A FINITE CHARACTERIZATION OF K-MATRICES IN DIMENSIONS LESS THAN FOUR

by

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(ABSTRACT)

The class of real $n \times n$ matrices M, known as K-matrices, for which the linear complementarity problem $w-Mz=q$, $w \geq 0$, $z \geq 0$, $w^t z=0$ has a solution whenever $w-Mz=q$, $w \geq 0$, $z \geq 0$ has a solution is characterized for dimensions $n<4$. The characterization is finite and "practical". Several necessary conditions, sufficient conditions, and counterexamples pertaining to K-matrices are also given. A finite characterization of completely K-matrices (K-matrices all of whose principal submatrices are also K-matrices) is proved for dimensions < 4.

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Chapter I

INTRODUCTION

Given a matrix $M \varepsilon E^{n\times n}$ and vector $q \varepsilon E^n$, the linear complementarity problem, denoted by (q,M), is to find vectors w,z, εE^n such that

$$w - Mz = q \qquad \text{(LCP)}$$

$$w \geq 0,\ z \geq 0,\ w^t z = 0.$$

The constraint $w^t z = 0$ is called the complementarity condition, since if w_i is positive then z_i is zero and vice versa. This problem arises in such diverse areas as economics, game theory, linear programming, mechanics, lubrication, numerical analysis, and nonlinear optimization. As an example of the wide range of applicability for the linear complementarity problem it will be shown that the quadratic programming problem is a special case of the LCP. The quadratic programming problem is used for the functional relationships occurring in the real world that are truly quadratic as well as countless other nonlinear relationships that can be adequately approximated locally by quadratic functions. The quadratic programming problem is

$$\min c^t x + \tfrac{1}{2}x^t Dx \quad \text{subject to}$$

$$Ax \geq b \text{ and } x \geq 0.$$

Letting u denote the dual variables on the constraints, the Kuhn-Tucker conditions for the quadratic programming problem are

$$y = Dx - A^t u + c$$

$$s = Ax - b$$
$$u^t s + y^t x = 0$$
$$x \geq 0,\ u \geq 0,\ y \geq 0,\ s \geq 0$$

To get the LCP let $w = (y,s)^t$, $z = (x,u)^t$, $q = (c,-b)^t$, and

$M = \begin{bmatrix} D & -A^t \\ A & 0 \end{bmatrix}$. With no assumptions about D, the Kuhn-Tucker conditions are necessary, but not sufficient, implying the possibility of multiple solutions to the LCP, including some that are not optimal. On the other hand, the matrix M is positive semidefinite if the matrix D is positive semidefinite, because $z^t Mz = x^t Dx$ implies $z^t Mz \geq 0$ for $z=(x,u)^t$ since $x^t Dx \geq 0$ for all x. In this case, the Kuhn-Tucker conditions are sufficient as well as necessary.

Generally in a particular application area the matrix M has a special structure (e.g., symmetric positive definite), and moderately efficient algorithms have been developed to solve these special linear complementarity problems. Nevertheless considerable attention has been devoted to the general linear complementarity problem, and in particular to special classes of matrices related to the problem.

The j^{th} column of a matrix M is referred to as M_j. A complementary set of column vectors is a set of column vectors $\{A_j \mid j = 1,2,\ldots,n\}$ where A_j is either the j^{th} column of -M or the j^{th} column of the identity matrix, I, for each $j = 1,2,\ldots,n$. The linear complementarity problem has a solution if and only if q can be represented as a nonnegative combination of some complementary set of column vectors.

If $A = \{A_j \mid j = 1,2,\ldots,n\}$ is some set of complementary column vectors, then $C(A)$ is called the complementary cone of A and $C(A) = \{\sum_{i}^{n} \alpha_i A_i \mid \alpha_i \geq 0, i = 1,2,\ldots,n\}$. So (q,M) has a solution if and only if $q \in C(A)$ where A is some set of complementary column vectors.

1.1 EXAMPLES IN TWO DIMENSIONS

E^2 is 2-dimensional Euclidean space. In E^2 there are only the following four sets of complementary column vectors: $\{I_1, I_2\}$, $\{I_1, -M_2\}$, $\{-M_1, I_2\}$, and $\{-M_1, -M_2\}$. The complementary cone generated by $\{I_1, I_2\}$ is just the positive quadrant as shown in figure 1.

If $M = \begin{bmatrix} -1 & 2 \\ 1 & 3 \end{bmatrix}$, then the complementary cones are as shown in figure 2. In this example (q_1,M) has two distinct solutions, since q_1 lies in two separate complementary cones, namely those generated by $\{I_1, I_2\}$ and $\{-M_1, I_2\}$. (q_2,M) also has two solutions, but (q_3,M) has no solution since q_3 does not lie in any complementary cone.

If $M = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$, then the complementary cones generated by the complementary sets of column vectors cover all of E^2 as shown in figure 3. Since the cones partition E^2, (q,M) has a unique solution for every $q \in E^2$.

As another example let $M = \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}$. The complementary cones for this example are given in figure 4. In this case (q,M) again has a solution for every $q \in E^2$. Let $q = (2,2)^t$. Then there are the following three distinct solutions for (q,M).

$$q = 2(I_1) + 2(I_2)$$
$$q = 2(-M_1) + 6(I_2)$$
$$q = 6(I_1) + 2(-M_2)$$

$q = 2(I_1) + 2(I_2)$ corresponds to $w = (2,2)^t$ and $z = (0,0)^t$. $q = 2(-M_1) + 6(I_2)$ corresponds to $w = (0,6)^t$ and $z = (2,0)^t$. $q = 6(I_1) + 2(-M_2)$ corresponds to $w = (6,0)^t$ and $z = (0,2)^t$.

Let $M = \begin{bmatrix} 1 & -4 \\ -3 & 2 \end{bmatrix}$. In this example, as shown in figure 5, (q_1,M) has a solution while there is no solution for (q_2,M).

In some cases a cone may have no volume. As an example of this let $M = \begin{bmatrix} 0 & 3 \\ -1 & 2 \end{bmatrix}$. The cones for this case are shown in figure 6. The complementary cone generated by $\{-M_1, I_2\}$ has an empty interior. If $q \in C(-M_1, I_2)$ then $q \in C(-M_1)$ and $q \in C(I_2)$.

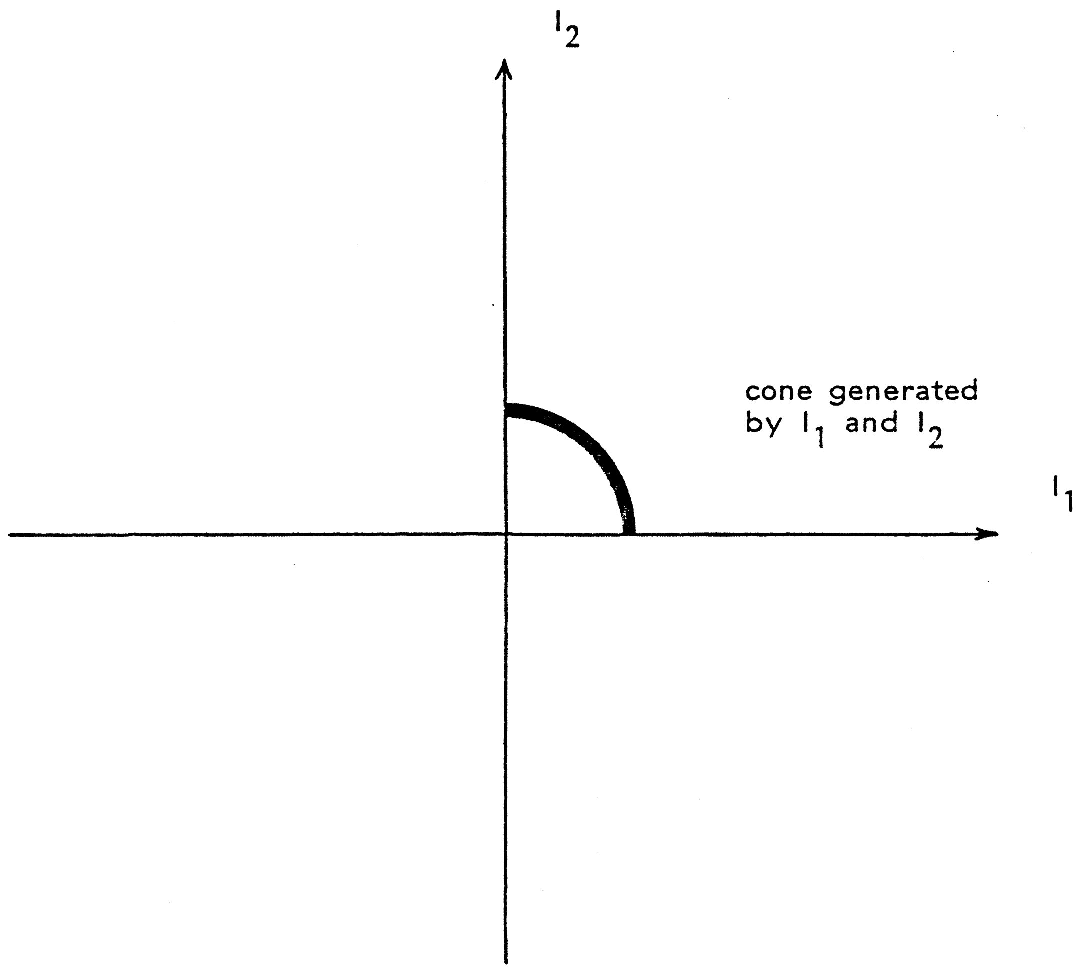

Figure 1

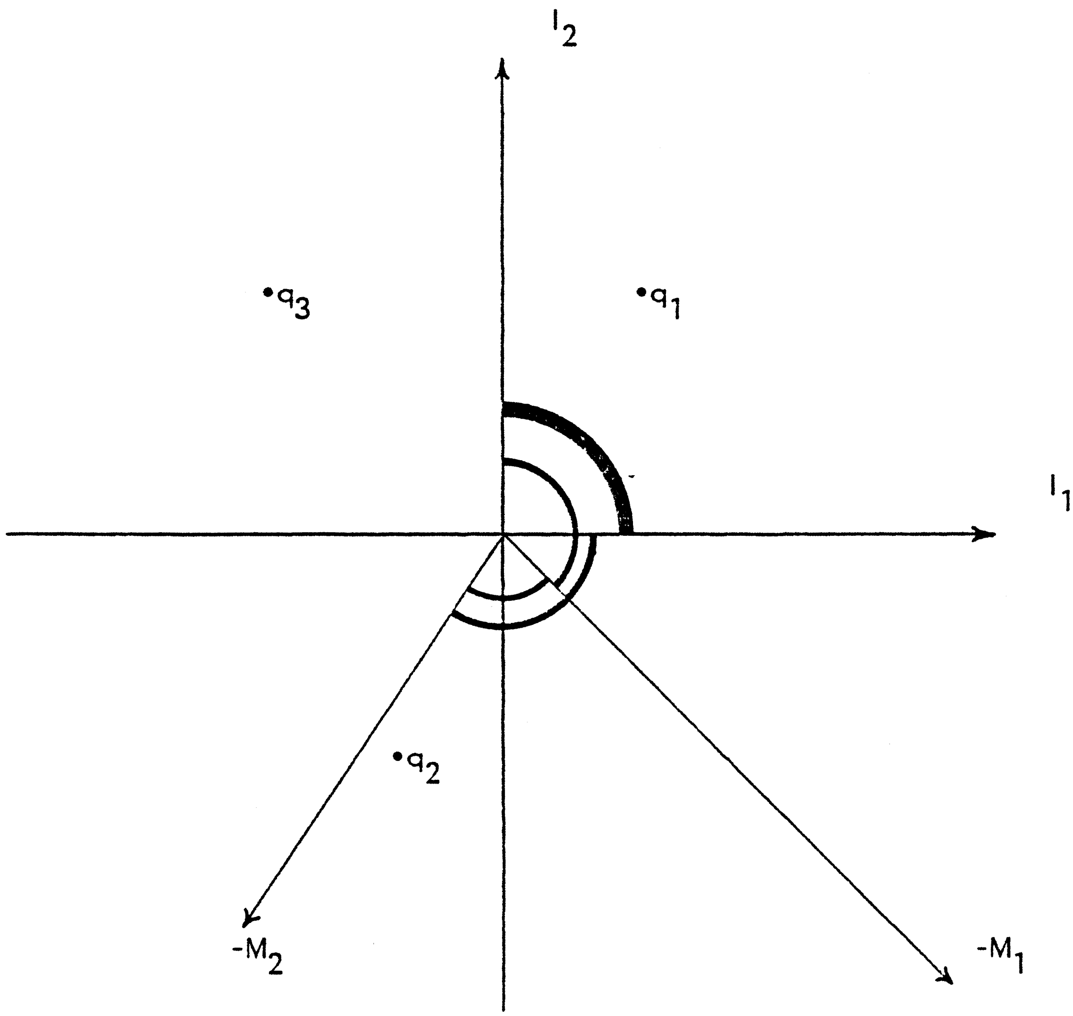

Figure 2

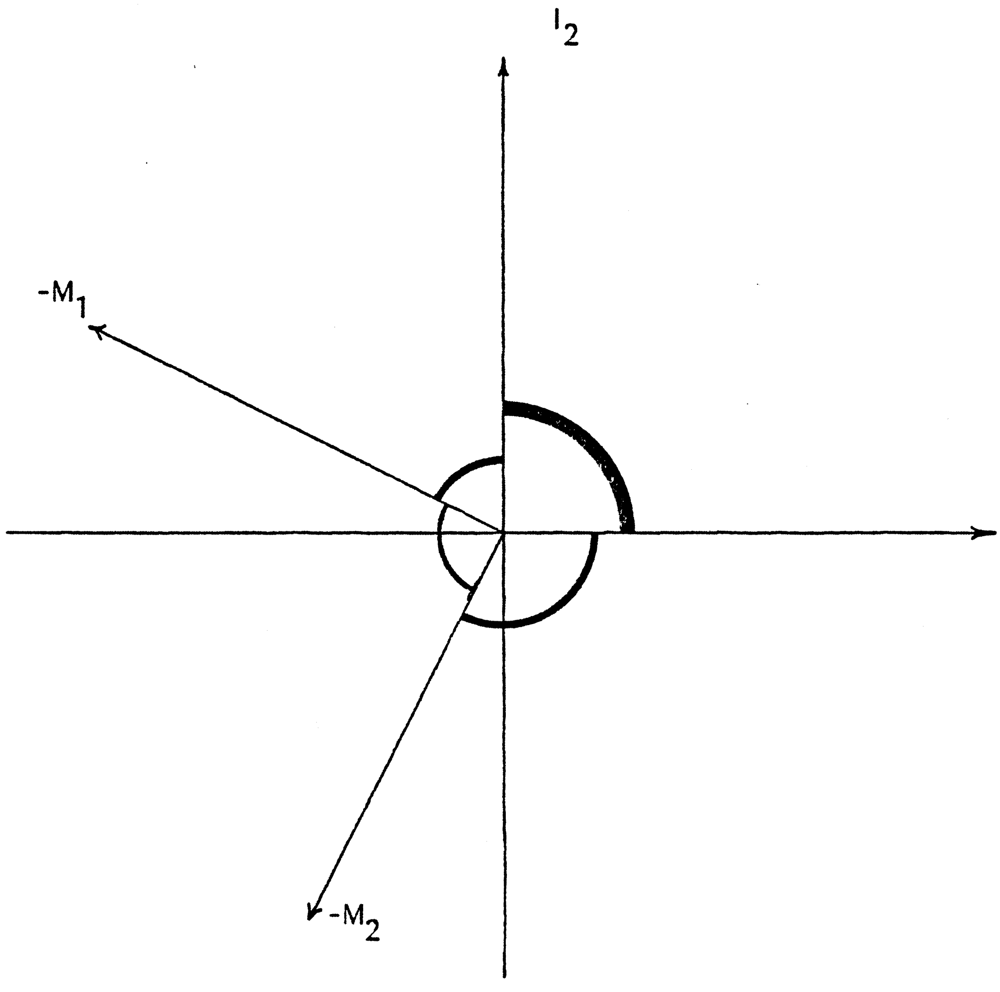

Figure 3

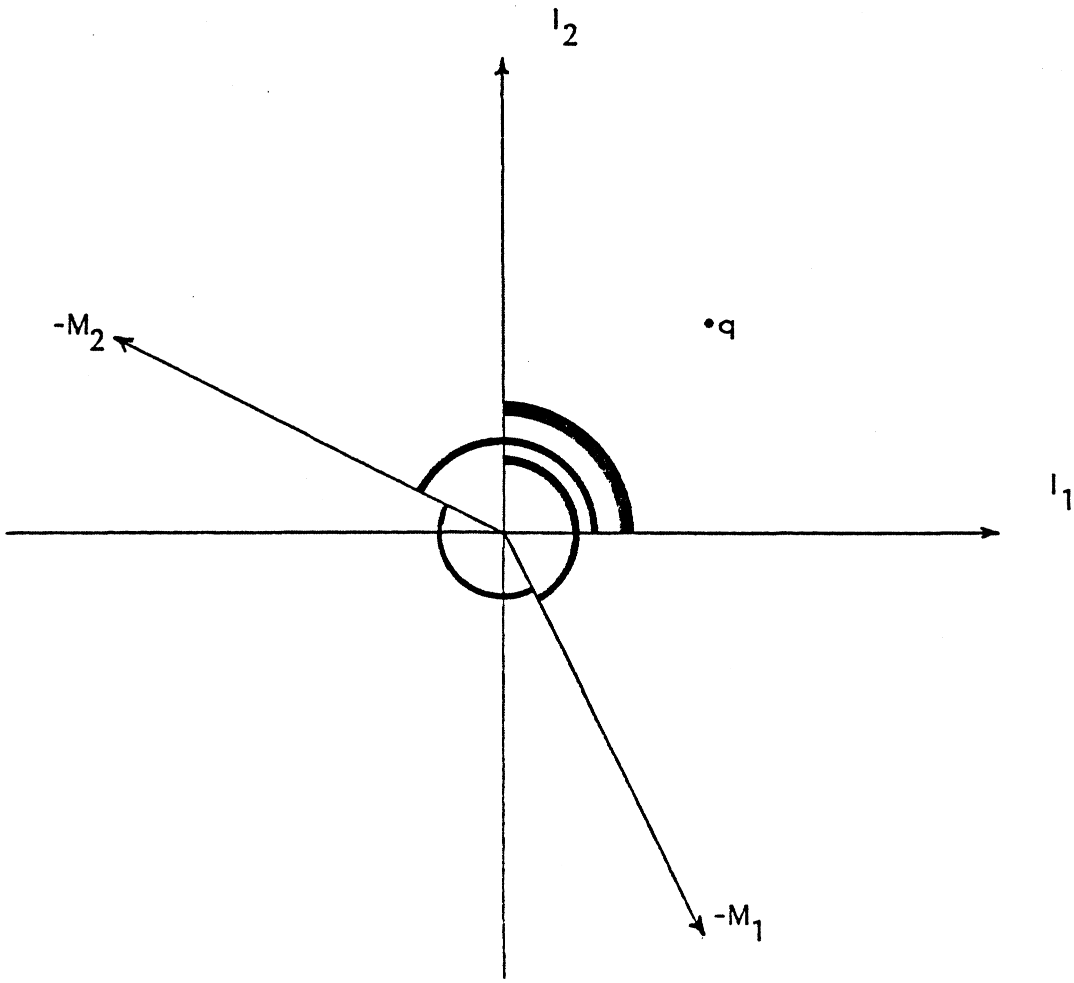

Figure 4

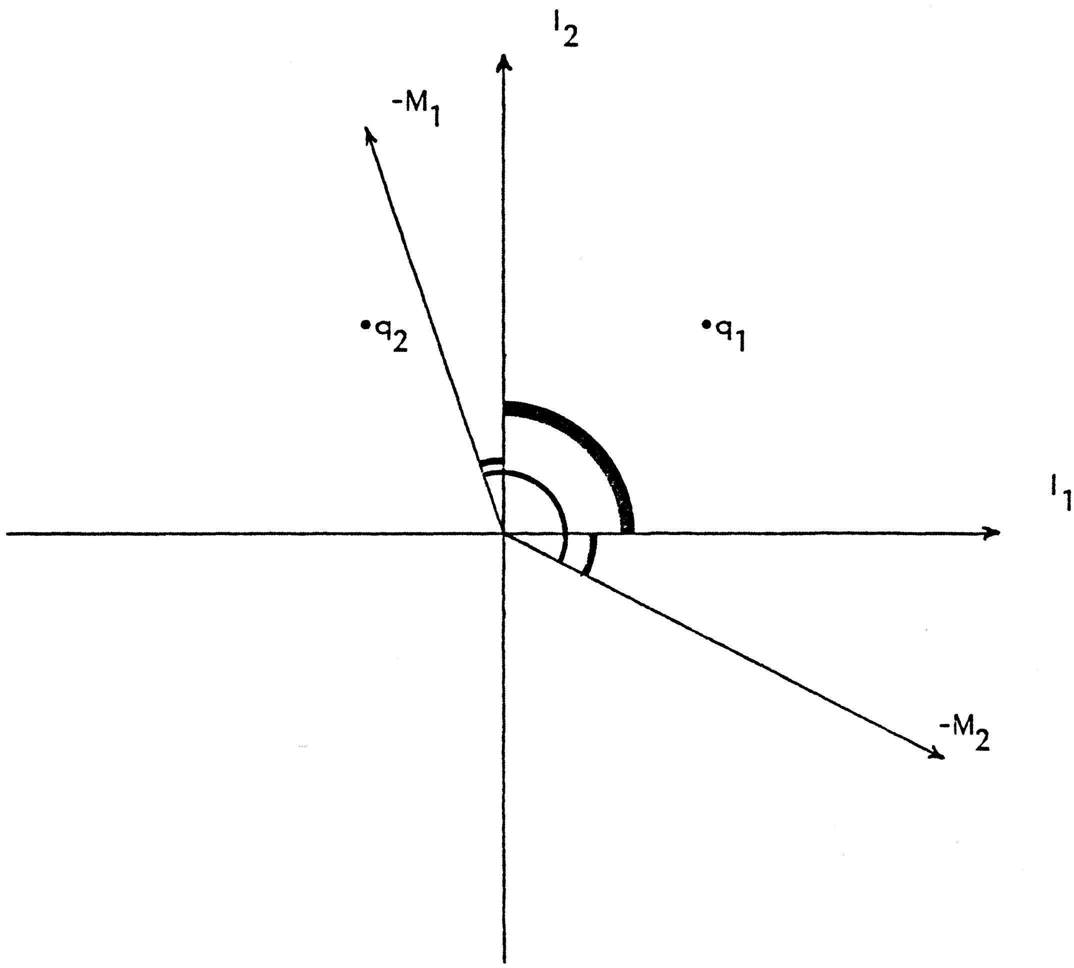

Figure 5

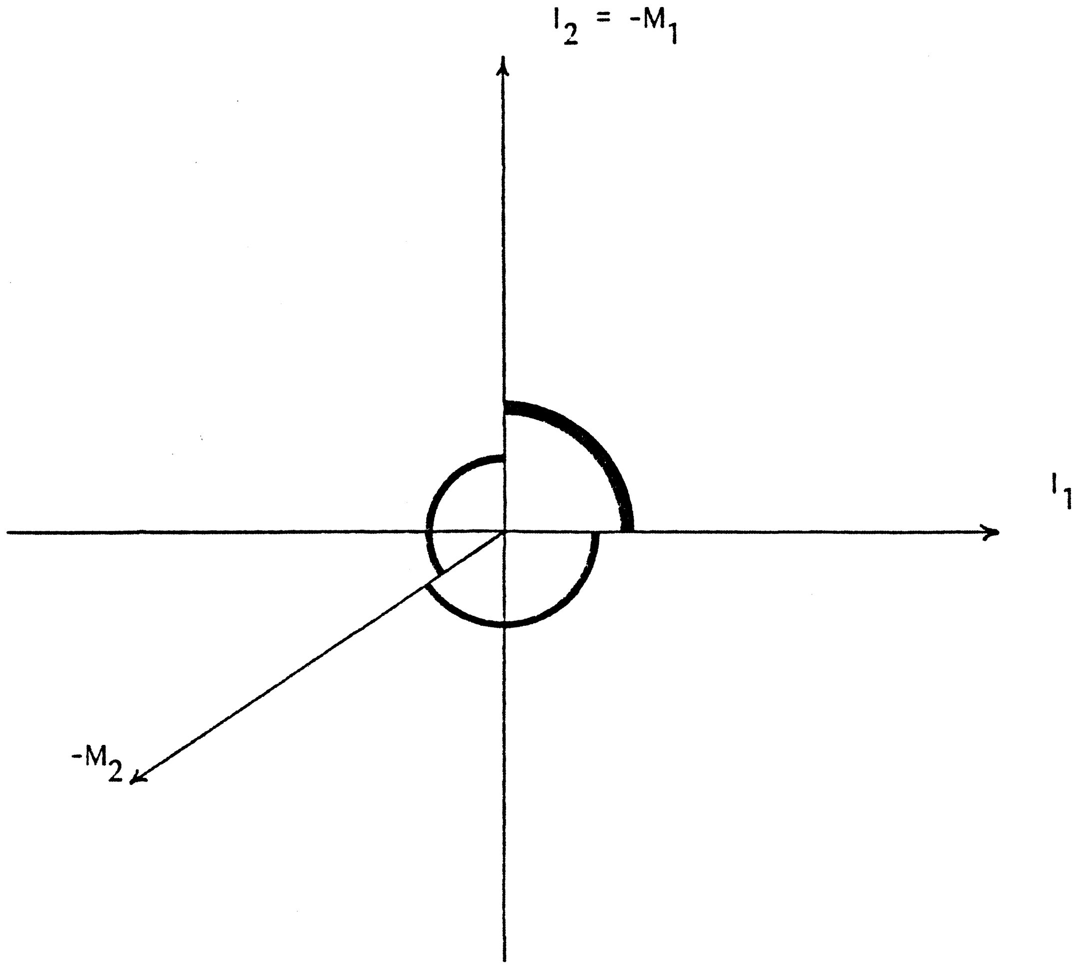

Figure 6

1.2 CLASSES OF MATRICES

The matrices $M \in E^{n \times n}$ for which (q,M) has a unique solution for every vector $q \in E^n$ are known as P-matrices, (the matrix used in figure 3 is an example of a P-matrix) and are characterized algebraically by having all their principal minors positive. The equivalence of (q,M) having a unique solution for all $q \in E^n$ and M being a P-matrix (all principal minors positive) has been proved in many different ways, most notably by Ingleton[11], Samelson, Thrall, and Wesler[19], Murty[18], Watson[21], and Kelly and Watson[12]. M is called a Q-matrix if (q,M) has at least one solution for all $q \in E^n$. For over ten years there have been serious attempts to characterize Q-matrices in a manner similar to P-matrices. (The matrices used in figures 3, 4, and 6 are all examples of Q-matrices. Also, note that the class of P-matrices is a subset of the class of Q-matrices.) Numerous necessary conditions or sufficient conditions have been found (see, e.g., Watson[21], Doverspike[4], Doverspike and Lemke[5]), but simple algorithmic necessary *and* sufficient conditions have remained elusive. Progress is being made, though, since Cottle[3] recently characterized completely Q-matrices (matrices whose principal submatrices are all Q-matrices).

M is called a K-matrix (following Lemke[14]) if the existence of a solution to $w - Mz = q$, $w \geq 0$, $z \geq 0$ implies the existence of a solution to the linear complementarity problem (q,M). (The class of

K-matrices is also referred to as the class of Q_0-matrices in some of the literature.) The classes of P-matrices and Q-matrices are both subsets of the class of K-matrices. The matrices used in figures 3, 4, 5, and 6 are all examples of K-matrices. The matrix used in figure 2 is not a K-matrix since q_3 is covered by the cone generated by I_2 and $-M_2$, but is not covered by any of the complementary cones of M.

The class of K-matrices is a larger class of matrices than the Q-matrices, but for various reasons, it may be easier to characterize than the class of Q-matrices. Using P-matrices as an ideal model, there are two types of desirable characterizations. One type would say that M is a K-matrix if and only if some linear algebraic statement about M holds, where the statement is in terms of eigenvalues, minors or some other algebraic quantity. The second type of characterization would say that M is a K-matrix if and only if (q,M) has a solution for all q in some finite "test set", which is permitted to depend on M and q. The exhaustive numerical examples in Watson[21] and Kelly and Watson[12] indicate it is unlikely that a characterization of either type exists for Q-matrices.

Attempts at an algorithmic characterization of K-matrices have been made by Garcia[9], Doverspike[4], Doverspike and Lemke[5], among others. In the paper by Garcia[9] several different classes of matrices are shown to be subsets of the class of K-matrices.

Some of the results presented by Garcia are taken from the work done by Eaves[6] and Lemke[14]. In all these cases, however, the classes of matrices discussed are all proper subsets of the class of K-matrices.

Doverspike[4] discusses some more subclasses of K-matrices and various properties of these subclasses. He also gives some topological results for K-matrices.

The paper by Doverspike and Lemke[5] analyzes the faces of the linear cones, and presents a labelling requirement for the faces that is necessary for a matrix to be in the class of K-matrices. Their results deal with the minors of the matrix and some additional conjectures in this area are given in their paper.

Chapter II

NOTATION

Let E^n be n-dimensional Euclidean space, S^{n-1} the unit sphere in E^n, and $E^{n\times n}$ the set of all real n×n matrices. For $I,J \subset \{1,2,\ldots,n\}$ and $M \in E^{n\times n}$, M_{ij} is the (i,j) entry in M, $M_{I.}$ is the submatrix of M consisting of the rows indexed by I, and $M_{.J}$ consists of the columns indexed by J. The jth column of M is $M_{.j}$ or simply M_j.

The cone generated by vectors $V_1,V_2,\ldots,V_k$ is $C(V) = C(V_1,V_2,\ldots,V_k) = \{\sum_1^k \alpha_i V_i \mid \alpha_i \geq 0, i = 1,2,\ldots,k\}$.

For given $M \in E^{n\times n}$, and $q \in E^n$, the linear complementarity problem is to find vectors $w \in E^n$ and $z \in E^n$ such that

$$w - Mz = q,$$

$$w \geq 0, z \geq 0, \text{ and } w^t z = 0.$$

The complementarity problem is denoted (q,M).

C(A), where $A_i \in \{I_i, -M_i\}, i = 1,2,\ldots,n$, and I represents the identity matrix, is called a *complementary cone formed from M.* (q,M) has a solution if and only if q is in some complementary cone.

C(A), where $\{A_1,\ldots,A_n\} \subset \{I_j, -M_j \mid j = 1,2,\ldots,n\}$, is called a *noncomplementary cone formed from M.* Note: The set of complementary cones formed from M is contained in the set of noncomplementary cones formed from M.

A matrix M such that (q,M) has a solution for all $q \varepsilon E^n$ is called a *Q-matrix.* A matrix M such that (q,M) has a solution for every $q \varepsilon C(A)$, where C(A) is any noncomplementary cone, is called a *K-matrix.* A matrix M is called $\bar{K}$ (completely K) iff it and all its principal submatrices are K.

For $q \varepsilon C(V) = C(V_1, V_2, \ldots, V_n)$, q is in the *interior* of C(V) if $q = \sum_{1}^{n-1} \alpha_i \beta_i$ has no solution with $\alpha_i \geq 0$ and $\beta_i \varepsilon \{V_1, V_2, \ldots, V_n\}$. If $q \varepsilon C(V)$ we say q is *covered* by C(V).

Chapter III

TWO-DIMENSIONAL THEOREMS

All 1×1 matrices are K-matrices. Therefore, a 2×2 K-matrix is also completely K. Hence the characterization of 2×2 K-matrices given in this chapter is also a characterization of completely K-matrices in two dimensions. The other results presented in this chapter are also true for both K-matrices and completely K-matrices.

<u>Theorem 1</u>. For $M \in E^{2\times 2}$, M is $\bar{K}$ iff $-M_1 + I_1$ and $-M_2 + I_2$ are covered by the complementary cones of M.

Proof: If M is $\bar{K}$ then by definition both $-M_1 + I_1$ and $-M_2 + I_2$ are covered by the complementary cones of M.

Assume $-M_1 + I_1$ and $-M_2 + I_2$ are covered by complementary cones. M is not $\bar{K}$ if and only if there is some portion of the cone generated by $-M_i$ and I_i, for i = 1 or 2, which is not covered by the complementary cones.

Case 1) Assume that the cone $C(-M_i, I_i)$ does not contain any I_j or $-M_j$, $j \neq i$. $-M_i + I_i$ is covered and since there are no generators in the interior of the cone $C(-M_i, I_i)$, the cone that covers $-M_i + I_i$ must cover the entire cone $C(-M_i, I_i)$.

Case 2) Assume that the cone $C(-M_i, I_i)$ does contain I_j or $-M_j$, $j \neq i$, and let A_j be this generator. Then the two cones $C(A_j, I_i)$ and $C(A_j, -M_i)$ will cover all of the cone $C(-M_i, I_i)$.

Q.E.D.

Counterexample to Theorem 1 for $M \varepsilon E^{3\times 3}$:

Let $M = \begin{bmatrix} 1 & -2 & 1 \\ 2 & -2 & 1 \\ 2 & -1 & -1 \end{bmatrix}$ and $q = \begin{bmatrix} 2 \\ 2 \\ -2 \end{bmatrix}$.

Then $3I_1 + 4I_2 - M_1 = q$,

$I_1 - M_1 = -2M_1 - M_2 = I_3$,

$I_2 - M_2 = 2I_1 + 3I_2 + I_3$,

and $I_3 - M_3 = I_1 + I_2 - 2M_3$, but q is not covered by any of the complementary cones of M.

<u>Theorem 2</u>. Let $M \varepsilon E^{2\times 2}$. If $\det(M) > 0$, then M is $\bar{K}$.

Proof: $\mathrm{Det}(M) > 0$ implies the arc from $-M_1$ to $-M_2$ is counterclockwise.

Case 1) $C(I_1,I_2) \cap C(-M_1,-M_2) \neq \{0\}$.

If the cone $C(I_1,I_2)$ contains $-M_i$, for i = 1 or 2, then $-M_i + I_i$ is covered by the cone $C(I_1,I_2)$. If the cone $C(I_1,I_2)$ does not contain $-M_i$, for i = 1 or 2, then $-M_i + I_i$ is covered by the cone $C(-M_1,-M_2)$. Therefore, by Theorem 1, M is $\bar{K}$.

Case 2) $C(I_1,I_2) \cap C(-M_1,-M_2) = \{0\}$.

The arc from I_1 to I_2 is also counterclockwise. Starting at I_1 and going counterclockwise the generators are encountered in the following order: $I_1, I_2, -M_1, -M_2$.

Since the generators are found in an alternating order (alternating order refers to the subscripts of the generators), if the cone $C(I_i,-M_i)$ has a nonempty interior it must contain $A_j = I_j$ or $-M_j$, $j \neq i$. In this case the cone $C(I_i,-M_i)$ is covered by the two cones $C(A_j,I_i)$ and $C(A_j,-M_i)$.

If the cone $C(I_i,-M_i)$ has an empty interior then $C(I_i,-M_i)$ is a straight line and is obviously covered by the complementary cones of M.

Q.E.D.

Counterexample to Theorem 2 for $M \varepsilon E^{3\times 3}$:

Let $M = \begin{bmatrix} 1 & -1 & 4 \\ 4 & -3 & 1 \\ 1 & .2 & -.05 \end{bmatrix}$.

Then $\det(M) = 13.95 > 0$, but $I_2 - M_2 = (1,4,-.2)^t$ is not covered by any of the complementary cones of M. This matrix is from Kelly and Watson[10].

<u>Theorem 3</u>. Let $M \varepsilon E^{2\times 2}$. If $M_{1.} < 0$ or $M_{2.} < 0$ then M is $\bar{K}$.

Proof:

Case 1) $M_{1.} < 0$. If $M_{1.} < 0$ then both $-M_1$ and $-M_2$ lie in the first or fourth quadrant of $E^{2\times 2}$. In this case $I_2 - M_2$ lies in the first quadrant and is covered by the cone $C(I_1,I_2)$.

Case 1a) $-M_1$ lies on I_1 or $-M_1$ is in the first quadrant. In this case $I_1 - M_1$ is covered by the cone $C(I_1, I_2)$. Then, by Theorem 1, M is $\bar{K}$.

Case 1b) $-M_1$ lies in the fourth quadrant. In this case I_1 is covered by the cone $C(I_2, -M_1)$, therefore $I_1 - M_1$ is covered by the same cone. Then, by Theorem 1, M is $\bar{K}$.

The case for $M_{2.} < 0$ can similarly be proved.

Q.E.D.

Counterexample to Theorem 3 for $M \varepsilon E^{3\times 3}$:

Let $M = \begin{bmatrix} -1 & -2 & -1 \\ -1 & -3 & 2 \\ 3 & 5 & -1 \end{bmatrix}$.

Then $M_{1.} < 0$ but $I_3 - M_3 = (1,-2,2)^t$ is not covered by any of the complementary cones of M.

<u>Theorem 4</u>. Let $M \varepsilon E^{2\times 2}$. If $C(I_1, -M_1)$ covers I_2 or $C(I_2, -M_2)$ covers I_1 then M is $\bar{K}$.

Proof: Assume $C(I_i,-M_i)$, $i = 1$ or 2, covers I_j, $j \neq i$. If $C(I_i,-M_i)$ covers I_j, $i \neq j$, then $I_i - M_i$ is covered by $C(I_i,I_j) \cup C(-M_i,I_j)$.

Case 1) $-M_j$ is covered by $C(I_i,-M_i)$.

Since I_j is also covered by $C(I_i,-M_i)$, $I_j - M_j$ is covered by $C(I_i,I_j) \cup C(-M_i,I_j)$. Therefore, by Theorem 1, M is $\bar{K}$.

Case 2) $-M_j$ is not covered by $C(I_i,-M_i)$.

Since I_j is covered by $C(I_i,-M_i)$, we know that if $C(I_j,-M_j)$ has a nonempty interior, then $C(I_j,-M_j)$ contains A, where $A = I_i$ or $A = -M_i$. So $I_j - M_j$ is covered by $C(A,I_j) \cup C(A,-M_j)$. Therefore, by Theorem 1, M is $\bar{K}$.

If $C(I_j,-M_j)$ has an empty interior then $I_j - M_j$ lies in either $C(I_j)$ or $C(-M_j)$ and is obviously covered by the complementary cones of M. Therefore, by Theorem 1, M is $\bar{K}$.

Q.E.D.

<u>Theorem 5</u>. Let $M \varepsilon E^{2\times 2}$. If $M_{11}M_{22} > 0$ then M is $\bar{K}$.

Proof:

Case 1) $M_{ii} > 0$ for $i = 1,2$.

$M_{ii} > 0$ for $i = 1,2$ implies that $-M_1$ lies in $C(-I_1)$ or the second or third quadrants and that $-M_2$ lies in $C(-I_2)$ or the third or fourth quadrants. For either $-M_1 \varepsilon C(-I_1)$ or $-M_2 \varepsilon C(-I_2)$, $\det(M) > 0$ so, by Theorem 2, M is $\bar{K}$. If $-M_1$ lies in the second quadrant, then $C(I_1,-M_1)$ covers I_2; if $-M_2$ lies in the fourth quadrant, then $C(I_2,-M_2)$ covers I_1. Therefore in either case, by Theorem 4, M is $\bar{K}$. If both $-M_1$ and $-M_2$ lie in the third quadrant, M is strictly positive, hence a Q-matrix [16], hence $\bar{K}$.

Case 2) $M_{ii} < 0$ for $i = 1,2$.

$M_{ii} < 0$ for $i = 1,2$ implies that $-M_1$ lies in $C(I_1)$ or the first or fourth quadrants and that $-M_2$ lies in $C(I_2)$ or the first or second quadrants. For either $-M_1 \varepsilon C(I_1)$ or $-M_2 \varepsilon C(I_2)$, $\det(M) > 0$ so, by Theorem 2, M is $\bar{K}$. If $-M_1$

or $-M_2$ lies in the first quadrant, then $M_{2.} < 0$ or $M_{1.} < 0$, respectively. Therefore, by Theorem 3, M is $\bar{K}$. For the remaining case, $-M_1$ lies in the fourth quadrant and $C(-M_1, I_2)$ covers I_1, so $C(-M_1, I_2)$ also covers $I_1 - M_1$. Also, since $-M_2$ lies in the second quadrant $C(I_1, -M_2)$ covers I_2 so $I_2 - M_2$ is covered by $C(I_1, -M_2)$. Therefore, by Theorem 1, M is $\bar{K}$.

Q.E.D.

Counterexamples to Theorem 5 for $M \varepsilon E^{3\times 3}$:

Let $M = \begin{bmatrix} 1 & -3 & 3 \\ -3 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix}$ and $q = \begin{bmatrix} -3 \\ -2 \\ -3 \end{bmatrix}$.

Then $M_{ii} > 0$ for i =1,2,3 but $I_3 - M_3 = q$ is not covered by any of the complementary cones of M.

Let $M = \begin{bmatrix} -1 & 1 & 2 \\ -1 & -3 & -2 \\ 3 & -2 & -4 \end{bmatrix}$ and $q = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}$.

Then $M_{ii} < 0$ for i = 1,2,3 but $I_1 - M_1 = q$ is not covered by any of the complementary cones of M.

<u>Theorem 6</u>. Let $M \in E^{2\times2}$. For M to be $\bar{K}$ it is necessary that $M_{11}M_{22} > 0$ or $M_{21} \le 0$ or $M_{12} \le 0$.

Proof: Assume $M_{11}M_{22} \le 0$ and $M_{21} > 0$ and $M_{12} > 0$. $M_{11}M_{22} \le 0$ implies that one of the following cases holds: $M_{11} = 0$, $M_{22} = 0$, $M_{11} < 0$ and $M_{22} > 0$, or $M_{11} > 0$ and $M_{22} < 0$.

Case 1) $M_{11} = 0$.

In this case $-M_1 = -\alpha I_2$, $\alpha > 0$, and $-M_2$ lies in either the second or third quadrant. If $-M_2$ lies in the second quadrant or on $C(-I_1)$, then $I_1 - M_1$ is not covered by any of the complementary cones of M; if $-M_2$ lies in the third quadrant, then $I_2 - M_2$ is not covered by any complementary cones of M. In either case, by Theorem 1, M is not $\bar{K}$.

Case 2) $M_{22} = 0$.

By symmetry, the proof of this case follows from Case 1.

Case 3) $M_{11} < 0$ and $M_{22} > 0$.

In this case $-M_1$ lies in the fourth quadrant and $-M_2$ lies in the third quadrant. Then $I_2 - M_2$ is not covered by any of the complementary cones of M and hence, by Theorem 1, M is not $\bar{K}$.

Case 4) $M_{11} > 0$ and $M_{22} < 0$.

By symmetry, this case follows from Case 3.

Therefore, in each case, M is not $\bar{K}$.

Q.E.D.

Theorem 7 Let $M \varepsilon E^{2\times 2}$. For M to be $\bar{K}$ it is necessary that $\det(M) \geq 0$ or for all i,j : $i \neq j$ $M_i \leq 0$ and $M_j > 0$ cannot hold simultaneously.

Proof: Assume $\det(M) < 0$ and $M_i \leq 0$ and $M_j > 0$ for $i,j \varepsilon \{1,2\}$ and $i \neq j$.

Case 1) $M_1 \leq 0$ and $M_2 > 0$.

$-M_2$ lies in the third quadrant and $-M_1$ lies in the first quadrant or on $C(I_1)$ or $C(I_2)$. Since $\det(M) < 0$, $I_2 - M_2$ is not covered by any of the complementary cones of M. Therefore, by Theorem 1, M is not $\bar{K}$.

Case 2) $M_1 > 0$ and $M_2 \leq 0$.

$-M_1$ lies in the third quadrant and $-M_2$ lies in the first quadrant or on $C(I_1)$ or $C(I_2)$. Since $\det(M) < 0$, $I_1 - M_1$ is not covered by any of the complementary cones of M. Therefore, by Theorem 1, M is not $\bar{K}$.

Q.E.D.

Theorem 8. Let $M \varepsilon E^{2\times 2}$ have all nonzero minors. Then M is $\bar{K}$ if and only if $M_{1.} < 0$ or $M_{2.} < 0$ or $M_{11}M_{22} > 0$ or $\det(M) > 0$.

Proof: By Theorems 2,3, and 5 the conditions

$$M_{1.} < 0$$
$$\text{or } M_{2.} < 0$$
$$\text{or } M_{11}M_{22} > 0$$
$$\text{or } \det(M) > 0$$

imply that M is $\bar{K}$.

The negation of the conditions of the theorem:

$$M_{11}M_{22} < 0$$
$$\text{and } \det(M) < 0$$
$$\text{and } M_{1.} \nless 0$$
$$\text{and } M_{2.} \nless 0$$

imply

$(M_{11}M_{22} < 0$ and $M_{21} > 0$ and $M_{12} > 0)$ or

$(\det(M) < 0$ and $M_i < 0$ and $M_j > 0,\ i \neq j)$,

which imply M is not $\bar{K}$ by Theorems 6 and 7.

Q.E.D.

It had been conjectured that a 3×3 matrix is $\bar{K}$ iff all of its 2×2 principal submatrices are $\bar{K}$. As a counterexample to this,

let $M = \begin{bmatrix} -3 & -2 & 3 \\ -1 & -4 & -2 \\ 2 & 1 & -1 \end{bmatrix}$.

Then all nine 2×2 submatrices of M are K-matrices but $I_3 - M_3$ is not covered by any of the complementary cones of M.

A matrix can be a K-matrix and not be $\bar{K}$.

Let $M = \begin{bmatrix} 4 & -4 & 0 \\ -1 & -1 & 3 \\ 2 & 0 & -1 \end{bmatrix}$ (Watson[16]).

Then M is a Q-matrix so it is a K-matrix. The submatrix

$$\begin{bmatrix} 4 & 0 \\ 2 & -1 \end{bmatrix}$$

is not a K-matrix so M is not $\bar{K}$.

Chapter IV

THREE-DIMENSIONAL THEOREMS

4.1 DEFINITIONS

The preceding chapter gave a finite characterization and several theorems for two dimensions which do not generalize to three dimensions. In this section a finite characterization for 3-dimensional K-matrices is given. For dimension ≤ 4 the spherical geometry approach of Kelly and Watson[10] is both convenient and rigorous, and will be employed here. For completeness, a few definitions from [10] are included here.

Definition: If in S^{n-1} (or E^n) U is a nonempty set and P a point, then Vis(P,U) is the union of the half-open segments [P,X) in S^{n-1} (or E^n) which lie entirely in the complement of U.

Definition: If in S^{n-1} (or E^n) U is a nonempty set and P a point, then the star St(P,U) is the union of closed segments [P,X], $X \in U$.

Segments in these two definitions and in the rest of the paper refer, of course, to spherical segments in S^{n-1}, that is, great circle arcs of length less than or equal to π, and Euclidean segments in E^n.

Let $M = \{M_i\}$, $N = \{N_i\}$ $i=1,2,\ldots,k$, be two k-tuples of linearly independent points on the unit sphere S^{n-1} in E^n. A spherical (k-1)-simplex with vertices $P_i \in \{M_i, N_i\}$ $i=1,2,\ldots,k$ is called a complementary simplex relative to M and N. The union of such simplices is denoted $C^k(M,N)$. $C_j^k(M,N)$ means $C^{k-1}(M',N')$ where $M' = M-\{M_j\}$, $N' = N-\{N_j\}$. The union of spherical simplices with verticies $P_i \in M \cup N$, $i=1,2,\ldots,k$, is denoted by $S^k(M,N)$. Note that $C^k(M,N) \subset S^k(M,N)$ always. Also note that no independence assumption is made for the sets M and N, hence some of the simplices may be degenerate.

Analagous to the definition of a Q-arrangement in [10] we have:

Definition: $C^n(M,N)$ is a *K-arrangement* on S^{n-1} if $C^n(M,N) = S^n(M,N)$.

4.2 CLASSIFICATION OF K-MATRICES IN THREE DIMENSIONS

Fix $i \in \{1,2,3\}$, let $A = M \cup N-\{M_i,N_i\}$, and denote the elements of A by A_r, $r = 1,2,3,4$. If $\overline{A_jA_k} \cap \overline{A_\ell A_m} = S \neq \phi$ for distinct j,k,ℓ,m, then let $R \in S$ and let P^i contain the elements $A_j + A_m + R$, $A_j + A_\ell + R$, $A_k + A_m + R$, $A_k + A_\ell + R$, and $M_i + N_i$. Else if $\overline{A_jA_k} \cap \overline{A_\ell A_m} = \phi$ for all distinct j,k,ℓ,m, then let $P^i = \{A_j + A_k + A_\ell,\ A_j + A_k + A_m,\ A_j + A_\ell + A_m,\ A_k + A_\ell + A_m,\ M_i + N_i\}$.

Theorem 9. $C^3(M,N)$ is a K-arrangement iff $P^1 \cup P^2 \cup P^3 \subset St(M_1,C_1^3(M,N)) \cup St(N_1,C_1^3(M,N))$.

Proof: The necessity is trivial, since clearly $P^1 \cup P^2 \cup P^3 \subset S^3(M,N) = C^3(M,N) = St(M_1,C_1^3(M,N)) \cup St(N_1,C_1^3(M,N))$.

For the sufficiency, let $P^1 \cup P^2 \cup P^3 \subset St(M_i, C_i^3(M,N)) \cup St(N_i, C_i^3(M,N)) = S$, where i is arbitrary. We wish to show that all simplices of each $S_i^3(M,N)$ are contained in S. For $j \neq i$ assume, without loss of generality, that for some $D \in \{M_i, N_i\}$, $M_j + N_j \in St(D, C_i^3(M,N))$. The line from D to -D through $M_j + N_j$ intersects a point of $C_i^3(M,N)$ at $M_j + N_j$ or beyond $M_j + N_j$; let X be the first such point.

Case 1) $M_j \neq N_j$ and X and D are separated by the great circle through M_j and N_j.

For some $F \in \{M_k, N_k \mid k \neq i,\ k \neq j\}$ the point X lies on either $\overline{FM_j}$ or $\overline{FN_j}$. Since X and D are separated by the great circle through M_j and N_j, F is also separated from D by the great circle through M_j and N_j. Therefore the triangle ΔFM_jN_j does not contain either D or -D in its interior. Any line from D to -D through a point on $\overline{M_jN_j}$ must also intersect either $\overline{FM_j}$ or $\overline{FN_j}$ of ΔFN_jM_j (Pasch's Theorem). Therefore, since $\{\overline{FM_j}, \overline{FN_j}\} \subset C_i^3(M,N)$, $\overline{M_jN_j} \subset St(D, C_i^3(M,N))$.

Case 2) $M_j = N_j$.

In this case $\overline{M_jN_j} = \{N_j\} \subset C_i^3(M,N)$ so $\overline{M_jN_j} \subset St(D, C_i^3(M,N))$.

Case 3) $M_j \neq N_j$ and D lies on the great circle through M_j and N_j.

In this case M_j, N_j, $M_j + N_j$, D, and X all lie on the same great circle.

Case 3a) $X = M_j$ or $X = N_j$.

If D lies on $\overline{M_jN_j}$ then $\overline{M_jN_j}$ is covered by the two lines $\overline{DM_j}$ and $\overline{DN_j}$, so $\overline{M_jN_j} \subset St(D, C_i^3(M,N))$.

If D does not lie on $\overline{M_jN_j}$ then without loss of generality assume $X = N_j$. The line from D to N_j covers $N_j * M_j$ so that line must also cover all of $\overline{M_jN_j}$. Therefore $\overline{M_jN_j} \subset St(D, C_i^3(M,N))$.

Case 3b) $X \neq N_j$ and $X \neq M_j$.

For some $F \in \{M_k, N_k \mid k \neq j,\ k \neq i\}$ the point X lies on either $\overline{FM_j}$ or $\overline{FN_j}$. Without loss of generality assume X lies on $\overline{FN_j}$. Since X and N_j lie on the great circle through M_j and N_j, the line $\overline{FN_j}$ must also lie on that great circle. Since X is the first point of $C_i^3(M,N)$ that the line from D to -D through $M_j * N_j$ intersects after intersecting $N_j * M_j$, X lies on $\overline{M_jN_j}$. Therefore $X \in (\overline{FN_j} \cap \overline{M_jN_j} - \{N_j\}) \neq \phi$. Since F, N_j, and M_j are all on the same great circle, this implies that F is between N_j and M_j or M_j is between N_j and F, so $\overline{M_jN_j} \subset \overline{FN_j} \cup \overline{FM_j} \subset St(D,\ C_i^3(M,N))$.

Case 4) $M_j \neq N_j$ and D does not lie on the great circle through M_j and N_j, but X does lie on this great circle.

In this case, $X = M_j * N_j$ and lies on $\overline{FM_j}$ or $\overline{FN_j}$ with F as before. Assume without loss of generality that $X \in \overline{FN_j}$. Since X, M_j, and N_j all lie on the same great circle, F must also lie on this great circle. Then $X = M_j *$

N_j on $\overline{FN_j}$ implies F is between N_j and M_j or M_j is between N_j and F. In both cases $\overline{M_jN_j} \subset \overline{FN_j} \cup \overline{FM_j} \subset St(D,C_i^3(M,N))$.

This completes the proof that $S_i^3(M,N) \subset S$, i=1,2,3. The simplices in $S_i^3(M,N)$ (or $C_i^3(M,N)$) create triangles on S^2 (some of which possibly degenerate to just line segments), refered to as *cells* of $S_i^3(M,N)$ (or $C_i^3(M,N)$).

The set P^i was defined so that any cell of $S_i^3(M,N)$ with nonempty interior contains a point of P^i in its interior. Take any cell B of $S_i^3(M,N)$.

Case 1) B is a cell of $C_i^3(M,N)$.

If B contains both $-M_i$ and $-N_i$ in its interior then the interior of B would not be covered by $St(N_i,C_i^3(M,N)) \cup St(M_i,C_i^3(M,N))$, a contradiction. So B does not contain both $-N_i$ and $-M_i$ in its interior, and $B \subset St(M_i,C_i^3(M,N)) \cup St(N_i,C_i^3(M,N))$.

Case 2) B is not a cell in $C_i^3(M,N)$ and B only has one edge contained in an element of $\{\overline{M_jN_j} \mid j \neq i\}$.

Let the cell B have one edge on $\overline{M_jN_j} \subset St(D,C_i^3(M,N))$, $D \in \{M_i,N_i\}$, $j \neq i$ (it has already been shown that $M_j + N_j$ and $\overline{M_jN_j}$ are in the same star).

Case 2a) B contains D.

A line from D through any point, say Y, in the interior of the cell B also intersects one side of B. If it intersects a side of B which lies on a part of $C_i^3(M,N)$, then $Y \in St(D,C_i^3(M,N))$ clearly. Also if the line intersects the side of B lying on $\overline{M_jN_j}$, then $Y \in St(D,C_i^3(M,N))$ because any line from D to -D through a point of $\overline{M_jN_j}$ also intersects a point of $C_i^3(M,N)$ after intersecting $\overline{M_jN_j}$.

Case 2b) B does not contain D.

Then since any line from D to -D through a point of $\overline{M_jN_j}$ intersects a point of $C_i^3(M,N)$ after intersecting $\overline{M_jN_j}$, it follows that B does not contain -D. Any line from D to -D through a point, say Y, in the interior of the cell intersects two sides of the cell. If the line doesn't intersect the side lying on $\overline{M_jN_j}$ second, then $Y \in St(D,C_i^3(M,N))$. If it intersects the side lying on $\overline{M_jN_j}$ second, then $Y \in St(D,C_i^3(M,N))$ since the line continues to a point of $C_i^3(M,N)$.

Case 3) B is not a cell of $C_i^3(M,N)$, and has one edge on $\overline{M_jN_j}$ and one edge on $\overline{M_kN_k}$, with i,j,k distinct.

Since the side of B on $\overline{M_jN_j}$ intersects the side of B on $\overline{M_kN_k}$, $\overline{M_jN_j}$ intersects $\overline{M_kN_k}$. So the great circle through M_j and N_j either separates M_k and N_k or one of them lies on that great circle. Without loss of generality assume that the third side of B lies on $\overline{N_kN_j}$. Then B lies entirely on one side of the great circle through M_j and N_j, the same side as N_k. B is also contained in the triangle $\Delta N_j M_j N_k$, which does not contain a point of $C_i^3(M,N)$ in its interior. By Case 2 the interior of triangle $\Delta N_j M_j N_k$, and hence also B, is contained in $St(D, C_i^3(M,N))$.

Therefore all the cells in $S_i^3(M,N)$ are covered by $St(N_i, C_i^3(M,N)) \cup St(M_i, C_i^3(M,N))$. It follows that $S^3(M,N)$ is the union of $C^3(M,N)$ and all the cells of $S_i^3(M,N)$ for i = 1,2,3, so

$$S^3(M,N) \subset \bigcup_{i=1}^{3} St(N_i, C_i^3(M,N)) \cup St(M_i, C_i^3(M,N)) = C^3(M,N),$$

and $C^3(M,N)$ is a K-arrangement.

Q.E.D.

The statement of Theorem 9 appears complicated, and a reasonable conjecture is that the following simpler formulation should suffice:

Let $\Gamma = \{ \sum_{r=1}^{k} A_{j_r} \mid 1 \le j_1 < \ldots < j_k \le 3,\ 1 \le k \le 3,\ A_{j_r} \in \{M_{j_r}, N_{j_r}\}\}$,

$Y = \{y_1 + \ldots + y_r \mid \{y_1, \ldots, y_r\} \subset \Gamma,\ 1 \le r \le 3\}$.

Then $C^3(M,N)$ is a K-arrangement if and only if $Y \subset C^3(M,N)$.

Unfortunately, this simple characterization is false, as illustrated by $M = \begin{bmatrix} -.05 & .15 & -.01 \\ .15 & -.05 & -.01 \\ 1 & 1 & -1 \end{bmatrix}$. All 2951 points in Y are covered by the complementary cones of M, but $q = (0,\ 0,\ -1)^t = -.25M_1 - .75M_2 + .1I_1$ is not. Therefore M is not a K-matrix, showing that the complexity of the statement of Theorem 9 is justified.

4.3 CLASSIFICATION OF COMPLETELY K-MATRICES IN THREE DIMENSIONS

A simple, finite, geometric characterization is:

Theorem 10. Let $M \varepsilon E^{3\times 3}$. Then M is $\bar{K}$ iff M is K,

$I_1 - M_1,\ I_2 - M_2 \ \varepsilon\ St(I_3, C_3^3(-M,I)) \cup St(-I_3, C_3^3(-M,I))$,

$I_1 - M_1,\ I_3 - M_3 \ \varepsilon\ St(I_2, C_2^3(-M,I)) \cup St(-I_2, C_2^3(-M,I))$,

and $I_2 - M_2,\ I_3 - M_3 \ \varepsilon\ St(I_1, C_1^3(-M,I)) \cup St(-I_1, C_1^3(-M,I))$.

Proof: For $J = K = \{1,2\}$, $N = M_{JK}$ is one of the three principal 2×2 submatrices of M. The first two elements of $I_1 - M_1$, $I_2 - M_2$ are equal to $I_1 - N_1$, $I_2 - N_2$, respectively. Since the first two elements of I_3 and $-I_3$ are zero,

$I_1 - M_1,\ I_2 - M_2 \ \varepsilon\ St(I_3, C_3^3(-M,I)) \cup St(-I_3, C_3^3(-M,I))$, <=>

the first two elements of $I_1 - M_1$, $I_2 - M_2$ are complementary combinations of $-N_1$, $-N_2$ and the columns from the two dimensional identity matrix <=>

$I_1 - N_1$ and $I_2 - N_2$ are both covered by complementary cones of N <=>

N is a K-matrix (by Theorem 1).

The equivalence of the last two conditions in the theorem to the other two principal 2×2 submatrices of M being K-matrices is proved similarly. Since all 1×1 matrices are K-matrices, the theorem follows.

Q.E.D.

An efficient characterization of completely K-matrices would exploit the K-matrix property of the lower dimensional principal submatrices to help characterize the K-matrix property of the higher dimensional principal submatrices. Thus for the case $n = 3$, it is natural to investigate the relationship between the 2×2 principal submatrices being K-matrices and the whole 3×3 matrix being a K-matrix. There is a weak relationship, as shown by

<u>Theorem 11</u>. Let $M \varepsilon E^{3\times 3}$. If all the principal 2×2 submatrices of M are not K-matrices, then M is not a K-matrix.

Proof: Assume all three principal 2×2 submatrices of M are not K-matrices. From Theorem 5 we know there can be at most one positive and at most one negative diagonal element of M.

For $N \varepsilon E^{2\times 2}$ not a K-matrix, it easily follows from the theorems of Chapter III that the following must hold:

If $N_{11} < 0$ and $N_{22} > 0$, then $N_{12} > 0$ and N_{21} must be such that $N_{12}N_{21} \geq N_{11}N_{22}$.

If $N_{11} < 0$ and $N_{22} = 0$, then $N_{12} > 0$ and $N_{21} \geq 0$.

If $N_{11} > 0$ and $N_{22} = 0$, then $N_{12} \geq 0$ and $N_{21} > 0$.

If $N_{11} = N_{22} = 0$, then $N_{12} \geq 0, N_{21} \geq 0$, and $N_{12} + N_{21} > 0$.

Using these facts and symmetry to reduce the number of cases, it immediately follows that M has one of the forms

$$\begin{bmatrix} - & + & + \\ -,0,+ & + & 0,+ \\ 0,+ & + & 0 \end{bmatrix} \qquad \begin{bmatrix} - & + & + \\ 0,+ & 0 & 0,+ \\ 0,+ & 0,+ & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0,+ & + \\ 0,+ & 0 & + \\ 0,+ & 0,+ & + \end{bmatrix} \qquad \begin{bmatrix} 0 & 0,+ & 0,+ \\ 0,+ & 0 & 0,+ \\ 0,+ & 0,+ & 0 \end{bmatrix}$$

where "+", "-", "0" indicate a positive, negative, or zero element respectively. For each of these forms it is easily verified that some part of $C(I_2, -M_2, I_3)$ is not covered by any of the complementary cones of M. Therefore, M is not a K-matrix.

Q.E.D.

If all the 2×2 principal submatrices of M are K-matrices, it should be easier to verify that M is also a K-matrix than if nothing were known about the 2×2 submatrices. Interestingly, this is not true, as shown by the following sequence of examples. All the 2×2 principal submatrices of

$$M = \begin{bmatrix} -.05 & .15 & -.01 \\ .15 & -.05 & -.01 \\ 1 & 1 & -1 \end{bmatrix}.$$

are K-matrices, but $q = -0.25M_1 - 0.75M_2 + 0.1I_1 = (0,0,-1)^t$ is not covered by complementary cones. Similarly for

$$M = \begin{bmatrix} -1 & 0 & -1 \\ 0 & 2 & -3 \\ -1 & 2 & 1 \end{bmatrix}$$

$q = I_2 - M_2 = (0,-1,-2)^t$ is not covered. These examples show that it is necessary to check <u>all</u> the points mentioned in Theorem 9, and thus knowing that the 2×2 principal submatrices are K-matrices is of no help (at least for this type of finite characterization). The algebraic signs of the minors of K-matrices, completely K-matrices, and non-K-matrices were investigated, and no patterns were apparent other than what have already been mentioned.

Chapter V

CONCLUSION

The two-dimensional K and $\bar{K}$-matrices have been characterized both algebraically, Theorem 8, and by a finite test set in terms of the linear complementarity problem, Theorem 1. In three dimensions the characterizations presented were in terms of a finite test set. Prior to this there were no characterizations of K or $\bar{K}$-matrices in any dimension.

The algebraic characterization and the other algebraic results for two dimensions provide some hope for algebraic results in three dimensions, but the numerous examples and counterexamples given indicate such results will not be trivial or easy to find.

For $\bar{K}$-matrices the fact that the set of lower dimensional K-matrices is so large makes it unlikely that the characterization of $\bar{K}$-matrices will be any less trivial than that for K-matrices.

The characterization for three-dimensional K-matrices does provide hope that such a characterization would be possible for higher dimensions.

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