Resolutions And Cohomology Of Finite Dimensional Algebras

by

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(ABSTRACT)

The purpose of this thesis is to develop machinery for calculating Hochschild cohomology groups of certain finite dimensional algebras. So let Λ be a finite dimensional quotient of a path algebra. A method of modeling the enveloping algebra Λ^e of Λ on a computer is presented. Adding the extra hypothesis that Λ is a monomial algebra, we construct a minimal projective resolution of Λ over Λ^e. The syzygies for this resolution exhibit an alternating behavior which is explained by the construction of a special sequence of paths from the quiver of Λ. Finally, a technique for calculating Hochschild cohomology groups from these resolutions is presented. An important application involving an invariant characterization for a certain class of monomial algebras is also included.

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Chapter 1

Introduction

The purpose of this thesis is to develop machinery for calculating homological invariants of finite dimensional algebras. In our case the invariants are Hochschild cohomology groups. Although the definition of these groups has been known for fifty years, actual calculations have been limited. With the introduction of computers and algorithms for computing minimal projective resolutions of modules over noncommutative rings, many of the computations of homological algebra are now more accessible. Recently emerging techniques from noncommutative Gröbner basis theory are also facilitating work in this area. As a result, symbolic computation has become a valuable tool for attacking problems from the representation theory of algebras.

Despite the recent advances in computational techniques, Hochschild cohomology groups are still very difficult to calculate in most cases. So it seems necessary at this point to justify the need for these groups. To do this, I refer to the origin of the subject. Cohomology was first used by algebraic topologists to assign algebraic invariants to topological spaces. The dimensions of many spaces can be described using cohomology. For example, it is known that the m^{th} cohomology group of an n-dimensional manifold vanishes for $m > n$, while these groups are nonzero for properly chosen coefficients and $m \leq n$. With these techniques in mind, algebraists have defined an analogous dimensional invariant for algebras. If A is an R-algebra ($A \neq 0$), then the *dimension* of A is

$$dim A = \sup\{n : \mathrm{H}^n_R(A, M) \neq 0 \quad \text{for some } A-\text{bimodule } M\}.$$

Here $\mathrm{H}^n_R(A, M)$ is the n^{th} Hochschild cohomology group of A with coefficients in the $A-$bimodule M. Details can be found in [22].

Another justification for Hochschild cohomology is that it has applications in other areas of mathematics. For example, there are established links between these groups and algebraic geometry. The first and second cohomology groups have been interpreted as the groups of infinitesimal automorphisms and infinitesimal deformations, respectively, of the given algebra (see [17]). Another application comes from algebraic topology. There is a theorem relating simplicial cohomology to Hochschild cohomology (see [16]). Given a simplicial complex K, one can construct the corresponding partially ordered set with the simplices of K ordered by inclusion. One can then define an *incidence algebra* from this poset. The Hochschild cohomology of this algebra is equivalent to the cohomology of the simplicial complex K. A more inclusive summary of the current work in Hochschild cohomology can be found in [20].

Now that the need for Hochschild cohomology has been established, we require a technique for computing these groups for a given algebra Λ. One approach is to construct a Λ^e-projective resolution of Λ. Here $\Lambda^e = \Lambda^{op} \bigotimes \Lambda$ is the enveloping algebra of Λ and k is an algebraically closed field. Then Λ is a right Λ^e-module. This is equivalent to saying that Λ is a $\Lambda - \Lambda$ bimodule (for details see [22]). The n^{th} Hochschild cohomology group of Λ is $\mathrm{H}^n(\Lambda, \Lambda) = \mathrm{Ext}^n_{\Lambda^e}(\Lambda, \Lambda)$. One can use the standard resolution for this purpose (see [7]). However, this resolution is too large to be of much use for many calculations. Smaller resolutions are desirable if one wants to pursue cohomology. In [9] Cibils gives a smaller resolution of Λ in the case where $\Lambda/\mathbf{r}$ is separable (here $\mathbf{r}$ is the Jacobson radical of Λ). Happel provides the projectives for a minimal projective resolution of a finite dimensional k-algebra over its enveloping algebra (see [20]). However, the maps are not given. The need for maps corresponding to minimal resolutions is clear.

The computer code used to calculate the resolutions for this thesis is based on [15] and requires one to use the language of path algebras. So let Γ denote a finite quiver (directed graph). We define the path algebra $k\Gamma$ over the field k to be the vector space with basis B consisting of all finite directed paths in Γ. Multiplication is defined by concatenation in the usual way. We denote the vertex set by $\Gamma_0 = \{v_1, v_2, ..., v_n\}$ and the arrow set by $\Gamma_1 = \{a_1, a_2, ..., a_m\}$. The algebras of interest to us have the form $\Lambda = k\Gamma/I$, where $J^N \subset I \subset J^2$ for some positive integer $N \geq 2$. Here J is the two-sided ideal generated by the arrows. An ideal I of this form is called *admissible.* Standard results from the representation theory of finite dimensional algebras show that this assumption is not too restrictive for our purpose. Note that, under the above hypotheses, Λ is finite dimensional over k. If we let Γ^{op} denote the quiver obtained from Γ by reversing all of the arrows, then $\Lambda^{op} \cong k\Gamma^{op}/I^{op}$. Here I^{op} is the obvious analogue of I in $k\Gamma^{op}$. The admissible ordering that we place on Λ induces an analogous ordering on Λ^{op} (see Chapter 3).

From now on when we refer to a path we mean a directed path. If p is a path we denote the origin of p by $o(p)$, the terminus of p by $t(p)$, and the length of p by $l(p)$. Here the length of a path means the number of arrows in the path. Each vertex will be regarded as a path of length zero. Finally, let us define the class of algebras that are of most interest throughout this thesis. We call Λ a *monomial algebra* if it can be written as a quotient of a path algebra, $\Lambda \cong k\Gamma/I$, where I is generated by a finite number of paths in Γ of length greater than or equal to two. Note that we are not saying every representation of Λ has resolution of Λ in the case where $\Lambda/\mathbf{r}$ is separable (here $\mathbf{r}$ is the Jacobson radical of Λ). Happel provides the projectives for a minimal projective resolution of a finite dimensional k-algebra over its enveloping algebra (see [20]). However, the maps are not given. The need for maps corresponding to minimal resolutions is clear.

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The heart of this paper is the construction of minimal projective resolutions for finite dimensional monomial algebras. Often monomial algebras are called zero relations algebras in the literature. It turns out that the syzygies for these resolutions exhibit an alternating behavior. This phenomenon is induced by certain properties of the associated sequence of paths which we will define in Chapter 5. Any even-odd behavior observed in $\mathrm{H}^n(\Lambda,\Lambda)$, for Λ monomial, follows naturally from this resolution. Other more general modules may also have properties linked to the combinatorics of these monomial algebras. For example, suppose M is a left Λ-module. If the minimal Λ^e-projective resolution of Λ is given by $\cdots P_n \xrightarrow{\phi_n} P_{n-1} \xrightarrow{\phi_{n-1}} \cdots P_0 \xrightarrow{\pi} \Lambda \longrightarrow 0$, then

$$\cdots \longrightarrow P_n \bigotimes_{\Lambda} M \xrightarrow{\phi_n \bigotimes 1_M} P_{n-1} \bigotimes_{\Lambda} M \xrightarrow{\phi_{n-1} \bigotimes 1_M} \cdots \xrightarrow{\phi_1 \bigotimes 1_M} P_0 \bigotimes_{\Lambda} M \longrightarrow M \longrightarrow 0$$

is a Λ-projective resolution of M. Although this resolution need not be minimal, the minimal resolution is a summand of this resolution. Other examples where this link may appear come from the resolutions of the simple A-modules given in [2]. Here A is a split basic k-algebra and there is a quiver Γ such that $A \cong k\Gamma/I$. Although I need not be a monomial ideal, a construction is given that produces what is called the associated monomial algebra A_{mon} of A. The complexity of this construction determines how much the resolutions of the simple modules over A_{mon} and A differ. So it may be possible, in some cases, for the alternating syzygy behavior of monomial algebras to be inherited, at least in part, by more general modules.

As a last prerequisite, we assume the reader is familiar with some of the terminology from noncommutative Gröbner basis theory. In particular, we freely use the term $Minsharp_<(I)$ which is the noncommutative analogue of a reduced Gröbner basis. For a monomial algebra $Minsharp_<(I)$ is just a finite collection of paths where, if $p \in Minsharp_<(I)$, then no proper subdivisors of p are in $Minsharp_<(I)$. For a more general description see [14]. Now, we need an admissible order $<$ on B. Assume first that $v_1 < \cdots v_n < a_1 < \cdots < a_m$ and let $<$ be length-lexicographic order reading origin to terminus for each directed path in B. This is the admissible order that we will use for B. Let $x = \sum_{i=1}^{t} \alpha_i p_i$, $p_i \in B$, where the p_i are distinct and $\alpha_i \in k \setminus 0$. We define the *support* of x to be $\{p_1, ..., p_t\}$ and *tip*(x) to be the largest path under this order in the support of x. We say x is *uniform* if there exists $u, v \in \Gamma_0$ such that $o(p_i) = u$ and $t(p_i) = v$ for $i = 1, ..., t$. In the literature two paths with the same origin and terminus are usually called *parallel*. Now, define *Tip*(I) to be the set of paths that are tips of elements of I. Let *NonTip*(I)=$B\setminus$Tip(I). It is known that the path algebra $k\Gamma$ has a k-vector space decomposition as follows:

$$k\Gamma = I \bigoplus \mathrm{Span}(\mathrm{NonTip}(I)).$$

So every nonzero element $x \in k\Gamma$ can be written $x = I(x) \bigoplus N(x)$ where $I(x) \in I$ and $N(x) \in$Span(NonTip(I)). $N(x)$ is called the *normal form* of x modulo I. Finally, if $x = \sum_{i=1}^{r} \beta_i p_i \bigotimes q_i$, $\beta_i \in k$, then reducing x to its *tensor normal form* modulo I will mean reducing each p_i and q_i to their respective normal forms modulo I. For a monomial algebra this will just mean that each p_i and q_i is not divisible by any path in $Minsharp_<(I)$. Further discussion of the terminology above can be found in [14] and [13].

To use the resolution package previously cited, one must be able to encode Λ and Λ^e on a computer. This task is not difficult for Λ. The quiver is encoded by its adjacency matrix and the relation set is described by k-linear combinations of paths. The purpose of the next two chapters is to provide a technique for modeling the enveloping algebra Λ^e of Λ on a computer. In chapter 2 we define the tensor quiver Γ^e of Λ^e. Then we use the adjacency matrices of Γ and Γ^{op} to give a "nice" matrix decomposition of the adjacency matrix for Γ^e. The main result of the first two chapters, namely that Λ^e can be written as the quotient of a path algebra, is Theorem 3.2. In proving this theorem we show that the isomorphism actually allows one to lift a special Gröbner basis of Λ to an analogous Gröbner basis of Λ^e. The author has written and implemented a program allowing one to model the enveloping algebra on a computer. This program is based on the results from Chapters 2 and 3.

The main result of this paper is the construction of the minimal projective resolution for a monomial algebra (see Theorem 6.1). Chapter 4 begins with the projective presentation of $\Lambda = k\Gamma/I$ where I is admissible but not necessarily monomial. The second projective and syzygy are also given in this general setting. Chapter 5 describes the associated sequence of paths which determine the higher projectives and syzygies for the monomial case. After stating the main theorem in Chapter 6 we extend the order $<$ above to the projectives P_n. Then we prove that the complex presented in Theorem 6.1 is exact. Following the evident alternating behavior of the maps, the proof is done inductively on pairs of maps. Since the second projective and map are given without the assumption that I is monomial, the induction begins with the third and fourth projectives and syzygies. The proof for the higher projectives and maps is then a generalization of this argument.

The last two chapters deal with Hochschild cohomology. In chapter 7 we show how to calculate these cohomology groups using the projective resolutions from Chapter 6. Using standard identifications from ring theory, the cohomology can be determined by a complex of "nice" vector spaces and induced boundary maps . A few examples are also presented. The alternating syzygy behavior is reflected in some of these examples. The last chapter shows how Hochschild cohomology can be used to give an invariant characterization for a special class of finite dimensional monomial algebras. A generalization of this problem has recently been described as one of the open problems in representation theory (see [3]). A few comments will be provided indicating how this result contributes to the more general solution.

Chapter 2

The Quiver of Λ^e

Our goal now is to write the enveloping algebra of Λ as the quotient of a path algebra. We first need to determine a quiver Γ^e associated with Λ^e. The primitive orthogonal idempotents of $\Lambda^{op} \bigotimes_k \Lambda$ are $v_i \bigotimes v_j$ for $1 \leq i, j \leq n$. So it seems natural to construct a quiver for Γ^e with n^2 vertices corresponding to these idempotents. Let us arrange these vertices into n columns and n rows. We obtain the arrow set of Γ^e by placing a copy of Γ along each column of vertices and a copy of Γ^{op} along each row of vertices. To determine an appropriate labeling scheme for Γ^e, let us enumerate the vertices $1, ..., n^2$ starting from the top left as follows:

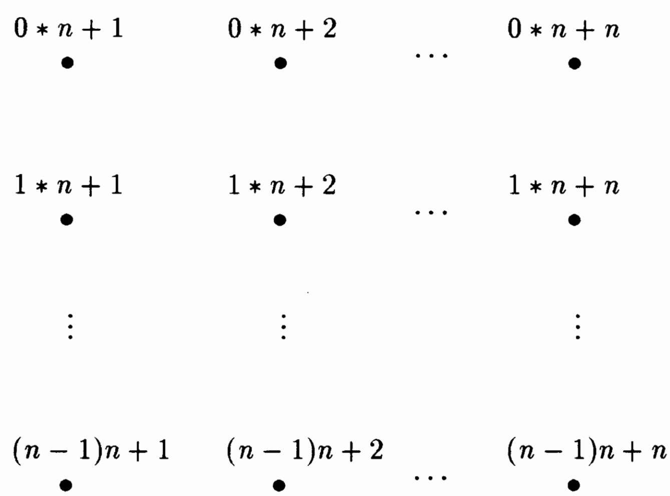

Figure 2.1

We have written each vertex as $h * n + k$, where $0 \leq h \leq n - 1$ and $1 \leq k \leq n$. So a fixed value of h gives a row of vertices and a fixed value of k gives a column of vertices. There is an arrow $a_{l,k}$ in Γ^e from $i * n + k$ to $j * n + k$ iff a_l is an arrow in Γ from v_{i+1} to v_{j+1}. Similarly, there is an arrow $a^{op}_{h,l}$ in Γ^e from $h * n + i$ to $h * n + j$ iff a^{op}_l is an arrow in Γ^{op} from v_i to v_j. Let us now relabel the vertices and arrows of Γ^e by replacing $h * n + k$ with $v_{h+1} \bigotimes v_k$, $a_{l,k}$ with $v_k \bigotimes a_l$, and $a^{op}_{h,l}$ with $a^{op}_l \bigotimes v_{h+1}$. Note that in our description of Γ^e, arrows of the form $v_k \bigotimes a_l$ are vertical whereas arrows of the form $a^{op}_l \bigotimes v_{h+1}$ are horizontal. The following example illustrates this construction.

Example: Let Γ be the following quiver:

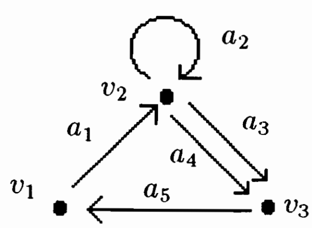

Figure 2.2

Then the adjacency matrices of Γ and Γ^{op} are the following:

$$\Gamma_{mat} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 2 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \Gamma^{op}_{mat} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 2 & 0 \end{bmatrix}.$$

The quiver Γ^e is the following:

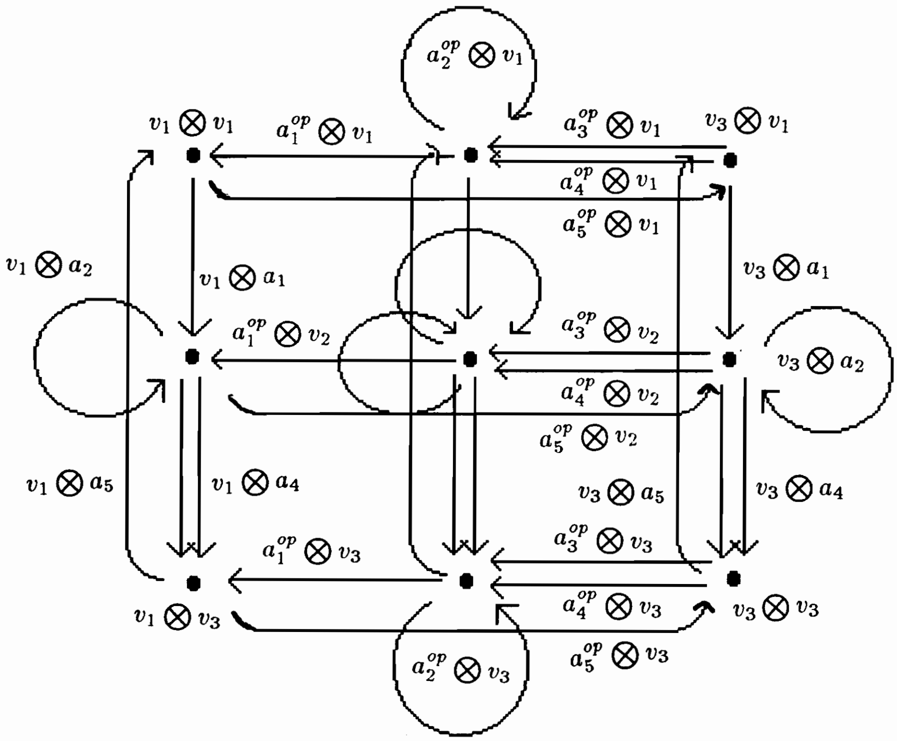

Figure 2.3

Thus, we obtain the corresponding adjacency matrix:

$$\Gamma^e_{mat} = \begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 & 2 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 2 & 0 \end{bmatrix}.$$

Notice that this matrix decomposes as follows:

$$\Gamma^e_{mat} = \begin{bmatrix} \Gamma^{op}_{mat} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \Gamma^{op}_{mat} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \Gamma^{op}_{mat} \end{bmatrix} + \begin{bmatrix} \mathbf{0} & \mathbf{I_3} & \mathbf{0} \\ \mathbf{0} & \mathbf{I_3} & \mathbf{2I_3} \\ \mathbf{I_3} & \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

As we will soon see, this is not a coincidence.

The Adjacency Matrix Decomposition of Γ^e

Let Γ be a finite quiver with n vertices and adjacency matrix

$$\Gamma_{mat} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} = [a_{ij}].$$

Then the adjacency matrix of the opposite graph is the transpose of Γ_{mat}, that is, $\Gamma^{op}_{mat} = [a_{ji}]$. A description of the tensor quiver was just given. Following that reasoning we obtain the following definition:

Definition 2.1 *The tensor quiver Γ^e is the finite directed graph with adjacency matrix defined as follows:*

$$\Gamma^e_{mat} = \left[d_{j*n+k,j'*n+k'}\right]$$

where $0 \le j, j' \le n-1$, $1 \le k, k' \le n$, and

$$d_{j*n+k,j'*n+k'} = \begin{cases} a_{j+1,j'+1} + a_{k',k} & \text{if } j = j' \text{ and } k = k' \\ a_{j+1,j'+1} & \text{if } k = k' \text{ and } j \ne j' \\ a_{k',k} & \text{if } j = j' \text{ and } k \ne k' \\ 0 & \text{otherwise.} \end{cases}$$

Define $M_1 = \left[e_{j*n+k,j'*n+k'}\right]$, where

$$e_{j*n+k,j'*n+k'} = \begin{cases} a_{k',k} & \text{if } j = j' \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, define $M_2 = \left[f_{j*n+k,j'*n+k'}\right]$, where

$$f_{j*n+k,j'*n+k'} = \begin{cases} a_{j+1,j'+1} & \text{if } k = k' \\ 0 & \text{otherwise.} \end{cases}$$

Then $\Gamma^e_{mat} = M_1 + M_2$.

Now, M_1 and M_2 are $n^2 \times n^2$ matrices which can be thought of as block matrices with n^2 blocks, each block B_{ij} being an $n \times n$ matrix. First consider the M_1 matrix. Note that $j = j'$ for entries in the diagonal blocks and $j \neq j'$ for entries in blocks that are not along the diagonal. So the only nonzero entries are in the blocks along the diagonal. Let $B_{j+1,j+1}$ be the $(j+1)^{th}$ block along the diagonal, $0 \leq j \leq n-1$, and consider the $(i,k)^{th}$ element $e_{j*n+i,j*n+k}$. By definition of the M_1 matrix this element has to be $a_{k,i}$. It follows that

$$B_{j+1,j+1} = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{bmatrix} = \Gamma^{op}_{mat}.$$

Thus, M_1 can be written as follows:

$$M_1 = \begin{bmatrix} \Gamma^{op}_{mat} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \Gamma^{op}_{mat} & \cdots & \mathbf{0} \\ \vdots & \vdots & & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \Gamma^{op}_{mat} \end{bmatrix}.$$

Now consider the M_2 matrix. Note that $k = k'$ along the main diagonal of each block $B_{j+1,j'+1}$, $0 \leq j, j' \leq n-1$. By definition of the M_2 matrix the $(k,k)^{th}$ element $f_{j*n+k,j'*n+k}$ of the block $B_{j+1,j'+1}$ must be equal to $a_{j+1,j'+1}$. It follows that

$$B_{j+1,j'+1} = \begin{bmatrix} a_{j+1,j'+1} & 0 & \cdots & 0 \\ 0 & a_{j+1,j'+1} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a_{j+1,j'+1} \end{bmatrix} = a_{j+1,j'+1}\mathbf{I_n},$$

where $\mathbf{I_n}$ is the $n \times n$ identity matrix. Thus, M_2 can be written as follows:

$$M_2 = \begin{bmatrix} a_{11}\mathbf{I_n} & a_{12}\mathbf{I_n} & \cdots & a_{1n}\mathbf{I_n} \\ a_{21}\mathbf{I_n} & a_{22}\mathbf{I_n} & \cdots & a_{2n}\mathbf{I_n} \\ \vdots & \vdots & & \vdots \\ a_{n1}\mathbf{I_n} & a_{n2}\mathbf{I_n} & \cdots & a_{nn}\mathbf{I_n} \end{bmatrix}.$$

We have shown the following:

Lemma 2.1 *The adjacency matrix of the tensor quiver Γ^e decomposes as*

$$\begin{bmatrix} \Gamma^{op}_{mat} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \Gamma^{op}_{mat} & \cdots & \mathbf{0} \\ \vdots & \vdots & & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \Gamma^{op}_{mat} \end{bmatrix} + \begin{bmatrix} a_{11}\mathbf{I_n} & a_{12}\mathbf{I_n} & \cdots & a_{1n}\mathbf{I_n} \\ a_{21}\mathbf{I_n} & a_{22}\mathbf{I_n} & \cdots & a_{2n}\mathbf{I_n} \\ \vdots & \vdots & & \vdots \\ a_{n1}\mathbf{I_n} & a_{n2}\mathbf{I_n} & \cdots & a_{nn}\mathbf{I_n} \end{bmatrix},$$

where the adjacency matrix of Γ is $\Gamma_{mat} = [a_{ij}]$ and $\Gamma^{op}_{mat} = [a_{ji}]$.

So we now have a nice matrix description of Γ^e. However, Λ^e is not isomorphic to $k\Gamma^e$. In the next chapter it will be shown that Λ^e is a quotient of $k\Gamma^e$. It is this quotient that we need to represent on the computer.

Chapter 3

Writing Λ^e as the Quotient of a Path Algebra

Let $\Lambda = k\Gamma/I$, $\Gamma_0 = \{v_1, ..., v_n\}$, and $\Gamma_1 = \{a_1, ..., a_m\}$. Let $<$ be length-lexicographic order defined on Λ reading from origin to terminus on each of the directed paths of Γ. Here we assume $v_1 < v_2 < \cdots < v_n < a_1 < a_2 < \cdots < a_m$. If $Minsharp_<(I) = \{r_1, ..., r_k\}$ then it is not hard to show that $Minsharp_{<'}(I^{op}) = \{r_1^{op}, ..., r_k^{op}\}$, where $<'$ is length-lexicographic order on Λ^{op} reading terminus to origin. As before, we assume $v_1 <' v_2 <' \cdots <' v_n <' a_1^{op} <' a_2^{op} <' \cdots <' a_m^{op}$. Now we need to determine the induced ordering $<^*$ on the tensor quiver Γ^e. First let us order the vertices. Suppose $u_1 \otimes w_1$ and $u_2 \otimes w_2$ are two distinct vertices of Γ^e. We say $u_1 \otimes w_1 <^* u_2 \otimes w_2$ if

$$\begin{aligned} &u_1 <' u_2 \quad \text{or} \\ &u_1 = u_2 \quad \text{and } w_1 < w_2. \end{aligned}$$

To order the arrows suppose $a_i^{op} \otimes w_1$, $a_j^{op} \otimes w_2$, $w_3 \otimes a_k$, and $w_4 \otimes a_l$ are four distinct arrows of Γ^e. Then we say $a_i^{op} \otimes w_1 <^* a_j^{op} \otimes w_2$ if

$$\begin{aligned} &a_i^{op} <' a_j^{op} \quad \text{or} \\ &a_i^{op} = a_j^{op} \quad \text{and } w_1 < w_2, \end{aligned}$$

$w_3 \otimes a_k <^* w_4 \otimes a_l$ if

$$\begin{aligned} &a_k < a_l \quad \text{or} \\ &a_k = a_l \quad \text{and } w_3 <' w_4, \end{aligned}$$

and $w_3 \otimes a_k <^* a_i^{op} \otimes w_1$. Now let p_1 and p_2 be finite directed paths in Γ^e, $p_1 \neq p_2$. Then we say $p_1 <^* p_2$ if $l(p_1) < l(p_2)$, or if $l(p_1) = l(p_2)$ and $p_1 <^* p_2$ using the lex condition on the arrows reading from origin to terminus on each directed path. Note than when we compare two subwords of the form $p_1^{op} \otimes v$ and $p_2^{op} \otimes v$ that have the same length, we read backwards - that is, from terminus to origin.

Now we are ready to show how the enveloping algebra can be written as the quotient the path algebra $k\Gamma^e$. Define

$$\phi : k\Gamma^e \to \Lambda^{op} \bigotimes_k \Lambda = \Lambda^e$$

first on the vertices and arrows:

$$v \otimes w \xrightarrow{\phi} (v + I^{op}) \otimes (w + I)$$
$$v \otimes a_i \xrightarrow{\phi} (v + I^{op}) \otimes (a_i + I)$$
$$a_j^{op} \otimes w \xrightarrow{\phi} (a_j^{op} + I^{op}) \otimes (w + I).$$

Here $v, w \in \Gamma_0$ and $a_i, a_j \in \Gamma_1$. Now, let $p = (c_1^{op} \otimes d_1)(c_2^{op} \otimes d_2) \cdots (c_l^{op} \otimes d_l)$ be a finite directed path of length l in $k\Gamma^e$, where $c_i^{op} \otimes d_i$ is an arrow in $k\Gamma^e$ for $i = 1, ..., l$. Define $\phi(p)$ as follows:

$$(c_1^{op} \otimes d_1) \cdots (c_l^{op} \otimes d_l) \xrightarrow{\phi} [c_1^{op} \cdots c_l^{op} + I^{op}] \otimes [d_1 \cdots d_l + I].$$

From now on we will abuse notation and write $\psi^{op} \otimes \theta$ instead of $(\psi^{op} + I^{op}) \otimes (\theta + I)$. Finally, let $\eta \in k\Gamma^e$. Then we can write $\eta = \sum_{i=1}^{t} \alpha_i p_i$, where $\alpha_i \in k \setminus 0$ and the p_i are chosen to be distinct finite directed paths in Γ^e. Define $\phi : k\Gamma^e \to \Lambda^{op} \otimes_k \Lambda$ by

$$\phi(\eta) = \sum_{i=1}^{t} \alpha_i \phi(p_i).$$

The reader can verify that ϕ is a k-map and a surjective ring homomorphism.

To see where ϕ maps the unity element, note that $\mathbf{1}_{k\Gamma^e} = \sum_{w,v\in\Gamma_0} w \otimes v$. So $\phi(\ \mathbf{1}_{k\Gamma^e})=$

$$
\begin{aligned}
&= \phi\left(\textstyle\sum_{w,v\in\Gamma_0} w \otimes v\right)\\
&= \textstyle\sum_{w,v\in\Gamma_0} \phi(w \otimes v)\\
&= \textstyle\sum_{w,v\in\Gamma_0} w \otimes v.
\end{aligned}
$$

Since $\mathbf{1}_\Lambda = \sum_{v\in\Gamma_0} v \pmod{I}$ and $\mathbf{1}_{\Lambda^{op}} = \sum_{w\in\Gamma_0} w \pmod{I^{op}}$, we have $\mathbf{1}_{\Lambda^{op}\otimes\Lambda} =$

$$
\begin{aligned}
&= \mathbf{1}_{\Lambda^{op}} \otimes \mathbf{1}_\Lambda\\
&= (\textstyle\sum_{w\in\Gamma_0} w) \otimes (\textstyle\sum_{v\in\Gamma_0} v)\\
&= \textstyle\sum_{w,v\in\Gamma_0} w \otimes v\\
&= \phi(\mathbf{1}_{k\Gamma^e}).
\end{aligned}
$$

Hence, ϕ preserves the unity element.

By the first isomorphism theorem $k\Gamma^e/\ker\phi \cong \Lambda^e$. So we just need the generators for $\ker\phi$. Fortunately, there is an easy description of these generators. Let $M= Minsharp_<(I) = \{r_1, ..., r_k\}$. Define $I_{rel} = \{v \otimes r_i : r_i \in M,\ v \in \Gamma_0\}$ and $I_{rel}^{op} = \{r_i^{op} \otimes v : r_i \in M,\ v \in \Gamma_0\}$. Here $v \otimes r_i = \sum_{j=1}^t \alpha_j (v \otimes p_j)$, where $r_i = \sum_{j=1}^t \alpha_j p_j$. Note that if $p = a_1 \cdots a_l$, then $v \otimes p$ means $(v \otimes a_1)(v \otimes a_2)\cdots(v \otimes a_l)$. We can similarly define $r_i^{op} \otimes v$. Thus, I_{rel} and I_{rel}^{op} contain relations in $k\Gamma^e$. There is one other type of relation that we need to define. Consider the following diagram:

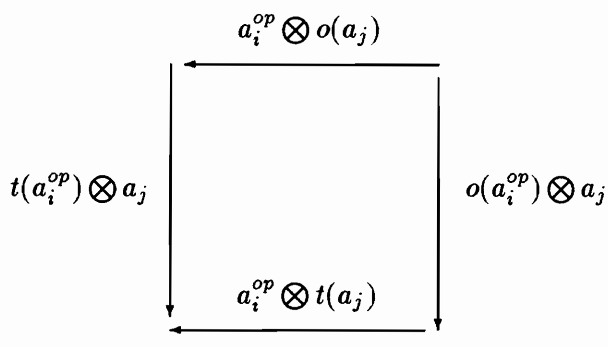

Figure 3.1

Since $\phi\,[(a_i^{op} \otimes o(a_j))(t(a_i^{op}) \otimes a_j) - (o(a_i^{op}) \otimes a_j)(a_i^{op} \otimes t(a_j))] = a_i^{op} \otimes a_j - a_i^{op} \otimes a_j = 0$, relations of this form must be included in the generating set. We call this type of relation a commutator relation. Let C be the collection of all commutator relations. Let I^e be the ideal in $k\Gamma^e$ generated by $\Im = I_{rel} \bigcup I_{rel}^{op} \bigcup C$. Then the following result is what we need:

Proposition 3.1 *$I^e = ker\,\phi$.*

Proof: Let us begin by showing $I^e \subset \ker\phi$. There are three types of generators that we need to check. First consider $w \otimes r_i \in I_{rel}$, where $r_i = \sum_{j=1}^{h} \alpha_j p_j \in M$. Recall that if $p_j = a_1 a_2 \cdots a_l$, then $w \otimes p_j = (w \otimes a_1)(w \otimes a_2) \cdots (w \otimes a_l)$ and $\phi(w \otimes p_j) = w \otimes a_1 a_2 \cdots a_l = w \otimes p_j$. So $\phi(w \otimes r_i) =$

$$
\begin{aligned}
&= \textstyle\sum_{j=1}^{h} \alpha_j \phi(w \otimes p_j) \\
&= \textstyle\sum_{j=1}^{h} \alpha_j w \otimes p_j \\
&= w \otimes \left[\textstyle\sum_{j=1}^{h} \alpha_j p_j\right] \\
&= w \otimes r_i \\
&= 0.
\end{aligned}
$$

Similarly, $I_{rel}^{op} \subset \ker\phi$. Since we already know that $C \subset \ker\phi$, the first inclusion follows.

To show the reverse inclusion, let $\bar{\eta} = \sum_{i=1}^{s} \bar{\alpha}_i \bar{p}_i \in \ker\phi$. Note that each path $\bar{p}_i$ in $k\Gamma^e$ can be reduced over the commutator relations to a path of the form $p_i = (v \otimes a_{i_1})(v \otimes a_{i_2}) \cdots \cdots (v \otimes a_{i_q})(a_{i_{q+1}}^{op} \otimes w)(a_{i_{q+2}}^{op} \otimes w) \cdots (a_{i_{q+t}}^{op} \otimes w)$ so that $p_i \equiv \bar{p}_i \pmod{I^e}$, where $v, w \in \Gamma_0$ and $a_{i_l}, a_{i_{q+k}} \in \Gamma_1$. If $\bar{\eta}$ reduces to 0, then $\bar{\eta} \in I^e$ and we are done. So assume $\bar{\eta}$ does not reduce to 0 over the commutators - say $\bar{\eta}$ reduces to $\eta = \sum_{i=1}^{d} \alpha_i p_i \neq 0$, where each $\alpha_i \neq 0$ and the p_i are distinct. Since $C \subset \ker\phi$, we have $\phi(\eta) = \sum_{i=1}^{d} \alpha_i \phi(p_i) = 0$. Without loss of generality assume that $\text{tip}(\eta) = p_1 = (v \otimes a_1)(v \otimes a_2) \cdots (v \otimes a_l)(a_{l+1}^{op} \otimes w) \cdots (a_{l+k}^{op} \otimes w)$. Then $\phi(p_1) = a_{l+1}^{op} a_{l+2}^{op} \cdots a_{l+k}^{op} \otimes a_1 a_2 \cdots a_l$.

Assume p_1 is in its normal form modulo I^e. Reduce $\eta - \alpha_1 p_1 = \sum_{i=2}^{d} \alpha_i p_i$ to its normal form η^* modulo I^e. Then $\phi(\alpha_1 p_1 + \eta^*) = 0$. If $\eta^* = 0$, then $\phi(p_1) = 0$, a contradiction. If $\eta^* = \sum_{i=2}^{f} \alpha_i^* p_i^*$, then $\{\phi(p_1), \phi(p_2^*), ..., \phi(p_f^*)\}$ is linearly independent. Since $\phi(\alpha_1 p_1 + \eta^*) = 0$, we conclude that $\alpha_1 = \alpha_2^* = \cdots = \alpha_f^* = 0$, a contradiction. So p_1 cannot be in its normal form modulo I^e. This means that there is some element of $\xi \in I_{rel} \bigcup I_{rel}^{op}$ such that tip(ξ) divides p_1. Thus, $\bar{\eta} \in \ker \phi$ can always be reduced over $\Im$. Since the k-basis for $k\Gamma^e$ is well-ordered, $\bar{\eta}$ can be reduced to 0 over $\Im$ with a finite number of reductions. We conclude that $\bar{\eta} \in I^e$ and the result follows.

Lifting the Gröbner Basis to Λ^e

Now we will see that $\Im$ is actually $Minsharp_<(I^e)$. First we need some terminology. Let $\tilde{\Gamma}$ be a quiver (under the usual hypotheses) and let $\tilde{B}$ be a k-basis of $k\tilde{\Gamma}$. The following two definitions and theorem are taken from [14].

Definition 3.1 *[14] : Let $\gamma \in k\tilde{\Gamma}$ be nonzero. A simple algebra reduction ρ for γ is determined by a 4-tuple $(\lambda, \psi, f, \theta)$ where $\lambda \in k \setminus 0$, $f \in k\tilde{\Gamma} \setminus 0$, and $\psi, \theta \in \tilde{B}$. It satisfies*
(1) ψ tip$(f)\theta \in$ support(γ)
(2) ψ tip$(f)\theta \notin$ support$(\gamma - \lambda\psi f\theta)$.
Then $\gamma - \lambda\psi f\theta$, written $\rho(\gamma)$, is the reduction of γ by ρ. A sequence of simple reductions $\rho_1, ..., \rho_n$ of this form is called an algebra reduction of γ. We say $f_1, ..., f_n$ are the subdivisors of the reduction. If each subdivisor for a reduction is in $S \subset k\tilde{\Gamma}$, then the reduction is said to be over S.

Definition 3.2 *[14]: Let $\psi, \theta \in \tilde{B}$. A $(\psi, \theta)-$overlap occurs when one can factor $\psi = \psi_1 \xi$, $\theta = \xi\theta_1$ where $\psi_1 \neq \psi$ and $\theta_1 \neq \theta$.Suppose $f, g \in k\Gamma$ such that tip(f)=ψ appears in f with coefficient α and tip(g)=θ appears in g with coefficient β. If a (ψ, θ)-overlap occurs, we say that f and g overlap and that $\beta f \theta_1 - \alpha \psi_1 g$ is the overlap difference for the factorization.*

The following theorem is used to determine $Minsharp_<(I^e)$.

Theorem 3.1 *[14]: Let S be a subset of nonzero uniform elements in $k\tilde{\Gamma}$ which generates the ideal $\tilde{I}$. Assume that*

(i) *the coefficient of the tip of each member of S is 1,*

(ii) *no member of S reduces over any other, and*

(iii) *every overlap difference for two (not necessarily distinct) members of S always reduces to zero over S. Then $S = Minsharp_<(\tilde{I})$.*

Let us now use Theorem 3.1 to show that $\Im = Minsharp_<(I^e)$. Define B^e to be the set of directed paths in Γ^e. First we check that the coefficient of the tip of each member of $\Im$ is 1. If $v \otimes r_i \in I_{rel}$, then $\text{tip}(v \otimes r_i) = v \otimes \text{tip}(r_i)$. Since $\text{tip}(r_i)$ has coefficient 1, so does $\text{tip}(v \otimes r_i)$. Similarly, the tip of each $r_i^{op} \otimes v \in I_{rel}^{op}$ has coefficient 1. Finally, it is clear that the coefficient of the tip of each commutator relation is 1.

Now we need to show that no member of $\Im$ reduces over any other. There are nine cases that need to be checked. We will check a few and leave the rest to the reader. First let us show that $v \otimes r_i$ does not reduce over $r_j^{op} \otimes w$, $r_i, r_j \in Minsharp_<(I)$. This follows from the fact that a path in the support of $v \otimes r_i$ has the form $v \otimes a_{i_1} \cdots v \otimes a_{i_y}$ and a path in the support of $r_j^{op} \otimes w$ has the form $a_{j_1}^{op} \otimes w \cdots a_{j_z}^{op} \otimes w$. So $\text{tip}(r_j^{op} \otimes w)$ cannot divide any path in the support of $v \otimes r_i$.

Next, let us show that $v \otimes r_i$ does not reduce over $w \otimes r_j$. Suppose the reduction does occur. Then there exists some $\lambda \in k \setminus 0$ and $\psi, \theta \in B^e$ such that

1) $\psi(\text{tip}(w \otimes r_j))\theta \in \text{support}(v \otimes r_i)$ and

2) $\psi(\text{tip}(w \otimes r_j))\theta \notin \text{support}(v \otimes r_i - \lambda\, \psi(w \otimes r_j)\theta)$.

Suppose $v \otimes p_i = v \otimes a_{i_1} \cdots v \otimes a_{i_q} \in \text{support}(v \otimes r_i)$ is the path satisfying (1), where $\text{tip}(w \otimes r_j) = a_{j_1} \cdots a_{j_r}$. Then $w \otimes a_{j_1} \cdots w \otimes a_{j_r}$ is a subword of $v \otimes a_{i_1} \cdots v \otimes a_{i_q}$. So $v = w$. Suppose $\psi = w \otimes c_1 \cdots w \otimes c_s$ and $\theta = w \otimes d_1 \cdots w \otimes d_t$. Then $(c_1 \cdots c_s)(a_{j_1} \cdots a_{j_r})(d_1 \cdots d_t) = a_{i_1} \cdots a_{i_q} = p_i$. Letting $C = c_1 \cdots c_s$ and $D = d_1 \cdots d_t$ we see that $C\text{tip}(r_j)D \in \text{support}(r_i)$. It is not hard to show that assumption (2) implies that $C\text{tip}(r_j)D \notin \text{support}(r_i - \lambda c r_i d)$. However, this means that r_i reduces over r_j, a contradiction since both are in $Minsharp_<(I)$.

Now, let $a_1^h a_1^v - a_2^v a_2^h$ denote a commutator relation. The superscripts h and v indicate whether the arrow $a \in k\Gamma^e$ is horizontal or vertical. As our last case we note that $a_1^h a_1^v - a_2^v a_1^h$ does not reduce over $v \otimes r_i$. This follows since the tip of $v \otimes r_i$ has the form $v \otimes p$, where p has length at least two, i.e. $v \otimes p$ cannot divide a path in the support of a commutator relation. The other cases are analogous to these first three and are left to the reader.

Finally, we need to check that the overlap difference of two members of $\Im$ reduces to 0 over $\Im$. If f and g are in $\Im$, then the overlap difference $f\theta_1 - \psi_1 g = \eta \in I^e \subset \ker \phi$. But by the proof of Proposition 3.1, we have already seen that η must reduce to 0 over $\Im$. It follows that $\Im = Minsharp_<(I)$. We have shown the following:

Theorem 3.2 *The enveloping algebra of $\Lambda \cong k\Gamma/I$ can be written as the quotient of a path algebra, $\Lambda^e = \Lambda^{op} \otimes_k \Lambda \cong k\Gamma^e/I^e$, where I^e is the ideal generated by the lifted Gröbner basis $\Im$ and Γ^e is the tensor quiver of Λ^e.*

This representation of Λ^e enables us to encode the enveloping algebra of Λ on a computer. Note that this result is not restricted to monomial algebras - is true for any finite dimensional quotient of a path algebra. Now we are almost ready to use the resolution package previously mentioned. The only thing missing is a projective presentation of Λ. This is provided in Chapter 4 along with the second projective and syzygy.

Chapter 4

The Projective Presentation and Second Syzygy

Let $\Lambda = K\Gamma/I$ where $I = \langle r_1, r_2, ..., r_m\rangle$ and each relation r_i does not have to be a path. We do assume each r_i is uniform. Throughout this paper $\otimes$ means $\otimes_k$. To construct a projective resolution of Λ as a right Λ^e-module we start with a projective presentation of Λ:

$$P_1 \xrightarrow{\phi_1} P_0 \xrightarrow{\pi} \Lambda \longrightarrow 0.$$

Here $\Lambda \cong \operatorname{coker}\phi_1$, that is, we choose the above sequence to be exact so that $\Lambda \cong P_0/\operatorname{im}\phi_1$. The projective modules we desire are

$$P_0 = \coprod_{v\in\Gamma_0} \Lambda v \bigotimes v\Lambda \qquad \text{and} \qquad P_1 = \coprod_{a\in\Gamma_1} \Lambda o(a) \bigotimes t(a)\Lambda$$

where π is the multiplication map and $\phi_1\left(o(a)\otimes t(a)\right) = a \otimes t(a) - o(a)\otimes a$.

Now, let

$$P_2 = \coprod_{i=1}^{m} \Lambda o(r_i) \bigotimes t(r_i)\Lambda$$

where each $r_i \in Minsharp_<(I)$ can be written $r_i = \sum_{j=1}^{k(i)} \alpha_{ij}p_{ij}$. Let p be a path in the support of r_i, say $p = a_{p_1}a_{p_2}\cdots a_{p_{l(p)}}$:

Figure 4.1

Define

$$x_p = \sum_{d=1}^{l(p)} a_{p_1} \cdots a_{p_{d-1}} \bigotimes a_{p_{d+1}} \cdots a_{p_{l(p)}}$$

where, for notational convenience, we let $a_{p_1} a_{p_0} = o(r_i)$ for $d = 1$ and $a_{p_{l(p)+1}} a_{p_{l(p)}} = t(r_i)$ for $d = l(p)$. Define $\phi_2 : P_2 \to P_1$ by

$$\phi_2 : o(r_i) \bigotimes t(r_i) \to \sum_{j=1}^{k(i)} \alpha_{ij} x_{p_{ij}}.$$

Proposition 4.1 *The sequence $P_2 \xrightarrow{\phi_2} P_1 \xrightarrow{\phi_1} P_0 \xrightarrow{\pi} \Lambda \longrightarrow 0$ is exact at P_1.*

Proof: If $p \in \text{support}(r_i)$ is the path we used in defining x_p we have $\phi_1(x_p) =$

$$\begin{aligned}
&= \textstyle\sum_{d=1}^{l(p)} a_{p_1} \cdots a_{p_{d-1}} [\phi_1(o(a_{p_d}) \otimes t(a_{p_d}))] a_{p_{d+1}} \cdots a_{p_{l(p)}} \\
&= \textstyle\sum_{d=1}^{l(p)} a_{p_1} \cdots a_{p_{d-1}} [a_{p_d} \otimes t(a_{p_d}) - o(a_{p_d}) \otimes a_{p_d}] a_{p_{d+1}} \cdots a_{p_{l(p)}} \\
&= a_{p_1} \cdots a_{p_{l(p)}} \otimes t(p) - o(p) \otimes a_{p_1} \cdots a_{p_{l(p)}} \\
&= p \otimes t(p) - o(p) \otimes p.
\end{aligned}$$

Then $\phi_1 \phi_2(o(r_i) \otimes t(r_i)) = \phi_1(\sum_{j=1}^{k(i)} \alpha_{ij} x_{p_{ij}}) =$

$$\begin{aligned}
&= \textstyle\sum_{j=1}^{k(i)} \alpha_{ij} \phi_1(x_{p_{ij}}) \\
&= \textstyle\sum_{j=1}^{k(i)} \alpha_{ij} [p_{ij} \otimes t(r_i) - o(r_i) \otimes p_{ij}] \\
&= \left(\textstyle\sum_{j=1}^{k(i)} \alpha_{ij} p_{ij}\right) \otimes t(r_i) - o(r_i) \otimes \left(\textstyle\sum_{j=1}^{k(i)} \alpha_{ij} p_{ij}\right) \\
&= r_i \otimes t(r_i) - o(r_i) \otimes r_i \\
&= 0.
\end{aligned}$$

It follows that $\text{im}\,\phi_2 \subset \ker \phi_1$.

To show the reverse inclusion we first need to extend our order $<$ to P_0 and P_1. So let $a_i, a_j \in \Gamma_1$. Suppose $p \otimes q \in \Lambda o(a_i) \otimes t(a_i) \Lambda$ and $r \otimes s \in \Lambda o(a_j) \otimes t(a_j) \Lambda$. Then

$p \otimes q > r \otimes s$ iff

1) $l(q) < l(s)$ or

2) $l(q) = l(s)$ and $l(p) > l(r)$; or

3) $l(q) = l(s)$, $l(p) = l(r)$, and $q < s$; or

4) $q = s$, $l(p) = l(r)$, and $p > r$; or

5) $q = s$, $p = r$, and $a_i > a_j$.

By replacing the arrows in the above order with two vertices and dropping the last requirement we obtain the desired order on P_0. Now let $x \in \ker \phi_1$ and reduce each term of x to its tensor normal form modulo I. Suppose $\text{tip}(x) = p \otimes q \in \Lambda o(a) \otimes t(a)\Lambda$. Then $\phi_1(p \otimes q) = p(a \otimes t(a) - o(a) \otimes a)q = pa \otimes q - p \otimes aq$ has tip $pa \otimes q$. Let $r \otimes s \in \text{support}(x)$, where $r \otimes s \in \Lambda o(\hat{a}) \otimes t(\hat{a})\Lambda$. Then $\phi_1(r \otimes s) = r\hat{a} \otimes s - r \otimes \hat{a}s$ has tip $r\hat{a} \otimes s$. So under the order on P_1, $p \otimes q > r \otimes s$ implies $pa \otimes q > r\hat{a} \otimes s$ in P_0. This means $pa \otimes q$ cannot cancel with any other term in the image of x. Since q is in its normal form modulo I it must be the case that $\text{tip}(r_i) = \hat{p}$ divides pa for some $r_i \in Minsharp_<(I)$. Since $\hat{p}$ does not divide p, the last arrow of $\hat{p}$ must be a:

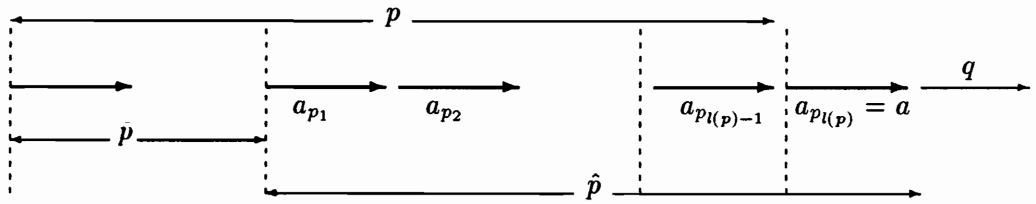

Figure 4.2

Without loss of generality suppose $r_i = \alpha_1 \hat{p} + \sum_{j=2}^{r} \alpha_j q_j$:

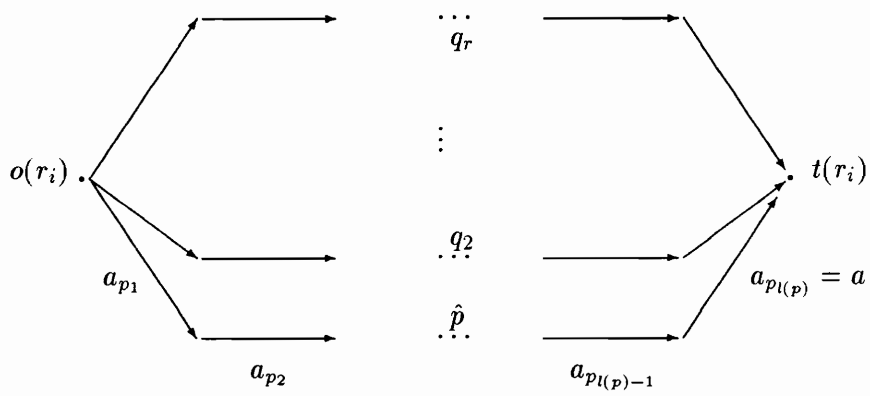

Figure 4.3

It follows that $\phi_2(o(r_i) \otimes t(r_i)) = \alpha_1 x_{\hat{p}} + \sum_{j=2}^{r} \alpha_j x_{q_j}$ has tip $a_{p_1} a_{p_2} \cdots a_{p_{l(p)-1}} \otimes t(r_i)$. So $a_{p_1} \cdots a_{p_{l(p-1)}} \otimes t(r_i)(\bar{p} \otimes q) = p \otimes q$, that is, tip $\phi_2(o(r_i) \otimes t(r_i))$ divides tip(x). Consider $x - \lambda/\alpha_1[\phi_2(o(r_i) \otimes t(r_i))(\bar{p} \otimes q)] = x - y$, where λ is the coefficient of $p \otimes q$ in x and $y = \lambda/\alpha_1[\bar{p} \sum_{j=2}^{r} \alpha_j x_{q_j} q + \bar{p} \alpha_1 x_{\hat{p}} q]$. Then $\bar{p} a_{p_1} \cdots a_{p_{l(p-1)}} \otimes q = p \otimes q$ =tip(y) which means $x - y < x$. Thus, x reduces over im ϕ_2 and, by induction, x reduces to zero over im ϕ_2. We conclude that $x \in$ im ϕ_2 and, hence, im ϕ_2 =ker ϕ_1.

Although the first and second syzygies have been given for path algebras modulo arbitrary admissible ideals, we turn our attention now to monomial algebras. Before we can define the higher projectives and syzygies for these algebras, we first need to introduce the associated sequence of paths. The description and properties of these paths are the subject of Chapter 5.

Chapter 5

The Associated Sequence of Paths

Now that a Λ^e-projective resolution of Λ has been constructed through the second projective and syzygy , it would be nice if we could describe the rest of the resolution. In Chapter 6 we provide this construction for monomial algebras. In the introduction it was stated that the syzygies for this resolution exhibit an "alternating behavior" induced by a special sequence of paths. The purpose of this chapter is to define these paths and illustrate their properties. Once the reader understands the results of this chapter, the construction in the next chapter should appear quite natural. So let us now introduce the associated sequence of paths.

For the first two definitions suppose $Minsharp_<(I) = \{p_1, p_2, ..., p_d\}$ is a finite collection of paths that lie along some directed path Υ.

Definition 5.1 *Let $p_i \in Minsharp_<(I)$. We define the associated sequence of paths corresponding to p_i inductively as follows: Let $r_2 \in Minsharp_<(I)$ be the path (if it exists) in $Minsharp_<(I)$ such that $o(p_i) < o(r_2) < t(p_i)$ and $o(r_2)$ is minimal with respect to this double inequality. Now assume $r_1, r_2, ..., r_j$ have been constructed, where $r_1 = p_i$. Let*

$$L_{j+1} = \{r \in Minsharp_<(I) : t(r_{j-1}) \leq o(r) < t(r_j)\}.$$

If $L_{j+1} \neq \emptyset$, let r_{j+1} be such that $o(r_{j+1})$ is minimal with respect to $r_{j+1} \in L_{j+1}$.

This sequence of paths was first described in [19]. We shall refer to this construction as the left construction for the associated paths. There is also a dual construction which we shall refer to as the right construction:

Definition 5.2 *Assuming the same hypotheses as in the previous definition let* $r_2 \in Minsharp_<(I)$ *be the path (if it exists) in* $Minsharp_<(I)$ *such that* $o(r_2) < o(p_i) < t(r_2)$ *and* $t(r_2)$ *is maximal with respect to this double inequality. Now assume* $r_1, r_2, ..., r_j$ *have been constructed. Let*

$$R_{j+1} = \{r \in Minsharp_<(I) : o(r_j) < t(r) \leq o(r_{j-1})\}.$$

If $R_{j+1} \neq \emptyset$, *let* r_{j+1} *be such that* $t(r_{j+1})$ *is maximal with respect to* $r_{j+1} \in R_{j+1}$.

Consider the left construction. Given an integer n, we refer to the sequence of the first n associated paths corresponding to $r_1 = p_i$ by $(r_1, r_2, ..., r_n)$ (if it exists). So far we have been doing this construction along the directed path Υ. However, if $p_i \in Minsharp_<(I)$ then p_i may be the starting point of many directed paths. In this case we will want to form the associated sequence of paths over all possible directed paths Υ_k beginning at p_i:

Definition 5.3 *Let* $r_1 = p_i$ *and define*

$$AS_i(n) = \{(r_1, ..., r_{n-1}) : (r_1, ..., r_{n-1}) \textit{ is an associated sequence of paths}\}.$$

For each $(r_1, ..., r_{n-1}) \in AS_i(n)$ define p_i^n to be the path from $o(p_i)$ to $t(r_{n-1})$ along the directed path Υ_k used in the construction of $(r_1, ..., r_{n-1})$. Let $AP_i(n)$ be the set of all p_i^n constructed from $AS_i(n)$. Suppose $Minsharp_<(I) = \{p_1, ..., p_m\}$. Then the following definition is what we really need:

$$AP(n) = \bigcup_{i=1}^{m} AP_i(n).$$

Now, we can also dualize the above definitions for the right construction and obtain $AP(n)^{op}$ which is the obvious analogue to $AP(n)$. Note that $AP(2) = AP(2)^{op} = Minsharp_<(I)$. The importance of using both constructions lies in the following result:

Lemma 5.1 *$AP(n) = AP(n)^{op}$ for $n \geq 2$.*

Proof: This is obvious for $n = 2, 3$ so assume $n \geq 4$. Suppose $p^n \in AP(n)$ and that p^n was obtained from the left construction using the $Minsharp_<(I)$ paths $p_1, p_2, ..., p_{n-1}$ (after re-indexing). Let us consider the right construction starting with p_{n-1}. For notational convenience we will denote the i^{th} $Minsharp_<(I)$ path in this construction by q_{n-i}. In particular, $q_{n-1} = p_{n-1}$. Now, $t(p_{n-2}) > o(q_{n-1})$. So $t(p_{n-2}) \leq t(q_{n-2})$. Since p_{n-1} is minimal with respect to $t(p_{n-3}) \leq o(p_{n-1}) < t(p_{n-2})$, it follows that $o(q_{n-2}) < t(p_{n-3})$. Hence, $o(p_{n-2}) \leq o(q_{n-2}) < t(p_{n-3})$. Continuing this argument inductively we see that, in general, $o(p_{n-k}) \leq o(q_{n-k}) < t(p_{n-(k+1)})$. In particular, $o(p_3) \leq o(q_3) < t(p_2)$ and $o(p_2) \leq o(q_2) < t(p_1)$. So now we just need q_1. Since $o(q_2) < t(p_1) \leq o(p_3) \leq o(q_3)$, we know $t(p_1) \leq t(q_1)$. However, $o(q_3) < t(p_2)$ so it must be the case that $o(q_1) < o(p_2)$, i.e. we know $t(q_1) \leq o(q_3)$. Since p_2 is minimal with respect to $o(p_1) < o(p_2) < t(p_1)$ we have $o(q_1) = o(p_1)$ and, thus, $q_1 = p_1$. Hence, the right construction using $q_{n-1}, q_{n-2}, ..., q_2, q_1$ gives a path $q^n \in AP(n)^{op}$ such that $q^n = p^n$. We conclude that $AP(n) \subset AP(n)^{op}$. A similar argument shows $AP(n)^{op} \subset AP(n)$, so the result follows.

Although the next lemma sounds reasonable, it is rather surprising that the conclusion fails when m is odd. This result is the first hint of the alternating behavior to come.

Lemma 5.2 *Suppose m is even and $p_1^m \neq p_2^m \in AP(m)$ overlap along a directed path Υ as follows:*

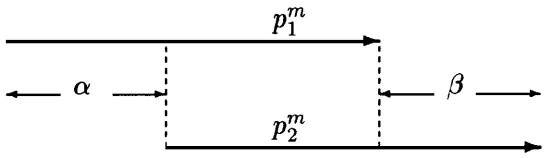

Figure 5.1

If α and β are not divisible by any path in $Minsharp_<(I)$, then there exists some $p^{m+1} \in AP(m+1)$ such that p^{m+1} divides the path from $o(p_1^m)$ to $t(p_2^m)$ along Υ.

Proof: The result is obvious for $m = 2$. So assume $m \geq 4$. Suppose p_1^m is constructed from the left using $p_1, p_2, ..., p_{m-1}$ and p_2^m is constructed from the right using $q_1, q_2, ..., q_{m-1}$. Assume there is no $p^{m+1} \in AP(m+1)$ dividing the path from $o(p_1^m)$ to $t(p_2^m)$. Since q_1 does not divide β, $o(q_1) < t(p_{m-1})$ and, thus, $o(q_1) < t(p_{m-2})$. Now, p_{m-1} intersects q_1 and q_2 is maximal with respect to $t(q_2) < t(q_1)$ so it follows that $o(p_{m-1}) \leq o(q_2) < o(q_1) < t(p_{m-2})$. Since $o(q_2) < t(q_3) \leq o(q_1)$, $o(q_3) < o(p_{m-2})$. However, $o(p_{m-2})$ is minimal with respect to $t(p_{m-4}) \leq o(p_{m-2}) < t(p_{m-3})$. Thus, $o(q_3) < t(p_{m-4})$. Since q_4 is maximal with respect to $o(q_3) < t(q_4) \leq o(q_2)$, we have $o(p_{m-3}) \leq o(q_4) < o(q_3) < t(p_{m-4})$. Continuing this process inductively we see $o(p_{m-(r-1)}) \leq o(q_r) < o(q_{r-1}) < t(p_{m-r})$ where r is even and $r \leq m-2$. In particular, letting $r = m-2$, we have $o(p_{m-(m-3)}) \leq o(q_{m-2}) < o(q_{m-3}) < t(p_{m-(m-2)})$, i.e. $o(p_3) \leq o(q_{m-2}) < o(q_{m-3}) < t(p_2)$. But this means $t(q_{m-1}) \leq o(q_{m-3}) < t(p_2)$. However, $o(p_2) \leq o(q_{m-1})$ and so q_{m-1} properly divides p_2, a contradiction. Hence, $t(p_{m-2}) \leq o(q_1) < t(p_{m-1})$ and the result follows.

The next result is crucial to the aforementioned alternating behavior. First need one more definition:

Definition 5.4 *Suppose $p^n \in AP(n)$. Define $Sub(p^n) = \{p^{n-1} \in AP(n-1) : p^{n-1}$ divides $p^n\}$.*

Lemma 5.3 *$Sub(p^n)$ contains two paths p_o^{n-1} and p_t^{n-1}, where $o(p_o^{n-1}) = o(p^n)$ and $t(p_t^{n-1}) = t(p^n)$. Furthermore, if n is odd then $Sub(p^n) = \{p_o^{n-1}, p_t^{n-1}\}$.*

Proof: The first part is immediate from the construction of $AP(n)$ and $AP(n)^{op}$ and the fact that these two constructions are the same. Note that p_o^{n-1} and p_t^{n-1} must properly divide p^n. Now suppose n is odd:

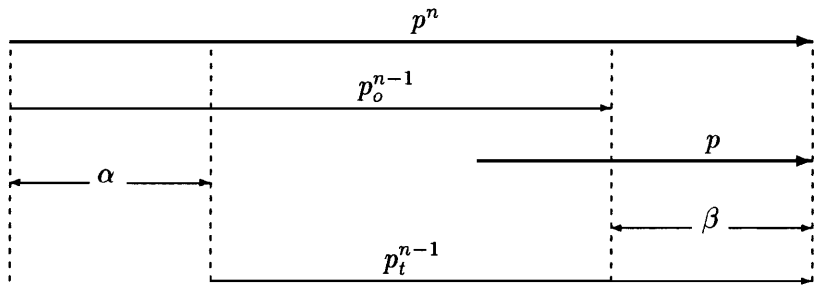

Figure 5.2

We know p_o^{n-1} and p_t^{n-1} overlap since, otherwise, p_t^{n-1} would have to properly divide the last path, call it p, in the left construction of p^n. For the same reason α and β cannot be zero. Now suppose there exists some $p^{n-1} \in AP(n-1)$ such that p^{n-1} divides p^n but is not p_o^{n-1} or p_t^{n-1}, that is, $o(p_o^{n-1}) < o(p^{n-1}) < o(p_t^{n-1})$. We have the following situation:

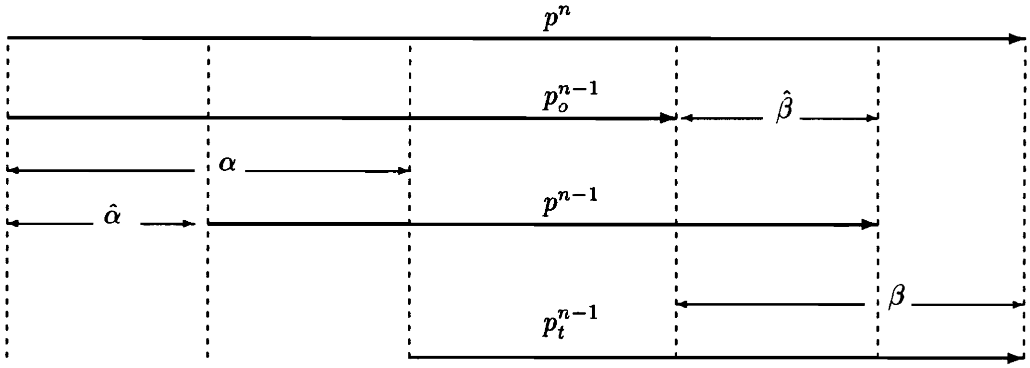

Figure 5.3

Since $\alpha, \beta \neq 0$, we know $\hat{\alpha}, \hat{\beta} \neq 0$. Now, $n-1$ is even so by Lemma 5.2 there exists some $q^n \in AP(n)$ so that q^n divides the path from $o(p_o^{n-1})$ to $t(p^{n-1})$. But this means q^n properly divides p^n, a contradiction.

Example: Suppose Γ is the directed path $a_1 a_2 \cdots a_{19}$ and, for notational convenience, we write each subpath of the form $a_i \cdots a_j$ as $a_{i \cdots j}$. Assume I is generated by $AP(2) = \{a_{1\cdots4}, a_{2\cdots5}, a_{3\cdots6}, a_{4\cdots8}, a_{6\cdots9}, a_{7\cdots10}, a_{8\cdots11}, a_{9\cdots12}, a_{11\cdots13}, a_{12\cdots15}, a_{13\cdots16}, a_{15\cdots17}, a_{16\cdots18}, a_{17\cdots19}\}$. Then $AP(n) = \emptyset$ for $n \geq 10$. The nonempty AP sets of paths are as follows:

$AP(4) = \{a_{2\cdots9}, a_{3\cdots10}, a_{4\cdots12}, a_{7\cdots13}, a_{8\cdots15}, a_{9\cdots16}, a_{11\cdots17}, a_{12\cdots18}, a_{13\cdots19}\}$.

$AP(5) = \{a_{2\cdots10}, a_{3\cdots12}, a_{4\cdots13}, a_{7\cdots15}, a_{8\cdots16}, a_{9\cdots17}, a_{11\cdots18}, a_{12\cdots19}\}$.

$AP(6) = \{a_{3\cdots13}, a_{4\cdots16}, a_{7\cdots17}, a_{8\cdots18}, a_{9\cdots19}\}$.

$AP(7) = \{a_{3\cdots16}, a_{4\cdots17}, a_{7\cdots18}, a_{8\cdots19}\}$.

$AP(8) = \{a_{3\cdots17}, a_{4\cdots19}\}$.

$AP(9) = \{a_{3\cdots19}\}$.

It is clear that $Sub(a_{3\cdots19}) = AP(8)$. However, the two $AP(8)$ paths each have a different number of divisors in $AP(7)$, i.e. $Sub(a_{3\cdots17}) = \{a_{3\cdots16}, a_{4\cdots17})$ but $Sub(a_{4\cdots19}) = \{a_{4\cdots17}, a_{7\cdots18}, a_{8\cdots19}\}$. We see that each $AP(7)$ path has exactly two divisors in $AP(6)$. Again, things are more complicated for $AP(6)$. We have $Sub(a_{3\cdots13}) = \{a_{3\cdots12}, a_{4\cdots13}\}$ but the rest of the $AP(6)$ paths each have three divisors in $AP(5)$. As expected, each $AP(5)$ path has two divisors in $AP(4)$. Most of the $AP(4)$ paths have three divisors in $AP(3)$. However, $Sub(a_{4\cdots12}) = \{a_{4\cdots9}, a_{6\cdots10}, a_{7\cdots11}, a_{8\cdots12}\}$. Finally, it is clear that each $AP(3)$ path (which we did not list) has only two divisors in $AP(2)$.

To facilitate future computations we introduce some new notation:

Definition 5.5 *Suppose m is even and $p_i^m, p_j^m \in AP(m)$ lie along the directed path Υ with $o(p_i^m) < o(p_j^m)$ (relative to Υ). Define L_i^j to be the subpath of Υ from $o(p_i^m)$ to $o(p_j^m)$ and R_j^i to be the subpath of Υ from $t(p_i^m)$ to $t(p_j^m)$.*

Note that it should be clear from context what m is when using the above definition. Now suppose n is odd and $p^n \in AP(n)$. Then by Lemma 5.3 we have $Sub(p^n) = \{p_1^{n-1}, p_2^{n-1}\}$. So we can write $p^n = L_1^2 p_2^{n-1} = p_1^{n-1} R_2^1$. Similarly, if $p^{n+1} \in AP(n+1)$ and $Sub(p^{n+1}) = \{p_1^n, p_2^n, ..., p_m^n\}$, then we can write $p^{n+1} = \theta_i p_i^n \mu_i$ for $i = 1, ..., m$. Here θ_i and μ_i are the obvious complements of p_i^n in p^{n+1}. This leads to one final result on the associated paths:

Lemma 5.4 *L_1^2, R_2^1, θ_i, and μ_i are not divisible by any path in $Minsharp_<(I)$.*

Proof: If p^n is constructed from the left using p_1^{n-1} and p_1, then R_2^1 must properly divide p_1. Similarly, if p^n is constructed from the right using p_2 and p_2^{n-1}, then L_1^2 must properly divide p_2:

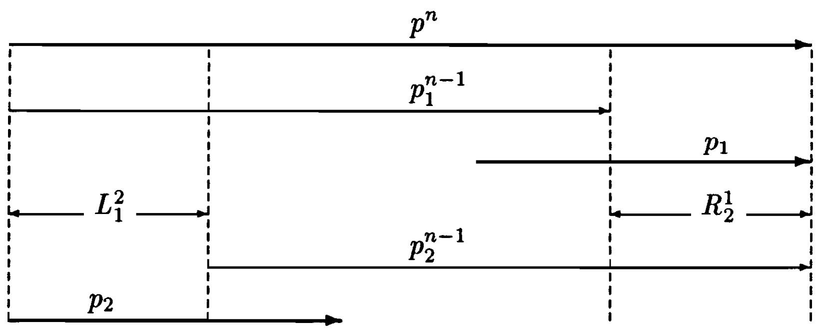

Figure 5.4

Thus, $L_1^2, R_2^1 \neq 0$.

We can use a similar argument for p^{n+1}. Using the left construction with p_1^n we see μ_1 cannot be divisible by any $Minsharp_<(I)$ path. Using the right construction with p_m^n we see θ_m cannot be divisible by any $Minsharp_<(I)$ path. The result follows since μ_i divides μ_1 for $i = 1, ..., m$ and θ_i divides θ_m for $i = 1, ..., m$.

We now have the machinery necessary for the construction of the minimal resolutions.

Chapter 6

The Minimal Resolutions

Now we need to define the projective P_n, $n \geq 2$, for the case where Λ is a monomial algebra. This is the motivation behind the associated sequence of paths in Chapter 5.

Definition 6.1

$$P_n = \coprod_{AP(n)} \Lambda o(p^n) \bigotimes t(p^n)\Lambda.$$

There is a different and more general description of these projectives in [20]. To define the maps recall from Lemma 5.3 that if n is odd and $p^n \in AP(n)$, then $Sub(p^n) = \{p_1^{n-1}, p_2^{n-1}\}$ and for $p^{n+1} \in AP(n+1)$, $Sub(p^{n+1}) = \{p_1^n, p_2^n, ..., p_m^n\}$. So we can write $p^n = L_1^2 p_2^{n-1} = p_1^{n-1} R_2^1$ and $p^{n+1} = \theta_i p_i^n \mu_i$ for $i = 1, ..., m$. Let us now define the following maps:

$$\phi_n \left(o(p^n) \bigotimes t(p^n) \right) = L_1^2 \bigotimes t(p^n) - o(p^n) \bigotimes R_2^1$$

$$\phi_{n+1} \left(o(p^{n+1}) \bigotimes t(p^{n+1}) \right) = \sum_{i=1}^{m} \theta_i \bigotimes \mu_i.$$

These maps show how the alternating behavior of the syzygies is inherited from the construction of the associated sequence of paths. The reader has probably noticed that the second part of Lemma 5.3 appears to contradict this construction. However, if we let $AP(0) = \Gamma_0$ and $AP(1) = \Gamma_1$, then this alternating behavior is clear through $AP(3)$. Given an arrow $a \in AP(1)$ ($p^3 \in AP(3)$), there are obviously only two vertices in $AP(0)$ (two paths in $AP(2)$) that divide a (divide p^3). Conversely, given a path $p \in AP(2)$, there are at least two arrows in $AP(1)$ that divide p. So even though the implications of Lemma 5.3 seem counterintuitive for $AP(n)$, $n > 3$, the analogue of this lemma for $AP(3)$, $AP(2)$, and $AP(1)$ is quite natural. We are now ready for the main theorem of this paper.

Theorem 6.1 *Let Γ be a finite quiver and suppose $\Lambda = k\Gamma/I$ is a monomial algebra. Furthermore, assume $J^N \subset I \subset J^2$ for some integer $N \geq 2$. Then the sequence*

$$\cdots \longrightarrow P_{n+1} \xrightarrow{\phi_{n+1}} P_n \xrightarrow{\phi_n} \cdots \xrightarrow{\phi_2} P_1 \xrightarrow{\phi_1} P_0 \xrightarrow{\pi} \Lambda \to 0$$

is a minimal projective resolution of Λ as a right Λ^e-module.

Recall that the above resolution is minimal if im $\phi_n \subset P_{n-1}\mathbf{r}^{\mathbf{e}}$, where $\mathbf{r}^{\mathbf{e}}$ is the Jacobson radical of Λ^e. By construction we know L_1^2 and R_2^1 are not vertices. Similarly, θ_i and μ_i are not simultaneously vertices for any given $i = 1, ..., m$. Since $\mathbf{r}^{\mathbf{e}}$ is generated by elements of the form $a_i^{op} \otimes v$ and $w \otimes a_j$, $a_i, a_j \in \Gamma_1$, $v, w \in \Gamma_0$, the minimality follows.

Before we proceed with the proof of Theorem 6.1 we first need to extend the order $<$ to P_n. Suppose $p_i^n, p_j^n \in AP(n)$ with $p \otimes q \in \Lambda o(p_i^n) \otimes t(p_i^n)\Lambda$ and $r \otimes s \in \Lambda o(p_j^n) \otimes t(p_j^n)\Lambda$. Then we say $p \otimes q > r \otimes s$ iff

1) $l(q) < l(s)$ or
2) $l(q) = l(s)$ and $l(pp_i^n) > l(rp_j^n)$; or
3) $l(q) = l(s)$, $l(pp_i^n) = l(rp_j^n)$, and $q < s$; or
4) $q = s$, $l(pp_i^n) = l(rp_j^n)$, and $pp_i^n > rp_j^n$.

Note that this order is just a generalization of the order we put on P_0 and P_1. The importance of this order is the following lemma.

Lemma 6.1 *Suppose $p \otimes q > r \otimes s$ in P_n. Then tip $\phi_n(p \otimes q) >$ tip $\phi_n(r \otimes s)$ in P_{n-1}.*

Proof: First let n be odd. Suppose $Sub(p_i^n) = \{p_1^{n-1}, p_2^{n-1}\}$ and $Sub(p_j^n) = \{p_3^{n-1}, p_4^{n-1}\}$, where $p_i^n = L_1^2 p_2^{n-1} = p_1^{n-1} R_2^1$ and $p_j^n = L_3^4 p_4^{n-1} = p_3^{n-1} R_4^3$. Then $\phi_n(p \otimes q) = pL_1^2 \otimes q - p \otimes R_2^1 q$ which has tip $pL_1^2 \otimes q$ and $\phi_n(r \otimes s) = rL_3^4 \otimes s - r \otimes R_4^3 s$ which has tip $rL_3^4 \otimes s$. Since $p \otimes q > r \otimes s$ in P_n there are four possibilities:

1) If $l(q) < l(s)$, then $pL_1^2 \otimes q > rL_3^4 \otimes s$;

2) If $l(q) = l(s)$ and $l(pp_i^n) > l(rp_j^n)$, then $l((pL_1^2)p_2^{n-1}) > l((rL_3^4)p_4^{n-1})$. This means $pL_1^2 \otimes q > rL_3^4 \otimes s$;

3) If $l(q) = l(s)$, $l(pp_i^n) = l(rp_j^n)$, and $q < s$, then $l((pL_1^2)p_2^{n-1}) = l((rL_3^4)p_4^{n-1})$ and we have $pL_1^2 \otimes q > rL_3^4 \otimes s$;

4) If $q = s$, $l(pp_i^n) = l(rp_j^n)$, and $pp_i^n > rp_j^n$, then $(pL_1^2)p_2^{n-1} > (rL_3^4)p_4^{n-1}$. This implies $pL_1^2 \otimes q > rL_3^4 \otimes s$.

Hence, tip $\phi_n(p \otimes q) >$ *tip* $\phi_n(r \otimes s)$ in P_{n-1} when n is odd. The proof for the even case is analogous.

Exactness at P_2

We need to show that im ϕ_3 =ker ϕ_2. So let $p^3 \in AP(3)$ and suppose $Sub(p^3) = \{p_i, p_j\}$. Composing the necessary maps gives $\phi_2\phi_3(o(p^3) \otimes t(p^3)) =$

$$\begin{aligned} &= \phi_2(L_i^j \otimes t(p^3) - o(p^3) \otimes R_j^i) \\ &= L_i^j \phi_2(o(p_j) \otimes t(p_j)) - \phi_2(o(p_i) \otimes t(p_i)) R_j^i \\ &= L_i^j x_{p_j} - x_{p_i} R_j^i. \end{aligned}$$

Now, suppose $p_i = a_1 a_2 \cdots a_{l-1} a_l \cdots a_{l+k}$ and $p_j = a_l a_{l+1} \cdots a_{l+k} \cdots a_{l+k+n}$:

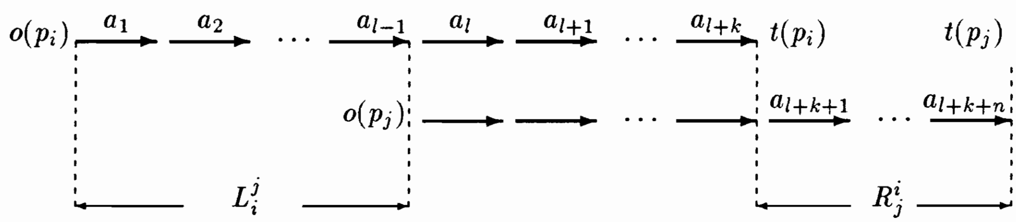

Figure 6.1

Then $L_i^j x_{p_j} =$

$$= L_i^j \sum_{d=l}^{l+k+n} a_l \cdots a_{d-1} \otimes a_{d+1} \cdots a_{l+k+n}$$
$$= L_i^j \sum_{d=l}^{l+k} a_l \cdots a_{d-1} \otimes a_{d+1} \cdots a_{l+k+n}$$
$$= \sum_{d=l}^{l+k} a_1 \cdots a_{d-1} \otimes a_{d+1} \cdots a_{l+k+n}.$$

Similarly, $x_{p_i} R_j^i = \sum_{d=l}^{l+k} a_1 \cdots a_{d-1} \otimes a_{d+1} \cdots a_{l+k+n}$. It follows that $\phi_2\phi_3 = 0$ and $\mathrm{im}\,\phi_3 \subset \ker \phi_2$.

To show the reverse inclusion let $x \in \ker \phi_2$ and reduce x to its tensor normal form modulo I. Suppose $\mathrm{tip}(x) = p \otimes q \in \Lambda o(p_i) \otimes t(p_i)\Lambda$, where $p_i = a_1 a_2 \cdots a_n \in Minsharp_<(I)$. Then $\phi_2(p \otimes q) = p x_{p_i} q$ has tip $p a_1 a_2 \cdots a_{n-1} \otimes q$. Now suppose $r \otimes s$ is in the support of x where $r \otimes s \in \Lambda o(p_j) \otimes t(p_j)\Lambda$, $p_j = b_1 b_2 \cdots b_m \in Minsharp_<(I)$. Then $\phi_2(r \otimes s) = r x_{p_j} s$ has tip $r b_1 b_2 \cdots b_{m-1} \otimes s$. So under the P_2 order, $p \otimes q > r \otimes s$ implies $p a_1 a_2 \cdots a_{n-1} \otimes q > r b_1 b_2 \cdots b_{m-1} \otimes s$ in P_1. Thus, $\mathrm{tip}\,\phi_2(x) = p a_1 a_2 \cdots \cdots a_{n-1} \otimes q$ and $p a_1 a_2 \cdots a_{n-1} \otimes q \notin \mathrm{support}(\phi_2(y))$ for any y in the support of x (except $p \otimes q$). Since q is in its normal form modulo I, it must be the case that p_k divides $p a_1 a_2 \cdots a_{n-1}$ for some $p_k \in Minsharp_<(I)$. But p_k does not divide p and p_k does not divide $a_1 a_2 \cdots a_{n-1}$, so we must have the following situation:

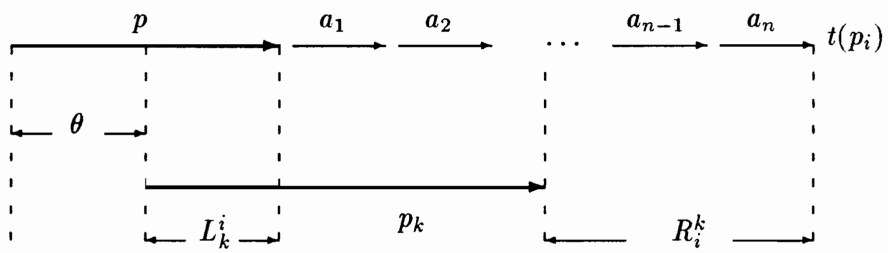

Figure 6.2

Without loss of generality we can choose p_k such that p_i is minimal with respect to $o(p_k) < o(p_i) < t(p_k)$. Then $\Lambda o(p_k) \bigotimes t(p_i)\Lambda$ is a summand of P_3 and $\phi_3(o(p_k) \bigotimes t(p_i)) = L^i_k \bigotimes t(p_i) - o(p_k) \bigotimes R^k_i$ has tip $L^i_k \bigotimes t(p_i)$. We see $L^i_k \bigotimes t(p_i)$ divides $p \bigotimes q$ since $(L^i_k \bigotimes t(p_i))(\theta \bigotimes q) = \theta L^i_k \bigotimes q = p \bigotimes q$. So x reduces over $\operatorname{im}\phi_3$ and it follows that $\operatorname{im}\phi_3 = \ker \phi_2$.

Exactness at P_3

To show that $\operatorname{im}\phi_4 = \ker\phi_3$ let $p^4 \in AP(4)$, $Sub(p^4) = \{p^3_1, ..., p^3_n\}$, and consider

$$\phi_3\phi_4(o(p^4) \bigotimes t(p^4)) = \sum_{i=1}^{n} \theta_i \phi_3(o(p^3_i) \bigotimes t(p^3_i))\mu_i.$$

To calculate this suppose p^3_1 is constructed from p_1 and p_2, p^3_2 is constructed from p_2 and p_3,..., and p^3_n is constructed from p_n and p_{n+1}:

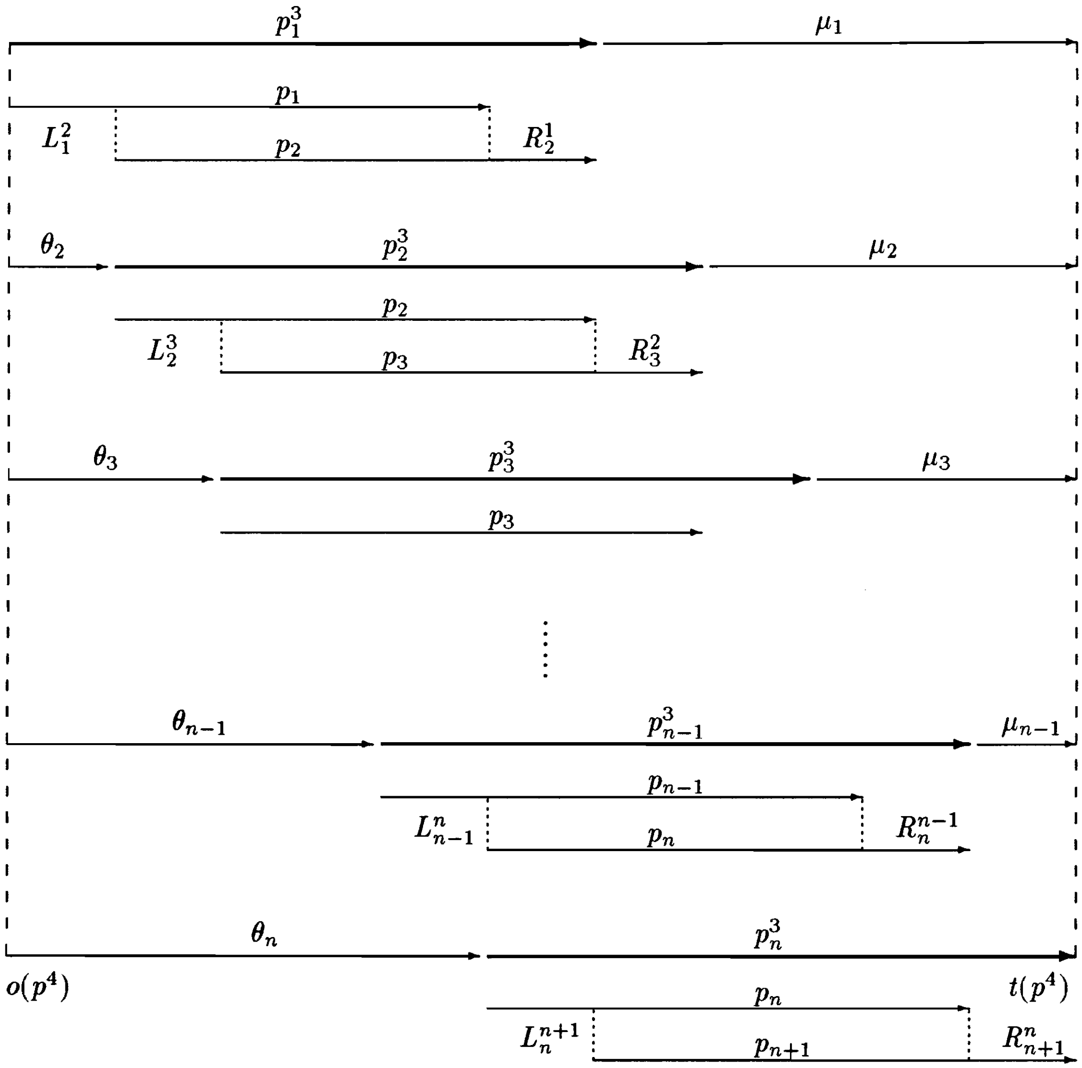

Figure 6.3

From this construction we have the following:

$$\theta_2 = L_1^2 \qquad \mu_{n-1} = R_{n+1}^n$$

$$\theta_3 = L_1^2 L_2^3 = L_1^3 \qquad \mu_{n-2} = R_n^{n-1} R_{n+1}^n = R_{n+1}^{n-1}$$

$$\vdots \qquad \vdots$$

$$\theta_n = L_1^2 L_2^3 \cdots L_{n-1}^n = L_1^n \quad \mu_1 = R_3^2 R_4^3 \cdots R_{n+1}^n = R_{n+1}^2$$

Since $\theta_1 = o(p^4) = L_1^1$ and $\mu_n = t(p^4) = R_{n+1}^{n+1}$, we conclude that $\theta_i = L_1^i$ and $\mu_i = R_{n+1}^{i+1}$ for $i = 1, ..., n$. Hence, $\phi_3\phi_4(o(p^4) \otimes t(p^4)) =$

$$
\begin{aligned}
&= \textstyle\sum_{i=1}^{n} \theta_i \phi_3(o(p_i^3 \otimes t(p_i^3))\mu_i \\
&= \textstyle\sum_{i=1}^{n} L_1^i (L_i^{i+1} \otimes t(p_i^3) - o(p_i^3) \otimes R_{i+1}^i) R_{n+1}^{i+1} \\
&= \textstyle\sum_{i=1}^{n} (L_1^{i+1} \otimes R_{n+1}^{i+1} - L_1^i \otimes R_{n+1}^i) \\
&= L_1^{n+1} \otimes t(p^4) - o(p^4) \otimes R_{n+1}^1.
\end{aligned}
$$

By the left construction of p^4 there must be some $p \in Minsharp_<(I)$ such that $t(p_1) \leq o(p) < t(p_2)$ and $t(p) = t(p^4)$. But this means that p divides R_{n+1}^1. Similarly, by the right construction of p^4, there must be some $q \in Minsharp_<(I)$ such that $t(q) \leq o(p_{n+1})$ and $o(q) = o(p^4)$. Thus, q divides L_1^{n+1}. It follows that $\phi_3\phi_4 = 0$ and $\operatorname{im}\phi_4 \subset \ker\phi_3$.

To show the reverse inclusion let $x \in \ker\phi_3$ and reduce x to its tensor normal form modulo I. Suppose $\operatorname{tip}(x) = p \otimes q \in \Lambda o(p_i^3) \otimes t(p_i^3)\Lambda$ where we assume p_i^3 is constructed from the $Minsharp_<(I)$ paths p_1 and p_2:

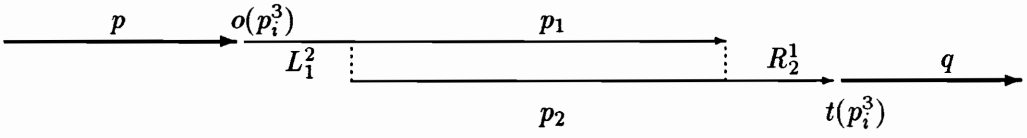

Figure 6.4

Then $\phi_3(p \otimes q) =$

$$
\begin{aligned}
&= p\phi_3(o(p_i^3) \otimes t(p_i^3))q \\
&= p(L_1^2 \otimes t(p_i^3) - o(p_i^3) \otimes R_2^1)q \\
&= pL_1^2 \otimes q - p \otimes R_2^1 q
\end{aligned}
$$

has tip $pL_1^2 \otimes q$. By Lemma 6.1 $\operatorname{tip}(\phi_3(x)) = pL_1^2 \otimes q$ and $pL_1^2 \otimes q$ is not an element of support$(\phi_3(y))$ for any $y \in$support(x) (except $p \otimes q$).

Now, q is in its normal form modulo I so it must be the case that p_k divides pL_1^2 for some $p_k \in Minsharp_<(I)$. Since p_k does not divide p and p_k does not divide L_1^2, we must have the following situation:

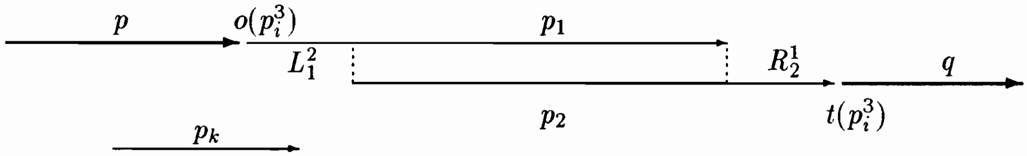

Figure 6.5

So $o(p) \leq o(p_k) < t(p)$ and $o(p_1) < t(p_k) \leq o(p_2)$. Without loss of generality we can assume p_k is maximal with respect to $o(p_1) < t(p_k) \leq o(p_2)$. Then $\Lambda o(p_k) \bigotimes t(p_2)\Lambda$ is a summand of P_4 - say p^4 is the path from $o(p_k)$ to $t(p_2)$ by the right construction. Suppose $Sub(p^4) = \{p_1^3, ..., p_n^3\}$:

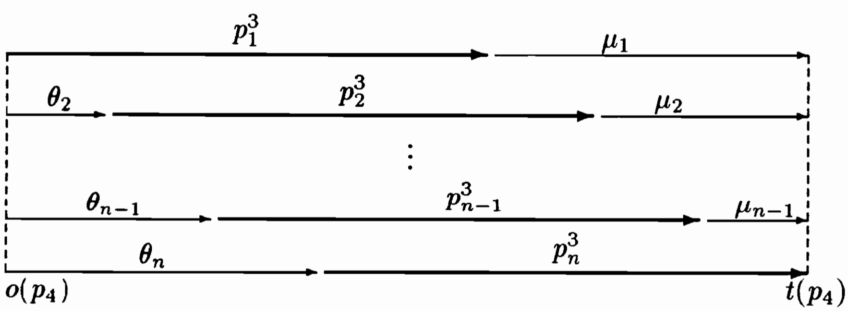

Figure 6.6

By construction $p_n^3 = p_i^3$. Then $\phi_4(o(p^4) \bigotimes t(p^4)) = \sum_{j=1}^{n} \theta_j \bigotimes \mu_j$ has tip $\theta_n \bigotimes t(p^4)$. We have the following:

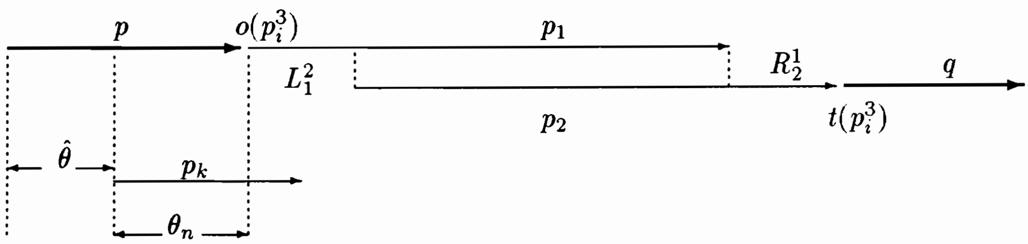

Figure 6.7

We see θ_n divides p and $(\theta_n \otimes t(p^4))\hat{\theta} \otimes q = \hat{\theta}\theta_n \otimes q = p \otimes q$, that is, tip $\phi_4(o(p^4) \otimes t(p^4))$ divides tip(x). Hence, x reduces over im ϕ_4 and it follows that im ϕ_4 =ker ϕ_3.

Exactness at P_n

We have seen that Theorem 6.1 is true for $P_4 \xrightarrow{\phi_4} P_3 \xrightarrow{\phi_3} P_2 \xrightarrow{\phi_2} P_1 \xrightarrow{\phi_1} P_0 \xrightarrow{\pi} \Lambda \longrightarrow 0$. So let us assume the result is true for $P_{n-1} \xrightarrow{\phi_{n-1}} P_{n-2} \xrightarrow{\phi_{n-1}} \cdots$, where n is odd and $n \geq 5$. Consider the sequence $P_{n+1} \xrightarrow{\phi_{n+1}} P_n \xrightarrow{\phi_n} P_{n-1} \xrightarrow{\phi_{n-1}} \cdots$ and let us first show that im ϕ_n =ker ϕ_{n-1}. To see $\phi_{n-1}\phi_n = 0$ let $p^n \in AP(n)$ and suppose $Sub(p^n) = \{p_1^{n-1}, p_2^{n-1}\}$:

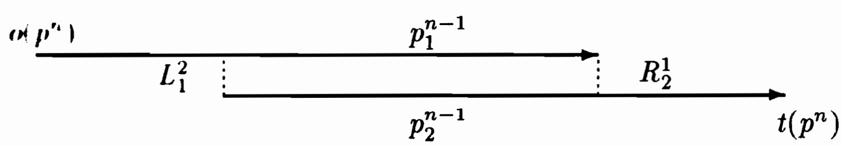

Figure 6.8

Then $\phi_{n-1}\phi_n(o(p^n) \otimes t(p^n)) = L_1^2\phi_{n-1}(o(p_2^{n-1}) \otimes t(p_2^{n-1})) - \phi_{n-1}(o(p_1^{n-1}) \otimes t(p_1^{n-1}))R_2^1$. Now suppose $Sub(p_1^{n-1}) = \{p_1^{n-2}, ..., p_{k+m}^{n-2}\}$ and $Sub(p_2^{n-1}) = \{p_k^{n-2}, ..., p_{k+m}^{n-2}, ..., p_r^{n-2}\}$. Here we are assuming $Sub(p_1^{n-1}) \bigcap Sub(p_2^{n-1}) \neq \emptyset$. The other case will be considered next. We have the following situation:

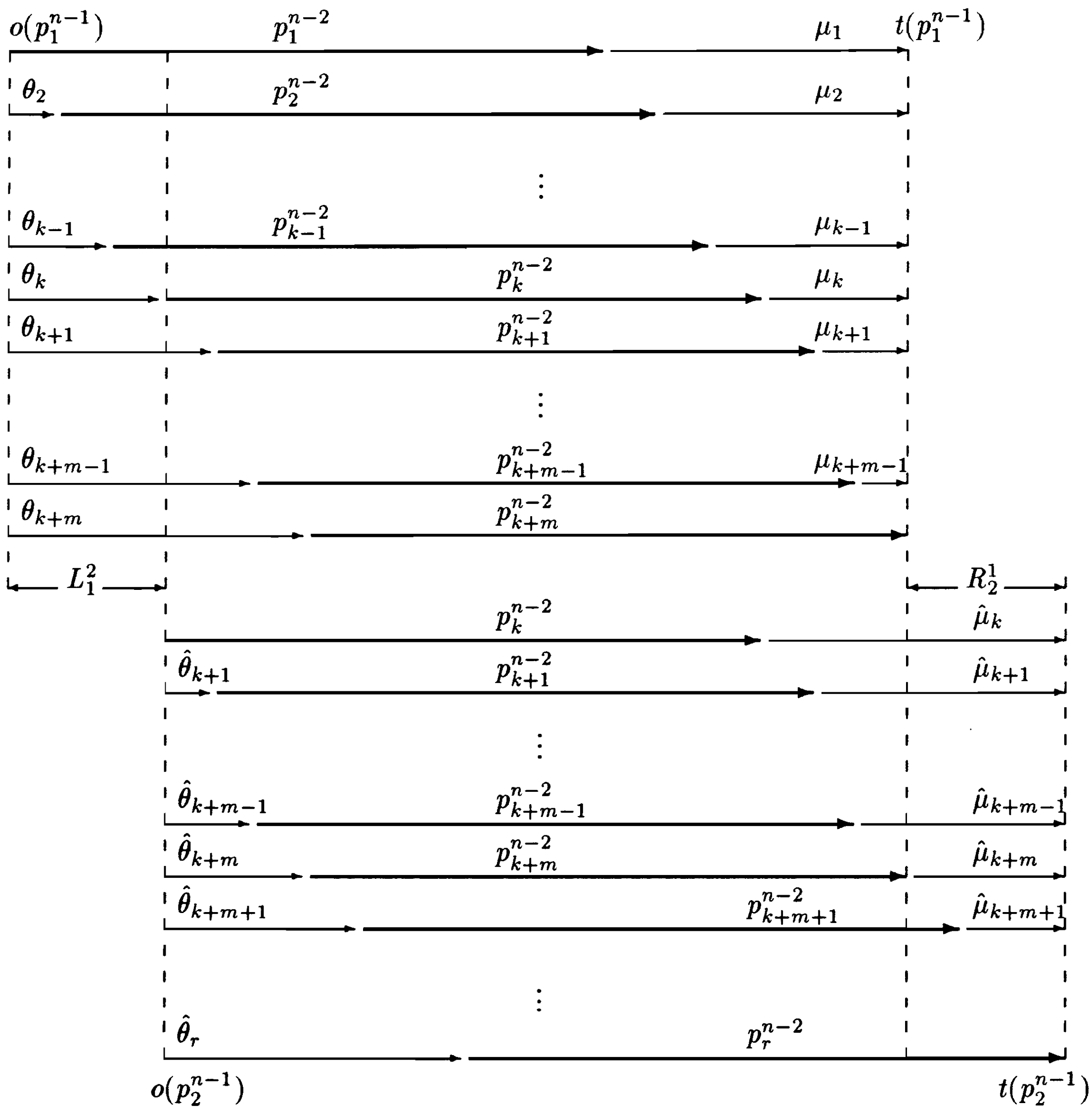

Figure 6.9

Then if $r = k + m + t$ we have $\phi_{n-1}\phi_n(o(p^n) \bigotimes t(p^n)) =$

$$= L_1^2 \left[\sum_{i=0}^{m+t} \hat{\theta}_{k+i} \bigotimes \hat{\mu}_{k+i} \right] - \left[\sum_{i=1}^{k+m} \theta_i \bigotimes \mu_i \right] R_2^1.$$

Breaking this sum up we obtain

$$L_1^2\left[\sum_{i=0}^{m}\hat{\theta}_{k+i}\bigotimes\hat{\mu}_{k+i}\right]+L_1^2\left[\sum_{i=m+1}^{m+t}\hat{\theta}_{k+i}\bigotimes\hat{\mu}_{k+i}\right]-\left[\sum_{i=1}^{k-1}\theta_i\bigotimes\mu_i\right]R_2^1-\left[\sum_{i=k}^{k+m}\theta_i\bigotimes\mu_i\right]R_2^1.$$

By construction

$$L_1^2\left[\sum_{i=0}^{m}\hat{\theta}_{k+i}\bigotimes\hat{\mu}_{k+i}\right]=\left[\sum_{i=k}^{k+m}\theta_i\bigotimes\mu_i\right]R_2^1.$$

Thus, $\phi_{n-1}\phi_n(o(p^n)\bigotimes t(p^n))=$

$$L_1^2\left[\sum_{i=m+1}^{m+t}\hat{\theta}_{k+i}\bigotimes\hat{\mu}_{k+i}\right]-\left[\sum_{i=1}^{k-1}\theta_i\bigotimes\mu_i\right]R_2^1.$$

To calculate this consider p_{k+m}^{n-2} and p_{k+m+1}^{n-2} from the previous diagram. Since $n-2$ is odd $Sub(p_{k+m}^{n-2})$ and $Sub(p_{k+m+1}^{n-2})$ both contain exactly two paths. Suppose $Sub(p_{k+m}^{n-2})=\{p_i^{n-3},p_{i+1}^{n-3}\}$ and $Sub(p_{m+k+1}^{n-2})=\{p_j^{n-3},p_{j+1}^{n-3}\}$. Using Lemma 5.2 it is not hard to show that $p_{i+1}^{n-3}=p_j^{n-3}$. After relabeling we have $Sub(p_{k+m}^{n-2})=\{p_i^{n-3},p_{i+1}^{n-3}\}$ and $Sub(p_{k+m+1}^{n-2})=\{p_{i+1}^{n-3},p_{i+2}^{n-3}\}$. So we have the following situation:

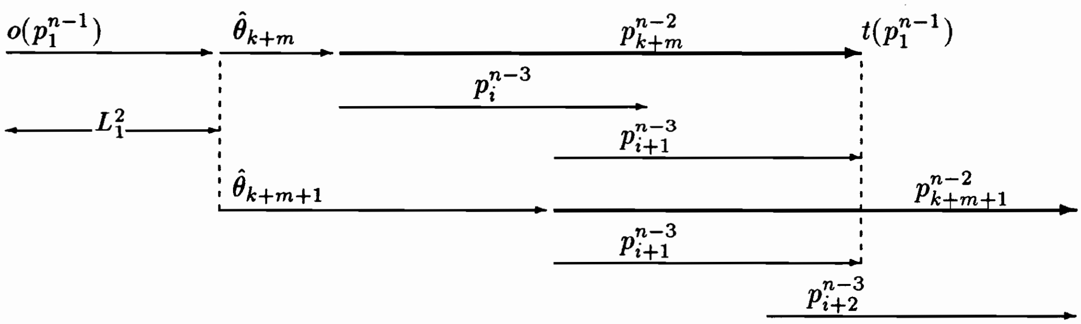

Figure 6.10

Let us consider the right construction of p_1^{n-1}. Since $t(p_{i+1}^{n-3}) = t(p_1^{n-1})$ we know p_{i+1}^{n-3} is part of this construction:

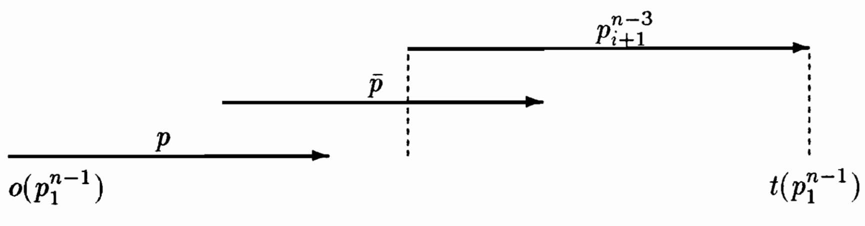

Figure 6.11

So there exists some path $\bar{p} \in Minsharp_<(I)$ such that $t(\bar{p}) > o(p_{i+1}^{n-3})$ and the path from $o(\bar{p})$ to $t(p_{i+1}^{n-3})$ is in $AP(n-2)$. Similarly, there must be some $p \in Minsharp_<(I)$ such that $o(\bar{p}) < t(p) \leq o(p_{i+1}^{n-3})$ and p_1^{n-1} is the path from $o(p)$ to $t(p_{i+1}^{n-3})$. Thus, p divides $L_1^2\hat{\theta}_{k+m+1}$. It follows that p divides $L_1^2\hat{\theta}_{k+m+i}$ for $i = 1, ..., t$ $(r = k + m + t)$. So

$$L_1^2 \left[\sum_{i=m+1}^{m+t} \hat{\theta}_{k+i} \bigotimes \hat{\mu}_{k+i} \right] = 0.$$

A similar argument using the left construction of p_2^{n-1} yields

$$\left[\sum_{i=1}^{k-1} \theta_i \bigotimes \mu_i \right] R_2^1 = 0.$$

Hence $\phi_{n-1}o_n = 0$ in this case.

Now we need to consider the case where $Sub(p_1^{n-1}) \bigcap Sub(p_2^{n-1}) = \emptyset$. By Lemma 5.4 we know L_1^2 and R_2^1 are not divisible by any $Minsharp_<(I)$ path so we must have the following situation:

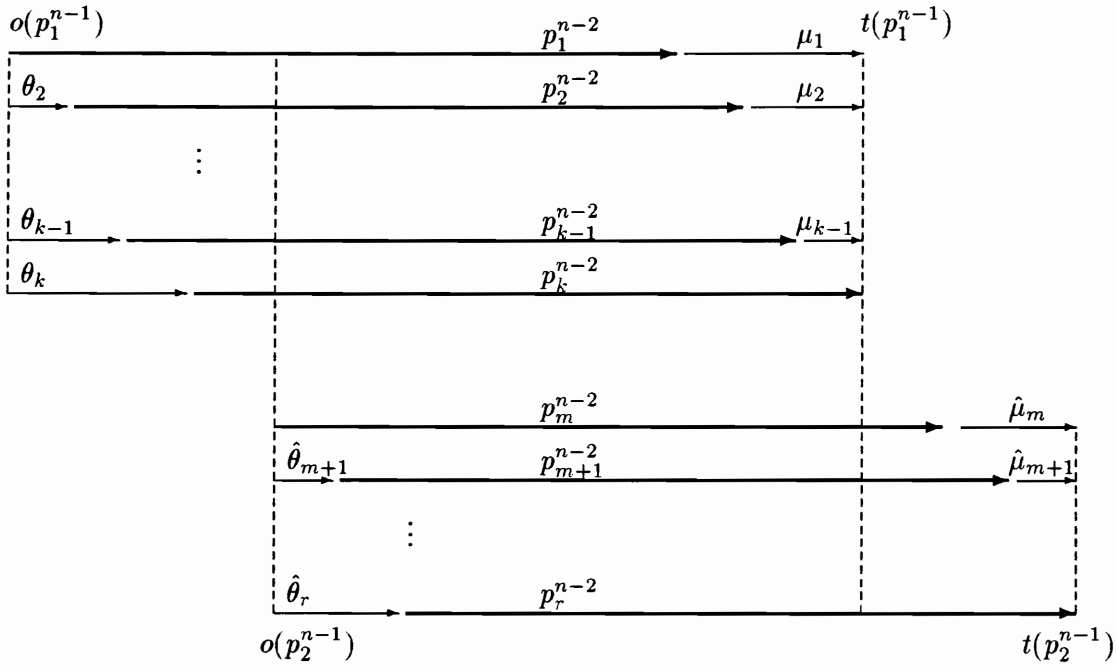

Figure 6.12

Suppose $Sub(p_k^{n-2}) = \{p_1^{n-3}, p_2^{n-3}\}$ and $Sub(p_m^{n-2}) = \{p_3^{n-3}, p_4^{n-3}\}$:

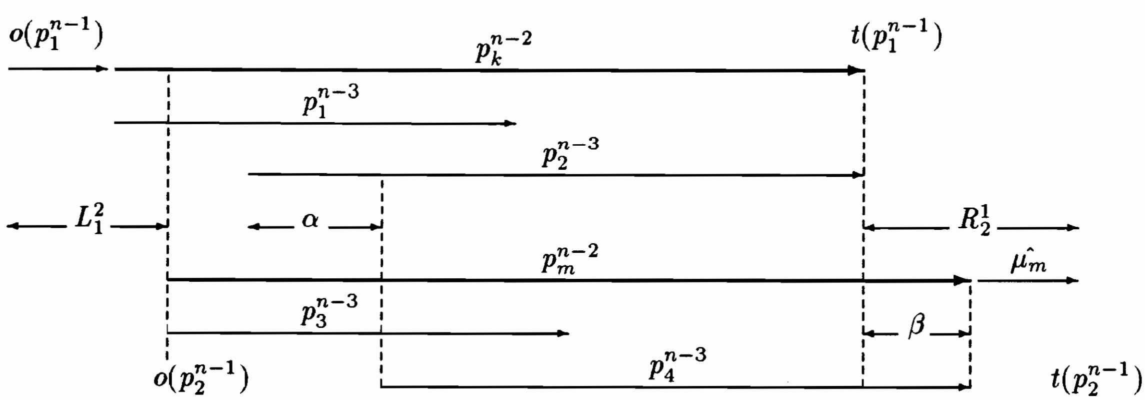

Figure 6.13

First assume $o(p_3^{n-3}) < o(p_2^{n-3})$. Then p_2^{n-3} and p_4^{n-3} overlap. Since α and β are not divisible by any $Minsharp_<(I)$ paths we know by Lemma 5.2 that there exists some

$p^{n-2} \in AP(n-2)$ such that p^{n-2} divides the path from $o(p_2^{n-3})$ to $t(p_4^{n-3})$. But then p^{n-2} properly divides p_m^{n-2}, a contradiction. So $o(p_2^{n-3}) \leq o(p_3^{n-3})$. However, by the right construction of p_1^{n-1} using p_2^{n-3}, there must be some $p \in Minsharp_<(I)$ such that $o(p) = o(p_1^{n-1})$ and $t(p) \leq o(p_2^{n-3})$. But then p divides L_1^2 contradicting Lemma 5.4. So the case $Sub(p_1^{n-1}) \bigcap Sub(p_2^{n-1}) = \emptyset$ is not possible and it follows that $\mathrm{im}\,\phi_n \subset \ker\phi_{n-1}$.

Now let $x \in \ker\phi_{n-1}$ and reduce x to its tensor normal form modulo I. Suppose $\mathrm{tip}(x) = p \bigotimes q \in \Lambda o(p_i^{n-1}) \bigotimes t(p_i^{n-1})\Lambda$. Furthermore, suppose $Sub(p_i^{n-1}) = \{p_1^{n-2}, p_2^{n-2}, ..., p_r^{n-2}\}$. Then $\phi_{n-1}(p \bigotimes q) = p(\sum_{i=1}^{r} \theta_i \bigotimes \mu_i)q$ which has tip $p\theta_r \bigotimes q \in \Lambda o(p_r^{n-2}) \bigotimes t(p_r^{n-2})\Lambda$. By Lemma 6.1 $\mathrm{tip}\,\phi_{n-1}(x) = p\theta_r \bigotimes q$. Since q is in its normal form modulo I there must be some $p_1 \in Minsharp_<(I)$ such that p_1 divides $p\theta_r$. But p_1 does not divide p and p_1 does not divide θ_r so we must have the following situation:

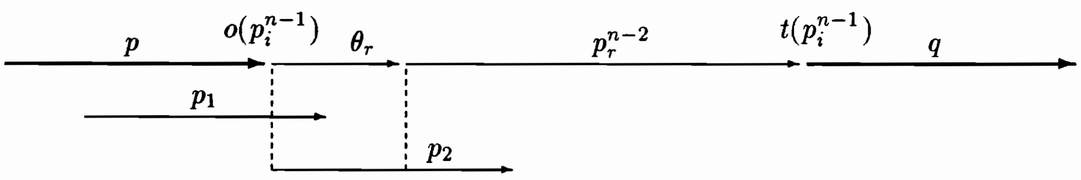

Figure 6.14

By the right construction of p_i^{n-1} using p_r^{n-2} we know there is some $p_2 \in Minsharp_<(I)$ such that $o(p_2) = o(p_i^{n-1})$ and $t(p_2) > o(p_r^{n-2})$. Without loss of generality we can assume we chose p_1 to be maximal with respect to $o(p_2) < t(p_1) \leq o(p_r^{n-2})$. Then there exists some $p^n \in AP(n)$ such that $o(p^n) = o(p_1)$ and $t(p^n) = t(p_i^{n-1})$:

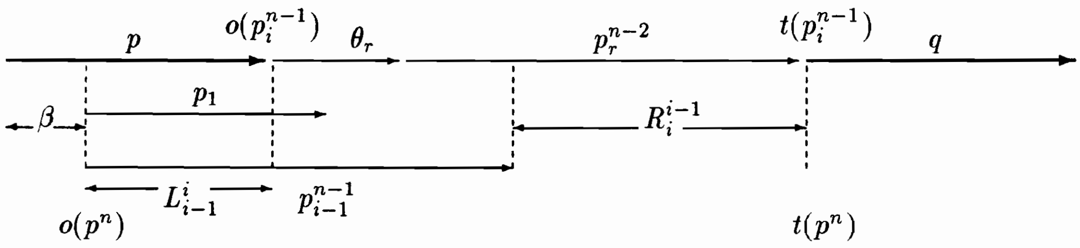

Figure 6.15

Suppose $Sub(p^n) = \{p_{i-1}^{n-1}, p_i^{n-1}\}$. Then $\phi_n(o(p^n) \otimes t(p^n)) = L_{i-1}^i \otimes t(p^n) - o(p^n) \otimes R_i^{i-1}$ which has tip $L_{i-1}^i \otimes t(p^n)$. Also $L_{i-1}^i \otimes t(p^n)$ divides $p \otimes q$ since $(L_{i-1}^i \otimes t(p^n))\,(\beta \otimes q) = \beta L_{i-1}^i \otimes q = p \otimes q$. So x reduces over $\operatorname{im} \phi_n$ and it follows that $\operatorname{im} \phi_n = \ker \phi_{n-1}$.

The proof that $\operatorname{im} \phi_{n+1} = \ker \phi_n$ is analogous to the argument showing $\operatorname{im} \phi_4 = \ker \phi_3$.

Now that the resolutions have been constructed we can define the complexes necessary for Hochschild cohomology. These will be presented in Chapter 7. Since the syzygies of the minimal resolutions exhibit an alternating behavior, the induced complexes of the next chapter should inherit, at least in part, some of these characteristics. We will see that this is indeed the case.

Chapter 7

Hochschild Cohomology

We are now ready to calculate the algebraic invariants mentioned in Chapter 1, that is, the Hochschild cohomology groups. So let Λ be a finite dimensional algebra and suppose

$$\cdots P_{n+1} \xrightarrow{\phi_{n+1}} P_n \xrightarrow{\phi_n} \cdots \xrightarrow{\phi_2} P_1 \xrightarrow{\phi_1} P_0 \xrightarrow{\pi} \Lambda \longrightarrow 0$$

is a Λ^e−projective resolution of Λ. Let us call this resolution P. Then we can apply the contravariant functor $\mathrm{Hom}_{\Lambda^e}(\ ,\Lambda)$ to P and obtain the following complex:

$$0 \longrightarrow \mathrm{Hom}_{\Lambda^e}(P_0,\Lambda) \xrightarrow{\hat{\phi}_1} \mathrm{Hom}_{\Lambda^e}(P_1,\Lambda) \xrightarrow{\hat{\phi}_2} \quad \cdots \quad \xrightarrow{\hat{\phi}_n} \mathrm{Hom}_{\Lambda^e}(P_n,\Lambda) \xrightarrow{\hat{\phi}_{n+1}} \cdots$$

Here $\hat{\phi}_n$ is the boundary map induced by ϕ_n, i.e if $f \in \mathrm{Hom}_{\Lambda^e}(P_{n-1},\Lambda)$, then $\hat{\phi}_n(f) = f\phi_n$. The n^{th} Hochschild cohomology group of Λ is $\mathrm{H}^n(\Lambda,\Lambda) = \mathrm{Ext}^n_{\Lambda^e}(\Lambda,\Lambda) = \mathrm{H}^n(\mathrm{Hom}_{\Lambda^e}(P,\Lambda))$. In other words, $\mathrm{H}^n(\Lambda,\Lambda) = \ker\hat{\phi}_{n+1}/\mathrm{im}\,\hat{\phi}_n$. Note that this definition does not depend on which projective resolution we use. In particular, we do not assume that P is minimal. The problem now is how to calculate the kernels of the induced boundary maps. Fortunately, we can use some of the standard identifications from ring theory to translate this problem into an equivalent statement about "nice" vector spaces and linear maps. To see this, let us denote $\mathrm{Hom}_{\Lambda^e}(A,B)$ by (A,B) and consider the following commutative diagram:

$$
\begin{array}{ccc}
(\coprod_{AP(n)} \Lambda o(p^n) \bigotimes t(p^n)\Lambda, \Lambda) & \xrightarrow{\hat{\phi}_{n+1}} & (\coprod_{AP(n+1)} \Lambda o(p^{n+1}) \bigotimes t(p^{n+1})\Lambda, \Lambda) \\
\uparrow \iota_i & & \downarrow \pi_i \\
\coprod_{AP(n)}(\Lambda o(p^n) \bigotimes t(p^n)\Lambda, \Lambda) & & \coprod_{AP(n+1)}(\Lambda o(p^{n+1}) \bigotimes t(p^{n+1})\Lambda, \Lambda) \\
\uparrow \mu & & \downarrow \mu\phi_{n+1} \\
\coprod_{AP(n)} o(p^n)\Lambda t(p^n) & \overset{\phi^*_{n+1}}{\dashrightarrow} & \coprod_{AP(n+1)} o(p^{n+1})\Lambda t(p^{n+1})
\end{array}
$$

Figure 7.1

Here ι_i and π_i are the standard canonical maps. Let $o(p^n)\lambda t(p^n) \in o(p^n)\Lambda t(p^n)$. Then $\mu(0, ..., o(p^n)\lambda t(p^n), ..., 0)$ is the map μ_λ in the p^n component of

$$\coprod_{AP(n)} \mathrm{Hom}_{\Lambda^e}(\Lambda o(p^n) \bigotimes t(p^n)\Lambda, \Lambda)$$

that takes $\alpha[o(p^n) \bigotimes t(p^n)]\beta$ to $\alpha\lambda\beta$. It is known that μ and $\mu\phi_{n+1}$ are isomorphisms (for more details see [1]). So the above commutative diagram shows that the p^{n+1} component of $\phi^*_{n+1}(0, ..., o(p^n)\lambda t(p^n), ..., 0)$ is $\mu_\lambda\phi_{n+1}(o(p^{n+1}) \bigotimes t(p^{n+1}))$. To illustrate these maps let us look at a few small examples.

Example 1: Let Γ be the following quiver:

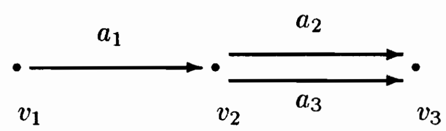

Figure 7.2

Let $I_1 = \langle a_1 a_2 \rangle$. Then the Hochschild cohomology of Λ is determined from the following complex:

$$0 \longrightarrow \coprod_{i=1}^{3} kv_i \xrightarrow{\phi_1^*} \coprod_{i=1}^{3} o(a_i)\Lambda t(a_i) \xrightarrow{\phi_2^*} o(a_1)\Lambda t(a_2) \longrightarrow 0.$$

Since $\phi_1(o(a_i) \otimes t(a_i)) = a_i \otimes t(a_i) - o(a_i) \otimes a_i$ for $i = 1, 2, 3$, we have $(v_1, 0, 0) \xrightarrow{\phi_1^*}$ $(\mu_{v_1}(a_1 \otimes t(a_1) - o(a_1) \otimes a_1), \mu_{v_1}(a_2 \otimes t(a_2) - o(a_2) \otimes a_2), \mu_{v_1}(a_3 \otimes t(a_3) - o(a_3) \otimes a_3)) =$ $(-a_1, 0, 0)$. Similarly,

$$(0, v_2, 0) \xrightarrow{\phi_1^*} (a_1, -a_2, -a_3)$$
$$(0, 0, v_3) \xrightarrow{\phi_1^*} (0, a_2, a_3).$$

Thus, $\dim_k \operatorname{im} \phi_1^* = 2$. Since $\phi_2(o(a_1) \otimes t(a_2)) = a_1 \otimes t(a_2) + o(a_1) \otimes a_2$, we have

$$(a_1, 0, 0) \xrightarrow{\phi_2^*} a_1 a_2 = 0$$
$$(0, \lambda_2, 0) \xrightarrow{\phi_2^*} a_1 \lambda_2$$
$$(0, 0, \lambda_3) \xrightarrow{\phi_2^*} 0.$$

Thus, $\phi_2^*(k_1 a_1, k_2 \lambda_2, k_3 \lambda_3) = k_2 a_1 \lambda_2$. If $k_2 \neq 0$ then this can only be zero if $\lambda_2 = a_2$. It follows that $\ker \phi_2^* = \{k_1(a_1, 0, 0) + k_2(0, a_2, 0) + k_3(0, 0, a_2) + k_4(0, 0, a_3) : k_i \in k\}$. Then $\dim_k \ker \phi_2^* = 4$ and $\dim_k \mathrm{H}^1(\Lambda, \Lambda) = 2$.

Now let us consider the same quiver with the ideal I_2 generated by $r = a_1 a_2 + a_1 a_3$. Then $\phi_2(o(r) \otimes t(r)) = o(a_1) \otimes a_2 + a_1 \otimes t(a_2) + a_1 \otimes t(a_3) + o(a_1) \otimes a_3$. The induced maps are as follows:

$$(a_1, 0, 0) \xrightarrow{\phi_2^*} a_1a_2 + a_1a_3 = 0$$
$$(0, \lambda_2, 0) \xrightarrow{\phi_2^*} a_1\lambda_2$$
$$(0, 0, \lambda_3) \xrightarrow{\phi_2^*} a_1\lambda_3.$$

Suppose $\phi_2^*(k_1a_1, k_2\lambda_2, k_3\lambda_3) = k_2a_1\lambda_2 + k_3a_1\lambda_3 = 0$. There are three possibilities. Either $k_2 = -k_3$ and $\lambda_2 = \lambda_3$, $k_2 = k_3$ with $\lambda_2 = a_2$ and $\lambda_3 = a_3$, or $k_2 = k_3 = 0$. So elements of $\ker \phi_2^*$ have the form $\alpha = k_1(a_1, 0, 0) + k_2(0, a_2, -a_2) + k_3(0, a_3, -a_3) + k_4(0, a_2, a_3)$. It follows that $\dim_k \ker \phi_2^* = 4$ and, thus, $\dim_k \mathrm{H}^1(\Lambda, \Lambda) = 2$. Since $k\Gamma/I_1$ and $k\Gamma/I_2$ are isomorphic as k-algebras, we expect to get the same dimension for $\mathrm{H}^1(\Lambda, \Lambda)$ in both cases. So, even though $k\Gamma/I_2$ does not appear to be a monomial algebra, it is by this isomorphism.

Example 2: Suppose that Λ is a monomial algebra with global dimension $n \geq 3$. Furthermore, let us assume that the n^{th} projective P_n is indecomposable. Then we can write $P_n = \Lambda o(p^n) \bigotimes t(p^n)\Lambda$, where $AP(n) = \{p^n\}$ and $AP(m) = \emptyset$ for $m > n$. Applying $\mathrm{Hom}_{\Lambda^e}(\ ,\Lambda)$ to the minimal projective resolution of Λ yields the following complex:

$$0 \longrightarrow \coprod_{v\in\Gamma_0} v\Lambda v \xrightarrow{\phi_1^*} \quad \cdots \quad \xrightarrow{\phi_{n-1}^*} \coprod_{AP(n-1)} o(p^{n-1})\Lambda t(p^{n-1}) \xrightarrow{\phi_n^*} o(p^n)\Lambda t(p^n) \longrightarrow 0$$

Then $\mathrm{H}^n(\Lambda, \Lambda) = o(p^n)\Lambda t(p^n)/\mathrm{im}\,\phi_n^*$. There are two possibilities at this point.

case 1: n is odd.

In this case p^n has exactly two divisors in $AP(n-1)$:

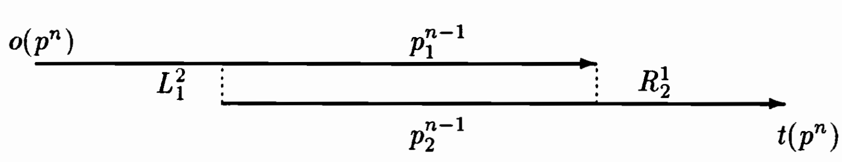

Figure 7.3

Since $\phi_n(o(p^n) \otimes t(p^n)) = L_1^2 \otimes t(p^n) - o(p^n) \otimes R_2^1$, we obtain the following induced map:

$$o(p_1^{n-1})\lambda_1 t(p_1^{n-1}) \xrightarrow{\phi_n^*} \mu_{\lambda_1}\phi_n(o(p^n) \otimes t(p^n)) = -o(p^n)\lambda_1 R_2^1$$
$$o(p_2^{n-1})\lambda_2 t(p_2^{n-1}) \xrightarrow{\phi_n^*} \mu_{\lambda_2}\phi_n(o(p^n) \otimes t(p^n)) = L_1^2\lambda_2 t(p^n).$$

Thus, $\mathrm{H}^n(\Lambda, \Lambda) = o(p^n)\Lambda t(p^n)/\langle L_1^2\lambda_2, \lambda_1 R_2^1\rangle$.

case 2: n is even.

In this case p^n has at least two divisors in $AP(n-1)$ - say $AP(n) = \{p_1^{n-1}, ..., p_r^{n-1}\}$:

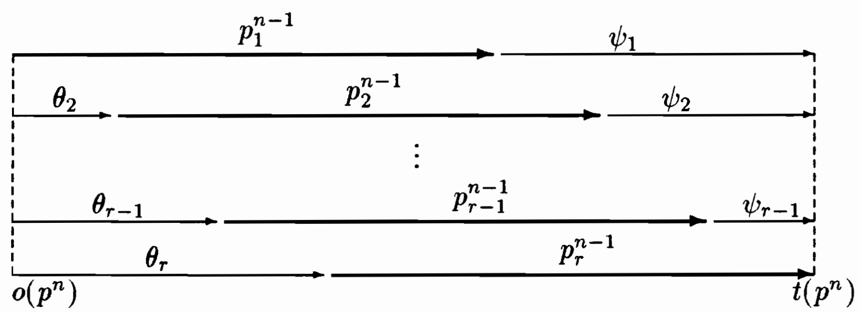

Figure 7.4

Since $\phi_n(o(p^n) \otimes t(p^n)) = \sum_{i=1}^{r} \theta_i \otimes \psi_i$, we obtain the following induced map:

$$o(p_1^{n-1})\lambda_1 t(p_1^{n-1}) \xrightarrow{\phi_n^*} \mu_{\lambda_1}\phi_n(o(p^n) \otimes t(p^n)) = \lambda_1\psi_1$$
$$\vdots$$
$$o(p_i^{n-1})\lambda_i t(p_i^{n-1}) \xrightarrow{\phi_n^*} \mu_{\lambda_i}\phi_n(o(p^n) \otimes t(p^n)) = \theta_i\lambda_i\psi_i$$
$$\vdots$$
$$o(p_r^{n-1})\lambda_r t(p_r^{n-1}) \xrightarrow{\phi_n^*} \mu_{\lambda_r}\phi_n(o(p^n) \otimes t(p^n)) = \theta_r\lambda_r.$$

Thus, $\mathrm{H}^n(\Lambda, \Lambda) = o(p^n)\Lambda t(p^n)/\langle \lambda_1\psi_1, ..., \theta_i\lambda_i\psi_i, ..., \theta_r\lambda_r\rangle$.

Note how the alternating behavior of the syzygies is reflected in this example.

Example 3. Let us now calculate the cohomology algebra for $\Lambda = k\Gamma/I$ where $I = \langle a_1a_2, a_2a_3, a_3a_1\rangle$ and Γ is the following quiver:

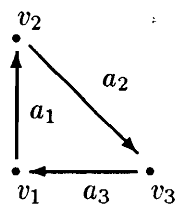

Figure 7.5

By constructing the associated sequence of paths for each relation in I we have the following:

$$P_0 = (\Lambda v_1 \otimes v_1\Lambda)\bigoplus(\Lambda v_2 \otimes v_2\Lambda)\bigoplus(\Lambda v_3 \otimes v_3\Lambda) = P_3 = P_6 = \cdots$$
$$P_1 = (\Lambda v_1 \otimes v_2\Lambda)\bigoplus(\Lambda v_2 \otimes v_3\Lambda)\bigoplus(\Lambda v_3 \otimes v_1\Lambda) = P_4 = P_7 = \cdots$$
$$P_2 = (\Lambda v_1 \otimes v_3\Lambda)\bigoplus(\Lambda v_2 \otimes v_1\Lambda)\bigoplus(\Lambda v_3 \otimes v_2\Lambda) = P_5 = P_8 = \cdots$$

Applying $\mathrm{Hom}_{\Lambda^e}(\ ,\Lambda)$ to the projective resolution of Λ yields the sequence

$$0 \to P_0^* \xrightarrow{\phi_1^*} P_1^* \xrightarrow{0} 0 \xrightarrow{0} P_0^* \xrightarrow{\phi_4^*} P_1^* \xrightarrow{0} 0 \to \cdots$$

where

$$P_0^* = v_1\Lambda v_1 \bigoplus v_2\Lambda v_2 \bigoplus v_3\Lambda v_3$$
$$P_1^* = v_1\Lambda v_2 \bigoplus v_2\Lambda v_3 \bigoplus v_3\Lambda v_1$$
$$P_2^* = v_1\Lambda v_3 \bigoplus v_2\Lambda v_1 \bigoplus v_3\Lambda v_2 = 0$$

and ϕ_i^* is the map induced by ϕ_i.

Since the new complex has a periodicity of six, the cohomology algebra of Λ is determined by $\mathrm{H}^0(\Lambda,\Lambda)$,...,$\mathrm{H}^5(\Lambda,\Lambda)$. Obviously $\mathrm{H}^2(\Lambda,\Lambda) = \mathrm{H}^5(\Lambda,\Lambda) = 0$. Let us first calculate $\mathrm{H}^0(\Lambda,\Lambda) = \ker\phi_1^*$. We know $P_0^* = \{(k_1v_1, k_2v_2, k_3v_3) : k_i \in k\}$ and

$$(v_1,0,0) \xrightarrow{\phi_1^*} (0,0,a_3) - (a_1,0,0)$$
$$(0,v_2,0) \to (a_1,0,0) - (0,a_2,0)$$
$$(0,0,v_3) \to (0,a_2,0) - (0,0,a_3).$$

So if $\alpha = (k_1v_1, k_2v_2, k_3v_3)$, then $\phi_1^*(\alpha) = (k_2 - k_1)(a_1, 0, 0) + (k_3 - k_2)(0, a_2, 0) + (k_1 - k_3)(0, 0, a_3)$. But then $\alpha \in \ker \phi_1^*$ iff $k_1 = k_2 = k_3$. Thus, $\mathrm{H}^0(\Lambda, \Lambda) = \{k_i(v_1, v_2, v_3) : k_i \in k\} \cong k$, as expected. Since $(a_1, a_2, a_3) \in P_1^* \backslash \mathrm{im}\, \phi_1^*$, we conclude that $\mathrm{H}^1(\Lambda, \Lambda) \neq 0$. To calculate $\mathrm{H}^3(\Lambda, \Lambda) = \ker \phi_4^*$ we note that

$$\begin{aligned}(v_1, 0, 0) &\xrightarrow{\phi_4^*} (a_1, 0, 0) + (0, 0, a_3)\\ (0, v_2, 0) &\rightarrow (a_1, 0, 0) + (0, a_2, 0)\\ (0, 0, v_3) &\rightarrow (0, a_2, 0) + (0, 0, a_3).\end{aligned}$$

If $\alpha = (k_1v_1, k_2v_2, k_3v_3) \in \ker \phi_4^*$, we have $\phi_4^*(\alpha) = (k_1 + k_2)(a_1, 0, 0) + (k_2 + k_3)(0, a_2, 0) + (k_1 + k_3)(0, 0, a_3) = 0$. But this is possible iff $k_1 = -k_2 = k_3 = -k_1 = 0$. So $\mathrm{H}^3(\Lambda, \Lambda) = 0$. Finally, let us calculate $\mathrm{H}^4(\Lambda, \Lambda) = P_1^*/\mathrm{im}\, \phi_4^*$. To see that ϕ_4^* is surjective, let $(l_1a_1, l_2a_2, l_3a_3) \in P_1^*$. If we assume that the characteristic of k does not equal 2 and let $k_1 = 1/2(l_1 - l_2 + l_3), k_2 = 1/2(l_2 + l_1 - l_3)$, and $k_3 = 1/2(l_2 - l_1 + l_3)$, then $\phi_4^*(k_1v_1, k_2v_2, k_3v_3) = (l_1a_1, l_2a_2, l_3a_3)$. It follows that $\mathrm{H}^4(\Lambda, \Lambda) = 0$. After re-indexing we obtain the cohomology algebra of Λ:

$$\mathrm{H}^*(\Lambda) = \coprod_1^\infty \mathrm{H}^0(\Lambda, \Lambda) \bigoplus \mathrm{H}^1(\Lambda, \Lambda).$$

We just looked at an oriented cycle with three arrows where I was the ideal generated by the three paths of length two. Now consider and oriented cycle Γ with four arrows and suppose the ideal I is generated by the four paths of length two. Then it is not hard to show that $\mathrm{H}^*(\Lambda)$ is determined by the following complex:

$$0 \rightarrow P_0^* \xrightarrow{\phi_1^*} P_1^* \rightarrow 0 \rightarrow 0 \rightarrow P_0^* \xrightarrow{\phi_1^*} P_1^* \rightarrow 0 \rightarrow \cdots$$

Note that this complex has a periodicity of four. As before, $\mathrm{H}^*(\Lambda)$ is determined by $\mathrm{H}^0(\Lambda, \Lambda)$ and $\mathrm{H}^1(\Lambda, \Lambda)$. However, in this example the periodicity of our complex equals the number of relations whereas in the last example the periodicity equals twice the number of relations.

This apparent discrepancy is a result of the fact that a periodic resolution must have an even period. Even though the projectives can have an odd period, the maps cannot. Otherwise the signs of the maps would not alternate in the manner previously described. So we see how the construction of the associated sequence of paths in the first example has led to a period doubling. This contrasts with the second example where the periodicity of the projectives equals the periodicity of the maps.

Chapter 8

An Application

Now that a method for calculating Hochschild cohomology groups has been presented, it is time to substantiate their need with an important application. In [3] a collection of open problems from the representation theory of algebras is listed. The fifth such problem reads as follows:

> "An artin algebra Λ is called a monomial algebra if Λ is
> isomorphic to a path algebra of a quiver with relations over a field k
> and where all the relations can be chosen to be paths. Give an invariant
> description, one that is independent of generators and relations, of
> when an artin algebra is a monomial algebra."

This problem is also addressed in [6].

Since Hochschild cohomology groups are invariants of a given algebra, it seems reasonable to compare these groups for monomial and non-monomial finite dimensional algebras. However, the resolutions past the second projective for this thesis are limited to monomial algebras. The only Hochschild cohomology groups that can be calculated in general are $\mathrm{H}^0(\Lambda,\Lambda)$ and $\mathrm{H}^1(\Lambda,\Lambda)$. Since $\mathrm{H}^0(\Lambda,\Lambda)$ is almost always k (for a connected quiver), it is obviously of no use for this problem. Fortunately, $\mathrm{H}^1(\Lambda,\Lambda)$ is far more useful. In this chapter we will see that $\dim_k \mathrm{H}^1(\Lambda,\Lambda)$ provides this invariant characterization for a fairly large class of algebras. Now let Λ be an arbitrary finite dimensional quotient of a path algebra. Ed Green proved that there is a covering algebra Λ^G that is contained in the previously mentioned class of algebras. He also showed that the problem can be "lifted" to the covering algebra and solved using a variation of the theorem from this chapter (see [4]).

To see this invariant characterization, let Γ be a finite connected quiver with vertex set Γ_0 and arrow set Γ_1. Let $k\Gamma$ be the corresponding path algebra. We are interested in algebras of the form $\Lambda = k\Gamma/I$, where I is an admissible ideal generated by the reduced Gröbner basis $Minsharp_<(I) = \{r_1, ..., r_m\}$. Here each r_i is assumed to be uniform. Define the reduced Euler characteristic of Γ to be $\chi(\Gamma) = 1 - |\Gamma_0| + |\Gamma_1|$. Before we can define the complex necessary for Hochschild cohomology, let us first recall the projective resolution of Λ through the second projective:

$$\cdots \longrightarrow \coprod_{i=1}^{m} \Lambda o(r_i) \bigotimes t(r_i)\Lambda \xrightarrow{\phi_2} \coprod_{\Gamma_1} \Lambda o(a) \bigotimes t(a)\Lambda \xrightarrow{\phi_1} \coprod_{\Gamma_0} \Lambda v \bigotimes v\Lambda \xrightarrow{\pi} \Lambda \longrightarrow 0.$$

Here $\phi_1(o(a) \bigotimes t(a)) = a \bigotimes t(a) - o(a) \bigotimes a$. If $p = a_{p_1} \cdots a_{p_{l(p)}}$, define $x_p = = \sum_{j=1}^{l(p)} a_{p_1} \cdots a_{p_{j-1}} \bigotimes a_{p_{j+1}} \cdots a_{p_{l(p)}}$. Let $r = \sum_{k=1}^{q} \alpha_k p_k \in Minsharp_<(I)$. Then $\phi_2(o(r) \bigotimes t(r)) = \sum_{k=1}^{q} \alpha_k x_{p_k}$.

Theorem 8.1 *Suppose $dim_k\, v\Lambda v = 1$ for all $v \in \Gamma_0$ and $dim_k\, o(a)\Lambda t(a) = 1$ for all $a \in \Gamma_1$. Then Λ is a monomial algebra if and only if $dim_k\, H^1(\Lambda, \Lambda) = \chi(\Gamma)$.*

Proof: First assume that Λ is a monomial algebra. Furthermore, suppose that I is generated by the paths $p_1, ..., p_m$. Under our hypotheses $\mathrm{H}^1(\Lambda, \Lambda)$ is determined by the following complex:

$$0 \longrightarrow \coprod_{\Gamma_0} kv \xrightarrow{\phi_1^*} \coprod_{\Gamma_1} ka \xrightarrow{\phi_2^*} \coprod_{i=1}^{m} o(p_i)\Lambda t(p_i) \longrightarrow \cdots$$

that is, $\dim_k \mathrm{H}^1(\Lambda, \Lambda) = \dim_k \ker \phi_2^* - \dim_k \mathrm{im}\, \phi_1^*$. Let $a \in \Gamma_1$ and let us determine the image of $(0, ..., a, ..., 0)$ under ϕ_2^* (a is in the a^{th} component). If we let μ_a denote multiplication by a, then $\phi_2^*(0, ..., a, ..., 0) = \{\mu_a \phi_2(o(p_i) \bigotimes t(p_i))\}_{i=1}^m = \{\mu_a x_{p_i}\}_{i=1}^m$. Since $\dim_k o(a)\Lambda t(a) = 1$, $\mu_a x_{p_i} = 0$ if a does not divide p_i. If a does divide p_i, then $\mu_a x_{p_i} = p_i = 0 \mod \Lambda$. Thus, $\phi_2^*(0, ..., a, ...0) = 0$ and it follows that $\dim_k \ker \phi_2^* = |\Gamma_1|$.

Let $|\Gamma_0| = n$ and $|\Gamma_1| = q$. To calculate $\dim_k \mathrm{im} \phi_1^*$, note that $\phi_1^*(0, ..., v, ..., 0) = \{\mu_v \phi_1(o(a_i) \bigotimes t(a_i))\}_{i=1}^q = \{\mu_v(a_i \bigotimes t(a_i) - o(a_i) \bigotimes a_i)\}_{i=1}^q = \{a_i v - v a_i\}_{i=1}^q$ for each $v \in \Gamma_0$. Consider $\{\phi_1^*(0, ..., v_j, ..., 0)\}_{j=1}^n$ and let us focus on the a_i^{th} component for each j. All but two of these have 0 in the a_i^{th} component . If v_k is the origin of a_i then $-a_i$ is the a_i^{th} component of $\phi_1^*(0, ..., v_k, ..., 0)$. Similarly, if v_l is the terminus of a_i, then a_i is the a_i^{th} component of $\phi_1^*(0, ..., v_l, ..., 0)$. It follows that $\sum_{j=1}^n \phi_1^*(0, ..., v_j, ...,0) = 0$ and $\dim_k \mathrm{im}\, \phi_1^* < n = |\Gamma_0|$. Now suppose $\sum_{j=1}^n \alpha_j \phi_1^*(0, ..., v_j, ...,0) = 0$ and $\alpha_1, ..., \alpha_s = 0$, $s < n$. Since Γ is connected there must be some arrow a connecting a vertex v_i, $1 \leq i \leq s$ to some vertex v_j, $s < j \leq n$. To keep the notation simple let us re-index the vertices and the appropriate coefficients so that v_s and v_{s+1} are the two vertices connected by a. Then in the a^{th} component of $\sum_{j=1}^n \alpha_j \phi_1^*(0, ..., v_j, ...,0)$ we have $\pm(\alpha_s - \alpha_{s+1})a$. Since $\alpha_s = 0$ it must be the case that $\alpha_{s+1} = 0$. By induction we conclude that that $\alpha_1 = \cdots = \alpha_n = 0$ and so $\dim_k \mathrm{im} \phi_1^* = |\Gamma_0| - 1$. Note that $\dim_k \mathrm{im} \phi_1^*$ is an invariant of the graph and does not depend on I.

Now suppose Λ has a non-monomial representation and $r = \sum_{j=1}^s \alpha_j p_j$, $s \geq 2$, is a uniform generator in $Minsharp_<(I)$. Without loss of generality let us assume that $\mathrm{tip}(r) = p_1$. Let us first show that there exists some arrow a that divides p_1 such that a does not divide p_j for all $j = 2, ..., s$. Consider the first arrow, call it a_1, in that path p_1. If a_1 divides p_j for all $j = 1, ..., s$, then all the p_j must begin with a_1 since $\dim_k o(r)\Lambda t(r) = 1$.

Similarly, if a_2 is the second arrow in p_1 and a_2 divides p_j, $j = 1, ..., s$, then all the p_j must begin with $a_1 a_2$. Continue is process inductively until either the desired arrow is found or we reach $t(p_1) = t(r)$. Then p_1 divides p_j for $j = 2, ..., s$. We know p_1 cannot properly divide any other p_j since r is uniform and $\dim_k t(r)\Lambda t(r) = 1$. Thus, $p_1 = \cdots = p_s$ and $r = (\sum_{j=1}^{s} \alpha_j)p_1$, contradicting the fact that r is a non-monomial relation.

So let us suppose that the arrow a divides p_1 but does not divide all the p_j. To keep the notation simple we will assume that a divides $p_1, ..., p_k$, $1 \leq k < s$ and a does not divide $p_{k+1}, ..., p_s$. Then $\mu_a \phi_2(o(r) \bigotimes t(r)) = \sum_{j=1}^{k} \alpha_j \mu_a x_{p_j} + \sum_{j=k+1}^{s} \alpha_j \mu_a x_{p_j}$. The second sum equals zero since a does not divide p_j, $j = k+1, ..., s$ and $\dim_k o(a_i)\Lambda t(a_i) = 1$ for all $a_i \in \Gamma_1$. Since a occurs exactly once in each p_j, $j = 1, ..., k$, the first sum equals $\sum_{j=1}^{k} \alpha_j p_j$. Thus, $\phi_2^*(0, ..., a, ..., 0) = \{\sum_{j=1}^{k} \alpha_j p_j\}_r + \{\mu_a \phi_2(o(r_l) \bigotimes t(r_l))\}_{l=1, r_l \neq r}^{m}$. Since the r^{th} component is nonzero, $a \notin \ker \phi_2^*$, $\dim_k \ker \phi_2^* < |\Gamma_1|$, and the result follows.

This is not the first time that $\dim_k \mathrm{H}^1(\Lambda, \Lambda)$ has been given in terms of $\chi(\Gamma)$. The reduced Euler characteristic has been used to describe the dimension of the first Hochschild cohomology group for certain classes of algebras described in [11] and [20]. The reader should note that Theorem 8.1 makes a strong statement about isomorphism properties of certain algebras. If Λ satisfies the hypotheses of the theorem and it has a non-monomial representation, then it cannot have a monomial representation. In other words, Λ cannot be a monomial algebra. This observation is not true in general.

Concluding Remarks

Although some examples of Hochschild cohomology have been presented, many of the computations for this project have been left out. For example, an alternate description of rigid monomial algebras has been found for the special case where all the paths in the generating set for I lie along a common directed path (for a more general description see [12]). This was accomplished by studying the vanishing of $\mathrm{H}^2(\Lambda, \Lambda)$. It is the author's hope to generalize this result in the near future by dropping the requirement that all generating paths lie along a common directed path. In addition, it would be nice to determine $\mathrm{H}^n(\Lambda, \Lambda)$ for more general classes of monomial algebras. Since the necessary complexes have already been constructed, this goal should be realizable. The alternating behavior of the syzygies reduces this problem to a few cases when $n \geq 2$. Of particular interest is case where the ideal of the algebra is generated by paths of length two. These algebras are Koszul. Much of the current work in representation theory is focused on Koszul algebras. Finally, we should not forget about homology. It may be possible to calculate $\mathrm{H}_n(\Lambda, \Lambda) = \mathrm{Tor}_n^{\Lambda^e}(\Lambda, \Lambda)$ for certain monomial algebras. Since homology groups are also invariants, they may be a worthwhile pursuit.

Another area of related interest is non-monomial algebras. Projective resolutions in this setting can be very difficult to calculate. Any results for these algebras will most likely rely heavily on noncommutative Gröbner basis theory. The next step would obviously be binomial algebras. In Chapter 1 the equivalence between the cohomology of a simplicial complex and the Hochschild cohomology of the associated incidence algebra was mentioned. Incidence algebras are binomial. The existing resolution package and the author's enveloping algebra program make it possible to calculate projective resolutions for these incidence algebras. It would be interesting to determine how the algebraic concept of rigidity manifests itself in simplicial cohomology.

Other possible extensions of this work include cyclic homology, the algebraic structure of the cohomology algebra $\mathrm{H}^*(\Lambda, \Lambda) = \coprod_{n=1}^{\infty} \mathrm{H}^n(\Lambda, \Lambda)$, and any possible generalizations of the alternating syzygy behavior. There is a paper on cyclic homology and the resolutions of path algebras (see [5]). It may be useful to see if the resolutions of Chapter 6 provide any new insight to topics from that paper or cyclic homology in general. Likewise, there is much to be learned about the generating sets for cohomology algebras. Determining when $H^*(\Lambda, \Lambda)$ is finitely generated could prove useful. Finally, the alternating behavior exhibited in the resolutions of monomial algebras has appeared in other areas. For example, there is evidence of a possible alternating phenomenon in the cohomology of incidence algebras. However, this observation is based on a different kind of resolution (see [10]). Since incidence algebras are not monomial, a generalization of the associated sequence of paths will be required if there is to be a connection between the cited work and the resolutions of this thesis. Certain periodicity results for the cohomology (not Hochschild) of Brauer tree algebras have also been established in [8]. Similar periodicity phenomenon have appeared in some of the calculations from this project. Even though the combinatorics of these two problems are different, it seems plausible that there is some underlying relationship. Looking for these relationships in this and other areas of mathematics should be the long range goal of this project.

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Vita

Michael Bardzell was born on January 25, 1967 in Washington D.C. He graduated from McLean High School in 1985 and then attended Shenandoah College for one year. After obtaining a B.S. in Physics from Mary Washington College in 1989, Michael married Mary Ellen Miller and resided in Fredericksburg for a year. There he worked part time and took undergraduate mathematics courses. The following year Michael and Maryellen moved to Blacksburg, Virginia. Soon after, they had their first son Benjamin. Michael will receive his Ph.D in Mathematics in May 1996 from Virginia Polytechnic Institute and State University. Starting in August 1996, he will be an Assistant Professor in the Mathematics and Computer Science Department at Salisbury State University in Maryland. In addition to running, Michael enjoys sledding and crabbing with his son. However, he greatly dislikes writing about himself in the third person.

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