A COMPARISON AND STUDY OF THE BORN AND RYTOV EXPANSIONS

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(ABSTRACT)

Since the introduction of the Born and Rytov approximations for use in random wave propagation some forty years ago, a controversy has boiled over the regions of validity and relative merits of the methods. Although the methods fail for strong fluctuations and distant path lengths, these two perturbation methods are the only approaches available for weak fluctuations in a random inhomogeneous media. The approximations have also been applied to the inverse problem for optical and acoustical tomography.

The intent of this thesis is to investigate the work of previous authors and attempt to clarify the distinctions of each method. The conclusion will be reached that neither approximation is necessarily better than the other in general for all applications. A careful consideration of the problem following the points given should point towards

the use of one approximation over the other.

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1. INTRODUCTION

The theory of wave propagation in a random media is concerned with the description of a wave traveling through a medium, which due to the randomness can only be described statistically. Propagation of waves through such media is enczountered in a wide variety of applications such as atmospheric physics, optical and radio astronomy, and remote sensing to name a few. Attempts to solve these problems, among others, spawned the use of two approximate methods, the Born and Rytov approximations, which are the focus of this thesis. More precisely, this thesis is concerned with the comparison of these two approximate methods.

1.1 A BRIEF HISTORY

A good deal of the early theoretical and experimental work in random wave propagation was performed in the Soviet Union in the 1950 and 1960's. The two initial pioneering monographs on wave propagation in random media were published by Chernov and Tatarski in the Soviet Union before 1960 [1,2]. The motivation for these two monographs was driven by an interest in propagation of starlight through the atmosphere and propagation of sound waves through the atmosphere and ocean.

Early approaches focused on geometrical optics. Given that the limitations in the geometrical optics approximations restrict the path lengths on the order of tens of meters, attention turned to other techniques. Two initial methods discussed by Chernov and Tatarski to circumvent this problem are known as the Born and Rytov methods. In the Russian literature, the Born and Rytov methods are referred to as the method of small perturbations and the method of smooth perturbations respectively. The assertions of Chernov and Tatarski regarding the superiority of the Rytov over the Born method stimulated a protracted controversy concerning the relative merits of each.

A major development occurred in the 1960's when Gracheva and Gurvich showed experimentally that the perturbation and other methods used to obtain the variance of the logarithm of the irradiance were completely inapplicable for long path lengths [3]. More sophisticated techniques evolved in attempts to reconcile the mystery of long path lengths and strong medium fluctuations.

For weak fluctuations in the medium and path lengths that are short enough such that multiple scattering events are minor, the Born and Rytov expansions are the only approximations that are available. These methods are quite easy and convenient to use to obtain a first order approximation. Another application of the approximations, which has received serious experimental and theoretical investigations, is the optical and acoustical inverse problem. In this problem the scattered fields are measured from which information about the scattering object is then inferred.

The controversy concerning the merits of the Born and Rytov method still exists today some thirty years later. Particular emphasis has been given to the conditions for the regions where each approximation is valid. It is the intent of this work to investigate the previous approaches over the years addressing this issue to clarify and refine the arguments in order to establish more precisely when one method is preferred over the other.

2. STARTING POINT FOR BOTH APPROXIMATIONS

Maxwell's equations will serve as the starting point for both approximations. The early applications of these two methods and comparison with experimental work was done with electromagnetic propagation through the atmosphere. Because of this fact, introduction of the two methods will be given in connection with this application.

2.1 MAXWELL'S EQUATIONS

The electromagnetic field is assumed to have a sinusoidal time dependence of $e^{i\omega t}$, which will render all the time dependent derivatives in Maxwell's equations into algebraic quantities:

$$\nabla\cdot\left(\vec{H}\right)=0 \qquad (2.1.1)$$

$$\nabla\times\vec{H}=i\omega\varepsilon\vec{E} \qquad (2.1.2)$$

$$\nabla \times \vec{E} = -i\omega\mu\vec{H} \qquad (2.1.3)$$

$$\nabla \cdot \left(n^2\vec{E}\right) = 0 \qquad (2.1.4)$$

where ε and μ are the permitivitty and permeability respectively. For this discussion of propagation in the atmosphere, the permeability is taken to be that of free space μ_0. The permitivitty is a random function dependent only on the space coordinate. The refractive index is defined by $n = \sqrt{\varepsilon\mu}$.

The usual approach in the development of the wave equation from Maxwell's equations is followed, starting with curl of (2.1.3)

$$\nabla \times \nabla \times \vec{E} = i\omega\mu_o \nabla \times \vec{H}\,. \qquad (2.1.5)$$

The following identity is used for the curl of the curl of a vector

$$\nabla \times \nabla \times \vec{E} = \nabla\left(\nabla \cdot \vec{E}\right) - \nabla^2\vec{E}\,. \qquad (2.1.6)$$

Substitution of (2.1.2) and (2.1.6) into (2.1.5) yields

$$\nabla\left(\nabla \cdot \vec{E}\right) - \nabla^2\vec{E} = -i\omega\mu_o\left(i\omega\varepsilon_o n^2\vec{E}\right). \qquad (2.1.7)$$

Expanding (2.1.4) produces

$$\nabla \cdot \left(n^2\vec{E}\right) = \vec{E} \cdot \nabla n^2 + n^2\left(\nabla \cdot \vec{E}\right) = 0 \qquad (2.1.8)$$

$$\nabla \cdot \vec{E} = -\vec{E} \cdot \frac{\nabla n^2}{n^2} = -\vec{E} \cdot \nabla \ln n^2 = -2\vec{E} \cdot \nabla \ln n\,. \qquad (2.1.9)$$

Putting (2.1.9) into (2.1.7) and rearranging, one arrives at

$$\nabla^2\vec{E} + k_o^2 n^2 \vec{E} + \nabla\left(2\vec{E}\cdot\nabla \ln n\right) = 0. \qquad (2.1.10)$$

The last term in (2.1.10) is related to the change in polarization of the wave, while propagating. This term has been extensively studied and surmised negligible for wavelengths much smaller than the smallest spatial scale [4,5,6]. In addition, Clifford has shown that this term can still be negligible even when the wavelength is greater than the smallest spatial scale [6]. So, dropping this last term results in the following

$$\nabla^2\vec{E} + k_o^2 n^2 \vec{E} = 0. \qquad (2.1.11)$$

Equation (2.1.11) may be decomposed into three scalar equations, representing one of the three electric field vector components. So, only one of the scalar equations needs to be solved thus circumventing the vector nature of (2.1.11) giving

$$\nabla^2 E + k_o^2 n^2 E = 0. \qquad (2.1.12)$$

Now, the respective perturbation techniques enter and are applied to (2.1.12).

2.2 METHOD OF SMALL PERTURBATIONS (BORN APPROXIMATION)

The method of small perturbations involves the expansion of the electric field into a series of ever decreasing terms of order ε, which may be either convergent or asymptotic

$$E = E_o + \varepsilon E_1 + \varepsilon^2 E_2 + \cdots = E_o + \sum_{n=1}^{\infty} \varepsilon^n E_n . \qquad (2.2.1)$$

The refractive index is decomposed into two parts

$$n(\bar{r}) = \langle n(\bar{r}) \rangle + \varepsilon n_1(\bar{r}) . \qquad (2.2.2)$$

The first term is the mean of the medium while the second represents the fluctuating part. The mean of the entire medium is assumed to be constant and equal to one. The parameter ε is a measure of the deviation of the refractive index from its mean. The series expression of E, along with (2.2.2) for the refractive index, is then applied to (2.1.12), the scalar wave equation, yielding

$$\sum_{n=0}^{\infty} \varepsilon^n \left[\Delta E_n + k_o^2 (1 + \varepsilon n_1(\bar{r}))^2 E_n \right] = 0 . \qquad (2.2.3)$$

The above equation can be written in such a manner that the coefficients of like powers of ε can be equated to zero

$$(\Delta E_0 + k_o^2 E_0) + \varepsilon \ (\Delta E_1 + k_o^2 E_1 + 2k_o^2 n_1 E_1) +$$

$$\varepsilon^2 (\Delta E_2 + k_o^2 E_2 + 2k_o^2 n_1 E_2 + k_o^2 n_1 E_2) + \cdots = 0 \qquad (2.2.4)$$

from which is obtained

$$\Delta E_0 + k_o^2 E_0 = 0 \tag{2.2.5}$$

$$\Delta E_1 + k_o^2 E_1 = -2k_o^2 n_1 E_0 \tag{2.2.6}$$

$$\Delta E_2 + k_o^2 E_2 = -2k_o^2 E_1 - k_o^2 n_1^2 E_0 \tag{2.2.7}$$

$$\vdots \;.$$

A solution for the electric field may then be obtained by solving the above equations recursively. Equation (2.2.5) represents a description of propagation in free space or the incident field. This incident field, satisfying the necessary conditions, is then the driving force for the rest of the nonhomogeneous equations. In general, the incident field is known for direct and inverse scattering applications. The above equations may be rewritten in integral form as

$$E_1 = \int_V G(\vec{r},\vec{r}\,')2k_o^2 n_1(\vec{r}\,')E_0(\vec{r}\,')d^3\vec{r}\,' \tag{2.2.8}$$

$$E_2 = \int_V G(\vec{r},\vec{r}\,')\left(2k_o^2 n_1(\vec{r}\,')E_1(\vec{r}\,') + k_o^2 n_1(\vec{r}\,')E_0(\vec{r}\,')\right)d^3\vec{r} \tag{2.2.9}$$

$$\vdots$$

The function $G(\vec{r},\vec{r}\,')$ is the Green's function given by

$$G(\vec{r},\vec{r}\,') = \frac{e^{ik_o|\vec{r}-\vec{r}\,'|}}{4\pi|\vec{r}-\vec{r}\,'|} \quad . \tag{2.2.10}$$

This function satisfies the following equation

$$\Delta G(\vec{r},\vec{r}\,') + k_o^2 G(\vec{r},\vec{r}\,') = -\delta(\vec{r}-\vec{r}\,') \,. \tag{2.2.11}$$

The usual physical interpretation for the higher-order terms is as follows. The zero-order term represents propagation of the wave if the index of refraction has no fluctuating part, (i.e. the unscattered wave). The first-order term represents a singly scattered wave, the second order represents double scattering and so on. It is rationalized that higher-order terms become increasingly important as the strength of fluctuations increases or the path length of propagation increases. Generally, however, only the first-order term is used and referred to as the first Born approximation or simply the Born approximation

$$E = E_0 + \varepsilon E_1 . \qquad (2.2.13)$$

The higher-order terms become increasingly difficult to calculate, even the second order term. Early workers then turned to more elaborate methods to describe strong fluctuations or distant path lengths. Under the conditions where first-order approximations are acceptable, the Born approximation can be an easy and descriptive calculation. The question of where the Born approximation is valid relative to the Rytov method is again the heart of the inquiry of this paper.

The wave equation, from which the treatment started is stochastic in nature with a random coefficient. Equation (2.1.12) represents a family of equations each with a different $n_1(r)$, which can be described by some probability density. No analytical treatment of the wave equation directly, with the random coefficient, exists to date. The Born approximation creates a series of coupled equations, where the random term now appears as a quasi-source term of the form

$$\Delta E_n(\bar{r}) + k_o^2 E_n(\bar{r}) = f(\bar{r}). \tag{2.2.13}$$

Then the equations of the form above can be rexpressed in integral form as in (2.2.8) and (2.2.9). Using the solution for the Green's function obtained from (2.2.10), a solution can be built up recursively, as each successive equation is dependent upon the solution of the preceding equation. The statistics of the refractive index are then applied to the integral equations, from which the average values of the field are found.

Because of the reasons mentioned above, the Born method was shown to be primarily a first-order approximation. Because the log amplitude is what is often measured experimentally, a bit of manipulation is often applied to separate the electric field's real and imaginary parts. So, the field $E = E_0 + \varepsilon E_1$ expressed in terms of amplitude and phase is

$$\frac{E}{E_0} = 1 + \frac{E_1}{E_0} = \frac{A}{A_0} e^{i(s - s_0)} \tag{2.2.14}$$

where

$$E(\bar{r}) = A(\bar{r})e^{is(\bar{r})} \tag{2.2.15}$$

$$E_0(\bar{r}) = A_0(\bar{r})e^{is_0(\bar{r})} \tag{2.2.16}$$

$$E_1(\bar{r}) = A_1(\bar{r})e^{is_1(\bar{r})} . \tag{2.2.17}$$

Taking the natural logarithm of both sides of (2.2.14) yields

$$\ln\left(1+\frac{E_1}{E_0}\right) = \ln\left(\frac{A}{A_0}\right) + i(s-s_0) = \ln\left(1+\frac{A_1}{A_0}\right) + i(s-s_0) . \tag{2.2.18}$$

Because for weak fluctuations $|E_1/E_0| << 1$ and $|A_1/A_0| << 1$, (2.2.14) can be expanded into a power series, while discarding all but the first term, such that

$$\frac{E_1}{E_0} \cong \frac{A_1}{A_0} + i(s-s_0) . \tag{2.2.19}$$

Because of (2.2.19) and the assumptions leading to it, the amplitude ratio and phase fluctuations can be obtained from the real and imaginary parts of (2.2.8) normalized to an incident plane wave, $A_0 e^{ik_o z}$, giving

$$\ln\left(\frac{A}{A_0}\right) \cong \frac{A_1}{A_0} = \frac{k_o^2}{4\pi}\int_V d\bar{r}' \cos\left[k_o\left(|\bar{r}-\bar{r}'| + (z'-z)\right)\right]\frac{n_1(\bar{r}')}{|\bar{r}-\bar{r}'|} \tag{2.2.20}$$

and

$$s - s_0 = \frac{k_o^2}{4\pi}\int_V d\bar{r}' \sin\left[k_o\left(|\bar{r}-\bar{r}'| + (z'-z)\right)\right]\frac{n_1(\bar{r}')}{|\bar{r}-\bar{r}'|} . \tag{2.2.21}$$

2.3 METHOD OF SMOOTH PERTURBATIONS (RYTOV APPROXIMATION)

The Rytov approximation takes a somewhat different approach. A solution of the wave equation, (2.1.12), is assumed to have the following form

$$E(\bar{r}) = e^{\psi(\bar{r})}. \tag{2.3.1}$$

An assumption is made that ψ may be represented in series form as

$$\psi(\bar{r}) = \psi(\bar{r}) + \varepsilon\,\psi_1(\bar{r}) + \varepsilon^2 \psi(\bar{r}) + \cdots = \sum_{n=0}^{\infty} \varepsilon^n \psi(\bar{r}) . \tag{2.3.2}$$

The electric field in the form of (2.3.1) is then substituted into the wave equation obtaining an equation for ψ

$$\Delta e^{\psi(\bar{r})} + k_o^2 \left(1 + \varepsilon n(\bar{r})\right)^2 e^{\psi(\bar{r})} = 0 \tag{2.3.3}$$

where

$$\Delta e^{\psi(\bar{r})} = \left[\Delta\psi(\bar{r}) + \nabla\psi(\bar{r}) \cdot \nabla\psi(\bar{r})\right] e^{\psi(\bar{r})}, \tag{2.3.4}$$

from which,

$$\Delta\psi(\bar{r}) + \nabla\psi(\bar{r}) \cdot \nabla\psi(\bar{r}) + k_o^2\left(1 + \varepsilon n_1(\bar{r})\right)^2 = 0 . \tag{2.3.5}$$

The equation above for ψ is known as Riccati's equation. Again as in the Born method, the random term has been isolated and moved from a coefficient to appearing in the nonhomogeneous term. However, this benefit has been at the cost of a nonlinear equation of which no exact solution exists. So, as a consequence, the perturbation of ψ is then applied to this equation yielding

$$\begin{aligned}&\Delta\left[\psi_0(\bar{r})+\varepsilon\psi_1(\bar{r})+\varepsilon^2\psi_2(\bar{r})+\cdots\right]+\nabla\left[\psi_0(\bar{r})+\varepsilon\psi_1(\bar{r})+\varepsilon_1^2\psi_2(\bar{r})+\cdots\right]\cdot\\&\nabla\left[\psi_0(\bar{r})+\varepsilon\psi_1(\bar{r})+\varepsilon^2\psi_2(\bar{r})+\cdots\right]=-k_o^2-2\varepsilon k_o n_1(\bar{r})-\varepsilon^2 n_1^2(\bar{r})\end{aligned} \tag{2.3.6}$$

Rearranging terms in powers of ε, with the sum of all ε^n terms equated to zero gives

$$\begin{aligned}&\Delta\left[\psi_0(\bar{r})+\varepsilon\psi_1(\bar{r})+\varepsilon^2\psi_2(\bar{r})+\cdots\right]+\\&\nabla\psi_0(\bar{r})\cdot\left[\nabla\psi_0(\bar{r})+\varepsilon\nabla\psi_1(\bar{r})+\varepsilon^2\nabla\psi(\bar{r})+\cdots\right]+\\&\varepsilon\nabla\psi_1(\bar{r})\cdot\left[\nabla\psi_0+\varepsilon\nabla\psi_1+\varepsilon^2\nabla\psi_2+\cdots\right]+\\&\varepsilon\nabla\psi_2(\bar{r})\cdot\left[\nabla\psi_0(\bar{r})+\varepsilon\nabla\psi_1(\bar{r})+\varepsilon^2\nabla\psi_2(\bar{r})+\cdots\right]+\cdots=\\&k_o^2-2\varepsilon k_o n_1(\bar{r})-\varepsilon n_1(\bar{r})\end{aligned} \tag{2.3.7}$$

$$\sum_{n=0}^{\infty}\varepsilon^n\left[\Delta\psi_n(\bar{r})+\sum_{p=0}^{n}\nabla\psi_p(\bar{r})\cdot\nabla\psi_{n-p}(\bar{r})\right]=-k-\varepsilon\,2k_o^2 n_1(\bar{r})-\varepsilon^2 n_1^2(\bar{r}) \tag{2.3.8}$$

$$\Delta\psi_0(\bar{r})+\nabla\psi_0(\bar{r})\cdot\nabla\psi_0(\bar{r})=-k_o^2 \tag{2.3.9}$$

$$\Delta\psi_1(\bar{r})+2\nabla\psi_0(\bar{r})\cdot\nabla\psi_1(\bar{r})=-2k_o^2 n_1(\bar{r}) \tag{2.3.10}$$

$$\Delta\psi_2(\bar{r})+2\nabla\psi_0(\bar{r})\cdot\nabla\psi_2(\bar{r})=-k_o^2 n_1(\bar{r})-\nabla\psi_1(\bar{r})\cdot\nabla\psi_1(\bar{r}) \tag{2.3.11}$$

$$\Delta\psi_n+2\nabla\psi_0\cdot\nabla\psi_n=-\sum_{p=1}^{n-1}\nabla\psi_p\cdot\nabla\psi_{n-p}\qquad n=3,4,5,\cdots. \tag{2.3.12}$$

If $E_0=\exp\psi_0$, then (2.3.9) is equivalent to (2.2.4)

$$\Delta E_0+k_o^2E_0=0\,. \tag{2.2.4}$$

Equations (2.3.9)-(2.3.12) can be solved recursively for ψ, similarly to the Born approximation. However, to facilitate this calculation it is convenient to introduce the substitution $W_n = E_0 \psi_n \qquad n = 1,2,3,\cdots.$

(2.3.13)

Utilizing the solution to (2.2.4) and expanding the Laplacian of W_n gives the following

$$\begin{aligned}\Delta W_n &= \Delta E_0 \psi_n \\ &= E_0 \left[\Delta \psi_n + 2 \nabla \psi_0 \cdot \nabla \psi_n + \psi_n \left[\Delta \psi_0 + \nabla \psi_0 \cdot \nabla \psi_0 \right] \right] \\ &= E_0 \left[\Delta \psi_n + 2 \nabla \psi_0 \cdot \nabla \psi_n \right] - k_o^2 W_n .\end{aligned} \tag{2.3.14}$$

Employing (2.3.10)-(2.3.12) and the above equation, an expression for W_n in the form of the wave equation is obtained as follows

$$\Delta W_1 + k_o^2 W_1 = -2 k_o^2 n_1(\bar{r}) E_0 \tag{2.3.15}$$

$$\Delta W_2 + k_o^2 W_2 = -\left[k_o^2 n_1^2(\bar{r}) + \nabla\left(\frac{W_1}{E_0}\right) \cdot \nabla\left(\frac{W_1}{E_0}\right) \right] E_0 \tag{2.3.16}$$

$$\Delta W_n + k_o^2 W_n = -\sum_{p=1}^{n-1} \left[\nabla\left(\frac{W_p}{E_0}\right) \cdot \nabla\left(\frac{W_{n-p}}{E_0}\right) E_0 \right] \qquad n = 3,4,5,\cdots \tag{2.3.17}$$

Using Green's function, again, the integral equations above are rewritten as

$$W_1(\bar{r}) = 2 k_o^2 \int_V G(\bar{r}, \bar{r}') n_1(\bar{r}') E_0(\bar{r}') d^3 \bar{r}' \tag{2.3.18}$$

$$W_2(\bar{r}) = \int_V G(\bar{r},\bar{r}')\left[k_o^2 n_1^2 + \nabla\left(\frac{W_1}{E_0}\right)\cdot\nabla\left(\frac{W_1}{E_0}\right)\right]E_0(\bar{r}')d^3\bar{r}' \qquad (2.3.19)$$

$$W_n(\bar{r}) = \sum_{p=1}^{n-1}\int_V G(\bar{r},\bar{r}')\nabla\left(\frac{W_p}{E_0}\right)\cdot\nabla\left(\frac{W_{n-p}}{E_0}\right)E_0(\bar{r}')d^3\bar{r}' \qquad n = 3,4,5\cdots \qquad (2.3.20)$$

Recovering ψ from the above equations with the use of (2.3.13), we get

$$\psi_1(\bar{r}) = 2k_o^2\int_V G(\bar{r},\bar{r}')n_1(\bar{r}')\frac{E_0(\bar{r}')}{E_0(\bar{r})}d^3\bar{r}' \qquad (2.3.21)$$

$$\psi_2(\bar{r}) = \int_V G(\bar{r},\bar{r}')\left[k_o^2 n_1^2 + \nabla\psi_1\cdot\nabla\psi_1\right]\frac{E_0(\bar{r}')}{E_0(\bar{r})}d^3\bar{r}' \qquad (2.3.22)$$

$$\psi_n(\bar{r}) = \sum_{p=1}^{n-1}\int_V G(\bar{r},\bar{r}')\left[\nabla\psi_p\cdot\nabla\psi_{n-p}\right]\frac{E_0(\bar{r}')}{E_0(\bar{r})}d^3\bar{r}' \qquad n = 3,4,5,\cdots \qquad (2.3.23)$$

As in the Born approximation, generally only the first term of the Rytov method is calculated and thus is referred to as a first order approximation. The first-order Rytov approximation and simply "the Rytov approximation" will be used interchangeably.

The preceding is the general procedure referred to as the Rytov approximation in the western literature. The Russian workers start off with the parabolic equation by discarding the second derivative in the direction of propagation in the wave equation

$$u(\bar{p},z) = v(\bar{p},z)e^{ikz} \qquad (2.3.24)$$

$$2ik\frac{\partial v}{\partial z}+\Delta_T v+k_o^2 n_1^2(\bar{p},z)=0 . \qquad (2.3.25)$$

The basis of the assumption lies in the restriction to changes in the index of refraction that are small on the scale of a wavelength in the direction of propagation.

In applying the Rytov approximation as in the Western literature directly to the wave equation, another assumption, known as the narrow cone approximation, is generally made to aid in the evaluation of the integral in (2.3.21). From the laws of diffraction, the angle of scattering will be largest for those scatterers of the smallest scale size of turbulence l_O. Also, the scattering angle will then be of the order λ/l_0. If an incident wave propagates along the z-axis, then after a first scattering it will propagate at an angle θ to the z-axis. If $z-z'$ is the longitudinal distance and $|\bar{p}-\bar{p}'|$ is the transverse displacement from the z axis, $z-z' >> |\bar{p}-\bar{p}'|$ is expected to be a good approximation. Then the higher order terms of the Taylor expansion of $|\bar{r}-\bar{r}'|$ can be dropped. So, by dropping the higher order terms in the Taylor series expansion

$$\begin{aligned}|\bar{r}-\bar{r}'| &= (z-z')\sqrt{1+\frac{|\bar{p}-\bar{p}'|^2}{(z-z')^2}} \\ &= (z-z')+\frac{1}{2}\frac{|\bar{p}-\bar{p}'|^2}{(z-z')}-\frac{1}{8}\frac{|\bar{p}-\bar{p}'|^4}{(z-z')^3}+\cdots\end{aligned} \qquad (2.3.26)$$

simplifies the Green's function to

$$G_s(\bar{r},\bar{r}') = -\frac{1}{4\pi(z-z')}\exp\left(ik(z-z') + \frac{ik|\bar{p}-\bar{p}'|}{2(z-z')}\right). \quad (2.3.27)$$

S.M. Rytov et al. show that this approximation is equivalent to dropping the second longitudinal derivative leading to the parabolic equation, (2.3.25) [8]. This approximation is much less restrictive than the requirements needed to hold for the Rytov expansion. So, the work on regions of applicability for the Rytov and Born approximation by the Russian and Western workers can be considered to be equivalent.

The assumed form of the field, e^{ψ}, is applied to the parabolic equation yielding a nonlinear equation by the Russian workers. Again however, the random function n_1 enters into this equation in an additive manner instead of as a coefficient. This approach is known as the method of smooth perturbations. In most cases in western literature, the method outlined above, starting directly with the wave equation, is called both the Rytov and method of smooth perturbations. This distinction should be made in reviewing the literature.

3. INTRODUCTION TO THE ACCURACY AND VALIDITY OF THE APPROXIMATIONS

The second chapter of this thesis addressed how the two perturbation techniques were applied to the Helmholtz equation. Up to this point, no real discussion has been made of the limitations of each method. The discussion following the presentation of the Born method gave the traditional physical explanation of the failure of this method (i.e. where multiple scattering was encountered). Generally, the belief has been held that the Rytov approximation is superior to the Born approximation. A good deal of literature over the years has been produced on this point. The precise conditions for validity of these approximations are difficult to appreciate since their mathematical expressions rely on strong inequalities holding. The intent of the following is not to establish a definitive conclusion to this problem, but by bringing all the literature together, a clarification might be made of the distinctions of each method.

3.1 KELLERS PAPER AND ENSUING LITERATURE

Keller (1970) began by considering a wave traveling in the x direction of the following form [9]

$$u(x,\varepsilon) = e^{ik(\varepsilon)x} . \tag{3.1.1}$$

And, the propagation constant k(ε), with ε sufficiently small, can be represented

$$k(\varepsilon) = \sum_{j=0}^{\infty} k_j \varepsilon^j . \tag{3.1.2}$$

The Born series is obtained by substituting (3.1.2) into (3.1.1) and expanding the exponential in a power series

$$u(x,\varepsilon) = e^{ik_o x} \exp\left[ix \sum_{j=1}^{\infty} k_j \varepsilon^j \right] = e^{ik_o x} \sum_{q=0}^{\infty} \left(ix \sum_{j=1}^{\infty} k_j \varepsilon^j \right)^q \frac{1}{q!} . \tag{3.1.3}$$

The n'th Born approximation is the sum of the first n+1 terms from (3.1.3), which is rearranged in orders of ε

$$u_B^n(x,\varepsilon) = e^{ik_o x} \sum_{s=0}^{\infty} \varepsilon^s \sum_{q=0}^{s} \frac{(ix)^q}{q!} \sum_{j_1+\cdots+j_q=s} k_{j_1} \cdots k_{j_q} . \tag{3.1.4}$$

The Rytov expansion is obtained likewise by the substitution of (3.1.2) into (3.1.1). Then, the nth Rytov approximation is obtained by using the first n+1 terms of

$$u_R^n(x,\varepsilon) = \exp\left(ix \sum_{j=0}^{n} k_j \varepsilon^j \right) . \tag{3.1.5}$$

The error of the nth Born approximation can be estimated from the n+1 term

$$u(x,\varepsilon) - u_B^n(x,\varepsilon) = e^{ik_o x} O\left(\varepsilon^{n+1} x^{n-1}\right). \qquad (3.1.6)$$

Dividing both sides of (3.1.6) gives

$$\frac{u(x,\varepsilon) - u_B^n(x,\varepsilon)}{u(x,\varepsilon)} = O\left(\varepsilon^{n+1} x^{n+1}\right). \qquad (3.1.7)$$

The function $\exp(ik_o x)$ differs from $u(x,\varepsilon)$ by terms of order ε, thus justifying the above expression.

The error of the nth Rytov approximation from (3.1.1) and (3.1.5) is given by

$$\begin{aligned} u(x,\varepsilon) - u_R^n(x,\varepsilon) &= e^{ix \sum_{j=0}^{\infty} k_j \varepsilon^j} - e^{ix \sum_{j=0}^{n} k_j \varepsilon^j} \\ &= u(x,\varepsilon)\left[1 - e^{-ix \sum_{j=n+1}^{\infty} k_j \varepsilon^j}\right]. \end{aligned} \qquad (3.1.8)$$

Investigation of the expression in brackets in (3.1.8) yields

$$\begin{aligned} 1 - e^{-ix \sum_{n-1}^{\infty} k_j \varepsilon^j} &= 1 - e^{-ix\varepsilon^{n+1}(k_{n+1} + \varepsilon k_{n+2} + \cdots)} \\ &= 1 - e^{-ix\varepsilon^{n+1}\alpha(\varepsilon)} \\ &= 1 - \left[1 - ix\varepsilon^{n+1}\alpha(\varepsilon) + \frac{1}{2!}\left(-ix\varepsilon^{n+1}\alpha(\varepsilon)\right)^2 + \cdots\right]. \end{aligned} \qquad (3.1.9)$$

Using the last expression in (3.1.9), (3.1.8) can be written as

$$\frac{u - u_R^n}{u} = O\left(\varepsilon^{n+1} x\right). \qquad (3.1.10)$$

Observing (3.1.7) and (3.1.10), for a fixed x the relative errors of the nth Born and nth Rytov approximations are $O(\varepsilon^{n+1})$. In this respect relative to ε, both approximations have the same degree of error. However, both errors exhibit a different dependence upon a function of x. For the nth Born approximation, the error increases as x^{n+1} as x increases. While the nth Rytov approximation only increases on the order of x as x increases.

So, Keller's argument is that the different rates of growth relative to x can be an advantage for the Rytov approximation. If the relative errors of (3.1.7) and (3.1.10) are denoted by δ, then the distance at which the approximations reach this value is

$$x_B = O\left(\delta^{1/(n+1)}/\varepsilon\right) \qquad (3.1.11)$$

$$x_R = O\left(\delta/\varepsilon^{n+1}\right). \qquad (3.1.12)$$

From this point of view, the distance at which the nth Rytov approximation accumulates an error of δ is much larger than for the nth Born.

Keller's final point is that for a single plane wave the nth Rytov approximation is valid for a much larger range of x than for the nth Born approximation. However due to the nature of the Rytov expression for the field, this advantage is lost for fields containing more than one plane wave. The extended range of validity will be lost if the Rytov expansion is not applied to each wave instead of the total field u. Therefore, the advantage gained by using the Rytov can only be exploited if there is a complete knowledge of all the waves that occur in the field u. Whereas no such apriori knowledge is required in the application of the Born.

Keller's paper rather neatly points out that the extra terms obtained in the Rytov expansion seem to increase the distance of equal accumulation of error relative to the Born. For the Born expansion the exponential expression is expanded into a infinite power series where for the nth approximation the n+1 terms are dropped. For the Rytov expansion, the propagation constant is again expressed as an infinite power series. The nth Rytov approximation drops the n+1 terms of the power series for $k(\varepsilon)$. If the nth Rytov is then expanded into a power series, an infinite number of terms is obtained.

3.2 RELATIONS BETWEEN TERMS IN THE APPROXIMATIONS

Following Keller's paper, Sancer and Varvatsis published a paper that came to the same conclusion that the Rytov and Born approximations have considerably different distances over which they are valid [10]. The paper begins by providing the expressions relating terms between the two methods. The expression for the electric field in the Rytov approximations, (2.3.1), is expanded as a Taylor series about ε=0. The coefficients of like powers of ε are equated between the Born and Rytov expressions as follows

$$u(x,\varepsilon)=\sum_{n=0}^{\infty}\varepsilon^n u_n=\sum_{n=0}^{\infty}\frac{f^n(\varepsilon=0)}{n!}\varepsilon^n \tag{3.2.1}$$

$$u_0=f(\varepsilon)\Big]_{\varepsilon=0}=e^{\psi_0+\varepsilon\psi_1+\varepsilon^2\psi_2+\cdots}\Big]_{\varepsilon=0}=e^{\psi_0} \tag{3.2.2}$$

$$u_1=(\psi_1+2\varepsilon\psi_2+\cdots)e^{\psi_0+\varepsilon\psi_1+\varepsilon^2\psi_2+\cdots}\Big]_{\varepsilon=0}=\psi_1 e^{\psi_0} \tag{3.2.3}$$

$$\begin{aligned}u_2=\frac{f^2(\varepsilon)}{2!}\Big]_{\varepsilon=0}&=\left.\begin{matrix}(\psi_1+2\varepsilon\psi_2+3\varepsilon^2\psi_3+\cdots)^2e^{\psi_0+\varepsilon\psi_1+\varepsilon^2\psi_2+\cdots}+\\(2\psi_2+6\varepsilon\psi_3+\cdots)e^{\psi_0+\varepsilon\psi_1+\varepsilon^2\psi_2+\cdots}\end{matrix}\right]_{\varepsilon=0}\\&=\left(\frac{1}{2!}\psi_1^2+\psi_2\right)e^{\psi_0}\end{aligned} \tag{3.2.4}$$

$$\vdots\ .$$

If $u_0=e^{\psi_0}$ and $p_n=u_n/u_0$, then the expressions for (3.2.1)-(3.2.4) can be written as

$$p_1=\psi_1,\quad p_2=\psi_2+\frac{\psi_1^2}{2!},\quad p_3=\psi_3+\psi_2\psi_1+\frac{\psi_1^3}{3!},\cdots. \tag{3.2.5}$$

If the representations of ψ_n are obtained from (3.2.5), a rather interesting reciprocity is encountered

$$\psi_1=p_1,\quad \psi_2=p_2-\frac{p_1^2}{2},\quad \psi_3=p_3-p_2p_1+\frac{p_1^3}{3},\cdots \tag{3.2.6}$$

The above relations in (3.2.5) and (3.2.6) can be proved in a different manner by inserting the integral forms for the ψ's and u's given above [11]. A question arises as to what the underlying factor is which, starting from dissimilar forms of the electric field assumed in the Born and Rytov approximations, leads to a highly symmetric relationship between the expressions given in (3.2.5) and (3.2.6).

A differential equation of the following form is assumed

$$\Delta u_n + k^2 u_n + f_n\left(\frac{u_n}{u_0}\right)u_0 = 0 \qquad (3.2.7)$$

in which the last term is a given function of E/u_o from (2.2.5)-(2.2.6)

$$f_n\left(\frac{u_n}{u_0}\right) = \sum_{n=1}^{\infty} 2k_o^2 n_1 \frac{u_{n-1}}{u_0} + k_o^2\left(1-\delta_{n,1}\right)\frac{u_{n-2}}{u_0}. \qquad (3.2.8)$$

As before the solution to (3.2.7) is given by the integral equation

$$u_n = -\int G(\bar{r},\bar{r}')f\left(\frac{u_n}{u_0}\right)u_0(\bar{r}')d^3\bar{r}'. \qquad (3.2.9)$$

Dividing (3.2.7) by u_o, the reduced form of u is defined as

$$p_n(\bar{r}) = \frac{u_n}{u_0} = -\int G(\bar{r},\bar{r}')f_n\left(\frac{u_n}{u_0}\right)\frac{u_0(\bar{r}')}{u_0(\bar{r})}d^3\bar{r}' \qquad (3.2.10)$$

If ∇p_n and Δp_n are calculated from u_n/u_o, then the following equation is obtained

$$\Delta p_n + 2\nabla\psi_0\cdot\nabla p_n = \frac{\left(\Delta p_n + k^2 p_n\right)}{u_0} \qquad (3.2.11)$$

Using (3.2.7), (3.2.9) can be rewritten as

$$\Delta p_n + 2\nabla\psi_0 \cdot \nabla p_n + f_n(p) = 0 \qquad (3.2.12)$$

If (3.2.12) is compared with (2.3.9) - (2.3.12) written in a more compact form

$$\Delta\psi_n + 2\nabla\psi_0 \cdot \nabla\psi_n + f_n(\psi) = 0 \qquad (3.2.13)$$

where

$$f_n(\psi) = (1-\delta_{n,1})\sum_{p=1}^{n-1}\nabla\psi_p \cdot \nabla\psi_{n-p} + 2k_o^2 n_1 \delta_{n,1} + k_o^2 n_1^2 \delta_{n,2} \qquad (3.2.14)$$

The forms of (3.2.12) and (3.2.13) are the reason for the symmetry between p_n and ψ_n found in the expressions of (3.2.5) and (3.2.6). From (3.2.12) and (3.2.13), it can be seen that $p_1=\psi_1$ and that for $n\geq 2$ $p_n \neq \psi_n$. The point to be made here is that there is a direct relationship between terms in the series. From this standpoint, the conclusion might be made that regions of validity would be the same or at least closely related. This assumption would seem very evident at least for the first order approximations. This kind of observation is part of the reason for the controversy over the regions of validity.

4. REGIONS OF VALIDITY IN A RANDOM MEDIA

After the treatises by Tatarski and Chernov [1,2], deWolf, in 1965, was one of the first workers to organize and compare the approximations for wave propagation in an random inhomogenous medium [12]. In this paper four approximations are considered, WKB, Geometric Optics, first Born and first Rytov expansions. However, the discussion in this thesis will only be concerned with the Born and Rytov methods. One of the novel items introduced in deWolf's is a graphical representation of the proposed regions of validity. Also unlike Keller's work, deWolf's discussion is based on a random medium, like Tatarskii's first and later works [2,13]. In addition, some of the difficulties of defining regions of validity and interpreting them in a random medium will become evident.

4.1 deWOLF'S DISCUSSION IN A RANDOM MEDIUM AND GRAPHICAL REPRESENTATION

The problem deWolf considered for the comparison of the methods was a plane wave traveling in the z direction normally incident upon a slab of thickness of d. To begin, the solution for the first order Born series is presented, as above in (2.2.13). Two conditions, which merely are deemed necessary, are presented:

(1) $|E_1|^2 << 1$. The first Born term must be small due to energy considerations.

(2) The scattered flux, I_{SC}, must be much less than the incident flux, I_O.

The first condition is a statement regarding the conservation of energy. The assumption is made that no absorption of the wave is taking place in the inhomogeneous slab. With this consideration, the amount of energy the wave carries into the slab should also exit the slab. The first Born approximation assumes

$$E = E_0 + E_1 . \tag{4.1.1}$$

The incident mean intensity is

$$\langle |E_0|^2 \rangle . \tag{4.1.2}$$

The mean intensity leaving the slab is

$$\langle |E|^2 \rangle = \langle |E_0 + E_1|^2 \rangle$$
$$= \langle |E_0|^2 \rangle + 2\,\mathrm{Re}\langle E_0^* E_1 \rangle + \langle |E_1|^2 \rangle . \qquad (4.1.3)$$

From the above argument (4.1.2) and (4.1.3) should be equal if energy is to be conserved. Since the mean of the fluctuations is zero, $\langle n_1(\vec{r}) \rangle = 0$, $2\,\mathrm{Re}\langle |E_0^* E_1| \rangle$ must also equal zero. However, this leaves a contradiction between (4.1.2) and (4.1.3), which leads to the first condition.

The second condition is shown to lead the same restrictions on the quantities to be discussed as the first condition. So, no further discussion will entail the second condition.

The first order Rytov approximation is then presented. The following equation is shown for the first order Rytov approximation

$$|\nabla \psi_1|^2 + \Delta \psi_1 + 2ik_o \bar{z} \cdot \nabla \psi_1 + k_o^2 n_1 = 0 . \qquad (4.1.4)$$

The following argument is presented for dropping the first term in (4.1.4). The inequalities expressed are

$$\lambda |\nabla \delta S| << 1 \qquad (4.1.5)$$

$$\lambda \left|\nabla \ln(A / A_o)\right| << 1 \qquad (4.1.6)$$

where $\psi_1 = \ln(A / A_0) + i\delta S$. If these inequalities hold, then the first term may be dropped relative to the others in (4.1.4). The first inequality implies that the wavefront never deviates from the initial direction. The second states that the derivative of amplitude never deviates much from that of the incident along a wavelength.

If the inequalities (4.1.5) and (4.1.6) are satisfied, then a solution is suggested of the form

$$\Psi = \alpha + \psi_1 . \tag{4.1.7}$$

The α is a real constant. The addition of the alpha into the solution for (4.1.4) is done in order to satisfy energy considerations. Again, examination of the mean intensity of the Rytov solution yields a violation of the law conservation of energy. For the Rytov approximation, the assumed form of the electric field is

$$E = e^{\psi_0 + \psi_1} = E_0 e^{\psi_1} . \tag{4.1.8}$$

where $\psi_1 = \chi + i\phi$. From (4.1.8), the magnitude squared of the electric field is then

$$|E|^2 = |E_0|^2 e^{2\chi} . \tag{4.1.9}$$

The mean intensity is then given by

$$\left\langle |E|^2 \right\rangle = |E_0|^2 \left\langle e^{2\chi} \right\rangle = |E_0|^2 e^{\langle \chi^2 \rangle} , \tag{4.1.10}$$

since χ follows a Gaussian distribution. However, since the assumption is made that the medium is nonabsorbing, the need for conservation of energy is encountered again, i.e.

$\langle |E|^2 \rangle = \langle |E_0|^2 \rangle$ the energy entering the slab. So, this constant was added in an attempt to account for conservation of energy, $\alpha = \langle \chi^2 \rangle$.

The second moments of the real and imaginary parts of E_1 are presented in order to use the conditions above to obtain a graphical representation of the proposed regions of validity with respect to the thickness of the slab. The moments are calculated using a Gaussian correlation function.

Due to the random nature of the medium, descriptions of the field are determined most usefully by statistics. So, regions of acceptable applicability must be calculated from averages. The nature of the form of the expressions for the conditions of applicability dictate the necessity of finding the second moments.

The first condition for the Born approximation averaged is now

$$\langle |E_1|^2 \rangle << 1. \tag{4.1.11}$$

Expanding E_1 into its real and imaginary parts

$$\langle E_{1r}^2 + E_{1i}^2 \rangle = \langle E_{1r}^2 \rangle + \langle E_{1i}^2 \rangle << 1. \tag{4.1.12}$$

The second moments for the real and imaginary part of E_1 are given by Ishimaru for a gaussian correlation function outside the geometric-optics realm, $(L >> l^2/\lambda)$, by [14]

$$\left\langle E_{1r}^2 \right\rangle = \left\langle E_{1i}^2 \right\rangle = \left\langle n_1^2 \right\rangle \frac{1}{2} k_o^2 \, d \, l \sqrt{\pi} \,. \tag{4.1.13}$$

Using the given expression for the second moments in (4.1.13), the condition of (4.1.12) reduces to

$$d << d_B \qquad d_B \equiv \lambda^2 / l \left\langle n_1^2 \right\rangle . \tag{4.1.14}$$

The condition of validity for the Rytov approximation formulated in a manner consistent with a statistical description is

$$\left\langle \left| \lambda \, \nabla \psi_1 \right|^2 \right\rangle << 1 . \tag{4.1.15}$$

Formulation of the above condition into a more meaningful form requires a bit more effort. Firstly, breaking ψ_1 into real and imaginery parts gives

$$\nabla \psi_1 = \nabla\left(\alpha + \mathrm{Re}(\psi_1) + i\,\mathrm{Im}(\psi_1)\right) = \nabla\left(\mathrm{Re}(\psi_1) + i\,\mathrm{Im}(\psi_1)\right) . \tag{4.1.16}$$

Inserting (4.1.16) into (4.1.15), will yield

$$\begin{aligned} \left\langle \lambda^2 \left| \nabla\left(\mathrm{Re}(\psi_1) + i\,\mathrm{Im}(\psi_1)\right) \right|^2 \right\rangle &= \lambda^2 \left\langle \left| \nabla \mathrm{Re}(\psi_1) \right|^2 \right\rangle \\ &+ \lambda^2 \left\langle 2i \nabla \mathrm{Re}(\psi_1) \cdot \nabla \mathrm{Im}(\psi_1) \right\rangle - \lambda^2 \left\langle \left| \nabla \mathrm{Im}(\psi_1) \right|^2 \right\rangle . \end{aligned} \tag{4.1.17}$$

From (4.1.17), the gradient of ψ_1 must be calculated, then the real and imaginery parts can separated. Begining with the gradient of ψ_1 from (2.3.22) and recalling that the incident disturbance is a plane wave

$$\nabla\psi_1(\bar{r}) = 2k_o^2 \int_V d^3\bar{r}\,\nabla\left[G(\bar{r},\bar{r}')\frac{E_0(\bar{r}')}{E_0(\bar{r})}\right] n_1(\bar{r}')$$

$$= -\frac{2k_o^2}{4\pi}\int_V d\bar{r}'\,\nabla'\left[\frac{e^{ik_o[|\bar{r}-\bar{r}'|+(z'-z)]}}{|\bar{r}-\bar{r}'|}\right] n_1(\bar{r}'). \tag{4.1.18}$$

Utilizing the following identity

$$\nabla(\phi\psi) = \phi\,\nabla\psi + \psi\nabla\phi \tag{4.1.19}$$

(4.1.18) can be written as

$$\nabla\psi_1 = -\frac{k_o^2}{4\pi}\int_v d\bar{r}'\,\nabla'\left[\frac{e^{ik_o(|\bar{r}-\bar{r}'|+(z'-z))}}{|\bar{r}-\bar{r}'|} n_1(\bar{r}')\right] +$$

$$\frac{k_o^2}{4\pi}\int_V d\bar{r}'\frac{e^{ik_o(|\bar{r}-\bar{r}'|+(z'-z))}}{|\bar{r}-\bar{r}'|}\nabla' n_1(\bar{r}'). \tag{4.1.20}$$

The first term in the summation of $\nabla\psi_1$ will tend to zero as $|\bar{r}-\bar{r}'|$ approaches infinity, leaving only the second term. This can be seen by expressing the volume integral as a surface integral

$$-\frac{k_o^2}{4\pi}\int_v d\bar{r}'\,\nabla'\left[\frac{e^{ik_o(|\bar{r}-\bar{r}'|+(z'-z))}}{|\bar{r}-\bar{r}'|} n_1(\bar{r}')\right] = \oiint_S d^2\bar{r}\,\frac{e^{ik_o(|\bar{r}-\bar{r}'|+(z'-z))}}{|\bar{r}-\bar{r}'|} n_1(\bar{r}'). \tag{4.1.21}$$

The real and imaginary parts of $\nabla\psi_1$ are obtained from the following expression

$$\mathrm{Re}[\nabla\psi_1] = \frac{1}{2}\left(\nabla\psi_1 + \nabla\psi_1^*\right) \tag{4.1.22}$$

$$\mathrm{Im}[\nabla\psi_1] = \frac{1}{2i}\left(\nabla\psi_1 - \nabla\psi_1^*\right). \tag{4.1.23}$$

Using (4.1.22) and (4.1.20), the real part of $\nabla\psi_1$ is

$$\mathrm{Re}[\nabla\psi_1]=\frac{k_o^2}{4\pi}\int_V d\bar{r}'2\cos\left[k_o\left(|\bar{r}-\bar{r}'|+(z'-z)\right)\right]\frac{\nabla n_1(\bar{r}')}{|\bar{r}-\bar{r}'|}. \qquad (4.1.24)$$

The imaginary part of $\nabla\psi_1$ is found with (4.1.23) and now (4.1.20)

$$\mathrm{Im}[\nabla\psi_1]=\frac{k_o^2}{4\pi}\int_v d\bar{r}'2\sin\left[k_o\left(|\bar{r}-\bar{r}'|+(z'-z)\right)\right]\frac{\nabla n_1(\bar{r}')}{|\bar{r}-\bar{r}'|}. \qquad (4.1.25)$$

Now the variance of the real and imaginary part of $\nabla\psi_1$ are

$$\left\langle \mathrm{Re}[\nabla\psi_1]^2\right\rangle=\frac{k_o^4}{16\pi^2}\int_V d^3\bar{r}'d^3\bar{r}'2\cos^2\left(k_o\left[|\bar{r}-\bar{r}'|+(z'-z)\right]\right)\frac{\left\langle|\nabla' n_1(\bar{r}')|^2\right\rangle}{|\bar{r}-\bar{r}'|} \qquad (4.1.26)$$

$$\left\langle \mathrm{Im}[\nabla\psi_1]^2\right\rangle=\frac{k_o^4}{16\pi^2}\iint_V d^3\bar{r}'d^3\bar{r}'4\sin^2\left(k_o\left[|\bar{r}-\bar{r}'|+(z'-z)\right]\right)\frac{\left\langle|\nabla' n_1(\bar{r}')|^2\right\rangle}{|\bar{r}-\bar{r}'|^2}. \qquad (4.1.27)$$

The expression for $\left\langle\nabla\psi_1(\bar{r}_1)\cdot\nabla\psi_1(\bar{r}_2)\right\rangle$ using any choice of correlation function is

$$\left\langle\nabla n_1(\bar{r}_1)\cdot\nabla n_1(\bar{r}_2)\right\rangle\approx l^2\left\langle n_1^2(\bar{r})\right\rangle C\left(|\bar{r}_1-\bar{r}_2|\right). \qquad (4.1.28)$$

Equation (4.1.28) can be obtained by a spatial Fourier transform of n_1 and then performing the gradient operation. Using the Gaussian correlation function, utilized in the paper, and (4.1.28), the following form is

$$\left\langle\nabla n_1^2(\bar{r})\right\rangle=\frac{1}{l^2}\left\langle n_1^2(\bar{r})\right\rangle. \qquad (4.1.29)$$

Substitution of (4.1.29) into (4.1.26) and (4.1.27) then gives

$$\left\langle \mathrm{Re}[\nabla\psi_1]^2 \right\rangle = \frac{k_o^2}{4\pi}\iint_V d\bar{r}'d\bar{r}'4\cos^2\left(k_o\left[|\bar{r}-\bar{r}'|+(z'-z)\right]\right)\frac{\left\langle n_1^2(\bar{r}')\right\rangle}{l^2|\bar{r}-\bar{r}'|^2} \qquad (4.1.30)$$

$$\left\langle \mathrm{Im}[\nabla\psi_1]^2 \right\rangle = \frac{k_o^2}{4\pi}\iint_V d\bar{r}'d\bar{r}'4\sin^2\left(k_o\left[|\bar{r}-\bar{r}'|+(z'-z)\right]\right)\frac{\left\langle n_1^2(\bar{r}')\right\rangle}{l^2|\bar{r}-\bar{r}'|^2}. \qquad (4.1.31)$$

Using (2.2.20) and (2.2.21), the variance of the real and imaginary parts of E_1/E_0 can be seen to be identical to (4.1.30) and (4.1.31) except for a factor of $1/l^2$. So, the above condition of (4.1.17) equates to the followin restraints

$$(\lambda^2/l^2)\left\langle E_{1r}^2\right\rangle << 1 \qquad (4.1.32)$$

$$(\lambda^2/l^2)\left\langle E_{1i}^2\right\rangle << 1 \qquad (4.1.33)$$

Using the expression in (4.1.13) for the moments of the real and imaginary parts of E_1 gives

$$d << D \qquad D \equiv l/\left\langle n_1^2\right\rangle \qquad (4.1.34)$$

Figure 4.1.1 depicts the fruit of the above efforts in a graphical representation of the proposed regions of validity. The y-axis is the inverse of the second moment of the refractive index. So, larger spreads in the fluctuation of the refractive index are found for smaller values of the y-axis. The x-axis is the depth of the slab, d. The diagram is sketched for $l/\lambda=2$.

Regions I and II denote the regions where the thickness of the slab is greater than the radius of the eddies, l, but

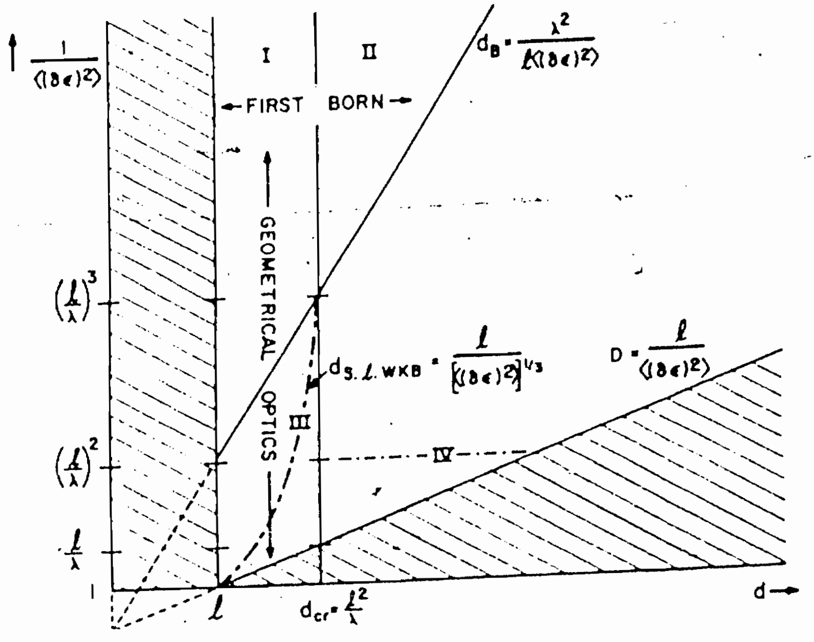


Figure 4.1.1. Parameter space for forward scattering through a slab of width d [12].

less than the condition stated for the Born approximation in (4.1.14), (i.e. $l \ll d \ll d_B$). Region IV, $d_B < d \ll D$, denotes the region where Rytov approximation is to hold. In this region , there seems to be a significant extension to the Born method. The Rytov approximation can handle larger path lengths for the same magnitude of refractive index fluctuations.

Again as mentioned in the introduction of these methods, it can be interpreted that the limiting factor of the approximations seems to be the onset of multiple scattering. Multiple scattering would become a factor when the fluctuations of the refractive index increase (i.e. small y-axis values) or for large path lengths (i.e. larger x-axis values). The interpretation of the regions is still hampered by the strong inequalities. Also, the investigation is limited to singular scale lengths, (size of eddies), and a Gaussian correlation function. In atmospheric and ocean turbulence, the Gaussian correlation function doesn't fully explain detailed characterics of actual scattering phenomena [14]. The conclusions drawn from the above arguments are tainted, due to the limiting simplifications.

4.2 A REVISED GRAPHICAL REPRESENTATION

Some years later in 1975, deWolf presented another graphical representation of parameter regimes similar to the one presented above [15]. A number of interesting items are presented in the paper, e.g., statistical behavior of the electric field in different regimes, and other approximations used outside the realm of the Born and Rytov approximations. However, only the points illustrating qualitative differences between the two methods will be presented, particularly where different applications fit in for the Born and Rytov .

Figure 4.2.1 shows the revised graph of the parameter regimes. The horizontal and vertical axis are somewhat different than those of Figure 4.1.1. The vertical or y-axis, instead of the variance of the dielectric permittivity as in Figure 4.1.1, is now the number of mean free paths of scattering, αL where

$$\alpha = const. \times k_o^2 \times \int_0^\infty dK\ K\ \Phi_\varepsilon(K) \tag{4.2.1}$$

where $\Phi_\varepsilon(K)$ is the spectral density of the correlation function of the permittivity. The phase variance is approximately equal to the number of free paths of scattering, $\langle \phi^2 \rangle \cong \alpha\, L$. Roughly speaking, α can be thought of the mean of the spectral number. So, $\alpha\, L$ would then be the likelihood on average of encountering a particular number of

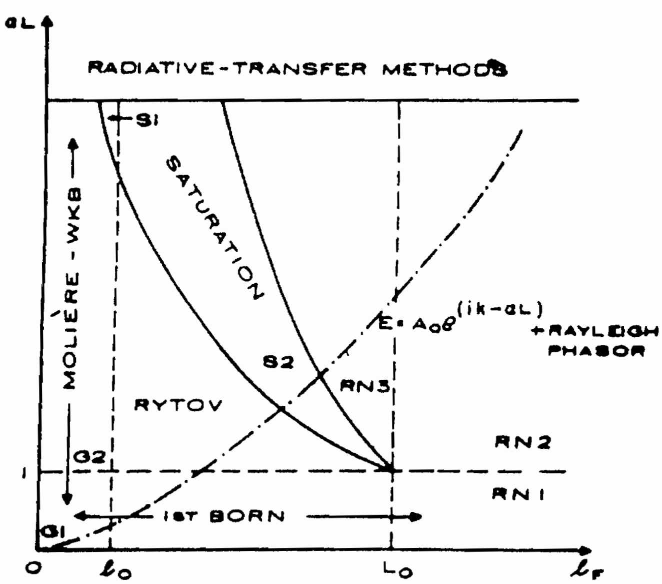


Figure 4.2.1. Parameter regimes with respect to electric field approximations (α^{-1}=mean free path, $l_F = (\lambda L)^{1/2}$) [15].

scattering events. The horizontal dashed line $\alpha L \cong 1$ denotes the distinction between the single scattering and multiple scattering regions. Or in other words, the probability that a wave has arrived at the receiver with only a single scattering event is pretty good. The horizontal or x-axis is $l_F \approx (\lambda L)^{1/2}$, the radius of the first Fresnel zone at the receiver. This radius distinguishes near field (refraction) effects of large scatterers from far-field (diffraction) effects of small scatterers. The two vertical lines give the inner and outer scales of the eddies in the media. A Kolmogorov spectrum is assumed for the spatial spectral density of the turbulence

$$\Phi_n(K) = 0.033 C_n^2 K^{-11/3} \qquad (4.2.2)$$

for $2\pi / L_0 < K < 2\pi / l_0$. $\Phi_n(K)$ is the Fourier transform of the correlation function for the refractive index. Outside these ranges the Kolmogorov spectrum is zero. C_n^2 is known as the structure constant, although not really a constant because it's a function of both time and space. This constant can be thought of as describing the strength of the refractive index turbulence. This is an improvement over the single scale size analysis of the first graphical representation.

Diffraction effects become important with increasing l_F (i.e. increasing path length), whereas multiple scattering effects dominate with increasing number of mean free paths of scattering. Where $\alpha L<1$, before multiple scattering is encountered, the Born approximation is able to handle the weak fluctuations from free space. The Rytov approximation, as illustrated in Figure 4.2.1, is shown to handle larger phase fluctuations than the Born approximation. The upper bound of the Rytov method is encountered when the saturation region is reached. Briefly, the saturation region is when the log-amplitude variance is observed to stop increasing linearly with C_n^2, as predicted by both the Born and Rytov approximations. The quantitative description involves the calculation of the angle of refraction variance $\langle \theta_r^2 \rangle$ [16]. The following discussion will investigate this upper bound on the Rytov approximation from experimental evidence.

Gracheva and Gurvich were the first to observe the saturation effect in 1965 [3]. The normalized variance of intensity, which has the form

$$\sigma_I^2 = \frac{\langle I^2 \rangle - \langle I \rangle^2}{\langle I \rangle^2} \qquad (4.2.3)$$

$$I = |E|^2, \qquad (4.2.4)$$

was measured along with the structure constant. From this quantity, the variance of the log amplitude was calculated from the Rytov approximation

$$\sigma_R^2 = .307 C_n^2 k_o^{7/6} L^{11/6}, \qquad (4.2.5)$$

where L is path length [14]. The square root of the normalized variance of the intensity, standard deviation, was then plotted against (4.2.5), which was calculated from the independent structure constant measurements. The resulting plot is illustrated in Figure 4.2.2. Two path lengths are shown for 250 meters and 1750 meters.

The square root of the normalized variance of the intensity, as predicted from the Rytov approximation, is plotted in Figure 4.2.2 for comparison with the observed values from measurements. The following discussion shows the calculation for the variance of the intensity using the Rytov method. The assumed form of the electric field for the Rytov approximation is

$$E_R = E_0 e^{\chi} e^{i\phi} . \qquad (4.2.6)$$

where $\psi_1 = \chi + i\phi$. The intensity is then

$$I = \left|E_R\right|^2 = e^{2\chi} . \qquad (4.2.7)$$

Also, the Rytov approximation predicts a log normal probability distribution of the intensity. So if $\langle\chi\rangle=0$, the average of (4.2.7) becomes

$$\langle I \rangle = \left\langle e^{2\chi} \right\rangle = e^{2\langle \chi^2 \rangle} . \qquad (4.2.8)$$

With the use of (4.2.8), the variance of the log-amplitude as calculated by the Rytov approximation becomes

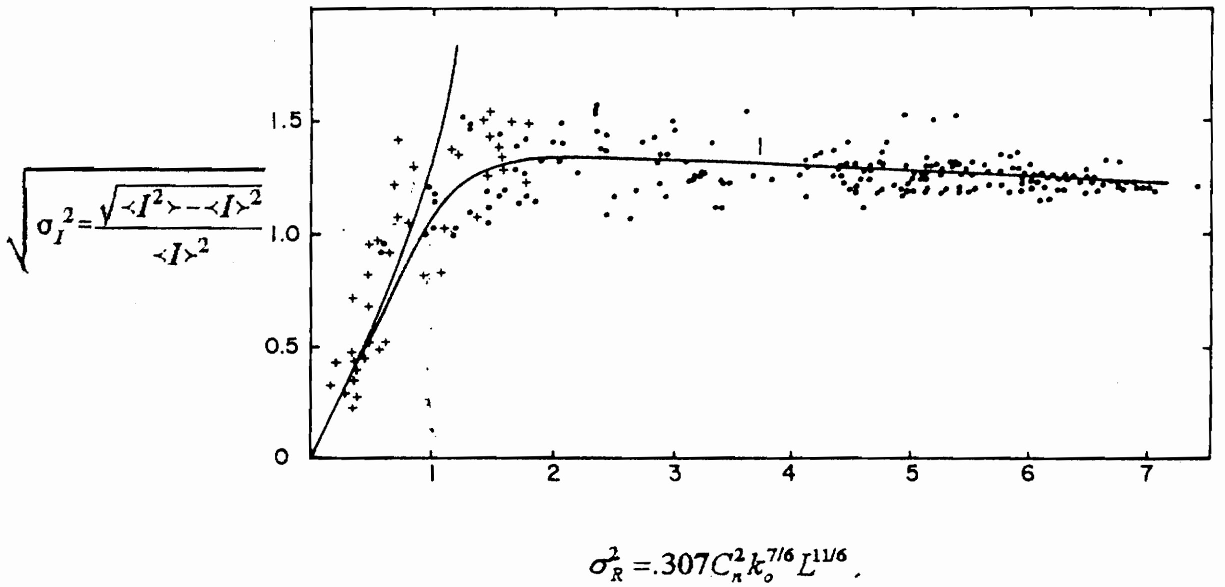


Figure 4.2.2. Normalized variance of intensity vs. $\sigma_R^2 = .307C_n^2 k_o^{7/6} L^{11/6}$ for L=250m(+) and for L=1750(.) [3].

$$\sigma_{I_R}^2 = \frac{\langle I^2 \rangle - \langle I \rangle^2}{\langle I \rangle^2} = \frac{e^{8\langle \chi^2 \rangle} - e^{4\langle \chi^2 \rangle}}{e^{4\langle \chi^2 \rangle}} = e^{4\langle \chi^2 \rangle} - 1 . \qquad (4.2.9)$$

The square root of this expression is labelled and plotted in Figure 4.2.2. For the path length of 250 meters, the Rytov approximation models variance of the intensity pretty well. For 1750 meters, the saturation effect is observed for $.307C_n^2 k_o^{7/6} L^{11/6} > 1$ where the upper limit of the Rytov approximation is then encountered.

From Figure 4.2.2, the conclusion can be drawn that $\sigma_R^2 << 1$ or $\langle |\mathrm{Re}\,\psi_1|^2 \rangle = \langle \chi^2 \rangle \leq 1$ must be satisfied. The condition stated above for the Rytov approximation, $\langle |\lambda\, \nabla \psi_1|^2 \rangle << 1$, is a condition that must hold at each point in the scattering medium (i.e. the angle of forward scattering must be small). However, as the wave propagates, small angles of forward scattering can have a significant cumulative effect, thus creating large-angle scattering. So, $\langle |\mathrm{Re}\,\psi_1|^2 \rangle = \langle \chi^2 \rangle \leq 1$ is generally a more limiting condition.

This new experimental condition for the Rytov approximation can be compared against the above condition for Born approximation, $|E_0|^2 = 1 >> |E_1|^2$. This condition averaged gives

$$\langle |E_1|^2 \rangle << 1 \qquad (4.2.10)$$

Using the above expression relating the terms of the Born and Rytov expansions, $\frac{E_1}{E_0} = \psi_1$ and $|E_0| = 1$, the condition for the Born approximation becomes

$$\left\langle |\psi_1|^2 \right\rangle << 1 \qquad (4.2.11)$$

The real and imaginary part of ψ_1 are denoted by $\psi_1 = \chi + i\phi$ making (4.2.11)

$$\left\langle |\chi|^2 + |\phi|^2 \right\rangle << 1 \qquad (4.2.12)$$

So, for the Born approximation to hold both the imaginary and real parts of the scattered field must be much less than 1. Whereas for the Rytov approximation, only the real part is required to meet that condition. From this analysis, it becomes evident that the Rytov approximation for irradiance can handle large phase fluctuations where the Born cannot.

As mentioned above and illustrated in Figure 4.2.1, for weak fluctuations the Born approximation may be valid where the Rytov approximation would fail. If the fluctuations are weak (i.e. very little energy scattered) but the fractional change of the dielectric permittivity on order of a wavelength is not small, creating a violation of $\left\langle |\lambda \nabla\psi_1|^2 \right\rangle << 1$, the Rytov approximation would fail. But, as the Born approximation only requires that the scattered field be weak, it should still provide a satisfactory approximation. As illustrated in Figure 4.2.1, as the x-axis, $(\lambda L)^{1/2}$, increases due to wavelength the less the likelihood of $\left\langle |\lambda \nabla\psi_1|^2 \right\rangle << 1$ occurring, or the fractional change of the permittivity being on the order of a wavelength being small, and thus limiting the Rytov approximation. Stated another way, the gradient of the amplitude or change in amplitude could then be expected to be of the same order as $k^2 n_1$ (i.e. see (4.1.4)). Consequently, the gradient squared term in the Riccati equation is not negligible. The same argument is used in the parabolic equation by Tatarski to drop the second longitudinal derivative [13].

Figure 4.2.3 illustrates the differences between optical and radio frequency propagation. Optical beams in the lower atmosphere are characterized by small l_F (path length relative to l^2/λ), on the order of centimeters. This limits the available regimes to a vertical band around $l_F \approx l_o$ (links with $l_F << l_o$ are too short to be useful). Optical propagation also is characterized by a large number of mean free paths of scattering. Due to the wavelength of optical frequencies relative to the size of the eddies, edge effects from diffraction are small with respect to refractive effects. Thus, the effects upon amplitude changes are diminished.

Radio waves in the troposphere are governed by path lengths usually less than 100km. The upper scale of tropospheric-layer turbulence is roughly 100m. At frequencies greater than 1GHz, αL can approach and perhaps

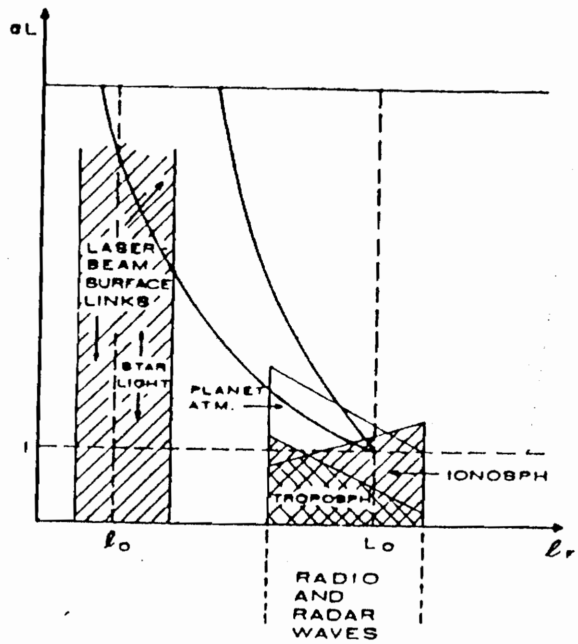


Figure 4.2.3. Propagation regimes for radio and optical regions over parameter regimes of Figure 4.2.1 [15].

exceed unity, but l_f will be less than L_O and $\alpha L<1$. For lower frequencies, where wavelengths are larger, l_F will be around L_O. Thus, there will not be multiple scattering.

5. APPLICATIONS TO PROBLEMS THAT COMPARE THE BORN AND RYTOV APPROXIMATIONS

This chapter discusses three different types of problems that have been used in attempts to gain insight between the differences of the Born and Rytov approximations. Wenzel discusses two real simplistic deterministic problems and one simple random problem. Hadden and Mintzer use the Epstein medium for a second comparison. Their discussion is quite a bit more complicated. Thirdly, Oristaglio considers oblique incidence of a plane wave on a slab. And finally, Lin and Fiddy investigate a deterministic slab.

5.1 WENZEL'S SIMPLE DETERMINISTIC AND RANDOM PROBLEMS

The first work to be investigated on the comparison of the two methods is that of Wenzel. His approach is to examine three simple one dimensional propagation problems that can be solved exactly, and to compare in each case the Born and Rytov approximations with the exact solution. Each of the three problems considered here is based on some version of the reduced wave equation in one dimension. The first two problems involve a deterministic medium, while the third involves a random medium.

The comparison of the two methods is based on their respective regions of validity. Wenzel defines validity in terms of the magnitude of the relative error. As developed in Keller's monograph, the approximation is considered valid only if $\frac{|u-\hat{u}|}{u}$ is much less than some maximum tolerable error. The range of validity of the approximation is then just the region of physical space for which the approximation is valid.

The first problem considered is a simple deterministic application defined by the wave equation

$$u''(x)+k_o^2(1+\varepsilon)^2 u=0 \qquad (5.1.1)$$

for $0 \le x < \infty$. The primes denote differentiation with respect to the variable x, k_o is a positive constant, and ε is a small constant parameter. The boundary conditions the solution is to satisfy are $u(0)=1$ and the outgoing radiation condition at infinity.

By use of the characteristic equation of (5.1.1), the exact solution can be written as

$$u = e^{ik_o(1+\varepsilon)x} . \qquad (5.1.2)$$

The n'th Born approximation is obtained by expanding the right hand side of (5.1.2) in powers of ε yielding

$$u_B^{(n)} = \left[1 + i\varepsilon k_o x - \frac{1}{2}(\varepsilon k_o x)^2 + \cdots\right] e^{ik_o x} . \qquad (5.1.3)$$

It is clear from the form of the right-hand side of (5.1.1) that the Rytov approximation of any order must agree with the exact expression for u.

For any order of the Born method, the approximation blows up for large x. Thus, the Born approximation has a finite range of validity. In contrast, any order of the Rytov approximation has an infinite range of validity, since it is identical to the exact solution.

The second deterministic problem involves the following equation

$$u''(x) + k_o^2\left[1 + \frac{2\varepsilon}{(1 + i\varepsilon k_o x)}\right] u(x) = 0 . \qquad (5.1.4)$$

for $0 \le x < \infty$. The first boundary condition is u(0)=1. Because the medium is dissipative, the second boundary condition is u(0)→0 as $x \to \infty$, which is the outgoing radiation condition at infinity.

By direct differentiation, the exact solution of (5.1.4) is

$$u(x) = (1 + \varepsilon k_o x)e^{ik_o x}. \tag{5.1.5}$$

This time the Born approximation matches the exact solution of (5.1.15). The n'th Rytov approximation is obtained by the following

$$u(x) = e^{ik_o x}e^{\psi} = e^{ik_o x}e^{\ln(1+\varepsilon k_o x)}. \tag{5.1.6}$$

Expanding $\ln(1 + i\varepsilon k_o x)$ by a power series yields

$$\ln(1 + i\varepsilon k_o x) = i\varepsilon k_o x + \frac{1}{2}(\varepsilon k_o x)^2 \cdots - (-i\varepsilon k_o x)^n / n. \tag{5.1.7}$$

Inserting (5.1.7) into (5.1.7), produces the following expression for the n'th Rytov approximation

$$u_R^{(n)} = e^{i\varepsilon k_o x + \frac{1}{2}(\varepsilon k_o x)^2 \cdots - (-i\varepsilon k_o x)^n / n} e^{ik_o x}. \tag{5.1.8}$$

Because any order of the Born approximation is the same as the exact solution, it has an infinite range of validity. The Rytov approximation, in this case, is the one with a finite range of validity. The relative error is then

$$\frac{\left|u - u_R^{(n)}\right|}{|u|} = \left|1 - (1 + i\varepsilon k_o x)^{-1} e^{i\varepsilon k_o x + \frac{1}{2}(\varepsilon k_o x)^2 \cdots - (-i\varepsilon k_o x)^n / n}\right|. \tag{5.1.9}$$

There are two possibilities for the exponential term in (5.1.9). For the first-order approximation, the relative error is bounded for all x and tends to 1 as x goes to infinity. For all other orders of $u_R^{(n)}$, the real elements in the exponent cause the relative error to blow up, as x goes to infinity. Thus, in this case the first Born approximation has a greater range of validity than any order of the Rytov approximation.

The last problem considers propagation in a random medium. Analysis begins with the wave equation of the following form:

$$u''(x) + k_o^2\left[1 + 2\varepsilon\mu(x)\right]u(x) = 0. \qquad (5.1.10)$$

This equation is then simplified by assuming a solution of the form $u(x) = v(x)e^{ik_o x}$ and dropping the second derivative term. The justification for dropping the second derivative is explained by Tatarskii in [8]. The one dimensional parabolic equation results for υ and is given by

$$v'(x) = i\varepsilon\, k_o\mu(x)v(x). \qquad (5.1.11)$$

for $0 \le x < \infty$. The randomness is contained in $\mu(x)$, which is assumed to be a homogeneous Gaussian random function with zero mean and unit variance. The boundary conditions are the same as the first problem. The goal is to calculate the mean field.

By direct integration, the solution to (5.1.11) is

$$v(x) = e^{i\varepsilon k_o \int_0^x \mu(x')dx'}. \qquad (5.1.12)$$

To find $\langle \nu(x) \rangle$, the ensemble average of (5.1.12) is taken, while making use of the Gaussian property of $\mu(x)$. Thus, the following is obtained

$$\langle \upsilon(x) \rangle = \exp\left\{ -\tfrac{1}{2}\varepsilon^2 k_o^2 \left\langle \left(\int_0 \mu(x')dx' \right)^2 \right\rangle \right\}. \quad (5.1.13)$$

A correlation function is introduced for $\mu(x)$, defined by

$$\rho(\xi) = \langle \mu(x)\mu(x+\xi) \rangle. \quad (5.1.14)$$

Now, the term in the exponent of (5.1.13) can be written as

$$\left\langle \left[\int_0^x \mu(x')dx' \right]^2 \right\rangle = \int_0^x \int_0^x \rho(x''-x')dx''dx' = 2\int_0^x (x-\xi)\rho(\xi)d\xi. \quad (5.1.15)$$

The last term in (5.1.15) is obtained by a transformation to integration variables $\xi = x'' - x'$, and $\eta = x'' + x'$. Equation (5.1.13) then gives with the substitution of (5.1.15)

$$\langle \nu(x) \rangle = e^{-\varepsilon^2 k_o^2 \int_0^x (x-\xi)\rho(\xi)d\xi}. \quad (5.1.16)$$

The Born approximation for $\langle \nu(x) \rangle$ from (5.1.12) is

$$\nu(x) = 1 + i\varepsilon k_o \int_0^x \mu(x')dx' - \tfrac{1}{2}k_o^2\varepsilon^2 \left[\int_0^x \mu(x')dx'\right]^2 + \cdots. \quad (5.1.17)$$

To facilitate a comparison between the two methods, the author takes the second Born approximation for quantitative illustration. Thus, taking the ensemble average of (5.1.17) and making use of the fact that $\langle \mu(x) \rangle = 0$ and (5.1.15) gives

$$\langle \nu_B^{(2)} \rangle = 1 - \varepsilon^2 k_o^2 \int_0^x (x-\xi)\rho(\xi)d\xi. \quad (5.1.18)$$

The Rytov approximation matches the exact expression for υ. Thus, the Rytov approximation for $\langle v(x)\rangle$ must also agree with the exact solution of the parabolic equation.

In order to facilitate the error analysis of the Born approximation, the discussion is limited to a single scale length of the random inhomogeneities of the medium. The scale length is defined by the following

$$l = \int_0^\infty \rho(\xi)d\xi \,. \tag{5.1.19}$$

Then, the assumption is made that x>>l. This assumption allows the following

$$\int_0^\infty \rho(\xi)d\xi \approx \int_0^x \rho(\xi)d\xi = l \tag{5.1.20}$$

$$\int_0^\infty \xi\rho(\xi)d\xi \approx \int_0^x \xi\rho(\xi)d\xi = al^2 \tag{5.1.21}$$

where a is a dimensionless constant. Upon combining (5.1.18), (5.1.20) and (5.1.21), the following is obtained:

$$\langle v(x)\rangle = e^{-\varepsilon^2 k_o^2 l(x-al)} \,. \tag{5.1.22}$$

The exponent is negative when x>l. The author attributes the decay of $\langle v(x)\rangle$ to random phase mixing.

The Born approximation yields, using (5.1.18), (5.1.20) and (5.1.21)

$$\langle v_B^{(2)}(x)\rangle = 1 - \varepsilon^2 k_o^2 l(x-al) \,. \tag{5.1.23}$$

From (5.1.22) and (5.1.23), the relative error can be found as

$$\frac{\left|\langle v\rangle - \langle v\rangle_B^{(2)}\right|}{\left|\langle v\rangle\right|} = 1-\left(1-\varepsilon^2 k_o^2 l(x-al)\right)e^{\varepsilon^2 k_o^2 l(x-al)} . \qquad (5.1.24)$$

Thus from (5.1.24), the second Born approximation is valid only if $\varepsilon^2 k_o^2 l(x-al) << 1$, which leads to $\varepsilon^2 k_o^2 lx << 1$, since $x >> 1$.

Since the Rytov approximation for the mean field is identical to the exact result of the parabolic equation, it possesses an infinite range of validity. The range of validity of the Born approximation for this case is $\varepsilon^2 k_o^2 lx << 1$. Thus, the first order Rytov approximation to the mean field for this case has a greater range of validity than does the second order Born approximation.

Wenzel goes on to point out that the range of validity differs considerably depending on the statistical quantity investigated. For example, the mean intensity from (5.1.18) for the second Born approximation is

$$\left|v_B^{(2)}\right|^2 = 1 + O(\varepsilon^4) \qquad (5.1.25)$$

$$\left\langle \left|v_B^{(2)}\right|^2 \right\rangle = 1 + O(\varepsilon^4) . \qquad (5.1.26)$$

From (5.1.18), the mean intensity of the exact field is

$$\left|v(x)\right|^2 = 1 \qquad (5.1.27)$$

$$\left\langle \left|v(x)\right|^2 \right\rangle = 1 . \qquad (5.1.28)$$

In comparing (5.1.26) and (5.1.28), the second Born approximation matches the exact result up to terms of order ε^4. Because the mean intensity is independent of x, the range of validity is infinite. In contrast, when x>>1 the second Born approximation is restricted by $\varepsilon^2 k_o^2 lx << 1$. The same variability is exhibited in the ranges of validity for the Rytov approximation.

The significance of the two deterministic problems lies in that problems can be thought of where the regions of validity differ. The random problem, besides exhibiting that the Rytov had a better range of validity, showed that care must be taken in comparing quantities because the regions of validity can vary.

5.2 THE EPSTEIN PROBLEM

In 1978 Hadden and Mintzer, published a paper in which the Born and Rytov approximations were compared by considering media with refractive-index profiles described by a model developed by Epstein [18]. The motivation was an attempt to find a test problem in which an exact solution exists and with nontrivial implications for modeling physical situations. The Epstein medium, which has been used to model a number of physical media, was selected in pursuit of such an end.

The refractive index of an Epstein medium is a function of only one space coordinate. In the most general case the index of refraction is described by

$$n^2(z) = n_1^2 + e^{kz/s}\left\{\left(n_2^2 - n_1^2\right)\left[e^{kz/s} + 1\right] + n_3^2\right\} \times \left[e^{kz/s} + 1\right]^{-2} \qquad (5.2.1)$$

where n_1, n_2, n_3, and s are adjustable parameters. Because of the flexibility and because an exact solution exists when inserted into the wave equation, the Epstein medium is widely used for testing approximation methods.

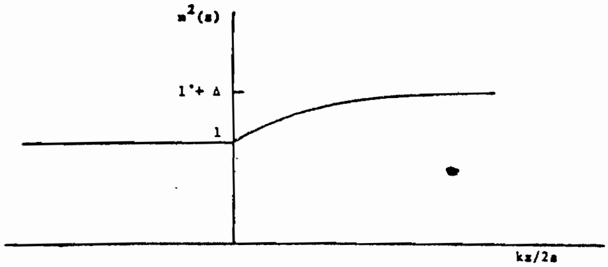
Three problems are considered by Hadden and Mintzer. The first two are half space problems with different parameters and the last a whole space problem. The first case (5.2.1) takes the following form

$$n^2(z) = 1 + \Delta \tanh(kz/s) \qquad (5.2.2)$$

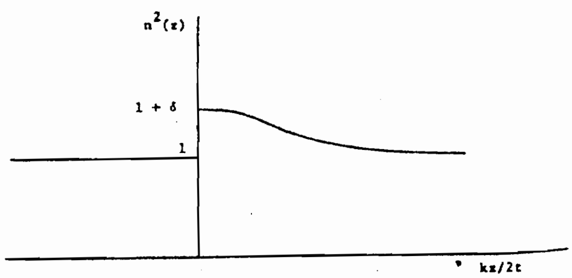
where $n_1^2=(1-\Delta)$, $n_2^2=(1+\Delta)$ and $n_3=0$; thus Δ is a measure of the strength of the refractive index change and s is the transition depth of the layer. Figure 5.2.1(a) illustrates a half space filled with such a medium for $z>0$. The second case, in which $n_1=n_2=1$, $n_3^2=4\delta$, leads to the symmetric refractive index

$$n^2(z) = 1 + \delta\left[\cosh(kz/2t)\right]^{-2} \qquad (5.2.3)$$

where δ and t are analagous to s and Δ in the first medium. Figure 5.2.1(b) illustrates a half space of $z>0$ filled with



(a)



(b)

Figure 5.2.1 (a) Refractive index profile for (5.2.2). (b) Refractive index profile for (5.2.3) [18].

a medium given by (5.2.3). The third case uses (5.2.3) to describe the entire medium (i. e. for all z).

Only the solution to the first half space problem will be presented as the same approach is used for all three test cases. The above forms were chosen because only two parameters are involved for $n^2(z)$: the strength of the scattering region Δ and δ, and the thickness of the region s and t. Because of the complex form of the solutions used by Hadden and Mintzer, a comparison of the Born and Rytov approximations with the exact solution in the form of relative error, which Keller and Wenzel used, was not accomplished. Their analysis was limited to a comparison of a series expansion of the exact solution with the Born and Rytov approximations.

The first test problem is a plane wave propagating from a half space z<0, for which n=1, into a half space (z>0) in which n(z) is given by (5.2.2). Equation (5.2.2) is used in the one dimensionl wave equation

$$\frac{d^2}{dz^2}p(z)+k_o^2n^2(z)p(z)=0 \qquad (5.2.4)$$

where p(z) is pressure. Applying the boundary conditions

$$p_{z<0}(z=0)=p_{z>0}(z=0) \qquad (5.2.5)$$

$$\frac{dp_{z<0}(z=0)}{dz}=\frac{dp_{z>0}(z=0)}{dz} \qquad (5.2.6)$$

across the interface, the exact solution takes the following form

$$P(z<0) = e^{ikz} + R_E e^{ikz} \tag{5.2.7}$$

and

$$P(z>0) = T_E e^{ikz} F\left[is(\mu-\tau); -is(\mu+\tau); 1-2is\tau; -e^{-kz/s}\right] \tag{5.2.8}$$

where

$$\tau \equiv (1+\Delta)^{1/2}, \quad \mu \equiv (1-\Delta)^{1/2} \tag{5.2.9}$$

and the function F in (5.2.8) is the hypergeometric function. Using α, β, γ for the arguments in (5.2.8), the transmission and reflection coefficients are

$$T_E = 2\left[(1+\tau)F(\alpha,\beta,\gamma,-1) + \frac{i}{s}\frac{\partial}{\partial\mu}F(\alpha,\beta,\gamma,-u)\Big|_{u=1}\right]^{-1} \tag{5.2.10}$$

$$R_E = T_E\left[(1-\tau)F(\alpha,\beta,\gamma,-1) - \frac{i}{s}\frac{\partial}{\partial\mu}F(\alpha,\beta,\gamma,-u)\Big|_{u=1}\right]. \tag{5.2.11}$$

The series representation for the hypergeometric function used is

$$F(\alpha,\beta,\gamma,-u) = 1 + \sum_{n=1}^{\infty}\frac{(-1)^n u^n}{n!}\prod_{m=0}^{n-1}\frac{(m+\alpha)(m+\beta)}{m+\gamma}. \tag{5.2.12}$$

First the Born and Rytov approximations to the solution of (5.2.4) will be presented. Then, a comparison of the two techniques with the exact solution will be discussed.

The Born approximation to any order may be represented as

$$P_B(z) = e^{ikz}\left[1 + \sum_{n=1}^{N} \varepsilon^n p_n(z)\right] \tag{5.2.13}$$

where ε is Δ for this problem (the strength of the scatterers).

Putting this expression into (5.2.4), the following is obtained

$$\frac{d^2 p_n(z)}{dz^2} + 2ik\frac{dp_n(z)}{dz} + k^2 \tanh\left(\frac{kz}{2s}\right)p_{n-1}(z>0) = 0 \,. \tag{5.2.14}$$

The solution to the above equation, utilizing the one dimensional Green's function, is

$$p_n(z) = \left(\frac{1}{2ik}\right)e^{-ikz}\int_0^{\infty} dz' e^{ik|z-z'|}e^{ik(z'-z)} \tanh\left(\frac{kz'}{2s}\right)p_{n-1}(z') \,. \tag{5.2.15}$$

The integrations in (5.2.15) can be found in Gradshteyn and Ryzhik.

Using the notation

$$v(n) = (n-2is)^{-1} \,, \tag{5.2.16}$$

the first order solution is

$$p_1(z<0) = e^{-2ikz}\left[-\tfrac{1}{4} + is\sum_{n=1}^{\infty}(-1)^n v(n)\right] \tag{5.2.17}$$

$$p_1(z>0) = \tfrac{1}{2}ik - \tfrac{1}{4} - is\ln 2 - 2s^2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}v(n)e^{-nkz/s} \,. \tag{5.2.18}$$

The second order solution is

$$p_2(z<0) = e^{-2ikz}\left\{\begin{array}{l}\frac{1}{16}-\left(\frac{1}{4}+is\ln 2\right)p_1(z<0) \\ +\frac{1}{2}s^2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\nu(n)\left[\frac{\nu(n)}{\nu(-n)}-4is\sum_{m=1}^{\infty}(-1)^m\nu(n+m)\right]\end{array}\right\}. \quad (5.2.19)$$

$$p_2(z\geq 0) = \frac{1}{16}-\frac{1}{8}(kz)^2-\frac{1}{8}ikz-\left(\frac{1}{4}+is\ln 2\right)p_1(z>0)-$$
$$is^2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\nu(n)e^{-nkz/s}\left[kz+2s\nu(n)\right]- \quad (5.2.20)$$
$$\frac{1}{2}s^2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\nu(n)\left[1+4is\sum_{m=1}^{\infty}\frac{(-1)^m}{(n+m)}\right]+$$
$$4s^4\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{(-1)^{n+m}}{n(n+m)}\nu(n)\,\nu(n+m)e^{-(n+m)kz/s}$$

With the above expressions available, the second-order Born approximation can be constructed.

The assumed form for the solution of (5.2.4) for the N'th Rytov approximation is

$$p_n(z) = \exp\left(ikz+\sum_{n=1}^{N}\varepsilon^n\psi_n(z)\right) \quad (5.2.21)$$

Putting this equation into (5.2.4) yields

$$\frac{d^2\psi_n}{dz^2}+2ikz\frac{d\psi_n}{dz}+\sum_{j=1}^{n-1}\frac{d\psi_n}{dz}\frac{d\psi_{n-j}}{dz}+\delta_{n,1}H(z)\tanh\left(\frac{kz}{2s}\right)=0 \quad (5.2.22)$$

where $\delta_{n,1}$ is the Kronecker delta and H(z) is a unit step function. The integral equation for (5.2.22) is

$$\psi_n(z) = -(2ik)^{-1} e^{ikz} \int_{-\infty}^{\infty} dz' e^{ik|z-z'|} e^{ikz'} \left[\delta_{n,1} H(z') \tanh\frac{kz'}{2s} + \sum_{j=1}^{n-1} \frac{d\psi_n}{dz}\frac{d\psi_{n-j}}{dz} \right]. \quad (5.2.23)$$

From (3.2.5) in chapter 2, the relation

$$\frac{u_1}{u_o} = p_1 = \psi_1 \quad (3.2.25)$$

was established by Sancer and Varvatis [10]. Since Hadden and Mintzer are only comparing the second-order approximations, ψ_2 is the highest-order quantity needed for the second order Rytov approximation. Using (5.2.17) and (5.2.18) to compute $(d\psi_1/dz)^2$ for use in (3.2.25) yields

$$\psi_2(z<0) = e^{-2ikz} \{ \tfrac{1}{16} + \left(1+\tfrac{1}{2}e^{-2ik}\right)\left[-\tfrac{1}{4} + is\sum_{n=1}^{\infty}(-1)^n \nu(n)\right]^2 - s^2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \nu(n)\left[1 - 2is\nu(n)\sum_{m=1}^{\infty}(-1)^m \nu(m)\nu(n+m)\right] \quad (5.2.24)$$

$$\psi_2(z>0) = \tfrac{3}{32} - \frac{1}{8}ikz - \frac{1}{4}is \times \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \nu(n)\{ \nu^{-1}(n-2is) + 8s^2 \nu(n) e^{-nkz/s} - 2isn\sum_{m=1}^{\infty}\frac{(-1)^m}{(n+m)}[\nu^{-1}(n+m-2is) + 8s^2 \nu(n+m) e^{-(m+m)kz/s}]\}. \quad (5.2.25)$$

The second-order Rytov approximation can now be constructed by use of (5.2.21). With the exact solution available along with both approximations, a comparison can now be investigated.

Even for this simple form of the Epstein medium, the exact solution is sufficiently complex to hinder direct comparison by the ideal use of relative error or similarly a plot for values Δ and s of the amplitude and phase versus kz. So, the exact solution is expanded in a power series of the strength parameter Δ and thickness s. As Δ defines the nature of the perturbations, the approximations are already expanded in Δ. However because Δ is in the exponent in the Rytov method and in τ in the exact solution, these expressions are also expanded as power series in Δ. By considering limiting values (small and large) of the thickness parameter s, a comparison of the exact and approximate solutions can finally be made for varying strength parameters Δ. This approach will enable regions of validity to be obtained.

The exact solution expanded in a power series of s to the second order gives

$$p_E(z<0) = e^{ikz} + \frac{(1-\tau)}{(1+\tau)} e^{-ikz}[1 - \frac{i4s\Delta \ln 2}{(1-\tau^2)} + \frac{2}{3}\frac{s^2\Delta\tau\pi^2}{(1-\tau^2)} - \frac{8s^2\Delta^2(\ln 2)^2}{(1+\tau)(1-\tau^2)} + O(s^3\Delta)] \quad (5.2.26)$$

$$p_E(z>0) = \frac{2}{(1+\tau)} e^{ikz}[1 - \frac{2is\Delta \ln 2}{(1+\tau)} - \frac{1}{6}\frac{s^2\Delta(1-\tau)}{(1+\tau)}\pi^2 - 2s^2\Delta\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}e^{-nkz/s} - 4\frac{s^2\Delta^2(\ln 2)^2}{(1+\tau)^2} + O(s^3\Delta)] \quad (5.2.27)$$

Similarly, expanding the Born approximation to the second order s gives

$$p_{B2}(z<0)=e^{ikz}+e^{-ikz}\{-\Delta[\frac{1}{4}+is\ln 2-\frac{1}{6}s^2\pi^2]+ \Delta^2[\frac{1}{8}+\frac{1}{2}is\ln 2-s^2(\ln 2)^2]+O(s^3\Delta)\} \quad (5.2.28)$$

$$p_{B2}(z>0)=e^{ikz}\left(1+\frac{1}{2}ikz\Delta[1-\frac{1}{2}\Delta]\right)-\frac{1}{8}(kz\Delta)^2- \Delta[\frac{1}{4}+is\ln 2+2s^2\sum_{n=1}^{\infty}(-1)^n n^{-2}e^{-nkz/s}]+ \quad (5.2.29)$$

$$\Delta^2\{\frac{1}{8}+\frac{1}{2}is\ln 2(1-ikz)-iskz\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}e^{-nkz/s}- s^2(\ln 2)^2+\frac{1}{2}s^2\sum_{n=1}^{\infty}\frac{(-n)^n}{n^2}[e^{-nkz/s}-1]\}+O(s^3\Delta)$$

Lastly, the expansion in terms of s for the Rytov approximation gives

$$p_{R2}(z<0)=\exp\{ikz-e^{-2ikz}\{\Delta[\frac{1}{4}+is\ln 2-\frac{1}{6}s^2\pi^2]+ \Delta^2[\frac{1}{8}+\frac{1}{2}is\ln 2-s^2(\ln 2)^2]\}-\frac{1}{2}\Delta^2e^{4ikz}[\frac{1}{16}+\frac{1}{2}is\ln 2- s^2(\ln 2)^2]+O(s^3\Delta)\} \quad (5.2.30)$$

$$p_{R2}(z<0)=\exp\{ikz[1+\frac{1}{2}\Delta-\frac{1}{8}\Delta^2]- \Delta[\frac{1}{4}+is\ln 2-2s^2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}e^{-nkz/s}]+ \Delta^2[\frac{3}{32}+\frac{1}{4}is\ln 2-\frac{1}{2}s^2(\ln 2)^2+\frac{1}{24}s^2\pi^2]+O(s^3\Delta) \quad (5.2.31)$$

Here, as mentioned above, when the exact solution and the Rytov approximation are expanded as power series in Δ, the Born and Rytov approximations are found to be in error by order Δ^3 for all z. For positive z, the Rytov approximation has an error of order $kz\Delta^3$, while the Born approximation acquires an error of order $(kz\Delta)^3$. This is similar to the findings of Keller, which led him to the conclusion that the Rytov approximation has a larger range of validity.

For negative z, Hadden and Mintzer compared the reflection coefficients, which were obtained from field expressions in $z<0$ by

$$R \equiv e^{ikz}\left[p(z<0) - e^{ikz}\right] \tag{5.2.32}$$

Again expanding the result for the exact and Rytov approximations in a power series of Δ, an error of the order of Δ^3 is found for both the Born and Rytov approximations. For the Rytov approximation, additional error terms of the form $\Delta^3\exp(-2ikz)$ and $\Delta^3\exp(-4ikz)$ are encountered. No additional error terms are found in the Born approximation. Hadden and Mintzer state that since the wave like terms in the Rytov expression have unit magnitude, the two methods yield equivalent expressions for the reflection coefficient.

Thus for $s<<1$ the Born approximation is valid for $-\infty<z<z_B$ with $(kz_B\Delta)^3<<1$, while the Rytov approximation is acceptable for $-\infty<z<z_R$ where $kz_R\Delta^3<<1$. In the other limiting case for s very large, but $s\Delta<<1$ (this must hold to insure proper behavior of the exact solution), the perturbation terms in the Born and Rytov approximations vanish ((5.2.17), (5.2.18), (5.2.19), (5.2.20), (5.2.24) and (5.2.25)). This result may also be found from the exact solution (5.2.7) and (5.2.8).

For the second half space problem, Hadden and Mintzer merely presented the complex fields for the exact solution and approximations. Following the same approach as above, both limiting cases for the thickness of the slab didn't yield any distinguishing features between each approximation and the exact solution. For the refractive index in the second case describing a whole space, again no preferential ranges for either method resulted.

Hadden and Mintzer state that the above results may also be inferred to hold in two and three dimensions. Their argument lies on the assumption that the medium may be modeled by one of the classifications covered by the Epstein model. This is legitimate for the media that may be modeled by the Epstein model, in which the index of refraction varies in only one space coordinate. However, no insight is gained into the general behavior of the approximations in two and three dimensions for the direct and the inverse problem no insight is gained. In many random medium applications, the change in the longitudinal direction is assumed small relative to a wavelength and a Fourier transform taken of the refractive index in the transverse coordinates. Also, greater analytical difficulty is pointed out for the second and higher order Rytov approximation due to the need to take derivatives in the integrand of the integral representation. It is then pointed out that this difficulty can be circumvented by utilizing the relationships between the Born and Rytov approximations mentioned earlier. These points are not of much significance since in most applications of each method only the first order approximations are used, due to the difficulty pointed out earlier of calculating higher order terms.

5.3 REFLECTION AND REFRACTION FROM A PLANE INTERFACE

The following work done by Oristaglio is a much simpler and insightful application for the comparison of the two approximations. A plane wave, whose polarization is perpendicular to the plane of incidence TE,

$$u_I = e^{ik\sin\theta_1 x + ik\cos\theta_1 z} \qquad (5.3.1)$$

is incident upon an interface at z=0 between two homogeneous half-spaces. Figure (5.3.1) illustrates the problem with

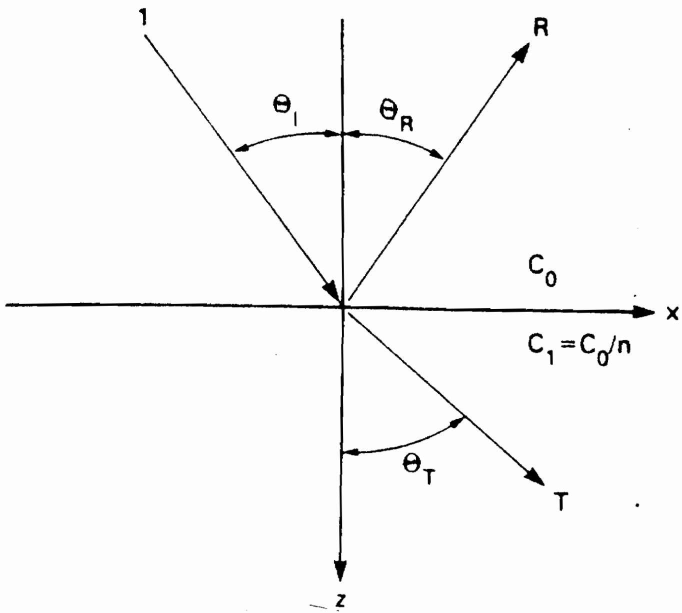


Figure 5.3.1. Geometry of the interface problem [19].

free space occupying z<0 and a non-unity refractive index occupying z>0. The exact solution to this problem can be found by simply applying the boundary conditions that u and du/dz must be continuous across the interface. The reflected and transmitted waves are

$$u_R(x,z) = R e^{ik \sin\theta_R x - ik \cos\theta_R z} \tag{5.3.2}$$

$$u_T(x,z) = T e^{ikn \sin\theta_T x - ikn \cos\theta_T z} \, . \tag{5.3.3}$$

The reflection and transmission coefficients for TE polarization are given by

$$R = \frac{\cos\theta_I - n\cos\theta_T}{\cos\theta_T + n\cos\theta_T} \tag{5.3.4}$$

$$T = 1 + R = \frac{2\cos\theta_I}{\cos\theta_I + n\cos\theta_T} \, . \tag{5.3.5}$$

The transmitted angle can be found from Snell's law

$$n \sin\theta_T = \sin\theta_I \tag{5.3.6}$$

and $\theta_R = \theta_I$.

With the exact solution available, the two first order approximations are now sought for comparison. The calculation for obtaining the expressions for the approximations is straightforward and given in Appendix A of the paper. Similar integrals are present in the work of Lin and Fiddy yet to be discussed, the integrations are abbreviated in both works, but shown more completely in the following review of their work. The result for reflected and transmitted fields from the first-order Born approximation is

$$u_R^{(B)} = -\frac{\alpha}{4} e^{ik\sin\theta_I + ik\cos\theta_I} \tag{5.3.7}$$

$$u_T^{(B)}(x,z) = \left(1 - \frac{\alpha}{4}(1 - 2ik\cos\theta_I z)\right) e^{ik\sin x\theta + ik\cos\theta z} \tag{5.3.8}$$

where

$$\alpha \equiv \frac{n^2 - 1}{\cos^2\theta_I}. \tag{5.3.9}$$

The total field in the upper half space and the transmitted field found from the Rytov approximation are

$$u_R^{(R)} = \exp\left(-\frac{\alpha}{4} e^{-2ik\cos\theta_I z}\right) e^{ik\sin\theta_I x + ik\cos\theta_I z} \tag{5.3.10}$$

$$u_T^{(R)} = e^{-\alpha/4} e^{ik\sin\theta_I x + ik(1-\alpha/2)\cos\theta_I z}. \tag{5.3.11}$$

To aid in the analysis of the reflected fields an alternate form of Snells law

$$n\cos\theta_T = \left(1 + \frac{n^2 + 1}{\cos^2\theta_I}\right)^{1/2} \cos\theta_I\,, \tag{5.3.12}$$

which has the following expansion

$$n\cos\theta_T = \cos\theta_I + \frac{1}{2}\frac{n^2-1}{\cos\theta_I} + \cdots, \tag{5.3.13}$$

allows the reflected field to written as

$$u_R(x,z) = \frac{1-(1+\alpha)^{1/2}}{1+(1+\alpha)^{1/2}} e^{ik\sin\theta_R x - ik\cos\theta_R z} \quad . \tag{5.3.14}$$

Comparison of (5.3.7) with (5.3.14) shows that the Born approximation gives a reflected wave at the correct angle, but wrong coefficient. Expanding the reflection coefficient in (5.3.14) by a Taylor series of α, the Born approximation is found to be just the first term in the expansion.

The Rytov approximation gives an expression for the total field on each side of the interface. To compare the Rytov approximation with the exact field, the total field for the exact solution is written as

$$u(x,z) = \left(1 + \frac{1-(1+\alpha)^{1/2}}{1+(1+\alpha)^{1/2}} e^{-2ik\cos\theta_I z}\right) e^{ik\sin\theta_R x - ik\cos\theta_R z} . \tag{5.3.15}$$

The second factor in (5.1.10), the Rytov approximation, and (5.1.15) are the same and is simply the incident field. In comparing the first factor, the Rytov approximation has replaced the exact solutions, which is a function of $(1+\alpha)^{1/2}$, with an exponential term whose argument is linear with respect to α.

The Rytov approximation for the total field is not a simple sum of an incident plane wave and a reflected plane wave. An expansion of the first exponential term in the Rytov approximation gives

$$u^{(R)} = \left(1 - \frac{\alpha}{4} e^{-2ik\cos\theta_I z} + \frac{\alpha^2}{32} e^{-4ik\cos\theta_I z} + \cdots\right) u_I \, . \qquad (5.3.16)$$

The Rytov approximation consists of an incident wave and an infinite series of reflected plane waves traveling at different angles away from the interface. The first reflected plane wave in this series is the Born approximation to the true reflected wave.

Finally, the Born and the Rytov approximations are independent of z for their validity. The error is dependent solely upon α and of the order α^2. Oristaglio concludes that the Born approximation, since it gives directly a single reflected plane wave, is a better approximation. These roles are reversed for the transmitted field.

By using both forms of Snell's law, the θ_T dependence in the exact expression for the transmitted field can removed to give

$$u_T = \frac{2}{1+(1+\alpha)^{1/2}} e^{ik\sin\theta_I x + ik(1+\alpha)^{1/2}\cos\theta_I z} \, . \qquad (5.3.17)$$

The Born approximation comes from three different series expansions of the exact expression. Firstly, the square root in Snells law is expanded (5.3.13). The transmission coefficient and exponential in (5.3.17) are expanded. Then if everything is truncated after the first linear term, the result is the Born approximation. The expansion of the exponential term produces a term in the coefficient of the Born approximation that depends on z. Because of this term, the approximation diverges with increasing z, no matter how small the perturbation is.

The expression for the field in the lower half space from the Rytov approximation is given in (5.3.11). The first exponential in the Rytov approximation agrees up to the first term of the corresponding expansion of the transmission coefficient. And, the argument in the plane wave term of (5.3.11), $(1+\alpha/2)$, is the first order approximation to the expansion of the $(1+\alpha)^{1/2}$ in the exponential of the exact solution.

Although the Rytov approximation gives a plane wave in the lower half space, the wave travels at a different angle from θ_T and the magnitude of the wave vector is incorrect. After some algebraic manipulation, the magnitude of the wave vector for the Rytov approximation can be expressed as

$$k\left(n^2+\alpha^2\cos\theta_I/4\right)^{1/2}. \qquad (5.3.18)$$

In effect, the Rytov approximation takes the lower half-space to have the modified index of refraction

$$n^{(R)} = \left(n^2 + \alpha^2 \cos\theta_I / 4\right)^{1/2} . \qquad (5.3.19)$$

and transmits a plane wave at the appropriate angle to satisfy Snell's law with n replaced by $n^{(R)}$. This is illustrated in Figure (2)

Because of the inverse problem, the phase of the Rytov approximation is also of interest. The difference between the exact phase and the Rytov approximation's phase is

$$k(1+\alpha)^{1/2} \cos\theta_I z - k(1+\alpha/2)\cos\theta_I z = -k\frac{\alpha^2}{8}\cos\theta_I z + \cdots . \qquad (5.3.20)$$

This difference grows unbounded as z increases. However, the relative phase error is bounded and is $O(\alpha^2)$.

The last part of the paper presents some numerical examples in order to illustrate the accuracy of both approximations in their preferred domain. The conclusion made was that the Rytov is generally much more accurate in the lower half space than the Born is in the upper half space. For example, with velocity contrasts less than 40% and incident angles less than 30°, the Rytov approximation to either the transmission coefficient or the transmitted angle is never more than 20% in error. The Born approximation requires much stronger restrictions to achieve the same accuracy for the reflection coefficient. Finally neither approximation handles an incident plane wave at the critical angle well, but the Rytov approximation is acceptable for angles just less than critical.

This simple problem illustrated that there can be considerable differences between the approximations. The extent to which generalizations can be made to more complex cases obviously is still left open. It is of interest in particular, that the Rytov approximation gives a reasonable result for the reflected field is interesting.

5.4 COMPARISON THROUGH NORMAL INCIDENCE ON SLABS

Lin and Fiddy performed an analysis similar to that of Oristaglio's work through comparing the approximations to the exact solution of the reflection and transmission of a normally incident plane wave from a homogeneous half space and slab. The problem of the normal incident plane wave on a half-space is a limited case of Oristaglio's more general work. The same argument is presented for the preference of the Born approximation for the reflected field of the Rytov approximation for the transmitted field. To avoid needless repetition of work, only the dielectric slab problem will be presented.

In free space a plane wave is incident upon a one dimensional homogeneous slab occupying the space $0<z\leq d$ as illustrated in Figure (5.4.1). The dielectric permittivity

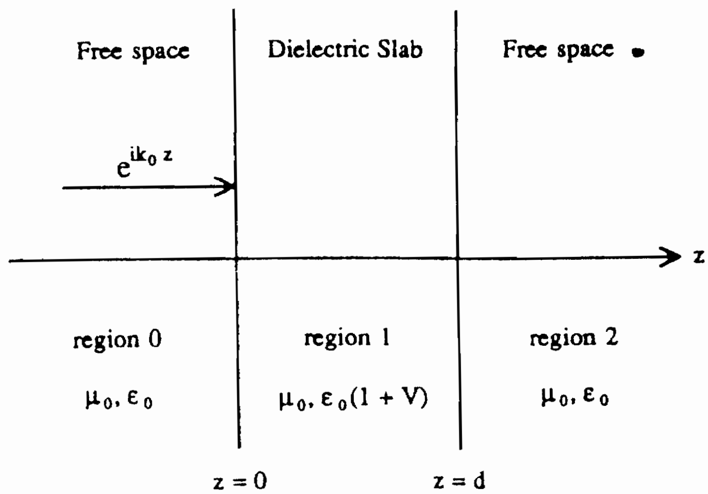


Figure 5.4.1. Geometry of the interface problem [20].

is given as $\varepsilon_o(1+V)$, where 1+V represents the relative dielectric permittivity causing scattering. The boundary conditions to be satisfied at z=0 and z=d are

$$u_0(z=0)=u_1(z=0) \tag{5.4.1}$$

$$u_1(z=d)=u_2(z=d) \tag{5.4.2}$$

$$\left.\frac{du_0(z)}{dz}\right|_{z=0}=\left.\frac{du_1(z)}{dz}\right|_{z=0} \tag{5.4.3}$$

$$\left.\frac{du_1(z)}{dz}\right|_{z=d}=\left.\frac{du_2(z)}{dz}\right|_{z=d}. \tag{5.4.4}$$

The wave number in the slab is denoted by $k_1=k_o(1+V)^{1/2}$. When the boundary conditions are satisfied, the total fields in the three regions are

$$u_0(z)=e^{ik_oz}+\frac{R(k_1)\left(1-e^{i2k_1d}\right)}{D(k_1)}e^{-ik_oz} \tag{5.4.5}$$

$$u_1(z)=\frac{T(k_1)}{D(k_1)}e^{ik_1z}-\frac{R(k_1)T(k_1)e^{i2k_{1d}}}{D(k_1)}e^{-ik_1z} \tag{5.4.6}$$

$$u_2(z)=\frac{T(k_1)\left(1-R\left(k_1e^{i(k_1-k_0)d}\right)\right)}{D(k_1)}e^{ik_oz} \tag{5.4.7}$$

where $D(k_1)=1-\left(R(k_1)\right)^2\exp(i2k_1d)$.

The analysis continues by expanding (5.4.5)-(5.4.7) with a Taylor series about V. The conditions for the validity of the series expansion are V<<1 and Vkd<<1. The expansions are

$$u_0(z) = e^{ik_o z} + \{\left(-\frac{V}{4}+\frac{V^2}{8}-\frac{5V^3}{64}-\frac{V^3}{64}\exp(i2k_0 d)\right)\left(1-e^{i2k_o d}\right)+ \\ i\left(\frac{V^2}{4}-\frac{3V^3}{16}\right)k_o d e^{i2k_o d} - \frac{V^3}{8}k_o^2 e^{i2k_o d} + \cdots\}e^{-ik_o z} \tag{5.4.8}$$

$$u_1(z) = \{1 - \frac{V}{4} + \frac{V^2}{8} - \frac{5V^3}{64} + \left(\frac{V^2}{16}-\frac{5V^3}{64}\right)e^{i2k_o d} + ik_o\frac{V^3}{16}\left(d+\frac{z}{2}\right)e^{i2k_o d} + \\ i\left(\frac{V}{2}-\frac{V^2}{4}+\frac{5V^3}{32}\right)k_o z + \left(\frac{V^2}{8}+\frac{3V^3}{32}\right)k_o^2 z^2 - i\frac{V^3}{48}k_o^3 z^3\}e^{ik_o z} + \\ \{\frac{V}{4} - \frac{3V^2}{16} + \frac{9V^3}{64} + \frac{V^3}{64}e^{i2k_o d} + i\left(\frac{V^2}{4}-\frac{V^3}{4}\right)k_o\left(d-\frac{z}{2}\right)\}e^{i2k_o d}e^{-ik_o z} + \cdots \tag{5.4.9}$$

$$u_2(z) = \{1 + \left(-\frac{V^2}{16}+\frac{V^3}{16}\right)\left(1-e^{i2k_o d}\right) + i\left(\frac{V}{2}-\frac{V^3}{8}+\frac{V^3}{32}\right)k_o d + i\frac{3V^3}{32}k_o d e^{i2k_o d} \\ + \left(-\frac{V^2}{8}+\frac{V^3}{16}\right)k_o^2 d^2 - i\frac{V^3}{48}k_o^2 d^3 + \cdots\}e^{ik_o z} \; . \tag{5.4.10}$$

In each region the solutions must satisfy

$$\frac{du_m(z)}{dz^2} + k_m^2 u_m(z) = 0 \tag{5.4.11}$$

where m=(1,2,3). The wave equation can be written in integral form as

$$u_m(z) = u_m^{(0)} - k_o^2 \int_0^d dz' G(z,z') V u_1(z') \, . \tag{5.4.12}$$

where $u_0(0)=u_1(0)=u_2(0)=e^{ik_o z}$ and G(z,z') is the one-dimensional Green's function given by

$$G(z,z') = \frac{-i}{2k_o} e^{ik_o|z-z'|} \, . \tag{5.4.13}$$

Doing the same calculation Orastaglio did for the first-order Born approximation, but continuing for higher orders, the Born series can be obtained from (5.4.12) replacing u_1 with the incident field exp(ik$_o$z). So, (5.4.12) is then approximated by

$$u_m(z) \approx e^{ik_o z} - k_o^2 \int_0^d dz' G(z,z') V e^{ik_o z'} + k_o^2 \int_0^d dz' G(z,z') V \int_0^d dz'' G(z',z'') V e^{ik_o z''} + \cdots \quad (5.4.13)$$

For region 1, the first integration becomes, noting that z<z',

$$u_1^{1BA}(z<0) = -k_o^2 \int_0^d dz' \frac{-i}{2k_o} V e^{-ik_o z} e^{2ik_o z'} = \frac{Vik_o}{2} e^{-ik_o z} \frac{1}{2i(k_o + i\delta)} e^{2i(k_o+i\delta)z'} \Big|_0^d = -\frac{V}{4} e^{ik_o z} \quad (5.4.14)$$

The principle of limiting absorption was invoked, assuming that the medium of the slab has a small imaginary part.

$$\int_0^d dz' e^{2ik_o z'} = \lim_{\substack{\delta\to 0 \\ d\to\infty}} \int_0^d dz' e^{2i(k_o+i\delta)z'} = \frac{-1}{2ik_o}. \quad (5.4.15)$$

The first order integrations for the other two regions become

$$u_1^{1BA}(0<z<d) = k_o^2 \int_0^z dz' \frac{i}{2k_o} V e^{ik_o z} + k_o^2 \int_z^d dz' \frac{i}{2k_o} V e^{-ik_o z} e^{2ik_o z'} = e^{ik_o z}\left(\frac{iVk_o z}{2} - \frac{V}{4}\right) \quad (5.4.16)$$

$$u_1^{1BA}(d<z) = -k_o^2 \int_0^d dz' \frac{-i}{2k_o} V e^{ik_o z} = -\frac{ik_o Vd}{2} e^{ik_o z} \quad (5.4.17)$$

The first order terms calculated from Born series are found to match the first order terms in the Taylor series expansions of the exact solutions. Carrying out the iterative integrations of (5.4.13), the higher order terms are also found to match those in (5.4.8)-(5.4.10) exactly. The first order integrations were not presented in the paper, but were added here for a more complete discussion.

For the Born series in all three regions to converge, the two above mentioned conditions must hold ($V \ll 1$ and $Vk_o d \ll 1$). These two constraints impose rather severe limitations on the applicability of the Born approximation.

The Rytov approximation is given the form as

$$u_m(z) = u_m^{(0)} e^{i\varphi_m(z)} \tag{5.4.18}$$

where $u_m(0) = e^{ik_o z}$ and φ_m are the unperturbed wave and phase functions in each region m=1,2,3 respectively. Performing the same operations as in preceding chapters, the wave equations with the insertion of (5.4.15) for each space become the following

$$\frac{d^2}{dz^2}\left(u_0(z)\varphi_0(z)\right) + k_o^2 u_9(z)\varphi_0(z) = -i\left(\frac{d\varphi_o(z)}{dz}\right)^2 u_o(z) \tag{5.4.19}$$

$$\frac{d^2}{dz^2}\left(u_0(z)\varphi_1(z)\right) + k_o^2 u_9(z)\varphi_1(z) = ik_o^2 V u_0(z) - i\left(\frac{d\varphi_1(z)}{dz}\right)^2 u_o(z) \tag{5.4.20}$$

$$\frac{d^2}{dz^2}\left(u_0(z)\varphi_2(z)\right) + k_o^2 u_2(z)\varphi_0(z) = -i\left(\frac{d\varphi_2(z)}{dz}\right)^2 u_o(z). \tag{5.4.21}$$

Following the same procedure applied above to the Born approximation, the exact solutions are used to obtain the complex phase functions in each space

$$\varphi_m(z) = -i\ln\left\{1+\frac{R(k_1)\left(1-e^{i2k_1d}\right)}{D(k_1)}\right\}e^{\cdot i2k_oz} \tag{5.4.22}$$

$$\varphi_2(z) = -i\ln\left\{\frac{T(k_1)}{D(k_1)}-\frac{R(k_1)T(k_1)e^{i2k_1d}}{D(k_1)}e^{-i2k_1z}\right\}+(k_1-k_0)z \tag{5.4.23}$$

$$\varphi_3(z) = -i\ln\left\{\frac{T(k_1)\left(1-R(k_1)\right)}{D(k_1)}\right\}+(k_1-k_1)d\,. \tag{5.4.24}$$

Expanding (5.4.22)-(5.4.24) into a Taylor series in terms of V, the following is obtained

$$\begin{aligned}\varphi_0(z) = \{&i\left(\tfrac{V}{4}-\tfrac{V^2}{8}+\tfrac{5V^3}{64}+\tfrac{V^3}{64}e^{i2k_od}\right)\left(1-e^{i2k_od}\right)+\\ &\left(\tfrac{V^2}{4}-\tfrac{3V^3}{16}\right)k_ode^{i2k_od}+i\tfrac{V^3}{8}k_c^{\hat{}}e^{i2k_od}\}e^{-i2k_oz}\\ &\left\{i\left(\tfrac{V^2}{32}-\tfrac{V^3}{32}\right)\left(1-e^{i2k_od}\right)+\tfrac{V^3}{16}k_ode^{i2k_od}\right\}\left(1-e^{2k_od}\right)e^{-ik_oz}\\ &i\tfrac{V^2}{192}\left(1-e^{i2k_od}\right)^3e^{-ik_oz}+\cdots\end{aligned} \tag{5.4.25}$$

$$\begin{aligned}\varphi_2(z) = {}&i\left(\tfrac{V}{4}-\tfrac{3V^2}{32}+\tfrac{5V^3}{96}\right)+\left(-i\tfrac{V^2}{16}+i\tfrac{V^3}{16}+\tfrac{V^3}{16}k_od\right)e^{i2k_od}+\left(\tfrac{V}{2}-\tfrac{V^2}{8}+\tfrac{V^3}{16}\right)k_oz\\ &\left\{i\left(-\tfrac{V}{4}+\tfrac{V^2}{8}-\tfrac{5V^3}{64}\right)+\left(\tfrac{V^2}{4}-\tfrac{V^3}{16}\right)k_o(d-z)+i\tfrac{V^3}{8}k_o^2(d-z)^2\right\}e^{i2k_od}e^{-i2k_oz}\\ &\left\{i\left(\tfrac{V^2}{32}-\tfrac{V^3}{32}\right)-\tfrac{V^3}{16}k_o(d-z)\right\}e^{i4k_od}e^{-i4k_oz}-i\tfrac{V^3}{192}e^{i6k_od}e^{-ik_oz}+\\ &O\left(V^4\right)+O\left(V^4k_o^2d^2\right)+O\left(V^4k_o^3(d-z)^3\right)+\cdots\end{aligned} \tag{5.4.26}$$

$$\begin{aligned}\varphi_2(z) = {}&\left(\tfrac{V}{2}-\tfrac{V^2}{8}+\tfrac{V^3}{16}-\tfrac{5V^4}{128}\right)k_od+i\left(\tfrac{V^2}{16}-\tfrac{V^3}{16}+\tfrac{29V^4}{512}\right)\\ &\left\{i\left(-\tfrac{V^2}{16}+\tfrac{V^3}{16}-\tfrac{7V^4}{128}\right)+\left(\tfrac{V^3}{16}-\tfrac{5V^4}{64}\right)k_od+i\tfrac{V^4}{32}k_o^2d^2\right\}e^{ik_od}-\,.\\ &i\frac{V^4}{512}e^{i4k_od}+\cdots\end{aligned} \tag{5.4.27}$$

The integral equations for the complex phase function in each region is

$$\begin{aligned}\varphi_m(z) &= -i\frac{1}{e^{ik_oz}}\int_{-\infty}^{0} dz' G(z,z')\left(\frac{d\varphi_0(z')}{dz'}\right)^2 e^{ik_oz'} \\ &+ \frac{i}{e^{ik_oz}}\int_0^d dz' G(z,z')\left\{k_o^2V - \left(\frac{d\varphi_1(z')}{dz'}\right)^2\right\}e^{ik_oz'} . \\ &- \frac{i}{e^{ik_oz}}\int_d^{\infty} dz' G(z,z')\left(\frac{d\varphi_2(z')}{dz'}\right)^2 e^{ik_oz'}\end{aligned} \tag{5.4.28}$$

where m=(1,2,3). The Rytov approximation approximates (5.4.28) as

$$\begin{aligned}\varphi_m(z) &= \frac{ik_o^2}{e^{ik_oz}}\int_0^d dz' G(z,z')Ve^{ik_oz} - \frac{i}{e^{ik_oz}} \\ &\times\int_0^d dz' G(z,z')\left(\frac{d\varphi_0^{1RA}(z')}{dz'} + \frac{d\varphi_0^{2RA}(z')}{dz'} + \frac{d\varphi_0^{3RA}(z')}{dz'} + \cdots\right)^2 e^{ik_oz'} - \frac{i}{e^{ik_oz}} \\ &\times\int_0^d dz' G(z,z')\left(\frac{d\varphi_1^{1RA}(z')}{dz'} + \frac{d\varphi_1^{2RA}(z')}{dz'} + \frac{d\varphi_1^{3RA}(z')}{dz'} + \cdots\right)^2 e^{ik_oz'} - \frac{i}{e^{ik_oz}} \\ &\times\int_0^d dz' G(z,z')\left(\frac{d\varphi_2^{1RA}(z')}{dz'} + \frac{d\varphi_2^{2RA}(z')}{dz'} + \frac{d\varphi_2^{3RA}(z')}{dz'} + \cdots\right)^2 e^{ik_oz'} .\end{aligned} \tag{5.4.29}$$

The integrations in (5.4.29) are the same as done for the Born series above, except for the coefficients in front of the integrals. Performing the same type of iterative calculation as before for the Born approximation, the expressions in (5.4.25)-(5.4.27) can be obtained.

If the Rytov series is to be a satisfactory approximation, the following conditions should hold: V<<1 and $Vk_0d<<1$. From (5.4.25)-(5.4.27), the Rytov complex phase functions diverge for $Vk_0d>1$ as in the Born series. So Lin and Fiddy conclude that in this case of the dielectric slab case, the Rytov method and the Born method have the same domain of validity. The statement is given that this conclusion is new and striking. In the past the suggestion had been made that the Rytov approximation is better than the Born approximation for an object of compact support. No reference is given for this statement. Also they state that, this conclusion can be extended to the three-dimensional case of a TE polarization normally incident on a slab, because analytically this case breaks down to the one-dimensional case presented.

6. THE INVERSE PROBLEM

The previous applications of the Born and Rytov approximations have been directed towards finding the scattered fields where the refractive index of the scattering medium is known or effectively modeled. This problem has been labeled the forward or direct scattering problem. Although computationally difficult, this problem is in principle quite straight-forward mathematically. The direct scattering problem, for which the Born and Rytov approximation are applied, leads to the evaluation of the integral equations (2.2.8) and (2.3.22).

The inverse problem involves approximately reconstructing a object's structure from measurements of the scattered fields. Generally, the scattered fields are obtained on a variety of planes normal to the incident fields direction. Then, the data obtained is related to the refractive index of semi-transparent objects by the Born or Rytov approximation. These steps are repeated varying the angles of illumination until a sufficient amount of data is obtained to put into a reconstruction algorithm. Fiddy discuss some of these different algorithms [28].

For semi-transparent objects, weak scattering approximations must be used. The Born and Rytov methods are the only weak scattering approximations available to date. At x-ray wavelengths, imaging techniques can use the geometrical optics approximation. However, at optical and acoustical wavelengths diffraction and scattering effects become important. Due to the wide possible applications (improved medical imaging, geophysical and optical problems), the inverse problem has received considerable attention over the past ten years. The following investigation hopes to illustrate some differences in an important application of the Born and Rytov approximations.

Wolf was the first to propose the possiblity of determining quantitatively the three-dimensional structure of a semi-transparent, weak scattering object [21]. Wolf also showed how both the amplitude and the phase of the scattered field could be obtained by holography [22]. Then with the complex scattered field known, the Born approximation related this quantity to the scattering potential of an object.

A basic discussion of Wolf's original approach to the inverse problem, which used the Born approximation, will be presented . Then, the approach will be formulated using the Rytov approximation and the differences between the two methods as utilized in this application will be discussed. Briefly two experiments, using these approximations, will be presented to point out the major difficulties in applying Wolf's ideas.

6.1 INVERSE WITH THE BORN APPROXIMATION

A scatterer of finite size is considered in Figure 6.1.1 illuminated by a monochomatic wave of frequency ω. For points outside the scatterer, the index of refraction is assumed to be one. The wave equation is rearranged in the following manner

$$\nabla^2 E_t + k_o^2 E_t = k_o^2\left(1 - n^2(\vec{r})\right)E_t . \qquad (6.1.1)$$

Wolf refers to the quantity on the right hand side of (6.1.1), not including E_t, as the scattering potential, denoted as $F(\vec{r})$.

The first Born approximation is now applied to (6.1.1).

$$E_t = E_i + E_s . \qquad (6.1.2)$$

Following the same approach as in chapter one, the approximation yields

$$\Delta E_i + k_o^2 E_i = 0 \qquad (6.1.3)$$

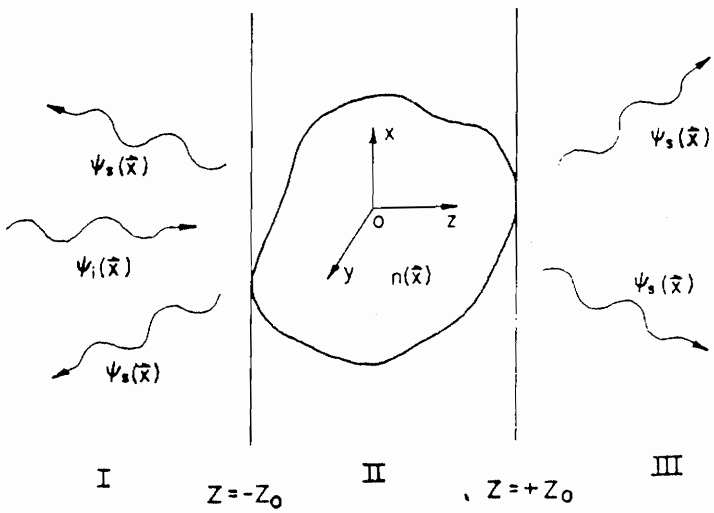


Figure 6.1.1. Illustration of scattering regions [27].

$$\Delta E_s + k_o^2 E_s = k_o^2\left(1 - n^2(\vec{r})\right)E_t. \qquad (6.1.4)$$

The Born method makes the assumption that the scattered field is so weak inside the scatterer relative to the incident that $E_t = E_i$. With this assumption (6.1.4) in integral form is

$$E_s(\vec{r}) = -\iiint_{\infty} F(\vec{r}\,')E_i(\vec{r}\,')\frac{e^{ik_o|\vec{r}-\vec{r}\,'|}}{4\pi|\vec{r}-\vec{r}\,'|}d^3\vec{r}\,'. \qquad (6.1.5)$$

The following representation of the Green's function due to Weyl is used

$$\frac{e^{ik_o|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|} = \frac{ik_o}{2\pi}\iint_{\infty}\frac{1}{m}e^{ik_o[p(x-x')+q(y-y')+m|z-z'|]}dpdq \qquad (6.1.6)$$

where

$$m = \left(1 - p^2 - q^2\right)^{1/2} \quad \text{if } p^2 + q^2 \le 1$$

$$= i\left(p^2 + q^2 - 1\right)^{1/2} \quad \text{if } p^2 + q^2 > 1 \qquad (6.1.7)$$

The scattering object is restricted to region II in Figure 6.1.1. Then, for region I, $(z-z')<0$; and for region III, $(z-z')>0$. Substituting (6.1.6) into (6.1.5), replacing $|z-z'|$ by $\pm(z-z')$, and interchanging the order of integration yields the following

$$E_s(\vec{r}) = \iint_{\infty} A^{\pm}(\vec{p})e^{ik_o(px+qy\pm mz)}dpdq \qquad (6.1.8)$$

where

$$A^{\pm}(\bar{p}) = \frac{-ik_o}{8\pi^2 m} \iiint_{\infty} F(\vec{r}\,')E_i(\vec{r}\,')e^{-ik_o(px'+qy'\pm mz')}d^3\vec{r}\,' \qquad (6.1.9)$$

is the angular spectrum of plane waves which make up the scattered field. The + sign is taken if the scattered field is measured in region III. The - sign is used if the scattered field is measured in region I. Also, according to (6.1.7), the plane waves in the angular spectrum are homogeneous for $p^2+q^2<1$, and evansecent for $p^2+q^2>1$. Since the evanescent waves die off rapidly with z, only the homogeneous waves will be considered in the following.

For inverse scattering calculations, (6.1.8) and (6.1.9) must be inverted. The inversion of (6.1.8) is easily done, because it has the form of a two-dimensional Fourier transform. So, inverting (6.1.8) is done by taking the Fourier transform with respect to the variables x and y, with z having the fixed values of $\pm z_s$,

$$A^{\pm}(\bar{p}) = \frac{k_o^2 e^{-ik_o z_s}}{(2\pi)^2} \iint_{\infty} E_s^{\pm}(\vec{r})e^{ik_o(px+qy)}dxdy\ . \qquad (6.1.10)$$

E_s is the scattered field measured over one of the two planes, $+z_s$ or $-z_s$ as labelled in Figure (6.1.1), in region I or II. So. $A^{\pm}$ is really the two dimensional Fourier transform of the scattered field minus some proportionality factors.

The inversion of (6.1.9) takes a bit more effort. The incident field is assumed to be a plane wave propagating in some arbitrary direction given by the direction cosines (p_o, q_o, m_o) such that

$$E_i(\bar{r}) = e^{ik_o(p_o x - q_o y + m_o z)} . \qquad (6.1.11)$$

Putting (6.1.11) into (6.1.9) produces

$$A^{\pm}(\bar{p}) = \frac{-ik_o}{8\pi^2}\tilde{F}(\bar{\xi}) . \qquad (6.1.12)$$

where $\tilde{F}(\bar{\xi})$ is a three dimensional transform of $F(\bar{r})$, the scattering potential, defined by

$$\tilde{F}(\bar{\xi}) = \iiint_{\mathscr{S}} F(\bar{r}')e^{-ik_o \bar{r}'\cdot\bar{\xi}} d^3\bar{r}' \qquad (6.1.13)$$

and where $\bar{\xi} = (\xi, \eta, \zeta)$ with

$$\xi = (p - p_o)$$

$$\eta = (q - q_o) \quad . \qquad (6.1.14)$$

$$\zeta = (\pm m - m_o)$$

Now substituting (6.1.10 into (6.1.12) gives

$$\tilde{F}(\bar{\xi}) = i2k_o m e^{ik_o z_s}\tilde{E}_s^{\pm}(\bar{p}) \qquad (6.1.15)$$

where

$$\tilde{E}_s^{\pm}(\bar{p}) = \iint_{\mathscr{S}} E_s^{\pm}(\bar{r})e^{-ik_o(px + qy)}dxdy . \qquad (6.1.16)$$

$\tilde{E}_s^{\pm}$ is the two dimensional Fourier transform over one of the planes in Figure 6.1.1. For a fixed direction of illumination p_o, the measurement of $\tilde{E}_s^{\pm}$ is Fourier transformed in two dimensions. This Fourier transform is then substituted into (6.1.14), which allows the calculation of the three dimensional Fourier transform of the scattering potential. Finally $F(\bar{r})$ can be obtained by the inversion of (6.1.12).

The scattering potential can be obtained in principle from experiments in which E_s is measured for all directions (p_o, q_o, m_o) and then $F(\bar{r})$ obtained by inverse Fourier transform.

6.2 INVERSE PROBLEM USING THE RYTOV APPROXIMATION

Devaney presented the formulation of the inverse problem with the Rytov approximation [23]. In the first Rytov approximation as stated in Chapter 2, the electric field is assumed to have the form

$$E_t = e^{ik_o \bar{s}_o \cdot \bar{r} + \psi_1(\bar{r})} . \qquad (6.2.1)$$

Starting with the form of the wave equation in (6.1.1), and following the procedure in chapter two, the solution to ψ_1 is given by

$$\psi_1(\bar{r}) = -\iiint_{\mathcal{V}} F(\bar{r}') \frac{E_0(\bar{r}')}{E_0(\bar{r})} \frac{e^{ik_o|\bar{r} - \bar{r}'|}}{4\pi|\bar{r} - \bar{r}'|} d^3\bar{r}' . \qquad (6.2.2)$$

The scattered field is obtained by measurement. So, ψ_1 must be related to the scattered field. This relation is

$$\psi_1(\bar{r}) = \ln\left(\frac{E_t^{\pm}}{E_0}\right). \quad (6.2.3)$$

The quantity in parenthesis represents the normalized scattered field. The same procedure is followed then for relating the measurement of the scattered field to the scattering potential as in the Born approximation. The two equations that must be inverted here are

$$E_0 \ln\left(\frac{E_t}{E_0}\right) = \iint_{\infty} A^{\pm}(\bar{p}) e^{ik_o(px+qy\pm mz)} dp dq \quad (6.2.4)$$

and

$$A^{\pm}(\bar{p}) = \frac{-ik_o}{8\pi^2 m} \iiint_{\infty} F(\bar{r}') E_0(\bar{r}') e^{-ik_o(px'+qy'\pm mz')} d^3\bar{r}' . \quad (6.2.5)$$

So, the inversion of (6.2.4) yields

$$A^{\pm}(\bar{p}) = \frac{k_o^2 e^{-ik_o z_s}}{(2\pi)^2} \iint_{\infty} E_0(\bar{r}) \ln\left(\frac{E_t^{\pm}}{E_0}\right) e^{ik_o(px+qy)} dx dy . \quad (6.2.6)$$

The inversion of (6.2.5) gives

$$\tilde{F}(\xi) = i2k_o m e^{ik_o z_s} \tilde{E}_{sr} \quad (6.2.7)$$

where

$$\tilde{E}_{sr} = F_T\left\{E_0 \ln\left(\frac{E_t^{\pm}}{E_0}\right)\right\} = \iint_{\infty} E_0 \ln\left(\frac{E_t^{\pm}}{E_0}\right) e^{-ik_o(px+qy)} dx dy . \quad (6.2.8)$$

The Fourier transform of a logarithm of a complex function introduces some further complications.

In practice only a finite number of scattering measurements can be made. So, (6.2.8) is really the Fourier transform of a complex sequence. To begin the explanation of the difficulties stirred up by (6.2.8), the z-transform is presented

$$Z[x(n)] = \sum_{n=-\infty}^{\infty} x[n] z^{-n} = X(z) . \qquad (6.2.9)$$

where z is

$$z = re^{i\omega} . \qquad (6.2.10)$$

As can be seen from (6.2.9), the Fourier transform is a special case of the z-transform for which r=1 [24]. A power series of the form of (6.2.9) is a Laurent series. A Laurent series and therefore the z-transform, represents an analytic function at every point inside the region of convergence. So, the z-transform and all its derivatives must be continuous functions of z within the region of convergence. If the Fourier series converges, then the Fourier transform and all its derivatives with respect to ω must be continuous functions of ω [25]. So, both the real and imaginary parts of the Fourier transform of the logarithm of a complex function

$$\hat{X}(e^{i\omega}) = \ln\left|X(e^{i\omega})\right| + i \arg X(e^{i\omega}) \qquad (6.2.11)$$

must be continuous functions of ω.

The complex field E_t is measured with phase angles limited to the range 0 to 2π. Thus 2π phase jumps are present in the data representing E_t and E_0. Because of the reasons given above, these discontinuities in phase must be removed. The removal of these discontinuities in phase is referred to as phase unwrapping. Figure 6.2.1 illustrates the graphical results of phase unwrapping. Schemes exist for unwrapping phase in one dimension [26]. However, no algorithms exist to date for unwrapping phase in more than one dimension. So, (6.2.8), which represents a two dimensional Fourier transform, presents a stumbling block for using the Rytov method in the inverse problem in more than one dimension.

6.3 EXPERIMENTAL LIMITATIONS TO THEORY

As discussed previously, the function F(r) can be obtained in principle from experiments by measuring the scattered fields from all different illumination angles. In practice this procedure is difficult. It has never been realized experimentally due to the problems of repeating a delicate interferometric measurement for the scattered field a great number of times with different directions of illumination while keeping a precisely constant phase reference.

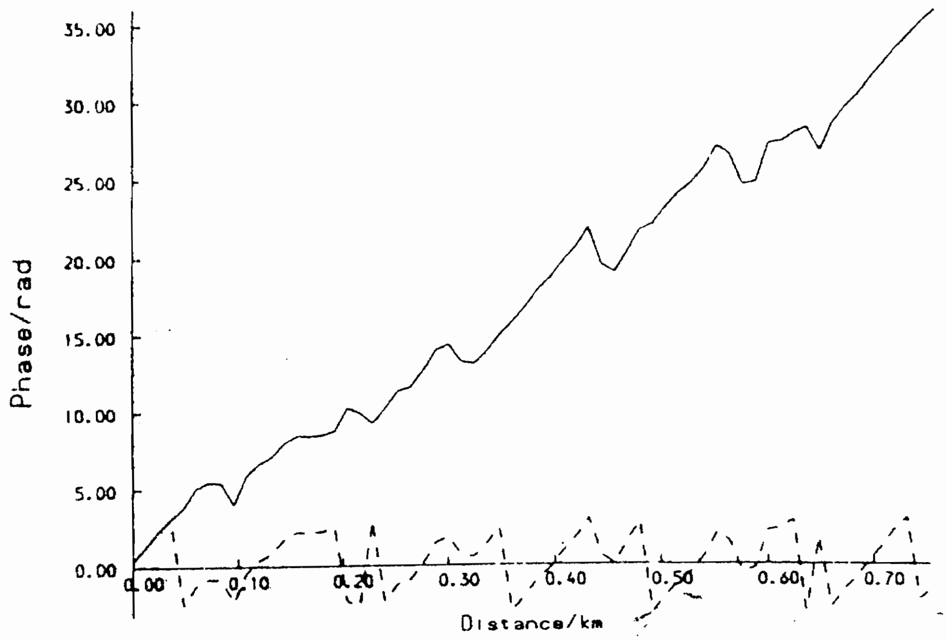


Figure 6.2.1. Illustration of phase unwrapping [26].

Carter has performed one of the few experimental investigations of Wolf's formulations [27]. The experiment was performed in the optical domain. A monochromatic laser source and holographic techniques were used to measure the complex scattered field. As mentioned above the difficulty of maintaining a precise phase reference while varying the direction of illumination, lead to using a symmetrical object such that the illuminating plane wave need only be made incident from a single direction.

Such a symmetrical object is shown in Figure 6.3.1. The object was made from two bars of different homogeneous materials. One bar was made of fused silica, and the other of extra dense flint glass. This object was illuminated by a monochromatic (632.8 nm) plane wave propagating in the +z direction. The scattered field was measured over a small region near the z-axis in plane of constant z (z_s=50 m) by use of a hologram. The results after reconstruction of the x variation of the scattering potential are shown in Figure 6.3.2. The amplitude of F_x agrees very well with the directly measured widths of the bar segments and known amplitude levels. The phase data in Figure 6.3.3 show some deviation from the constant phase for the real scattering potential. The phase errors were attributed to variations in the refractive index of the bar that the Born approximation couldn't handle.

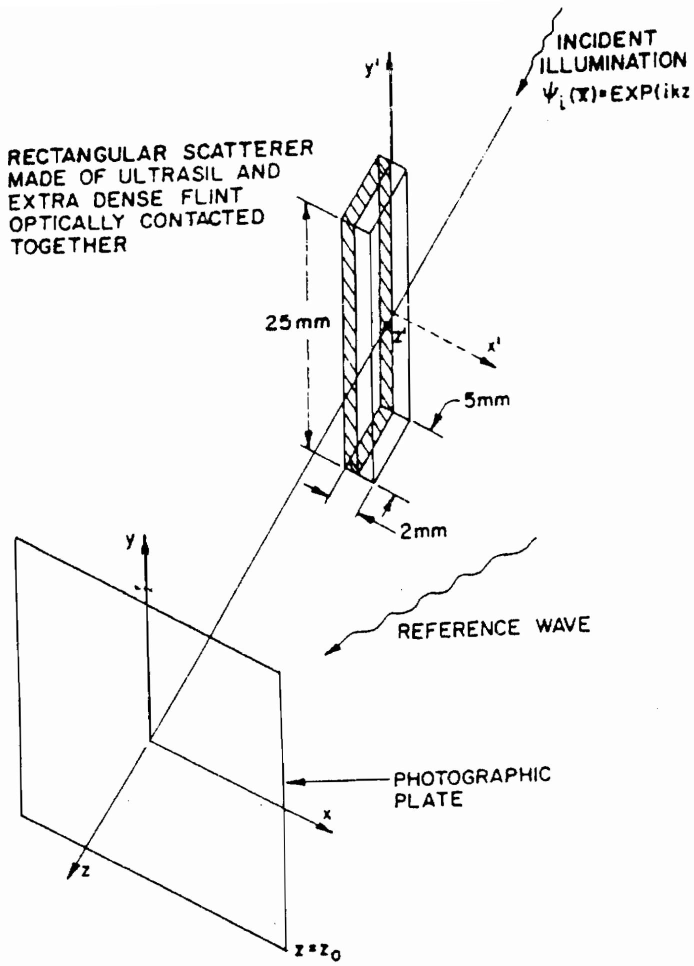


Figure 6.3.1. Experimental setup [27].

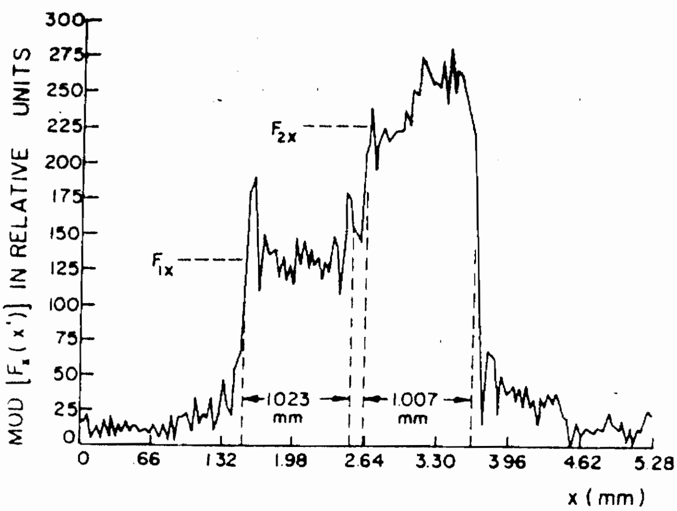


Figure 6.3.2. The magnitude of the scattering potential [27].

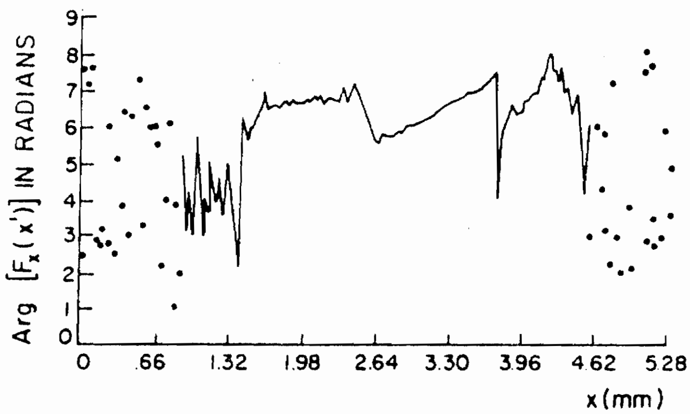


Figure 6.3.3. The phase of the scattering potential [27].

Because of the symmetry, the Fourier transform of the scattered field is only one dimensional. This permits the use of the Rytov approximation. In a later experiment using a cylindrical bar, Carter utilized the Rytov approximation to compare against the results found with the Born approximation [28]. The phase unwrapping was done in a direct manner and checked against a procedure described by Tribolet. In summary, both approximations yielded rather poor results. The edge of the bar was not as well defined in the Rytov approximation as in the Born approximation. However, the phase, which should be constant inside the bar, varied much more slowly in the Rytov approximation. The poor results were attributed to refraction, which the Born and Rytov approximations were unable to handle satisfactorily, at the curved surfaces of the cylinder.

Another group, M. Kaveh et al., beginning in the late 1970's, applied the approximations for use in ultrasonic diffraction tomography [29, 30]. The experimental setup is illustrated in Figure 6.3.4. Gelatin molds are placed on a turntable immersed in a water tank and insonified with a 3 MHz acoustic plane wave. Phase recovery for the complex scattered field can be obtained using the incident field for coherent demodulation. This procedure because of the limits of the electronics is'nt possible in the optical domain. So, the molds are rotated on the turntable and the scattered fields are measured and stored on a magnetic recorder.

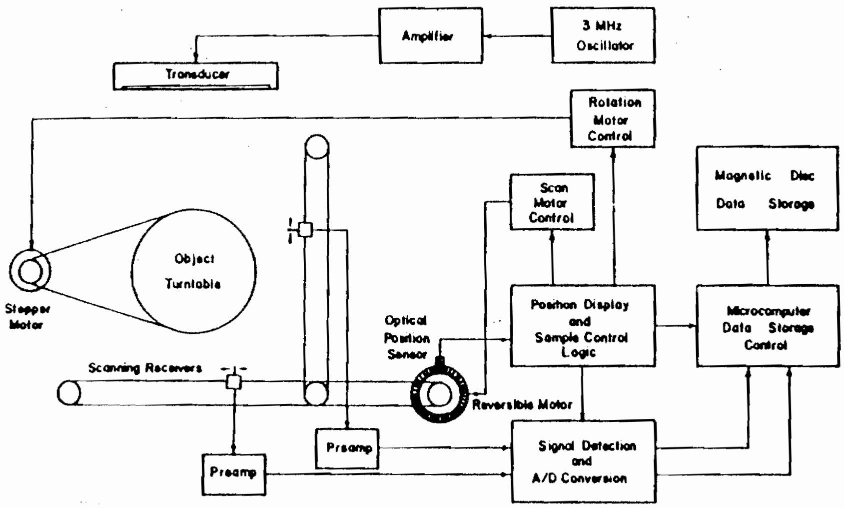


Figure 6.3.4. The experimental setup. [30].

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In a later publication, the authors use the experimental results of [30] to compare the Born and Rytov approximations [31]. A matrix of 64x64 was constructed of 64 samples/profiles with 64 profiles. The comparison of the approximations were based solely on the distortions of their respective reconstructions. A complex annular cylinder with a scattering potential of

$$f(r) = \begin{cases} -0.1 + j0.19 & \lambda < r \leq 2\lambda \\ -.01 & r \leq \lambda \end{cases} . \qquad 6.3.1$$

Figure 6.3.5 shows the reconstruction of the cylinders. So, the maximum velocity perturbation is 2%. The Rytov reconstruction shows good match with quantitative fidelity. The problem with Born's reconstruction is attributed to the mixing of the real and imaginery parts of the scattering potential from the error neglected in dropping the second order and higher terms (i.e. error). The real part of the tomogram, Figure 6.3.5(c), washed out because of the strength of the error introduced in the first order approximation, whereas the imaginery part contains a mixture of the real and imaginery parts of the scattering potential. The authors trace the superior performance of the Rytov approximation to the error merely depending on the gradient of the scattered field, and thus the distortion is'nt directly related to the scattering potential.

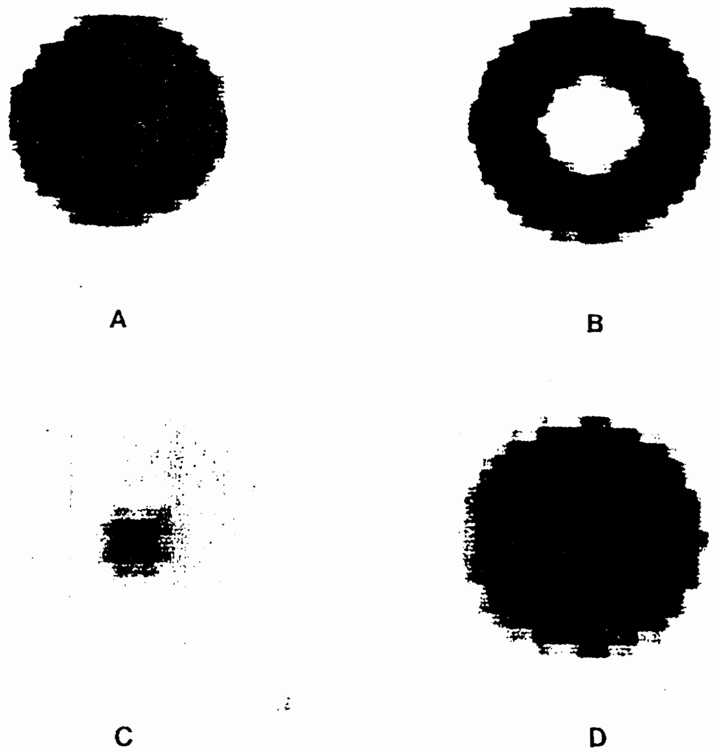

Figure 6.3.5. Complex annular cylinder. The real and imaginery parts of the scattering potential are $f_r=-.1$ for $r \le 2\lambda$, $f_i=0.19$ for $\lambda \le r \le 2\lambda$. The Rytov (a) f_r (b) f_i; The Born (c) f_r (d) f_i [31].

The phase unwrapping wasn't a problem since profiles were measured and only one-dimensional inverse Fourier transforms were necessary. An important trade off encountered in both approximations was between measurement noise and interpolation noise. Fewer profiles reduce low frequency noise power. However, interpolation errors increase due to the increase in the distance of measured values of the scattered field for the inverse Fourier transform. The experiment showed that the Rytov method can yield superior images over the Born method, where forward scattering is predominent. The performance of this method on more complex media such as tissue remains to be demonstrated.

Even for simple symmetric objects the experimental procedure is complicated. Further developements are required if the potential value of inverse scattering in many fields is to be realized. The developement of an algorithm enabling the calculation of the scattered field from its measured intensity distribution without the need of a reference wave would solve the big problem encountered by Carter. Much continuing work has been done on the problem of phase retrieval [29]. The goal is to find a relationship between the phase and magnitude of the scattered field, knowing some analytical properties of the scattered field. Then the phase could be calculated directly from the magnitude of the scattered field. Even though the phase measurement in the acoustic problem is not a stumbling block, phase unwrapping is still a problem for more complicated objects with the Rytov method. There still exists the challenge of obtaining a solution which is computationally efficient and subject to reasonable interpretation.

6.4 CONCLUDING REMARKS OF THE APPROXIMATIONS IN THE INVERSE PROBLEM

For the inverse problem, the performance of the approximations depends strongly on the object being investigated. For objects that aren't symmetric to the degree that the Fourier transform can be reduced to a one-dimensional transform, the Rytov approximation simply can't be used at the present, because of the need to unwrap the phase of the scattered field. However for certain geometries, it seems likely that the Rytov approximation may offer better results than the Born, as follows from results obtained from the direct scattering problem.

7. THE CONTROVERSY

The use of the approximations arose out of attempts to describe wave propagation in random media. This application generated the beginning of the controversy dealing with the region of validities. The controversy lies in the use of the conditions utilized for the determination of the region of validity in the Rytov approximation. The following describes these conditions and then a comparison is made with the condition used for the Born approximation in a random medium.

As stated before, the Born approximation involves the expansion of the field into a series of the form

$$U = U_0 + U_1 + U_2 + \cdots = \sum_{n=0}^{\infty} n_1^n u_n \ . \qquad (7.1.1)$$

Each term in the Born series is assumed to be of the order of the strength of the fluctuations of the index of refraction $(n_1)^n$. The condition generally given for dropping higher order terms is $|U_{n+1}| << |U_n|$. As previously mentioned, most analysis with the Born method are only concerned with the first order terms as the higher order terms for random media are difficult to calculate and complicated for deterministic cases.

The Rytov approximation assumes the field can be represented by

$$U = e^{\Psi_0+\Psi_1+\Psi_2+\cdots} = e^{\psi} = e^{\sum_{n=0}^{\infty} n_1^n \psi_n} \tag{7.1.2}$$

where the term in the exponent is now perturbed by orders of $(n_1)^n$ and $\psi = \chi + i\phi$ is the complex phase. The conditions for the validity of the Rytov approximation are difficult to express. It is here in the definition of the region of validity of the Rytov approximation where the heart of the controversy has boiled.

There have been three conditions that have been used separately for the determination of the region of validity of the Rytov approximation:

1. $|\lambda \nabla \psi_1| << 1$ (7.1.3)
2. $\langle \chi_1^2 \rangle << 1$ (7.1.4)
3. $|\psi_2|^2 << |\psi_1|^2$. (7.1.5)

For ease of examination and for the reason mentioned above for the Born series, the discussion will focus on the first order approximation.

The first condition of (7.1.3) originates from an approximation to the wave equation after (7.1.2) substituted for the field. So, for $\psi = \psi_0 + \psi_1$, the two equations from chapter 1 are obtained

$$\Delta\psi_0 + \nabla\psi_0 \cdot \nabla\psi_0 = -k_o^2 \qquad (7.1.6)$$

$$\Delta\psi_1 + 2\nabla\psi_0 \cdot \nabla\psi_1 = -2k_o^2 n_1 - \nabla\psi_1 \cdot \nabla\psi_1 . \qquad (7.1.7)$$

Rytov simplified (7.1.7) by reasoning if $|\nabla\psi_1 \cdot \nabla\psi_1|$ is small compared to $|2i\vec{k}_o \cdot \nabla\psi_1|$ and $|2k_o^2 n_1|$ then this nonlinear term could be dropped. Hufnagle and Stanley questioned this argument due to the fact that $|\nabla\psi_1 \cdot \nabla\psi_1|$ can be comparable to $\nabla^2\psi_1$ [32]. The meaning of this argument is difficult to understand, since different terms play different roles in governing the growth of a solution. However, if the integral representation of (7.1.7) is examined

$$\psi_1(\vec{r}) = \int_V d^3\vec{r} G(\vec{r},\vec{r}\,')\left(2k_o^2 n_1 + \nabla\psi_1 \cdot \nabla\psi_1\right)\frac{U_0(\vec{r}\,')}{U(\vec{r})} . \qquad (7.1.8)$$

then it is clear that dropping $|\nabla\psi_1 \cdot \nabla\psi_1|$ depends on whether this term is large relative to $|2k_o^2 n_1|$. And, the term's magnitude relative to $\nabla^2\psi_1$ may have no significance. Because $\nabla\psi_1$ is roughly of magnitude n_1, $|\nabla\psi_1 \cdot \nabla\psi_1|$ being smaller than $|2i\vec{k}_o \cdot \nabla\psi_1|$ is a stronger condition relative to $|2k_o^2 n_1|$. This reasoning leads to the first condition of (7.1.3) from $(2i\vec{k}_o + \nabla\psi_1)\cdot\nabla\psi_1$.

If ψ_1 is written as $\psi_1 = \chi_1 + i\phi_1$, then the above condition reduces to

$$\lambda|\nabla\phi_1| << 1 \qquad \lambda|\nabla\chi_1| << 1 . \tag{7.1.9}$$

Physically, these conditions state that the amplitude the wave of the doesn't change much over a wavelength and the wave front doesn't deviate far from the direction of the incident wave. The actual usefulness of this condition for determining whether the Rytov approximation is valid or not is difficult to ascertain.

While this condition is difficult to quantify more usefully, it does lead to some generalizations. If a scattering object has sharp edges or other features, which would contribute strongly to a rapidly varying complex phase, then the Rytov approximation may be expected to be less appropriate than the Born. This is of some usefulness in the inverse problem. Gretzula and Carter attributed this to the superior performance of the Born approximation over the Rytov approximation in their optical imaging experiment presented earlier partly to this reason [28].

The second condition follows from the consideration of conservation of energy for the first order approximation. So, the energy flux density after scattering is

$$\left\langle |U|^2 \right\rangle = \left\langle \left| e^{ik_o x} e^{\chi_1 + i\phi_1} \right|^2 \right\rangle = \left\langle e^{2\chi_1} \right\rangle = e^{2\langle \chi_1^2 \rangle} . \tag{7.1.10}$$

This results follow from the probability distribution of χ_1 approaching a normal distribution by virtue of the central-limit theorem of probability theory and $\langle \chi_1 \rangle = 0$. However in the absence of absorption, the energy flux density should be

$$\left\langle |U_o|^2 \right\rangle = \left\langle \left|e^{ik_o x}\right|^2 \right\rangle = 1 . \qquad (7.1.11)$$

The second condition of (7.1.4) should hold to keep the energy lost by the incident wave to the scattered wave small,

Condition two is different from one and three. The difference is that conditions one and three are limits on the complex phase itself. If the components of the complex phase do not satisfy these two conditions, it is expected that the averaged quantities (intensity, phase and amplitude variance, and etc...) calculated from ψ will not be very good approximations. However, condition two is more like an expectation. It is expected that little energy is transferred to the scattered wave. Condition two really doesn't indicate a condition on other quantities of interest.

From experiments investigating strong fluctuations of the irradiance of a laser beam in the atmosphere, it was found by Gracheva and Gurvich that the first order Rytov approximation holds up to $\left\langle \chi_1^2 \right\rangle \approx 1$ [3]. These experiments were discussed earlier in conjunction with deWolf's work [12]. As illustrated in Figure 4.2.2, the Rytov approximation fails for the normalized intensity variance when $\left\langle \chi_1^2 \right\rangle \geq 1$.

The third condition for the validity of the Rytov series, (7.1.5), is centered in the confines of the perturbation of the complex phase function. Specifically, a comparison is made of the second order term with the first order term of the Rytov expansion. Barabanenkov, et al. and Yura et al. state that condition 3 leads to the following inequality for ψ_1 [33, 35]:

$$\left\langle |\psi_1|^2 \right\rangle << 1. \tag{7.1.12}$$

Barabanenkov et al. state that (7.1.12) follows from condition 3 for a medium of single scale. These authors then state that the necessity of condition 3 is far from evident.

However, Yura et al. claim that (7.1.12) follows from condition 3 more generally with the following relations:

$$\psi_2 = u_2 - \frac{\psi_1^2}{2} \tag{7.1.13}$$

$$2\,\mathrm{Re}\left\langle \frac{u_2}{u_0} \right\rangle + \left\langle |\psi_1|^2 \right\rangle = 0 \tag{7.1.14}$$

$$\left| \mathrm{Im}\frac{u_2}{u_0} \right| << \left| \mathrm{Re}\frac{u_2}{u_0} \right|. \tag{7.1.15}$$

Equation (7.1.13) is the relation between the second order terms of the Born and Rytov expansions [10]. Equation (7.1.14) follows from the conservation of energy applied to the Born approximation. The energy flux density with the Born expansion is

$$\left\langle |U|^2 \right\rangle = \left\langle |U_0|^2 \right\rangle + 2\,\mathrm{Re}\left\langle U_0^* U_1 \right\rangle + \left\langle |U_1|^2 \right\rangle + 2\,\mathrm{Re}\left\langle U_0^* U_2 \right\rangle + \left\langle |U_2|^2 \right\rangle + \cdots. \tag{7.1.16}$$

As mentioned above to satisfy (7.1.11), all the orders of one and higher must equate to zero. Equating the terms in (7.1.16) of second order to zero lead to (7.1.14). Equation (7.1.5) holds for all cases of interest [13, 14].

Yura et al. state, although not clearly, that (7.1.12) follows directly from (7.1.13)-(7.1.14) and condition 3. The condition imposed by (7.1.12) is equivalent to that stated for the energy consideration used for the first order Born approximation

$$\left\langle |U_1|^2 \right\rangle << 1. \tag{7.1.17}$$

This inequality follows from $|u_1| = |\chi_1 + i\phi_1| << |u_0| = 1$. From condition 3, then it would follow that the Born and Rytov approximation would entertain the same regions of validity.

The arguments purporting that the Born and Rytov approximation have the same regions of validity have centered around condition three (i.e. (7.1.5)). However, the work presented throughout this monograph provides evidence that there can exist situations where the Born and Rytov approximations can entertain different regions of validity. Yura, and Brown used this approach for their basis of evaluating the Rytov approximation [34, 35]. However, this type of criterion, as pointed out above by Barabanenkov et al. and problems discussed herewith, can lead to excessively strict validity conditions for the approximation.

It is beneficial to make a brief comparison of conditions two and three. From the integral expressions for u_1 and ψ_1 ((2.2.8) and (2.3.22)), it can be seen that the magnitudes of these two expressions are equal (i.e. $|u_1| = |\psi_1| = \sqrt{\chi_1^2 + \phi_1^2}$). So, from (7.1.12), the following for the Born approximation is obtained

$$\left\langle \chi_1^2 + \phi_1^2 \right\rangle << 1 . \tag{7.1.18}$$

Whereas, condition two states that only the variance of the amplitude is required to stay small. From this comparison, the conclusion can be drawn that the Rytov approximation can handle large phase fluctuations while the Born cannot.

Continuing with the random medium, there is one definite experimental result that favors the Rytov approximation. In the region of weak fluctuations, the probability distribution of the irradiance is very close to a log normal distribution. This result agrees with the prediction of the Rytov method. As the following shows, the Born approximation predicts a Rayleigh probability distribution.

The first Born approximation assumes the solution of the form $E = E_0 + E_1 = C(\bar{r}) + iD(\bar{r})$. The probability distribution of C and D are nearly normal by virtue of the central limit theorem. So, the amplitude (i.e. $A^2 = (C^2 + D^2)^{1/2}$) of the wave is normally distributed. The probability density function of the amplitude or intensity ($I = A^2$) is

$$P(I) = \frac{1}{2\pi\langle I\rangle} e^{-I/2\langle I\rangle}. \qquad (7.1.19)$$

Equation (7.1.19) is known as a Rayleigh distribution.

The Rytov approximation is given by $E = e^{C+iD}$. In the Rytov approximation C and D are the log amplitude and phase of the wave. Again from the central limit theorem, the log amplitude (ln(C)) and thus intensity tends towards a normal distribution. Gracheva et al. found that experimental data was described quite closely by a log normal distribution of the intensity Figure 7.1.1.

Besides the probability distribution of intensity, Gracheva's et al. experimental data for intensity didn't really delineate any differences for the approximations. Expressions derived for the variance of amplitude using the Born approximation do not appear to have a much different range of validity than the expression for the variance of the log amplitude.

Because exact solutions aren't available with random media, attention was directed toward deterministic problems to gain some insight into the differences between the approximations. However, the deterministic cases were still plagued by the same strong inequalities. In order to keep the exact solutions manageable for comparison, the problems had to be kept pretty simple. This simplicity removed the analysis too far from the actual applications of the approximations and experimental evidence. The important

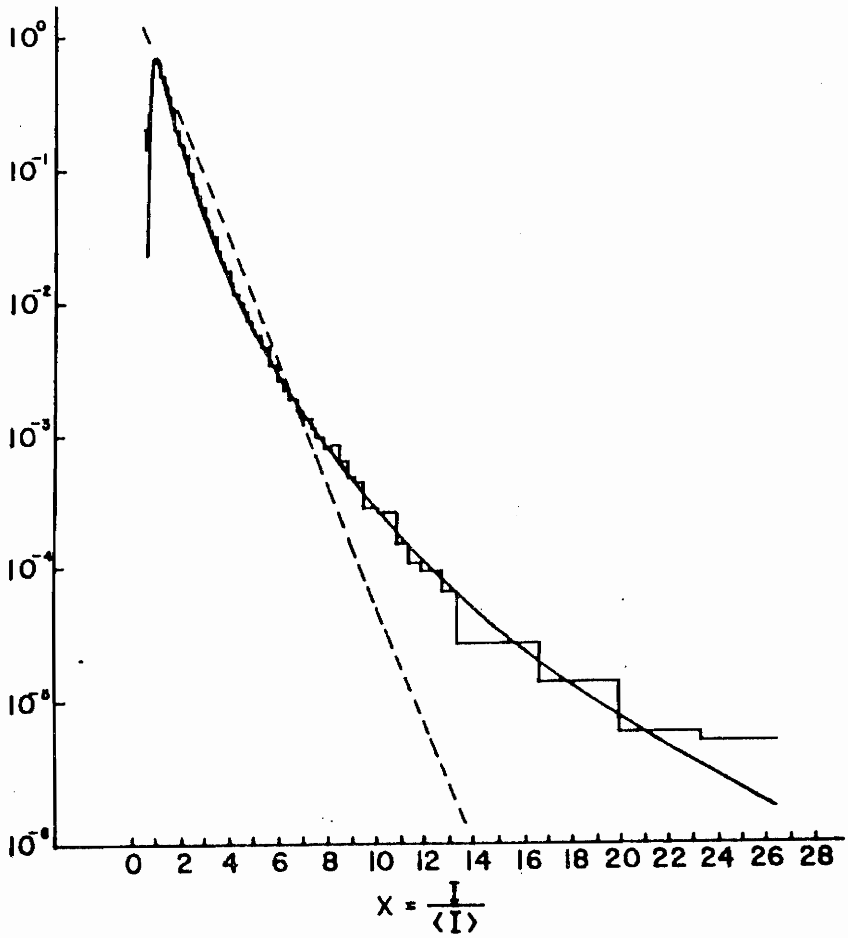

Figure 7.1.1. The probability density of the normalized intensity fluctuations [3].

discussions of the deterministic tests have been discussed earlier in this monograph. The following is a brief history of the approximations, beginning with the controversy up to the deterministic problems.

7.2 THE HISTORY

The discussion will proceed in chronological order. After Tatarskii's and Chernov's initial application of the methods to wave propagation, deWolf was one of the first of the western workers to really organize and attempt to point out the advantages and limitations of the different weak fluctuating approximations [12]. Basically, he showed that the limitations on propagation distance and magnitude of the inhomogeneities were less severe for the expressions which result from the first Rytov method than those for the first Born approximation. DeWolf based his conclusions on a one dimensional slab, where only one scale length was encountered and the correlation function of the turbulence was a Gaussian.

In 1966, Brown concluded by the comparison of successive terms in the Rytov approximation that the Born and Rytov approximation have the same regions of validity [35]. The problems with this analysis were discussed above. In 1969, Keller compared the exponentiated Born series (see later) and the Rytov series with an exact expression in a one dimensional deterministic problem [9]. Keller found that both approximations have the same degree of error relative to the magnitude of fluctuations of the medium. However, the errors exhibited different dependence upon the distance x. The Born series showed error of the complex field increasing as x^{n+1} with x, while in the Rytov series the error increases on the order of x as x increases. Keller attributes the extended region of validity to the extra terms that the Rytov approximation carries, which are observed upon series expansion of the Rytov approximation.

In 1969, Yura supported Brown position that the approximations have the same region of validity based again upon term comparison in the Rytov series [35].

Hadden and Mintzer (1978) used the Epstein medium for which an exact solution was available for comparison, and in which, hopefully, nontrivial implications for the physical situations were alleviated [18]. In the first test problem, which corresponds to an extended inhomogeneous region (with a discontinuity in the slope of the index of refraction at the interface) the same conclusions reached by Keller were observed for $z>0$ for small thickness s. Similar dependence on the order of the error relative to the strength of scattering was encountered, but stronger range dependence was found in the Born approximation. The Rytov approximation yielded advantageous range dependence for the field, amplitude and phase. For large values of s (transition layer bigger), the advantage was lost. For $z<0$, additional wave-like terms were encountered in the complex field but since the terms are of unit magnitude Hadden and Mintzer showed there was no advantage for either method. In the second test problem, which corresponds to a localized inhomogeneity (with a discontinuity in the index of refraction at the interface), there are no preferential ranges for either method for either small or large values of the thickness parameter. No comment is made of this result in the paper. However, this result at least deserves some comment due to a peculiarity. Because of the smoothness of the media required by the Rytov approximation (i.e. $\lambda|\nabla\psi_1|<<1$), the expectation could be held that the Rytov approximation should not perform as well as the Born method. No difference is found in the range dependence for the continuous case of the localized perturbation of the refractive index.

The problem with Hadden and Mintzer's paper is centered around the complexity of the exact solution. This complexity required the exact solutions to be expanded in series form around the strength parameters in refractive index (Δ and δ) for comparison with the approximations. Hadden and Mintzer tried to circumvent this problem by considering limiting values of the thickness parameter s, whereby comparisons of the exact and approximate solutions could be made for any values of strength. This approach still enables information about regions of validity to be extracted. Although their problem gives a nontrivial test of the Born and Rytov approximations, the complexity of the exact solution clouds the analysis especially in the last two problems.

Oristaglio gave a rather simple and elegant test of the approximations through oblique incidence of a plane wave upon a planar surface [19]. This analysis allowed investigation of the performance of the approximations with varying angles of incidence. In the reflected domain, again the wave-like terms were encountered for the Rytov approximation. However, for the numerical examples performed for differing velocity contrasts, the Rytov approximation gave results no worse than the Born approximation did for the reflection coefficient. The Rytov approximation performed better than the Born for the transmitted field, since the Born series had a range dependent term (i.e. the scattered field had first order dependence on x). The Rytov approximation performed better for the transmitted field than the Born series for the reflected field.

Fin and Liddy (1992) perform two one-dimensional deterministic problems [20]. The first problem is a limiting case of Oristaglio's work, where a plane wave is normally incident upon a dielectric half-space. Their results are the same as Oristaglio's. However, because of the additional wave-like terms in the Rytov expression for the reflected field, Lin and Fiddy conclude that the Born approximation is superior to the Rytov approximation even though the regions of validity are the same for the reason stated above. The second problem is that of a plane wave normally incident upon a slab. Both approximations were found to have the same domain of validity for the complex field in this case.

7.3 ENTER THE INVERSE PROBLEM

The inverse problem introduces several different considerations for the Born and Rytov approximations. Wolf (1969) was the originator of the theory of using the Born approximation to obtain information about a scattering object from the scattered waves [21]. Basically, Wolf treated the integral equation for the scattered field as a spatial Fourier transform. So if the scattered fields were known, an inverse Fourier transform would yield the scattering object. Devaney (1981) formulated the inverse problem for the Rytov approximation [23]. However, because of the need to take the inverse Fourier transform of the phase, the measurement of the phase retrieval must be "unwrapped" to be made continuous. For one-dimension, phase unwrapping schemes exist. However, the problem of unwrapping the phase in more than one dimension is still an unsolved problem. So, for objects that aren't symmetric to the degree that the Fourier transform can be reduced to an one-dimensional transform, the Rytov approximation can't be used at the present. Optical experiments by Carter showed the complexity of describing even simple symmetric objects [27]. The only real distinguishing feature observed between the performance of the approximations in the experiments was that the Born method denoted the edge of a cylindrical bar a little better than the Rytov approximation. Kaveh et al. used the approximations in acoustic tomography to image gelatin cylinders immersed in water [30]. It was found that the Rytov approximation yielded quite detailed images whereas the Born's approximation images were useless. Phase measurement is still a problem in these measurements, but not as bad as in the optical case.

The acoustical problem had smaller velocity profiles from the surrounding medium than the optical problem. This would mean that there would more appreciable forward scattering where the Rytov approximation would be expected to perform better. However, for more than one dimensional direct object retrieval, the Rytov approximation simply can't be used. Even, with the one-dimensional case, the phase still has to be unwrapped adding another computational difficulty.

The work reviewed in this thesis show that the Born and Rytov approximations may have significantly different ranges of validity, depending upon the particular problem under consideration. Specifically, for certain problems the Rytov approximation has a greater range of validity than the Born approximation, whereas for other problems the opposite is true. And then, even in the same problem the range of validity of either method may vary considerably, depending on the particular field quantity being calculated.

The claims by Yura et al. and Brown that the regions of validity were the same were based on their definition of validity. These authors based their validity for the approximations upon the relative magnitudes of successive terms in the Born and Rytov approximations. Although this may be a sufficient condition for the Rytov approximation, it need not be a necessary condition as shown above.

8. CONCLUSIONS

The choice of an approximation is really dependent upon the problem being investigated. For many forward scattering problems, the Rytov method will probably give a better approximation based on the test problems (i.e. longer path lengths and stronger fluctuations) given above and the extra terms that are carried due to the exponential form as pointed out by Keller. However, if $|\nabla\psi|$ is large anywhere due to a local sharp change in the medium, Rytov's approximation can rapidly deteriorate over the first Born approximation. This situation has no quantitative justification at present. But evidence of this problem was possibly experienced in Carter's optical inverse problem of imaging a cylinder.

The Born method in other circumstances can exhibit advantages over the Rytov method. In the inverse problem, there is no concern about phase unwrapping as in the Rytov approximation. If the reflected field is of interest in a semi-transparent medium, Oristaglio showed that the Born expansion performed a little better in expressing the magnitude of the reflection coefficient. The Born approximation gave the exact angle of reflection, whereas the Rytov series did not. Also, if more than one plane wave is involved in the problem, the advantage of the Rytov approximation will be lost if the method is not applied to each wave separately and not to the total field. On the other hand, the Born expansion can be used without such knowledge, as the condition of its validity is applied to the total scattered field.

The distinguishing differences between the Born and Rytov approximations is less quantitative than qualitative. Concluding, for weak fluctuating media, the choice of perturbation that will account for some diffraction effects depends largely on the problem of interest.

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