

A Novel Continuum Approximation to Power System Electromechanical Dynamics

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Abstract—This paper proposes a novel two-dimensional continuum model that aims to capture the electromechanical wave phenomenon observed in power systems after a disturbance. The model is derived by a homogenization process of the discrete parameters of the conventional, discrete transient analysis model. The implementation of the proposed model is detailed. The paper also demonstrates the accuracy of the proposed model by comparing its results with those obtained from the discrete model. It is observed that the proposed model is able to provide a similar characterization of system electromechanical dynamics as the discrete model from a continuum point of view.

Index Terms—Power system dynamics, Power system analysis computing,

I. INTRODUCTION

Managing a low-inertia grid is one of the most important challenges faced as we the share of inverter-based renewable energy generation increases [1]. In a low-inertia grid, electromechanical disturbances propagate much faster, which requires an enhancement of protection and control paradigms. It has been observed that electromechanical disturbances propagate in a wave-like manner through the power system with a velocity far slower than the speed of light [2], [3]. Although the conventional model, based on a differential-algebraic equation (DAE) set, is able to describe the electromechanical process, it fails to quickly convey insight into the wave characteristics. To design effective protection and control schemes for the future low-inertia grid, it is necessary to understand the spatio-temporal characteristics of the electromechanical wave.

To clearly model and analyze the wave-like characteristics of electromechanical dynamics in large power grids, as far back as 1974, a continuum approximation of the power system was introduced by [4]. In this work, power systems are modeled by one-dimensional (1-D) or two-dimensional (2-D) second-order linear hyperbolic partial differential equations (PDEs) with constant coefficients under strict simplifying assumptions. Cresap and Hauer [5] derive a 1-D continuum model, similar to the one proposed by Semlyen [4], to explain a new low-frequency oscillation mode in the US western power system. Later, Thorp *et al.* [2] propose a new continuum model that takes into account losses in power systems, yielding

a second-order nonlinear PDE. However, this model is not able to deal with non-uniformity in parameter distributions [6]. Recent work by Li *et al.* [7] that uses the continuum model of [2] to design control schemes is limited by these uniform assumptions. The continuum models proposed in [2], [4], [5] can only be applied to idealized chain-like, ring-like or power systems with homogeneous parameters. The geometric structure of the idealized power systems is inconsistent with those real-world ones, which typically have meshed or radial structures. Moreover, these models do not account for the generator internal impedance.

An improved continuum model is proposed by Parashar *et al.* [3], which consists of a spatial PDE that describes power flow in the continuum and a nonlinear second-order wave equation describing the swing dynamics. This model is able to handle anisotropy and nonhomogeneity in parameter distributions; it also considers the internal impedances of the generators. The model is applied to a power system that has a meshed geometric structure; parameters of the studied system are distributed onto the plane with a Gaussian filter via convolution. Although simulation results are presented, the model is in fact never validated with field measurements or the conventional discrete model.

Another viewpoint on electromechanical wave modeling is to consider the power system as a set of interconnected 1-D continua [8] [9] [10] [11] [12]. Each involved 1-D continuum is constructed by distributing the generator parameters to adjacent transmission lines. These transmission line-based 1-D continua are then connected to busses to form a description of the typically meshed or radial power systems. However, this viewpoint is out of the scope of the current paper.

This paper aims at developing and validating a novel 2-D continuum model of power systems, resulting in the following main contributions. Firstly, the process of obtaining the continuum parameters from the discrete network topology and parameters is rigorously derived. A novel tensor-based formulation that takes into account the direction of the lines in the 2-D space allows for an accurate mapping from the continuum to the discrete. Although the model has a similar form as that in [3], the meaning and construction of the parameters are different. Secondly, the implementation of the proposed continuum model is introduced, which facilitates

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the numerical solution of the model. Thirdly, the proposed continuum model is validated with the conventional discrete model via numerical case studies.

The remainder of this paper is organized as follows. Section II details the way to obtain parameter distributions on the plane according to the conventional discrete model. Section III derives the proposed continuum model based on the parameter distributions. Section IV introduces the implementation of the proposed continuum model so as to perform numerical computations. Section V presents numerical case studies that demonstrates the accuracy of the proposed continuum model. Finally, Section VI concludes the paper and points out some directions for future research.

II. FILTERING DISCRETE QUANTITIES

The 2-D continuum model is derived from the conventional discrete model of power systems. The first step in the derivation is to distribute the parameters of the conventional discrete model onto a plane. The parameters can be classified into two categories: point parameters and line parameters. A point parameter is associated with power apparatus which is located at a single spatial point. For example, the inertia of synchronous generators is a point parameter. A line parameter is associated with transmission lines, which form line segments. For example, the transmission line reactance is a line parameter.

The process of distributing point and line parameters onto the plane is named *filtering*. A Gaussian filter, such as the one introduced in [3], is adopted in this paper. The Gaussian filter centered at $(x_0, y_0)^T$ with standard deviation σ is

$$f_{(x_0, y_0)}(x, y) = \frac{1}{2\pi\sigma^2} \exp\left(-\frac{(x - x_0)^2 + (y - y_0)^2}{2\sigma^2}\right), \quad (1)$$

where x and y are the spatial coordinates; $f_{(x_0, y_0)}$ denotes the Gaussian filter centered at $(x_0, y_0)^T$.

A. Distributing Point Parameters

Without loss of generality, the synchronous generator inertia is taken as a concrete example here to introduce how point parameters are distributed. Suppose a synchronous generator with inertia M_i is located at $(x_0, y_0)^T$. The inertia is distributed onto the plane as

$$m_i(x, y) = f_{(x_0, y_0)}(x, y)M_i, \quad (2)$$

where m_i is the inertia density of the generator, which is defined on the plane. This filtering process guarantees the consistency between the continuum model and the conventional discrete model in the sense that

$$M_i = \iint_{R^2} m_i(x, y) dx dy. \quad (3)$$

The total system inertia density m is defined as the summation of the inertia densities of all the individual generators. Namely,

$$m(x, y) = \sum_i^{N_{gen}} m_i(x, y). \quad (4)$$

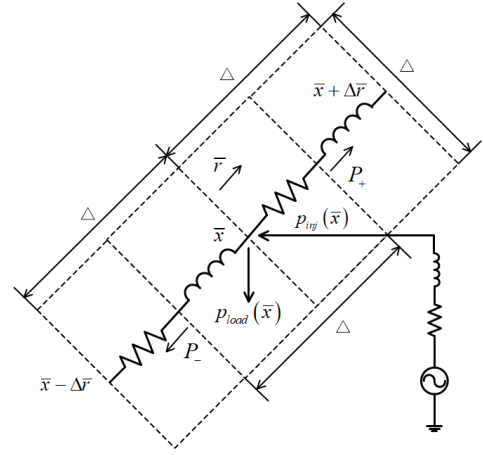


Fig. 1. Infinitesimal element.

Other point-quantities such as the rotational damping d , the generator internal susceptance b_{int} , the mechanical power p_{mech} , and the load injection are also distributed in the same fashion. In summary, from discrete point-quantities pertaining to generators and loads we obtain system-wide continuum quantities on the plane.

B. Distributing Line Parameters

Suppose that a transmission line spans from $(x_f, y_f)^T$ to $(x_t, y_t)^T$ on the plane; the reactance is X_j . The length of the transmission line is

$$L_j = \sqrt{(x_t - x_f)^2 + (y_t - y_f)^2}. \quad (5)$$

The reactance density of the transmission line at an arbitrary point $(x_0, y_0)^T$ of the plane is computed by

$$\rho_{X,j}(x_0, y_0) = \frac{X_j}{L_j} \int_{(x_f, y_f)}^{(x_t, y_t)} f_{(x_0, y_0)}(x, y) dl, \quad (6)$$

where $\rho_{X,j}$ is the reactance density of the transmission line, which is defined on the plane. Logically, $\rho_{X,j}$ on the whole plane can be obtained by performing the calculation (6) at every point. This filtering process guarantees the consistency between the continuum model and the conventional discrete model in the sense that

$$X_j = \iint_{R^2} \rho_{X,j}(x, y) dx dy. \quad (7)$$

III. DERIVATION OF THE 2-D CONTINUUM MODEL

A. Continuum Power Flow

Consider a single transmission line j , we would like to define a distributed power function, $p^j(x, y) = p_{inj}^j(x, y) - p_{load}^j(x, y)$, that takes values everywhere on the plane. This can be done by taking a point $(\bar{x} = (x_0, y_0)^T)$ in such line and studying an infinitesimal element along its direction (Fig.

1). By conservation of energy and assuming unity voltage magnitude [2], [3]:

$$\begin{aligned} p^j(\bar{x}) \Delta^2 &= \frac{\Delta^2 \rho_{R,j}(\bar{x})}{(\Delta r_j)^2 + (\Delta x_j)^2} (1 - \cos(\theta(\bar{x}) - \theta(\bar{x} + \Delta \bar{r}))) \\ &+ \frac{\Delta^2 \rho_{X,j}(\bar{x})}{(\Delta r_j)^2 + (\Delta x_j)^2} \sin(\theta(\bar{x}) - \theta(\bar{x} + \Delta \bar{r})) \\ &+ \frac{\Delta^2 \rho_{R,j}(\bar{x})}{(\Delta r_j)^2 + (\Delta x_j)^2} (1 - \cos(\theta(\bar{x}) - \theta(\bar{x} - \Delta \bar{r}))) \\ &+ \frac{\Delta^2 \rho_{X,j}(\bar{x})}{(\Delta r_j)^2 + (\Delta x_j)^2} \sin(\theta(\bar{x}) - \theta(\bar{x} - \Delta \bar{r})) \end{aligned} \quad (8)$$

where Δ is the side length of the infinitesimal element; r_j and x_j are respectively the per unit length resistance and reactance of the transmission line; $\theta(\cdot)$ is the voltage angle distribution which is a real-valued function defined on the plane; \bar{r} is the unit direction vector of the transmission line, namely

$$\bar{r} = (x_t - x_f, y_t - y_f)^T / L_j. \quad (9)$$

Performing a second-order Taylor expansion at \bar{x} on $\theta(\bar{x} + \Delta \bar{r})$ and $\theta(\bar{x} - \Delta \bar{r})$ in (8) while considering small-angle approximation of trigonometric functions yields

$$\begin{aligned} p^j(\bar{x}) &= -\frac{\rho_{X,j}(\bar{x})}{(r_j)^2 + (x_j)^2} (\bar{r}^T \nabla^2 \theta(\bar{x}) \bar{r}) \Delta^2 \\ &+ \frac{\rho_{R,j}(\bar{x})}{(r_j)^2 + (x_j)^2} (\bar{r}^T \nabla \theta(\bar{x}))^2 \Delta^2 + O(\Delta^3) \end{aligned} \quad (10)$$

Dropping the small $O(\Delta^3)$ term and eliminating the Δ^2 term on both sides of (10) result in

$$\begin{aligned} p^j(\bar{x}) &= -\frac{\rho_{X,j}(\bar{x})}{(r_j)^2 + (x_j)^2} \bar{r}^T \nabla^2 \theta(\bar{x}) \bar{r} \\ &+ \frac{\rho_{R,j}(\bar{x})}{(r_j)^2 + (x_j)^2} (\bar{r}^T \nabla \theta(\bar{x}))^2 \end{aligned} \quad (11)$$

In (11),

$$\bar{r}^T \nabla^2 \theta(\bar{x}) \bar{r} = \nabla^T \bar{r} \bar{r}^T \nabla \theta(\bar{x}), \quad (12)$$

$$(\bar{r}^T \nabla \theta(\bar{x}))^2 = \nabla^T \theta(\bar{x}) \bar{r} \bar{r}^T \nabla \theta(\bar{x}). \quad (13)$$

Therefore, (11) can be rewritten as

$$p^j(\bar{x}) = -\nabla^T (B_j \nabla \theta(\bar{x})) + \nabla^T \theta(\bar{x}) G_j \nabla \theta(\bar{x}), \quad (14)$$

where

$$\begin{cases} B_j(\bar{x}) = \frac{\rho_{X,j}(\bar{x})}{(r_j)^2 + (x_j)^2} \bar{r} \bar{r}^T \\ G_j(\bar{x}) = \frac{\rho_{R,j}(\bar{x})}{(r_j)^2 + (x_j)^2} \bar{r} \bar{r}^T \end{cases} \quad (15)$$

B_j and G_j are respectively called the susceptance tensor and the conductance tensor of the transmission line in this paper. Note that they are defined on the plane because $\rho_{X,j}$ and $\rho_{R,j}$ are defined on the plane.

The previous analysis carries on to a system with an arbitrary number of lines. Owing to the energy conservation, the following equation must hold:

$$p(\bar{x}) = \sum_j^{N_j} (-\nabla^T (B_j \nabla \theta(\bar{x})) + \nabla^T \theta(\bar{x}) G_j \nabla \theta(\bar{x})), \quad (16)$$

where the summation is made over the N_j transmission lines. Equation (16) is equivalent to

$$\begin{aligned} p(\bar{x}) &= -\nabla^T \left(\left(\sum_j^{N_j} B_j \right) \nabla \theta(\bar{x}) \right) \\ &+ \nabla^T \theta(\bar{x}) \left(\sum_j^{N_j} G_j \right) \nabla \theta(\bar{x}) \end{aligned} \quad (17)$$

Here, $B = \sum_j^{N_j} B_j$ and $G = \sum_j^{N_j} G_j$ are the system susceptance tensor and the system conductance tensor respectively. Both of them associate a 2^{nd} order tensor to every point of the 2-D plane. Substituting into (17), yields

$$p(\bar{x}) = -\nabla^T (B \nabla \theta(\bar{x})) + \nabla^T \theta(\bar{x}) G \nabla \theta(\bar{x}). \quad (18)$$

Equation (18) forms a continuum power flow description.

B. Including Inertial Dynamics

Looking back at the infinitesimal element at \bar{x} (Fig. 1), according to the swing equation of a generator expressed as

$$\begin{aligned} \Delta^2 m \frac{\partial^2 \delta}{\partial t^2} + \Delta^2 d \frac{\partial \delta}{\partial t} \\ = \Delta^2 p_{mech} - \Delta^2 g_{int} (1 - \cos(\delta(\bar{x}) - \theta(\bar{x}))) \\ - \Delta^2 b_{int} \sin(\delta(\bar{x}) - \theta(\bar{x})) \end{aligned} \quad (19)$$

where δ is the rotor angle distribution, which is a real-valued function defined on the plane. Equivalently, we have

$$\begin{aligned} m \frac{\partial^2 \delta}{\partial t^2} + d \frac{\partial \delta}{\partial t} \\ = p_{mech} - g_{int} (1 - \cos(\delta(\bar{x}) - \theta(\bar{x}))) \\ - b_{int} \sin(\delta(\bar{x}) - \theta(\bar{x})) \end{aligned} \quad (20)$$

Equation (20) forms the continuum swing equation.

According to basic power flow relations,

$$\begin{aligned} -\Delta^2 p_{inj}(\bar{x}) \\ = \Delta^2 \bar{g}_{int} (1 - \cos(\theta(\bar{x}) - \delta(\bar{x}))) \\ + \Delta^2 \bar{b}_{int} \sin(\theta(\bar{x}) - \delta(\bar{x})) \end{aligned} \quad (21)$$

or equivalently,

$$\begin{aligned} -p_{inj}(\bar{x}) \\ = g_{int} (1 - \cos(\theta(\bar{x}) - \delta(\bar{x}))) \\ + b_{int} \sin(\theta(\bar{x}) - \delta(\bar{x})) \end{aligned} \quad (22)$$

Combining (18) and (22), yields

$$\begin{aligned} -\nabla^T (B \nabla \theta(\bar{x})) + \nabla^T \theta(\bar{x}) G \nabla \theta(\bar{x}) = \\ -b_{int} \sin(\theta(\bar{x}) - \delta(\bar{x})) \\ - g_{int} (1 - \cos(\theta(\bar{x}) - \delta(\bar{x}))) \\ - p_{load}(\bar{x}) \end{aligned} \quad (23)$$

Equations (20) and (23) constitute a description of the electromechanical wave.

IV. IMPLEMENTATION OF THE 2-D CONTINUUM MODEL

A. Truncating the Continuum Model and Enforcing Boundary Conditions

When distributing the point parameters of individual pieces of equipment, the value of the Gaussian filter in (2) at a point of the plane is ignored if it is smaller than a threshold. When distributing the reactance and resistance of an individual transmission line, the value of the integral term in (6) is ignored if it is smaller than a given threshold.

According to (18), the real power flow density at an arbitrary point of the continuum, including the boundary, is given by $-B \nabla \theta$ [3]. Denoting the unit outward normal at a boundary point by \bar{n} , the real power flow density in the direction of \bar{n}

at that point is given by $-\bar{n}^T B \nabla \theta$. The following condition should be enforced [3]

$$-\bar{n}^T B \nabla \theta = 0, \quad (24)$$

so that real power cannot flow out of the continuum. Nevertheless, the boundary condition (24) can fail at certain boundary points. Specifically, if

$$\bar{n}^T (B / \|B\|_2) = 0, \quad (25)$$

then $\nabla \theta$ is not constrained by (24) and multiple solutions are possible. Note that in this case, (24) is automatically satisfied so real power will not flow out. This paper proposes a heuristic resolution to the multiple solutions issue by enforcing another boundary condition

$$-\bar{n}^T \nabla \theta = 0 \quad (26)$$

at those boundary points where (25) holds.

B. Numerical Solutions

1) *Finite Difference Method*: This paper adopts the finite difference method described in [13] to solve the proposed continuum model. A uniform square mesh is adopted to spatially discretize the continuum. Midpoint rules are used to calculate the partial derivatives with a 9-point stencil for (23) at interior points of the continuum. At boundary points where a 9-point stencil is not feasible, however, first order approximation to the partial derivatives are used for (24) and (26) according to the availability. 2-step backward differentiation formula (BDF2) is used to perform the temporal discretization for (20). The implicit trapezoidal method is used to start BDF2 at the beginning of a simulation run and immediately after the algebraic time step due to a discontinuity event, such as a fault application or a generator trip. Ensuring sufficient mesh points is necessary for the numerical solution process. It is thus recommended that the mesh width is no larger than the standard deviation of the Gaussian filter.

2) *Image Gradient*: According to (24) and (26), the unit outward normal is required at boundary points of the continuum. These vectors can be obtained by calculating the image gradients for the 2-norm of the system susceptance tensor. In this paper, the Sobel–Feldman operator [14] is used for the calculation.

3) *Bilinear Interpolation*: It is quite often that the location of a bus does not coincide with a mesh point. In this situation, 2-D bilinear interpolation [15] is used to calculate the values of voltage angle, rotor angle and speed at the bus according to the values at adjacent mesh points.

4) *Time-Stepping*: After the spatial and temporal discretization, the continuum model is converted into a nonlinear algebraic equation set at each time step. In this paper, the equation set is solved with Newton’s method. The numerical solution to the proposed model is thus carried out in a time-step-by-time-step manner. Whenever a discontinuity event occurs, an algebraic time step is inserted. At this time step, the timer is not updated; generator speed and rotor angle are frozen at the previous values; only the voltage angle is updated according to (23) as a result of the event.

TABLE I
POWER FLOW COMPUTATION COMPARISON

Bus (#)	Discrete	Existing		Proposed	
	Value (rad)	Value (rad)	Relative Error (%)	Value (rad)	Relative Error (%)
1	0.000	0.000	0.00	0.000	0.00
2	0.162	0.145	10.38	0.174	7.74
3	0.081	0.076	6.28	0.091	11.76
4	-0.039	-0.023	40.54	-0.037	3.48
5	-0.070	-0.054	21.81	-0.068	2.19
6	-0.064	-0.050	22.82	-0.063	1.87
7	0.065	0.068	4.48	0.076	16.44
8	0.013	0.019	53.57	0.022	72.40
9	0.034	0.039	12.84	0.042	22.91

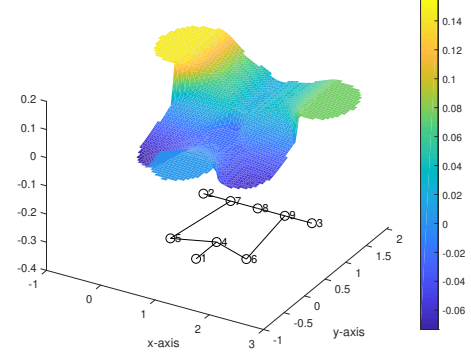


Fig. 2. Voltage angle distribution from the proposed continuum model.

V. NUMERICAL CASE STUDIES

In this section, the proposed continuum model and the one in [3] are applied to the IEEE 9-bus system [16] to study their fidelity, by comparing their respective results to those from the conventional discrete model. For the two continuum models, the mesh width is 0.05 while the Gaussian filter standard deviation is 0.1.

Power flow computation results from the two continuum models, namely the voltage angles at busses, are compared to the results from the discrete model in Table I. It can be observed that the fidelity of both continuum models is similar regarding power flow computation. Fig. 2 shows voltage angle distribution from the proposed continuum model. While the power flow approximation does not achieve greater fidelity than [3], the model achieves greater accuracy capturing the electromechanical process.

Time domain simulation is performed on the test system with different models. Simulation runs start from the power flow solution from 0.0 s. From 0.1 to 0.2 s, real power injection of 1.0 is imposed at Bus 2. Generator speeds from different models are compared in Fig. 3. It can be observed that the accuracy of the proposed continuum model is higher in that its results are significantly closer to those from the discrete model.

VI. CONCLUSION AND FUTURE WORK

As the penetration of inverter-based resources increases, it is necessary to develop models to understand the wide-area iner-

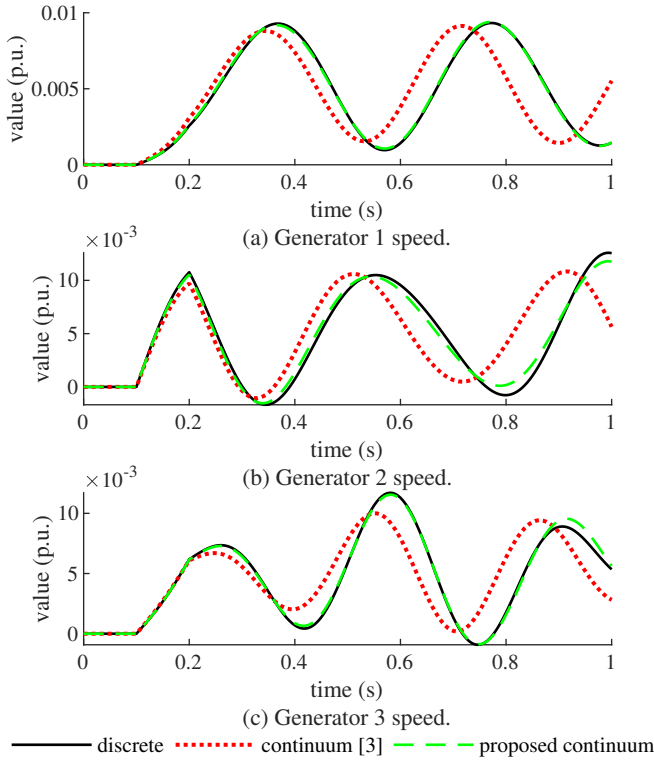


Fig. 3. Comparison of different models. Black solid line: the discrete model. Red dotted line: the continuum model in [3]. Green dashed line: the proposed continuum model.

tial behavior of the power system without the need of focusing on the individual components. The continuum model of the power grid results from a process of *homogenization* to obtain a model of the grid that will allow a better understanding of the spatiotemporal characteristics of grid disturbances.

In this paper, we have introduced a novel two-dimensional continuum model of power systems. Its implementation is detailed and its fidelity is validated against the conventional discrete model. Results show that the proposed model provides a similar dynamic response as the classical DAE model and improves over previously proposed models. In addition, the new representation of the power grid dynamics as a hyperbolic partial differential equation will allow the use of techniques in other fields such as mechanics of structures or computational fluid dynamics for the wide-area optimization and control of power grid dynamics.

In future work, we will assess the accuracy of the proposed model in terms of system parameters and we will explore the applications of protection and control schemes that consider the wave propagation speed and direction. One of the limitations of this model is the lack of generator and inverter controls modeling, which directly impact the electromechanical behavior of the system. In future work, we will augment this model to account for the impact of control devices.

One of the applications of the model is the design of

protection and control devices to arrest the electromechanical wave. Because of the inherent variability of the system (e.g. system loading, changes in the network), it is necessary to make this approach robust. Future work will combine the continuum model with PMU data using statistical methods to allow for a better generalization of the model predictions.

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