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# Probability-One Homotopy Algorithms for Robust Controller Synthesis with Fixed-Structure Multipliers

by

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## Summary

Continuation algorithms that avoid multiplier-controller iteration have been developed earlier for fixed-architecture, mixed structured singular value controller synthesis. These algorithms have only been formulated for the special case of Popov multipliers and rely on an *ad hoc* initialization scheme. In addition, the algorithms have not used the prediction capabilities obtained by computing the Jacobian matrix of the continuation (or homotopy) map, and have assumed that the homotopy zero curve is monotonic. This paper develops probability-one homotopy algorithms based on the use of general fixed-structure multipliers. These algorithms can be initialized using an arbitrary (admissible) multiplier and a stabilizing compensator. In addition, as with all probability-one algorithms, the homotopy zero curve is not assumed to be monotonic and prediction is accomplished by using the homotopy Jacobian matrix. This approach also appears to have some advantages over the bilinear matrix inequality (BMI) approaches resulting from extensions of the LMI framework for robustness analysis.



**Keywords**: Fixed-structure multipliers, multivariable stability margin, LMI's, BMI's, homotopy algorithms


**Running Title**: Robust Controller Synthesis with Fixed-Structure Multipliers

## 1. Introduction

During the past two decades, major advancements have been made in robust control theory. Building upon $H_\infty$ theory, the structured singular value (SSV)[1, 2] was defined as a nonconservative robustness measure for the analysis of linear systems with dynamic, arbitrary phase, multiple-block uncertainty. The supremum of the structured singular value over nonnegative frequencies is the inverse of the multivariable stability margin (see References 3, 4 and the references therein). The initial developments in structured singular value theory focussed on dynamic uncertainty with arbitrary phase (often called "complex uncertainty") and hence, although less conservative than $H_\infty$ theory, could still yield very conservative robustness bounds for systems with parametric uncertainty. This led to the development of mixed (i.e., real and complex) structured singular value (MSSV) theory[5, 6] which considers block-diagonal uncertainty with both dynamic and real scalar parametric elements.

Parallel research addressed the issue of real parameter uncertainty using absolute stability theory such as Popov analysis[7, 8, 9, 10, 11, 12] and was developed by recognizing the relationship between sector bounded nonlinearities and interval bounds on linear uncertainties. This work was soon seen to provide an upper bound for the MSSV.[8, 10, 13] In fact, in contrast to the initial work on the MSSV, this research provided the first *fixed-structure* multiplier versions of MSSV theory. A unique contribution of some of this work is that it led to the development of upper bounds on an $H_2$ cost functional over the uncertainty set under consideration. By optimizing this upper bound and using a Riccati equation constraint, continuation algorithms have been developed for MSSV controller synthesis.[14, 15, 16] A related algorithm for CSSV controller synthesis is given in Reference 17. Note that the $H_2$ approach allows the direct design of fixed-architecture (e.g., reduced-order or decentralized) controllers and the simultaneous optimization of the controller and (fixed-structure) multipliers, hence avoiding $M$-$K$ (i.e, multiplier-controller) iteration schemes. However, to date the synthesis algorithms have been formulated only for the case of the the Popov multiplier. In addition, the algorithms rely on an *ad hoc* initialization scheme, have not used the prediction capabilities obtained by computing the Jacobian of the homotopy (or continuation) map, and have assumed that the homotopy curve is monotonic.

A similar line of research has been developed independently in Reference 18, 19, 20. This work also provides a fixed-structure multiplier version of the MSSV but, unlike the approach described in References 7, 8, 9, 10, 12, this approach develops multipliers for strictly linear uncertainties. The associated robustness analysis was originally formulated in terms of a convex, frequency-domain optimization problem but has recently been reformulated in terms of a (convex) linear-matrix-inequality (LMI) problem.[21, 19] These results have led to the recognition that robust control design can be approached via solving a (nonconvex) "bilinear matrix inequality" (BMI).[22, 23, 24] This approach allows the design of fixed-architecture controllers and can be implemented without using $M$-$K$ iteration. To obtain a reasonably sized BMI, the multiplier set must be restricted to lie in the span of a stable basis.[22] However, the choice of this basis is unclear and can potentially introduce a high degree of conservatism. If the less conservative LMI formulation, requiring the use of unstable multipliers, is used, the resultant BMI is of very high dimension due to the introduction of a Lyapunov inequality of the dimension of the closed-loop system to ensure closed-loop stability.[24] In contrast, the robustness analysis results using a Riccati equation formulation easily extend to robust control design without placing any basis restrictions on the multipliers or introducing high dimensionality.

This paper uses a Riccati equation constraint to formulate fixed-architecture, robust control design methods that use general forms of the fixed-structure multipliers. The proposed method relies on the development of an artificial cost function. This cost function also includes barrier functions to enforce positive definite constraints (e.g., on the Riccati solution $P$) which allows the constrained optimization problem (the constraints including $P > 0$) to be converted into an unconstrained optimization problem. The cost function is not physically meaningful so we do not encounter the normal problems associated with making the barrier functions small as the last step of the optimization process. (See Reference 25 for a discussion of this negative feature of standard barrier function methods.) If the barrier terms are ignored and a certain term is added to the cost function, the cost function becomes an $H_2$ upper bound.

Due to the positive definite constraint on the Riccati solution, it is not possible to approach the solution to the optimization problem using standard descent methods. Hence, we develop probability-one homotopy algorithms[26, 27] to find the solution. This class of homotopy algorithms is distinct from classical continuation algorithms[28] in that they follow the zero curve using the arc length parameter and not the actual homotopy parameter $\lambda$. This allows the zero curve to be nonmonotonic in $\lambda$ and provides additional numerical robustness. In addition, the algorithms developed here can be initialized with any stabilizing compensator and admissible multiplier, in contrast to the algorithms of References 17, 14, 15, 16.

### 1.1. Notation and Definitions

Let $\mathcal{R}$ and $\mathcal{C}$ denote the real and complex numbers, $\mathcal{R}^{m\times m}$ and $\mathcal{C}^{m\times m}$ the real and complex $m \times m$ matrices, let $(\cdot)^T$ and $(\cdot)^*$ denote transpose and complex conjugate transpose, let "Re" and

"Im" denote real and imaginary part, and let $I_n$ or $I$ denote the $n \times n$ identity matrix. Furthermore we write $\sigma_{\max}(\cdot)$ for the maximum singular value, "tr" for the trace operator, and $M \geq 0$ ($M > 0$) to denote the fact that the Hermitian matrix $M$ is nonnegative (positive) definite. The Hermitian and skew-Hermitian parts of an arbitrary complex square matrix $G$ are defined by He $G \triangleq \frac{1}{2}(G + G^*)$ and Sh $G \triangleq \frac{1}{2}(G - G^*)$, respectively. Finally, vec$(\cdot)$ denotes the standard column stacking operator.

Next, we establish certain key definitions used later in the paper. Let $n(s)$ and $d(s)$ be polynomials in $s$ with real coefficients. A function $g(s)$ of the form $g(s) = n(s)/d(s)$ is called a *real rational function*. The function $g(s)$ is called *proper* (resp., *strictly proper*) if deg $n(s) \leq$ deg $d(s)$ (resp., deg $n(s) <$ deg $d(s)$), where "deg" denotes the degree of the respective polynomials. A *real-rational matrix function* is a matrix whose elements are rational functions with real coefficients. Furthermore, a *transfer function* $G(s)$ is called *proper* (resp., *strictly proper*) if every element of $G(s)$ is proper (resp., strictly proper). In this paper we assume all transfer functions are real-rational matrix functions. Also, we define $G^*(s) \triangleq G^T(-s)$ for transfer functions $G(s)$.

An *asymptotically stable transfer function* is a transfer function each of whose poles is in the open left half plane. Finally, a *Lyapunov stable transfer function* is a transfer function each of whose poles is in the closed left half plane with semi-simple poles on the imaginary axis. Let $G(s) \sim \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ denote a state space realization of a transfer function $G(s)$, that is, $G(s) = C(sI - A)^{-1}B + D$.

A square transfer function $G(s)$ is called *positive real* (resp., *generalized positive real*)[29] if $G(s)$ is Lyapunov stable and He $G(s)$ is nonnegative definite for Re$[s] > 0$ (resp., $G(s)$ has no imaginary poles and He $G(j\omega)$ is nonnegative definite for all $\omega \in \mathcal{R}$). A square transfer function is called *strictly positive real* [30] (resp., *strictly generalized positive real*) if $G(s)$ is asymptotically stable and He $G(j\omega)$ is positive definite for $\omega \in \mathcal{R}$ (resp., $G(s)$ has no imaginary poles and He $G(j\omega)$ is positive definite for $\omega \in \mathcal{R}$). A square transfer function $G(s)$ is *strongly positive real* (resp., *strongly generalized positive real*) if it is strictly positive real (resp., strictly generalized positive real) and $D + D^T > 0$, where $D \triangleq G(\infty)$. Note that although a minimal realization of a positive real transfer function is stable in the sense of Lyapunov, a minimal realization of a generalized positive real transfer function may be unstable.

## 1.2. Paper Organization

Section 2 presents the general framework for robustness analysis with fixed-structure multipliers and briefly discusses the BMI approach to fixed-architecture, robust control synthesis. Section 3 demonstrates that an alternative formulation to robust synthesis is in terms of a Riccati equation feasibility problem. Section 4 develops probability-one homotopy algorithms to solve the Riccati equation feasibility problem. Section 5 specializes the results to the special case of the Popov multiplier and presents a numerical example. Finally, Section 6 presents conclusions and directions for further research.

## 2. Multiplier Methods in Robust Analysis

In this section we review the framework for mixed uncertainty robustness analysis with fixed-structure multipliers. The exposition generally follows that presented in References 21, 12, 19, 20. We begin by considering the standard uncertainty feedback configuration of Figure 1, where $G(s) \in \mathcal{C}^{m \times m}$ is asymptotically stable and $G(s) \sim \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$. It is assumed that the uncertainty $\Delta \in \mathcal{C}^{m \times m}$ belongs to the set

$$\boldsymbol{\Delta}_\gamma \triangleq \{\Delta = \text{block-diag}(\Delta_1, ..., \Delta_p) : \Delta_i \in \mathcal{I}_i,\ ||\Delta_i||_\infty \le \gamma, i = 1, ..., p, \sum_{i=1}^{p} k_i = m\}, \tag{2.1}$$

where $\mathcal{I}_i \subseteq \mathcal{C}^{k_i \times k_i}$ denotes the internal structure of the uncertainty block $\Delta_i$ and $\gamma > 0$. For example $\mathcal{I}_i$ may be given by any of the following five sets:

$$\boldsymbol{\Delta}^{\text{I}} \triangleq \mathcal{C}^{k_i \times k_i}, \tag{2.2}$$

$$\boldsymbol{\Delta}^{\text{II}} \triangleq \{\Delta_i = \delta_i I_{k_i} : \delta_i \in \mathcal{C}\}, \tag{2.3}$$

$$\boldsymbol{\Delta}^{\text{III}} \triangleq \{\Delta_i = \delta_i I_{k_i} : \delta_i \in \mathcal{R}\}, \tag{2.4}$$

$$\boldsymbol{\Delta}^{\text{IV}} \triangleq \{\Delta_i \in \mathcal{R}^{k_i \times k_i} : \Delta_i = \Delta_i^T\}, \tag{2.5}$$

$$\boldsymbol{\Delta}^{\text{V}} \triangleq \{\Delta_i = \begin{bmatrix} -\delta_\nu & \delta_\omega \\ -\delta_\omega & -\delta_\nu \end{bmatrix} : \delta_\nu, \delta_\omega \in \mathcal{R}\}. \tag{2.6}$$

Note that $\boldsymbol{\Delta}^{\text{I}}, \boldsymbol{\Delta}^{\text{II}}$, and $\boldsymbol{\Delta}^{\text{III}}$ are standard in the literature, corresponding respectively to complex matrix block uncertainty, repeated complex scalar uncertainty, and repeated real scalar uncertainty. $\boldsymbol{\Delta}^{\text{IV}}$ is symmetric, real matrix block uncertainty, while $\boldsymbol{\Delta}^{\text{V}}$ can be used to describe uncertainty in the imaginary and real parts of a structural system represented in real normal form. If the uncertainty is of the form described by $\boldsymbol{\Delta}^{\text{IV}}$ or $\boldsymbol{\Delta}^{\text{V}}$, it is possible to represent the uncertainty by $\boldsymbol{\Delta}^{\text{III}}$. However, as discussed in Reference 12 this reformulation leads to increased conservatism and numerical complexity. The ensuing discussion is not restricted to these forms of uncertainty, but they are important special cases and will be used to provide concrete illustrations of the subsequent concepts.

To state the multivariable absolute stability criterion for $\Delta \in \mathbf{\Delta}_\gamma$ we define the sets of Hermitian, frequency-dependent, scaling matrix functions by

$$\begin{aligned}\mathcal{D}_i \triangleq \{&D_i : j\mathcal{R} \cup \infty \to \mathcal{C}^{k_i \times k_i} : D_i(j\omega) \geq 0,\\ &D_i(j\omega)\Delta_i = \Delta_i D_i(j\omega),\ \omega \in \mathcal{R},\ \Delta_i \in \mathcal{I}_i,\ i = 1, ..., p\},\end{aligned} \tag{2.7}$$

$$\begin{aligned}\mathcal{N}_i \triangleq \{&N_i : j\mathcal{R} \cup \infty \to \mathcal{C}^{k_i \times k_i} : N_i(j\omega) = N_i{}^*(j\omega),\\ &N_i(j\omega)\Delta_i = \Delta_i{}^* N_i(j\omega),\ \omega \in \mathcal{R},\ \Delta_i \in \mathcal{I}_i,\ i = 1, ..., p\}.\end{aligned} \tag{2.8}$$

Furthermore, define the sets $\mathcal{M}_i$ and $\mathcal{M}$ of multiplier transfer functions by

$$\mathcal{M}_i \triangleq \{M_i(s) = D_i(s) + Q_i(s) : D_i(j\omega) \in \mathcal{D}_i,\ Q_i(j\omega) = j\omega N_i(j\omega),\ N_i(j\omega) \in \mathcal{N}_i,\ i = 1, ..., p\}, \tag{2.9}$$

$$\mathcal{M} \triangleq \{M(s) \in \mathcal{C}^{m \times m} : M(s) = \text{block-diag}(M_1(s), ..., M_p(s)),\ M_i(s) \in \mathcal{M}_i,\ i = 1, ..., p\}. \tag{2.10}$$

Note that in (2.9), $D_i(j\omega) = \text{He } M_i(j\omega) \geq 0$ and $N_i(j\omega) = -j\text{Sh } M_i(j\omega)$. Furthermore, $M(s) \in \mathcal{M}$ satisfies

$$\text{He } M(j\omega) \geq 0,\ \ \omega \in \mathcal{R} \cup \infty, \tag{2.11}$$

and is not necessarily stable.

**Theorem 1.** Suppose that $G(s)[I - \gamma G(s)]^{-1}$ is asymptotically stable. If there exists $M(s) \in \mathcal{M}$ such that

$$\text{He}[M(j\omega)T_\gamma(j\omega)] > 0,\ \omega \in \mathcal{R} \cup \infty, \tag{2.12}$$

where

$$T_\gamma(s) \triangleq [I + \gamma G(s)][I - \gamma G(s)]^{-1}, \tag{2.13}$$

then the negative feedback interconnection of $G(s)$ and $\Delta$ is asymptotically stable (or, equivalently, $\det(I + G(j\omega)\Delta) \neq 0,\ \omega \geq 0$) for all $\Delta \in \mathbf{\Delta}_\gamma$.

**Proof.** A rigorous proof of this result is given in Reference 12. Similar results are presented in References 21, 19, 20. $\triangle$

**Remark 1.** Note that[21]

$$T_\gamma(s) \sim \left[\begin{array}{c|c} A + \gamma B(I - \gamma D)^{-1}C & \sqrt{2\gamma}B(I - \gamma D)^{-1} \\ \hline \sqrt{2\gamma}(I - \gamma D)^{-1}C & (I + \gamma D)(I - \gamma D)^{-1} \end{array}\right]. \tag{2.14}$$

Using the coprime factorization result presented in Reference 32, it follows that $M(s)$ can be factored as

$$M(s) = [M_{\rm B}^*(s)]^{-1} M_{\rm A}(s), \tag{2.15}$$

where both $M_{\rm A}(s)$ and $M_{\rm B}(s)$ are asymptotically stable and nonunique. In practice, stable $M_{\rm A}(s)$ and $M_{\rm B}(s)$ satisfying (2.15) can be computed using the approach pioneered in Reference 31 which is also given in Reference 32. In Reference 20 (2.15) is used to prove the following important corollary to Theorem 1.

**Corollary 1.** Assume $M(s) \in \mathcal{M}$, and $M_{\rm A}(s)$ and $M_{\rm B}(s)$ are asymptotically stable transfer function matrices satisfying (2.15). Then (2.12) in Theorem 1 holds if and only if

$$\mathrm{He}[M_{\rm A}(j\omega) T_\gamma(j\omega) M_{\rm B}(j\omega)] > 0, \ \omega \in \mathcal{R} \cup \infty. \tag{2.16}$$

Corollary 1 allows us to develop robust controllers based on positive real theory. Such a method is developed in Section 3.

We now characterize $\mathcal{M}_i$ for $\mathcal{I}_i$ equal to the sets defined by (2.2)-(2.6). These characterizations are restatements of results given in Reference 12. Multiplier sets corresponding to $\boldsymbol{\Delta}^{\rm I}, \boldsymbol{\Delta}^{\rm II}$, and $\boldsymbol{\Delta}^{\rm III}$ are given in References 19, 20. The set corresponding to $\boldsymbol{\Delta}^{\rm II}$ [20] differs from that given here. The following characterizations are useful in constructing state space realizations of the multipliers. In particular, for $I_i = \boldsymbol{\Delta}^{\rm I}$

$$\mathcal{M}_i = \{m_i(s) I_{k_i} : \mathrm{Re}[m_i(j\omega)] \geq 0, \ \mathrm{Im}[m_i(j\omega)] = 0, \ \omega \in \mathcal{R}\}, \tag{2.17}$$

while for $\mathcal{I}_i = \boldsymbol{\Delta}^{\rm II}$

$$\mathcal{M}_i = \left\{ M_i(s) \in \mathcal{C}^{k_i \times k_i} : \mathrm{He}\ M_i(j\omega) \geq 0, \ M(j\omega) = M^*(j\omega), \ \omega \in \mathcal{R} \right\}. \tag{2.18}$$

Note that if we denote $\mathcal{M}_i(s) = [m_{jk}(s)]_{(j,k=1,\ldots,k_i)}$, $M(j\omega) = M^*(j\omega)$ implies

$$\mathrm{Im}[m_{jj}(j\omega)] = 0, \quad m_{jk}(j\omega) = m_{kj}(-j\omega), \ j \neq k.$$

Furthermore, for $\mathcal{I}_i = \boldsymbol{\Delta}^{\rm III}$

$$\mathcal{M}_i = \left\{ M_i(s) \in \mathcal{C}^{k_i \times k_i} : \ \mathrm{He}\ M(j\omega) \geq 0, \ \omega \in \mathcal{R} \right\}, \tag{2.19}$$

while for $\mathcal{I}_i = \boldsymbol{\Delta}^{\rm IV}$

$$\mathcal{M}_i = \{m_i(s) \mathcal{I}_{k_i} : m_i(s) \in \mathcal{C}, \ \mathrm{Re}[m_i(j\omega)] \geq 0, \ \omega \in \mathcal{R}\}. \tag{2.20}$$

Finally, for $\mathcal{I}_i = \mathbf{\Delta}^{\rm V}$

$$\begin{aligned}
\mathcal{M}_i &= \{M_i(s) = D(s) + Q(s) : D(s) = \begin{bmatrix} d_{11}(s) & d_{12}(s) \\ -d_{12}(s) & d_{11}(s) \end{bmatrix}, \\
Q(s) &= \begin{bmatrix} q_{11}(s) & q_{12}(s) \\ q_{12}(s) & -q_{11}(s) \end{bmatrix}, \text{He } D(j\omega) > 0, \\
\text{Re } Q(j\omega) &= 0, \text{Im}[d_{11}(j\omega)] = \text{Re}[d_{12}(j\omega)] = 0\}.
\end{aligned} \tag{2.21}$$

### 2.1. The Structured Singular Value and Robust Performance

For a multiple block-structured uncertainty set $\mathcal{I}$, with possibly repeated scalar elements, complex scalar elements, real blocks, and complex blocks,[12] the structured singular value of a complex matrix $G(j\omega)$ is defined by

$$\mu(G(j\omega)) \triangleq \left( \inf_{\Delta \in \mathcal{I}} \{\sigma_{\max}(\Delta) : \det(I + G(j\omega)\Delta) = 0\} \right)^{-1},$$

where by convention $\mu(G(j\omega)) = 0$ if there does not exist $\Delta \in \mathcal{I}$ such that $\det(I + G(j\omega)\Delta) = 0$. The structured singular value nonconservatively characterizes the robust stability of the uncertainty feedback system of Figure 1, as stated by the following theorem.[12, 2]

**Theorem 2.** Suppose $G(s)$ is asymptotically stable. Then the negative feedback interconnection of $G(s)$ and $\Delta$ is asymptotically stable for all $\Delta \in \mathbf{\Delta}_\gamma$ if and only if

$$\mu(G(j\omega)) < \gamma^{-1}, \ \omega \in \mathcal{R} \cup \infty. \tag{2.22}$$

**Remark 2.** The parameter $\gamma_{\rm m} \triangleq 1/\sup_{\omega \geq 0} \mu(G(j\omega))$ is the multivariable stability margin.[4]

Next define

$$\mu_{\rm abs}(G(j\omega)) \triangleq \inf\{\gamma > 0 : \text{ there exists } M(\cdot) \in \mathcal{M} \text{ such that } \text{He}[M(j\omega)T_\gamma(j\omega)] > 0, \ \omega \in \mathcal{R} \cup \infty\}. \tag{2.23}$$

Then the following holds.

**Theorem 3.** For $\omega \in \mathcal{R} \cup \infty$, let $G(j\omega)$ be a complex matrix. Then

$$\mu(G(j\omega)) \leq \mu_{\rm abs}(G(j\omega)).$$

**Proof** A rigorous proof is given in Reference 12. Similar results are stated in References 19, 20. $\triangle$

The significance of the above theorem is that it allows us to consider both robust stability and robust performance in the same setting.[6] Hence, as was proved for structured singular value analysis with purely complex uncertainty,[2] robust performance can be ensured by appropriate inclusion of a "fictitious" full complex uncertainty block.

## 2.2. Linear Matrix Inequality and Riccati Equation Characterizations of (Generalized) Positive Real Matrix Functions

Theorem 1 characterizes robustness in terms of the strictly generalized positive real condition (2.12) while Corollary 1 relies on the strictly positive real condition (2.16). Hence, to implement the robustness test of Theorem 1 using state-space computations requires state-space characterizations of strictly generalized positive real and strictly positive real transfer functions.

State-space conditions for *strictly* positive real transfer functions are given in Reference 33, but include observability and rank conditions which are difficult to incorporate into numerical schemes. Hence, the lemma below provides state-space characterizations of strongly generalized positive real matrices and strongly positive real matrices, special cases respectively of strictly generalized positive real matices and strictly positive real matrices.

**Lemma 1.** Let $G(s)$ be square with $G(s) \sim \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ where $(A, B)$ is controllable. Then the following statements are equivalent:

1. $G(s)$ is strongly generalized positive real.

2. There exist $\epsilon > 0$ and symmetric $P$ such that

$$\begin{bmatrix} -A^T P - PA & -PB + C^T \\ -B^T P + C & (D - \epsilon I) + (D - \epsilon I)^T \end{bmatrix} \geq 0.$$

Furthermore, if (1) and (2) hold, then:

4. $(A, C)$ is observable if and only if $P$ is nonsingular.

5. $G(s)$ is strongly positive real if and only if $A$ is asymptotically stable and $P \geq 0$. In this case, $(A, C)$ is observable if and only if $P > 0$.

6. If $(A, C)$ is observable, $G(s)$ is strongly positive real if and only if $D + D^T > 0$ and there exist $P > 0$ and $\epsilon > 0$ such that

$$0 = A^T P + PA + (B^T P - C)^T (D + D^T)^{-1} (B^T P - C) + \epsilon I. \tag{2.24}$$

**Proof.** The equivalence of statements 1 and 2 is shown in Reference 19. Statements 3 and 4 are proved in Reference 29 and statement 5 is proved in Reference 9. $\triangle$

Lemma 1 shows that strongly generalized positive reality and generalized positive reality is equivalent to the existence of a certain solution to an LMI. Furthermore, the latter condition (under the assumption that $(A, B, C, D)$ is a minimal state space realization) is equivalent to the existence of a certain solution to a Riccati equation. This Riccati equation is subsequently used to develop probability-one homotopy maps for robust controller synthesis with fixed-structure multipliers.

## 2.3. Bilinear Matrix Inequality Approaches to Robust Controller Synthesis

Both References 21, 19 describe ways of using the LMI's corresponding to generalized positive real tests to develop robustness tests expressed in terms of the existence of a solution to an LMI. The key is to represent the multiplier $M(s)$ in the strictly generalized positive real test (2.12) of Theorem 1 in such a way that its parameters appear linearly in the corresponding state space test of Lemma 1. When the LMI robustness *analysis* results are generalized to fixed-architecture, robust control *synthesis*, a BMI results.

A straightforward way of achieving the desired representation of $M(s)$ is described in Reference 21 and is based on earlier ideas given in Reference 20. This approach requires expressing the multiplier $M(s)$ in terms of a basis expansion. In particular,

$$M(s) = \sum_{j=1}^{r} \beta_j \bar{M}^{(j)}(s), \tag{2.25}$$

where $\beta_j \geq 0$ and $\bar{M}^j(s) \in \mathcal{M},\ j = 1, ..., r$. A substantial weakness of this approach is the difficulty of choosing the basis $\{\bar{M}^{(1)}(s), ..., \bar{M}^{(r)}(s)\}$. Specifically, for a given multiplier $M(s) \in \mathcal{M}$ and a given basis, the approach in Reference 34 does not guarantee that there exists nonnegative $\beta_j$ such that $M(s)$ is given by (2.25).

The approach proposed in Reference 19 does not suffer from these weaknesses. It is based on the following result.

**Lemma 2.** [20] Equation (2.12) is satisfied for some transfer matrix $M^{(0)}(s) \in \mathcal{M}$ if and only if there exist a real polynomial matrix $M(s) \in \mathcal{M}$ for which (2.12) holds. Furthermore, if $M^{(0)}(s)$ is factored as $M^{(0)}(s) = \frac{1}{d^{(0)}(s)} N^{(0)}(s)$ where $N^{(0)}(s)$ is a real polynomial matrix and $d^{(0)}(s)$ is a scalar-valued real polynomial, then the degree of $M(s)$ need not be greater than the sum of degrees of $N^{(0)}(s)$ and $d^{(0)}(s)$, and in fact one can choose $M(s) = d^{(0)}(-s)N^{(0)}(s)$. In addition, if $M(s)$ is of order $2n$, then for all $n^{\text{th}}$-order real polynomials $d(s)$ having no zeros on the $j\omega$ axis,

$\tilde{M}(s) = \frac{M(s)}{d(-s)d(s)} \in \mathcal{M}$ and satisfies (2.12).

**Remark 3.** Lemma 2 allows one to restrict the multiplier search to $2n^{\mathrm{th}}$ order, real polynomial matrices $M(s)$. To obtain state space realizable transfer functions we can consider

$$\tilde{M}(s) = \frac{M(s)}{d(s)d(-s)}, \tag{2.26}$$

where $d(s)$ is an arbitrary $n^{\mathrm{th}}$-order polynomial having no zeros on the $j\omega$-axis.

**Remark 4.** Note that since the zeros of $d(-s)$ are the mirror images of the zeros of $d(s)$ about the imaginary axis, $\tilde{M}(s)$ given by (2.26) is *always* an unstable multiplier.

This latter approach is powerful but as eluded to in Remark 4 always produces an unstable multiplier. When extended to fixed architecture, robust control design, this approach results in a BMI with very high dimension since it requires the introduction of a Lyapunov inequality of dimension equal to that of the closed-loop system to ensure closed-loop stability. We hence begin the development of an alternative scheme without these limitations.

## 3. A Riccati Equation Approach to Robust Controller Synthesis with Fixed-Structure Multipliers

We now give exclusive attention to robustness tests that are expressed in terms of positive real conditions, as opposed to generalized positive real conditions. Hence, we will consider robustness tests corresponding to Corollary 1. The developments here differ dramatically from those in the previous section since we use the Riccati equation characterization of strong positive reality given in Lemma 1 instead of the LMI characterization of strong generalized positive reality which is also given in Lemma 1.

We focus on the uncertainty structures corresponding to the sets $\mathbf{\Delta}^{\mathrm{I}}$ and $\mathbf{\Delta}^{\mathrm{III}}$, defined respectively by (2.2) and (2.4). Hence, we develop fixed-structure multiplier tests for the complex, block-structured uncertainty considered by classical complex structured singular value analysis [1, 2] and the real, diagonal uncertainty considered by classical real structured singular value analysis.[5, 6] Although not detailed here, the uncertainty structures corresponding to the sets $\mathbf{\Delta}^{\mathrm{II}}$, $\mathbf{\Delta}^{\mathrm{IV}}$, and $\mathbf{\Delta}^{\mathrm{V}}$, may also be considered in the framework of this section. In addition, the results may be extended in a straightforward manner to mixed uncertainty sets.

The resulting numerical algorithms are probability-one homotopy algorithms. A detailed discussion of the numerical algorithms is reserved for the next section. Here, we lay the necessary foundation for these algorithms. We begin by developing a constructive characterization of multiplier factors $M_{\rm A}(s)$ and $M_{\rm B}(s)$ corresponding to complex, block-structured uncertainty and real, diagonal uncertainty. These constructions are essential to the subsequent developments.

### 3.1. Complex, Block-Structured Uncertainty

From (2.10) and (2.17) it follows that a multiplier corresponding to complex block-diagonal uncertainty is given by

$$M(s) = \text{block-diag}\left(m_1(s)I_{k_1}, ..., m_p(s)I_{k_p}\right), \tag{3.1}$$

$$\text{Re } m_i(j\omega) > 0, \ \omega \in \mathcal{R}, \tag{3.2}$$

$$\text{Im } m_i(j\omega) = 0, \ \omega \in \mathcal{R}. \tag{3.3}$$

Note that we have replaced the weak inequality in (2.17) with a strict inequality in (3.2).

**Lemma 3.** For uncertainty structures corresponding to (2.2) there exists $\tilde{M}(s)$ with no zeros or poles on the imaginary axis and satisfying the compatibility conditions (3.1)-(3.3) and the robustness test (2.12), if and only if there exists $M(s)$ satisfying (2.12) and (3.1) such that

$$M(s) = \bar{M}(-s)\bar{M}(s), \tag{3.4}$$

where $\bar{M}(s)$ has no poles or zeros in the closed right half plane.

**Proof.** From Lemma 2 it follows that without loss of generality we can restrict our attention to real polynomial $m_i(s)$ satisfying (3.1)-(3.3) and (2.12). Furthermore, the restriction (3.3) also requires that $m_i(s)$ have only even powers of $s$, such that for some integer $n$, $m_i(s) = \sum_{j=0}^{n} m_{ij}s^{2j}$. This implies that

$$m_i(s) = m_{in}\prod_{k=1}^{n}(s^2 - a_k) = m_{in}\prod_{k=1}^{n}(s - b_k)(s + b_k) \tag{3.5}$$

where $b_k = \sqrt{a_k}$ and here $\sqrt{\cdot}$ denotes the square root in the right half plane.

For $s \to \infty$ (3.2) implies that for some real positive scalar $c$, $(-1)^n m_{in} = c$, which in turn implies that

$$m_i(s) = c^2\prod_{k=1}^{n}(-s + b_k)(s + b_k) = n_i(-s)n_i(s) \tag{3.6}$$

where $n_i(s)$ has no zeros in the closed right half plane. Note that if $\text{Im}(b_k) \neq 0$ there exists $j \neq k$ such that $b_j = \bar{b}_k$. It follows that $n_i(s) \triangleq c\prod_{k=1}^{n}(s + b_k)$ is a real polynomial. It follows from (3.6) that $M(s) = N(-s)N(s)$ where $N(s)$ is a real polynomial matrix with no zeros in the closed right half plane. If $d(s)$ is any $n^{th}$ order, real polynomial with no zeros in the closed right half plane and $M'(s) \triangleq \bar{M}(-s)\bar{M}(s)$ where $\bar{M}(s) \triangleq \frac{N(s)}{d(s)}$, then $\bar{M}'(s)$ has no zeros or poles on the imaginary axis, satisfies (3.1)-(3.3) and from Lemma 2 also satisfies (2.12). Finally, note that each $M(s)$ given by (3.1) and (3.4), where $\bar{M}(s)$ has no poles or zeros in the closed right half plane, satisfies (3.2) and (3.3). $\triangle$

**Remark 5.** Equation (3.4) corresponds to the stable coprime factorization of $M(s)$ given by (2.15) with $M_{\mathrm{A}}(s) = \bar{M}(s)$ and $M_{\mathrm{B}}(s) = \bar{M}^{-1}(s)$. Furthermore, if $\bar{M}(s)$ is strictly proper then $M(s)$ given by (3.4) is strongly positive real and $\bar{M}(s)$ and $\bar{M}^{-1}(s)$ both have state space realizations.

**Remark 6.** $\bar{M}(s)$ is precisely an asymptotically stable, minimum phase transfer function representation of the classical $D$-scales from complex structured singular value analysis.[1, 2]

From Remark 5, it follows that if we denote the state space realization of strictly proper $M_{\mathrm{A}}(s)$ in (2.15) by

$$M_{\mathrm{A}}(s) \sim \left[\begin{array}{c|c} \bar{A} & \bar{B} \\ \hline \bar{C} & \bar{D} \end{array}\right], \tag{3.7}$$

then, we can simply choose $M_{\mathrm{B}}(s) = M_{\mathrm{A}}{}^{-1}(s)$ and hence using a standard state space realization inversion formula [35]

$$M_{\mathrm{B}}(s) \sim \left[\begin{array}{c|c} \bar{A} - \bar{B}\bar{D}^{-1}\bar{C} & \bar{B}\bar{D}^{-1} \\ \hline -\bar{D}^{-1}\bar{C} & \bar{D}^{-1} \end{array}\right]. \tag{3.8}$$

Details on how to choose $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$ to enforce the block-diagonal structure of $M_{\mathrm{A}}(s)$ are given in Reference 36.

Now, if we let $T_\gamma(s)$ have the state space realization $(A_\gamma, B_\gamma, C_\gamma, D_\gamma)$ as in (2.14), then, referring to (2.16),

$$M_{\mathrm{A}}(s)T_\gamma(s)M_{\mathrm{B}}(s) \sim \left[\begin{array}{c|c} \tilde{A}_\gamma & \tilde{B}_\gamma \\ \hline \tilde{C}_\gamma & \tilde{D}_\gamma \end{array}\right],$$

where

$$\tilde{A}_\gamma = \begin{bmatrix} \bar{A} - \bar{B}\bar{D}^{-1}\bar{C} & 0 & 0 \\ -B_\gamma\bar{D}^{-1}\bar{C} & A_\gamma & 0 \\ -\bar{B}D_\gamma\bar{D}^{-1}\bar{C} & \bar{B}B_\gamma & \bar{A} \end{bmatrix}, \quad \tilde{B}_\gamma = \begin{bmatrix} \bar{B}\bar{D}^{-1} \\ B_\gamma\bar{D}^{-1} \\ \bar{B}D_\gamma\bar{D}^{-1} \end{bmatrix}, \tag{3.9}$$

$$\tilde{C}_\gamma = \begin{bmatrix} -\bar{D}D_\gamma\bar{D}^{-1}\bar{C} & \bar{D}D_\gamma & \bar{C} \end{bmatrix}, \qquad \tilde{D}_\gamma = \bar{D}D_\gamma\bar{D}^{-1}. \tag{3.10}$$

The next theorem which considers complex, block-structured uncertainty, follows immediately from Corollary 1 and Lemma 1.

**Theorem 4.** Let $\mathcal{I}_i = \mathbf{\Delta}^{\mathrm{I}}$, $i = 1, \ldots, p$, and suppose $G(s)$ is asymptotically stable. If there exist $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$, $P > 0$, and $\epsilon > 0$ such that $\tilde{D}_\gamma + \tilde{D}_\gamma^T > 0$ and

$$0 = \tilde{A}_\gamma{}^T P + P\tilde{A}_\gamma + (\tilde{B}_\gamma{}^T P - \tilde{C}_\gamma)^T (\tilde{D}_\gamma + \tilde{D}_\gamma{}^T)^{-1} (\tilde{B}_\gamma{}^T P - \tilde{C}_\gamma) + \epsilon I,$$

where $\tilde{A}_\gamma, \tilde{B}_\gamma, \tilde{C}_\gamma, \tilde{D}_\gamma$ are given by (3.9) and (3.10), then the negative feedback interconnection of $G(s)$ and $\Delta$ is asymptotically stable for all $\Delta \in \mathbf{\Delta}_\gamma$.

## 3.2. Real, Diagonal Uncertainty

Recall from (2.10) and (2.19) that a multiplier corresponding to real, diagonal uncertainty (with possible repeated elements) is given by

$$M(s) = \text{block-diag}(M_1(s), ..., M_p(s)),$$

where

$$\mathrm{He}[M(j\omega)] \geq 0 \text{ for } \omega \in \mathcal{R}. \tag{3.11}$$

Now, if we let $M(s) = M_{\mathrm{B}}{}^*(s)^{-1} M_{\mathrm{A}}(s)$ as in (2.15), then it follows from Corollary 1 that if we consider only strict inequality in (3.11), then we can replace (3.11) with

$$\mathrm{He}[M_{\mathrm{A}}(j\omega) M_{\mathrm{B}}(j\omega)] > 0, \ \omega \in \mathcal{R}. \tag{3.12}$$

Next let the state space realizations of $M_{\mathrm{A}}(s)$ and $M_{\mathrm{B}}(s)$ be denoted, respectively, by

$$M_{\mathrm{A}}(s) \sim \left[\begin{array}{c|c} A_{\mathrm{A}} & B_{\mathrm{A}} \\ \hline C_{\mathrm{A}} & D_{\mathrm{A}} \end{array}\right], \ M_{\mathrm{B}}(s) \sim \left[\begin{array}{c|c} A_{\mathrm{B}} & B_{\mathrm{B}} \\ \hline C_{\mathrm{B}} & D_{\mathrm{B}} \end{array}\right].$$

If we let $T_\gamma(s)$ have the state space realization $(A_\gamma, B_\gamma, C_\gamma, D_\gamma)$ as in (2.14), then referring to (2.16)

$$M_{\mathrm{A}}(s) T_\gamma(s) M_{\mathrm{B}}(s) \sim \left[\begin{array}{c|c} \tilde{A}_{\gamma,1} & \tilde{B}_{\gamma,1} \\ \hline \tilde{C}_{\gamma,1} & \tilde{D}_{\gamma,1} \end{array}\right],$$

where

$$\tilde{A}_{\gamma,1} = \begin{bmatrix} A_{\mathrm{B}} & 0 & 0 \\ B_\gamma C_{\mathrm{B}} & A_\gamma & 0 \\ B_{\mathrm{A}} D_\gamma C_{\mathrm{B}} & B_{\mathrm{A}} C_\gamma & A_{\mathrm{A}} \end{bmatrix}, \ \tilde{B}_{\gamma,1} = \begin{bmatrix} B_{\mathrm{B}} \\ B_\gamma D_{\mathrm{B}} \\ B_{\mathrm{A}} D_\gamma D_{\mathrm{B}} \end{bmatrix},$$
$$\tilde{C}_{\gamma,1} = \begin{bmatrix} D_{\mathrm{A}} D_\gamma D_{\mathrm{B}} & D_{\mathrm{A}} C_\gamma & C_{\mathrm{A}} \end{bmatrix}, \quad \tilde{D}_{\gamma,1} = D_{\mathrm{A}} D_\gamma D_{\mathrm{B}}.$$

Similarly, referring to (3.12),

$$M_{\mathrm{A}}(s) M_{\mathrm{B}}(s) \sim \left[\begin{array}{c|c} \tilde{A}_{\gamma,2} & \tilde{B}_{\gamma,2} \\ \hline \tilde{C}_{\gamma,2} & \tilde{D}_{\gamma,2} \end{array}\right],$$

where

$$\tilde{A}_{\gamma,2} = \begin{bmatrix} A_{\rm B} & 0 \\ B_{\rm A}C_{\rm B} & A_{\rm A} \end{bmatrix}, \ \tilde{B}_{\gamma,2} = \begin{bmatrix} B_{\rm B} \\ B_{\rm A}D_{\rm B} \end{bmatrix},$$
$$\tilde{C}_{\gamma,2} = \begin{bmatrix} D_{\rm A}C_{\rm B} & C_{\rm A} \end{bmatrix}, \quad \tilde{D}_{\gamma,2} = D_{\rm A}D_{\rm B}.$$

Now, let

$$\tilde{A}_\gamma = \text{block-diag}\{\tilde{A}_{\gamma,1}, \tilde{A}_{\gamma,2}\}, \quad \tilde{B}_\gamma = \text{block-diag}\{\tilde{B}_{\gamma,1}, \tilde{B}_{\gamma,2}\}, \tag{3.13}$$

$$\tilde{C}_\gamma = \text{block-diag}\{\tilde{C}_{\gamma,1}, \tilde{C}_{\gamma,2}\}, \quad \tilde{D}_\gamma = \text{block-diag}\{\tilde{D}_{\gamma,1}, \tilde{D}_{\gamma,2}\}. \tag{3.14}$$

The next theorem, follows immediately from Corollary 1 and Lemma 1.

**Theorem 5.** Let $\mathcal{I}_i = \mathbf{\Delta}^{\rm III}$, $i = 1, ..., p$, and suppose $G(s)$ is asymptotically stable. If there exist $(A_{\rm A}, B_{\rm A}, C_{\rm A}, D_{\rm A})$, $(A_{\rm B}, B_{\rm B}, C_{\rm B}, D_{\rm B})$, $P > 0$, and $\epsilon > 0$ such that $\tilde{D}_\gamma + \tilde{D}_\gamma^T > 0$ and

$$0 = \tilde{A}_\gamma{}^T P + P\tilde{A}_\gamma + (\tilde{B}_\gamma{}^T P - \tilde{C}_\gamma)^T (\tilde{D}_\gamma + \tilde{D}_\gamma{}^T)^{-1} (\tilde{B}_\gamma{}^T P - \tilde{C}_\gamma) + \epsilon I,$$

where $\tilde{A}_\gamma, \tilde{B}_\gamma, \tilde{C}_\gamma, \tilde{D}_\gamma$ are given by (3.13) and (3.14), then the negative feedback interconnection of $G(s)$ and $\Delta$ is asymptotically stable for all $\Delta \in \mathbf{\Delta}_\gamma$.

### 3.3. Problem Formulation

Both Theorems 4 and 5 provide robustness tests in terms of the following feasibility problem.

**Riccati Equation Feasibility Problem (REFP).** Find $\theta \in \mathcal{R}^q$, $\epsilon > 0$, and $P \in \mathcal{R}^{r\times r}$ such that

$$0 = \tilde{A}_\gamma{}^T(\theta)P + P\tilde{A}_\gamma(\theta) + (\tilde{B}_\gamma{}^T(\theta)P - \tilde{C}_\gamma(\theta))^T (\tilde{D}_\gamma(\theta) + \tilde{D}_\gamma{}^T(\theta))^{-1} (\tilde{B}^T(\theta)P - \tilde{C}_\gamma(\theta)) + \epsilon I, \tag{3.15}$$

$$P > 0, \quad \tilde{D}_\gamma(\theta) + \tilde{D}_\gamma^T(\theta) > 0, \tag{3.16}$$

where the dimension $q$ is determined by the multiplier and $r$ is determined by both the multiplier and the nominal plant size. In Theorems 4 and 5, $\theta$ corresponds to the free parameters of the matrices providing a state-space representation of the multiplier factors $M_{\rm A}(s)$ and $M_{\rm B}(s)$. For example, considering Theorem 4, if all of the elements of the matrices $\bar{A}, \bar{B}, \bar{C}$, and $\bar{D}$ in (3.7) and (3.8) are free, then $\theta$ is defined by $\theta = \left(\text{vec}^T(\bar{A}), \text{vec}^T(\bar{B}), \text{vec}^T(\bar{C}), \text{vec}^T(\bar{D})\right)^T$.

If we are considering control design for a plant $(A_p, B_p, C_p, D_p)$ under a feedback controller $(A_c, B_c, C_c)$, then, assuming negative feedback, $A$ in (2.14) is given by

$$A = \begin{bmatrix} A_p & -B_pC_c \\ B_cC_p & A_c - B_cD_pC_c \end{bmatrix}. \tag{3.17}$$

Hence, in Theorems 4 and 5 $\tilde{A}_\gamma$ is linear in the controller matrices. The controller matrices essentially provide extra degrees of freedom to satisfy the Riccati equation constraint (3.15). To illustrate, if all of the elements of the matrices $\bar{A},\ \bar{B},\ \bar{C}$, and $\bar{D}$ are free, and all of the matrices $A_c,\ B_c,\ C_c$ are also free then $\theta$ is defined by $\theta = (\mathrm{vec}^T(\bar{A}), \mathrm{vec}^T(\bar{B}), \mathrm{vec}^T(\bar{C}), \mathrm{vec}^T(\bar{D}), \mathrm{vec}^T(A_c), \mathrm{vec}^T(B_c), \mathrm{vec}^T(D_c))^T$. Note that $\tilde{A}_\gamma(\theta),\ \tilde{B}_\gamma(\theta),\ \tilde{C}_\gamma(\theta)$, and $\tilde{D}_\gamma(\theta)$ are generally nonlinear functions of $\theta$. Hence it is not possible to convert the REFP to an LMI feasibility problem.

We approach the development of a solution technique by defining the following artificial cost function

$$J(\theta, \epsilon, P) = \alpha\theta^T\theta + \alpha\epsilon^2 + r_d \operatorname{tr}\left[\tilde{D}_\gamma(\theta) + \tilde{D}_\gamma^T(\theta)\right]^{-1} + r_p \operatorname{tr} P^{-1} + r_\epsilon \frac{1}{\epsilon}, \tag{3.18}$$

where $\alpha,\ r_d,\ r_p$, and $r_\epsilon$ are positive scalars. This barrier cost function has the nice properties that

1. $J(\theta, \epsilon, P)$ becomes unbounded if one of the eigenvalues of $P$ or $\tilde{D}_\gamma(\theta) + \tilde{D}_\gamma^T(\theta)$ approaches zero or $\epsilon$ approaches zero; and

2. the second derivative of the first two terms with respect to the vector $[\theta, \epsilon]^T$ is $2\alpha I$.

Property (1) is used to enforce the constraints $P > 0,\ \tilde{D}_\gamma(\theta) + \tilde{D}_\gamma^T(\theta) > 0$, and $\epsilon > 0$. Property (2) is used to improve the numerical conditioning of the Hessian matrix, which sometimes improves the numerical robustness of the associated numerical algorithms.

**Remark 7.** Replace $\epsilon I$ in (3.15) with $\epsilon R$ where $R = E^T E$ and $E$ is the output matrix corresponding to the performance variable and let $V = D_w D_w^T$ and $D_w$ is the input matrix corresponding to the plant disturbance. Then, if we choose $J(\theta, \epsilon, P) = \operatorname{tr} PV$ + an additional nonnegative, multiplier-dependent term, then $J(\cdot)$ is an upper bound on an $H_2$ cost functional. This type of cost functional is the basis for the continuation and homotopy algorithms of References 34, 37, 17, 14, 15, 16.

Let $\mathcal{S}$ denote the set of solutions $(\theta, \epsilon, P)$ to the REFP and consider the optimization problem

$$\min_{\theta,\epsilon}\ J(\theta, \epsilon, P) \text{ subject to (3.15).} \tag{3.19}$$

We will attempt to find $(\theta, \epsilon, P) \in \mathcal{S}$ by solving the optimization problem (3.19). That is, we will search for $(\theta, \epsilon, P) \in \mathcal{S}$ that is an extremal to (3.19).

Note that we are assuming that if $\mathcal{S}$ is nonempty, (3.19) has a solution. However, this has not been proven. A sufficient condition to guarantee a solution is that the feasible level set

$$\Pi(\beta) = \{(\theta, \epsilon, P) \in \mathcal{S} : J(\theta, \epsilon, P) \leq \beta\},$$

is compact for $\beta \geq 0$. Clearly, $\Pi(\beta)$ is bounded due to the terms $\alpha\theta^T\theta + \alpha\epsilon^2$ and the positivity of each of the terms in (3.18). However, whether $\Pi(\beta)$ is closed depends on the closure of the set $\mathcal{S}$. The characterization of $\mathcal{S}$ is a subject of future research.

To characterize the extremals, we define the Lagrangian

$$\mathcal{L}(\theta, \epsilon, P, Q) = J(\theta, \epsilon, P) + \operatorname{tr} QW(\theta, \epsilon, P), \tag{3.20}$$

where $W(\theta, \epsilon, P)$ denotes the right hand side of (3.15) and $Q$ is the symmetric Lagrange multiplier. Note that the constraints (3.16) are absorbed into $J$ as barriers. The necessary conditions for a solution to (3.19) are given by[25]

$$0 = \frac{\partial \mathcal{L}}{\partial \theta}, \;\; 0 = \frac{\partial \mathcal{L}}{\partial \epsilon}, \tag{3.21}$$

$$0 = \frac{\partial \mathcal{L}}{\partial Q} = \tilde{A}_\gamma{}^T(\theta)P + P\tilde{A}_\gamma(\theta) + \left(\tilde{B}_\gamma{}^T(\theta)P - \tilde{C}_\gamma(\theta)\right)^T \left(\tilde{D}_\gamma(\theta) + \tilde{D}_\gamma{}^T(\theta)\right)^{-1} \left(\tilde{B}^T(\theta)P - \tilde{C}_\gamma(\theta)\right) + \epsilon I, \tag{3.22}$$

$$0 = \frac{\partial \mathcal{L}}{\partial P} = \tilde{F}_\gamma Q + Q\tilde{F}_\gamma^T - r_p(P^{-2}) + \frac{\beta}{\epsilon}V^T, \tag{3.23}$$

where

$$\tilde{F}_\gamma = \tilde{A}_\gamma - \tilde{B}_\gamma\left[\tilde{D}_\gamma + \tilde{D}_\gamma^T\right]^{-1}\left[\tilde{B}_\gamma P - \tilde{C}_\gamma\right].$$

Although (3.21)-(3.23) characterizes extremals to (3.19), we have not yet developed a reliable method to compute these extremals. Note that standard interior point descent methods (e.g., steepest descent, conjugate gradient, or quasi-Newton methods) cannot be directly applied due to the nature of the constraints. For example, suppose we attempt to initialize one of these methods with a multiplier (in the class of multipliers for the given uncertainty set) represented by $\theta_0$ and also choose an initial $\epsilon$ denoted by $\epsilon_0$. Then, if there exists a positive-definite solution $P_0$ to (3.15), the REFP is solved and there is no need for further computations. However, suppose there is no positive definite solution $P_0$ to (3.15). Then, $(\theta_0, \epsilon_0, P_0)$ cannot be used to initialize an interior point descent method to find a solution to the optimization problem (3.19) since this class of methods requires an initial feasible interior point. What is needed is a numerical technique that is able to find a solution to (3.19) by starting with a nonfeasible point $(\theta_0, \epsilon_0, P_0)$. This is accomplished in the next section using a probabililty-one homotopy algorithm.

## 4. Probability-One Homotopy Algorithms for Robust Controller Synthesis

Homotopies have traditionally been studied as a part of topology and have found significant application in nonlinear functional analysis and differential geometry. However, it has only been in the last two decades that homotopies have been used for practical numerical computation. The algorithms described here are probability one, globally convergent homotopy algorithms.[26, 27] This class of algorithms is related to, but distinct from, other homotopy methods such as continuation methods.[28] An advantage of probability-one algorithms is that under certain conditions, it is possible to guarantee convergence of the algorithm from an arbitrary starting point with probability one. (This does not imply that the algorithms are guaranteed to converge to global optimum, only that they can converge to a local extremum from arbitrary starting points. These algorithms also relieve some of the technical difficulties sometimes encountered when implementing alternative homotopy methods. For example, they allow the zero curve of the homotopy map to be nonmonotonic in the homotopy parameter $\lambda$, and avoid singularities along the zero curve.

Consider a function $F : \mathcal{R}^N \times \mathcal{R} \to \mathcal{R}^N$ that is $\mathcal{C}^2$. Given $\gamma_f \in \mathcal{R}$, it is desired to find $x \in \mathcal{R}^N$ such that

$$0 = F(x, \gamma_f). \tag{4.1}$$

This is a standard zero finding problem. In the context of the robustness analysis results of the previous section

$$x = (\theta, \epsilon), \quad N = q + 1, \tag{4.2}$$

$$F(x, \gamma) = \left(\frac{\partial \mathcal{L}}{\partial \theta}, \frac{\partial \mathcal{L}}{\partial \epsilon}\right), \tag{4.3}$$

and $\gamma_f$ corresponds to some desired lower bound on the multivariable stability margin. Note that $0 = \frac{\partial \mathcal{L}}{\partial Q}$ and $0 = \frac{\partial \mathcal{L}}{\partial P}$ are implicitly satisfied by choosing $P$ as the (maximal) solution of the Riccati equation (3.15) and $Q$ as the solution of the Lyapunov equation (3.23).

Let $x_0 = [\theta_0, \epsilon_0]$ represent a feasible multiplier, a stablizing compensator and a positive $\epsilon$. Furthermore let $\gamma_0$ be chosen small enough such that there exists a positive-definite solution $P_0$ to (3.15). (It is assumed that $\gamma_0 < \gamma_f$ such that the robustness problem is not trivial.)

We let

$$\gamma(\lambda) = (1 - \lambda)\gamma_0 + \lambda \gamma_f \tag{4.4}$$

and define the probability-one homotopy map $\rho : [0, 1) \times \mathcal{R}^N \to \mathcal{R}^N$ by

$$\rho(\lambda, x) = \lambda F(x, \gamma(\lambda)) + (1 - \lambda)(x - x_0). \tag{4.5}$$

Obviously, $0 = \rho$ has the unique solution $x_0$ and $\rho = F(x, \gamma_f)$. These are necessary conditions for the homotopy map. In the context of the robustness problem of Section 3, this homotopy map has the desirable property that it can be initialized with any feasible multiplier. In addition, for any $\lambda \in [0, 1)$ the corresponding point on the zero curve $(x, \lambda)$ corresponds to a controller and multiplier that guarantees the level of robustness corresponding to $\gamma(\lambda)$ since the Riccati equation (3.15) (or

(3.22)) with the constraints (3.16) are satisfied with $\gamma = \gamma(\lambda)$. Hence, each point on the zero curve $(0 = \rho(\lambda, x),\ \lambda \in [0, 1))$, is physically meaningful even though $F(x, \gamma(\lambda)) \neq 0$ for $0 < \lambda < 1$.

### 4.1. Probability-One Homotopy Algorithm

Complete details of the numerical algorithm are in Reference 27. A sketch follows.

1. Set $\lambda \triangleq 0,\ x \triangleq x_0$.

2. Evaluate the homotopy map $\rho$ and the Jacobian of the homotopy map $D\rho$.

3. Predict the next point $Z^{(0)}$ on the homotopy zero curve using, e.g., a Hermite cubic interpolant.

4. For $k = 0, 1, 2, ...$ until convergence do

$$Z^{(k+1)} = Z^{(k)} - [D\rho(Z^{(k)})]^{\dagger} \rho(Z^{(k)}),$$

   where $[D\rho(Z)]^{\dagger}$ is the Moore-Penrose pseudoinverse of $D\rho(Z)$. Let $(x_1, \lambda_1) = \lim_{k\to\infty} Z^k$.

5. If $\lambda_1 < 1$, then set $x = x_1,\ \lambda = \lambda_1$, and go to to step (2).

6. If $\lambda_1 > 1$, compute the solution $x$ at $\lambda = 1$ using, e.g., inverse linear interpolation.[27]

## 5. The Popov Multiplier and an Illustrative Numerical Example

In this section we consider the special case of the Popov multiplier. In particular we let $M(s) = H^2 + Ns$, where $H$ and $N$ are real and diagonal. This multiplier is in the class of multipliers for real parametric uncertainty and can also be applied to complex uncertainty by choosing $N = 0$, in which case $H$ is the constant $D$-scale matrix considered in Reference 38, 17. Note that if $M(s)$ is the Popov multiplier and if $T_\gamma(s)$ is asymptotically stable, then $M(s)T_\gamma(s)$ is asymptotically stable. Hence, the framework of Section 2 applies with $M_{\mathrm{A}}(s) = M(s)$ and $M_{\mathrm{B}}(s) = I$. Next we use the Popov multiplier to consider the case of nonrepeated, real parametric, scalar uncertainty (i.e., $\mathcal{I}_i = \mathbf{\Delta}^{\mathrm{III}}$ and $k_i = 1$ for $i = 1, ..., p$).

### 5.1. Nonrepeated, Real Parametric, Diagonal Uncertainty

It is important to note that for real parametric uncertainty, $D$, the direct feedthrough term in the state space realization of $G(s)$ in Figure 1, is always zero. Hence, the state-space realization of

$T_\gamma(s)$ (see (2.13) and (2.14)) becomes

$$T_\gamma(s) \sim \left[\begin{array}{c|c} A+\gamma BC & \sqrt{2\gamma}B \\ \hline \sqrt{2\gamma}C & I \end{array}\right], \tag{5.1}$$

or, equivalently,

$$T_\gamma(s) = T_{\rm sp}(s) + I, \tag{5.2}$$

where $T_{\rm sp}(s)$ is the strictly proper transfer function given by

$$T_{\rm sp}(s) \sim \left[\begin{array}{c|c} A+\gamma BC & \sqrt{2\gamma}B \\ \hline \sqrt{2\gamma}C & 0 \end{array}\right]. \tag{5.3}$$

Note that for nonrepeated, real parametric, diagonal uncertainty the Popov multiplier (with $H$ and $N$ diagonal) is a compatible multiplier. In this case the robustness condition (2.12) becomes

$$\mathrm{He}[(H^2 + j\omega N)(T_{\rm sp}(j\omega) + I)] > 0, \quad \omega \in \mathcal{R}, \tag{5.4}$$

or, equivalently,

$$\mathrm{He}[(H^2 + j\omega N)T_{\rm sp}(j\omega)] + \mathrm{He}[H^2 + j\omega N] > 0, \quad \omega \in \mathcal{R}. \tag{5.5}$$

It follows from Lemma 3 of Reference 19 that

$$(H^2 + Ns)T_{\rm sp}(s) \sim \left[\begin{array}{c|c} A+\gamma BC & \sqrt{2\gamma}B \\ \hline \sqrt{2\gamma}(H^2C + NC(A+\gamma BC)) & 2\gamma NCB \end{array}\right]. \tag{5.6}$$

Then, since $\mathrm{He}[H^2 + j\omega N] = H^2 = \mathrm{He}\ H^2$, the robustness condition (5.4) is equivalent to

$$\mathrm{He}\ T_{\rm H}(j\omega) > 0, \quad \omega \in \mathcal{R}, \tag{5.7}$$

where

$$T_{\rm H}(s) \sim \left[\begin{array}{c|c} A+\gamma BC & \sqrt{2\gamma}B \\ \hline \sqrt{2\gamma}(H^2C + NC(A+\gamma BC)) & H^2 + 2\gamma NCB \end{array}\right]. \tag{5.8}$$

The next theorem follows immediately from (5.7), (5.8), and Lemma 1.

**Theorem 6.** Let $\mathcal{I}_i = \mathbf{\Delta}^{\rm IV}$, $k_i = 1, ..., p$, assume $G(s)$ is asymptotically stable, and define $A_\gamma \triangleq A + \gamma BC$.

1. If there exist real diagonal $H$ and $N$, $P > 0$ and $\epsilon > 0$ such that $NCB + B^TC^TN + \gamma^{-1}H^2 > 0$ and

$$\begin{aligned} 0 \ =\ & {A_\gamma}^T P + PA_\gamma \\ & + [B^TP - H^2C - NCA_\gamma]^T[NCB + B^TC^TN + \gamma^{-1}H^2]^{-1}[B^TP - H^2C - NCA_\gamma] \\ & + \epsilon I, \end{aligned} \tag{5.9}$$

or,

2. if there exist positive definite, diagonal $S(\triangleq H^2)$ and $N$, $P$, and $\epsilon > 0$ such that

$$\begin{bmatrix} -A_\gamma{}^T P - PA_\gamma & -\sqrt{2\gamma}PB + \sqrt{2\gamma}C^T S + \sqrt{2\gamma}A_\gamma{}^T C^T N \\ -\sqrt{2\gamma}B^T P + \sqrt{2\gamma}SC + \sqrt{2\gamma}NCA_\gamma & 2\gamma NCB + 2\gamma NB^T C^T + 2S - 2\epsilon I \end{bmatrix} \geq 0, \tag{5.10}$$

then the negative feedback interconnection of $G(s)$ and $\Delta$ is asymptotically stable for all $\Delta \in \boldsymbol{\Delta}_\gamma$.

## 5.2. Robust Control Synthesis Using the Popov Multiplier for a Benchmark Problem

To illustrate robust control synthesis with the probability-one homotopy algorithm, we consider the two-mass/spring benchmark system shown in Figure 2 with uncertain stiffness $k$. A control force acts on the body 1 and the position of body 2 is measured, resulting in a noncolocated control problem. This benchmark problem is discussed in detail in Reference 39.

The open-loop plant (for $m_1 = m_2 = 1$) is given by

$$\dot{x}_p(t) = A_p(k)x_p(t) + B_p u(t) + D_1 w(t), \tag{5.11}$$

$$y(t) = C_p x_p(t) + D_2 w(t), \tag{5.12}$$

$$z(t) = E_1 x_p(t), \tag{5.13}$$

where $z = x_2$ is the performance variable, $y$ is a noise corrupted measurement of $x_2$,

$$x_p^T = [x_1, x_2, x_3, x_4], \quad A_p(k) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -k & k & 0 & 0 \\ k & -k & 0 & 0 \end{bmatrix}, \quad B_p = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix},$$

$$D_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad C_p = E_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 0 & 1 \end{bmatrix}.$$

We desire to design a constant gain linear feedback compensator of the form

$$\dot{x}_c(t) = A_c x_c(t) + B_c y(t), \tag{5.14}$$

$$u(t) = -C_c x_c(t), \tag{5.15}$$

such that the closed-loop system is stable for $0.5 < k < 2.0$ and for a unit impulse disturbance at $t = 0$, the performance variable $z$ has a settling time of about 15 s for the nominal system (with $k = k_{\rm nom} = 1$).

We begin by constructing the uncertainty feedback system as shown in Figure 1. It is assumed that $k = k_{\text{nom}} + \Delta k$. The perturbation in $A_p(k)$ due to a change $\Delta k$ in the stiffness element $k$ from the nominal value $k_{\text{nom}}$ is given by

$$A_p(k) - A_p(k_{\text{nom}}) \triangleq \Delta A_p = -B_o \Delta k C_o, \tag{5.16}$$

where $B_o = \begin{bmatrix} 0 & 0 & 1 & -1 \end{bmatrix}^T$, and $C_o = \begin{bmatrix} 1 & -1 & 0 & 0 \end{bmatrix}$. Assuming negative feedback, the closed-loop state matrix is given by

$$A(k) = \begin{bmatrix} A_p(k) & -B_p C_c \\ B_c C_p & A_c \end{bmatrix}. \tag{5.17}$$

Next, define $B = \begin{bmatrix} B_o & 0_{4\times 1} \end{bmatrix}$, and $C = \begin{bmatrix} C & 0_{1\times 4} \end{bmatrix}$ so that $G(s)$ in Figure 1 is given by $G(s) \sim \left[\begin{array}{c|c} A(k_{\text{nom}}) & B \\ \hline C & 0 \end{array}\right]$ and $\Delta = \Delta k$.

The upper bound on the $H_2$ cost functional discussed in Remark 7 is given by

$$\frac{\beta}{\epsilon}\text{tr}[(P + 2\gamma C^T N C)V], \tag{5.18}$$

where $V$ is given by

$$V = \begin{bmatrix} V_1 & V_{12} {B_c}^T \\ B_c V_{12}^T & B_c V_2 B_c^T \end{bmatrix}, \tag{5.19}$$

with $V_1 = D_1 {D_1}^T$, $V_2 = \rho D_2 {D_2}^T$, and $V_{12} = \sqrt{\rho}\, D_1 {D_2}^T$. The Riccati equation in (5.9) is modified slightly in that $\epsilon I$ is replaced by $\epsilon R$ where $R$ is given by

$$R = \begin{bmatrix} R_1 & -R_{12} C_c \\ -C_c^T R_{12}^T & C_c^T R_2 C_c \end{bmatrix}, \tag{5.20}$$

where $R_1 = E_1^T E_1$, $R_2 = \rho$, and $R_{12} = \sqrt{\rho}\, E_1^T$. Here the parameter $\rho$ is used as a design parameter to increase or decrease the authority of the controller.

It can be seen that the diagonal $H$ and $N$ of the Popov multiplier, reduce to scalars for this particular example. The variable $x$ in (4.2) is given by

$$x = \begin{bmatrix} H & N & \epsilon & \text{vec}^T(A_c) & \text{vec}^T(B_c) & \text{vec}^T(C_c) \end{bmatrix}^T, \tag{5.21}$$

and consequently, the function $F$ in (4.3), is given by

$$F(x,\gamma) = \begin{bmatrix} \frac{\partial \mathcal{L}}{\partial H} & \frac{\partial \mathcal{L}}{\partial N} & \frac{\partial \mathcal{L}}{\partial \epsilon} & \text{vec}^T\left(\frac{\partial \mathcal{L}}{\partial A_c}\right) & \text{vec}^T\left(\frac{\partial \mathcal{L}}{\partial B_c}\right) & \text{vec}^T\left(\frac{\partial \mathcal{L}}{\partial C_c}\right) \end{bmatrix}^T. \tag{5.22}$$

The initial point $x_0$ is chosen in the following manner. $H_0$, $N_0$, and $\epsilon_0$, are chosen arbitrarily as 10, 10, and 1. The initial controller $(A_{c,0}, B_{c,0}, C_{c,0})$ is an LQG controller for the nominal plant corresponding to $\rho = 0.001$. No robustness is expected of this controller and hence $\gamma_0$ in (4.4) is chosen close to zero ($\gamma_0 = 0.01$).

The controller transfer function obtained by the probability-one homotopy algorithm in subsection 4.1 is given by

$$H(s) = \frac{2819\,(s + 0.2079)[(s - 0.0978)^2 + 0.8063^2]}{[(s + 4.004)^2 + 1.8294^2][(s + 3.4747)^2 + 9.9745^2]}. \tag{5.23}$$

This controller is guaranteed by the theory to be robust for the range $0.5 < k < 2.0$ and this was also verified by a direct search. The settling time for the system was chosen to be the time required for the displacement of mass 2 to reach and stay within the interval [-0.1m, 0.1m]. The controller is seen to satisfy the settling time objective when connected to the nominal model corresponding to $k = 1$ N/m, as can be seen from the impulse response in Figure 3.

## 6. Conclusions

This paper has demonstrated that fixed-architecture, robust control design using general fixed-structure multipliers can be formulated as a Riccati equation feasibility problem; a nonlinear, algebraic feasibility problem. Probability-one homotopy algorithms were proposed to solve this feasibility problem. These algorithms differ from previously developed continuation algorithms, developed exclusively for the case of the Popov multiplier, in that they can be initialized with any admissible multiplier and stabilizing compensator. Like other probability-one homotopy algorithms they also tend to be better conditioned than the alternative continuation algorithms. The results were specialized to the special case of Popov multipliers and fixed-architecture robust control design was illustrated using a standard benchmark problem.

The key step in the practical implementation of these homotopy algorithms is the numerical implementation of the computation of the Jacobian of the homotopy map. Analytical expressions for the Jacobian tend to be complex and difficult to derive due to this complexity. For the practical implementation of homotopy algorithms for more general forms of the multipliers it is important to develop reliable symbolic matrix differentiation routines. The special matrix structures that arise in the computation of the Jacobian must also be exploited to speed up the algorithms. An illustration of this is given in Reference 40.

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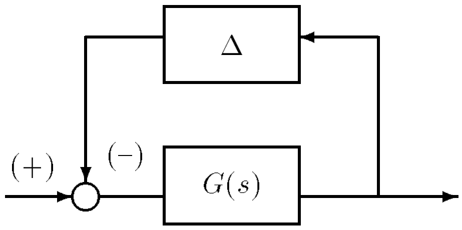


**Figure 1**: Standard Uncertainty Feedback Configuration

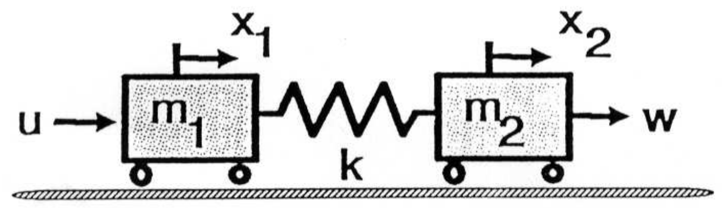


**Figure 2**: Two Mass/Spring System

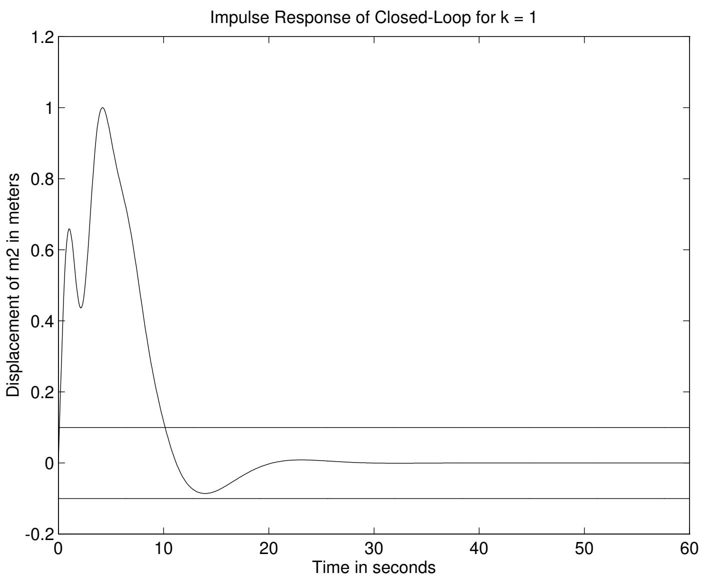


**Figure 3**: Impulse Response of Closed-Loop System

**Figure Captions**

1. Standard Uncertainty Feedback Configuration
2. Two Mass/Spring System
3. Impulse Response of Closed-Loop System