5.1.1.C. Variable bound factor based constraints

By using the above constraints (5.3), (5.4), (5.6), and (5.8) on the variables $x$, $y$, $\alpha$ and $w$, we generate several implied constraints based on various products using the variable-bound factors as described below. Consider the following set of equations defined previously.

$$\Psi_{0k} - \Psi_{1k}\, x_i - \Psi_{2k}\, y_i \geq 0 \qquad \forall(i,k) \qquad \text{(A1)}$$

$$\alpha_{ij} \geq 0 \qquad \forall(i,j) \qquad \text{(A2)}$$

$$u_{\alpha_{ij}} - \alpha_{ij} \geq 0 \qquad \forall(i,j) \qquad \text{(A3)}$$

$$\alpha_{ij} - \max\{|x_i - a_j|, |y_i - b_j|\} \geq 0 \; \forall(i,j) \qquad \text{(A4)}$$

$$\sqrt{2}\;\alpha_{ij} - |x_i - a_j| - |y_i - b_j| \geq 0 \;\; \forall(i,j) \qquad \text{(A5)}$$

$$u_{ij} - w_{ij} \geq 0 \qquad \forall(i,j) \qquad \text{(A6)}$$

$$w_{ij} - l_{ij} \geq 0 \qquad \forall(i,j). \qquad \text{(A7)}$$

Now, we add to the reformulated problem the following sets of implied or valid constraints:

1. Constraint set

$$w_{ij}\Psi_{0k} - u_{ij}(\Psi_{0k} - \Psi_{1k}\, x_i - \Psi_{2k}\, y_i) \leq w_{ij}(\Psi_{1k}\, x_i + \Psi_{2k}\, y_i) \leq w_{ij}\Psi_{0k} - l_{ij}(\Psi_{0k} - \Psi_{1k}\, x_i - \Psi_{2k}\, y_i)$$
$$\forall(i,j,k) \qquad (5.11)$$

obtained by multiplying (A1) with (A6) and (A7).

3. Constraint set $u_{\alpha_{ij}}\alpha_{ij} \geq (x_i - a_j)^2 + (y_i - b_j)^2 \qquad \forall(i,j) \qquad (5.12)$

   obtained by multiplying (A2) with (A3) and using the value of $\alpha_{ij}^2$ from (5.2.b).

4. Constraint set $l_{ij}\alpha_{ij} \leq w_{ij}\alpha_{ij} \leq u_{ij}\alpha_{ij} \;\; \forall(i,j) \qquad (5.13)$

obtained by multiplying the constraint set (A2) with (A6) and (A7).

5. Constraint set $u_{\alpha_{ij}} w_{ij} + \alpha_{ij} u_{ij} - u_{\alpha_{ij}} u_{ij} \leq w_{ij}\alpha_{ij} \leq u_{\alpha_{ij}} w_{ij} + \alpha_{ij} l_{ij} - u_{\alpha_{ij}} l_{ij} \;\; \forall(i,j) \qquad (5.14)$

obtained by multiplying the constraint set (A3) with (A6) and (A7).

6.. Constraint set

$$w_{ij}(x_i - a_j) + l_{ij}(\alpha_{ij} - x_i + a_j) \leq w_{ij}\alpha_{ij} \leq w_{ij}(x_i - a_j) + u_{ij}(\alpha_{ij} - x_i + a_j),$$

$$w_{ij}(a_j - x_i) + l_{ij}(\alpha_{ij} + x_i - a_j) \le w_{ij}\alpha_{ij} \le w_{ij}(a_j - x_i) + u_{ij}(\alpha_{ij} + x_i - a_j),$$

$$w_{ij}(y_i - b_j) + l_{ij}(\alpha_{ij} - y_i + b_j) \le w_{ij}\alpha_{ij} \le w_{ij}(y_i - b_j) + u_{ij}(\alpha_{ij} - y_i + b_j),$$

$$w_{ij}(b_j - y_i) + l_{ij}(\alpha_{ij} + y_i - b_j) \le w_{ij}\alpha_{ij} \le w_{ij}(b_j - y_i) + u_{ij}(\alpha_{ij} + y_i - b_j) \qquad \forall(i, j) \quad (5.15)$$

obtained by multiplying the four equivalent linear inequalities that are represented by constraint set (A4) with (A6) and (A7).

7. Constraint set

$$w_{ij}(x_i - a_j + y_i - b_j) + l_{ij}(\sqrt{2}\,\alpha_{ij} - x_i + a_j - y_i + b_j) \le \sqrt{2}\,w_{ij}\alpha_{ij} \le w_{ij}(x_i - a_j + y_i - b_j) + u_{ij}(\sqrt{2}\,\alpha_{ij} - x_i + a_j - y_i + b_j),$$

$$w_{ij}(x_i - a_j - y_i + b_j) + l_{ij}(\sqrt{2}\,\alpha_{ij} - x_i + a_j + y_i - b_j) \le \sqrt{2}\,w_{ij}\alpha_{ij} \le w_{ij}(x_i - a_j - y_i + b_j) + u_{ij}(\sqrt{2}\,\alpha_{ij} - x_i + a_j + y_i - b_j),$$

$$w_{ij}(a_j - x_i + y_i - b_j) + l_{ij}(\sqrt{2}\,\alpha_{ij} + x_i - a_j - y_i + b_j) \le \sqrt{2}\,w_{ij}\alpha_{ij} \le w_{ij}(a_j - x_i + y_i - b_j) + u_{ij}(\sqrt{2}\,\alpha_{ij} + x_i - a_j - y_i + b_j),$$

$$w_{ij}(a_j - x_i - y_i + b_j) + l_{ij}(\sqrt{2}\,\alpha_{ij} + x_i - a_j + y_i - b_j) \le \sqrt{2}\,w_{ij}\alpha_{ij} \le w_{ij}(a_j - x_i - y_i + b_j) + u_{ij}(\sqrt{2}\,\alpha_{ij} + x_i - a_j + y_i - b_j),$$

$$\forall(i, j) \qquad (5.16)$$

obtained by multiplying the four equivalent linear inequalities that are represented by constraint set (A5) with (A6) and (A7).

5.1.1.D. Location Cuts

In light of the form of the objective function and (1a), we can generate several valid cuts in the space of the location variables $\alpha$ that can also be added to the reformulated problem. These constraints are stated below followed by an explanation of their derivation.

$$\sum_i \sum_j c_{ij} w_{ij}^* \alpha_{ij} \ge \nu^* \qquad (5.17)$$

$$\sum_i \sum_j \alpha_{ij} \ge \nu_1 \qquad (5.18)$$

$$\sum_i \sum_j c_{ij} \overline{w}_{ij} \alpha_{ij} \ge \overline{\nu}\,. \qquad (5.19)$$

**Remark 5.3** In constraint (5.17), $w^*$ is the incumbent solution for EDLAP and $\nu^*$ is its objective value. This constraint is updated each time a new incumbent is found. Its validity follows from the fact that a minimizing feasible solution $\alpha$ on the left-hand side yields a value of $\nu^*$.

**Remark 5.4** In constraint (5.18), $n_1$ is the location problem value with all weights "$w_{ij}$"≡1. $\forall(i, j)$, that is,

$$n_1 = \min \sum_i \sum_j \{(x_i - a_j)^2 + (y_i - b_j)^2\}^{1/2} \quad . \tag{5.20}$$

**Remark 5.5** Constraint (5.19) is similar to (5.17) and based on $\bar{w}$ and $\bar{v}$ that satisfy

$$l_{ij} \le \bar{w}_{ij} \le u_{ij} \; \forall(i, j) \text{ ,and } \bar{n} = \min \sum_i \sum_j c_{ij} \, \bar{w}_{ij} \, \{(x_i - a_j)^2 + (y_i - b_j)^2\}^{1/2} . \tag{5.21}$$

This constraint plays a key role in the convergence argument as well as in tightening the relaxation. The particular value of $\bar{w}_{ij}$ selected for this purpose is described in Section 5.3.

5.1.1.E. Other types of generated constraints

Additionally, we generate the constraint set obtained by multiplying the capacity constraint $\sum_j w_{ij} = s_i$ with its corresponding $x_i$ and $y_i$ variables, respectively, $\forall i$. The resulting constraints are

$$\begin{aligned} (\sum_j w_{ij} - s_i) \; x_i &= 0 \quad \forall i \\ (\sum_j w_{ij} - s_i) \; y_i &= 0 \quad \forall i \, . \end{aligned} \tag{5.22}$$

**Remark 5.6** Note that similar products with demand constraints are not generated since this will create additional nonlinear product terms of the type $w_{pj} \, x_q$ and $w_{pj} \, y_q$ for $p \neq q$. This will increase the problem size substantially, and is therefore avoided.

Now, by adding the previous generated constraints to the equivalent problem and by excluding the constraints (5.3), (5.4), (5.5), and (5.7), since they are implied by constraints (5.11), (5.14), (5.15), (5.16), respectively, but replacing (5.2b) with its relaxation (5.12) the following reformulated relaxation is obtained.

$$\text{Minimize } \sum_i \sum_j c_{ij}\ w_{ij}\ \mathrm{a}_{ij} \tag{5.23a}$$

subject to

$$l_{ij}\,\mathrm{a}_{ij} \le w_{ij}\,\mathrm{a}_{ij} \le u_{ij}\,\mathrm{a}_{ij}\ \forall(i, j) \tag{5.23b}$$

$$u_{\mathrm{a}_{ij}}\, w_{ij} + \mathrm{a}_{ij}\, u_{ij} - u_{\mathrm{a}_{ij}}\, u_{ij} \le w_{ij}\,\mathrm{a}_{ij} \le u_{\mathrm{a}_{ij}}\, w_{ij} + \mathrm{a}_{ij}\, l_{ij} - u_{\mathrm{a}_{ij}}\, l_{ij}\ \forall(i, j) \tag{5.23c}$$

$$\sum_i \sum_j c_{ij} w_{ij}^{*} \mathrm{a}_{ij} \ge \mathrm{n}^{*} \tag{5.23d}$$

$$\sum_i \sum_j \mathrm{a}_{ij} \ge \mathrm{n}_1 \tag{5.23e}$$

$$\sum_i \sum_j c_{ij} \overline{w}_{ij} \mathrm{a}_{ij} \ge \overline{\mathrm{n}} \tag{5.23f}$$

$$\begin{aligned}
&w_{ij}\,(x_i - a_j) + l_{ij}(\mathrm{a}_{ij} - x_i + a_j) \le w_{ij}\mathrm{a}_{ij} \le w_{ij}\,(x_i - a_j) + u_{ij}(\mathrm{a}_{ij} - x_i + a_j),\\
&w_{ij}\,(a_j - x_i) + l_{ij}(\mathrm{a}_{ij} + x_i - a_j) \le w_{ij}\mathrm{a}_{ij} \le w_{ij}\,(a_j - x_i) + u_{ij}(\mathrm{a}_{ij} + x_i - a_j),\\
&w_{ij}\,(y_i - b_j) + l_{ij}(\mathrm{a}_{ij} - y_i + b_j) \le w_{ij}\,\mathrm{a}_{ij} \le w_{ij}\,(y_i - b_j) + u_{ij}(\mathrm{a}_{ij} - y_i + b_j),\\
&w_{ij}\,(b_j - y_i) + l_{ij}(\mathrm{a}_{ij} + y_i - b_j) \le w_{ij}\mathrm{a}_{ij} \le w_{ij}\,(b_j - y_i) + u_{ij}(\mathrm{a}_{ij} + y_i - b_j)\\
&\qquad \forall(i, j)
\end{aligned} \tag{5.23g}$$

$$\begin{aligned}
&w_{ij}(x_i-a_j+y_i-b_j)+l_{ij}(\sqrt{2}\,\mathrm{a}_{ij}-x_i+a_j-y_i+b_j \le \sqrt{2}\,w_{ij}\mathrm{a}_{ij} \le w_{ij}(x_i-a_j+y_i-b_j)+u_{ij}(\sqrt{2}\,\mathrm{a}_{ij}-x_i+a_j-y_i+b_j),\\
&w_{ij}(x_i-a_j-y_i+b_j)+l_{ij}(\sqrt{2}\,\mathrm{a}_{ij}-x_i+a_j+y-b_j) \le \sqrt{2}\,w_{ij}\mathrm{a}_{ij} \le w_{ij}(x_i-a_j-y_i+b_j)+u_{ij}(\sqrt{2}\,\mathrm{a}_{ij}-x_i+a_j+y_i-b_j),\\
&w_{ij}(a_j-x_i+y_i-b_j)+l_{ij}(\sqrt{2}\,\mathrm{a}_{ij}+x_i-a_j-y_i+b_j) \le \sqrt{2}\,w_{ij}\mathrm{a}_{ij} \le w_{ij}(a_j-x_i+y_i-b_j)+u_{ij}(\sqrt{2}\,\mathrm{a}_{ij}+x_i-a_j-y_i+b_j),\\
&w_{i}(a_j-x_i-y+b_j)+l_{ij}(\sqrt{2}\,\mathrm{a}_{ij}+x_i-a_j+y_i-b_j) \le \sqrt{2}\,w_{ij}\mathrm{a}_{ij} \le w_{ij}(a_j-x_i-y_i+b_j)+u_{ij}(\sqrt{2}\,\mathrm{a}_{ij}+x_i-a_j+y_i-b_j),\\
&\qquad \forall(i, j)
\end{aligned} \tag{5.23h}$$

$$\Big(\sum_j w_{ij} - s_i\Big)\ x_i = 0 \quad \forall i \tag{5.23i}$$

$$\Big(\sum_j w_{ij} - s_i\Big)\ y_i = 0 \quad \forall i \tag{5.23j}$$

$$w_{ij}\,\mathrm{Y}_{0k} - u_{ij}\,(\mathrm{Y}_{0k} - \mathrm{Y}_{1k}\, x_i - \mathrm{Y}_{2k}\, y_i) \le w_{ij}\,(\mathrm{Y}_{1k}\, x_i + \mathrm{Y}_{2k} y_i) \le w_{ij}\,\mathrm{Y}_{0k} - l_{ij}\,(\mathrm{Y}_{0k} - \mathrm{Y}_{1k}\, x_i - \mathrm{Y}_{2k}\, y_i) \quad \forall(i, j, k) \tag{5.23k}$$