z}_2$.

It will be useful to note that D_3 has three irreducible representations, of dimensions 1, 1, and 2. In this case, decomposition [1] predicts

$$\mathrm{QFT}([X/D_6]) = \mathrm{QFT}\left(\coprod_3 [X/\mathbb{Z}_2]\right). \tag{3.28}$$

Specifically, genus g partition functions are predicted by (3.2) to have the form

$$Z_g([X/D_6]) = |K|^{2g-2}\left(1 + 1 + (2)^{2-2g}\right)Z_g([X/\mathbb{Z}_2]), \tag{3.29}$$

since the three representations of D_3 have dimension 1, 1, 2.

Now, let us verify this in the partition function for several genera. The genus g D_6 orbifold partition function is already a highly intricate combinatorics computation; the fact that it matches the expression above in examples of genera $g > 1$ will provide a strong test of the conjecture (3.2) (as well as of decomposition itself).

To that end, it will be useful to characterize commutators $[a,b] = aba^{-1}b^{-1}$.

Table 1. Characterization of ordered pairs of elements of D_6. The first column lists to which pair in $D_6/D_3 = \mathbb{Z}_2$ they project. The second column is the value of their commutators. The last column counts the number of entries in the row. We have omitted ordered pairs corresponding to $(\xi, 1)$, as they can be obtained straightforwardly from the other entries.

Projects to	$[a,b]$	Pairs	Count
$(1,1)$	1	$(b^{\text{even}}, b^{\text{even}})$, (ab^{2n}, ab^{2n}),$(ab^{2n}, 1)$, $(1, ab^{2n})$	18
$(1,1)$	b^2	(ab^{2n}, b^2), (b^4, ab^{2n}),(ab^{2i}, ab^{2j}) $(2i-2j \equiv 2 \mod 6)$	9
$(1,1)$	b^4	(b^2, ab^{even}), (a, b^4), (a, ab^2),(ab^2, ab^4), (ab^4, a), (ab^2, b^4), (ab^4, b^4)	9
$(1,\xi)$	1	$(b^{\text{even}}, b^{\text{odd}})$, $(1, ab^{\text{odd}})$,(ab^{2n}, b^3), $(ab^{2n}, ab^{2n}z)$	18
$(1,\xi)$	b^2	(b^4, ab^{odd}), (a, b^5), (a, ab), (ab^4, ab^5), (ab^2, ab^3), $(ab^2, b^5$, (ab^4, b^5)	9
$(1,\xi)$	b^4	(b^2, ab^{odd}), (a, b), (a, ab^5), (ab^2, ab), (ab^4, ab^3), (ab^2, b), (ab^4, b)	9
(ξ,ξ)	1	$(b^{\text{odd}}, b^{\text{odd}})$, (ab^{2k+1}, ab^{2k+1}),$(ab^{2n}z, z)$, $(z, ab^{2n}z)$	18
(ξ,ξ)	b^2	(b, ab^{odd}), (ab, ab^5), (ab^3, b^5),(ab^3, ab), (ab^5, ab^3), (ab^5, b^5), (ab, b^5)	9
(ξ,ξ)	b^4	(b^5, ab^{odd}), (ab^5, ab), (ab^3, b),(ab^3, ab^5), (ab, ab^3), (ab^5, b), (ab, b)	9

Using table 1, the genus-one partition function is given by

$$Z_1([X/D_6]) = \frac{1}{|D_6|} \sum_{gh=hg} Z_{g,h}, \tag{3.30}$$

$$= \frac{18}{12}\left(Z_{1,1} + Z_{1,\xi} + Z_{\xi,1} + Z_{\xi,\xi}\right), \tag{3.31}$$

$$= (3)\, Z_1([X/\mathbb{Z}_2]), \tag{3.32}$$

consistent with the prediction (3.28).

Now, to see dilaton shifts, we have to compute a partition function at genus different from one, and we will work through the combinatorics here for $g = 2$. Every sector is determined by group elements $a_{1,2}$, $b_{1,2}$ such that

$$[a_1, b_1]\,[a_2, b_2] \;=\; 1. \tag{3.33}$$

Let $(a_1|b_1|a_2|b_2)$ denote group elements, then sectors that contribute to the $(1|1|1|1)$ sector of $[X/\mathbb{Z}_2]$ at genus two are, schematically,

$$([a_1, b_1] = 1)\,([a_2, b_2] = 1) \tag{3.34}$$

$$+\;\left([a_1, b_1) = b^2\right)\left([a_2, b_2] = b^4\right) \tag{3.35}$$

$$+\;\left([a_1, b_1] = b^4\right)\left([a_2, b_2 0 = b^2\right), \tag{3.36}$$

of which there are

$$(18)\,(18) \;+\; (9)\,(9) \;+\; (9)\,(9) \;=\; 486. \tag{3.37}$$

The counting of other contributions is very similar, and so we find for the genus two partition function

$$Z_2([X/D_6]) = \frac{1}{|D_6|}\sum_{a_i,b_i} Z(a_1,b_1,a_2,b_2), \tag{3.38}$$

$$= \frac{486}{12}|\mathbb{Z}_2| Z_2([X/\mathbb{Z}_2]), \tag{3.39}$$

$$= (81)\, Z_2([X/\mathbb{Z}_2]). \tag{3.40}$$

Now, let us compare to the prediction of (3.2). Here, the three universes are associated to representations of dimensions 1, 1, and 2, so (3.2) predicts that at genus g,

$$Z_g([X/D_6]) \;=\; |D_3|^{2g-2}\left(1+1+(2)^{2-2g}\right) Z_g([X/\mathbb{Z}_2]). \tag{3.41}$$

At genus $g=2$,

$$Z_2([X/D_6]) = |D_3|^2\left(1+1+(2)^{-2}\right) Z_2([X/\mathbb{Z}_2]), \tag{3.42}$$

$$= (36)\left(2+\frac{1}{4}\right) Z_2([X/\mathbb{Z}_2]), \tag{3.43}$$

$$= (36)\left(\frac{9}{4}\right) Z_2([X/\mathbb{Z}_2]) \;=\; (81)\, Z_2([X/\mathbb{Z}_2]), \tag{3.44}$$

matching the result for the genus two partition function above.

Next, let us outline the same consistency test for genus $g=3$. Here, the partition function is defined by 6-tuples $a_{1,2,3}$, $b_{1,2,3}$, such that

$$[a_1,b_1]\,[a_2,b_2]\,[a_3,b_3] \;=\; 1. \tag{3.45}$$

The contributions to the D_6 orbifold that contribute to any sector of the $\mathbb{Z}_2$ orbifold are of the form

$$([a_1,b_1]=1)\,([a_2,b_2]=1)\,([a_3,b_3]=1) \tag{3.46}$$

$$+\;\left([a_1,b_1]=b^2\right)\left([a_2,b_2]=b^2\right)\left([a_3,b_3]=b^2\right) \tag{3.47}$$

$$+\;\left([a_1,b_1]=b^4\right)\left([a_2,b_2]=b^4\right)\left([a_3,b_3]=b^4\right) \tag{3.48}$$

$$+\;([a_1,b_1]=1)\left([a_2,b_2]=b^2\right)\left([a_3,b_3]=b^4\right) \tag{3.49}$$

$$+\;([a_1,b_1]=1)\left([a_2,b_2]=b^4\right)\left([a_3,b_3]=b^2\right) \tag{3.50}$$

$$+\;\left([a_1,b_1]=b^2\right)([a_2,b_2]=1)\left([a_3,b_3]=b^4\right) \tag{3.51}$$

$$+\;\left([a_1,b_1]=b^2\right)\left([a_2,b_2]=b^4\right)([a_3,b_3]=1) \tag{3.52}$$

$$+\;\left([a_1,b_1]=b^4\right)([a_2,b_2]=1)\left([a_3,b_3]=b^2\right) \tag{3.53}$$

$$+\;\left([a_1,b_1]=b^4\right)\left([a_2,b_2]=b^2\right)([a_3,b_3]=1), \tag{3.54}$$

and from table 1, there are

$$(18)^3 + (9)^3 + (9)^3 + (18)(9)(9)(6) = 16038 \tag{3.55}$$

such 6-tuples. We find for the genus three partition function

$$Z_3([X/D_6]) = \frac{1}{|D_6|} \sum_{a_i, b_i} Z(a_1, b_1, a_2, b_2, a_3, b_3), \tag{3.56}$$

$$= \frac{16038}{12} |\mathbb{Z}_2| Z_3([X/\mathbb{Z}_2]) = (2673) Z_3([X/\mathbb{Z}_2]). \tag{3.57}$$

Now, let us compare to the prediction of (3.2). Here, the three universes are associated to representations of dimensions 1, 1, and 2, so (3.2) predicts that at genus three,

$$Z_3([X/D_6]) = |D_3|^{2(3)-2} \left(1 + 1 + (2)^{2-2(3)}\right) Z_3([X/\mathbb{Z}_2]), \tag{3.58}$$

$$= (6)^4 \left(2 + \frac{1}{16}\right) Z_3([X/\mathbb{Z}_2]), \tag{3.59}$$

$$= (1296) \left(\frac{33}{16}\right) Z_3([X/\mathbb{Z}_2]) = (81)(33) Z_3([X/\mathbb{Z}_2]), \tag{3.60}$$

$$= (2673) Z_3([X/\mathbb{Z}_2]), \tag{3.61}$$

matching the result for the genus three partition function above. This provides another, rather intricate, test of the dilaton shift conjecture (3.2).

3.2.7. Two-dimensional Dijkgraaf-Witten theory . Two-dimensional Dijkgraaf-Witten theory is the theory of an orbifold of a point. The partition function of two-dimensional Dijkgraaf-Witten theory with orbifold group G is[4] (see [51–53] and also e.g. [54–57])

$$Z_g(G) = |G|^{2g-2} \sum_R (\dim R)^{2-2g}. \tag{3.62}$$

If there is no discrete torsion, the sum is over all ordinary irreducible representations R, and the partition function above can also be written

$$\frac{1}{|G|} |\mathrm{Hom}(\pi_1, G)|. \tag{3.63}$$

In the presence of discrete torsion, the form of the partition function Z_g is unchanged, though the sum runs over irreducible projective representations (determined by the discrete torsion), rather than over ordinary irreducible representations, see e.g. [57, section C.1, equation (C.28)]. We do not include discrete torsion for the moment, but will return to Dijkgraaf-Witten theory with discrete torsion later in section 3.3.2.

Now, let us consider the decomposition of two-dimensional Dijkgraaf-Witten theory for group G. As all of the orbifold group G acts trivially, it decomposes, into a disjoint union

[4] Up to an overall normalization, as previously discussed. Our conventions here are consistent with usage elsewhere in this paper.

of points indexed by irreducible representations of G. The partition function sum (3.62) precisely matches that of the dilaton shift conjecture (3.2), with universes indexed by irreducible representations of G, and with each X_U a point.

In particular, two-dimensional Dijkgraaf-Witten theory provides a general test of the dilaton shift conjecture (3.2), in cases in which $K = G$ is nonabelian. (Later in section 3.3.2 we will apply it as a test in cases in which the restriction of discrete torsion to K is nontrivial.)

In passing, the dilaton shift factors at genus zero, namely

$$\frac{(\dim R)^2}{|G|^2}, \tag{3.64}$$

are proportional to the Plancherel measure on the set of irreducible representations

$$\frac{(\dim R)^2}{|G|}, \tag{3.65}$$

a point to which we shall return in section 6.

3.3. Examples in orbifolds with discrete torsion

In this section we will study decomposition in orbifolds with discrete torsion. Briefly, we will see in numerous examples that the dilaton shift factors on universes have the same form as previously discussed.

3.3.1. Conjecture, split into three cases . Decomposition in orbifolds with discrete torsion was discussed in [58], and has a somewhat more complex form. Consider an orbifold $[X/\Gamma]_\omega$, where $\omega \in H^2(\Gamma, U(1))$ (with trivial action on the coefficients), and where a subgroup $K \subset \Gamma$ acts trivially. Describe G by the extension

$$1 \longrightarrow K \stackrel{\iota}{\longrightarrow} \Gamma \stackrel{\pi}{\longrightarrow} \Gamma/K \longrightarrow 1. \tag{3.66}$$

Then, decomposition is described in terms of the three cases, which we briefly review next.

1. In the case $\iota^*\omega \neq 0$, then,

$$\mathrm{QFT}\left([X/\Gamma]_\omega\right) = \mathrm{QFT}\left(\left[\frac{X \times \hat{K}_{\iota^*\omega}}{\Gamma/K}\right]_{\hat{\omega}}\right), \tag{3.67}$$

   where $\hat{K}_{\iota^*\omega}$ denotes the set of irreducible projective representations of K, twisted by $\iota^*\omega$, and $\hat{\omega}$ is discrete torsion on the factors.
2. Suppose that $\iota^*\omega = 0$, then there is an exact sequence

$$H^2\left(\Gamma/K, U(1)\right) \stackrel{\pi^*}{\longrightarrow} L \stackrel{\beta}{\longrightarrow} H^1\left(\Gamma/K, H^1\left(K, U(1)\right)\right), \tag{3.68}$$

   where

$$L = \mathrm{Ker}\, \iota^* : H^2\left(\Gamma, U(1)\right) \longrightarrow H^2\left(K, U(1)\right). \tag{3.69}$$

If $\beta(\omega) \neq 0$ and, for simplicity, K is in the center of Γ, then

$$\mathrm{QFT}\left([X/\Gamma]_{\omega}\right) = \mathrm{QFT}\left(\left[\frac{X \times \mathrm{Coker}\left(\beta\left(\omega\right)\right)}{\mathrm{Ker}\left(\beta\left(\omega\right)\right)}\right]_{\hat{\omega}}\right), \tag{3.70}$$

where we interpret $\beta(\omega)$ as a homomorphism $\Gamma/K \to \hat{K}$, and $\hat{\omega}$ denotes discrete torsion on factors.

3. The final case is that $\iota^*\omega = 0$ and $\beta(\omega) = 0$. Then, there exists $\overline{\omega} \in H^2(\Gamma/K, U(1))$ such that $\omega = \pi^*\overline{\omega}$, and

$$\mathrm{QFT}\left([X/\Gamma]_{\omega}\right) = \mathrm{QFT}\left(\left[\frac{X \times \hat{K}}{\Gamma/K}\right]_{\hat{\omega}}\right). \tag{3.71}$$

In this case, the effect of $\overline{\omega}$ is to shift the discrete torsion $\hat{\omega}$ on factors relative to the case $\omega = 0$.

We conjecture that the dilaton shift factors have the same form in the case in which the orbifold has discrete torsion, namely that the genus g partition function is

$$Z_g\left([X/\Gamma]_{\omega}\right) = \sum_U \left(\frac{\dim R_U}{|K|}\right)^{2-2g} Z_g\left(X_U\right), \tag{3.72}$$

where X_U denotes the theory for Universe U, as determined by the statements above, and where Z_g is normalized in the same way discussed in section 3.2.1. (As discussed in that section, we could also phrase this more invariantly in terms of relative dilaton shifts.) However, the interpretation of R_U differs between the three cases, as we describe below:

1. In the case $\iota^*\omega \neq 0$, R_U denotes a $\iota^*\omega$-twisted projective irreducible representation in $\hat{K}_{\iota^*\omega}$ in the orbit of Γ/K.
2. In the case $\iota^*\omega = 0$ and $\beta(\omega) \neq 0$, R_U denotes an ordinary irreducible representation in $\mathrm{Coker}(\beta(\omega)) \subset \hat{K}$ in the orbit of $\mathrm{Ker}(\beta(\omega))$, and in the trivial case $\mathrm{Coker}(\beta(\omega)) = 0$, we take $\dim R_U = 1$.
3. In the case $\iota^*\omega = 0$ and $\beta(\omega) = 0$, R_U denotes an ordinary irreducible representation of K, an element of $\hat{K}$, in the orbit of Γ/K.

In passing, note that the case of vanishing discrete torsion is part of case (3), and it is easy to see that the conjecture for that case correctly specializes to previous results.

We will check the conjecture above in examples of each type given, in the next several subsections.

3.3.2. Example of case (1): two-dimensional Dijkgraaf-Witten theory .

Previously in section 3.2.7 we discussed two-dimensional Dijkgraaf-Witten theory (without discrete torsion) as an example in which the conjecture for dilaton shift factors could be checked.

Since the entire orbifold group acts trivially, two-dimensional Dijkgraaf-Witten theory with discrete torsion is an example of case (1).

Our analysis is nearly identical to the case without discrete torsion. Here, the partition function of Dijkgraaf-Witten theory for finite group G is well-known to be given by

$$Z_g\left(G\right) = \sum_R \left(\frac{\dim R}{|G|}\right)^{2-2g} Z_g\left(\mathrm{point}\right), \tag{3.73}$$

where the sum is over irreducible projective representations (twisted by the $\omega \in H^2(G, U(1))$ corresponding to discrete torsion). This immediately reproduces the prediction of the conjecture of section 3.3.1.

3.3.3. Example of case (2): $[X/\mathbb{Z}_2 \times \mathbb{Z}_2]$*,* $K = \mathbb{Z}_2$*.* Consider the case of the orbifold $[X/\mathbb{Z}_2 \times \mathbb{Z}_2]$ with nontrivial discrete torsion, and with one trivially-acting $\mathbb{Z}_2$ factor. This was discussed (at genus one) in [58, section 5.1]. Briefly, the analysis there predicted

$$\mathrm{QFT}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]_\omega\right) \,=\, \mathrm{QFT}\,(X)\,. \tag{3.74}$$

In terms of the conjecture, it was argued in [58, section 5.1] that $\iota^* \omega = 0$ and $\beta(\omega)$ is an isomorphism, hence

$$\mathrm{Ker}\,(\beta\,(\omega)) \,=\, 0 \,=\, \mathrm{Coker}\,(\beta\,(\omega))\,. \tag{3.75}$$

Discrete torsion phase factors on Riemann surfaces of genus greater than one can be found in e.g. [59, equation (15)], [60]. In the case that the orbifold group G is abelian, those phase factors reduce to a product of genus-one phase factors, so that on a genus g Riemann surface, the phase factor is

$$\prod_{i=1}^{g} \frac{\omega\,(a_i, b_i)}{\omega\,(b_i, a_i)}, \tag{3.76}$$

where $\{a_1, b_1, a_2, b_2, \ldots, a_g, b_g\}$ are the elements of G defining a given genus g twisted sector.

Now, write the two generators of $\mathbb{Z}_2 \times \mathbb{Z}_2$ as x, y, where x acts trivially on X, and y acts nontrivially on X.

Let $Z(a_1, b_1, a_2, b_2, \ldots, b_g)$ denote the twisted sector arising from a polygon with sides labelled by the group elements $a_1, \ldots, b_g$. Generalizing the computations of [58], it is straightforward to check that contributions from $Z(a_1, b_1, \ldots, b_g)$ with any $a_i \notin \{1, x\}$ or $b_i \notin \{1, x\}$ cancel out. That leaves 4^g remaining twisted sectors, and since x acts trivially, they can all be identified with $Z_g(X)$. Including the overall normalization of $1/|\mathbb{Z}_2 \times \mathbb{Z}_2|$, we find that the genus g partition function is given by

$$Z_g\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]_\omega\right) \,=\, \frac{4^g}{|\mathbb{Z}_2 \times \mathbb{Z}_2|} Z_g\,(X) \,=\, 4^{g-1} Z_g\,(X)\,. \tag{3.77}$$

Now, let us compare to the prediction of section 3.3.1. Here, since $\mathrm{Coker}(\beta(\omega)) = 0$, we take $\dim R_U = 1$, so the dilaton shift factor on each Universe is

$$\left(\frac{\dim R_U}{|K|}\right)^{2-2g} \,=\, \left(|K|^2\right)^{g-1} \,=\, 4^{g-1}, \tag{3.78}$$

as $K = \mathbb{Z}_2$. Since both the kernel and cokernel of β vanish, there is only one Universe, so the prediction is predicted to be

$$Z_g\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]_\omega\right) \,=\, 4^{g-1} Z_g\,(X)\,, \tag{3.79}$$

which matches the result above.

3.3.4. Example of case (2): $[X/\mathbb{Z}_2 \times \mathbb{Z}_4]$, $K = \mathbb{Z}_4$. Consider the case of the orbifold $[X/\mathbb{Z}_2 \times \mathbb{Z}_4]$ with nontrivial discrete torsion, and with trivially-acting $\mathbb{Z}_4$. This was discussed (at genus one) in [58, section 5.3]. Briefly, the analysis there predicted

$$\mathrm{QFT}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_4]_\omega\right) = \mathrm{QFT}\left(X \coprod X\right). \tag{3.80}$$

In terms of the conjecture, it was argued in [58, section 5.3] that

$$\mathrm{Ker}\left(\beta\left(\omega\right)\right) = 0, \;\; \mathrm{Coker}\left(\beta\left(\omega\right)\right) = \mathbb{Z}_2. \tag{3.81}$$

As in the previous subsection, since the orbifold group is abelian, discrete torsion on a genus g Riemann surface is the product of discrete torsion on factorized genus one surfaces.

Write the generator of $\mathbb{Z}_2$ as a, and the generator of $\mathbb{Z}_4$ as b. We assume that b acts trivially.

Let $Z(a_1, b_1, \ldots, b_g)$ denote the twisted sector arising from a polygon with sides labelled by group elements $a_1, \ldots, b_g \in \mathbb{Z}_2 \times \mathbb{Z}_4$. Generalizing the computations of [58, section 5.3], it is straightforward to check that most sectors cancel out (due to signs arising from discrete torsion phase factors), with the exception of sectors in which all of the a_i, b_i are powers of b. At genus g, there are $2g$ such factors, and as b acts trivially, the partition function is then

$$Z_g\left([X/\mathbb{Z}_2 \times \mathbb{Z}_4]_\omega\right) = \frac{4^{2g}}{|\mathbb{Z}_2 \times \mathbb{Z}_4|} Z(X) = 4^{2g-2} Z\left(X \coprod X\right). \tag{3.82}$$

Now, let us compare to the prediction of section 3.3.1. Here, since $\mathrm{Coker}(\beta(\omega)) = \mathbb{Z}_2$, dim $R_U = 1$, so the dilaton shift factor on each Universe is predicted to be

$$\left(\frac{\dim R_U}{|K|}\right)^{2-2g} = \left(|K|\right)^{2g-2} = 4^{2g-2}, \tag{3.83}$$

as $K = \mathbb{Z}_4$. Thus, the prediction is

$$Z_g\left([X/\mathbb{Z}_2 \times \mathbb{Z}_4]_\omega\right) = 4^{2g-2} Z\left(X \coprod X\right), \tag{3.84}$$

which matches the result above.

3.3.5. Example of case (2): $[X/\mathbb{Z}_2 \times \mathbb{Z}_4]$, $K = \mathbb{Z}_2$. Consider the case of the orbifold $[X/\mathbb{Z}_2 \times \mathbb{Z}_4]$ with nontrivial discrete torsion, and with trivially-acting $\mathbb{Z}_2$. This was discussed in [58, section 5.4]. Briefly, the analysis there predicted

$$\mathrm{QFT}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]_\omega\right) = \mathrm{QFT}\left([X/\mathbb{Z}_2]\right). \tag{3.85}$$

In terms of the conjecture, it was argued in [58, section 5.4] that

$$\mathrm{Ker}\left(\beta\left(\omega\right)\right) = \mathbb{Z}_2, \;\; \mathrm{Coker}\left(\beta\left(\omega\right)\right) = 0. \tag{3.86}$$

As in the previous subsection, since the orbifold group is abelian, discrete torsion on a genus g Riemann surface is the product of discrete torsion on factorized genus one surfaces.

Write the generator of $\mathbb{Z}_2$ as a, and the generator of $\mathbb{Z}_4$ as b. We assume that a acts trivially.

Let $Z(a_1, b_1, \ldots, b_g)$ denote the twisted sector arising from a polygon with sides labelled by group elements $a_1, \ldots, b_g \in \mathbb{Z}_2 \times \mathbb{Z}_4$. First, consider a genus one computation. From the discrete torsion phase factors in [58, table D.2]. it is straightforward to check that, for example,

$$\epsilon\left(ab^i, b^j\right) = (-)^j, \ \ \epsilon\left(b^i, ab^j\right) = (-)^i, \ \ \epsilon\left(ab^i, ab^j\right) = (-)^{i+j}. \tag{3.87}$$

Given that a acts trivially, it is then straightforward to compute that the genus g partition function is given by

$$Z_g\left([X/\mathbb{Z}_2 \times \mathbb{Z}_4]_\omega\right) = \frac{1}{|\mathbb{Z}_2 \times \mathbb{Z}_4|} \sum_{i_1, j_1, i_2, \ldots, j_g = 0}^{3} Z\left(b^{i_1}, b^{j_1}, b^{i_2}, \ldots, b^{j_g}\right) \times \left(1 + (-)^i + (-)^j + (-)^{i+j}\right)^g. \tag{3.88}$$

Now,

$$1 + (-)^i + (-)^j + (-)^{i+j} = \begin{cases} 4 & i \text{ even and } j \text{ even}, \\ 0 & \text{else}, \end{cases} \tag{3.89}$$

hence

$$Z_g\left([X/\mathbb{Z}_2 \times \mathbb{Z}_4]_\omega\right) = \frac{4^g}{|\mathbb{Z}_2 \times \mathbb{Z}_4|} |\mathbb{Z}_2| Z_g\left(\left[X/\langle b^2 \rangle\right]\right), \tag{3.90}$$

$$= 4^{g-1} Z_g\left([X/\mathbb{Z}_2]\right). \tag{3.91}$$

Now, let us compare to the prediction of section 3.3.1. Here, since $\mathrm{Coker}(\beta(\omega)) = 0$, dim $R_U = 1$, so the dilaton shift factor on each Universe is predicted to be

$$\left(\frac{\dim R_U}{|K|}\right)^{2-2g} = (|K|)^{2g-2} = 2^{2g-2} = 4^{g-1}, \tag{3.92}$$

as $K = \mathbb{Z}_2$. Thus, the prediction is

$$Z_g\left([X/\mathbb{Z}_2 \times \mathbb{Z}_4]_\omega\right) = 4^{g-1} Z_g\left([X/\mathbb{Z}_2]\right), \tag{3.93}$$

which matches the result above.

3.3.6. Examples of case (3): trivially-acting central subgroups . Previously in section 3.2.2 we discussed examples of dilaton shift factors in decomposition in orbifolds in which the trivially-acting subgroup K is central in the orbifold group Γ. In this section we extend those remarks to the case in which the decomposing orbifold has discrete torsion, in case (3), so that the discrete torsion in the Γ orbifold is a pullback from discrete torsion in a Γ/K orbifold.

Let $[\omega] \in H^2(\Gamma, U(1))$ denote discrete torsion in the Γ orbifold, and assume that $\omega = \pi^* \tilde{\omega}$ for some $[\tilde{\omega}] \in H^2(G, U(1))$. As discussed in [58], in this case, the effect of ω is merely to shift the discrete torsion on universes by $\tilde{\omega}$; the decomposition is otherwise unchanged. If we let $\epsilon_g(a_i, b_i)$ denote the genus-g discrete torsion phases in a given sector defined by $a_i, b_i \in \Gamma$, then in this case,

$$\epsilon_g\left(a_i, b_i\right) = \epsilon_g\left(\overline{a}_i, \overline{b}_i\right) \tag{3.94}$$

for $\overline{a}_i, \overline{b}_i \in G$, and so ε_g can be completely absorbed into $Z(\overline{a}_i, \overline{b}_i)$. In this case, we see immediately that the dilaton shifts are identical to those without the discrete torsion. For example, assuming again that K is central, the genus g partition function is

$$Z_g \;=\; |K|^{2g-2} \sum_U Z'_g\left(X_U\right), \tag{3.95}$$

where $Z'_g(X_U)$ is the genus-g partition function of Universe X_U, with discrete torsion $\tilde{\omega}$. This confirms our conjecture of section 3.3.1 for this case of discrete torsion in the Γ orbifold, since all representations R_U of K are one-dimensional for K central.

Next, we will discuss some concrete examples of this form.

3.3.7. Example of case (3): $[X/\mathbb{Z}_2 \times \mathbb{Z}_4]$, $K = \mathbb{Z}_2$. Consider the orbifold $[X/\mathbb{Z}_2 \times \mathbb{Z}_4]$ with nontrivial discrete torsion, and one trivially-acting $\mathbb{Z}_2$ factor. Write $\mathbb{Z}_2 = \langle x \rangle$, $\mathbb{Z}_4 = \langle y \rangle$, and take the trivially-acting $K = \langle y^2 \rangle \cong \mathbb{Z}_2$. This was discussed (at genus one) in [58, section 6.1]. Briefly, the analysis there argued that $\iota^* \omega = 0$, $\omega = \pi^* \overline{\omega}$ for $\overline{\omega} \in H^2(\mathbb{Z}_2 \times \mathbb{Z}_2, U(1))$, and that

$$\mathrm{QFT}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_4]_\omega\right) \;=\; \mathrm{QFT}\left(\coprod_2 [X/\mathbb{Z}_2 \times \mathbb{Z}_2]_{\overline{\omega}} \right). \tag{3.96}$$

As noted previously, in abelian orbifolds, discrete torsion phases on a genus g Riemann surface are the product of discrete torsion on factorized genus one surfaces. The genus one phases can be found in [58, table D.2], and from the table there it is easy to see that multiplication by y^2 does not change the discrete torsion phase. It is then straightforward to compute

$$Z_g\left([X/\mathbb{Z}_2 \times \mathbb{Z}_4]_\omega\right) = \frac{2^{2g}}{|\mathbb{Z}_2 \times \mathbb{Z}_4|} |\mathbb{Z}_2 \times \mathbb{Z}_2| \, Z_g\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]_{\overline{\omega}}\right), \tag{3.97}$$

$$= \frac{2^{2g}}{2} Z_g\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]_{\overline{\omega}}\right), \tag{3.98}$$

$$= 2^{2g-2} Z_g\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]_{\overline{\omega}}\right). \tag{3.99}$$

Now, let us compare to the prediction of section 3.3.1. Here, $|K| = 2$ and $\dim R_U = 1$ for each Universe, so the dilaton shift factor is predicted to be

$$\left(\frac{\dim R_U}{|K|} \right)^{2-2g} \;=\; \left(|K|\right)^{2g-2} \;=\; 2^{2g-2}, \tag{3.100}$$

as $K = \mathbb{Z}_2$. Thus, the prediction is

$$Z_g\left([X/\mathbb{Z}_2 \times \mathbb{Z}_4]_\omega\right) \;=\; 2^{2g-2} Z_g\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]_{\overline{\omega}}\right), \tag{3.101}$$

which matches the result above.

3.3.8. Example of case (3): $[X/\mathbb{Z}_4 \rtimes \mathbb{Z}_4]$, $K = \mathbb{Z}_2$. Consider next the case of the orbifold $[X/\mathbb{Z}_4 \rtimes \mathbb{Z}_4]$ (the semidirect product of two copies of $\mathbb{Z}_4$), with discrete torsion, and a trivially-acting $\mathbb{Z}_2$ in the center of $\mathbb{Z}_4 \rtimes \mathbb{Z}_4$. This was discussed (at genus one) in [58, section 6.2]. We will use the same notation as in [58, appendix D.4]; for example, we let x, y denote the generators of the two copies of $\mathbb{Z}_4$, and $K = \langle y^2 \rangle$, so that $\mathbb{Z}_4 \rtimes \mathbb{Z}_4 / K = D_4$. Briefly, the analysis of [58, section 6.2] predicted that

$$\mathrm{QFT}\left([X/\mathbb{Z}_4 \rtimes \mathbb{Z}_4]_\omega\right) = \mathrm{QFT}\left(\coprod_2 [X/D_4]_{\overline{\omega}}\right), \tag{3.102}$$

where $\overline{\omega} \in H^2(D_4, U(1))$ and $\omega = \pi^* \overline{\omega}$.

Following the analysis of [58, section 6.2], and using the fact that K lies in the center, it is straightforward to compute that

$$Z_g\left([X/\mathbb{Z}_4 \rtimes \mathbb{Z}_4]_\omega\right) = \frac{2^{2g}}{|\mathbb{Z}_4 \rtimes \mathbb{Z}_4|} |D_4| Z_g\left([X/D_4]_{\overline{\omega}}\right), \tag{3.103}$$

$$= \frac{2^{2g}}{2} Z_g\left([X/D_4]_{\overline{\omega}}\right), \tag{3.104}$$

$$= 2^{2g-2} Z_g\left(\coprod_2 [X/D_4]_{\overline{\omega}}\right). \tag{3.105}$$

Now, let us compare to the prediction of section 3.3.1. Here, $|K| = 2$ and $\dim R_U = 1$ for each Universe, so the dilaton shift factor is predicted to be

$$\left(\frac{\dim R_U}{|K|}\right)^{2-2g} = (|K|)^{2g-2} = 2^{2g-2}, \tag{3.106}$$

as $K = \mathbb{Z}_2$. Thus, the prediction is

$$Z_g\left([X/\mathbb{Z}_4 \rtimes \mathbb{Z}_4]_\omega\right) = 2^{2g-2} Z_g\left(\coprod_2 [X/D_4]_{\overline{\omega}}\right), \tag{3.107}$$

which matches the result above.

3.4. Examples with quantum symmetries

Next, let us consider examples with quantum symmetries in the sense of [26]. Consider orbifolds $[X/\Gamma]$ where a central subgroup $K \subset \Gamma$ acts trivially, and $G = \Gamma/K$. The quantum symmetry, as the term is used in [26], is an element $B \in H^1(G, H^1(K, U(1))$ (generalizing older notions of quantum symmetries), that provides a relative phase between sectors. It was argued in [26] that in the presence of such B,

$$\mathrm{QFT}\left([X/\Gamma]_B\right) = \mathrm{QFT}\left(\left[\frac{X \times \widehat{\mathrm{Coker}\, B}}{\mathrm{Ker}\, B}\right]\right), \tag{3.108}$$

This notion of quantum symmetries specializes to both ordinary quantum symmetries and to results on discrete torsion, as discussed in [26]. For example, as discussed in [58, section 5.1], the ordinary quantum symmetry of a $\mathbb{Z}_2$ orbifold, and the fact that orbifolding by the quantum symmetry returns the original theory, can be understood as decomposition in a $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold with discrete torsion. In terms of orbifolds with discrete torsion, the pertinent decomposition is of case (2) in the classification of section 3.3. The decomposition statement above therefore generalizes those results.

We conjecture that the dilaton shift factors have the same form as we have seen in the previous examples, namely

$$Z_g\left([X/\Gamma]_\omega\right) = \sum_U \left(\frac{\dim R_U}{|K|}\right)^{2-2g} Z_g\left(X_U\right), \tag{3.109}$$

where X_U denotes the theory for universe U, and R_U is an irreducible representation of Coker B, in the orbit of Ker B. (More invariantly, these dilaton shifts should be understood as relative dilaton shifts between the partition functions for Universe U and for the ambient theory, as discussed in section 3.2.1.)

We will work through several examples.

For use in computing examples, we make an observation next. Describe the contributions to a genus g partition function of $[X/\Gamma]$ as

$$(a_1|b_1|a_2|b_2|\cdots|a_g|b_g) \tag{3.110}$$

for $a_{1-g}, b_{1-g} \in \Gamma$, where

$$\prod_{i=1}^{g}[a_i, b_i] = 1, \tag{3.111}$$

then a quantum symmetry $B \in H^1(G, H^1(K, U(1)))$ relates sectors as

$$(a_1 z|b_1|a_2|b_2|\cdots|b_g) = B(\pi(b_1), z)\,(a_1|b_1|a_2|b_2|\cdots|a_g|b_g), \tag{3.112}$$

$$(a_1|b_1 z|a_2|b_2|\cdots|b_g) = B(\pi(a_1), z)^{-1}\,(a_1|b_1|a_2|b_2|\cdots|a_g|b_g), \tag{3.113}$$

$$(a_1|b_1|a_2 z|b_2|\cdots|b_g) = B(\pi(b_2), z)\,(a_1|b_1|a_2|b_2|\cdots|a_g|b_g), \tag{3.114}$$

$$(a_1|b_1|a_2|b_2 z|\cdots|b_g) = B(\pi(a_2), z)^{-1}\,(a_1|b_1|a_2|b_2|\cdots|a_g|b_g), \tag{3.115}$$

and so forth, where $z \in K$.

Now, consider the case of a $\mathbb{Z}_4$ orbifold with trivially-acting $K = \mathbb{Z}_2 \subset \mathbb{Z}_4$, and a nontrivial quantum symmetry in Hom$(\mathbb{Z}_2, \hat{\mathbb{Z}}_2)$. This was discussed in [26, section 4.1.1], which argued that

$$\mathrm{QFT}([X/\mathbb{Z}_4]) = \mathrm{QFT}(X). \tag{3.116}$$

From the general prediction (3.109), since there is only one Universe, which is associated to a representation of dimension 1, it should be the case that

$$Z_g([X/\mathbb{Z}_4]) = |K|^{2g-2} Z(X) = 2^{2g-2} Z(X). \tag{3.117}$$

Label the elements of $\mathbb{Z}_4$ by $i \in \{0, \ldots, 3\}$ as in [26, section 4.1.1]. Then, for example,

$$(a_1 + 2|b_1|a_2|b_2|\cdots|b_g) = (-)^{b_1}\,(a_1|b_1|a_2|b_2|\cdots|b_g). \tag{3.118}$$

It is straightforward to check that if any of a_{1-g}, $b_{1-g} \in \{0, \ldots, 3\}$ are odd, then contributions from that sectors cancel out. For example, if a_1 is odd, the contributions from the sector with $b_1 + 2$ cancels out the contribution from that sector. As a result, the only sectors that contribute to the partition function have all of a_{1-g}, b_{1-g} even, and these all match the contribution from the $(0|0|0|\cdots|0)$ sector, which is exactly $Z(X)$.

Counting contributions, we have

$$Z_g\left([X/\mathbb{Z}_4]_B\right) \;=\; \frac{1}{|\mathbb{Z}_4|} 2^{2g} Z(X) \;=\; 2^{2g-2} Z(X)\,, \tag{3.119}$$

which confirms the prediction (3.109).

Next, let us consider the orbifold $[X/\mathbb{Z}_{2k}]_B$, for k even, as discussed in [26, section 4.1.2]. Here, $K = \mathbb{Z}_k \subset \mathbb{Z}_{2k}$ acts trivially, with a nontrivial quantum symmetry $B \in H^1(G, H^1((K, U(1))))$ such that, if x denotes the generator of $\mathbb{Z}_{2k}$, so that x^2 generates $K = \mathbb{Z}_k$, then

$$x^2 \mathop{\square}_{x} \;=\; -\left(1 \mathop{\square}_{x} \right). \tag{3.120}$$

For this theory, decomposition predicts

$$\mathrm{QFT}\left([X/\mathbb{Z}_{2k}]_B\right) \;=\; \mathrm{QFT}\left(\coprod_{k/2} X \right). \tag{3.121}$$

Furthermore, the dilaton shift conjecture (3.109) predicts

$$Z_g\left([X/\mathbb{Z}_{2k}]_B\right) \;=\; |K|^{2g-2} Z_g\left(\coprod_{k/2} X \right), \tag{3.122}$$

since each universe is associated with a dimension-one irreducible representation of K.

It is straightforward to check this prediction explicitly, as the details are not dissimilar to the previous example. Much as happened there, if we enumerate group elements by integers in $\{0, \ldots, 2k-1\}$, then twisted sectors

$$(a_1|b_1|a_2|\cdots|b_g) \tag{3.123}$$

cancel out whenever any of the a_i or b_i are odd. The only surviving sectors are those for which all a_i and b_i are even—in which case the boundary conditions are trivial, so that the sector is equivalent to the partition function of a sigma model on X. Since there are a total of k^{2g} such sectors in which all the a_i and b_i are even, this means that

$$Z_g\left([X/\mathbb{Z}_{2k}]_B\right) = \frac{1}{|\mathbb{Z}_{2k}|} \sum_{a_i, b_i} (a_1|b_1|a_2|\cdots|b_g)\,, \tag{3.124}$$

$$= \frac{1}{|\mathbb{Z}_{2k}|} k^{2g} Z(X)\,, \tag{3.125}$$

$$= k^{2g-2} \frac{k}{2} Z(X) \;=\; k^{2g-2} Z\left(\coprod_{k/2} X \right), \tag{3.126}$$

again matching the prediction of (3.109).

3.5. Examples in gauge theories

3.5.1. Pure Yang-Mills: decomposition along center symmetries . In this section we will look at the decomposition of two-dimensional pure Yang-Mills theory along a center one-form symmetry, as described in detail in [61, section 2.4]. Two-dimensional pure Yang-Mills theories have a second decomposition, to invertible field theories [35, 36], which we will discuss in this context in section 3.5.2. We will recover the same dilaton shifts as before, and also find a connection between those dilaton shifts and moduli space volumes.

Consider pure Yang-Mills theory in two dimensions with gauge group G, which we will assume to be semisimple, and which has (finite) center K. As discussed in [61, section 2.4], this decomposes into a sum of G/K gauge theories with variable discrete theta angles $\lambda \in \hat{K}$, schematically as

$$\mathrm{QFT}(G) = \oplus_{\lambda\in\hat{K}} \mathrm{QFT}(G/K,\lambda). \tag{3.127}$$

Now, let us compare partition functions. Up to overall factors, the partition function of pure G Yang-Mills on a Riemann surface Σ of genus g is [47–49, 62–69]

$$Z(G) \propto \sum_R (\dim R)^{2-2g} \exp(-AC_2(R)). \tag{3.128}$$

For our purposes, we need to fix a normalization, which was determined in [63], [62, equation (4.18)] through a careful analysis of the zero-area limit and Reidemeister-Ray-Singer torsion to be

$$Z(G) = \left(\frac{\mathrm{Vol}(G)}{(2\pi)^{\dim G}}\right)^{2g-2} \sum_R (\dim R)^{2-2g} \exp(-AC_2(R)). \tag{3.129}$$

Now, define $Z(G/K,\lambda)$ to be the partition function of the corresponding G/K Yang-Mills theory with discrete theta angle λ (for which the sum is restricted to representations of corresponding n-ality, as discussed in [70], [61, section 2.4]).

Then, as discussed in [61, section 2.4]), decomposition becomes the statement

$$Z(G) = \left(\frac{\mathrm{Vol}(G)}{\mathrm{Vol}(G/K)}\right)^{2g-2} \sum_{\lambda\in\hat{K}} Z(G/K,\lambda), \tag{3.130}$$

$$= |K|^{2g-2} \sum_{\lambda\in\hat{K}} Z(G/K,\lambda). \tag{3.131}$$

Since K is central and so abelian, in the language of (3.2), every irreducible representation R_U of K necessarily has dimension one, and so we see that the dilaton shift factor encoded by the Reidemeister-torsion-based normalization, namely

$$|K|^{2g-2} = \left(\frac{1}{|K|}\right)^{2-2g}, \tag{3.132}$$

is of the form predicted by the dilaton shift conjecture (3.2).

3.5.2. Pure Yang-Mills: decomposition to invertibles . In the previous subsections, we looked at decompositions of two-dimensional pure gauge theories with trivially-acting subgroups K for K finite, along the center BK symmetry. In this section we consider the decomposition in which K is no longer finite, and in fact $K = G$ because it is a pure gauge theory. The resultig decomposition pure Yang-Mills in two dimensions by noninvertible one-form symmetries yields universes which are invertible field theories, as discussed in [35, 36]. We will see that the dilaton shift factors involve the same factors of $(\dim R)^2$ that we have seen elsewhere.

Specifically, the proposal of [35, 36] is that pure G Yang-Mills in two dimensions decomposes into the following union of invertible field theories, in the notation of appendix A:

$$\text{Pure } G \text{ Yang-Mills} = \coprod_R \text{Inv}\left(-C_2(R), \ln\left(\frac{(2\pi)^{\dim G}}{\text{Vol}(G)} \dim R\right)\right), \tag{3.133}$$

where the disjoint union is over irreducible representations R. This is reflected in the fact that the partition function of pure G Yang-Mills on a Riemann surface Σ of genus g is [47–49, 62–69]

$$Z(G) = \left(\frac{\text{Vol}(G)}{(2\pi)^{\dim G}}\right)^{2g-2} \sum_R (\dim R)^{2-2g} \exp(-AC_2(R)), \tag{3.134}$$

for A the area of Σ, as discussed earlier.

Since this decomposition is derived by observing that the entire gauge group G acts trivially, and $|G|$ is infinite, our previous expression for dilaton shifts (3.2) does not quite apply. However, if we modify it, by taking the universes to have dilatons

$$\left(\frac{(2\pi)^{\dim K}}{\text{Vol}(K)} \dim R\right)^{2-2g} \exp(-AC_2(R)) \tag{3.135}$$

for $K = G$, instead of

$$\left(\frac{\dim R_U}{|K|}\right)^{2-2g}, \tag{3.136}$$

then we get a close analogue of (3.2).

3.5.3. Nonabelian BF theory . Now we turn to nonabelian BF theory, or specifically BF theory for a nonabelian gauge group G, which for simplicity we take to be connected and simply-connected, at level one.

Now, BF theory is the zero-area limit of two-dimensional pure Yang-Mills (see e.g. [63, section 2]). As that zero-area limit, it also has a decomposition to countably many invertible field theories, indexed by irreducible representations of the gauge group, just as pure Yang-Mills in two dimensions.

For simplicity, in this section we will assume the genus of the Riemann surface $g > 1$. (For smaller g, the exact expression for the partition function as a series does not converge in the zero-area limit, and must be regularized, see e.g. [49, section 2.5].)

From the exact expression (3.129) for two-dimensional pure Yang-Mills, we see that the partition function of two-dimensional nonabelian *BF* theory is

$$Z(G) = \left(\frac{\mathrm{Vol}(G)}{(2\pi)^{\dim G}} \right)^{2g-2} \sum_R (\dim R)^{2-2g} \tag{3.137}$$

on a Riemann surface of genus g.

This is in the expected form

$$Z = \sum_R f(R)^\chi \tag{3.138}$$

for some function $f(R)$.

For connected and simply-connected G, *BF* theory at level one is equivalent to a disjoint union

$$\coprod_R \mathrm{Inv}\left(0, \ln\left(\frac{(2\pi)^{\dim G}(\dim R)}{\mathrm{Vol}(G)}\right)\right) \cong \coprod_R \mathrm{Inv}(0, \ln(\dim R)), \tag{3.139}$$

where the disjoint union is over all irreducible representations of G. As before, this matches the common form (2.11) mentioned in the introduction (in the sense that the representation-dependence is identical, omitting normalization constants).

BF theory recently made an appearance in [43], which reviewed the form of the partition function above, as a special case of our general conjecture (3.2).

3.5.4. Aside: moduli space volumes . The results above for decomposition in *BF* theory are related to the symplectic volume of the moduli space of flat connections. We shall review results of [62, 63] (see also e.g. [71–76]), and describe their understanding in terms of decomposition and dilaton shift factors.

First, let us review some results of [62]. Let $\mathrm{Vol}(\mathcal{M}, G)$ denote the symplectic volume of the moduli space of flat G connections over a fixed Riemann surface Σ. From [62, equation (4.19)], the partition function Z of *BF* theory is related to the volume as

$$Z(G) = \frac{\mathrm{Vol}(\mathcal{M}, G)}{|\mathcal{Z}(G)|}, \tag{3.140}$$

$$= \left(\frac{\mathrm{Vol}(G)}{(2\pi)^{\dim G}} \right)^{2g-2} \sum_R (\dim R)^{2-2g}, \tag{3.141}$$

where $\mathcal{Z}(G)$ denotes the center of G, and in the second line, Vol(G) denotes the volume of G itself, rather than the moduli space.

Note that decomposition of *BF* theory into invertibles gives a simple physical explanation for why the symplectic volume of the moduli space of flat connections can be written in the form of a sum over irreducible representations in (3.141), and the fact that each irreducible representation contributes a term proportional to $(\dim R)^\chi$ is a consequence of the dilaton shift factors.

Now, let us reconcile decomposition with the results of [62] above, by applying the analysis to two-dimensional *BF* theories and pure Yang-Mills with gauge group G/K, for K a subgroup of the center, with discrete theta angles $\lambda \in \hat{K}$. Recall that the effect of adding a discrete theta angle is to weight contributions to partition functions defined by G/K bundles with characteristic class $w \in H^2(\Sigma, K)$ (Σ the two-dimensional space) by phase factors $\exp(\langle w, \lambda \rangle)$, with $\langle , \rangle$ denoting the natural pairing. Schematically, if Z_w denotes the part of a partition function obtained by summing over bundles of fixed characteristic class w, then the whole partition function $Z(\lambda)$ for fixed discrete theta angle $\lambda \in \hat{K}$ is

$$Z(\lambda) = \sum_{w \in H^2(\Sigma, K)} \exp(\langle w, \lambda \rangle) \, Z_w. \tag{3.142}$$

The analysis in [63, section 2] relating operator determinant ratios to the measure defined by the symplectic form is local in nature, so we expect it to also apply in this case to the individual components of the moduli space, with the phase factors $\exp(\langle w, \lambda \rangle)$ just 'going along for the ride.' Thus, at least naively, the analysis of [63, section 2] seems to suggest that the partition function is related to a 'weighted volume' which adds volumes of different moduli space components (of fixed w) weighted by the same phase factors $\exp(\langle w, \lambda \rangle)$.

To that end, define $\mathrm{Vol}(\mathcal{M}, G/K, \lambda)$ to be the weighted volume of the moduli space of flat G/K connections, K a subset of the center of G, G simple and simply-connected, in which a component of the moduli space with characteristic class $w \in H^2(\Sigma, K)$ is weighted by the phase $\exp(\langle w, \lambda \rangle)$ for $\lambda \in \hat{K}$. Schematically, if V_w is the volume of a component with fixed characteristic class w, then in other words,

$$\mathrm{Vol}((\mathcal{M}, G/K, \lambda) = \sum_{w \in H^2(\Sigma, K)} \exp(\langle w, \lambda \rangle) \, V_w. \tag{3.143}$$

Now, the partition function of two-dimensional BF theory (the zero-area limit of pure Yang-Mills) with gauge group G/K and discrete theta angle λ is [70]

$$Z(G/K, \lambda) = \left(\frac{\mathrm{Vol}(G/K)}{(2\pi)^{\dim \mathrm{G}}} \right)^{2g-2} \sum_{R:\lambda} (\dim R)^{2-2g}, \tag{3.144}$$

where the sum is over irreducible representations of G (not necessarily G/K) of n-ality λ, and where we have used the normalization conventions of [62, equation (4.19)]. Applying (3.140) suggests that

$$\mathrm{Vol}(\mathcal{M}, G/K, \lambda) = |\mathcal{Z}(G/K)| \left(\frac{\mathrm{Vol}(G/K)}{(2\pi)^{\dim \mathrm{G}}} \right)^{2g-2} \sum_{R:\lambda} (\dim R)^{2-2g}. \tag{3.145}$$

As before, the form of this expression is a consequence of the decomposition of the BF theory or pure Yang-Mills to invertibles, and the terms $(\dim R)^{\chi}$ are a reflection of dilaton shifts.

Furthermore, decomposition (along the center BK symmetry of two-dimensional BF or pure Yang-Mills) then makes a prediction relating weighted moduli space volumes. Specifically, from the decomposition (3.131), namely

$$Z(G) = |K|^{2g-2} \sum_{\lambda \in \hat{K}} Z(G/K, \lambda), \tag{3.146}$$

and the relation (3.140) between partition functions and moduli space volumes, we have

$$\frac{\mathrm{Vol}(\mathcal{M}, G)}{|\mathcal{Z}(G)|} = |K|^{2g-2} \sum_{\lambda \in \hat{K}} \frac{\mathrm{Vol}(\mathcal{M}, G/K, \lambda)}{|\mathcal{Z}(G/K)|}, \tag{3.147}$$

where $\mathcal{Z}(G)$ denotes the center of G. If we make the further simplifying assumption that $K = \mathcal{Z}(G)$ (so $\mathcal{Z}(G/K) = \{1\}$), then

$$\mathrm{Vol}\left(\mathcal{M}, G\right) = |K|^{2g-1} \sum_{\lambda \in \hat{K}} \mathrm{Vol}\left(\mathcal{M}, G/K, \lambda\right). \tag{3.148}$$

Next, we study the abstract statements above in a simple concrete example, namely $G = SU(2)$ and $K = \mathbb{Z}_2$. In this case, we will be able to explicitly check the result above for the weighted moduli space volume for $SO(3)$.

First, let us quickly review results of [63] on (unweighted) moduli space volumes for $SU(2)$ and $SO(3)$. In the conventions of [63, section 4.5],

$$\mathrm{Vol}\left(SU(2)\right) = 2^{5/2}\pi^2, \tag{3.149}$$

hence, using (3.141), the symplectic volume $\mathrm{Vol}(\mathcal{M}, SU(2))$ of the moduli space of flat $SU(2)$ connections on a Riemann surface of genus g is [63, equations (3.11) and (4.73)]

$$\mathrm{Vol}\left(\mathcal{M}, SU(2)\right) = \frac{2}{(2\pi^2)^{g-1}} \sum_{n=1}^{\infty} n^{2-2g} = \frac{2}{(2\pi^2)^{g-1}} \zeta\left(2g-2\right). \tag{3.150}$$

The values of n above are simply the dimensions of irreducible representations of $SU(2)$, corresponding to spins j, where $n = 2j+1$. Similarly, using the fact that in the conventions of [63, section 4.5],

$$\mathrm{Vol}\left(SO(3)\right) = 2^{3/2}\pi^2, \tag{3.151}$$

the volume of the moduli space of flat $SO(3)$ connections is [63, equations (3.29) and (4.74)]

$$\mathrm{Vol}\left(\mathcal{M}, SO(3)\right) = \frac{1}{(8\pi^2)^{g-1}} \sum_{n=1,3,5,\cdots} n^{-(2g-2)}. \tag{3.152}$$

Here, n indexes dimensions of irreducible representations of $SO(3)$. To distinguish this quantity from the weighted moduli space volume, and to follow notation for discrete theta angles, we will use the notation

$$\mathrm{Vol}\left(\mathcal{M}, SO(3)_+\right) = \mathrm{Vol}\left(\mathcal{M}, SO(3)\right). \tag{3.153}$$

(Technically, the $+$ subscript indicates that the corresponding QFT does not have a discrete theta angle, to distinguish it from the next case we consider.)

Now, let us turn to weighted moduli space volumes. Here, the characteristic class $w \in H^2(SO(3), \mathbb{Z}_2)$ is the second Stiefel-Whitney class, so we will denote it w_2 in the remainder of this section. Let $\mathrm{Vol}(\mathcal{M}, SO(3))_-)$ denote this weighted sum of $SO(3)$ moduli space component volumes. To make the description above more concrete, define

- $V_0 =$ the sum of volumes of components of the moduli space of flat $SO(3)$ connections with w_2 trivial, and
- $V_1 =$ the sum of volumes of components of the moduli space of flat $SO(3)$ connections with w_2 nontrivial.

The phase factors $\exp(\langle w, \lambda \rangle)$ simply reduce to signs in this case. Then,

$$\mathrm{Vol}\left(\mathcal{M}, SO(3)_+\right) = V_0 + V_1, \tag{3.154}$$

$$\mathrm{Vol}\left(\mathcal{M}, SO(3)_-\right) = V_0 - V_1. \tag{3.155}$$

The partition function of the two-dimensional $SO(3)$ gauge theory with nontrivial discrete theta angle is [70]

$$Z\left(SO(3)_-\right) = \frac{1}{(8\pi^2)^{g-1}} \sum_{n=2,4,6,\cdots} n^{-(2g-2)}. \tag{3.156}$$

As the center of $SO(3)$ is trivial, equation (3.145) reduces to

$$\mathrm{Vol}\left(\mathcal{M}, SO(3)_-\right) = Z\left(SO(3)_-\right) = \frac{1}{(8\pi^2)^{g-1}} \sum_{n=2,4,6,\cdots} n^{-(2g-2)}. \tag{3.157}$$

Here, n indexes dimensions of allowed irreducible representations—of of $SU(2)$ representations that do not descend to $SO(3)$, following [70]. This is easily checked to be true by algebraically solving equations (3.154) and (3.155), verifying that for $SO(3)$, our expression for weighted moduli space sums is correct.

Decomposition along the central BK symmetry implied equation (3.147), which in this case specializes to

$$\mathrm{Vol}\left(\mathcal{M}, SO(3)_+\right) + \mathrm{Vol}\left(\mathcal{M}, SO(3)_-\right) = \frac{1}{(8\pi^2)^{g-1}} \sum_{n=1}^{\infty} n^{-2g-2}, \tag{3.158}$$

$$= \frac{1}{(2)\,4^{g-1}} \mathrm{Vol}\left(\mathcal{M}, SU(2)\right), \tag{3.159}$$

or more simply

$$\mathrm{Vol}\left(\mathcal{M}, SU(2)\right) = (2)(2)^{2g-2}\left(\mathrm{Vol}\left(\mathcal{M}, SO(3)_+\right) + \mathrm{Vol}\left(\mathcal{M}, SO(3)_-\right)\right). \tag{3.160}$$

This is also easily checked to be true just from the existing results (3.149) and (3.150).

For $SO(3)$, the single weighted moduli space volume $\mathrm{Vol}(\mathcal{M}, SO(3)_-)$ can be deduced using the fact that the corresponding volumes for both $SU(2)$ and $SO(3)_+$ were known. For higher-rank cases, there can be additional weighted moduli space volumes (correspoinding to all of the $\lambda \in \hat{K}$), so knowledge of moduli space volumes for just G, G/K alone do not always suffice to algebraically determine the remainder.

3.6. Examples in TFTs

Further examples of decomposition are provided by unitary topological field theories with semisimple local operator algebras, such as two-dimensional Dijkgraaf-Witten theory, BF theory, and the G/G model. We have already examined some of these theories; in this section, we will provide a systematic construction as topological field theories, and examine those theories as special cases of that construction.

Existence of a decomposition of these theories is a consequence of the fact, well-known in the TFT community, that the operator algebras of such TFTs admit a complete set of projectors. This implies that these theories are equivalent to disjoint unions of invertible field theories, which is a special case[5] of decomposition, see e.g. [44], [45, section 3.1], and more recently, [24, appendix C.1], [38].

We will see that in such two-dimensional topological field theories, the partition function has the form

$$\sum_U \left(S_{00}\,\dim_q \Pi_U\right)^{2-2g}, \tag{3.161}$$

where here $\dim_q$ indicates the quantum dimension. The fact that the dilaton shift on each Universe is proportional to $(\dim_q \Pi_U)^{2-2g}$ is closely comparable to results discussed in previous sections.

We will compute this in examples and compare to previous results where applicable.

In section 3.6.1, we begin by giving a general analysis of the decomposition and dilaton shifts of two-dimensional unitary topological field theories. We then work out the details in two examples. In section 3.6.2, we return to two-dimensional Dijkgraaf-Witten theory, which we now view as an example of a unitary topological field theory, and recover the previous description of dilaton shifts from sections 3.2.7 and 3.3.2 as a special case of general aspects of topological field theories. Then, in section 3.6.3, we consider G/G models, and analyze their decompositions and dilaton shifts in examples.

Other examples of unitary topological field theories also exist, including nonabelian BF theory and abelian BF theory at various levels. These also decompose, as has been discussed elsewhere (see the zero-area limit of [35, 36] for nonabelian BF theory, [78] for abelian BF theory), but for the purpose of outlining topological field theories in this framework, Dijkgraaf-Witten theory and the G/G models will suffice.

We hasten to add that we have deliberately restricted to unitary TFTs, meaning we exclude cohomological field theories (obtained by a topological twist of a supersymmetric theory). In these theories, the topological subsector may also admit a decomposition, but the topological subsector is only one small slice of a much larger QFT, and typically in those examples, the full QFT does not decompose, only the topological subsector.

In passing, the fact that unitary TFTs decompose into invertible field theories was utilized recently in studies of factorization issues in AdS/CFT, see e.g. [54, 79–82].

3.6.1. General remarks . We can describe dilaton shifts in the decomposition of unitary two-dimensionsl TFTs in a simple general fashion, as we now review, following e.g. [83–85], [24, appendix C].

Briefly, the local operators form a commutative Frobenius algebra F, meaning (among other things) that there is a linear trace map $\theta : F \to \mathbb{C}$. This trace map defines the metric and correlation functions. For example, if $\mathcal{O}_i$ denotes local operators in F, then correlation functions on S^2 are

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle_0 = \theta\left(\mathcal{O}_1 \cdots \mathcal{O}_n\right), \tag{3.162}$$

where the subscript 0 emphasizes that this is a correlation function in genus zero. Similarly, the topological metric is defined to be a two-point function on S^2

$$g_{ij} = \langle \mathcal{O}_i \mathcal{O}_j \rangle_0 = \theta\left(\mathcal{O}_i \mathcal{O}_j\right). \tag{3.163}$$

[5] Decompositions go hand-in-hand with global one-form symmetries in two-dimensional theories. For unitary semisimple TFTs, the one-form symmetries responsible for their description as a decomposition are described in e.g. [38, 77].

Let C_{ijk} denote the three-point function on S^2

$$C_{ijk} = \langle \mathcal{O}_i \mathcal{O}_j \mathcal{O}_k \rangle_0 \tag{3.164}$$

or, equivalently,

$$\mathcal{O}_i \cdot \mathcal{O}_j = \sum_k C_{ij}{}^k \mathcal{O}_k, \tag{3.165}$$

and define the handle-attaching operator $H : F \to F$ by (see e.g. [86, section 4.1])

$$H = \eta^{ij} C_{ij}{}^k \mathcal{O}_k, \tag{3.166}$$

or in components,

$$H^i_j = C^{ik\ell} C_{k\ell j}, \tag{3.167}$$

where the indices are raised with the topological metric above. Then, the partition function on a genus g surface is

$$Z(\Sigma_g) = \langle H^g \rangle_0 = \theta(H^g). \tag{3.168}$$

Semisimplicity implies that there is a basis $\{\Pi_i\}$ of the local operators such that

$$\Pi_i \Pi_j = \delta_{ij} \Pi_i, \quad \sum_i \Pi_i = 1. \tag{3.169}$$

We will refer to the elements of this basis as projectors, for obvious reasons. Unitarity requires that the one-point function

$$\theta_i = \langle \Pi_i \rangle_0 = \theta(\Pi_i) \tag{3.170}$$

be a positive real number [24, appendix C.1].

Now, let us compute the partition function from the expression (3.168). If we work in a basis of projectors, then the topological metric is diagonal, and we write

$$g_{ij} = \langle \Pi_i \Pi_j \rangle_0 = \langle \Pi_i \rangle_0 \delta_{ij} = \theta_i \delta_{ij}. \tag{3.171}$$

Similarly,

$$C_{ijk} = \begin{cases} \theta_i & i = j = k, \\ 0 & \text{else.} \end{cases} \tag{3.172}$$

As a result, the handle-attaching operator is

$$H = \sum_i (\theta_i)^{-1} \Pi_i, \tag{3.173}$$

or in components,

$$H^i_j = (\theta_i)^{-1} \delta_{ij}. \tag{3.174}$$

A correlation function on a genus g Riemann surface is then

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle_g = \theta \left(\mathcal{O}_1 \cdots \mathcal{O}_n H^g \right), \tag{3.175}$$

and more pertinently, the partition function is (see e.g. [24, equation (C.4)])

$$Z(\Sigma_g) = \langle H^g \rangle_0 = \sum_i (\theta_i)^{1-g}. \tag{3.176}$$

Then, the dilaton shift in the ith Universe is given by

$$\frac{1}{2} \ln \theta_i, \tag{3.177}$$

where $\theta_i = \langle \Pi_i \rangle$. (As in section 3.2.1, this may be more invariantly understood as a shift relative to that of the ambient theory.)

To understand this more elegantly, let us briefly review the role of the modular S-matrix. (In the remainder of this section, we will also restrict to TCFTs (topological conformal field theories), for simplicity.)

In conventions in which $\mathcal{O}_0 = 1$, the fusion rules are (famously in RCFT) diagonalized by the modular S-matrix (see e.g. [87, equation (3.11)], [88, equation (9.57)], [89, 90])

$$\mathcal{O}_i \mathcal{O}_j = \sum_{mn} S_{im} S_{jm} \left(\frac{(S^\dagger)_{mn}}{S_{0m}} \right) \mathcal{O}_n. \tag{3.178}$$

Given the fusion rules above, one can define projectors [24, equation (C.16)]

$$\Pi_i = S_{0i} \sum_p \left(S^\dagger \right)_{ip} \mathcal{O}_p, \tag{3.179}$$

and it is straightforward to check that

$$\Pi_i \Pi_j = \delta_{ij} \Pi_i, \quad \sum_i \Pi_i = 1, \tag{3.180}$$

which are both a consequence of the unitarity of S, $S^\dagger = S^{-1}$.

Note that conversely, given the projectors above, we can compute the fusion rules. First, inverting (3.179), one quickly finds

$$\mathcal{O}_p = \sum_m S_{pm} \left(\frac{\Pi_m}{S_{0m}} \right), \tag{3.181}$$

which implies

$$\mathcal{O}_p\mathcal{O}_q = \left(\sum_i S_{pi}\frac{\Pi_i}{S_{0i}}\right)\left(\sum_j S_{pj}\frac{\Pi_j}{S_{0j}}\right), \tag{3.182}$$

$$= \sum_m S_{pm}S_{qm}\frac{\Pi_m}{(S_{0m})^2}, \tag{3.183}$$

$$= \sum_m S_{pm}S_{qm}\sum_n \frac{(S^\dagger)_{mn}}{S_{0m}}\mathcal{O}_n, \tag{3.184}$$

giving another perspective on the diagonalization of the fusion rules.

The partition function at genus g is then (see e.g. [24, equation (C.4)])

$$Z_g = \sum_i (S_{0i})^{2-2g}, \tag{3.185}$$

where S_{0i} is the matrix element for 0 denoting the operator corresponding to the identity. From [24, appendix C.2], in terms of the S matrix above, the quantum dimension $\dim_q$ of the operator $\mathcal{O}_i$ is

$$\dim_q \mathcal{O}_i = \frac{S_{0i}}{S_{00}}, \tag{3.186}$$

hence in terms of the quantum dimension $\dim_q$, we see that

$$Z_g = \sum_i (S_{00}\dim_q \mathcal{O}_i)^{2-2g}. \tag{3.187}$$

Thus, we see that the contribution from the ith Universe is proportional to $(\dim_q \Pi_i)^{2-2g}$, hence the dilaton shift in the ith Universe is proportional to $\ln \dim_q \Pi_i$.

3.6.2. Two-dimensional Dijkgraaf-Witten theory . Previously in section 3.2.7 we discussed decomposition in two-dimensional Dijkgraaf-Witten theories, viewed as orbifolds of points. In this section we return to Dijkgraaf-Witten theories, now viewing them as examples of unitary topological field theories, in which to apply the abstract machinery described above.

First, we note that the projector onto a Universe corresponding to irreducible (projective) representation R is given by [57, equation (2.17)], [91, section 7.3]

$$\Pi_R = \frac{\dim R}{|G|}\sum_{g\in G}\frac{\chi_R\left(g^{-1}\right)}{\omega\left(g,g^{-1}\right)}\, g, \tag{3.188}$$

where $[\omega] \in H^2(G, U(1))$ denotes the choice of discrete torsion (normalized so that $\omega(1,g) = \omega(g,1) = 1$) in the G orbifold.

Now, correlation functions have the form (see e.g. [57])

$$\langle U_1 \cdots U_b\rangle = \sum_R \left(\frac{\dim R}{|G|}\right)^{\chi(\Sigma)} \frac{\chi_R(U_1)}{\dim R}\cdots\frac{\chi_R(U_b)}{\dim R}, \tag{3.189}$$

for U_i elements of the center of the group algebra $\mathbb{C}[G]$ (which forms the space of operators in the Dijkgraaf-Witten theory), and

$$\chi_R(\Pi_S) = \frac{\dim S}{|G|} \sum_{g \in G} \frac{\chi_S(g^{-1})}{\omega(g, g^{-1})} \chi_R(g), \tag{3.190}$$

$$= \delta_{R,S} \dim R, \tag{3.191}$$

using identity (B.1).

As a result,

$$\langle \Pi_R \rangle = \left(\frac{\dim R}{|G|} \right)^{2-2g} \tag{3.192}$$

on a Riemann surface Σ of genus g.

In the abstract language of section 3.6.1, we reviewed how the partition function of any unitary semisimple TFT should have the form

$$Z = \sum_R (\theta_R)^{1-g} \tag{3.193}$$

for $\theta_R = \langle \Pi_R \rangle_0$ the expectation value of the projector on S^2. Here, we can confirm that explicitly, as the partition function of two-dimensional Dijkgraaf-Witten theory on a surface of genus g is

$$Z(\Sigma_g) = \sum_R \left(\frac{\dim R}{|G|} \right)^{2-2g} = \sum_R (\theta_R)^{1-g} \tag{3.194}$$

for

$$\theta_R = \langle \Pi_R \rangle_0 = \left(\frac{\dim R}{|G|} \right)^2 \tag{3.195}$$

(the expectation value at genus zero). This matches the earlier description (3.62). There, we viewed Dijkgraaf-Witten theory as an orbifold of a point; here, we have approached it abstractly as a topological field theory. Just as there, the factor

$$\left(\frac{\dim R}{|G|} \right)^{2-2g} \tag{3.196}$$

is interpreted in the decomposition in terms of a dilaton shift.

In the special case of vanishing discrete torsion, S matrix elements can be written explicitly for two-dimensional Dijkgraaf-Witten theory, following [92, exercise 10.18]. (We will recover the partition function up to an overall R-independent constant factor.) For a finite group G with representation R and character χ_R, (and no discrete torsion,) one can define a group S-matrix on any conjugacy class $[g]$ as [93, equation (A.8)], [92, equation (10.277)] (see also [94, section 2.2])

$$S_R([g]) = \left(\frac{|\,[g]\,|}{|G|} \right)^{1/2} \chi_R(g) \tag{3.197}$$

that diagonalizes products of irreducible representations, meaning [92, equation (10.276)]

$$C_{RS}^{T} = \frac{1}{|G|} \sum_{[g]} |[g]| \chi_R(g) \chi_S(g) \overline{\chi}_T(g), \tag{3.198}$$

$$= \sum_{[g]} S_R(g) \frac{S_S(g)}{S_1(g)} \overline{S}_T(g). \tag{3.199}$$

for R, S, T irreducible representations. In particular, in this fashion we can recover the Dijkgraaf-Witten projectors, as described in the general analysis of section 3.6.1. Following (3.179), and letting $\mathcal{O}_{[g]}$ denote the twist field associated to conjugacy class $[g]$, normalized as

$$\mathcal{O}_{[g]} = \frac{1}{|[g]|^{1/2}} \sum_{h \in [g]} h, \tag{3.200}$$

we have

$$\Pi_R = S_{0R} \sum_{[g]} \left(S^{\dagger}\right)_R ([g]) \, \mathcal{O}_{[g]}, \tag{3.201}$$

$$= \chi_R(1) \sum_{[g]} \frac{|[g]|^{1/2}}{|G|} \overline{\chi}_R(g) \, \mathcal{O}_{[g]}, \tag{3.202}$$

$$= \frac{\dim R}{|G|} \sum_{[g]} \chi_R\left(g^{-1}\right) |[g]|^{1/2} \mathcal{O}_{[g]}, \tag{3.203}$$

$$= \frac{\dim R}{|G|} \sum_{g \in G} \chi_R\left(g^{-1}\right) g, \tag{3.204}$$

where in the last line we have accounted for the multiplicity in conjugacy class elements. This matches the Dijkgraaf-Witten projectors (3.188) (for vanishing discrete torsion).

In terms of quantum dimensions, recall from (3.186) that the quantum dimension of the projector Π_R is given by

$$\dim_q \Pi_R = \frac{S_{0R}}{S_{00}}. \tag{3.205}$$

From the definition (3.197),

$$S_{0R} = S_R(1) = \frac{1}{|G|^{1/2}} \dim R, \quad S_{00} = \frac{1}{|G|^{1/2}} \chi_1(1) = \frac{1}{|G|^{1/2}}, \tag{3.206}$$

hence

$$\dim_q \Pi_R = \dim R, \tag{3.207}$$

and the predicted partition function for the universe R is

$$(S_{0R})^{2-2g} = \left(\frac{\dim R}{|G|^{1/2}}\right)^{2-2g}, \tag{3.208}$$

which matches previous results up to an overall convention-dependent R-independent factor involving $|G|$.

We only discuss S matrices for two-dimensional Dijkgraaf-Witten theory without discrete torsion, because the analogue of $C_{ij}{}^{k}$ for projective representations is more complicated, due to the fact that the tensor product of projective representatives for a fixed twist does not close onto itself. (The product of an α-twisted representation and a β-twisted representation is an $\alpha\beta$-twisted representation.)

3.6.3. G/G model. The G/G model is a (bosonic[6]) gauged WZW model G/H (see e.g. [95]) for the special case that $H = G$. In this special case, the gauged WZW model is a topological field theory (see e.g. [96, section 4]). This is a standard example in two dimensions, with relations to other two-dimensional topological gauge theories. We outline in this section its decomposition to invertible field theories as a unitary topological field theory, as in section 3.6.1, and discuss dilaton shifts.

The G/G model is discussed in detail in e.g. [24, 96–98], to which we refer the reader. As described there, the physical states of the G/G model at level k correspond to conformal primaries of the G WZW model at level k, i.e. integrable representations of the corresponding Kac–Moody algebra, with OPEs corresponding to fusion rules in the same WZW model.

The physical states all have dimension zero [97, section 4]. Since there exist multiple dimension-zero states, one expects a decomposition, and since they are all dimension-zero, one expects a decomposition to a disjoint union of invertible field theories, whose form we shall outline momentarily.

Since the states are dimension-zero, one can construct a complete set of projection operators, which can be done using the modular S matrix from the general formula (3.179) for any two-dimensional topological field theory. We will apply this in examples later.

For G connected and simply-connected, the partition function of the G/G model at level k equals the dimension of the corresponding Chern–Simons Hilbert space (see e.g. [49, section 3.4]), which at genus g is [87, equation (3.15)], [99],

$$Z_g = \sum_i (S_{0i})^{2-2g}, \tag{3.209}$$

the same result described earlier in equation (3.185) for any two-dimensional unitary topological field theory, where S_{0i} is proportional to the quantum dimension of the integrable representation i, and the sum is again over integrable representations i of the Kac–Moody algebra at level k.

As a result, the G/G model at level k decomposes into (is equivalent to) the following disjoint union of invertible field theories (see e.g. [24, section C.1]):

$$(G/G)_k = \coprod_i \mathrm{Inv}(0, \ln S_{0i}) \cong \coprod_i \mathrm{Inv}(0, \ln(\dim_q R_i)), \tag{3.210}$$

indexed by the integrable representations of the G Kac–Moody algebra at level k, and where $\dim_q R_i$ denotes the quantum dimension, S_{0i}/S_{00}. This matches the common form (2.11) described in the general analysis of section 3.6.1.

[6] One can supersymmetrize gauged WZW models and topologically twist. However, that is not our intent here—we are describing ordinary bosonic gauged WZW models, without fermions.

In passing, in addition to local operators, the G/G model also contains 'Verlinde' line operators L_p, dimensional reductions of Wilson lines in three-dimensional Chern–Simons theory, which obey the same fusion relations as the local operators x_p above. As a result, one can form an identical projector from the Verlinde lines,

$$\Pi_i^L = S_{0i} \sum_p \left(S^\dagger\right)_{ip} L_p, \tag{3.211}$$

which acts as a (nondynamical) domain wall separating universes.

Next, for completeness, we outline two examples. First, we consider $SU(n)$ algebras. From [100, section 8.3],

- $SU(n)$ integrable reps at level 1 are antisymmetric powers of the fundamental,
- $SU(n)$ integrable reps at level k are Young diagrams of width bounded by the level.

For example, at level 2, the adjoint representation of $SU(n)$ becomes integrable.

For integrable $SU(n)$ representations at level 1, the fusion algebra is [101, equation (3.7)]

$$\left[\wedge^i \mathbf{n}\right] \times \left[\wedge^j \mathbf{n}\right] = \left[\wedge^{i+j \mod n} \mathbf{n}\right], \tag{3.212}$$

which can be expressed more compactly as the ring

$$\mathbf{C}\left[x_1, \ldots, x_{n-1}\right] / \left(x_i x_j - x_{i+j \mod n}\right). \tag{3.213}$$

For $n > 2$, there are more than $n-1$ constraints, so generically one might expect no solutions, but it is straightforward to check that solutions always exist of the form

$$x_1^n = 1, \; x_i = (x_1)^i, \tag{3.214}$$

and so describe n points.

To compare to the modular S-matrix, let us specialize to $SU(3)_1$. The integrable representations are $\mathbf{1}$, $\mathbf{3}$, $\overline{\mathbf{3}} = \wedge^2 \mathbf{3}$, and the fusion rules are

$$[\mathbf{3}] \times [\mathbf{3}] = \overline{\mathbf{3}}, \; [\mathbf{3}] \times [\overline{\mathbf{3}}] = [\mathbf{1}], \; [\overline{\mathbf{3}}] \times [\overline{\mathbf{3}}] = [\mathbf{3}]. \tag{3.215}$$

If we identify x_1 with $[\mathbf{3}]$ and x_2 with $[\overline{\mathbf{3}}]$, then the fusion algebra is the ring

$$\mathbb{C}\left[x_1, x_2\right] / \left(x_1^2 - x_2, x_1 x_2 - 1, x_2^2 - x_1\right). \tag{3.216}$$

This is a system of three equations in two unknowns—an overdetermined system. Nevertheless, it does admit solutions, corresponding to a set of three points, located at

$$x_1^3 = 1, \; x_2 = x_1^{-1}. \tag{3.217}$$

The fusion ring for $SU(3)_1$ is encoded in the modular S-matrix [92, equation (14.222)]

$$S = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \xi & \xi^2 \\ 1 & \xi^2 & \xi \end{bmatrix}, \tag{3.218}$$

for $\xi = \exp(2\pi i/3)$. (Note that $S^\dagger = S^{-1}$, as expected for a unitary matrix.)

Given the modular S-matrix, from the general formula (3.179) we have the projection operators

$$\Pi_1 = \frac{1}{3}(1 + x_1 + x_2), \tag{3.219}$$

$$\Pi_2 = \frac{1}{3}\left(1 + \xi^2 x_1 + \xi x_2\right), \tag{3.220}$$

$$\Pi_3 = \frac{1}{3}\left(1 + \xi x_1 + \xi^2 x_2\right), \tag{3.221}$$

(which in this case can also be easily computed directly, without knowledge of the S-matrix). These are easily checked to obey

$$\Pi_i \Pi_j = \delta_{ij} \Pi_i, \quad \sum_i \Pi_i = 1, \tag{3.222}$$

as expected for projection operators.

From equation (3.185), the partition function is then

$$Z_g = \sum_i (S_{0i})^{2-2g} = \sum_i \left(\frac{1}{\sqrt{3}}\right)^{2-2g}, \tag{3.223}$$

hence each Universe is weighted by a factor of

$$\left(\frac{1}{\sqrt{3}}\right)^{2-2g}, \tag{3.224}$$

reflecting a dilaton shift of

$$-\frac{1}{2}\ln 3. \tag{3.225}$$

Furthermore, from (3.186), the quantum dimension of each projector is

$$\dim_q \Pi_i = \frac{S_{0i}}{S_{00}} = 1. \tag{3.226}$$

Next, we outline the example of G_2/G_2 at level one. From [100, section 6], [101, section 3], there are two integrable representations of G_2 at level 1, namely [**1**], [**7**], which obey

$$[\mathbf{7}] \times [\mathbf{7}] = [\mathbf{1}] + [\mathbf{7}]. \tag{3.227}$$

We can write the ring as

$$\mathbb{C}[x] / \left(x^2 - 1 - x\right), \tag{3.228}$$

where we have identified x with [**7**]. (In passing, F_4 at level 1 also has only two integrable representations, which obey a fusion algebra of the same form.) Geometrically, this ring describes two points, located at

$$x = \frac{1 \pm \sqrt{5}}{2}, \tag{3.229}$$

corresponding to the two universes (and to the roots of the quadratic polynomial $x^2 - x - 1$). The S matrix is [92, equation (16.64)]

$$S = \sqrt{\frac{4}{5}} \left[\begin{array}{cc} \sin(\pi/5) & \sin(3\pi/5) \\ \sin(3\pi/5) & -\sin(\pi/5) \end{array} \right]. \tag{3.230}$$

It is straightforward[7] to check from the general formula (3.179) (or directly) that the two projectors are

$$\Pi_{\pm} = \frac{1}{2}\left(1 \pm \frac{\sqrt{5}}{5}\right) \mp \frac{\sqrt{5}}{5} x. \tag{3.232}$$

From the S matrix (3.230), From the general expression for partition functions in topological field theories (3.185), we have that

$$Z_g = \sum_i (S_{0i})^{2-2g} = \left(\sqrt{\frac{4}{5}}\sin(\pi/5)\right)^{2-2g} + \left(\sqrt{\frac{4}{5}}\sin(3\pi/5)\right)^{2-2g}, \tag{3.233}$$

which is interpreted to mean that one Universe is weighted by

$$\left(\sqrt{\frac{4}{5}}\sin(\pi/5)\right)^{2-2g} \tag{3.234}$$

and the other by

$$\left(\sqrt{\frac{4}{5}}\sin(3\pi/5)\right)^{2-2g}, \tag{3.235}$$

with corresponding dilaton shifts. We can also read off from equation (3.186) that the quantum dimensions of the two projectors are

$$1, \quad \frac{\sin(3\pi/5)}{\sin(\pi/5)}. \tag{3.236}$$

Finally, let us relate the G/G model to other models discussed in this overview. First, we have already provided S matrix elements for two-dimensional Dijkgraaf-Witten theory that enable it to be treated in a fashion closely related to the G/G model. Also, it is believed that in the limit of large level, the G/G model reduces to BF theory (see e.g. [49, section 3.3]), which implies that in that limit, the decomposition of the G/G model should become the decomposition of BF theory. To that end, we note that in the limit of large level, all representations become integrable, and the quantum dimensions of the integrable representations become the ordinary dimensions (see e.g. [92, section 16.3]). Thus, the decomposition of the G/G model

[7] The following may be helpful:

$$\sin(\pi/5) = \frac{\sqrt{10-2\sqrt{5}}}{4}, \quad \sin(3\pi/5) = \frac{\sqrt{10+2\sqrt{5}}}{4}. \tag{3.231}$$

into invertibles given by (3.210) reduces to the decomposition of BF theory, up to an irrelevant overall dilaton shift, as expected.

In passing, we have only mentioned G/G cosets, but there exist more general G/H cosets. Since H acts by adjoints, if H has a center $Z(H)$ then the gauged G/H WZW model has an $Z(H)$ one-form symmetry, and so decomposes, as is discussed in examples in [24].

4. Noninvertible symmetries and asymptotic densities of states

So far in this paper we have discussed dilaton shift (Euler counterterm) factors arising in the universes of decomposition, and how they have a more or less canonical form, proportional to the dimension of the representation corresponding to the Universe (with other factors that are convention-dependent).

That same $\dim R$ dependence was also recently discussed in [102], in the context of gapped theories, which in the IR can be thought of as special cases of decomposition. Briefly, they argue in those special cases that the fact that the dilaton shifts (Euler counterterms) have a canonical form is due to the presence of a (noninvertible) symmetry, see e.g. [102, section 2]. More precisely, they argue that due to linking between one-dimensional interfaces between vacua, there are relative Euler terms, essentially arising as the quantum dimensions of those interfaces, see in particular [102, section 2.3]. One-dimensional interfaces also exist more generally between universes, see e.g. [37, 103], and using linking between those interfaces and the (local) univese projection operators, the same argument applies and one immediately reaches the same conclusion, that there is a relative Euler counterterm shift between contributions from different universes, with Euler counterterm proportional to

$$\ln\left(\frac{\dim R_i}{\dim R_j}\right), \tag{4.1}$$

where $R_{i,j}$ are the representations associated with either Universe. Similar ideas also appear in [104].

We note for our purposes in this paper that in an orbifold or gauge theory in which a subgroup K acts trivially, there is a $\mathrm{Rep}(K)$ quantum symmetry [105], (possibly a subset of a larger quantum symmetry,) which will be noninvertible if K is nonabelian. We therefore interpret the form of these dilaton shifts in gauge theories[8] in terms of the presence of such symmetries in the decomposition.

The remarks above are meant to be specific to gauge theories. For example, two-dimensional unitary topological field theories also decompose [44, 45], but we are not aware of relevant corresponding symmetries applicable to all such.

In passing, related notions have arisen in discussion of asymptotic state densities, see e.g. [43, 106–108]. For example, [43, section 4] related asymptotic state densities on boundaries of two-dimensional G gauge theories to $(\dim R)^2$ factors in bulk partition functions. In such a case, the bulk theory has a global $\mathrm{Rep}(G)$ symmetry, but the boundary theory has a global G symmetry.

[8] We have discussed a variety of theories, and it is not c.

5. General argument via coupling to a TFT

We have just outlined how one way to understand the dilaton shifts appearing in decomposition is through the presence and properties of interfaces linking the different universes. In this section we shall outline another way to understand the dilaton shift conjecture (3.2) in the case of orbifolds and gauge theories. This alternative understanding uses the fact that such theories (with 1-form symmetries) can be represented formally as theories coupled to a topological field theory, specifically, two-dimensional Dijkgraaf-Witten theory—the prototypical example of an orbifold with a trivially-acting subgroup.

Now, to be clear, one should be careful when talking about coupling physical theories to topological field theories. Although TFT's are fantastically useful for mathematics applications, as physical theories they violate basic axioms of field theory such as unitarity and spin-statistics. As a result, one expects that coupling a physical theory to a TFT ordinarily would ordinarily not not yield a perfectly well-behaved physical theory—or at least, the resulting theory may violate some of the standard axioms of quantum field theory. As a result, given any theory that is described as the result of coupling to a TFT, it behooves one to check to understand which axioms are violated. In the present circumstances, two-dimensional Dijkgraaf-Witten theory is a unitary theory, so unitarity is unbroken, but cluster decomposition is violated, which ultimately is one way of thinking about the existence of a decomposition.

Let us make this intuition more precise. We compare the partition function of the general conjecture (3.2) on a Riemann surface of genus g, namely

$$Z_g\left([X/\Gamma]\right) \,=\, |K|^{2g-2} \sum_U \left(\dim R_U\right)^{2-2g} Z_g\left(X_U\right), \tag{5.1}$$

to the analogous partition function of a two-dimensional Dijkgraaf-Witten theory with orbifold group K and twisting $\omega \in H^2(K, U(1))$ on the same Riemann surface, namely

$$Z_{\mathrm{DW},g} \,=\, |K|^{2g-2} \sum_R \left(\dim R\right)^{2-2g}. \tag{5.2}$$

In the Dijkgraaf-Witten partition function $Z_{\mathrm{DW},g}$, the sum is over all irreducible representations of K.

Comparing the Dijkgraaf-Witten partition function to that of the general conjecture, they clearly have basic parallels—both involve a sum over irreducible representations R of K, both involve factors of

$$\left(\frac{\dim R}{|K|}\right)^{2-2g}. \tag{5.3}$$

To construct the partition function of the general conjecture from that of Dijkgraaf-Witten, we restrict the sum over irreducible representations to a subset (the orbits of a group action on $\hat{K}$), and we multiply the contribution to each Dijkgraaf-Witten universe by a factor of the partition function $Z_g(X_R)$ of a coupled theory.

In passing, note that dilaton shifts in other topological field theories should be understood a bit differently. For example, in the G/G model, the universes are indexed by *integrable* irreducible representations of G.

6. Dilaton shifts versus probability measures

Dilaton shifts often arise in ways that can be interpreted in terms of probability densities. We have already discussed their appearance in asymptotic densities of states in [43] in section 4. Another simple example arises in two-dimensional Dijkgraaf-Witten theory. There, the dilaton shift factors are (see e.g. section 3.2.7)

$$\left(\frac{\dim R}{|G|} \right)^{2-2g}, \tag{6.1}$$

which are clearly related to the Plancherel measure on the set of irreducible representations of a finite group G, a normalized probability density on the set of irreducible representations whose value for any irreducible representation R is

$$\frac{(\dim R)^2}{|G|}, \tag{6.2}$$

see e.g. [109, 110]. Related ideas also appear in [111], which discusses superselection sectors associated to irreducible representatiosn of a gauge group, and probability densities related to the square of the dimensions of those representations.

All that said, however, decompositions are not the same as ensembles.

In this section we will do a more careful comparison of the two notions. We begin in section 6.1 by showing how to distinguish the two, by working on a spacetime with multiple connected components. We further pursue that difference in section 6.2, in a comparison of fields appearing in mirrors to decompositions, compared to stochastic variables appearing in e.g. the SYK construction. The notions appear to be more closely related on connected spacetimes, and later in section 6.3 we observe how, at least on connected spacetimes, dilaton shift factors have been interpreted as probability densities in some entanglement entropy computations. We also discuss a generalization of those entanglement entropy computations.

6.1. Fundamental distinction

It is tempting to relate dilaton shifts to some sort of probability measure, interpreting universes as events in a probabilistic ensemble over a space of couplings. Let R index universes, and for any one spacetime X over which a quantum field theory is defined, let $\rho_X(R)$ denote the dilaton shift associated with Universe R. At least on a *connected* spacetime, correlation functions in the theory then have the form

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle = \sum_R \rho_X(R) \langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle_R, \tag{6.3}$$

where $\langle \cdots \rangle_R$ denotes a correlation function in Universe R without a dilaton shift.

However, there is a fundamental difficulty in such an interpretation which manifests itself on disconnected spacetimes. Briefly,

- in a decomposition, one sums over all universes on each component of the spacetime, as one has a disjoint union of QFTs,
- whereas in an ensemble, there is one fixed sum over the different events (universes), independent of the number of components of spacetime.

This distinction is visible in e.g. partition functions.

To be concrete, consider an example of a QFT that decomposes into n separate universes, schematically

$$X = Y_1 \coprod Y_2 \coprod \cdots \coprod Y_n. \tag{6.4}$$

Suppose further that our spacetime Σ also decomposes into two pieces,

$$\Sigma = \Sigma_1 \coprod \Sigma_2. \tag{6.5}$$

Then, in a decomposition, the partition function of theory X on spacetime Σ has the form

$$Z(\Sigma, X) = \left(\sum_{i=1}^{n} Z(\Sigma_1, Y_i) \right) \left(\sum_{i=1}^{n} Z(\Sigma_2, Y_i) \right), \tag{6.6}$$

whereas if the decomposition was interpreted as a probabilistic ensemble, with a probability distribution determined by the dilaton shift, then the partition function would be

$$Z(\Sigma, X) = \sum_{i=1}^{n} Z(\Sigma_1, Y_i) Z(\Sigma_2, Y_i). \tag{6.7}$$

In short, the difference between the two interpretations is visible in the (non)existence of cross terms in the partition function.

6.2. Decomposition mirror fields versus SYK stochastic parameters

To try to further illuminate the difference, we will compare a couple of closely related examples, namely,

- the mirror to a GLSM for a gerbe, which admits a locally constant field valued in roots of unity,
- Landau–Ginzburg models with a stochastic parameter as in the SYK model.

As before, differences arise over spacetimes with multiple components.

First, we discuss mirrors to two-dimensional gauged linear sigma models in which a subgroup of the gauge group acts trivially (technically, GLSMs for gerbes). Examples are discussed in [2, 112]. Briefly, when one computes the mirror to such a theory, following the usual prescriptions of [112, 113], the mirrors all take the form of Landau–Ginzburg models with locally-constant 'finite-valued fields.' An example of the superpotential in these Landau–Ginzburg models has the form

$$W = x_1^3 + x_2^3 + x_3^3 + \Upsilon x_1 x_2 x_3 \tag{6.8}$$

where x_{1-3} are ordinary chiral superfields, and Υ is locally constant, valued in kth roots of unity, so that $\Upsilon^k = 1$.

The path integral's sum over values of Υ is the sum over universes, with different values of Υ corresponding to different universes, each of which is, individually, an ordinary Landau–Ginzburg model, with a different complex structure.

The intuition for this result is straightforward. The original theory decomposed, into a disjoint sum of theories with different theta angles/B fields. The mirror to such a decomposition is a disjoint sum of theories with different complex structures, which is precisely the effect of the Υ term in the superpotential above.

Because Υ is locally constant, if the spacetime is not connected, it can take different values on different components, and so on a disconnected space, the partition function is a product of sums—we get cross terms, exactly as described earlier in section 6.1.

Now, naively, at least on a connected spacetime, the field Υ appears to be a finite version of a stochastic variable as has been utilized in the SYK model (see e.g. [114, 115], and [116] for a GLSM version)), in the sense that when one computes correlation functions, the theory sums over its values, weighted by probability densities (corresponding to dilaton shifts). The difference is that in the SYK model, there is one sum, independent of the number of connected components of spacetime.

For completeness, let us pursue this a bit further, to describe Landau–Ginzburg models with stochastic variables, as in the SYK model.

Consider an ensemble indexed by $\psi \in \mathbb{C}$, over Landau–Ginzburg models with superpotential of the form

$$W = x_1^3 + x_2^3 + x_3^3 + \psi x_1 x_2 x_3, \tag{6.9}$$

and the stochastic variable ψ is weighted by probability $\rho(\psi)$. Since ψ is not a field, varying over the worldsheet, but instead an index for universes, the path integral contains only a single ordinary integral over its values (independent of the number of connected components of spacetime).

To be more specific, consider a B-twisted Landau–Ginzburg model with such a field. Assume further for simplicity that the vacua are isolated, and the worldsheet is connected, of genus g, then, correlation functions have the schematic form[9]

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_n \psi \cdots \psi \rangle = \int \mathrm{d}\psi \, \rho(\psi) \int [D\phi] \exp(-S) \, \mathcal{O}_1 \cdots \mathcal{O}_n \psi \cdots \psi, \tag{6.10}$$

$$= \int \mathrm{d}\psi \, \rho(\psi) \sum_{\mathrm{d}W=0} \mathcal{O}_1 \cdots \mathcal{O}_n \psi \cdots \psi H^{g-1}, \tag{6.11}$$

where H is the determinant of the matrix of second derivatives $\partial_i \partial_j W$, evaluated at critical loci.

On a connected worldsheet, the mirror to a decomposition, described by a finite-valued Υ, has correlation functions computed in essentially the same form, with $\rho(\Upsilon) = 1$. On a disconnected worldsheet, a correlation function is a product of correlation functions on each component, with a different Υ on each component, instead of a single overall Υ as would happen in an ensemble.

6.3. Entropy and dilaton shifts

In computations of entropy, dilaton shift between universes are sometimes interpreted as probability measures – with all the caveats discussed earlier in this section. In this section we will briefly describe how some entropy computations appearing in the literature can be understood in terms of dilaton shifts.

Entanglement entropies have been computed in many references. Our discussion below will follow the framework of [118–121], and more specifically, [118, section 1.4]. (A handful of

[9] This is a trivial extension of results in [117].

additional references include [122–126], and we emphasize that the literature contains numerous others.) Since various entropies have been discussed in detail in the literature, we shall just summarize pertinent computations, highlighting the specific relation to dilaton shifts and decompositions.

Begin with a sphere S^2. Slice it along $n \geqslant 1$ intervals, and let $\pm$ denote either side of the cut. Take $q > 1$ copies of this cut sphere, and let $(i, \pm)$ denote either side of the cut on the ith copy (independently of the choice of interval, which will all be treated in parallel). Glue $(i, -)$ to $(i+1, +)$ (and permuted cycically). Label the result $X^{(q)}$.

Geometrically, $X^{(q)}$ is a branched q-fold cover of S^2, branched over $2n$ points (the endpoints of the n intervals), at each of which the branching is maximal (all sheets of the cover participate). From the Riemann-Hurwitz theorem,

$$\chi\left(X^{(q)}\right) = q\chi\left(\mathbb{P}^1\right) - \sum_{i=1}^{2n}(q-1) = 2q - 2n(q-1). \tag{6.12}$$

Geometrically, $X^{(q)}$ is a curve of genus

$$g = (1-n)(1-q). \tag{6.13}$$

The total area of $X^{(q)}$ is qA for A the area of a single S^2.

Let $Z(q)$ denote the partition function of a theory on $X^{(q)}$. Following e.g. [118], define the density matrix ρ by

$$\mathrm{Tr}\,\rho^q = \frac{Z(q)}{Z(1)^q}, \tag{6.14}$$

and then the replica trick yields the von Neumann entropy as a limit of the Rényi entropy [118, equation (31)]

$$S = \lim_{q\to 1}\frac{1}{1-q}\ln\frac{Z(q)}{Z(1)^q} = -\left.\frac{\partial}{\partial q}\frac{Z(q)}{Z(1)^q}\right|_{q=1}. \tag{6.15}$$

Now, let us apply this to a theory which decomposes. Write the partition function on a connected spacetime Σ in the form

$$Z(\Sigma) = \sum_R f(R)^{\chi(\Sigma)} Z_R(\Sigma), \tag{6.16}$$

where R indexes universes, Z_R is the partition function for the theory in Universe R, and $f(R)$ encodes the dilaton shift. Then, in the present case,

$$Z(q) = \sum_R f(R)^{2q-2nq+2n} Z_R(q). \tag{6.17}$$

Define

$$p(R) = \frac{f(R)^2 Z_R(1)}{Z(1)}, \tag{6.18}$$

so that

$$\sum_R p(R) = 1. \tag{6.19}$$

(In effect, $p(R)$ acts analogously to a probability, though in the spirit of this paper we can also interpret it in terms of dilaton shifts.)

Then, plugging into (6.15), it is straightforward to compute, at least formally,

$$S = -\frac{\partial}{\partial q}\frac{Z(q)}{Z(1)^q}\bigg|_{q=1}, \tag{6.20}$$

$$= \sum_R p(R)\left(-\ln p(R) + S_R + 2n\ln f(R)\right), \tag{6.21}$$

where S_R represents the result from just universe R:

$$S_R = -\frac{\partial}{\partial q}\frac{Z_R(q)}{Z_R(1)^q}\bigg|_{q=1}, \tag{6.22}$$

$$= \ln Z_R(1) - \frac{Z_R'(1)}{Z_R(1)}. \tag{6.23}$$

The reader should note that

- the result has the form of a sum over universes,
- the probability $p(R)$ is proportional to the genus-zero dilaton shift.

This result is also closely related[10] to an analogous result for entropy in the presence of superselection sectors, where the entropy can be described as a sum of a contributions from separate sectors plus a Shannon contribution arising from just the probabilitiy densities, see e.g. [127, section 3.2, equations (27) and (28)], [128, 129]. The good reason for this relationship is that in deep IR/infinite volume limits, superselection becomes decomposition. The difference is that at finite energies and finite volumes, in a decomposition one still has a disjoint union of quantum field theories, which is not true of superselection sectors. Since superselection has a limit in which it becomes decomposition, it is natural to expect entropy formulas, for example, to have a similar form, as we have observed here. (In passing, see also [130] for a discussion in terms of higher-form symmetries.)

In the special case of a decomposition to invertible field theories, where

$$Z_R(q) = \exp\left(-qAf_2(R)\right), \tag{6.24}$$

it is straightforward to check that $S_R = 0$, so that

$$S = \sum_R p(R)\left(-\ln p(R) + 2n\ln f(R)\right). \tag{6.25}$$

[10] We would like to thank O Parrikar for pointing this out to us.

Now, let us compare to particular cases.

- Two-dimensional pure Yang-Mills. This is discussed in e.g. [118–121]. Here, the universes are indexed by irreducible representations R of the gauge group, and

$$f(R) = \dim R, \ f_2(R) = C_2(R). \tag{6.26}$$

The expression for the entropy,

$$S = \sum_R p(R)\left(-\ln p(R) + 2n \ln \dim R\right), \tag{6.27}$$

matches e.g. [118, equation (37)].

- Nonabelian BF theory. This is just the zero-area limit of two-dimensional pure Yang-Mills (see e.g. [63, section 2]), and so results for entropy follow immediately from those above.
- Two-dimensional Dijkgraaf-Witten theory. Here, the universes are indexed by irreducible (projective) representations R of the orbifold group G, and

$$f(R) = \frac{\dim R}{|G|}, \ f_2(R) = 0, \tag{6.28}$$

so

$$p(R) = f(R)^2 = \left(\frac{\dim R}{|G|}\right)^2, \ Z_R(q) = \exp\left(-qAf_2(R)\right) = 1, \tag{6.29}$$

and

$$S = \sum_R p(R)\left(-\ln p(R) + 2\pi \ln\left(\frac{\dim R}{|G|}\right)\right). \tag{6.30}$$

- Two-dimensional unitary topological field theories. In a modification of the notation of section 3.6.1, if we index constituent universes by R to write the partition function of a two-dimensional topological field theory on a Riemann surface of genus g as (3.176)

$$\sum_R (\theta_R)^{1-g}, \tag{6.31}$$

then we take $f(R) = \sqrt{\theta_R} \propto \sqrt{\dim_q \Pi_R}, f_2(R) = 0$, so that

$$p(R) = \frac{\theta_R}{Z(1)}. \tag{6.32}$$

7. Conclusions

In this paper we have studied dilaton shifts (Euler counterterms) weighting the different universes of decompositions in two-dimensional quantum field theories. Although these shifts are just counterterms, they arise in a more or less canonical form, determined by the dimension of representations indexing the universes, whose form we have discussed in detail, and as is expected from global symmetries. We have outlined consequences for volumes of moduli spaces of flat connections, and also discussed distinctions with and relations to probability measures in several contexts.

Data availability statement

No new data were created or analysed in this study.

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Appendix A. Review of invertible field theories

The notion of invertible field theories arise when discussing tensor products of quantum field theories. In a product $A \otimes B$ of quantum field theories A, B, (distinguished from a disjoint union or sum that play a role in decomposition), the Fock space is a tensor product of the Fock spaces of A and B separately, and the partition function of $A \otimes B$ is a product of the partition functions of A and B separately.

An invertible field theory is a quantum field theory that is invertible under such a product operation, which implies, for example, that its Fock space is one-dimensional—the only states are scalar multiples of the vacuum.

A prototype for an invertible field theory in two dimensions is a sigma model whose target space is a single point, with vanishing action. Given such a sigma model, we can still add counterterms. For example, we can consider a two-parameter family of counterterms described by the action

$$S = \int_{\Sigma} \mathrm{d}^2 x \sqrt{g} \left(\lambda_1 + \lambda_2 \frac{R}{4\pi} \right), \tag{A.1}$$

where g is the (classical, nondynamical) metric on the worldsheet Σ, and R is the Ricci sclar of Σ, so that the partition function at genus g is

$$Z = \exp\left(\lambda_1 \left(\text{area of } \Sigma\right) + \lambda_2 \chi\left(\Sigma\right)\right), \tag{A.2}$$

where $\chi(\Sigma)$ is the Euler characteristic of Σ.

Elsewhere in this paper, we denote this family of invertible field theories by $\mathrm{Inv}(\lambda_1, \lambda_2)$.

Appendix B. Finite group representation theory identities

In this appendix we collect some identities arising in the representation theory of a finite group G, twisted by a cocycle $[\omega] \in H^2(G, U(1))$, normalized so that $\omega(1,g) = \omega(g,1) = 1$. These identities can be found in [57, appendix B], [37, appendix B], and references therein:

$$\frac{1}{|G|} \sum_{g \in G} \frac{\omega(a,g)\,\omega\left(g^{-1},b\right)}{\omega\left(g,g^{-1}\right)} \chi_R(ag)\, \chi_S\left(g^{-1}b\right) = \frac{\delta_{R,S}}{\dim R} \omega(a,b)\, \chi_R(ab). \tag{B.1}$$

$$\frac{1}{|G|} \sum_{g \in G} \frac{\omega(g,a)\,\omega\left(b,g^{-1}\right)}{\omega\left(g,g^{-1}\right)} \chi_R(ga)\, \chi_S\left(bg^{-1}\right) = \frac{\delta_{R,S}}{\dim R} \omega(a,b)\, \chi_R(ab). \tag{B.2}$$

$$\frac{1}{|G|}\sum_{g\in G}\frac{\omega\left(g,a\right)\omega\left(g^{-1},b\right)\omega\left(ga,g^{-1}b\right)}{\omega\left(g,g^{-1}\right)}\chi_R\left(gag^{-1}b\right) = \frac{1}{\dim R}\chi^R\left(a\right)\chi_R\left(b\right). \tag{B.3}$$

$$\frac{1}{|G|}\sum_{g\in G}\frac{\omega\left(a,g\right)\omega\left(b,g^{-1}\right)\omega\left(ag,bg^{-1}\right)}{\omega\left(g,g^{-1}\right)}\chi^R\left(agbg^{-1}\right) = \frac{1}{\dim R}\chi_R\left(a\right)\chi_R\left(b\right) \tag{B.4}$$

and also, for $[g]$, $[h]$ both[11] ω-regular conjugacy classes,

$$\sum_R\frac{\chi_R\left(g\right)\chi_R\left(h^{-1}\right)}{\omega\left(h,h^{-1}\right)} = \begin{cases} 0 & g,h \text{ not conjugate,}\\ \frac{|G|}{|[g]|} & g=h,\\ \frac{\omega(a,g)}{\omega(h,a)}\frac{|G|}{|[g]|} & g=a^{-1}ha,\end{cases} \tag{B.5}$$

where R, S are irreducible projective representations (with respect to ω).

Appendix C. Two-dimensional Dijkgraaf-Witten theory, from triangulations

It is a standard result that two-dimensional pure Yang-Mills theory can be described by associating data to a triangulation of a Riemann surface, and then gluing along edges. In this appendix we will review the analogous construction for two-dimensional Dijkgraaf-Witten theory. (See also e.g. [54, appendix A] for a related state sum construction.)

Specifically, we will describe computations in Dijkgraaf-Witten theory in terms of data assigned to cylinders, disks, and pairs-of-pants, and how they are glued, close to the spirit of pure Yang-Mills.

Recall that the partition function[12] of a two-dimensional (possibly twisted) Dijkgraaf-Witten theory with b boundary components has the form

$$Z(\Sigma,U_1,\ldots,U_b) = \sum_R\left(\frac{\dim R}{|G|}\right)^{\chi(\Sigma)}\chi_R\left(U_1\right)\cdots\chi_R\left(U_b\right), \tag{C.2}$$

where the sum is over irreducible projective representations R of G, twisted by the cocycle $[\omega]\in H^2(G,U(1))$ (itself normalized by $\omega(1,g)=\omega(g,1)=1$). This can be described axiomatically in the same sense as two-dimensional pure Yang-Mills, with gluing accomplished via

$$\frac{1}{|G|}\sum_{U\in G}\frac{1}{\omega\left(U,U^{-1}\right)}. \tag{C.3}$$

Listed below are the partition functions for some standard examples:

[11] If either is not an ω-regular conjugacy class, then the corresponding characters vanish, and the sum equals zero.

[12] This is closely related to, but slightly different from, results for Dijkgraaf-Witten correlation functions. Specifically, a correlation function has the form (see e.g. [57])

$$\langle U_1\cdots U_b\rangle = \sum_R\left(\frac{\dim R}{|G|}\right)^{\chi(\Sigma)}\frac{\chi_R\left(U_1\right)}{\dim R}\cdots\frac{\chi_R\left(U_b\right)}{\dim R}, \tag{C.1}$$

which differs from the result above for partition functions with boundary components by factors of $\dim R$, reflecting differences in the normalizations of states. The different factors of $\dim R$ in the boundary case make the gluing construction possible.

1. Disk:

$$Z_{\text{disk}}(U) = \sum_R \left(\frac{\dim R}{|G|}\right) \chi_R(U). \tag{C.4}$$

2. Cylinder:

$$Z_{\text{cylinder}}(U_1, U_2) = \sum_R \chi_R(U_1)\chi_R\left(U_2^{-1}\right). \tag{C.5}$$

3. Pair of pants:

$$Z_{\text{pants}}(U_1, U_2, U_3) = \sum_R \left(\frac{\dim R}{|G|}\right)^{-1} \chi_R(U_1)\chi_R(U_2)\chi_R(U_3). \tag{C.6}$$

The key identity needed to implement the gluing is

$$\frac{1}{|G|}\sum_{U\in G}\frac{1}{\omega(U,U^{-1})}\chi_R(U)\chi_S\left(U^{-1}\right) = \delta_{R,S}. \tag{C.7}$$

Further identities of this form are given in appendix B.

As a consistency check, let us formally glue a cylinder to a disk:

$$\begin{aligned}
&\frac{1}{|G|}\sum_{U\in G}\frac{1}{\omega(U,U^{-1})}\left[\sum_R \chi_R(U_1)\chi_R\left(U^{-1}\right)\right]\left[\sum_S\left(\frac{\dim S}{|G|}\right)\chi_S(U)\right] \\
&\quad = \frac{1}{|G|}\sum_{R,S}\chi_R(U_1)\left(\frac{\dim S}{|G|}\right)|G|\delta_{R,S}, &(\text{C.8})\\
&\quad = \sum_R\left(\frac{\dim R}{|G|}\right)\chi_R(U_1), &(\text{C.9})
\end{aligned}$$

which is precisely the partition function of a disk, as expected.

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