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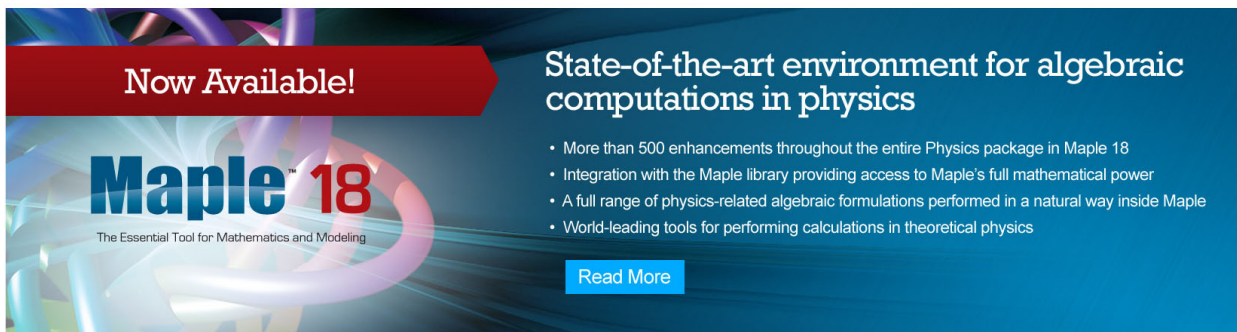
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Asymptotic behavior of Jost functions near resonance points for Wigner–von Neumann type potentials

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In this work are considered radial Schrödinger operators $-\psi'' + V(r)\psi = E\psi$, where $V(r) = a \sin br/r + W(r)$ with $W(r)$ bounded, $W(r) = O(r^{-2})$ at infinity (a,b real). The asymptotic behavior of the Jost function and the scattering matrix near the resonance point $E_0 = b^2/4$ are studied. If $|a| > |b|$, then this point may be an eigenvalue embedded in the continuous spectrum. The leading behavior of the Jost function for all values of a and b were determined. Somewhat surprisingly, situations were found where the Jost function becomes singular as $E \to E_0$ even if E_0 is an embedded eigenvalue. Moreover, it is found that the scattering matrix is always discontinuous at E_0 except in a few special cases. It is also shown that the asymptotics for the Jost function and the scattering matrix hold under weaker assumptions on $W(r)$, in particular an angular momentum term $l(l+1)r^{-2}$ may be incorporated into $W(r)$. The results are also applied to a whole line problem with a potential $V(x)$ such that $V(x) = 0$ for $x < 0$ and $V(x)$ of Wigner–von Neumann type for $x > 0$, and the behavior of the transmission and reflection coefficients as $E \to E_0$ is also studied.

I. INTRODUCTION

This paper is concerned with radial Schrödinger equations of the form

$$-\psi'' + V(r)\psi = E\psi, \quad \psi(0) = 0 \tag{1.1}$$

where

$$V(r) = a \sin br/r + W(r) \tag{1.2}$$

with nonzero real numbers a, b and $W(r)$ a real short-range perturbation satisfying

$$|W(r)| \leqslant C/(1+r)^2. \tag{1.3}$$

A particular case of (1.2) is the Wigner–von Neumann potential which is of the form $V(r) = -8 \sin 2r/r + O(r^{-2})$.[1] It has the property that it supports a positive eigenvalue embedded in the continuous spectrum. In general, if such an embedded eigenvalue occurs for (1.1), then it can only occur at $E = E_0 = b^2/4$. This follows from the large-r behavior of the solutions of (1.1), for it is only at E_0 that (1.1) has a fundamental set of solutions ψ_1, ψ_2 such that

$$\psi_1(r) = r^{-\rho}\cos(b/2)r + o(r^{-\rho})$$
$$(r \to \infty) \tag{1.4}$$
$$\psi_2(r) = r^\rho \sin(b/2)r + o(r^\rho)$$

where $\rho = |a/2b|$. At all the other positive energies there exist two linearly independent solutions with asymptotic form $e^{\pm ikr}, k = \sqrt{E}$. So we can define Jost solutions $f(k,r)$ such that

$$f(k,r)e^{-ikr} \to 1, \quad r \to \infty \quad (k^2 \neq E_0). \tag{1.5}$$

The point E_0 is called a resonance point for (1.1). It is a bound state if and only if $\rho > \frac{1}{2}$ and $\psi_1(0) = 0$. If $\rho \leqslant \frac{1}{2}$ and $\psi_1(0) = 0$ we call E_0 a half-bound state [in analogy to the case where V obeys $\int_0^\infty (1+r)|V(r)|dr < \infty$ and $E = 0$ is called a half-bound state if the solution satisfying $\psi(0) = 0$ is bounded at infinity but not square integrable].

In addition to $f(k,r)$ we introduce a solution $\varphi(k,r)$ that satisfies

$$\varphi(k,0) = 0, \quad \varphi'(k,0) = 1. \tag{1.6}$$

Then the Jost function, phase shift, and scattering matrix are given by

$$F(k) = W[f,\varphi] = f\varphi' - f'\varphi \quad (k \neq k_0), \tag{1.7}$$

$$\delta(k) = -\arg F(k), \tag{1.8}$$

and

$$S(k) = \overline{F}(k)/F(k) = e^{2i\delta(k)}. \tag{1.9}$$

The Jost solution and Jost function can be continued analytically to Im $k > 0$. The S matrix is defined by (1.9) only for real $k > 0$, $k \neq k_0$.

Furthermore, we define the phase jump Δ at k_0 by

$$\Delta = \lim_{\epsilon \to 0}(\delta(k_0 + \epsilon) - \delta(k_0 - \epsilon)). \tag{1.10}$$

Here the difference on the right is to be evaluated by going from $k_0 - \epsilon$ to $k_0 + \epsilon$ along a small semicircle of radius ϵ in the upper half k plane.

The primary goal of this paper is to analyze the behavior of $F(k)$ as $k \to k_0$ through values in Im $k \geqslant 0$. Table I gives an overview of the main results. Let $A = 2a/b$, $\gamma = |A|/2 (= 2\rho)$. In Table I an entry in the column labeled $F(k)$ means that $F(k)$ divided by the entry approaches a constant as $k \to k_0$ uniformly in $0 \leqslant \arg(k - k_0) \leqslant \pi$ [e.g., $F(k)(k - k_0)^{\gamma/2} \to c$, when $\psi_1(0) \neq 0$, $0 < \gamma < 1$]. The constant c is not real in general. The entries for $S(k)$ show the limits from the right and left of $S(k)$ as $k \to k_0$. The limits for $A < 0$ are just the negatives of those for $A > 0$. Note that $\gamma \leqslant 1$, $\psi_1(0) = 0$ corresponds to a half-bound state and $\gamma > 1$, $\psi_1(0) = 0$ corresponds to a bound state.

Table I shows that the behavior of $F(k)$ is quite different from what one might expect by analogy with the familiar case of potentials satisfying $(1 + r)V(r) \in L_1$. For such po-

TABLE I. Overview of the main results of this paper showing the behavior of $F(k)$, $S(k)$, and Δ as $k \to k_0$ (see text for details).

	$F(k)$	If $A>0$ $S(k_0^+)$	If $A>0$ $S(k_0^-)$	If $A<0$ $S(k_0^+)$	If $A<0$ $S(k_0^-)$	Δ
$0<\gamma<1$						
$\psi_1(0)\neq0$	$(k-k_0)^{-\gamma/2}$	$e^{-i(\pi/2)\gamma}$	$e^{i(\pi/2)\gamma}$	$-e^{-i(\pi/2)\gamma}$	$-e^{i(\pi/2)\gamma}$	$-\gamma\pi/2$
$\psi_1(0)=0$	$(k-k_0)^{\gamma/2}$	$-e^{i(\pi/2)\gamma}$	$-e^{-i(\pi/2)\gamma}$	$e^{i(\pi/2)\gamma}$	$e^{-i(\pi/2)\gamma}$	$\gamma\pi/2$
$\gamma=1$						
$\psi_1(0)\neq0$	$(k-k_0)^{-1/2}$	$-i$	i	i	$-i$	$-\pi/2$
$\psi_1(0)=0$	$(k-k_0)^{1/2}$. $\ln(k-k_0)$	$-i$	i	i	$-i$	$\pi/2$
$\gamma>1$						
$\psi_1(0)\neq0$	$(k-k_0)^{-\gamma/2}$	$e^{-i(\pi/2)\gamma}$	$e^{i(\pi/2)\gamma}$	$-e^{-i(\pi/2)\gamma}$	$-e^{i(\pi/2)\gamma}$	$-\gamma\pi/2$
$\psi_1(0)=0$	$(k-k_0)^{1-\gamma/2}$	$e^{-i(\pi/2)\gamma}$	$e^{i(\pi/2)\gamma}$	$-e^{-i(\pi/2)\gamma}$	$-e^{i(\pi/2)\gamma}$	$(1-\gamma/2)\pi$

tentials a half-bound state may occur at $k=0$ and then $F(k)$ vanishes linearly. If an angular momentum term is added to $V(r)$, then $k=0$ may be a bound state in which case $F(k)$ vanishes quadratically.[2] For potentials of type (1.1) the Jost function can never vanish linearly or quadratically as $k \to k_0$. In fact, when $\gamma>2$ and $\psi_1(0)=0$ then E_0 is a bound state and $F(k)$ *diverges* like $(k-k_0)^{1-\gamma/2}$.

In the literature the behavior of $F(k)$ near an embedded eigenvalue has been studied in a special case by Jain and Shastry in Ref. 3. We are afraid to say that, for reasons given below, our results do *not* agree with those of Jain and Shastry. Moreover, we note a discrepancy with a statement in Ref. 4 where for the same potentials as in Ref. 3 a phase jump of π is said to occur at an embedded eigenvalue. We find a jump of $-\pi$. In Ref. 3 a class of potentials with parameter $A=-8$, i.e., $\gamma=4$, were investigated both analytically and numerically (note that in Ref. 3 there is a factor $\frac{1}{2}$ in front of the Laplacian). Jain and Shastry found that $F(k)\to0$ and $S(k)\to1$ as $k \to k_0$, while we find that $F(k)\sim c(k-k_0)^{-1}$ and $S(k)\to-1$. The reason for this discrepancy lies in the fact that the two limits $k \to k_0$ and $r \to \infty$ are not interchangeable in this problem.

To make this point more precise we introduce a cut-off at $r=R$ and let $V_R(r)=V(r)$ if $r<R$, $V_R(r)=0$ if $r>R$. Then the Jost function associated with V_R, call it $F_R(k)$, is exactly given by $F_R(k)=W[e^{ikr},\varphi(k,r)]_{r=R}$. Hence one may ask the question

$$\text{Is } \lim_{R\to\infty}\lim_{k\to k_0}F_R(k)=\lim_{k\to k_0}\lim_{R\to\infty}F_R(k)? \quad (1.11)$$

Of course we must be prepared for the possibility that one or both sides are infinite. Note that the limits $\lim_{k\to k_0}F_R(k)=F_R(k_0)$ and $\lim_{R\to\infty}F_R(k)=F(k)$ ($k\neq k_0$) both exist. If $\psi_1(0)\neq0$, then $\varphi(k_0,r)$ behaves like $r^\rho \sin k_0 r$ as $r \to \infty$ by (1.4) and hence the left-hand side of (1.11) is infinite ($F_R(k_0)\sim\text{const } R^\rho$). The right-hand side diverges also, like $(k-k_0)^{-\gamma/2}$, as $k \to k_0$, and so we may view (1.11) as being correct. If $\psi_1(0)=0$ [i.e., $\varphi(k_0,r)$ decays] and $\gamma<2$ then both sides are zero and hence (1.11) holds true. But if $\psi_1(0)=0$ and $\gamma\geqslant2$, then the left-hand side

is zero while the right-hand side diverges if $\gamma>2$ (or goes to a nonzero constant if $\gamma=2$). This happens in the examples considered in Ref. 3. There the left-hand side of (1.11) is evaluated as $W[e^{ik_0r},\varphi(k_0,r)]$ for $r=\infty$ and zero is obtained. However, it is really the right-hand side of (1.11) we are interested in.

In their numerical work Jain and Shastry rely on the formula

$$F(k)=1+\int_0^\infty e^{ikr}V(r)\varphi(k,r)dr, \quad (1.12)$$

which is well established if $V\in L^1$ (Ref. 5) (in Ref. 3 the factor e^{-ikr} appears instead of e^{ikr} because the convention $\text{Im } k \leqslant 0$ is implicit there). Relation (1.12) is still valid for our class of potentials (a point that was questioned in Ref. 3). Indeed, it follows from

$$\varphi(k,r)=(2ik)^{-1}(\overline{F(k)}f(k,r)-F(k)\overline{f}(k,r))$$
$$=[|F(k)|/k]\sin(kr+\delta(k))+O(r^{-1})$$
$$(k\neq k_0) \quad (1.13)$$

that the integral in (1.12) converges conditionally and that the right-hand side of (1.12) represents $F(k)$ for $k\neq k_0$. The $O(r^{-1})$ remainder in (1.13) is established in Ref. 6 [Theorem XI.67 (b)] [or it can easily be deduced from the estimates for $f(k,r)$ given in the Appendix]. Now the trouble with using (1.12) numerically is, of course, that one has to introduce a cut-off at some large R and so one is effectively looking at $F_R(k)$ with R fixed and $k \to k_0$. If $F(k)$ blows up as $k \to k_0$ then $\varphi(k,r)$ is large for r large and k near k_0 by (1.13) and hence the weak fall-off of $V(r)$ is counteracted by the large amplitude of φ. All this suggests that a numerical approach to $F(k)$ via (1.12) may be difficult. However, at this point I would like to mention that my colleague George Hagedorn has done a direct numerical integration of the Schrödinger equation which to some extent has confirmed the result that the large-r oscillations of $\varphi(k,r)$ may indeed increase as $k \to k_0$.

Returning to the table we further observe that in general $S(k)$ will be discontinuous at k_0 (except when $\gamma=2n$,

$n = 1, 2, \ldots$). This phenomenon is possibly of interest in connection with results in the recent book by Pearson[7] about discontinuities of $S(k)$ at singular points of the continuous spectrum (see Ref. 7, Chap 13 and p. 505 for definitions and results). For the operators considered in this paper E_0 is not a singular point in the sense of Ref. 7 (see the Corollary to Theorem 10.2) yet $S(k)$ may be discontinuous at k_0.

The entries for the phase jump Δ show that neither the sign nor the value of Δ allow us to say, in general, which alternative, $\psi_1(0) = 0$ or $\psi_1(0) \neq 0$, prevails at E_0. However, $\Delta \geqslant 0$ implies $\psi_1(0) = 0$ and here are two values of γ, γ_1 and γ_2, such that $0 < \gamma_1 < 1$, $1 < \gamma_2 < 2$, which correspond to a given $\Delta \in (0, \pi/2)$ (if $\Delta = \pi/2$ or 0 then $\gamma = 1$ or 2 uniquely). If $\Delta < 0$ and $\Delta \neq -\pi/2$ then there exists two corresponding γ values, $\gamma_1 < \gamma_2$, such that $\psi_1(0) \neq 0$ for γ_1 and $\psi_1(0) = 0$ for γ_2. If $\Delta = -\pi/2$ then $\gamma = 1$ and $\psi_1(0) \neq 0$ uniquely. A knowledge of the phase jump Δ is relevant for the derivation of Levinson's theorem. Since a full discussion of this theorem also requires a study of $F(k)$ near $k = 0$ which, at present, we have not yet completed, we refrain here from formulating a Levinson theorem. We note in this connection that Levinson's theorem for *nonlocal* potentials which support embedded eigenvalues has been discussed by several authors (see Refs. 8 and 9, and references therein). In that case a phase jump of π can be associated with such an eigenvalue. As the present paper shows, this simple behavior does not carry over to the case of local potentials.

When $\gamma > 1$ and $\psi_1(0) \neq 0$ there is a result (due to Aronszajn) in Eastham–Kalf[10] (Lemma 2.7.1) which makes it clear why $F(k)$ has to diverge at $k = k_0$. Let $\rho(E)$ denote the spectral function associated with (1.1). Since $\gamma > 1$ we have that $\psi_1(r)$ is an eigenfunction for (1.1) corresponding to some boundary condition other than $\psi(0) = 0$. This fact implies that $J = \int_{-\infty}^{\infty} (E_0 - E)^{-2} d\rho(E) < \infty$. Since $d\rho(E)/dE = \pi^{-1} k |F(k)|^{-2}$ $(E \geqslant 0)$ by general principles[11] we see that $F(k)$ must diverge sufficiently fast as $k \to k_0$ in order to render J finite and, in fact, a behavior of the form $F(k) \sim c(k - k_0)^{\gamma/2}$ is sufficient for this.

As an aside we take a brief look at the equation $\ddot{x} + (E - V(t))x = 0$ which represents a mass-spring system with a time-dependent spring constant $k(t) = E - V(t)$. Consider the solution $x(t;E)$ defined by $x(0;E) = 0$, $\dot{x}(0;E) = 1$. Let $H(t;E) = \frac{1}{2}(\dot{x}^2 + k(t)x^2)$ be the total energy stored in the system. Computing the time average of $H(t;E)$ we obtain by using (1.13)

$$\overline{H}(E) \equiv \lim_{T \to \infty} T^{-1} \int_0^T H(t;E)dt$$

$$= \frac{1}{2}|F(k)|^2 \quad (k^2 = E \neq E_0). \quad (1.14)$$

Now suppose that the parameters of V are such that $\gamma > 2$ and $x(t;E_0) \to 0$ as $t \to \infty$ [i.e., $x(t;E_0)$ is a multiple of $\psi_1(t)$]. Then it is remarkable that $\overline{H}(E) \to \infty$ as $E \to E_0$ but $\overline{H}(E_0) = 0$. In other words, the system can absorb arbitrarily large amounts of energy as the parameter $E \to E_0$ but at E_0 itself if looses all its energy.

The methods used in this paper have recently been developed to a large extent in joint work with D. B. Hinton and K. Shaw.[12] There the goal was to determine the behavior of

the Titchmarsh–Weyl m coefficient, $m(E)$, near a resonance point. This was done by expressing $m(E)$ as a ratio of certain Jost functions. A special property of $m(E)$ [Im $m(E) > 0$ for Im $E > 0$] allowed us to obtain the asymptotics of $m(E)$ as $E \to E_0$ through *complex* values from corresponding results, involving *real* k only, for the Jost functions. In this paper we extend the results about the Jost functions to the complex plane. This is accomplished by means of an *a priori* bound, Lemma (2.1), followed by a Phragmén–Lindelöf-type argument. This strategy is the only way in which we are able to deal with complex k as $k \to k_0$ in this problem. It seems to be very difficult to get sufficient control on $F(k)$ directly by using the standard integral equations. The situation here is much different from the familiar one where $V \in L^1$ say. In Ref. 12 only the case $\gamma < 1$ was considered [without a term $W(r)$ but with a more general oscillatory term] since this case is of particular interest in connection with $m(E)$. It turns out that each of the cases $\gamma < 1$, $\gamma = 1$, and $\gamma > 1$ requires separate arguments although the general line of the approach is the same.

We should also mention that the scattering theory for potentials (1.2) has been worked out in detail (see, e.g., Refs. 13–15). However, basic questions like existence of wave operators and asymptotic completeness can be answered without making a detailed study of resonance points; it suffices to exclude these points when limits down to the real axis are being taken (see Refs. 14 and 15).

The paper is organized as follows. Section II is devoted to the analysis of $F(k)$ as $k \to k_0$. For simplicity and without loss we do the analysis for potentials of the form $A \sin 2r/(r+1) + W(r)$ with $A > 0$. The case $A < 0$ requires some modifications and it is fitting to discuss these together with other generalizations in Sec. III. In Sec. II we also consider the case of a general self-adjoint boundary condition at zero because this will be used in Sec. III. There we extend the results of Sec. II in various directions. We show how to treat local singularities in $W(r)$ and how to include more general oscillatory terms in place of $A \sin 2r/(r+1)$. We also consider a solvable example and use it to test our analysis. Finally, we apply the results to study the transmission and reflection coefficients of a whole line problem with a potential $V(x)$ such that $V(x) = 0$ for $x < 0$ and $V(x)$ of type (1.2) for $x > 0$. We find that the transmission coefficient approaches zero like $(k - k_0)^{\gamma/2}$ as $k \to k_0$ [irrespective of whether $\psi_1(0) = 0$ or $\psi_1(0) \neq 0$].

II. ASYMPTOTICS OF F(k)

We first convert (1.1) to a form which makes it easier to quote results from Ref. 12. Let $X(r) = \psi(\beta r)$, $\widetilde{E} = \beta^2 E$, $\beta = 2/|b|$. Then X obeys

$$-X'' + [A \sin 2r/(r+1)]X + \widetilde{W}(r)X = \widetilde{E}X, \quad (2.1)$$

where $\widetilde{W}(r) = Ar^{-1}(r+1)^{-1} \sin 2r + \beta^2 W(\beta r)$ obeys the bound (1.3). The resonance point for (2.1) is now at $\widetilde{E} = 1$. Reverting to the notation of Sec. I we study in this section the equation (1.1) with potential

$$V(r) = A \sin 2r/(r+1) + W(r) \quad (2.2)$$

near the resonance point $E_0 = 1$. We further assume $A > 0$,

deferring discussion of the case $A < 0$ to Sec. III. Moreover, we first suppose that $k < k_0 = 1$ and introduce

$$\epsilon = 2(1 - k), \quad \epsilon > 0. \tag{2.3}$$

The changes when $\epsilon < 0$ will be obvious. To indicate k dependence we use a subscript ϵ. However, it will sometimes be suppressed if no emphasis on ϵ is intended. We continue to use k and ϵ simultaneously. Following Ref. 12 we define functions h_1, h_2 by

$$\varphi(k,r) = kh_1(k,r)\cos kr + h_2(k,r)\sin kr, \tag{2.4}$$

$$\varphi'(k,r) = -k^2 h_1(k,r)\sin kr + kh_2(k,r)\cos kr. \tag{2.5}$$

Then h_1 and h_2 satisfy

$$h_1' + (V/2k)\sin(2kr)h_1 = (V/2k^2)(\cos 2kr - 1)h_2, \tag{2.6}$$

$$h_2' - (V/2k)\sin(2kr)h_2 = (V/2)(\cos 2kr + 1)h_1, \tag{2.7}$$

with $h_1(0) = 0$, $h_2(0) = k^{-1}$. Let

$$S_\epsilon(r) = \int_0^r \frac{V}{2k}\sin(2kt)dt \tag{2.8}$$

and

$$\hat{h}_1 = e^{S_\epsilon}h_1 \tag{2.9}$$

$$\hat{h}_2 = e^{-S_\epsilon}h_2. \tag{2.10}$$

In terms of $\hat{h}_1$ and $\hat{h}_2$ (2.6) and (2.7) become

$$\hat{h}_1' = (V/2k^2)(\cos 2kr - 1)e^{2S_\epsilon}\hat{h}_2, \tag{2.11}$$

$$\hat{h}_2' = (V/2)(\cos 2kr + 1)e^{-2S_\epsilon}\hat{h}_1 \tag{2.12}$$

with $\hat{h}_1(0) = 0$, $\hat{h}_2(0) = k^{-1}$. Equivalently

$$\hat{h}_{1,\epsilon} = \int_0^r \frac{V}{2k^2}(\cos 2kt - 1)e^{2S_\epsilon}\hat{h}_{2,\epsilon}(t)dt, \tag{2.13}$$

$$\hat{h}_{2,\epsilon} = k^{-1} + \int_0^r \frac{V}{2}(\cos 2kt + 1)e^{-2S_\epsilon}\hat{h}_{1,\epsilon}(t)dt. \tag{2.14}$$

On substituting (2.13) in (2.14) we get

$$\hat{h}_{2,\epsilon}(r) = k^{-1} + \int_0^r \frac{V}{2}(\cos 2kt + 1)e^{-2S_\epsilon}$$
$$\times \left(\int_0^t \frac{V}{2k^2}(\cos 2ks - 1)e^{2S_\epsilon}\hat{h}_{2,\epsilon}(s)ds\right)dt,$$
$$\equiv k^{-1} + (\mathcal{K}_\epsilon \hat{h}_{2,\epsilon})(r), \tag{2.15}$$

where the definition of the operator $\mathcal{K}_\epsilon$ is obvious. In Ref. 12 we went further and interchanged the order of integration on the right of (2.15). The ensuing integral equation is the basis for studying $\hat{h}_{2,\epsilon}(r)$. We will not repeat these steps here but merely quote some results from Ref. 12. We also leave it to the reader to check that the additional term $W(r)$ does not cause any difficulties; it can be handled by the estimates used in Ref. 12. It follows from (2.15) that the functions $\hat{h}_{k,\epsilon}(r)$, $h_{k,\epsilon}(r)$ $(k = 1,2)$, and $S_\epsilon(r)$ have finite limits as $r \to \infty$ which will be denoted by $\hat{h}_{k,\epsilon}(\infty)$, $h_{k,\epsilon}(\infty)$, and $S_\epsilon(\infty)$. This entails by using (1.7), (2.4), (2.5), (2.13), and (2.14) that

$$F(k) = k\hat{h}_{2,\epsilon}(\infty) - ik^2\hat{h}_{1,\epsilon}(\infty). \tag{2.16}$$

This expression could be transformed into (1.12) by using the relations (2.4)–(2.15) but this is not how we proceed. Instead we take (2.16) along with (2.13)–(2.15) as the basis for our analysis of $F(k)$. Using (2.15) we will first study $\hat{h}_{2,\epsilon}(\infty)$ as $\epsilon \to 0$ and then obtain information about $\hat{h}_{1,\epsilon}(\infty)$ from (2.13). Before we go into this analysis we list a few relations involving $S_\epsilon(r)$ that are needed later:

(i)

$$S_0(r) = \int_0^r \frac{V(t)}{2}\sin(2t)dt$$
$$= \frac{A}{4}\ln(r+1) - \frac{A}{4}\int_0^r \frac{\cos 4t}{t+1}dt + \frac{1}{2}$$
$$\times \int_0^r W(t)\sin(2t)dt,$$

$$\lim_{r \to \infty} r^{\mp \gamma/2}e^{\pm S_0(r)}$$
$$= \exp\left(\mp \frac{A}{4}\int_0^\infty \frac{\cos 4t}{t+1}dt \pm \frac{1}{2}\int_0^\infty W(t)\sin(2t)dt\right)$$
$$\equiv e^{\pm \sigma}, \tag{2.17}$$

(ii)

$$S_\epsilon(\infty) = -\frac{A}{4k}\ln \epsilon + \frac{A}{4k}\ln(1+\epsilon) + T_1(\epsilon),$$

where

$$T_1(\epsilon) = \frac{A}{4k}\int_0^1 \frac{\cos t - 1}{t+\epsilon}dt + \frac{A}{4k}\int_1^\infty \frac{\cos t}{t+\epsilon}dt$$
$$- \frac{A}{4k}\int_0^\infty \frac{\cos(4-\epsilon)t}{t+1}dt$$
$$+ \frac{1}{2k}\int_0^\infty W(t)\sin(2kt)dt.$$

Thus

$$\lim_{\epsilon \to 0} \epsilon^{\pm \gamma/2}e^{\pm S_\epsilon(\infty)} = e^{\pm T_1(0)}. \tag{2.18}$$

(iii)

$$S_\epsilon\left(\frac{r}{\epsilon}\right) = -\frac{A}{4k}\ln \epsilon + \frac{A}{4k}\ln(r+\epsilon)$$
$$+ \frac{A}{4k}\int_0^r \frac{\cos t - 1}{t+1}dt$$
$$- \frac{A}{4k}\int_0^{r/\epsilon} \frac{\cos(4-\epsilon)t}{t+1}dt$$
$$+ \frac{1}{2k}\int_0^{r/\epsilon} W(t)\sin(2kt)dt$$

so that

$$\lim_{\epsilon \to 0} \epsilon^{\pm \gamma}e^{\pm 2S_\epsilon(r/\epsilon)} = r^{\pm \gamma}e^{\pm T(r)}, \tag{2.19}$$

where

$$T(r) = \frac{A}{2}\int_0^r \frac{\cos t - 1}{t}dt - \frac{A}{2}\int_0^\infty \frac{\cos 4t}{t+1}dt$$
$$+ \int_0^\infty W\sin(2t)dt. \tag{2.20}$$

The quantities $T_1(0)$ and $T(r)$ are the same as in Ref. 12 except that the contribution due to $W(r)$ has been included here. Now, we continue our discussion of the function $\hat{h}_{2,\epsilon}(r)$ and first recapitulate the basic steps from Ref. 12. Return to (2.15) and set $r = \infty$. Then there is a term on the right which requires special attention a $\epsilon \to 0$. To see this write $V(t)\cos 2kt = (A/2)(t+1)^{-1}\sin \epsilon t + (A/2) \times (t+1)^{-1}\sin(2t(1+k)) + W(t)\cos 2kt$ and substitute it in (2.15). Then the following integral appears:

$$\frac{A^2}{16k^2}\int_0^\infty \frac{\sin \epsilon t}{t+1}e^{-2S_\epsilon(t)}\left(\int_0^t \frac{\sin \epsilon s}{s+1}e^{2S_\epsilon(s)}\hat{h}_{2,\epsilon}(s)ds\right)dt. \tag{2.21}$$

This term does not go to zero with ϵ. For if we let $w = \epsilon s$, $u = \epsilon t$ we get

$$\frac{A^2}{16k^2}\int_0^\infty \frac{\sin u}{u+\epsilon}e^{-2S_\epsilon(u/\epsilon)}$$
$$\times\left(\int_0^u \frac{\sin w}{w+\epsilon}e^{2S_\epsilon(w/\epsilon)}\hat{h}_{2,\epsilon}\frac{w}{\epsilon}dw\right)du. \tag{2.22}$$

As in Ref. 12 we can prove that

$$\lim_{\epsilon\to0}\hat{h}_{2,\epsilon}(r/\epsilon) = \tilde{h}_{2,0}(r) \tag{2.23}$$

exists [the presence of $W(r)$ does not affect the proof significantly] and that we may pass to the limit $\epsilon \to 0$ in the integrand of (2.22). By using (2.19) we obtain in the limit

$$\frac{A^2}{16}\int_0^\infty \frac{\sin u}{u^{1+\gamma}}e^{-T(u)}\left(\int_0^u \frac{\sin w}{w^{1-\gamma}}e^{T(w)}\tilde{h}_{2,0}(w)dw\right)du. \tag{2.24}$$

This term will play an important role in the sequel. All the other integrals that contribute to (2.15) behave nicely as $\epsilon \to 0$, that is, we may take their pointwise limits. This again can be justified as in Ref. 12. To simplify the notation it is proper to define two operators $\mathcal{K}_0$ and $\mathcal{I}_0$,

$$(\mathcal{K}_0 f)(r) = \frac{1}{4}\int_0^r V(\cos 2t+1)e^{-2S_0}$$
$$\times\left(\int_0^t V(\cos 2s-1)e^{2S_0}f(s)ds\right)dt, \tag{2.25}$$

$$(\mathcal{I}_0 f)(r) = \frac{A^2}{16}\int_0^r \frac{\sin u}{u^{1+\gamma}}e^{-T(u)}$$
$$\times\left(\int_0^u \frac{\sin w}{w^{1-\gamma}}e^{T(w)}f(w)dw\right)du. \tag{2.26}$$

The kernel $\mathcal{K}_0$ is simply the pointwise limit of the kernel $\mathcal{K}_\epsilon$ defined in (2.15). Then, from (2.15) and Ref. 12,

$$\lim_{\epsilon\to0}\hat{h}_{2,\epsilon}(\infty) \equiv \hat{h}_{2,\epsilon\to0}(\infty)$$
$$= 1 + (\mathcal{K}_0\hat{h}_{2,0})(\infty) + (\mathcal{I}_0\tilde{h}_{2,0})(\infty) \tag{2.27}$$

and

$$1 + (\mathcal{K}_0\hat{h}_{2,0})(\infty) = \hat{h}_{2,0}(\infty) \quad (= \lim_{r\to\infty}\tilde{h}_{2,0}(r)). \tag{2.28}$$

Relation (2.28) follows from (2.15) for $\epsilon = 0$. Moreover, by

replacing r by r/ϵ in (2.15) and letting $\epsilon \to 0$ we arrive at the following integral equation for $\tilde{h}_{2,0}(r)$:

$$\tilde{h}_{2,0}(r) = \hat{h}_{2,0}(\infty) + (\mathcal{I}_0\tilde{h}_{2,0})(r). \tag{2.29}$$

Hence from (2.27) and (2.29)

$$\hat{h}_{2,\epsilon\to0}(\infty) = \tilde{h}_{2,0}(\infty) \quad (= \lim_{r\to\infty}\tilde{h}_{2,0}(r)). \tag{2.30}$$

Equations (2.27) and (2.28) show that the term $(\mathcal{I}_0\tilde{h}_{2,0})(\infty)$ accounts for the difference between $\lim_{\epsilon\to0}\lim_{r\to\infty}\hat{h}_{2,\epsilon}(r)$ and $\lim_{r\to\infty}\lim_{\epsilon\to0}\hat{h}_{2,\epsilon}(r)$.

We now divide our discussion into two parts according to whether $\psi_1(0) \neq 0$ or $\psi_1(0) = 0$.

$\psi_1(0) \neq 0$: It follows as in Ref. 12 that in this case $\hat{h}_{2,0}(\infty) \neq 0$. This ensures that $\tilde{h}_{2,0}(r)$ does not vanish identically for $r > 0$ [by (2.29)]. Thus by (2.10), (2.18), (2.27), and (2.30),

$$h_{2,\epsilon}(\infty) = e^{T_1(0)}\tilde{h}_{2,0}(\infty)\epsilon^{-\gamma/2} + o(\epsilon^{-\gamma/2}). \tag{2.31}$$

It may happen that $\tilde{h}_{2,0}(\infty) = 0$ (see the example in Sec. III). Next we look at $\hat{h}_{1,\epsilon}(\infty)$. Following Ref. 12, using (2.13), we see that the leading behavior comes from the resonant integral

$$\frac{A}{4k^2}\int_0^\infty \frac{\sin \epsilon t}{t+1}e^{2S_\epsilon}\hat{h}_{2,\epsilon}(t)dt.$$

Substitute $u = \epsilon t$ and use (2.18), (2.19), (2.23), and (2.13) to get

$$h_{1,\epsilon}(\infty) = e^{-T_1(0)}\epsilon^{-\gamma/2}\frac{A}{4}$$
$$\times\int_0^\infty \frac{\sin u}{u^{1-\gamma}}e^{T(u)}\tilde{h}_{2,0}(u)du + o(\epsilon^{-\gamma/2}). \tag{2.32}$$

Define

$$\mathcal{Z}(f) = e^{T_1(0)}\tilde{f}(\infty) - ie^{-T_1(0)}\frac{A}{4}$$
$$\times\int_0^\infty \frac{\sin u}{u^{1-\gamma}}e^{T(u)}\tilde{f}(u)du, \tag{2.33}$$

where $\tilde{f}(u) = \lim_{\epsilon\to0}f(u/\epsilon)$ and insert (2.31) and (2.32) in (2.16). Thus

$$F(k) = \eta\epsilon^{-\gamma/2} + o(\epsilon^{-\gamma/2}), \quad \eta = \mathcal{Z}(\tilde{h}_{2,0}). \tag{2.34}$$

It was proved in Ref. 12 that $\mathcal{Z}(\tilde{h}_{2,0}) \neq 0$ follows from $\hat{h}_{2,0}(\infty) \neq 0$ [$\mathcal{Z}(\tilde{h}_{2,0}) = 0$ would imply $\tilde{h}_{2,0}(r) = 0$ for $r > 0$ via (2.29) and hence $\hat{h}_{2,0}(\infty) = 0$]. When $\epsilon < 0$ the imaginary part of $\mathcal{Z}(\tilde{h}_{2,0})$ changes sign and ϵ must be replaced by $|\epsilon|$ (see Ref. 12). Thus

$$F(k) = \bar{\eta}|\epsilon|^{-\gamma/2} + o(|\epsilon|^{-\gamma/2}), \quad \epsilon\to0^-. \tag{2.35}$$

Now we want to extend (2.34) to the complex upper half k plane [which by (2.3) corresponds to the lower half ϵ plane] by using a Phragmén-Lindelöf argument. For this we need an estimate.

Lemma (2.1): Let $D_\delta = \{k: |k-1| \leq \delta, \text{Im } k \geq 0, k \neq 1\}$, where $0 < \delta < 1$. Then for any β, $2/3 < \beta < 1$, there exist constants $c_1, c_2 > 0$ (depending on β and δ but not on k) such that

$$|F(k)| \leqslant c_1 \exp c_2 |k-1|^{-\beta}, \quad k \in D_\delta. \tag{2.36}$$

For the proof see the Appendix.

The following reasoning is justified by the results in Titchmarsh[15] (Sect. 5.6). Inequality (2.36) implies that $\epsilon^{\gamma/2} F(k)$ is bounded for ϵ small and complex with Im $\epsilon \leqslant 0$. Since $\epsilon^{\gamma/2} F(k)$ has finite limits both as $\epsilon \to 0 +$ and $\epsilon \to 0 -$ these limits must agree and hence be equal to η. Moreover, $\epsilon^{\gamma/2} F(k) \to \eta$ as $\epsilon \to 0 (k \to 1)$ through complex values with Im $\epsilon \leqslant 0$ (Im $k \geqslant 0$) uniformly in arg ϵ (arg $(k-1)$). This fact and (2.35) imply that

$$\bar{\eta} = e^{i\pi(\gamma/2)}\eta. \tag{2.37}$$

Using (2.34) and the definitions (1.8), (1.9), and (1.10) it is now straightforward to deduce the properties of $F(k)$, $S(k)$, and Δ listed in Table I under the heading $\psi_1(0) \neq 0 (A > 0)$.

$\psi_1(0) = 0$: In this case $\hat{h}_{2,0}(\infty) = 0$ and $\hat{h}_{2,0}(r) = 0$ for all $r > 0$. This implies $\eta = 0$ in (2.34). The starting point for our analysis is the equation

$$\hat{h}_{2,\epsilon}(r) - \hat{h}_{2,0}(r) = (k^{-1} - 1) + ((\mathscr{K}_\epsilon - \mathscr{K}_0)\hat{h}_{2,0})(r)$$
$$+ (\mathscr{K}_\epsilon(\hat{h}_{2,\epsilon} - \hat{h}_{2,0}))(r). \tag{2.38}$$

Here $\mathscr{K}_\epsilon$ and $\mathscr{K}_0$ are the operators defined by (2.15) and (2.25). Equation (2.38) follows immediately from (2.15) by subtracting the corresponding equation for $\epsilon = 0$. Since $\hat{h}_{2,0}(\infty) = 0$, (2.38) represents $\hat{h}_{2,\epsilon}(\infty)$ when r is set equal to infinity. For r fixed and finite it is clear that $\hat{h}_{2,\epsilon}(r) - \hat{h}_{2,0}(r)$ is of order ϵ, but this is no longer true if $r = \infty$. In fact, it turns out that the leading behavior of the right-hand side of (2.38) depends on γ and that the leading-order contributions come from different sources according to whether $\gamma < 1$, $\gamma = 1$, or $\gamma > 1$. Unfortunately, once the leading-order terms have been identified, it is a very tedious job to estimate all the other terms and to show that they are of lower order. Since the case $\gamma < 1$ [with $W(r) = 0$] has been worked out in detail in Ref. 12, we will be satisfied here with giving a few typical estimates in the case $\gamma \geqslant 1$ in order to illustrate the method and to comment on some of the modifications that are caused by the presence of $W(r)$. We begin by recapitulating from Ref. 12 the main steps in the case $\gamma < 1$.

$\gamma < 1$: It follows as in Ref. 12 that when $r = \infty$ the leading behavior of the right-hand side of (2.38) comes from a term in $((\mathscr{K}_\epsilon - \mathscr{K}_0)\hat{h}_{2,0})(\infty)$, namely $g_\epsilon(\infty)$ where

$$g_\epsilon(r) = \frac{A}{4}\left(\int_0^r \frac{\sin \epsilon t}{t+1} e^{-2S_\epsilon} dt\right)$$
$$\times \left(\int_0^\infty \frac{V}{2k^2}(\cos 2ks - 1)e^{2S_\epsilon}\hat{h}_{2,0}(s)ds\right). \tag{2.39}$$

(The second integral comes from an integral of the form $\int_0^r \cdots$, which was written as $\int_0^\infty \cdots - \int_r^\infty \cdots$.) The first integral, for $r = \infty$, vanishes like ϵ^γ when $\epsilon \to 0$. This follows on substituting $u = \epsilon t$ and invoking (2.19). The second integral approaches a finite limit as $\epsilon \to 0$. This suggests that we introduce

$$\widehat{V}_{2,\epsilon}(r) = \epsilon^{-\gamma}(\hat{h}_{2,\epsilon}(r) - \hat{h}_{2,0}(r)). \tag{2.40}$$

Following Ref. 12 one then proves by working with (2.38) that

$$\lim_{\epsilon \to 0} \widehat{V}_{2,\epsilon}(\infty) = \widehat{V}_{2,\epsilon \to 0}(\infty) \quad \text{and}$$
$$\lim_{\epsilon \to 0} \widehat{V}_{2,\epsilon}(r/\epsilon) = \widetilde{V}_{2,0}(r) \tag{2.41}$$

both exist [compare (2.23)]. Let $\tilde{g}_0(r) = \lim_{\epsilon \to 0} \epsilon^{-\gamma} g_\epsilon(r/\epsilon)$ which by (2.39), (2.19), and (2.13) is equal to

$$\tilde{g}_0(r) = \frac{A}{4}\left(\int_0^r \frac{\sin u}{u^{1+\gamma}} e^{-T(u)} du\right)\hat{h}_{1,0}(\infty). \tag{2.42}$$

Then $\widehat{V}_{2,\epsilon \to 0}$ and $\widetilde{V}_{2,0}$ obey

$$\widehat{V}_{2,\epsilon \to 0}(\infty) = \tilde{g}_0(\infty) + (\mathscr{I}_0 \widetilde{V}_{2,0})(\infty) \tag{2.43}$$

and

$$\widetilde{V}_{2,0}(r) = \tilde{g}_0(r) + (\mathscr{I}_0 \widetilde{V}_{2,0})(r). \tag{2.44}$$

Thus

$$\widetilde{V}_{2,\epsilon \to 0}(\infty) = \widetilde{V}_{2,0}(\infty). \tag{2.45}$$

These properties of $\widetilde{V}_{2,\epsilon}$ parallel those of $\hat{h}_{2,\epsilon}$ mentioned in (2.27)–(2.30), but note the following differences. Instead of the constant $\hat{h}_{2,0}(\infty)$ in (2.29) we encounter the function $\tilde{g}_0(r)$ at the corresponding place in (2.44). Moreover, there is no term corresponding to $(\mathscr{K}_0 \hat{h}_{2,0}(\infty)$ [see (2.27)] in (2.43) since for fixed r, $\widehat{V}_{2,\epsilon}(r) = O(\epsilon^{1-\gamma})$ and thus $\widehat{V}_{2,\epsilon}(r)$ vanishes as $\epsilon \to 0$. The fact that $\lim_{r \to \infty} \tilde{g}_0(r) = \lim_{\epsilon \to 0} \epsilon^{-\gamma} g_\epsilon(\infty)$ has also been used in deriving (2.43). By (2.40) and (2.45) $\hat{h}_{2,\epsilon}(\infty) = \widetilde{V}_{2,0}(\infty)\epsilon^\gamma + o(\epsilon^\gamma)$ and hence

$$h_{2,0}(\infty) = e^{T_1(0)}\widetilde{V}_{2,0}(\infty)\epsilon^{\gamma/2} + o(\epsilon^{\gamma/2}). \tag{2.46}$$

Combining this with the result for $h_{1,\epsilon}(\infty)$ [see Ref. 12 and (2.32)] we get by (2.16), as $\epsilon \to 0 +$,

$$F(k) = \mu\epsilon^{\gamma/2} + o(\epsilon^{\gamma/2}), \quad \mu = \mathscr{I}(\widetilde{V}_{2,0}), \tag{2.47}$$

where $\mathscr{I}(\widetilde{V}_{2,0})$ is defined in (2.33). As in Ref. 12 one shows that $\mu \neq 0$. If $\epsilon < 0$, then $g_\epsilon(r)$ changes sign and therefore, as $\epsilon \to 0 -$,

$$F(k) = -\bar{\mu}|\epsilon|^{\gamma/2} + o(|\epsilon|^{\gamma/2}). \tag{2.48}$$

Thanks to Lemma (2.1) the relation (2.47) extends to the complex plane. The analog of (2.37) is

$$\bar{\mu} = -\mu e^{-i\pi\gamma/2}. \tag{2.49}$$

From this the entries in the table for $\psi_1(0) = 0, 0 < \gamma < 1$ $(A > 0)$ follow.

$\gamma = 1$: In this case it is still the term $g_\epsilon(\infty)$ [see (2.39)] which contains the leading behavior. The relevant integral has the following behavior:

$$\int_0^\infty \frac{\sin \epsilon t}{t+1} e^{-2S_\epsilon(t)} dt = \int_0^\infty \frac{\sin u}{u+\epsilon} e^{-2S_\epsilon(u/\epsilon)} du$$
$$= e^{-2\sigma}\epsilon \ln(\epsilon^{-1}) + o(\epsilon \ln \epsilon^{-1}). \tag{2.50}$$

The $\epsilon \ln(\epsilon^{-1})$ term comes from the behavior of the integrand near $u = 0$ where it is of the form $4(u+\epsilon)^{-1-k^{-1}} \sin u$. We also need (2.19) and the definition of σ from (2.17). Now one has to show that all the other terms in $((\mathscr{K}_\epsilon - \mathscr{K}_0)\hat{h}_{2,0})(\infty)$ are $o(\epsilon \ln(\epsilon^{-1}))$. In fact, one can argue that they are all $O(\epsilon)$. For some of them this

was already done in Ref. 12 but others were only shown to be $o(\epsilon^{\beta})$ with suitable $0 < \beta < 1$ because this was sufficient for the purposes of that paper. The latter terms must be reinspected and shown to be $O(\epsilon)$. Owing to the use of trigonometric identities and the presence of $W(r)$ the number of such terms is large (roughly 10). As already mentioned above, we therefore find it reasonable to look here only at some typical terms in order to illustrate the estimates that are involved. First consider

$$I_1 = \int_0^\infty \frac{\rho(t)}{t+1} \sin\left(\frac{\epsilon}{2} t\right) e^{-2S_\epsilon(t)} dt, \qquad (2.51)$$

where $\rho(t)$ is a sum of trigonometric functions of the form $\sin \alpha(\epsilon) t$ with $\alpha(\epsilon) \to \alpha(0) \neq 0$ as $\epsilon \to 0$. For the origin of this integral see Ref. 12 [Eqs. (4.14)–(4.19)]. Since $\rho(t)$ has a bounded antiderivative $\tilde{\rho}(t)$, we can integrate by parts and get

$$I_1 = -\int_0^\infty \tilde{\rho}(t)\left(\frac{(\epsilon/2)\cos[(\epsilon/2)t\,]e^{-2S_\epsilon}}{t+1} \right.$$
$$- \frac{2\sin[(\epsilon/2)t\,]S'_\epsilon e^{-2S_\epsilon}}{t+1}$$
$$\left. - \frac{\sin[(\epsilon/2)t\,]e^{-2S_\epsilon}}{(t+1)^2} \right) dt. \qquad (2.52)$$

Consider the third term on the right and split the integral as $\int_0^{1/\epsilon} \cdots + \int_{1/\epsilon}^\infty \cdots$. In the former use $e^{-2S_\epsilon(t)} \leqslant c(t+1)^{-1}$ (since $\gamma = 1$) and $|\sin(\epsilon/2)t| \leqslant (\epsilon/2)t$, in the latter use $e^{-2S_\epsilon} \leqslant c\epsilon$ (these estimates were established in Ref. 12). So the third term in (2.52) is $O(\epsilon)$. Since $S'_\epsilon(t) = O((t+1)^{-1})$ the same reasoning applies to the middle term in (2.52). In the first integral use the fact that also $\tilde{\rho}\cos(\epsilon/2)t$ has a bounded antiderivative which allows us to integrate by parts once more. In view of the extra factor ϵ this term is $O(\epsilon)$ also. Hence $I_1 = O(\epsilon)$.

Secondly, consider

$$I_2 = \int_0^\infty W(t)(e^{-2S_\epsilon} - e^{-2S_0}) dt. \qquad (2.53)$$

For the origin of this term see again Ref. 12. Write $e^{-2S_\epsilon} - e^{-2S_0} = e^{-2S_\epsilon}(1 + e^{2(S_\epsilon - S_0)})$ and recall that $|1 - e^{2(S_\epsilon - S_0)}| \leqslant c|S_\epsilon - S_0| \leqslant c\epsilon(t+1)$, on appealing to Lemma 4.1 (i) of Ref. 12 with $\gamma = 1$ [note that this Lemma is still valid if W is present; only the constant c in the term $c\ln(r+1)$ is affected by W]. Now split I_2 in the same way as I_1 above. On $0 \leqslant t \leqslant \epsilon^{-1}$ use $e^{-2S_\epsilon(t)} \leqslant c(t+1)^{-1}$, so this part is $O(\epsilon)$. On $\epsilon^{-1} < t < \infty$ simply use $e^{-2S_\epsilon} \leqslant c\epsilon$ and $|1 - e^{2(S_\epsilon - S_0)}| < c$. Hence $I_2 = O(\epsilon)$. This much about estimates. Now we introduce, in analogy to (2.40), $\hat{V}_{2,\epsilon}(r) = [\hat{h}_{2,\epsilon}(r) - \hat{h}_{2,0}(r)]/(\epsilon\ln(\epsilon^{-1}))$ and establish formulas analogous to (2.41)–(2.45). Since, as we have mentioned in connection with (2.50), the vicinity of $u = 0$ determines the behavior of such integrals, it follows that for any $r > 0$,

$$\int_0^{r/\epsilon} \frac{\sin \epsilon t}{t+1} e^{-2S_\epsilon} dt = e^{-2\sigma}\epsilon \ln \epsilon^{-1} + o(\epsilon \ln(\epsilon^{-1})). \qquad (2.54)$$

Hence (with $A = 2$, $\gamma = 1$)

$$\lim_{\epsilon \to 0}(\epsilon \ln(\epsilon^{-1}))^{-1}g_\epsilon(r/\epsilon) = \tfrac{1}{2}e^{-2\sigma}\hat{h}_{1,0}(\infty) \equiv d \qquad (2.55)$$

is independent of r. So, let $\tilde{V}_{2,0}(r) = \lim_{\epsilon \to 0}\hat{V}_{2,\epsilon}(r/\epsilon)$, $\lim_{\epsilon \to 0}\hat{V}_{2,\epsilon}(\infty) = \hat{V}_{2,\epsilon \to 0}(\infty)$. Then

$$\hat{V}_{2,\epsilon \to 0}(\infty) = d + (\mathcal{I}_0\tilde{V}_{2,0})(\infty), \qquad (2.56)$$

$$\tilde{V}_{2,0}(r) = d + (\mathcal{I}_0\tilde{V}_{2,0})(r). \qquad (2.57)$$

Also, $\hat{V}_{2,\epsilon \to 0}(\infty) = \tilde{V}_{2,0}(\infty)$. Hence

$$h_{2,\epsilon}(\infty) = e^{T_1(0)}\tilde{V}_{2,0}(\infty)\epsilon \ln(\epsilon^{-1}) + o(\epsilon \ln(\epsilon^{-1})). \qquad (2.58)$$

To analyze $\hat{h}_{1,\epsilon}(\infty)$ replace $\hat{h}_{2,\epsilon}(r)$ by $[\hat{h}_{2,\epsilon}(r) - \hat{h}_{2,0}(r)] + \hat{h}_{2,0}(r)$ in (2.13) and use the fact that $\hat{h}_{2,\epsilon}(r/\epsilon) - \hat{h}_{2,0}(r/\epsilon) = O(\epsilon \ln(\epsilon^{-1}))$ and $\hat{h}_{2,0}(r) = O((r+1)^{-2})$ (see Corollary 2.2 in Ref. 12). The leading term in $\hat{h}_{1,\epsilon}(\infty)$ is given by the integral

$$\frac{A}{4k^2}\int_0^\infty \frac{\sin \epsilon t}{t+1} e^{2S_\epsilon}(\hat{h}_{2,\epsilon} - \hat{h}_{2,0}) dt$$
$$= \left[\frac{A}{4}\int_0^\infty (\sin u)e^{T(u)}\tilde{V}_{2,0}(u) du \right] \ln(\epsilon^{-1})$$
$$+ o(\ln(\epsilon^{-1})). \qquad (2.59)$$

Therefore, as $\epsilon \to 0+$,

$$F(k) = \tau\epsilon^{1/2}\ln \epsilon^{-1} + o(\epsilon^{1/2}\ln(\epsilon^{-1})), \quad \tau = \mathcal{L}(\tilde{V}_{2,0}) \qquad (2.60)$$

and, as $\epsilon \to 0-$,

$$F(k) = -\bar{\tau}|\epsilon|^{1/2}\ln(|\epsilon|^{-1}) + o(|\epsilon|^{1/2}\ln(|\epsilon|^{-1})). \qquad (2.61)$$

Moreover, $\bar{\tau} = i\tau$. This explains the entries in the table when $\psi_1(0) = 0$ and $\gamma = 1$.

$\gamma > 1$: In this case the leading term in $((\mathcal{K}_\epsilon - \mathcal{K}_0)\hat{h}_{2,0})(\infty)$ is of order ϵ and hence the term $k^{-1} - 1 = \epsilon/2 + O(\epsilon)$ must also be taken into account. The fall-off of $\hat{h}_{2,0}(r)$ which is $O((r+1)^{-1-\gamma})$ allows us to expand the difference $((\mathcal{K}_\epsilon - \mathcal{K}_0)\hat{h}_{2,0})(r)$ as $\epsilon(\dot{\mathcal{K}}_0\hat{h}_{2,0})(r) + o(\epsilon)$ where the dot denotes differentiation with respect to ϵ. The explicit expression for $\dot{\mathcal{K}}_0$ can be obtained by differentiating the kernel in (2.15). We omit writing it out. Let us introduce $\hat{V}_{2,\epsilon}(r) = \epsilon^{-1}(\hat{h}_{2,\epsilon}(r) - \hat{h}_{2,0}(r))$, $\tilde{V}_{2,0}(r) = \lim_{\epsilon \to 0}\hat{V}_{2,\epsilon}(r/\epsilon)$, and $c = \tilde{V}_{2,0}(r \to \infty) = \lim_{r \to \infty}\lim_{\epsilon \to 0}\hat{V}_{2,\epsilon}(r)$. Here c can also be interpreted as

$$c = \lim_{r \to \infty}\dot{\hat{h}}_{2,0}(r). \qquad (2.62)$$

Now set $r = \infty$ in (2.38), divide by ϵ, and let $\epsilon \to 0$ using

$$\lim_{\epsilon \to 0}\epsilon^{-1}(\mathcal{K}_\epsilon(\hat{h}_{2,\epsilon} - \hat{h}_{2,0}))(\infty)$$
$$= (\dot{\mathcal{K}}_0\hat{h}_{2,0})(\infty) + (\mathcal{I}_0\tilde{V}_{2,0})(\infty).$$

This yields

$$\hat{V}_{2,\epsilon \to 0}(\infty) = c + (\mathcal{I}_0\tilde{V}_{2,0})(\infty) \qquad (2.63)$$

and

$$\tilde{V}_{2,0}(r) = c + (\mathcal{I}_0\tilde{V}_{2,0})(r). \qquad (2.64)$$

Now c can be expressed in a different form which then shows that $c \neq 0$. Since $\varphi(1,r)$ is a multiple of $\psi_1(r)$, let

$\varphi(1,r) = L_\varphi \psi_1(r)$ (remember that $k_0 = 1$). Also, let $\phi(1,r) = L_\varphi^{-1} \psi_2(r)$ [see (1.4)]. Then $W[\varphi,\phi] = 1$. First we can compute $\dot{\varphi}(1,r)$ by using $\varphi(1,r)$ and $\phi(1,r)$ in the variation of constants formula for $\varphi(k,r)$. Then we can relate $\lim_{r\to\infty} \dot{\varphi}(1,r)$ to $\lim_{r\to\infty} \hat{h}_{2,0}(r) = c$ by using (2.4), (2.5) and elementary facts about S_0 and $\dot{S}_0$ in the limit as $r \to \infty$ (for example, that $\dot{S}_0 e^{-2S_0} h_{2,0} \to 0$ as $r \to \infty$). The details of this calculation are omitted. We find that

$$c = L_\varphi^{-1} e^{-\sigma} \int_0^\infty \varphi^2(1,t)dt. \tag{2.65}$$

Then $\hat{h}_{2,\epsilon}(\infty) \sim \widetilde{V}_{2,0}(\infty)\epsilon$ and $h_{2,\epsilon}(\infty) \sim e^{T_1(0)} \widetilde{V}_{2,0}(\infty)\epsilon^{1-\gamma/2}$. Finally, after inspecting $h_{1,\epsilon}(\infty)$, we get the result that

$$F(k) = \nu \epsilon^{1-\gamma/2} + o(\epsilon^{1-\gamma/2}), \quad \nu = \mathscr{L}(\widetilde{V}_{2,0}) \tag{2.66}$$

as $\epsilon \to 0 +$, and

$$F(k) = -\bar{\nu}|\epsilon|^{1-\gamma/2} + o(|\epsilon|^{1-\gamma/2}) \tag{2.67}$$

as $\epsilon \to 0 -$. Again, (2.65) extends to complex ϵ (resp. complex k), and

$$\bar{\nu} = \nu e^{i\pi(\gamma/2)}. \tag{2.68}$$

This yields the entries in the table when $\gamma > 1$, $\psi_1(0) = 0 (A > 0)$.

The case $A < 0$ can be handled either in the way described at the end of Ref. 12 which effectively involves switching the roles of $\hat{h}_{1,\epsilon}$ and $\hat{h}_{2,\epsilon}$, or it can be considered as a particular case of a larger class of potentials to which our theory applies and which will be described in Sec. III.

We further remark that the boundary condition $\psi(0) = 0$ can be replaced by a general self-adjoint boundary condition of the form $\psi(0)\cos\alpha - \psi'(0)\sin\alpha = 0 (0 \leqslant \alpha < \pi)$. If φ_α is the solution satisfying $\varphi_\alpha(k,0) = \sin\alpha, \varphi_\alpha'(k,0) = \cos\alpha$ then $F_\alpha(k) = W[f,\varphi_\alpha]$ is the Jost function associated with the above boundary condition. It contains the information about the spectrum of the Schrödinger operator satisfying this boundary condition. The alternative $\psi_1(0) = 0$ or $\psi_1(0) \neq 0$ is now replaced by the alternative whether or not ψ_1 satisfies the α-boundary condition (i.e., whether or not ψ_1 is a multiple of φ_α). The properties of $F_\alpha(k)$ can be deduced from integral equations similar to those in (2.13), (2.14) [and (2.15)] except that we must add the constant terms $k^{-1}\sin\alpha$ and $k^{-1}\cos\alpha$ on the right of (2.13) and (2.14), respectively. We refer to Ref. 12 where such additional terms were discussed when $\gamma < 1$. The final asymptotic formulas for $F_\alpha(k)$ are essentially like those for the $\alpha = 0$ boundary condition except for one minor change. If $\gamma < 1$ and φ_α is a multiple of ψ_1 we have that

$$h_{1,\epsilon,\alpha}(\infty) = e^{-S_\epsilon(\infty)}\left(k^{-1}\sin\alpha + \int_0^\infty \frac{V}{2k^2}(\cos 2kt - 1) \right.$$
$$\times e^{2S_\epsilon}(\hat{h}_{2,\epsilon,\alpha}(t) - \hat{h}_{2,0,\alpha}(t))dt$$
$$\left. + \int_0^\infty \frac{V}{2k^2}(\cos 2kt - 1)e^{2S_\epsilon}\hat{h}_{2,0,\alpha}(t)dt \right). \tag{2.69}$$

If $\gamma < 1$ then all three terms in the parenthesis have limits as $\epsilon \to 0$, so the term $k^{-1}\sin\alpha$ contributes to the coefficient $\mu = \mu_\alpha$ in (2.47) (also see Ref. 12). If $\gamma \geqslant 1$, then the middle

term dominates the others as $\epsilon \to 0$; it is of order $\ln(\epsilon^{-1})$ if $\gamma = 1$ and of order $\epsilon^{1-\gamma}$ if $\gamma > 1$. The third term is $O(1)$ always. Hence when $\gamma \geqslant 1$ the formulas for the coefficients τ_α in (2.60) and ν_α in (2.66) are obtained simply by replacing $\widetilde{V}_{2,\epsilon}(r)$ with $\widetilde{V}_{2,\epsilon,\alpha}(r)$. When φ_α is not a multiple of ψ_1, then formula (2.34) applies with $\hat{h}_{2,0,\alpha}$ replacing $\hat{h}_{2,0}$.

As a preparation for Sec. III we go back to (2.34), (2.47), (2.60), and (2.66) and derive a representation for the coefficients η, μ, τ, and ν which makes it easier to see their implicit dependence on W. Note that in the integrand for $\mathscr{I}_0$ in (2.26) there is no dependence on W since the integral containing W cancels out [use (2.20)]. Let $M_\gamma(r)$ be a solution of

$$M_\gamma(r) = 1 + (\mathscr{I}_0 M_\gamma)(r). \tag{2.70}$$

The constant 1 amounts to a normalization. Suppose $\psi_1(0) \neq 0$. Then $\hat{h}_{2,0}(r) = \hat{h}_{2,0}(\infty)M_\gamma(r)$ by (2.29). Since $\varphi(1,r)$ is unbounded, $\varphi(1,r) \sim D_\varphi r^{\gamma/2}\sin r$ as $r \to \infty$. From $h_{2,0}(r) = \varphi(1,r)\sin r + \varphi'(1,r)\cos r$ it follows that $h_{2,0}(r) \sim D_\varphi r^{\gamma/2}$. By (2.17) $\hat{h}_{2,0}(r) = e^{-S_0(r)} h_{2,0}(r) \to D_\varphi e^{-\sigma}$ as $r \to \infty$, i.e., $\hat{h}_{2,0}(\infty) = D_\varphi e^{-\sigma}$. Then $e^{T_1(0)}\hat{h}_{2,0}(\infty) = D_\varphi e^{-(A/4)C}M_\gamma(\infty)$, where $C = \int_0^1 (1 - \cos t)t^{-1}dt - \int_1^\infty t^{-1}(\cos t)dt$ is Euler's constant. This implies

$$\eta = \mathscr{L}(\hat{h}_{2,0})$$
$$= D_\varphi e^{-A/4)C}\left[M_\gamma(\infty) - i(A/4)e^{(A/2)C} \right.$$
$$\times \int_0^\infty \frac{\sin u}{u^{1-\gamma}}\exp\left(\frac{A}{2}\int_0^u \frac{\cos t - 1}{t}dt \right)M_\gamma(u)du \right]$$
$$\equiv D_\varphi \kappa(A)Q(M_\gamma), \tag{2.71}$$

where $\kappa(A) = e^{-(A/4)C}$. In this formula only D depends (implicitly) on W. Now suppose that $\psi_1(0) = 0$. Then $\varphi(1,r) \sim L_\varphi r^{-\gamma/2}\cos r$. If $\gamma < 1$ let $N_\gamma(r)$ satisfy

$$N_\gamma(r) = \frac{A}{4}\int_0^r \frac{\sin u}{u^{1+\gamma}}\exp\left(-\frac{A}{2}\int_0^u \frac{\cos t - 1}{t}dt \right)du$$
$$+ (\mathscr{I}_0 N_\gamma)(r). \tag{2.72}$$

Thus $\widetilde{V}_{2,0}(r) = e^{-2\sigma}\hat{h}_{1,0}(\infty)N_\gamma(r)$ by (2.42) and (2.44). Since $h_{1,0}(r) = \varphi(1,r)\cos r - \varphi'(1,r)\sin r \sim L_\varphi r^{-\gamma/2}$, $\hat{h}_{1,0}(r) = e^{S_0(r)}h_{1,0}(r) \to L_\varphi e^\sigma$ as $r \to \infty$. So $\widetilde{V}_{2,0}(r) = L_\varphi e^{-\sigma}N_\gamma(r)$ and

$$\mu = \mathscr{L}(\widetilde{V}_{2,0}) = L_\varphi \kappa(A)Q(N_\gamma). \tag{2.73}$$

If $\gamma \geqslant 1$, then $\widetilde{V}_{2,0}(r) = dM_\gamma(r)(\gamma = 1)$ and $\widetilde{V}_{2,0}(r) = cM_\gamma(r)(\gamma > 1)$. Therefore

$$\tau = \mathscr{L}(\widetilde{V}_{2,0}) = \tfrac{1}{2}L_\varphi \kappa(2)Q(M_\gamma) \quad (A = 2, \quad \gamma = 1), \tag{2.74}$$

$$\nu = \mathscr{L}(\widetilde{V}_{2,0}) = L_\varphi^{-1}\kappa(A)\left(\int_0^\infty \varphi(1,t)^2dt \right)Q(M_\gamma). \tag{2.75}$$

Representations (2.71), (2.73)–(2.75) will be useful when we discuss local singularities of W in Sec. III.

III. EXTENSIONS, EXAMPLES, APPLICATIONS

As a first extension we consider potentials of the form

$$V(r) = A\sin 2(r + r_0)/(r + 1), \quad 0 < r_0 < \pi, \tag{3.1}$$

where without loss we have set $W(r) = 0$. If $r_0 = \pi/4$ we get $\cos 2r$ instead of $\sin 2r$, if $r_0 = \pi/2$ we get $-\sin 2r$ which means A has been replaced by $-A$. If $\sin 2r$ in (2.2) were replaced by $a \sin 2r + b \cos 2r$ then this would also be of the above form with $A = \sqrt{a^2 + b^2}$, $2r_0 = \sin^{-1} \times (b/\sqrt{a^2 + b^2})$. Combining this with the method of Ref. 12 we can treat potentials of the form $V(r) = p(r)/(r + 1)$ where $p(r)$ is a periodic function. Thus (3.1) is the prototype for a variety of further generalizations. In order to deal with (3.1) we return to (2.4), (2.5) and change the definitions of h_1 and h_2 as follows. Let

$$\varphi(k,r) = k h_1(k,r) \cos(kr + r_0)$$
$$+ h_2(k,r) \sin(kr + r_0), \tag{3.2}$$

$$\varphi'(k,r) = -k^2 h_1(k,r) \sin(kr + r_0)$$
$$+ k h_2(k,r) \cos(kr + r_0). \tag{3.3}$$

Then $S_\epsilon(r)$ becomes

$$S_\epsilon(r) = \int_0^r \frac{V(t)}{2k} \sin 2(kt + r_0) \, dt \tag{3.4}$$

and we can see that resonance occurs due to

$$\sin 2(t + r_0) \sin 2(kt + r_0)$$
$$= \tfrac{1}{2}[\cos \epsilon t - \sin(2t(1 + k) + 4r_0)].$$

This means S_ϵ has similar asymptotic properties as in the case $r_0 = 0$. The Jost function for (3.1) picks up a phase factor and is given by

$$F(k) = e^{-ir_0}(k h_{2,\epsilon}(\infty) - ik^2 h_{1,\epsilon}(\infty)). \tag{3.5}$$

The analysis of $F(k)$ remains essentially unchanged, we need only replace the term $\cos 2kt$ by $\cos 2(kt + r_0)$ in the integral equations. The quantity r_0 will explicitly enter into the expressions for σ, $T_1(0)$ and $T(r)$ in (2.17)–(2.20).

The results concerning $S(k)$ given in the table when $A < 0$ follow from (3.5) taking into account that $r_0 = \pi/2$ implies $e^{-ir_0} = -i$.

This extension to potentials of the form (3.1) in conjunction with what we said about the general boundary conditions at the end of Sec. II can be used to incorporate an angular momentum term $l(l + 1)r^{-2}(l = 0,1,...)$ in (1.2) resp. (2.2). Such a term still obeys (1.3) for r away from zero.

In this case $F_l(k)$ is defined as $W[f_l, \varphi_l]$, where φ_l is the standard regular solution of the Schrödinger equation corresponding to angular momentum l and f_l is the Jost solution.[2,5] To keep the notation simple we omit the subscript l in the following. Pick an arbitrary point $r_0 > 0$. Then $\varphi(k,r_0)$ and $\varphi'(k,r_0)$ are analytic functions of k. Denote differentiation with respect to k by a dot. Then

$$F(k) = f(k,r_0)\varphi'(1,r_0) - f'(k,r_0)\varphi(1,r_0)$$
$$+ [f(k,r_0)\dot\varphi'(1,r_0) - f'(k,r_0)\dot\varphi(1,r_0)](k - 1)$$
$$+ O((k - 1)^2). \tag{3.6}$$

Put $r = r_0 + \xi$ and consider the Schrödinger equation

$$-\psi'' + V(r_0 + \xi)\psi = k^2\psi \quad (\xi \geqslant 0). \tag{3.7}$$

Note that $V(r_0 + \xi)$ obeys (1.3) on $\xi \geqslant 0$. Define solutions ϕ

and χ of (3.7) such that $\phi(k,0) = \varphi(1,r_0)$, $\phi'(k,0) = \varphi'(1,r_0)$, $\chi(k,0) = \dot\varphi(1,r_0)$, $\chi'(k,0) = \dot\varphi'(1,r_0)$. Also, let $\tilde{f}(k,\xi)$ denote the Jost solution for (3.7). Clearly $\tilde{f}(k,\xi) = e^{-ikr_0}f(k,r)$. Therefore we can rewrite (3.6) as

$$F(k) = e^{ikr_0}W[\tilde{f},\phi] + e^{ikr_0}W[\tilde{f},\chi](k - 1)$$
$$+ O((k - 1)^2)$$
$$= e^{ikr_0}\tilde{F}_\phi(k) + e^{ikr_0}\tilde{F}_\chi(k)(k - 1)$$
$$+ O((k - 1)^2), \tag{3.8}$$

where the meaning of $\tilde{F}_\phi$ and $\tilde{F}_\chi$ is obvious. We fully understand the behavior of $\tilde{F}_\phi$ and $\tilde{F}_\chi$ as $k \to 1$ since $V(r_0 + \xi)$ is of type (3.1). The simplest case is when $\phi(1,\xi)$ is unbounded [i.e., $\varphi(1,r)$ is unbounded]. Then $\tilde{F}_\phi(k)$ diverges like $e^{-ir_0}\mathscr{L}(\bar{h}_{2,\epsilon,\phi}) \cdot \epsilon^{-\gamma/2}$ while $\tilde{F}_\chi(k)(k - 1)$ is in all cases at least $O(\epsilon^{1 - \gamma/2})$. The coefficient $\mathscr{L}(\bar{h}_{2,\epsilon,\phi})$ must be independent of r_0. So we can let $r_0 \to 0$ which means that $\bar{h}_{2,\epsilon,\phi}$ goes over into $\bar{h}_{2,\epsilon,\varphi}$ and, by (2.71), $\mathscr{L}(\bar{h}_{2,\epsilon,\varphi}) = D_\varphi \kappa(A)Q(M_\gamma)$, where D_φ is the constant in $\varphi(1,r) \sim D_\varphi r^{\gamma/2} \sin r$. If $\phi(1,\xi)$ is bounded, i.e., $\phi(1,\xi) \sim L_\phi(\xi + r_0)^{-\gamma/2} \cos(\xi + r_0)$ and $\varphi(1,r) \sim L_\phi r^{-\gamma/2} \times \cos r$, then $\tilde{F}_\phi(k)$ behaves according to (2.47), (2.60), or (2.66), depending on the value of γ. Since $\tilde{F}_\chi(k)(k - 1)$ is no larger than $O(\epsilon^{1 - \gamma/2})$ we get that $\tilde{F}_\phi(k)$ is the leading term when $\gamma \leqslant 1$. Then $\tilde{F}_\phi(k)$ vanishes like $\epsilon^{\gamma/2}$ if $\gamma < 1$ and $\epsilon^{1/2}\ln(\epsilon^{-1})$ if $\gamma = 1$. On letting $r_0 \to 0$ the corresponding coefficients μ and τ are given by (2.73), (2.74) with L as above. When $\gamma > 1$ then $\tilde{F}_\phi(k)$ and $\tilde{F}_\chi(k)(k - 1)$ are of the same order and so both contribute in leading order. Note that the two vectors $(\varphi(1,r_0),\varphi'(1,r_0))$ and $(\dot\varphi(1,r_0),\dot\varphi'(1,r_0))$ are linearly independent because

$$\varphi(1,r_0)\dot\varphi'(1,r_0) - \varphi'(1,r_0)\dot\varphi(1,r_0) = -2\int_0^{r_0} |\varphi(1,r)|^2 \, dr$$

which follows from the differential equation for φ and its dotted version. This means χ is unbounded if ϕ is bounded. So

$$\chi \sim D(\xi + r_0)^{\gamma/2} \sin(\xi + r_0),$$

$$W[\phi,\chi] = LD = -2\int_0^{r_0} |\varphi(1,x)|^2 \, dx.$$

Thus we have from (2.74), (2.71), and (3.5)

$$e^{ikr_0}\tilde{F}_\phi(k) \sim L^{-1}\kappa(A)Q(M_\gamma)\left(\int_0^\infty \phi(1,\xi)^2 \, d\xi\right)\epsilon^{1 - \gamma/2}, \tag{3.9}$$

$$e^{ikr_0}\tilde{F}_\chi(k)(k - 1) \sim -\frac{1}{2}D\kappa(A)Q(M_\gamma)\epsilon^{1 - \gamma/2}$$
$$= L^{-1}\kappa(A)Q(M_\gamma)$$
$$\times\left(\int_0^{r_0} \varphi(1,r)^2 \, dr\right)\epsilon^{1 - \gamma/2}. \tag{3.10}$$

Adding both terms yields

$$F(k) \sim L^{-1}\kappa(A)Q(M_\gamma)\left(\int_0^\infty \varphi(1,r)^2 \, dr\right)\epsilon^{1 - \gamma/2}. \tag{3.11}$$

This shows that (2.66) is valid in the presence of an angular momentum term when $E_0 = 1$ is an eigenvalue.

There is only a very narrow class of potentials (see Refs. 5, 10, and 16) for which it seems possible to verify explicitly some of our formulas. These potentials have the property that their spectral function $\rho(E)$ is identical to the unperturbed ($V=0$) spectral function except for the addition of a jump discontinuity at a prescribed energy E_0. Explicitly, $\rho(E) = 2(3\pi)^{-1}E^{3/2} + \rho_0\Theta(E-E_0)$, where $0 < \rho_0 < 1$ and $\Theta(E) = 0$ for $E < 0$, $\Theta(E) = 1$ for $E \geqslant 0$. The corresponding potential can be found via the Gelfand–Levitan equations. The result is

$$V(r) = -2\rho_0 \frac{d}{dr}\left(\frac{\sin^2 k_0 r}{k_0^2 + \rho_0 I(r)}\right), \tag{3.12}$$

where $I(r) = \int_0^r \sin^2 k_0 t\, dt$. The solution $\varphi(k,r)$ is given by

$$\varphi(k,r) = \frac{\sin kr}{k} - \frac{\rho_0}{2k}\frac{\sin k_0 r}{k_0^2 + \rho_0 I(r)}$$
$$\times\left(\frac{\sin(k-k_0)r}{k-k_0} - \frac{\sin(k+k_0)r}{k+k_0}\right) \tag{3.13}$$

and

$$\varphi(k_0,r) = k_0 \sin k_0 r / [k_0^2 + \rho_0 I(r)] \tag{3.14}$$

which is in L^2. The large-r behavior of $V(r)$ is given by

$$V(r) = -4k_0 \sin 2k_0 r/r + O(r^{-2}), \tag{3.15}$$

so we have a case with $A = -4$, $\gamma = 2$, and an embedded eigenvalue at k_0^2. From (3.13) it follows immediately that $F(k) \equiv 1$. We can now verify formula (2.66) in this case. Set $k_0 = 1$. From the relations (3.2), (3.3) with $r_0 = \pi/2$ we get $h_2(k,r) = \varphi(k,r)\cos kr - k^{-1}\varphi'(k,r)\sin kr$ and after a straightforward calculation using (3.13) we find

$$\lim_{\epsilon \to 0} h_{2,\epsilon}(r/\epsilon) = -(2/r)\sin^2(r/2) \tag{3.16}$$

and thus [since $\epsilon^{-1}\hat{h}_{2,0}(r/\epsilon) \to 0$]

$$\lim_{\epsilon \to 0}(1/\epsilon)(\hat{h}_{2,\epsilon}(r/\epsilon) - \hat{h}_{2,0}(r/\epsilon))$$
$$= \tilde{V}_{2,0}(r) = -2e^{-1/2T(r)}r^{-2}\sin^2(r/2). \tag{3.17}$$

In this case we have found a solution to (2.64). (It is an amusing exercise to verify this directly by computing $(\mathscr{I}_0 \tilde{V}_{2,0}(r))$. Inserting (3.17) in the integral in (2.33) and doing it (the trick is to use integration by parts and to replace the lower limits of the integrals by δ and then let $\delta \to 0$ at the end) we get $\mathscr{I}(\hat{V}_{2,0}) = i$. Multiplying this by $e^{-i\pi/2}$ [see (3.15)] we find that $\nu = 1$, hence $F(k) \to 1$ as $k \to 1$. This agrees with the table and the fact that $F(k) \equiv 1$.

Our next remark concerns the Titchmarsh–Weyl m coefficient. It is given by Ref. 12

$$m(E) = -F_\theta(k)/F_\varphi(k). \tag{3.18}$$

Here F_φ and F_θ are the Jost functions associated with the boundary condition $\psi(0) = 0$ and $\psi'(0) = 0$ respectively, i.e., $\varphi(k,r)$ obeys (1.6) and $\theta(k,r) = 0$ obeys $\theta(k,0) = 1$, $\theta'(k,0) = 0$. If $\psi_1(0) \neq 0$, then by (2.34) and (2.71)

$$\lim_{E \to 1} m(E) = -D_\theta/D_\varphi \tag{3.19}$$

and D_φ was defined below (2.70) and D_θ is such that $\theta(1,r) \sim D_\theta r^{\nu/2}\sin r$ as $r \to \infty$. If θ is a multiple of ψ_1 then

(3.19) is valid with $D_\theta = 0$ [the behavior of $F_\theta(k)$ is then governed by relations of the form (2.47), (2.60), or (2.66)]. If $\psi_1(0) = 0$, then $D_\theta L_\varphi = W[\theta,\varphi] = 1$, and

$$m(E) \sim -\frac{Q(M_\gamma)}{L_\varphi^2 Q(N_\gamma)}\epsilon^{-\gamma}, \quad \gamma < 1; \tag{3.20}$$

$$m(E) \sim -\frac{2}{L_\varphi^2 \epsilon \ln(\epsilon^{-1})}, \quad \gamma = 1; \tag{3.21}$$

$$m(E) \sim -\frac{1}{(\int_0^\infty \varphi(k_0,t)^2\, dt)(E-1)}, \quad \gamma > 1. \tag{3.22}$$

These relations are true if $E \to 1$ from the upper half-plane. Equation (3.20) was already derived in Ref. 12. Relation (3.22) is in agreement with general results.[11] Strictly speaking, the blow-up of the form $(E-E_0)^{-1}$ near an embedded eigenvalue is in general only known to hold for approach in a sector $\delta \leqslant \arg(E-E_0) \leqslant \pi - \delta (\delta > 0)$. Here we also get it along the real axis.

Lastly we consider a full-line Schrödinger equation with a potential $V(x) = 0$ for $x < 0$ and $V(x)$ of type (2.2) for $x > 0$. We are interested in the behavior of the transmission and reflection coefficients near the resonance point. In the whole line problem we have the two Jost solutions $f_\pm(k,x)$ obeying $e^{-ikx}f_+(k,x) \to 1$ as $x \to \infty$ and $e^{ikx}f_-(k,x) \to 1$ as $x \to -\infty$. Then the transmission and reflection coefficients are given by (see, e.g., Refs. 5 and 17)

$$T(k) = -2ik/W[f_+, f_-], \tag{3.23}$$

$$R_1(k) = [T(k)/2ik]W[\bar{f}_+, f_-], \tag{3.24}$$

$$R_2(k) = [T(k)/2ik]W[f_+, \bar{f}_-], \tag{3.25}$$

$R_1(R_2)$ is the reflection coefficient for a wave incident from the right(left). For our potential, $f(k,x) = e^{-ikx}$ if $x < 0$. Also $f'_+(k,0) = -W[f,\theta] = -F_\theta(k)$ and $f_+(k,0) = F_\varphi(k)$, where θ is the solution introduced above (3.19). Then

$$W[f_+, f_-] = -F_\varphi(k)[ik + m(E)]. \tag{3.26}$$

Since we know the behaviors of $F_\varphi(k)$ and $m(E)$ in all cases it is straightforward to figure out the behavior of $T(k)$, $R_1(k)$, and $R_2(k)$ as $k \to k_0$. The results are as follows:

$$T(k) \sim c_\gamma(k-k_0)^{\gamma/2}, \quad k \to k_0 \,(\mathrm{Im}\, k \geqslant 0, c_\gamma \neq 0); \tag{3.27}$$

$$R_1(k) = -\frac{\bar{F}_\varphi}{F_\varphi}\frac{ik + \bar{m}}{ik + m} \to -e^{\mp i(\gamma/2)\pi},$$
$$k \to k_0 \pm (k \text{ real}); \tag{3.28}$$

$$R_2(k) = \frac{ik - m}{ik + m} \to -\frac{D_\theta + iD_\varphi}{D_\theta - iD_\varphi}, \quad k \to k_0 \,(\mathrm{Im}\, k \geqslant 0). \tag{3.29}$$

Note that (3.28) holds for real k only because, in general, $R_1(k)$ does not have an analytic continuation into the upper half-plane. In (3.29) we get $R_2(k_0) = -1$ if $\psi_1(0) = 0$ ($D_\varphi = 0$) since $m(E)$ blows up at E_0. This is not unexpected because the solution which describes a plane wave coming in from the left is given by $e^{ikx} + R_2(k)e^{-ikx}$ for $x < 0$ [since $V(x) = 0$ for $x < 0$]. If $R_2(k_0) = -1$, then this solution continues to the right as a multiple of $\psi_1(x)$ and hence represents a damped transmitted wave. In any

case, the above relations show that at $E = 1$ we have total reflection. For similar results in "half-crystals" see Ref. 18.

ACKNOWLEDGMENTS

APPENDIX: PROOF OF LEMMA (2.1)

The idea of the proof is the following. We evaluate the Wronskian $F(k) = W[f, \varphi]$ at a point $r = R(\epsilon)$, where $R(\epsilon)$ tends to infinity as $|\epsilon| \to 0$ in a manner to be specified below. On the interval $[R(\epsilon), \infty)$ we construct $f(k, r)$ by adapting a method of[19] to complex k. On $[0, R(\epsilon)]$ we construct $\varphi(k, r)$ by using the standard integral equation.

Let

$$s(k,r) = \frac{1}{2k} \int_r^\infty V(t)dt - \frac{e^{-2ikr}}{2k} \int_r^\infty V(t)e^{2ikt}dt, \quad \text{(A1)}$$

$$\eta(k,r) = s'(k,r) = ie^{-2ikr} \int_r^\infty V(t)e^{2ikt}dt, \quad \text{(A2)}$$

$$f_0(k,r) = \exp(ikr + is(k,r)), \quad \text{(A3)}$$

$$\psi_0(k,r) = f_0(k,r) \int_{R(\epsilon)}^r f_0^{-2}(k,t)dt. \quad \text{(A4)}$$

Rewrite (1.1) as

$$-\psi'' + (V(r) - k^2 - \eta(k,r)^2)\psi = -\eta(k,r)^2\psi \quad \text{(A5)}$$

and note that f_0 and ψ_0 are linearly independent solutions of the corresponding homogeneous equation ($W[f_0, \psi_0] = 1$). Using variation of parameters we may define the Jost solution as the solution to

$$f(k,r) = f_0(k,r) + \int_r^\infty G(k;r,t)f(k,t)dt, \quad \text{(A6)}$$

where

$$G(k;r,t) = [f_0(k,r)\psi_0(k,t) - \psi_0(k,r)f_0(k,t)]\eta(k,t)^2. \quad \text{(A7)}$$

Looking at the second integral in (A1) we observe that resonance occurs as $k \to 1$ owing to the product $(\sin 2t)e^{2ikt}$. This causes the factor $|\epsilon|^{-1}$ in the next estimate which follows easily from (A1) and (A2) by integrating by parts

$$|s(k,r)|, |\eta(k,r)| \leq C|\epsilon|^{-1}(r+1)^{-1}. \quad \text{(A8)}$$

The constant C is independent of k for $k \in D_\delta$. It is now convenient to impose a first restriction on $R(\epsilon)$, namely,

$$R(\epsilon) \geq |\epsilon|^{-1}. \quad \text{(A9)}$$

Then $|s(k,r)| \leq C$ for $r \geq R(\epsilon)$ and also

$$|f_0(k,r)| \leq Ce^{-(\operatorname{Im} k)r}, \quad k \in D_\delta. \quad \text{(A10)}$$

Considering $\psi_0(k,r)$ we expand $e^{is(k,r)}$ so that $[r \geq R(\epsilon)]$

$$\int_{R(\epsilon)}^r f_0(k,t)^{-2}dt = \int_{R(\epsilon)}^r e^{-2ikt}dt$$
$$- 2i \int_{R(\epsilon)}^r e^{-2ikt}s(k,t)dt$$
$$+ \int_{R(\epsilon)}^r e^{-2ikt}O(s(k,t)^2)dt. \quad \text{(A11)}$$

The first integral on the right is $O(e^{2(\operatorname{Im} k)r}), k \in D_\delta$. The second integral we integrate by parts

$$\int_{R(\epsilon)}^r e^{2ikt}s(k,t)dt = \frac{i}{2k} s(k,t)e^{2ikt}\Big|_{R(\epsilon)}^r$$
$$- \frac{i}{2k} \int_{R(\epsilon)}^r e^{-2ikt}s'(k,t)dt. \quad \text{(A12)}$$

The integrated term is again $O(e^{2(\operatorname{Im} k)r})$. In the integral on the right we substitute (A2) for s' and integrate by parts

$$\int_{R(\epsilon)}^r e^{-2ikt}s'(k,t)dt = -\frac{e^{-4ikt}}{4k} \int_t^\infty V(s)e^{2iks} ds\Big|_{R(\epsilon)}^r$$
$$- \frac{1}{4k} \int_{R(\epsilon)}^r V(t)e^{-2ikt} dt. \quad \text{(A13)}$$

The integrated term is $O(e^{2(\operatorname{Im} k)r})$ and this is also true of the integral, again on integrating by parts. We come to the third term on the right of (A11). It can be estimated by using (A8)

$$\left|\int_{R(\epsilon)}^r e^{-2ikt}O(s(k,t)^2)dt\right| \leq Ce^{2(\operatorname{Im} k)r}|\epsilon|^{-2}R(\epsilon)^{-1}. \quad \text{(A14)}$$

It is crucial to leave the right-hand side in this form and not to simplify it by using (A9). From (A11)–(A14) and (A10) we conclude that

$$|\psi_0(k,r)| \leq Ce^{(\operatorname{Im} k)r}(1 + |\epsilon|^{-2}R(\epsilon)^{-1}),$$
$$k \in D_\delta, \quad r \geq R(\epsilon), \quad \text{(A15)}$$

and

$$|G(k;r,t)| \leq C(1 + |\epsilon|^{-2}R(\epsilon)^{-1})|\epsilon|^{-2}(t+1)^{-2}$$
$$\times \exp(\operatorname{Im} k)(t-r), \quad t \geq r \geq R(\epsilon). \quad \text{(A16)}$$

Put $B(\epsilon) = (1 + |\epsilon|^{-2}R(\epsilon)^{-1})|\epsilon|^{-2}$ and $\hat{f}(k,r) = e^{(\operatorname{Im} k)r}f(k,r)$. Then

$$|\hat{f}(k,r)| \leq C + CB(\epsilon)\int_r^\infty (t+1)^{-2}|\hat{f}(k,t)|dt \quad \text{(A17)}$$

and hence

$$|f(k,r)| \leq Ce^{-(\operatorname{Im} k)r}e^{CB(\epsilon)/r}, \quad k \in D_\delta, \quad r \geq R(\epsilon), \quad \text{(A18)}$$

by an application of Gronwall's inequality. On differentiating (A6) it follows that $f'(k,r)$ obeys a similar estimate [with a different c but the same constant $B(\epsilon)$].

Now we turn to the solution $\varphi(k,r)$ on $[0, R(\epsilon)]$. Let ψ_1, ψ_2 be the solutions defined in (1.4) with $b = 2$ (so that $k_0 = 1$). Then, by using variation of parameters, we have that

$$\varphi(k,r) = \varphi(1,r) + (k^2 - 1)\int_0^r (\psi_1(r)\psi_2(t) - \psi_2(r)\psi_1(t))\varphi(k,t)dt. \quad \text{(A19)}$$

Put $\hat{\varphi}(k,r) = (r+1)^{-\gamma/2}\varphi(k,r)$. Then

$$|\hat{\varphi}(k,r)| \leq C + C|\epsilon|\int_0^r |\hat{\varphi}(k,t)|dt \quad \text{(A20)}$$

and so

$$|\varphi(k,r)| \leq Ce^{C|\epsilon|r}(r+1)^{\gamma/2}, \quad k \in D_\delta, \quad r \leq R(\epsilon). \quad \text{(A21)}$$

Also, $\varphi'(k,r)$ obeys a similar estimate. Combining (A18)

and (A21), setting $r = R(\epsilon)$, yields for $F(k)$

$$|F(k)| \leqslant Ce^{C(B(\epsilon)R(\epsilon)^{-1} + |\epsilon|R(\epsilon))}(R(\epsilon) + 1)^{\gamma/2}. \quad (A22)$$

Now put $R(\epsilon) = |\epsilon|^{-\sigma}$. Then $B(\epsilon)R(\epsilon)^{-1} + |\epsilon|R(\epsilon)$ $= |\epsilon|^{-2+\sigma} + |\epsilon|^{-4+2\sigma} + |\epsilon|^{1-\sigma}$. In order that all three exponents are between -1 and 0 it is necessary that $\frac{3}{2} < \sigma < 2$. Then (A9) is automatically satisfied. If $\frac{3}{2} < \sigma < 2$ we have that min$\{-2+\sigma, -4+2\sigma, 1-\sigma\}$ is equal to $-4 + 2\sigma$ when $\frac{3}{2} < \sigma \leqslant \frac{5}{3}$ and equal to $1 - \sigma$ when $\frac{5}{3} \leqslant \sigma < 2$. This means that the exponential factor in (A22) can be bounded by $e^{C|\epsilon|^{-\beta}}$ with any β, $\frac{2}{3} \leqslant \beta < 1$. In order to absorb the factor $(R(\epsilon) + 1)^{\gamma/2}$ we discard the value $\frac{2}{3}$. This proves Lemma (2.1).

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