THE ANALYSIS OF ARCHED TRUSSED BENTS

BY MOMENT AND THRUST DISTRIBUTION

by

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LIST OF SYMBOLS

L = Length of Truss

S = Stress in any Member

U = Strain Energy

A = Cross-Sectional Area

E = Young's Modulus of Elasticity

h = Thrust at any Point

δ = Distance from Neutral Point to Knee-brace Joint

M = Moment at any Point

I = Moment of Inertia

W = Loading on Any Member

Y = Horizontal Deflection

Δ = Unit Displacement

ϕ = Unit Rotation

J = Thrust Stiffness

r_h = Thrust Carry-over Factor

K = Moment Stiffness

r_m = Moment Carry-over Factor

d_t = Distance from Neutral Point to Thrust Line for Distributed Thrust

d_m = Distance from Neutral Point to Thrust Line for Distributed Moment

d_c = Distance from Neutral Point to Thrust Line for Carried Over Moment

I. INTRODUCTION

It is the usual practice for engineers, when analyzing certain types of structures, to make certain simplifying assumptions and approximations which will lead to fairly accurate results that may be used in design of these structures. The time saved and the accuracy of the results obtained for the most part justify the use of these approximate methods rather than the more time consuming classical analysis.

In recent years, new methods of analysis have been developed which have shortened the time and labor involved in the analysis of many statically indeterminate structures. These newer methods are, basically, systems of successive approximations, and if the processes are carried out far enough, "exact" solutions could be obtained without having to resort to the tedious classical method of solution.

In the analysis of trussed bents, one simplifying assumption often made was that the thrust was negligible and could be ignored. There was reason to believe, however, that this assumption was not always valid. It was with this in view that the investigation, Application of Moment Distribution of Trussed Bents [3] was undertaken by Joseph E. Spagnuolo on parallel chord trusses.

The results obtained showed that for some applications the thrust was negligible but in other cases, depending primarily on the end fixity of the columns, the thrust was an important factor and could not be neglected.

In the analysis of Fink Trussed Bents by Moment and Thrust Distribution,[11] by Vincent J. Vitagliano, it was found that since the bent approached an arch in configuration the thrust became very important and could not be neglected. This fact concerning arches was brought out by Professor Hardy Cross in his book Continuous Arches in Reinforced Concrete,[2] and so the analysis of Fink trussed bents was treated accordingly by his method of moment and thrust distribution about the joint, by Mr. Vitagliano. The primary difficulty of the analysis of arched trussed bents by moment and thrust distribution about the joint is the extremely slow convergence of the solution.

It is the purpose of this thesis to carry on the investigation of trussed bents started by Spagnuolo and Vitagliano, and to determine if moment and thrust distribution about the neutral point can be applied to Fink trussed bents and to see whether or not such a method is more practical than the conventional method of balancing moments and thrusts about a joint.

II. REVIEW OF LITERATURE

The method of Moment Distribution had been known and used by a few people prior to 1930, but it was not until 1932 that the method was formally published by Professor Hardy Cross, in the 1932 Transactions [1] of the American Society of Civil Engineers. Since then, many text books have been published, and many papers have been written on the subject, which have discussed the method in general and suggested certain refinements and short cuts which make the method more practical.

Moment Distribution has been applied to arches and arched structures, [1] but balancing moments only is not sufficient for a complete analysis in such applications, since large thrust forces are developed under the applied loads. For this reason the method was extended to cover thrust distribution as well as moment distribution. The method of moment and thrust distribution is discussed in detail by Professor Cross in his book, Continuous Frames in Reinforced Concrete. [2]

One of the main difficulties of applying moment and thrust distribution to the analysis of a structure is the large unbalanced moments created by the thrust distribution, and the large unbalanced thrusts due to the moment distribution, when balanced about the joint. This difficulty has been somewhat overcome by balancing moments and thrusts about a hypothetical point in the structure called the neutral point. The neutral point is defined as that point about which no unbalanced moment is created if the joint is translated; and no unbalanced thrust is created if the structure is rotated, about that point.

This method is explained in detail by Professor D. H. Pletta in his Notes on Multiple Arches. [7]

One type of problem which has been overlooked by most writers in the past, is the application of moment and thrust distribution to trussed bents. Professor L. C. Maugh, in his book, Statically Indeterminate Structures [5] appears to have done the most on this subject. He has, however, consistently ignored the effects of thrusts; and, the solutions obtained by his methods have shown a certain discrepancy from the solutions obtained from the classical methods.

In 1940, and in 1948, theses on the application of moment and thrust distribution to trussed bents with parallel chord trusses were written by Spagnuolo [3] and Rogers [4], respectively, and in 1950, a thesis on the application of moment and thrust distribution to a Fink trussed bent was written by Vitagliano. [11] It was found that moment distribution was satisfactory for girder type trussed bents, but both moment and thrust distribution was required as the bent approached more nearly to the configuration of an arch.

III. INVESTIGATION

A. General

The purpose of this investigation, as stated previously, is primarily to determine whether moment and thrust distribution about the neutral point of a multiple-aisled, trussed-bent could be done, and secondly, whether the method would be practical to use.

The investigation, therefore, has been carried out for a single aisled hinged-end Fink trussed-bent and a triple aisled hinged-end Fink trussed-bent, with a ratio of height to span of one-quarter. The ratio of area of web members to area of chord members was taken as one-half. The load chosen was a unit panel load. (See Plate I)

The reason for choosing the type of truss and loading was to correlate the results obtained with this method with results obtained by Mr. Vitagliano, who had previously obtained the solution for the same truss and loading by moment and thrust distribution about the joint.

An exact solution, by the least work method, has been worked out for both cases, and these reults have been compared with those obtained by the neutral point method.

Before a problem can be solved by moment and thrust distribution, certain constants must be found for the elements of the structure being analyzed. These constants are:

1. Truss fixed-end moment and thrust
2. Truss thrust stiffness factor
3. Truss thrust carry-over factor
4. Column thrust stiffness factor
5. Column thrust carry-over factor

6. Neutral point of trussed bent
7. Truss moment stiffness factor
8. Truss moment carry-over factor
9. Column moment stiffness factor
10. Column moment carry-over factor

The method of least work has been used to determine the fixed end moment and thrust constants for the truss, and the three moment theorem has been used for the column constants.

B. Moment and Thrust Distribution Constants

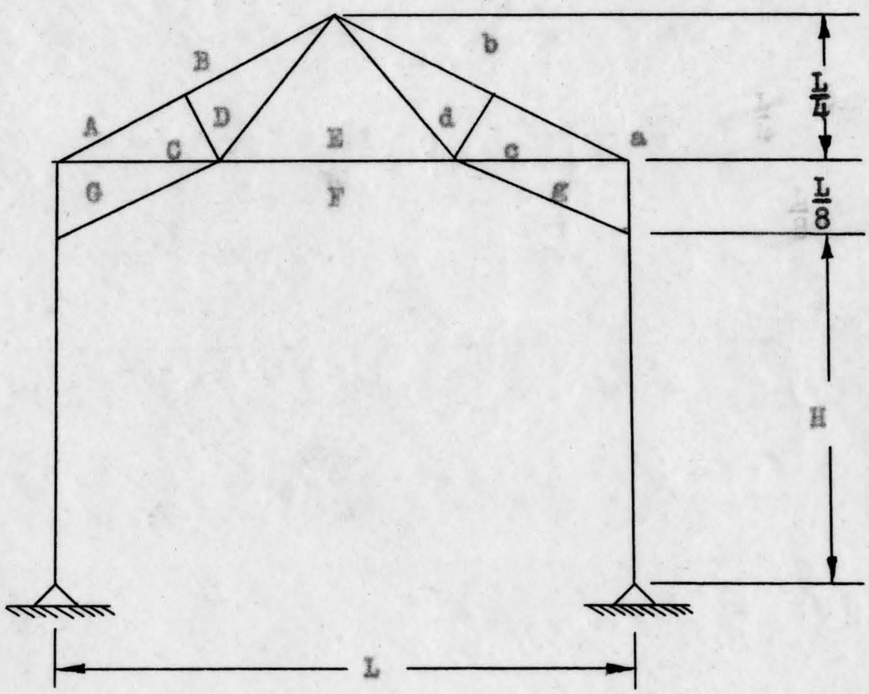
1. Fixed-End Moments And Fixed-End Thrusts

Fixed end moments and thrusts are those moments and thrusts exerted upon the truss by the supports, if the supports do not deflect or rotate under the applied loads on the truss.

For a fixed-end Fink truss, however, it is not possible to solve for these values of moment and thrust directly, since the point at which the thrust acts is not known. Therefore, it was found necessary to solve for fixed end reactions which could be combined into a thrust and moment acting at a particular point, to be determined later.

PLATE I

LENGTHS, AREAS AND $\frac{L'}{A}$ RATIOS OF TRUSS MEMBERS



MEMBER	L'	A	$\frac{L'}{A}$
AC	0.2795 L	a	0.2795 L/a
BD	0.2795 L	a	0.2795 L/a
bd	0.2795 L	a	0.2795 L/a
ac	0.2795 L	a	0.2795 L/a
CG	0.3125 L	a	0.3125 L/a
cg	0.3125 L	a	0.3125 L/a
EF	0.3750 L	a	0.3750 L/a
CD	0.1398 L	[illegible]a	0.0699 L/a
DE	0.3125 L	[illegible]a	0.1563 L/a
de	0.3125 L	[illegible]a	0.1563 L/a
cd	0.1398 L	[illegible]a	0.0699 L/a
GF	0.3366 L	a	0.3366 L/a
gf	0.3366 L	a	0.3366 L/a

A unit panel load was chosen for the external loading. The loading and reactions which are shown in Figure (I), were determined using the method of least work.

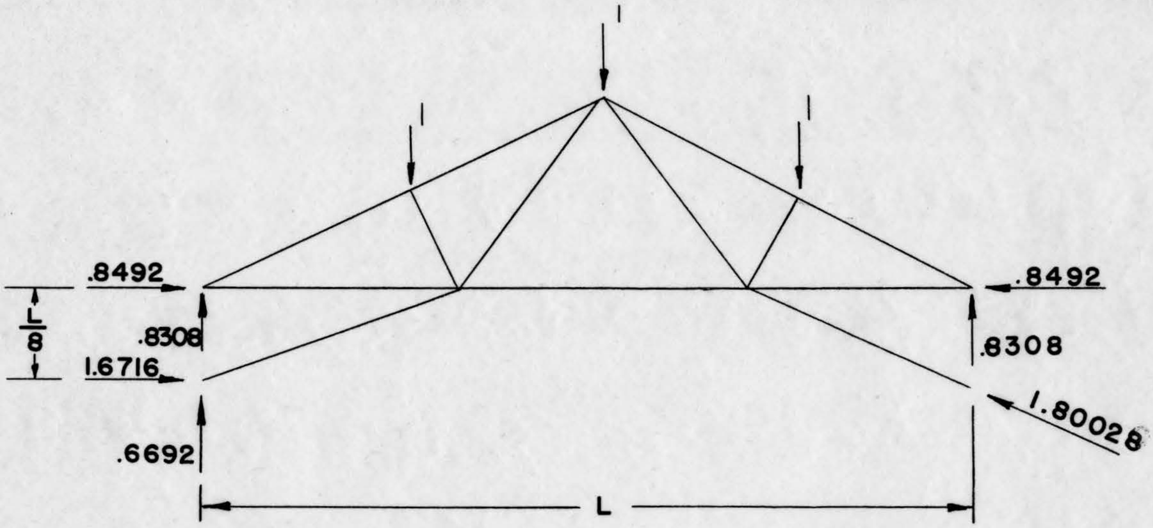


Figure I.

2. SOLUTION OF FIXED-END FINK TRUSS

Castigliano's principle of Least Work was used for the solution. This method is as follows:

I. Sufficient reactions were eliminated to make the structure statically determinate.

2. The stresses, S_o, of all members were then determined using the original loading.

3. Unit forces were applied, in turn, at all eliminated reactions and the corresponding stresses determined due the unit forces.

4. By the laws of superposition, the actual stresses in any member are

$$S = S_o + XS_x + YS_y + ZS_z \qquad (1)$$

Using Least Work, the strain energy of the system is the summation of strain energy of individual members, or:

$$U = \sum \frac{S^2 L}{2AE} \qquad (2)$$

Where U = strain energy of the system

S = stress in any member

L = length of that member

A = cross-sectional area of that member

E = Youngs Modulus of Elasticity

The partial derivative of the strain energy with respect to any force will give the deflection of the member at that point, in the direction of the force. For redundant forces, we know the deflections are zero, therefore:

$$\frac{\partial U}{\partial X} = 0 = \sum \frac{S_o S_x L}{A E} + \sum \frac{S_x^2 L}{A E} + \sum \frac{S_x S_y L}{A E} + \sum \frac{S_x S_z L}{A E} \quad (3)$$

$$\frac{\partial U}{\partial Y} = 0 = \sum \frac{S_o S_y L}{A E} + \sum \frac{S_x S_y L}{A E} + \sum \frac{S_y^2 L}{A E} + \sum \frac{S_y S_z L}{A E} \quad (4)$$

Using the above equations, with (1) and (2), a system of N linear equations in N unknowns is obtained, the solution of which is routine. Table A shows the calculations for a fixed-end Fink truss. The results are shown in Figure (1).

3. TRUSS THRUST STIFFNESS

Truss thrust stiffness is defined as the thrust required to produce a unit translation, without rotation, of one end of a truss, the other end held fixed.

Since the point of application of this thrust was now known, the truss reactions necessary to produce a unit translation, without rotation, were determined. These reactions are shown in Figure 2, and were determined by the principle of least work. (See Table B)

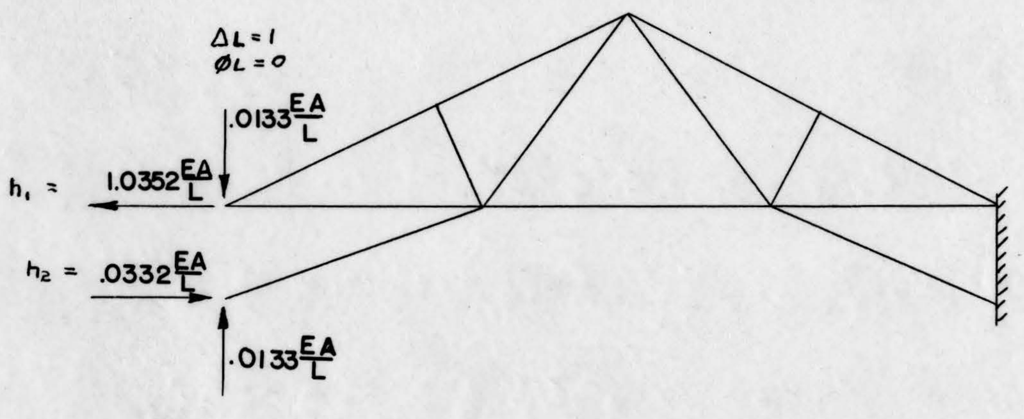


Figure 2

TABLE A

FIXED END REACTIONS FOR UNIT PANEL LOAD

MEMBER	$\frac{L'}{A} \cdot \frac{a}{L}$	S_o	S_x	S_y	S_z	$S_oS_x \frac{L'}{A} \cdot \frac{a}{L}$	$S_oS_y \frac{L'}{A} \cdot \frac{a}{L}$	$S_oS_z \frac{L'}{A} \cdot \frac{a}{L}$	$S_x \frac{2L'}{A} \cdot \frac{a}{L}$	$S_xS_y \frac{L'}{A} \cdot \frac{a}{L}$	$S_xS_z \frac{L'}{A} \cdot \frac{a}{L}$	$S_y \frac{2L'}{A} \cdot \frac{a}{L}$	$S_yS_z \frac{L'}{A} \cdot \frac{a}{L}$	$S_z \frac{2L'}{A} \cdot \frac{a}{L}$
AC	0.2795	-3.3541	0	-0.5709	-0.2595	0	0.5352	0.2433	0	0	0	0.0911	0.0414	0.0188
BD	0.2795	-2.9069	0	-0.5709	-0.2595	0	0.4638	0.2108	0	0	0	0.0911	0.0414	0.0188
bd	0.2795	-2.9069	0	-0.2595	-0.5709	0	0.2108	0.4638	0	0	0	0.0188	0.0414	0.0911
ac	0.2795	-3.3541	0	-0.2595	-0.5709	0	0.2433	0.5352	0	0	0	0.0188	0.0414	0.0911
CG	0.3125	3.0000	1.0000	0.5107	0.2321	0.9375	0.4788	0.2176	0.3125	0.1596	0.0725	0.0815	0.0370	0.0168
cg	0.3125	3.0000	1.0000	1.1606	-0.4178	0.9375	1.0881	-0.3917	0.3125	0.3627	-0.1306	0.4209	-0.1515	0.0546
EF	0.3750	2.0000	1.0000	1.1606	0.2321	0.7500	0.8705	0.1741	0.3750	0.4352	0.0870	0.5051	0.1010	0.0202
CD	0.0699	-0.8944	0	0	0	0	0	0	0	0	0	0	0	0
DE	0.1563	1.0000	0	0.4642	0	0	0.0726	0	0	0	0	0.0337	0	0
de	0.1563	1.0000	0	0	0.4642	0	0	0.0726	0	0	0	0	0	0.0337
cd	0.0699	-0.8944	0	0	0	0	0	0	0	0	0	0	0	0
GF	0.3366	0	0	1.0000	0	0	0	0	0	0	0	0.3366	0	0
gF	0.3366	0	0	0	1.0000	0	0	0	0	0	0	0	0	0.3366
					Σ	2.6250	3.9631	1.5257	1.0000	0.9575	0.0289	1.5976	0.1521	0.6817

TABLE B

TRUSS THRUST STIFFNESS

MEMBER	$\frac{L'}{A} \cdot \frac{a}{L}$	S_x	S_y	S_z	$S_x^2 \frac{L'}{A} \cdot \frac{a}{L}$	$S_x S_y \frac{L'}{A} \cdot \frac{a}{L}$	$S_x S_z \frac{L'}{A} \cdot \frac{a}{L}$	$S_y^2 \frac{L'}{A} \cdot \frac{a}{L}$	$S_y S_z \frac{L'}{A} \cdot \frac{a}{L}$	$S_z^2 \frac{L'}{A} \cdot \frac{a}{L}$
AC	0.2795	0	-0.6149	-0.2595	0	0	0	0.1057	0.0446	0.0188
BD	0.2795	0	-0.6149	-0.2595	0	0	0	0.1057	0.0446	0.0188
bd	0.2795	0	-0.2795	-0.5709	0	0	0	0.0218	0.0446	0.0911
ac	0.2795	0	-0.2795	-0.5709	0	0	0	0.0218	0.0446	0.0911
CG	0.3125	1.0000	0.5500	0.2321	0.3125	0.1719	0.0725	0.0945	0.0399	0.0168
cg	0.3125	1.0000	1.2500	-0.4178	0.3125	0.3906	-0.1306	0.4883	-0.1632	0.0546
EF	0.3750	1.0000	1.2500	0.2321	0.3750	0.4688	0.0870	0.5859	0.1088	0.0202
CD	0.0699	0	0	0	0	0	0	0	0	0
DE	0.1563	0	0.5000	0	0	0	0	0.0391	0	0
dE	0.1563	0	0	0.4642	0	0	0	0	0	0.0337
cd	0.0699	0	0	0	0	0	0	0	0	0
GF	0.3366	0	1.0770	0	0	0	0	0.3905	0	0
gf	0.3366	0	0	1.0000	0	0	0	0	0	0.3366
				$\sum$	1.0000	1.0313	0.0289	1.8533	1.1639	0.6817

Due to symmetry of the loading and of the structure it is obvious that the reactions at the opposite end of the truss are identical.

By a summation of forces in the horizontal direction of either end, the thrust stiffness was found to be $1.002 \frac{Ea}{L}$. The thrust carry-over factor was found to be unity.

4. TRUSS NEUTRAL POINT

It should be noted that if the thrust is applied at any arbitrarily chosen point there will be an unbalanced moment created about that point, unless that point is chosen to eliminate the moment. The method used to locate that point, which is called the neutral point for the truss, is as follows:

If a force of $1.002 \frac{Ea}{L}$ is applied to a rigid bracket as shown, a distance δ above the knee brace joint, so that the unbalanced moment created will just equal the unbalanced moment of the given system, the two systems will be equivalent, since they produce the same stress in each member.

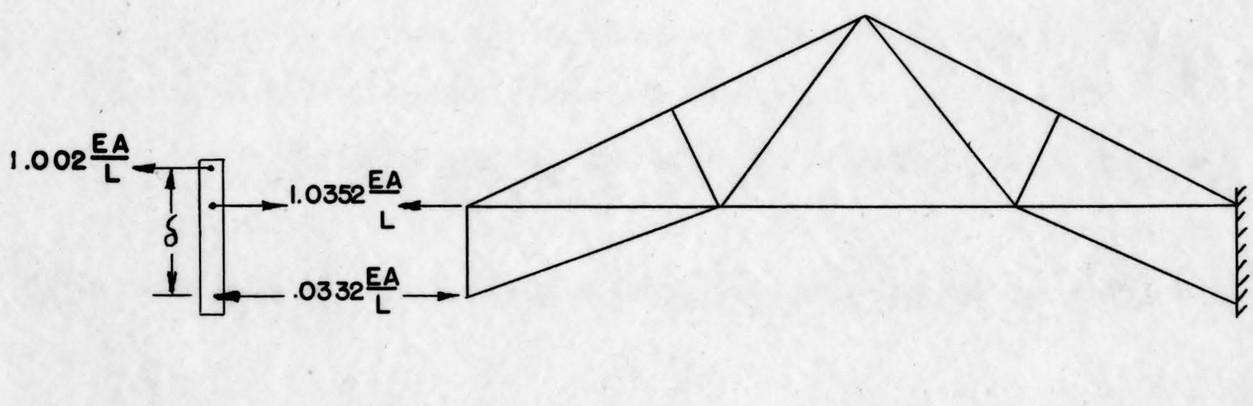


Figure 3

Taking a moment balance about the knee brace joint in each case

$$\sum F_h = 1.002 \frac{Ea}{L} + .0332 \frac{Ea}{L} - 1.0352 \frac{Ea}{L} = 0$$

$$\sum M = 1.0352 \frac{Ea}{L} \times \frac{L}{8} - 1.002 \frac{Ea}{L} \times \delta = 0$$

$$\delta = \frac{1.0352 \frac{Ea}{L} \times \frac{L}{8}}{1.002 \frac{Ea}{L}} = .1291417 \text{ L}$$

The neutral point then is .1291417 L above the knee brace. Therefore, if a load of 1.002 $\frac{Ea}{L}$ is applied a distance of .1291417 L above the knee brace joint, the truss will elongate a unit distance without rotation.

If a line were drawn connecting the points of application of the thrust, this would correspond to the thrust line for distributed thrust in the analysis of a continuous arch.

5. COLUMN THRUST STIFFNESS

The dimensions of the truss fixes the length of the column from the knee brace to the upper joint. This distance is $\frac{L}{8}$.

The lower portion of the column was chosen as K times the truss length, or KL. Therefore, the total column length is $(\frac{L}{8} + KL)$.

The thrust stiffness for the column was calculated using the three moment theorem. A pin connected base for the column was decided upon; however, the calculations for any type and restraint can be easily made.

The general three moment equation is:

$$\frac{M_1\ L_1}{I_1} + \frac{2\ M_2\ L_1}{I_1} + \frac{2\ M_2\ L_2}{I_2} + \frac{M_3\ L_2}{I_2} =$$

$$\frac{W_1\ L_1{}^3}{4\ I_1} + \frac{W_2\ L_2{}^3}{4\ I_2} + \frac{6\ E}{L_1}\ (Y_1 + Y_2) + \frac{6\ E}{L_2}\ (Y_3 - Y_2)$$

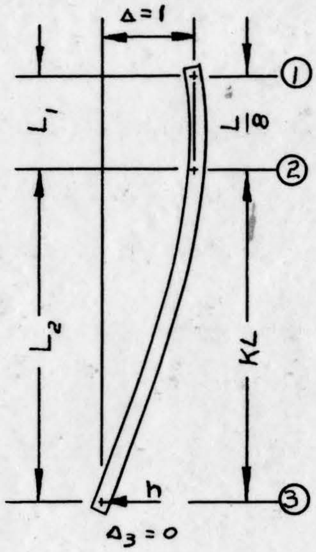


Figure 4

Where:

L = Length of truss.

$(\frac{L}{8} + KL)$ = Length of column.

h = Thrust at base of column produced by Δtop = 1.

M_1 = Moment at Point 1.

M_2 = Moment at point 2.

M_3 = Moment at point 3.

W_1 = Load on L_1.

W_2 = Load on L_2.

Y_1 = Horizontal deflection of Point 1.

Y_2 = Horizontal deflection of Point 2.

Y_3 = Horizontal deflection of Point 3.

If the moment of inertia of the column is constant and if the knee brace joint and top joint deflect equally, as they must do for unit translation, without rotation, the general equation reduces to :

$$\frac{2\ M_2}{I}\ (L_1 + L_2) = \frac{6\ E\ \Delta}{L_2}$$

$$\frac{2KLh}{I}\ (\frac{L}{8} + KL) = \frac{6\ E\ (1)}{KL}$$

Solving for h

$$h = \frac{24\ E\ I}{K^2\ L^3\ (8K + 1)}$$

For $L = 16$; $K = 3/4$; and $I = 10\ a$; the equation reduces to:

$$h = 0.01488Ea.$$

6. LOCATION OF THE NEUTRAL POINT FOR A SINGLE-AISLED FINK TRUSS BENT

The neutral point is defined as the point in a trussed bent about which no unbalanced moment is created if the joint is displaced from its equilibrium position. Conversely, a rotation of the bent about the neutral point will produce no total unbalanced thrust.

The neutral point of a single trussed bent has been determined and is 2.06627 feet above the knee brace. The thrust stiffness was found to be 1.002 $\frac{Ea.}{L}$

The neutral point for a pin ended column is located at the pinned end. The thrust stiffness was found to be o.01488 Ea. Using the individual neutral points and the individual thrust-stiffness, the neutral point for any combination of columns and bents can be found by the laws of statics.

When a truss and column are connected at a joint, the neutral point for the joint must be located so that a translation of the neutral point will produce no unbalanced moment.

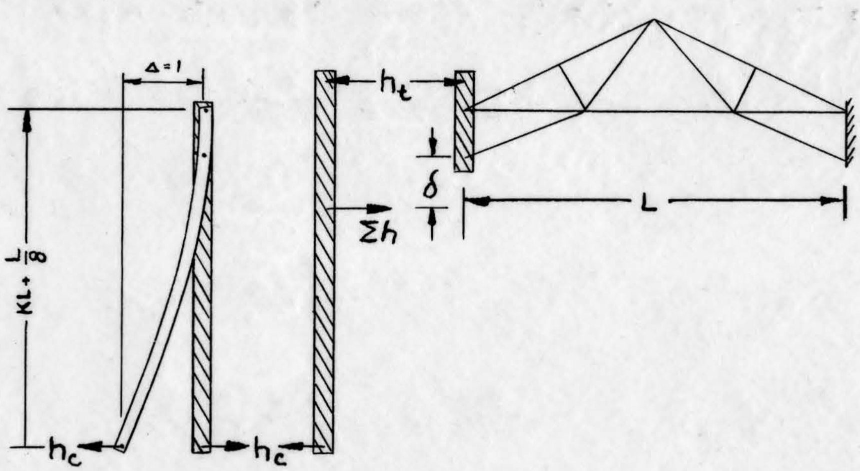


Figure 5

The thrust stiffness (h_c) for the column is .01488 Ea and the thrust-stiffness (h_t) for the truss is 1.002 $\frac{Ea}{L}$. The truss thrust-stiffness must be changed to 1.002 $\frac{Ea}{L} \times \frac{L}{16}$ = .06263 Ea so that the equations will be dimensionally equal.

Taking moments about the knee brace we have

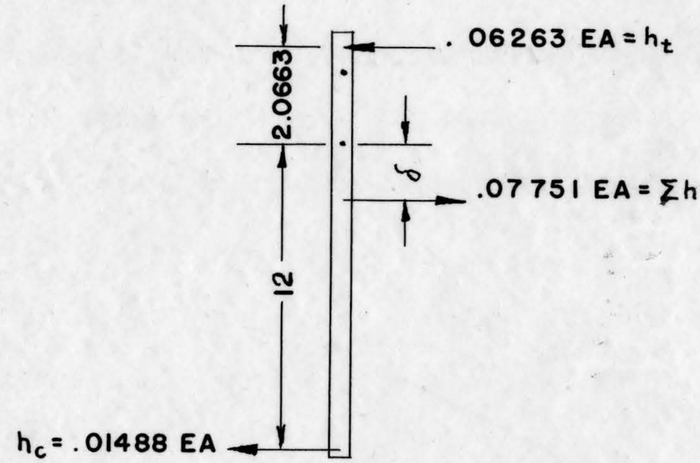


Figure 6

$$\Sigma M = .062625 \text{ Ea} \times 2.00627' - .01488 \text{ Ea} \times 12' + .077505 \text{ Ea} \times \delta = 0$$

δ = 0.6340 feet below the knee brace as shown.

7. TRUSS MOMENT STIFFNESS

Truss moment stiffness is defined as the moment necessary to rotate one end of the truss through a unit angle, without translation, while the other end remains fixed. The point about which the truss is rotated is the neutral point for the joint.

Figure 7 shows the free body diagram of the truss for determining the value of the moment stiffness. The force, T, must be located at the neutral point to prevent translation, while the moment, M, produces a unit rotation about the neutral point.

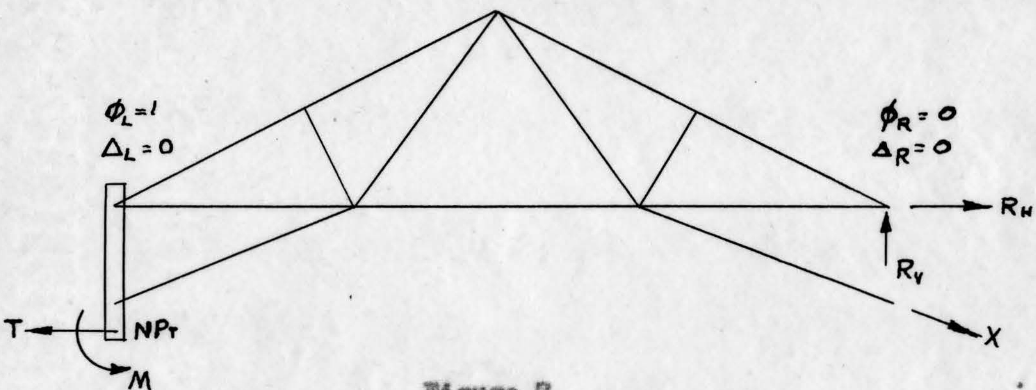


Figure 7

By using Castigliano's Principle, the total strain energy is

$$U = \sum \frac{S^2 L}{2 A E}$$

where

$$S = M S_m + T S_t + X S_x$$

S_m is the stress in any member of the statically determinate truss, due to a unit moment applied at the neutral point. S_t is the stress in any member due to a unit load applied at the neutral point in the direction of T, and S_x is the load in any member due to a unit load applied at X in the direction of X.

Taking partial derivatives with respect to M, T, and X respectively we obtain three equations in three unknowns whereby we arrive at the following equations:

$$\frac{\partial U}{\partial M} = 1 = M\sum\frac{S_m^2 L}{A E} + T\sum\frac{S_m S_t L}{A E} + X\sum\frac{S_m S_x L}{A E}$$

$$\frac{\partial U}{\partial T} = 0 = M\sum\frac{S_m S_t L}{A E} + T\sum\frac{S_t^2 L}{A E} + X\sum\frac{S_t S_x L}{A E}$$

$$\frac{\partial U}{\partial X} = 0 = M\sum\frac{S_m S_x L}{A E} + T\sum\frac{S_t S_x L}{A E} + X\sum\frac{S_x^2 L}{A E}$$

In TABLE C are found the values of the constants involved, with their solution. These values are shown in Figure 8.

Since the truss length chosen was 16 feet the length parameter will be eliminated giving:

M_L = .0.7831 Ea M_R = .3959 Ea

T_L = .16893 Ea T_R = .16893 Ea

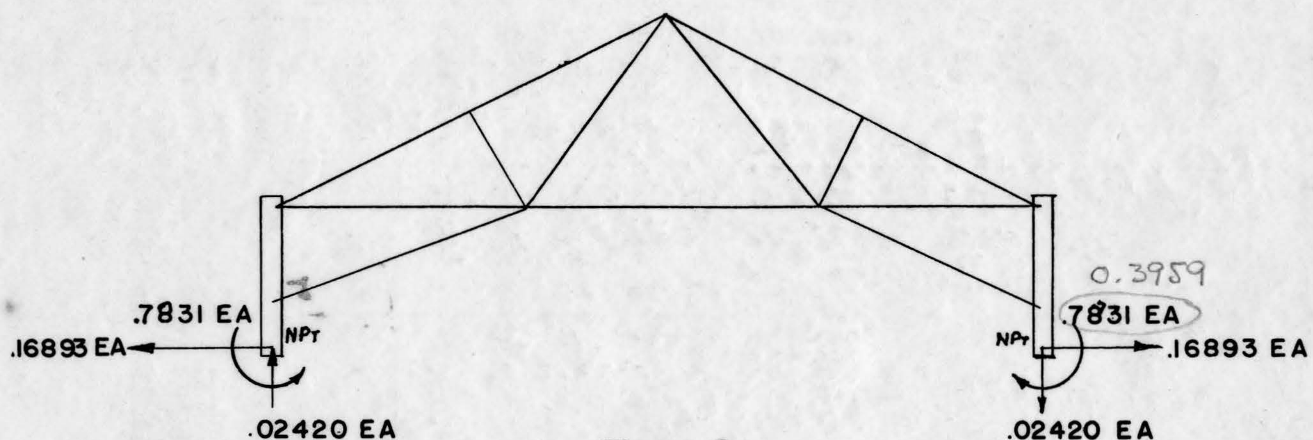


Figure 8

The force and moment at each end may be replaced by a single force displaced from the neutral point a distance necessary to give the same reactions as shown in Figure 9.

TABLE C

TRUSS MOMENT STIFFNESS AT THE NEUTRAL POINT

MEMBER	$\frac{L'}{A} \cdot \frac{a}{L}$	S_m	S_t	S_x	$S_m^2 \frac{L'}{A} \cdot \frac{a}{L}$	$S_m S_t \frac{L'}{A} \cdot \frac{a}{L}$	$S_m S_x \frac{L'}{A} \cdot \frac{a}{L}$	$S_t^2 \frac{L'}{A} \cdot \frac{a}{L}$	$S_t S_x \frac{L'}{A} \cdot \frac{a}{L}$	$S_x^2 \frac{L'}{A} \frac{a}{L}$
AC	0.2795	0.3074	- .8098	-0.2595	0.0264	-0.0696	-0.0223	0.1833	0.0587	0.0188
BD	0.2795	.3074	- .8098	-0.2595	0.0264	-0.0696	-0.0223	0.1833	0.0587	0.0188
bd	0.2795	.1397	- .3681	- .5709	0.0055	-0.0144	-0.0223	0.0379	0.0587	0.0911
ac	0.2795	.1397	- .3681	- .5709	0.0055	-0.0144	-0.0223	0.0379	0.0587	0.0911
CG	0.3125	.2250	.4074	.2321	0.0158	0.0286	0.0163	0.0519	0.0295	0.0168
cg	0.3125	-0.1250	1.3293	- .4178	0.0049	-0.0519	0.0162	0.5522	-0.1736	0.0546
EF	0.3750	-0.1250	1.3293	.2321	0.0059	-0.0623	-0.0109	0.6627	0.1157	0.0202
CD	0.0699	0	0	0	0	0	0	0	0	0
DE	0.1563	-.2500	0.6585	0	0.0098	- .0257	0	0.0678	0	0
de	0.1563	0	0	0.4642	0	0	0	0	0	0.0337
cd	0.0699	0	0	0	0	0	0	0	0	0
GF	0.3366	-0.5385	1.4185	0	0.0976	-.2571	0	0.6773	0	0
gF	0.3366	0	0	1.0000	0	0	0	0	0	0.3366
				Σ	0.1978	-0.5364	-0.0675	2.4543	0.2064	0.6817

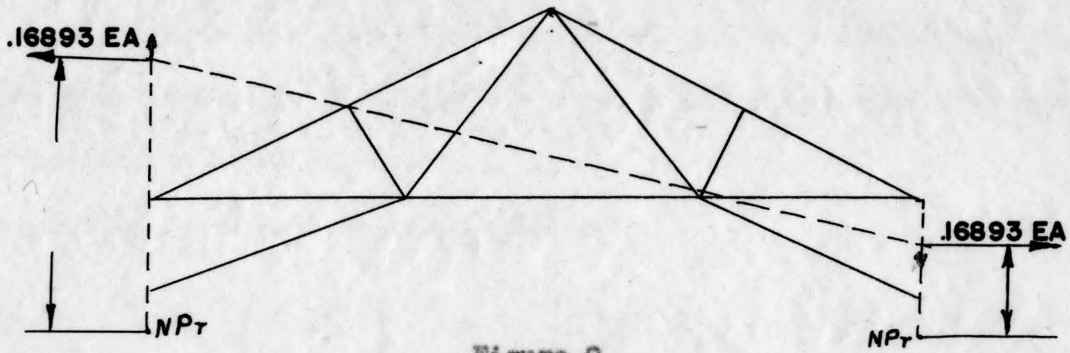


Figure 9

This is essentially a two force member and the line connecting the two forces corresponds to the thrust line for distributed moments of continuous arch analysis.

8. TRUSS MOMENT CARRY-OVER FACTOR

The carry-over factor for moment is defined as the ratio of the moment at the fixed end of the truss to that at the end which is rotated through a unit angle. The value of carry-over for the truss used is -0.50573.

9. COLUMN MOMENT STIFFNESS

The column moment stiffness is the moment necessary to produce a unit rotation, without translation, of the top of the column about the neutral point, with the base of the column pinned.

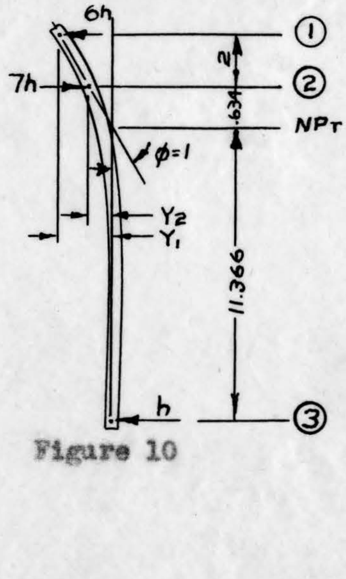


Figure 10

The theorem of three moments will be used to calculate the moment stiffness since the only "loads" on the column are essentially support translations, assuming points 1, 2, and 3 are supports. In this case the general three moment equation reduces to:

$$2M_2 \left(\frac{L_1}{I_1} + \frac{L_2}{I_2}\right) = 6E \left[\left(\frac{Y_1 - Y_2}{L_1}\right) - \left(\frac{Y_2 - Y_3}{L_2}\right)\right]$$

Since the angle of rotation is unity and the moment of inertia is constant, the equation becomes:

$$\frac{2M_2}{I}(14) = 6E \left[\frac{(2.634 - .634)}{2} - \frac{(.634-0)}{12}\right]$$

$$28M_2 = 6EI\ (1-.05284)$$

assuming $I = 10a$

$$M2 = \frac{60}{28}\ Ea\ (.94716)$$

$$M_2 = 2.02963\ Ea.$$

Since the moment at the neutral point is $\frac{11.366}{12}$ as large as that at point 2, the required moment is

$$2.02963Ea \times \frac{11.366}{12} = 1.9224Ea$$

Since the bottom joint is pinned the carry-over factor for the column is zero.

10. SIGN CONVENTION

The sign convention used in working out the solutions to the cases discussed is shown in Figure (11).

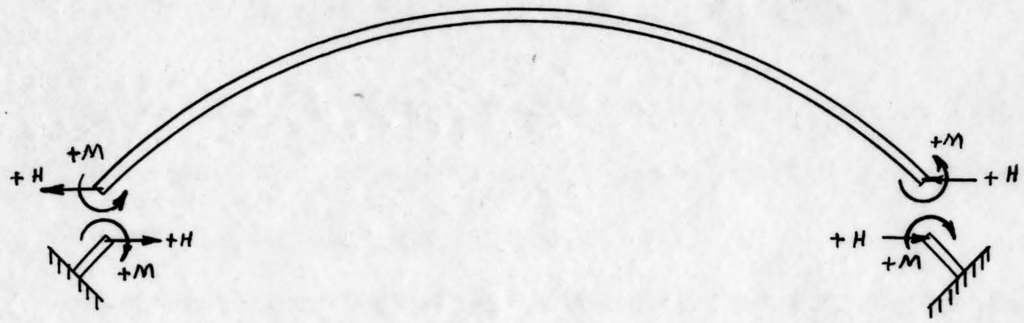


Figure 11

Moments that tend to rotate the joint clockwise are assumed positive and thrusts that tend to move the joint to the right are assumed positive.

C. Distribution of Moments and Thrusts

When an unbalanced thrust or moment exists at a joint, it must be distributed among the members framing into that joint in proportion to the relative stiffness of the members.

For an unbalanced thrust the amount distributed to any member is proportional to the thrust stiffness of that member divided by the sum of the thrust stiffness of all members framing into the joint.

Using the same symbols used in analysis of continuous arches the thrust stiffness would be (J) and the proportional part would be $\left(\frac{J}{\Sigma J}\right)$. The thrust carry-over factor would **also** be r_h.

The unbalanced moments are distributed in a similar manner, the symbol for moment stiffness is (K) and the proportional part is $\left(\frac{K}{\Sigma K}\right)$. The moment carry-over symbol is r_m.

The solution for a single-aisled hinged-end Fink trussed bent is shown in Table D. There are some additional symbols shown, which should be defined. These are:

d_t = Distance from neutral point to thrust line for distributed thrust.

d_m = Distance from neutral point to thrust line for distributed moment.

d_c = Distance from neutral point to thrust line for carried over moment.

The procedure used in the moment and thrust distribution is as follows:

(1) Determine the fixed-end moment (M_f) about the neutral point and fixed-end thrust (H_f) in the truss due to the applied loads or deformations.

(2) Distribute the unbalanced thrust at each joint to the connecting members in proportion to $\left(\frac{J}{\Sigma J}\right)$ and write down only that carried over to the far end of each member. The thrust carried over equals $(B_h)\left(-\frac{J}{\Sigma J}\right) = (-1)\left(-\frac{J}{\Sigma J}\right) = \left(+\frac{J}{\Sigma J}\right)$. Continue until all joints are balanced and find the total thrust distributed.

(3) Balancing thrusts will create an unbalanced moment about the neutral point equal to the distributed thrust times the distance from the thrust line for distributed thrust to the neutral point. Convert this distributed thrust into moment about the neutral point by multiplying by (d_t). Add this converted moment to the original fixed-end moment (M_f).

(4) Distribute the unbalanced moment about each neutral point to each of the connecting members in proportional to their individual $\left(\frac{K}{\Sigma K}\right)$, multiply this by the carry-over factor (r_m) and write down the moment carried over to the far end of each member. Continue this operation until all joints are balanced and find the total moment distributed.

(5) As in step 3 the distribution of moments create unbalanced thrusts. These moments are converted to thrusts by dividing by (d_c).

(6) Repeat steps (2) to (5) until the desired accuracy is obtained.

(7) When the desired accuracy is obtained the final moment and thrust can be obtained as shown in lines (20) through (36) of Table D.

TABLE D

MOMENT AND THRUST DISTRIBUTION

ABOUT THE NEUTRAL POINT

Lines	Operation	Truss	
1	d_t	2.70055	
2	d_m	4.6356	
3	d_c	2.3440	
4	r_h $\frac{J}{J}$ = (-1) $\frac{.062625}{.062625 \quad .01488}$	.8080	
5	r_m $\frac{K}{K}$ = - $\frac{3959}{.78307}$ $\frac{.78307}{1.9224 \quad .78307}$ =	.14634	
6	Fixed End Hp	-2.5208	2.5208
7	Distribute H		-2.0368
		..3910	
			.3161
		.2550	
			.2064
		.1665	
			.1348
		.1086	
			. 0880
		.0709	
			.0575
		.0463	
			.0376
		.0302	
			.0246
		.0197	
			.0160
		.0128	
			.0104
		.0084	
			.0068
		. 0055	
			.0044
		.0036	
			.0029
		.0023	
			.0019
		.0015	
			.0012
		.0010	
			.0008
		.0006	
			.0005
		.0004	
			.0003
		.0002	
			.0001
8	Total H	1.1265	-1.1265

TABLE D (Continued)

9	Convert to M. $H \times d_t$	3.0422	-3.0422
10	Fixed end M_f	-3.2980	3.2980
11	Total M_f at Neutral Point	- .2558	.2558
12	Distribute M		- .0374
		.0320	
			.0047
		.0007	
			.0000
13	Total M	.0327	-.0327
14	Convert to H $M \div d_c$	.0140	-.0140
7	Distribute H		
8	Total H	-.0063	-.0063
9	Convert to M $H \times d_t$	-.0170	.0170
12	Distribute M		
13	Total M	.0021	-.0021
14	Convert to H M d_c	.0009	-.0009
20	H due to Translation		
21	H due to H carried over (8)	1.1202	-1.1202
22	H due to H distributed	1.1202	-1.1202
23	Total H	2.2404	-2.2404
24	H due to rotation of Neutral Point.		
25	H due to M carried over (13)	.0149	- .0149
26	H due to M distributed	.0149	- .0149
27	Total H	.0298	- .0298
28	Fixed end H_f	-2.5208	2.5208
29	Final H (23 27 28)	-.2506	.2506
30	Moment at the Neutral Point due to		
31	H due to N.Pt. Translation $(23 \cdot d_t)$	6.0503	-6.0503
32	H due to M carried over $(25 \cdot d_c)$	.0349	- .0349
33	H due to M distributed $(26 \cdot d_m)$	.0691	- .0691
34	Fixed end M_f at Neutral Point	-3.2980	3.2980
35	Final M at Neutral Point	2.8563	-2.8563
36	Thrust at column base calculated from Final M = (35 $\div$ 11.366)	.2513	-.2513

D. Solution by Least Work

Advantage is taken of the fact that Mr. Vitagliano has already worked out the solution by least work. Only the results will be shown here. Since the structure is redundant to only the first degree, it is sufficient to determine the thrust at the base of the column. This value is

H = 0.2513.

E. Solution of Triple-Aisled Hinged-end Bent

All dimensions in tripled-aisled hinged-end bent will be taken the same as the dimensions for the single-aisled hinged-end bent. These dimensions are taken since it is desired to keep the routine work to a minimum. A diagram of the structure is shown in Figure 12, with a unit load at each panel point.

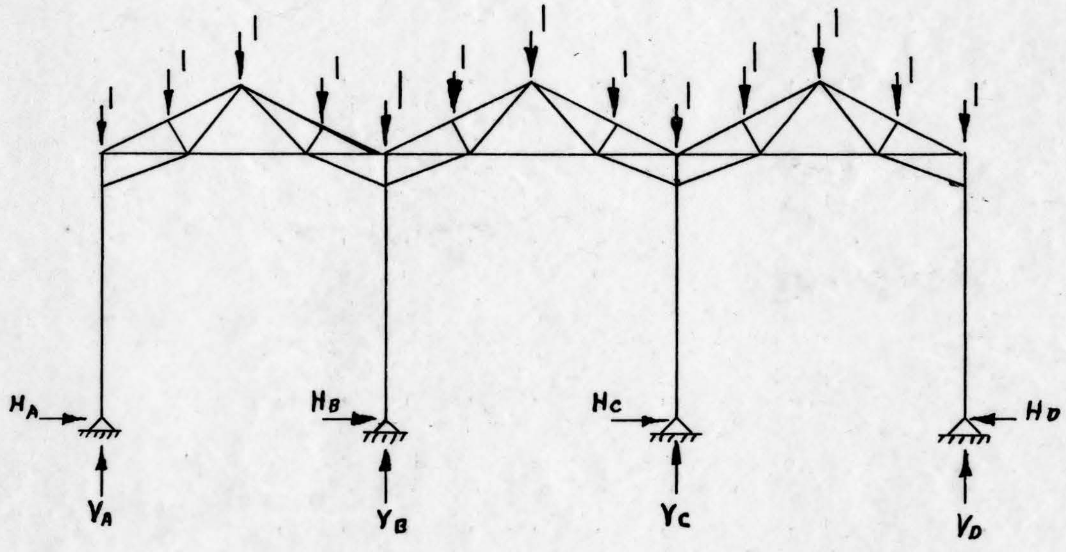


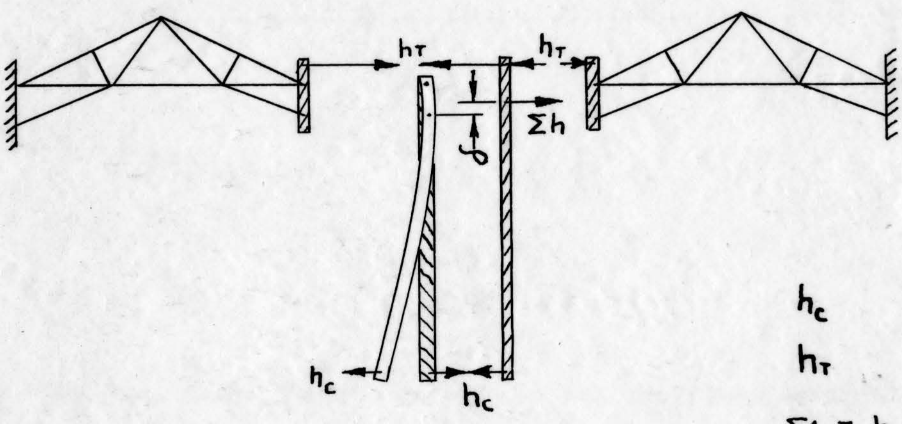
Figure 12

The solution by the neutral point method requires all the constants evaluated for the single-aisled bent, plus those constants necessary for the center columns,- since the neutral points for the center columns will not be the same as those for the end columns.

1. Location of the Neutral Points

The location of the neutral point for the end columns is the same as that determined for the single-aisled bent, which was 0.634 feet below the knee brace joint.

The location of the neutral point for the center column is determined as follows:



h_c

h_T

$\Sigma h = h_c + 2 h_T$

Figure 13

Taking a moment balance about the knee brace joint of Figure 13 gives:

$$\Sigma M = h_c \times 12' - 2\,h_t \times 2.06627 + (2\,h_t + h_c)\;\delta\,npt = 0$$

$$\Sigma M = .01488\ Ea \times 12' - .12525\ Ea \times 2.06627' + .14013\ Ea \times \delta = 0$$

$$\delta = .5734\ ft.$$

2. Truss Moment Stiffness

The moment stiffness for the outer end of the outer truss is the same as that calculated for the single-aisled bent as shown below:

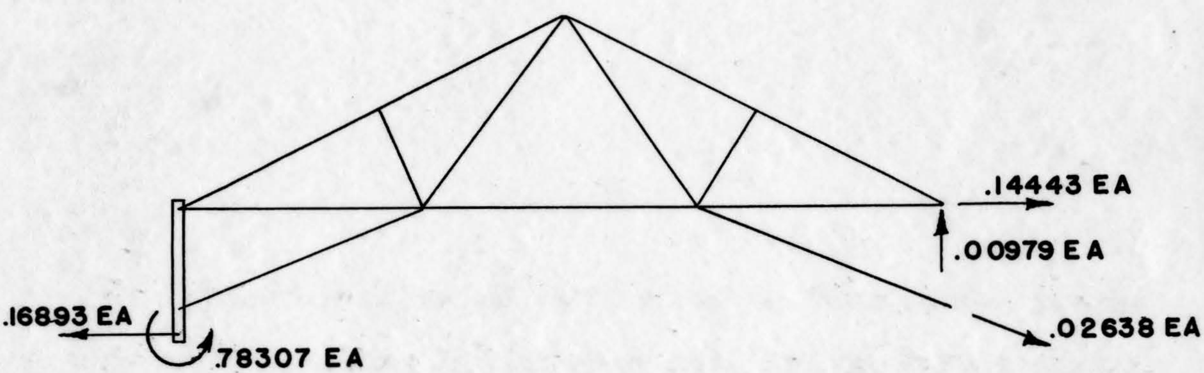


Figure 14

The moment carry-over factor is not the same, however, since the neutral point for the center column is different. The most satisfactory procedure for calculating the moment carry-over factor was found to be as follows: calculate moment stiffness of one end and fixed-end reactions at the opposite end as shown in Figure 14.

Then calculate the moment created about the neutral point caused by those same reactions. Then the moment carry-over factor is the moment created by the fixed-end reactions divided by the moment at the opposite end, which is created by the unit rotation. These values are shown below in Figure 15.

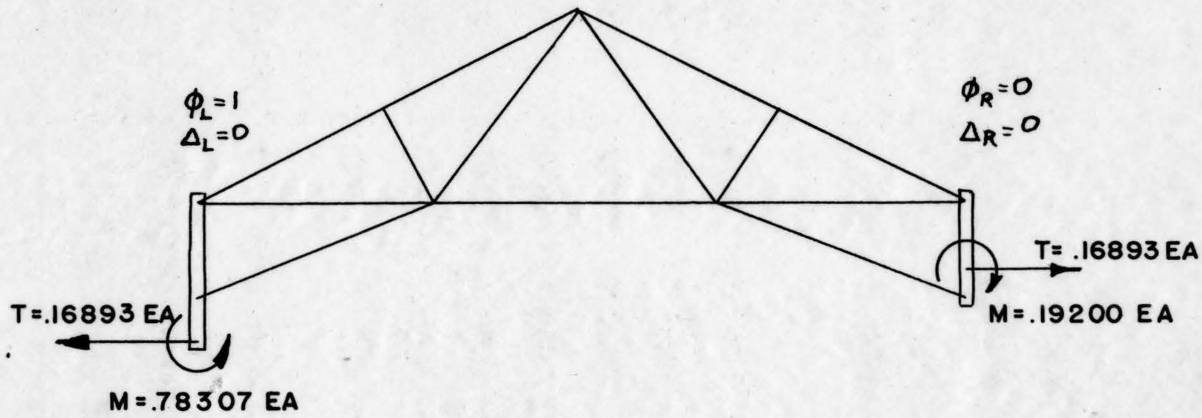


Figure 15

The calculation for moment stiffness at the inner columns is made in the same manner, the only difference being that the neutral point is in a different location. For the inner columns, the neutral point is 0.5734 feet above the knee brace, and the moment stiffness, as calculated in Table E, is shown in Figure 16.

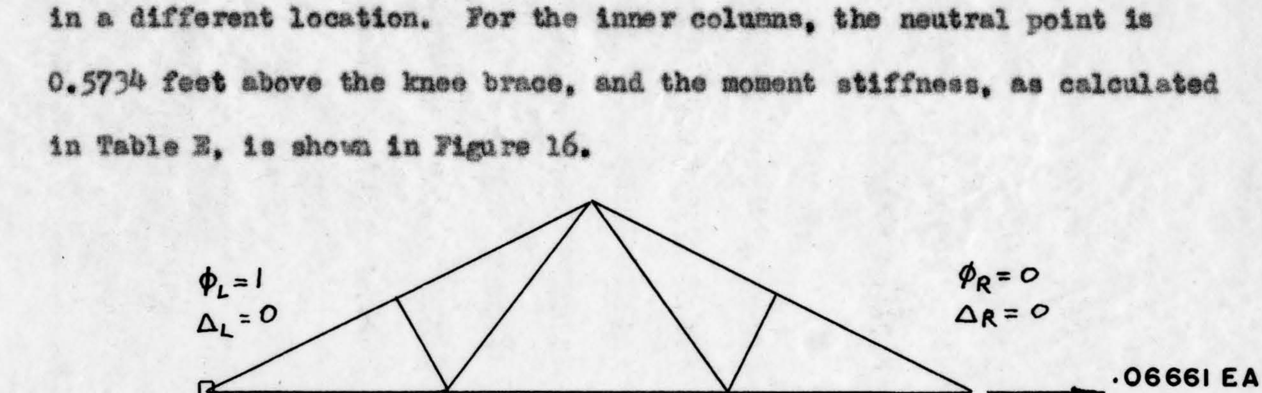


Figure 16

TABLE E

TRUSS MOMENT STIFFNESS NEUTRAL POINT = .5734

MEMBER	$\frac{L'}{A} \cdot \frac{a}{L}$	S_m	S_t	S_x	$S_m^2 \frac{L'}{A} \cdot \frac{a}{L}$	$S_m S_t \frac{L'}{A} \cdot \frac{a}{L}$	$S_m S_x \frac{L'}{A} \frac{a}{L}$	$S_t^2 \frac{L'}{A} \cdot \frac{a}{L}$	$S_t S_x \frac{L'}{A} \cdot \frac{a}{L}$	$S_x^2 \frac{L'}{A} \cdot \frac{a}{L}$
AC	0.2795	.3074	- .4386	- .2595	0.0264	- .0377	- .0223	.0538	.0318	.0188
BD	0.2795	.3074	- .4386	- .2595	0.0264	- .0377	- .0223	.0538	.0318	.0188
bd	0.2795	.1397	- .1994	- .5709	0.0055	- .0078	- .0223	.0111	.0318	.0911
ac	0.2795	.1397	- .1994	- .5709	0.0055	- .0078	- .0223	.0111	.0318	.0911
CG	0.3125	.2250	.6790	.2321	0.0158	.0477	.0163	.1441	.0493	.0168
cg	0.3125	- .1250	1.1783	- .4178	0.0049	- .0460	.0162	.4339	- .1538	.0546
EF	0.3750	- .1250	1.1783	.2321	0.0059	- .0552	- .0109	.5207	.1026	.0202
CD	0.0699	0	0	0	0	0	0	0	0	0
DE	0.1563	- .2500	.3567	0	0.0098	- .0140	0	.0199	0	0
de	0.1563	0	0	.4642	0	0	0	0	0	.0337
cd	0.0699	0	0	0	0	0	0	0	0	0
GF	0.3366	- .5385	.7682	0	0.0976	- .1393	0	.1987	0	0
gF	0.3366	0	0	1.0000	0	0	0	0	0	.3366
				Σ	.1978	- .2978	- .0675	1.4471	.1253	.6817

For the center truss the neutral point is 0.5734 feet above the knee-brace at each end. This gives a moment stiffness at each end of 0.46669 Ea and a moment carry-over factor of 0.17045 for each end. For the outer trusses the neutral point at the inner side is 0.5734 feet above the knee-brace joint, and at the outer side is 0.6340 feet below the knee-brace joint. The moment carry-over factor, from the inner to the outer side is 0.4123, and from the outer to the inner side, it is 0.2542, as calculated previously.

3. Column Moment Stiffness

The column moment stiffness has been determined using the three moment equation. Referring to Figure 17.

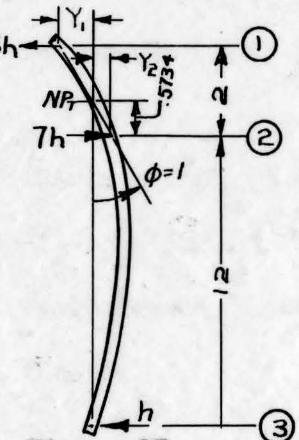


Figure 17

$$\frac{2 M_2}{I} (L_1 + L_2) = 6E \frac{(Y_1 - Y_2)}{L_1} + 6E \frac{(Y_3 - Y_2)}{L_2}$$

Substituting values and solving for M_2

$$M_2 = 2.24524 \text{ Ea}$$

The moment at point 1 is zero, therefore the moment at the neutral point is proportional to its distance from point 1, or

$$M_{npt} = 2.24524 \text{ Ea} \times \frac{1.4266}{2} = 1.60193 \text{ Ea}$$

The moment stiffness for the outer column, as previously calculated for the single-aisled bent is 1.9224 Ea.

4. Distribution of Moments and Thrusts

Distribution of moments and thrusts for the triple-aisled hinged-end Fink trussed bent is done in the same manner as was done for the single-aisled bent. At the outer columns it is exactly the same. At the inner columns, the only difference is an additional truss is present and therefore the distribution factors are smaller.

The calculations are omitted in Table F for individual distributions; only the totals are shown. This was done solely to shorten the table.

The distribution constants are shown at the top of Table F. Their derivation is given in previous paragraphs or they were obtained from Table D where applicable.

There are certain small discrepancies in the results as calculated in Table F; it is believed that these are due to accumulated errors resulting from carrying only four decimal places in the calculations, and not due to errors in the system.

TABLE F

MOMENT AND THRUST DISTRIBUTION

OF TRIPLE AISLED BENT ABOUT THE NEUTRAL POINT

Lines	Operation	Outer Truss		Inner Truss		Outer Truss	
1.	d_t	2.7006	1.4930	1.4930	1.4930	1.4930	2.7006
2.	d_m	4.6356	4.9898	4.9898	4.9898	4.9898	4.6356
3.	d_c	2.0574	1.1367	.8505	.8505	1.1367	2.0574
4.	$r_h \frac{J}{J}$	.8080	.4469	.4469	.4469	.4469	.8080
5.	$r_m \frac{K}{K}$	.0710	.0760	.0314	.0314	.0760	.0710
6.	Fixed end H_F	-2.5208	2.5208	-2.5208	2.5208	-2.5208	2.5208
7.	Distribute H						
8.	Total H	- .8389	-2.7146	.8378	- .8389	2.7135	.8378
9.	Convert to M H x d_t	-2.2655	-4.0546	1.2514	-1.2530	4.0530	2.2625
10.	M_F	-3.2953	.2532	- .2532	.2532	- .2532	3.2953
11.	Total M at Neutral Point	-5.5608	-3.8014	.9882	-.9998	3.7998	5.5578
12.	Distribute M						
13.	Total M	- .2369	- .4116	.0978	- .0978	.4114	.2366
14.	Convert to H M ÷ d_c	- .1151	- .3621	.1150	- .1150	.3619	.1150
7.	Distribute H						
8.	Total H	- .1400	- .2059	.1395	- .1400	.2057	.1395
9.	Convert to M H x d_t	- .3781	- .3074	.2083	- .2090	.3071	.3767
12.	Distribute M						
13.	Total M	- .0093	- .0274	.0038	- .0039	.0274	.0093
14.	Convert to H M ÷ d_c	- .0045	- .0241	.0045	- .0045	.0241	.0045
7.	Distribute H						
8.	Total H	- .0089	- .0106	.0103	- .0089	.0119	.0103
20.	H due to Translation						
21.	H due to M carried over	- .9878	-2.9311	.9876	- .9878	2.9311	.9876
22.	H due to H distributed	2.9311	.9878	.9878	- .9876	- .9876	-2.9311
23.	Total H	1.9433	-1.9433	1.9754	-1.9754	1.9435	-1.9435
24.	H due to rotation of N. Pt.						
25.	H due to M carried over	- .1196	- .3862	.1195	- .1195	.3860	.1195
26.	H due to M distributed	.3862	.1196	.1195	- .1195	- .1195	- .3860
27.	Total H	.2666	- .2666	.2390	- .2390	.2665	- .2665
28.	Fixed end H_F	-2.5208	2.5208	-2.5208	2.5208	-2.5208	2.5208
29.	Final H (23 27 28)	- .3109	.3109	- .3664	.3664	- .3108	.3108
30.	Moment at the N. Pt. due to						
31.	H due to N. Pt. translation	5.2480	-2.9013	2.9493	-2.9493	2.9017	-5.2485
32.	H due to M carried over	- .2461	- .4390	.1016	- .1016	.4388	.2459
33.	H due to M distributed	1.7903	.5968	.5963	- .5963	- .5963	-1.7893
34.	Fixed end M_F at N. Pt.	-3.2980	.2532	- .2532	.2532	- .2532	3.2980
35.	Final M at N. Pt.	3.4942	-2.4903	3.3940	-3.3940	2.4910	-3.4903

IV. COMPARISON OF RESULTS

The results obtained by moment and thrust distribution about the neutral point for the single-aisled Fink trussed bent check very closely with that obtained by both least work and moment and thrust distribution about the joint. It is obvious that the least work solution is the simpler and quicker of the three for a single-aisled bent. However, it is known that as the number of redundant reactions increase the least work method becomes too unwieldly for use. It is for these situations that this method is designed.

In comparing the neutral point method with the joint method it is believed that the neutral point method is superior for the following reasons, for the convergence is much more rapid for the neutral point method. The number of calculations which must be made are less and fewer numbers must be recorded. Since the truss analyzed was symmetric it was actually necessary to deal with only one-half of it; however, for clarity the author used the whole truss. A lesser advantage which might be important in volume production is the fact that calculations can be made on any modern calculator without clearing the machine each time.

The neutral point method is also believed to be superior for structures which are highly redundant. For example, in the analysis of the triple-aisled bent the convergence was complete after two cycles using the neutral point method. Mr. Vitagliano, using moment and thrust distribution about the joint, required 6 complete cycles for convergence.

A comparison of the results obtained by the neutral point method with the other two methods is given in Table C.

TABLE G

COMPARISON OF RESULTS

Method of Analysis	Single-Aisled Bent	Triple-Aisled Bent		
	h	h_a	h_b	$V_b = V_c$
Least Work Method	.2513	.3281	.0603	3.9145
Joint Method				
Maugh Column	.2602	.3169	.0593	3.8930
Spagnuolo Column	.2521	.2846	.0190	4.0057
Neutral Point Method	.2513	.3109	.0555	3.9141

V. CONCLUSIONS

From the preceding investigations, the following conclusions may be drawn:

1. That moment and thrust distribution about the neutral point can be applied to arched trussed bents.

2. That the neutral point can be determined easily from the thrust stiffness of the truss and column.

3. That the method is practical and can be carried out with comparative ease as compared with the classical methods. This conclusion is brought out strongly where a high degree of redundancy exists.

4. That the results obtained check very closely with those obtained by the method of least work or the method of moment and thrust distribution about the joint for both single aisled and triple-aisled bents. The calculations can be made in a much shorter time however, since the convergence is much more rapid about the neutral point than it is about the joint.

5. One draw back noted is that the final answers depend upon sums and differences of numbers which are fairly large compared to the final answer. This is not as serious, however, if a calculator is used instead of a slide rule.

VI. SUMMARY

The neutral point of a truss-column joint can be determined very easily, using the thrust stiffnesses. Once the neutral point is determined the moment and thrust distribution about the neutral point may be performed quite rapidly, since the thrusts and moments converge more rapidly about the neutral point than about the joint.

The actual process of distribution of moments and thrusts follows the pattern outlined by Professor D. H. Pletta in his Notes on Statically Indeterminate Structures [7].

VII. ACKNOWLEDGEMENT

Grateful appreciation is here expressed for the advice and suggestions given me in the preparation of this thesis by Professor D. H. Pletta. His generous help and constructive criticism proved to be a most valuable aid in accomplishing this work.

Sincere thanks are also expressed to   for the use of his work which had been done previously on the same subject and which has been used quite freely; and also to all those members of the Applied Mechanics Staff who have helped make this thesis possible.

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