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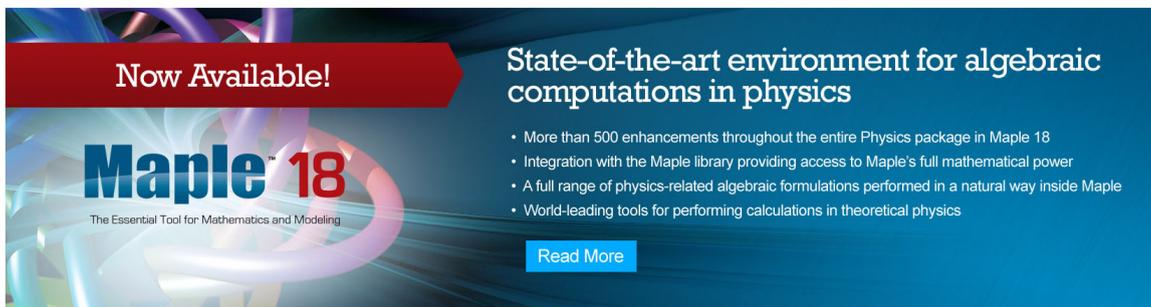
Horst Lange and P. F. Zweifel

Dissipation in Wigner–Poisson systems

Horst Lange
Fachbereich Mathematik, Universität Köln, Weyertal, 5000 Köln 41, Germany

P. F. Zweifel
Center for Transport Theory and Mathematical Physics, Virginia Polytechnic Institute and State University, Blacksburg, Virginia 24061-0435

The Wigner–Poisson (WP) system (or quantum Vlasov–Poisson system) is modified to include dissipative terms in the Hamiltonian. By utilizing the equivalence of the WP system to the Schrödinger–Poisson system, global existence and uniqueness are proved and regularity properties are deduced. The proof differs somewhat from that for the nondissipative case treated previously by Brezzi–Markowich and Illner *et al.*; in particular the Hille–Yosida Theorem is used since the linear evolution is not unitary, and a Liapunov function is introduced to replace the energy, which is not conserved.

I. INTRODUCTION

In some approximations, the magneto fluid dynamics of quantum plasmas can be described by adjoining to the usual Hamiltonian terms representing dissipation (or viscosity). We refer to a previous publication[1] and references therein for a review of progress on the dissipation-free version of the Wigner–Poisson (WP) system, which is a model for transport in collisonless charged quantum systems.

The WP system is nothing other than the well-known evolution equation for the Wigner distribution function[2] (also known as the quantum Liouville equation) in a mean-field approximation with a Coulomb force acting between particles.[3] This force may be attractive or repulsive.[2] As is suggested by the definition of the Wigner transform,[2,3] the WP system is equivalent to an infinite set of coupled, nonlinear Schrödinger equations (the SP system); this equivalence has been demonstrated by Markowich.[4] Because of this equivalence, the WP system may also be thought of as a statistical version of Hartree–Fock, a fact which was exploited extensively in Ref. 1. In the dissipative case considered here, the Schrödinger system is of the Ginzburg–Landau type.[5]

In Sec. II, we derive the evolution equation obeyed by the Wigner function in the presence of the dissipation. We do this by using the definition of the Wigner function,[3] as well as its Fourier transform.

The dissipation introduced in this paper is purely linear; the nonlinearity is the same as in Ref. 1. However, it is necessary to seek solutions in a different function space because of the dissipation. Thus, the proof that the nonlinearity is locally Lipschitz must be modified somewhat from that of Ref. 1. This result, along with the proof that the linear portion of the Hamiltonian generates a contraction semigroup (proved by utilizing the Hille–Yosida–Phillips theorem) implies local (in time) existence and uniqueness. All of this is the content of Sec. III—the result is valid for either attractive or repulsive potentials.

In Sec. IV, we go on to prove global existence. In Ref. 1, this was accomplished by taking advantage of energy conservation. In our model, energy is not conserved, but the same thing is accomplished by utilizing an energylike Liapunov functional, but this function is monotonic only for the repulsive case, so global existence has not been proved for attractive forces.

In Sec. V we make some concluding remarks concerning regularity and time-asymptotic behavior.

II. DERIVATION OF THE WIGNER EQUATION

We begin with Eq. (17) of Ref. 3a (but set $\hbar = m = 1$, then $p = v$):

$$\rho_W(x,v,t) = \sum_{m \in I} \lambda_m \int e^{ivz} \psi_m\left(x - \frac{z}{2}\right) \bar{\psi}_m\left(x + \frac{z}{2}\right) dz. \tag{2.1}$$

(Throughout this paper, all integrals and all function spaces are over $\mathbb{R}^3$.) As in Ref. 1, we suppose without loss of generality that $\lambda_m > 0$ ($\forall m \in I \subset \mathbb{N}$).

We recall[3] that the λ_m are determined by the initial value $\rho_W(x,v,0) := \rho_{WI}(x,v)$ or, alternatively, by the initial value of the density matrix ρ; ρ_W is the Wigner transform of ρ.[3] Here we choose to specify ρ_{WI}. We consider the Fourier transform of ρ_W with respect to v to be the kernel of an integral operator A; then the λ_m are the eigenvalues of A corresponding to eigenvectors ϕ_m. [In Ref. 3(b) it has been discussed that arbitrary functions ρ_{WI} cannot serve as initial values for the Wigner equation—they must satisfy certain conditions which guarantee that ρ_{WI} is indeed the Wigner transform of a positive trace-class operator.] We have already stated that the λ_m are positive; we further assume the normalizations $\Sigma \lambda_m = 1$ and $\|\phi_m\|_{L^2} = 1$ ($\forall m$).

The ψ_m in Eq. (2.1) are solutions to the SP system of equations

$$i\partial_t \psi_m = -\tfrac{1}{2}\Delta \psi_m + V\psi_m + iH_1\psi_m, \tag{2.2a}$$

$$\psi_m(x,0) = \phi_m(x), \quad x \in \mathbb{R}^3, \tag{2.2b}$$

$$\Delta V = -\varepsilon n \quad (\varepsilon = \pm 1), \tag{2.2c}$$

$$n(x,t) = \sum_m \lambda_m |\psi_m|^2(x,t). \tag{2.2d}$$

Also, H_1 represents the dissipative part of the Hamiltonian which has been introduced in the present paper; we choose

$$H_1 = \alpha\Delta + \beta x^2 + \gamma, \tag{2.3}$$

with α, β, γ real constants. The specific form of H_1 was chosen to model kinetic friction (the first term), loss of energy to a reservoir (the second term), and static friction (the third term). Other authors have considered similar models.[6]

The Wigner equation in the absence of H_1 has the form

$$\partial_t \rho_W + v \cdot \nabla_x \rho_W(x,v,t) - i\Theta(V)\rho_W(x,v,t) = 0, \tag{2.4a}$$

where $\Theta(V)$ is the pseudo-differential operator with symbol

$$\text{sym } \Theta(V) = V(x + \tau/2) - V(x - \tau/2). \tag{2.4b}$$

For convenience, we abbreviate Eq. (2.4a) as

$$\partial_t \rho_W = W\rho_W. \tag{2.5}$$

We can now state the following.

Proposition 2.1: With H_1 as in (2.3), ρ_W obeys the evolution equation

$$\partial_t \rho_W = W\rho_W - \alpha(2v^2 - \tfrac{1}{2}\Delta_x)\rho_W + \beta(2x^2 - \tfrac{1}{2}\Delta_v)\rho_W + 2\gamma\rho_W. \tag{2.6}$$

We need the following lemma.

Lemma 2.2:

$$\rho_W = \sum_m \lambda_m \int e^{-ix\cdot\xi} \chi_m\left(v-\frac{\xi}{2}\right)\bar{\chi}_m\left(v+\frac{\xi}{2}\right)d\xi, \qquad (2.7a)$$

where χ_m is the Fourier transform of ψ_m:

$$\psi_m(x,t) = \left(\frac{1}{2\pi}\right)^{3/2}\int e^{ikx}\chi_m(k,t)dk. \qquad (2.7b)$$

Proof of Lemma 2.2: Substituting (2.7b) in (2.1) and integrating over k' gives $\delta(p-k/2-k'/2)$. Subsequent integration over k gives the stated result.

Proof of Proposition 2.1: Differentiating Eq. (2.1) and using (2.2) gives

$$\partial_t \rho_W = \frac{1}{i}\sum_m \lambda_m \int e^{ipz}\left[\bar{\psi}_m\left(x+\frac{z}{2}\right)(H_0+iH_1)\psi_m\left(x-\frac{z}{2}\right)-\psi_m\left(x-\frac{z}{2}\right)(H_0-iH_1)\bar{\psi}_m\left(x+\frac{z}{2}\right)\right]dz,$$

where we have abbreviated

$$H_0 = -\tfrac{1}{2}\Delta + V.$$

The terms involving H_0 lead to $W\rho_W$ exactly as in Ref. 5. Thus

$$\partial_t \rho_W = (W+2\gamma)\rho_W + 2\beta\sum_m \lambda_m \int e^{ivz}\left(x^2+\frac{z^2}{4}\right)\bar{\psi}_m\left(x+\frac{z}{2}\right)\psi_m\left(x-\frac{z}{2}\right)dz$$

$$+\alpha\sum_m \lambda_m \int e^{ivz}\left[\bar{\psi}_m\left(x+\frac{z}{2}\right)\Delta\psi_m\left(x-\frac{z}{2}\right)+\psi_m\left(x-\frac{z}{2}\right)\Delta\bar{\psi}_m\left(x+\frac{z}{2}\right)\right]dz. \qquad (2.8)$$

The term proportional to β reduces to

$$(2x^2-\tfrac{1}{2}\Delta_v)\rho_W; \qquad (2.9)$$

the term proportional to α is somewhat more complicated, but can be evaluated using the Lemma. We find it to be equal to

$$-(2p^2-\tfrac{1}{2}\Delta_x). \qquad (2.10)$$

Putting together (2.8)–(2.10) gives the proposition.

III. LOCAL EXISTENCE

As in Ref. 1, we prove existence (and uniqueness) for the SP system which implies the same for the WP equations. The notation is similar to that of Ref. 1—for convenience we summarize it below.

The spaces (all over $\mathbb{R}^3$) introduced in Ref. 1 were

$$X := \left\{\Gamma = (\gamma_m)_{m\in\mathbb{N}}; \gamma_m \in L^2 \forall m, \|\Gamma\|_X^2 = \sum_m \lambda_m \|\gamma_m\|_{L^2}^2 < \infty\right\}, \qquad (3.1a)$$

$$Y := \left\{\Gamma = (\gamma_m)_{m\in\mathbb{N}}; \gamma_m \in H^1 \forall m, \|\Gamma\|_Y^2 = \sum_m \lambda_m \|\gamma_m\|_{H^1}^2 < \infty\right\}, \qquad (3.1b)$$

$$Z := \left\{ \Gamma = (\gamma_m)_{m \in \mathbb{N}} ; \gamma_m \in H^2 \forall m, \|\Gamma\|_Z^2 = \sum_m \lambda_m \|\gamma_m\|_{H^2}^2 < \infty \right\}. \tag{3.1c}$$

Additionally, we shall need the spaces

$$\widetilde{X} := \{ \Gamma = (\gamma_m)_{m \in \mathbb{N}} ; \|\Gamma\|_{\widetilde{X}}^2 = \|x^2 \Gamma\|_X^2 + \|x \otimes \nabla \Gamma\|_X^2 < \infty \}, \tag{3.2}$$

where

$$\|x \otimes \nabla \Gamma\|_X^2 = \sum_m \lambda_m \sum_{j,k=1}^3 \|x_j \partial_{x_k} \psi_m\|_{L^2}^2,$$

and

$$\widetilde{Z} = Z \cap \widetilde{X}.$$

The norm in $\widetilde{Z}$ is the natural norm induced by the definitions of Z and $\widetilde{X}$. Furthermore, in all spaces, natural inner products compatible with the norms are subsumed.

Let us further define the operator

$$T : D(T) \subset X \to X$$

by $D(T) = \widetilde{Z}$ and $T(\Gamma) = ([-\frac{1}{2}\Delta + iH_1]\gamma_m)_{m \in \mathbb{N}}$. By abuse of notation we shall frequently use the same symbol for operators on sequences and on their components; thus,

$$\Delta\Gamma = (\Delta\gamma_m)_{m \in \mathbb{N}},$$

etc.

We now state the following.

Lemma 3.1: The operator $S = -iT$ generates a contraction semigroup in X for $\alpha \geqslant 0$, $\beta \leqslant 0$, $\gamma < 0$.

Proof: We use the Hille–Yosida theorem.[7] First, it is clear that $D(T)$ is dense in X. We need to prove that S is closed and

$$\|(\mu I + iT)^{-1}\|_X \leqslant 1/\mu, \quad \mu > 0. \tag{3.3}$$

Equation (3.3) follows from the estimate:

$$\mathrm{Re} \sum_m \lambda_m ([\mu I + iT]\gamma_m, \gamma_m)_{L^2} = \sum_m \lambda_m \left\{ \mu \int |\gamma_m|^2 \, dx + \mathrm{Re} \int \left[-\frac{i}{2}\Delta\gamma_m \cdot \overline{\gamma}_m - \alpha\Delta\gamma_m \cdot \overline{\gamma}_m \right. \right.$$

$$\left. \left. -\beta x^2 |\gamma_m|^2 - \gamma |\gamma_m|^2 \right] dx \right\} \geqslant \mu \|\Gamma\|_X^2.$$

The inequality above follows by partial integration (valid in H^2) and the sign assumptions on α, β, γ.

For the proof that S (and T) are closed we may assume without loss of generality that $\alpha, \beta \neq 0$, $\gamma = 0$. The desired property is implied from the following identity [which we first consider for functions $\Psi = (\psi_m)$, $\psi_m \in C_0^\infty(\mathbb{R}^3)$ only]:

$$\|S\Psi\|_X^2 = \left(\frac{1}{4} + \alpha^2\right) \|\Delta\Psi\|_X^2 + \beta^2 \|x^2\Psi\|_X^2 + 2\beta \, \mathrm{Re} \sum_m \lambda_m \int \Delta\psi_m \cdot x^2 \overline{\psi}_m \, dx$$

$$-\beta \, \mathrm{Im} \sum_m \lambda_m \int \Delta\psi_m \cdot x^2 \overline{\psi}_m \, dx. \tag{3.4}$$

The result now follows after partial integration, the use of Young's inequality in the form $2ab \leqslant \varepsilon a^2 + C_\varepsilon b^2$, and some density arguments.

Remark: Equation (3.4) also implies that $D(T) = D(S) = \{\Psi \in X \,|\, \Psi \in X^1,\ \Delta\Psi$ exists, $x \cdot \Psi \in X, S\Psi \in X\}$.

Next we have the following.

Lemma 3.2: We define the nonlinear portion of the SP problem as in Ref. 1:

$$J(\Psi)_m = V(\Psi)\psi_m,$$

where $V(\Psi)$ is the solution of (2.2c) and (2.2d). Then J is locally Lipschitz on $\widetilde{Z}$.

Proof: The property that J is locally Lipschitz on Z is proved in Ref. 1. The proof of the local Lipschitz property on $\widetilde{X}$ is similar.

Lemma 3.1 and 3.2 along with Theorems 1.2 and 1.7 of Ref. 7 give the following.

Theorem 3.3: Let the initial datum $\Phi = (\phi_m) \in \widetilde{Z}$ and $\varepsilon = \pm 1$. Then the SP system (2.2) has a unique solution $\Phi(t)$ on a time interval $S = [0, \tau]$ for some $\tau > 0$. The solution has the property

$$\Phi \in C(S; \widetilde{Z}) \cap C^1(S; X).$$

IV. GLOBAL EXISTENCE

The extension of the local existence theorem proved in Sec. III to a global theorem was accomplished in Ref. 1 from energy conservation arguments. In the present case, we do not have a conservation law, but we can work alternatively with a Liapunov functional which corresponds to the dissipativity of the SP or WP system we consider; this approach works out only for the repulsive case $\varepsilon = +1$ (which we assume now). Let $\delta > 0$ and

$$E(t) = E(t, \Psi) = \int \{ |\nabla\Psi|^2 + |\nabla V|^2 + \delta |\Psi|^2 \} dx. \tag{4.1}$$

Note that $\nabla V(\Psi)$ for a local solution Ψ of SP is in L^2 according to Lemma 3.4 of Ref. 1.

We now can state the following lemma.

Lemma 4.1: Let Ψ be the local solution of SP from Theorem 3.3 on $S = [0, \tau]$. Then for $t \in S$,

$$\frac{\partial E(t)}{\partial t} = -2\alpha \int |\Delta\Psi|^2 \, dx + 2(\delta - \delta\alpha) \int |\nabla\Psi|^2 \, dx + 2\delta\beta \int x^2 |\Psi|^2 dx$$

$$+ 2(\delta\gamma - 3\beta) \int |\Psi|^2 \, dx + 2\beta \int x^2 |\nabla\Psi|^2 \, dx - 4\alpha \int V(\Psi) |\nabla\Psi|^2 \, dx$$

$$- 2\alpha \int |\Psi|^4 dx + 4\beta \int x^2 V(\Psi) |\Psi|^2 \, dx + 4\gamma \int V(\Psi) |\Psi|^2. \tag{4.2}$$

Proof: Let us first remark that for any solution Ψ of SP according to Theorem 3.3 we have $\Psi \in \widetilde{Z}$. This implies that all terms on the right-hand side exist, and that all the calculations which lead to (4.2) are justified. We write SP in the form

$$\partial_t \psi_m = \frac{i}{2} \Delta\psi_m + H_1\psi_m - iV\psi_m, \quad \Delta V = -\sum_{m \in \mathbb{N}} \lambda_m |\psi_m|^2. \tag{4.3}$$

Then we set (after partial integration)

$$\frac{\partial}{\partial t} \int |\nabla\Psi|^2 \, dx = 2 \, \mathrm{Re} \, \sum \lambda_m \int \nabla\bar{\psi}_m \cdot \nabla\psi_{m,t} \, dx$$

$$= -2 \, \mathrm{Re} \, \sum \lambda_m \int \Delta\bar{\psi}_m \cdot \psi_{m,t} \, dx$$

$$= -2\alpha \sum \lambda_m \int |\Delta\psi_m|^2 \, dx - 2\beta \, \mathrm{Re} \, \sum \lambda_m \int \Delta\bar{\psi}_m \cdot x^2\psi_m \, dx$$

$$-2\gamma \, \mathrm{Re} \, \sum \lambda_m \int \Delta\bar{\psi}_m \cdot \psi_m \, dx + 2 \, \mathrm{Im} \, \sum \lambda_m \int V |\nabla\psi_m|^2 \, dx$$

$$+2 \, \mathrm{Im} \, \sum \lambda_m \int \psi_m \nabla V \cdot \nabla\bar{\psi}_m \, dx$$

$$= -2\alpha \int |\Delta\Psi|^2 \, dx + 2\beta \int x^2 |\nabla\Psi|^2 \, dx$$

$$+2\gamma \int |\nabla\Psi|^2 \, dx - 6\beta \int |\Psi|^2 \, dx + J_1, \qquad (4.4)$$

where

$$J_1 = 2 \, \mathrm{Im} \, \sum \lambda_m \int \psi_m \nabla V \cdot \nabla\bar{\psi}_m \, dx.$$

Furthermore

$$\frac{\partial}{\partial t} \int |\nabla V|^2 \, dx = -2 \int V\Delta V_t \, dx$$

$$= 4 \, \mathrm{Re} \, i \sum \lambda_m \int V\bar{\psi}_m \psi_{m,t} \, dx$$

$$= 2 \, \mathrm{Re} \, i \sum \lambda_m \int V\bar{\psi}_m \Delta\psi_m \, dx - 4\alpha \int V |\Delta\Psi|^2 \, dx$$

$$-2\alpha \sum \lambda_m \int \nabla V \cdot \nabla |\psi_m|^2 \, dx + 4\beta \int x^2 V |\Psi|^2 \, dx + 4\gamma \int V |\Psi|^2 \, dx$$

$$= -J_1 - 4\alpha \int V |\nabla\Psi|^2 \, dx - 2\alpha \int |\Psi|^4 \, dx + 4\beta \int x^2 V |\Psi|^2 \, dx$$

$$+4\gamma \int V |\Psi|^2 \, dx, \qquad (4.5)$$

$$\frac{\partial}{\partial\psi} \delta \int |\Psi|^2 \, dx = -2\delta\alpha \int |\nabla\Psi|^2 \, dx + 2\delta\beta \int x^2 |\Psi|^2 \, dx + 2\delta\gamma \int |\Psi|^2 \, dx. \qquad (4.6)$$

Adding up (4.4)–(4.6) gives the result.

The next Lemma expresses the existence of an *a priori* bound of the local solution of SP from Theorem 3.3.

Lemma 4.2: Let Ψ be as in Lemma 4.1, and let $\alpha \geqslant 0$, $\beta \leqslant 0$, $\gamma \leqslant 0$, $\Phi \in \tilde{Z}$. Then there is a constant C depending only on $\|\phi\|_{\tilde{Z}}$ and on τ such that

$$\|\Psi(t)\|_{\tilde{Z}} \leqslant C$$

for any $t \in S = [0, \tau]$.

Proof: By Lemma 4.1 the sign conditions on the coefficients α, β, γ imply that for an appropriate $\delta > 0$ one obtains

$$\frac{\partial}{\partial t} E(\Psi) \leqslant 0.$$

This gives an *a priori* bound of $\|\psi(t)\|_Y$ on S. Now we write SP in the form (see Lemma 3.1)

$$\partial_t \psi_m = (-iT)\psi_m - iJ(\Psi)_m. \tag{4.7}$$

Integrating (4.7) and using Lemma 3.1 gives

$$\Psi(t) = G(t)\phi - i \int_0^t G(t-s)J(\Psi(s))ds, \tag{4.8}$$

where $\{G(t)\}_{t \geqslant 0}$ is the contraction semigroup generated by $-iT$.

We now introduce a regularization procedure for $\Psi(t) = (\psi_m(t))_{m \in \mathbb{N}}$, e.g., a set of regularization functions $(\phi_\varepsilon(x))_{\varepsilon > 0}$ such that all regularity properties of Ψ are preserved for the convolution $\psi * \phi_\varepsilon$ and $T(\psi * \phi_\varepsilon) \in D(T)$ and $\psi * \phi_\varepsilon \to \psi$ in any L^p-norm. We then can write the formulas

$$x \otimes (\nabla\Psi * \phi_\varepsilon) = G(t)(x \otimes \nabla\Phi * \phi_\varepsilon) + \int_0^t G(t-s)\{\sigma\nabla \otimes \nabla(\Psi * \phi_\varepsilon)$$

$$+ 2i\beta x \otimes x(\Psi * \phi_\varepsilon) - ix \otimes (J(\Psi) * \nabla\phi_\varepsilon)\}ds, \tag{4.9}$$

where $\sigma = -1 + 2i\alpha$ and we have used the commutators $[x \otimes \nabla, \Delta] = -2\nabla \otimes \nabla$, $[x \cdot \nabla, x^2] = -2x \otimes x$. Letting $\varepsilon \to 0$ in (4.9) we obtain

$$x \otimes \nabla\Psi(t) = G(t)(x \otimes \nabla\Phi) + \int_0^t G(t-s)\{\sigma\nabla \otimes \nabla\Psi(s) + 2i\beta x \otimes x\Psi(s) - ix \otimes \nabla J(\Psi(s))\}ds. \tag{4.10}$$

Similarly by using $[\Delta, x^2] = 6I - 4x \otimes \nabla$ we can derive

$$\Delta\Psi(t) = G(t)\Delta\Phi - \int_0^t G(t-s)\{6i\beta\Psi(s)t - 4i\beta x \cdot \nabla\Psi(s) - i\Delta J(\Psi(s))\}ds, \tag{4.11}$$

and

$$x^2\Psi(t) = G(t)x^2\Phi + \int_0^t G(t-s)[3\sigma\Psi(s) - 2\sigma x \cdot \nabla\Psi(s) - ix^2J(\Psi(s))]ds. \tag{4.12}$$

By using Lemmas 3.3 and 3.9 of Ref. 1 and the boundedness of $\|\psi(x)\|_Y$ one easily sees that an application of formulas (4.10)–(4.12) will finish the proof of Lemma 4.2.

Theorem 4.3: Suppose $\Phi \in \widetilde{Z}$, $\alpha \geqslant 0, \beta \leqslant 0, \gamma \leqslant 0$. Then the Schrödinger–Poisson system SP has a unique global strong solution (Ψ, n, V) on $[0, \infty)$ such that

$$\Psi \in C([0, \infty); \widetilde{Z}) \cap C^1([0, \infty); X),$$

$$n, \Delta V \in C^1([0, \infty); L^1) \cap C([0, \infty); W^{2,1}),$$

$$V \in C([0, \infty); L^\infty(\mathbb{R}^3 \times [0, \infty))),$$

$$\Delta V \in C([0, \infty); L^2) \cap L^\infty([0, \infty); L^p) \quad (2 \leqslant p \leqslant \infty).$$

Proof: The existence and uniqueness follow from Lemma 4.2 in much the same way as Theorem 3.10 of Ref. 1 was proved.

The regularity properties of the solution are accomplished as in Ref. 1.

V. CONCLUDING REMARKS

As in Ref. 1, the existence of a unique solution to the SP system implies the same for the WP system. In particular, the following theorem is almost the same as Theorem 4.2 of Ref. 1.

Theorem 5.1: Assume that $\rho_{WI} \in L^2(\mathbb{R}^3 \times \mathbb{R}^3)$ and that the representation obtained from Eq. (2.1) by setting $t = 0$,

$$\rho_{WI}(x, v, 0) = \sum \lambda_m \int e^{ivt} \phi_m\left(x - \frac{z}{2}\right) \bar{\phi}_m\left(x + \frac{z}{2}\right) dz,$$

is valid with $\Phi = (\phi_m)_{m \in \mathbb{N}} \in \widetilde{Z}$. Then the evolution equation (2.6) has a unique global classical solution ρ_W satisfying

$$\rho_W \in C([0, \infty); L^2(\mathbb{R}^3 \times \mathbb{R}^3)) \cap L^\infty([0, \infty); L^s(\mathbb{R}^3 \times \mathbb{R}^3)), \quad 2 \leqslant s \leqslant \infty,$$

$$\partial_t \rho_W, v \cdot \nabla \rho_W, \Theta(V) \rho_W \in C([0, \infty); L^2(\mathbb{R}^3 \times \mathbb{R}^3)).$$

Additional regularity conditions on the number density n and the potential V can be read from Theorem 3.11 of Ref. 1.

Time decay estimates for the solutions to SP and WP cannot be obtained so easily as for the nondissipative case.[1] However, we do have a decay result for the functional

$$F(t) = \int (|\nabla \Psi|^2 + \delta |\Psi|^2) dx, \quad \delta > 0. \tag{5.1}$$

(Note this is not the same functional used in the global existence proof.)

Proposition 5.1: Let $\Psi(t)$ be the global solution of Theorem 4.3 with $\gamma < 0$ and with $F(t)$ as in Eq. (5.1). Then for δ sufficiently large, there is $c > 0$ such that $F(t) \leqslant F(0) e^{-ct}$.

Proof: As in the proof of Lemma 4.1 one obtains

$$\frac{\partial F(t)}{\partial t} = 2(\gamma - \delta\alpha) \int |\nabla \Psi|^2 \, dx + 2(\delta\gamma - \beta) \int |\Psi|^2 \, dx + J_1, \tag{5.2}$$

where

$$J_1 = 2 \, \mathrm{Im} \, \sum \lambda_m \int \psi_m \nabla V \cdot \nabla \bar{\psi}_m \, dx.$$

Now since $\|\nabla V(t)\|_{L^\infty}$ is bounded (see Theorem 3.3) we have for any $\varepsilon > 0$

$$J_1 \leqslant 2C_1 \|\psi\|_X \|\nabla\psi\|_X \leqslant \varepsilon\|\nabla\psi\|_X^2 + (C_1^2/\varepsilon)\|\psi\|_X^2.$$

Thus from (5.2) we obtain

$$\frac{\partial F}{\partial t} \leqslant [2(\gamma - \delta\alpha) + \varepsilon] \int |\nabla\Psi|^2\, dx + \left[2(\delta\gamma - \beta) + \frac{C_1^2}{\varepsilon}\right] \int |\Psi|^2\, dx.$$

Since $\gamma < 0$ by assumption there is a sufficiently large $\delta > 0$ and a sufficiently small $\varepsilon > 0$ such that

$$2(\gamma - \delta\alpha) + \varepsilon \leqslant -c, \quad 2(\delta\gamma - \beta) + C_1^2/\varepsilon \leqslant -c$$

for an appropriate $c > 0$; note that $\alpha \geqslant 0, \beta \leqslant 0$.

It would be interesting to prove the existence of global attractors for the dissipative WP systems. Such studies, as well as numerical calculations, are in progress.

ACKNOWLEDGMENTS

[1] R. Illner, H. Lange, and P. F. Zweifel, "Global Existence, Uniqueness and Asymptotic Behavior of Solutions of the Wigner-Poisson and Schrödinger-Poisson Systems," to be published in Math. Meth. Appl. Sci.
[2] E. Wigner, Phys. Rev. 40, 749 (1932).
[3] (a) P. F. Zweifel, "The Wigner Transform and the Wigner–Poisson System," Transport Theory and Statistical Physics 22, 459 (1993). (b) C. S. Bohun, R. Illner, and P. F. Zweifel, Le Matematiche XLVI, 429 (1991).
[4] P. A. Markowich, Math. Meth. Appl. Sci. 11, 459 (1989).
[5] D. R. Tilley and J. Tilley, Superfluidity and Superconductivity (Hilger, Bristol, 1986).
[6] J. Messer, Acta. Phys. Austriaca 50, 75 (1979); R. W. Hasse, J. Math. Phys. 16, 2005 (1975); K. Albrecht, Phys. Lett. B 56, 127 (1975); B. K. Skagerstam, ibid. 58, 21 (1975); H. Lange, Nonlin. Anal. Theory Meth. Appl. 9, 1115 (1985).
[7] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations (Springer-Verlag, Berlin, 1983).