A SIMILARITY SOLUTION FOR PLANAR ATMOSPHERIC REENTRY WITH AERODYNAMIC FORCE MODULATION

by

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LIST OF SYMBOLS

$\tilde{A}$ = reference area of the vehicle

a_i = constants used in drag modulation function

c_j = constants of integration

C_D = drag coefficient

C_{D_o} = profile drag coefficient

C_L = lift coefficient

F = nondimensional density based on $\left(\frac{k_1 \tilde{A}}{m} \right)_s$

H = non-dimensional altitude measured from initial radius = $\frac{\tilde{h}_o - \tilde{h}}{\varepsilon^2 \tilde{r}_o}$

H' = non-dimensional altitude measured from planet radius = $\frac{\tilde{h}_o - \tilde{h}}{\varepsilon^2 r_e}$

$\tilde{h}$ = altitude above earth or planet surface = $\tilde{r} - \tilde{r}_e$

K = constant in drag polar equation

k_o = profile drag coefficient at initial conditions

k_1 = reference drag coefficient of the vehicle (in this analysis, $k_1 = 1$, for convenience)

k_2 = lift coefficient at initial conditions (in this solution, $\lambda = 1$; hence $k_2 = C_L$)

M = Flight Mach number

M_1 = Special modulation function [equation (116-d)]

$\tilde{m}$ = mass of the vehicle

O( ) = order of magnitude symbol

$\tilde{r}$ = radius from the center of the earth or planet to the vehicle

$\tilde{r}_o$ = initial radius from the center of the earth or planet to the vehicle

$\tilde{r}_e$ = radius of earth or planet

r = non-dimensional radius = $\tilde{r}/\tilde{r}_o$

$\tilde{t}$ = time

t = nondimensional time = $\dfrac{\tilde{v}_{co}\tilde{t}}{\tilde{r}_o}$

$\tilde{v}$ = speed of vehicle

v = nondimensional speed of vehicle = $\tilde{v}/\tilde{v}_{co}$

$\tilde{v}_a$ = speed of sound

$\tilde{v}_{co}$ = circular orbit speed = $(\dfrac{\mu}{\tilde{r}_o})^{1/2}$

w = vehicle weight

X_1 and X_2 = composite expressions defined by equations (115-b) and (115-c)

x = nondimensional speed squared (in basic differential equations) = v^2

Y = $\ln \cos\Theta$

Z = $\ln x$

Z_2 = order of magnitude transformed speed function = $O(1)$ for aerodominated flight regime

α = exponent for Broglio's density Law; ($\simeq \dfrac{\tilde{v}_{co}^2}{\tilde{v}_a^2} \simeq 900$ for earth atmosphere)

ε = perturbation parameter $(1/\alpha)^{1/2}$ ($\varepsilon \simeq 1/30$ for earth atmosphere)

η = nondimensional drag modulation function [see equation (2-a) and Chapter III]

η_0 = nondimensional profile drag modulation function [see equation (11)]

Θ = flight path angle (measured positive downward from the horizontal)

Θ_2 = flight path angle for the aerodominated flight regime.

λ = nondimensional lift modulation function [see equation (2-b) and Chapter III]

μ = universal gravity constant times the mass of the earth (or planet considered)

$\tilde{\rho}$ = atmospheric density

ρ = nondimensional density = $\dfrac{k_1 \tilde{A} \tilde{r}_o \tilde{\rho}}{\tilde{m}}$

ϕ = range angle (measured from the initial position).

Φ_k = special integration functions, k = 1, 2, 3. [see equations (116-a), (116-b) and (118)]

ω = $\varepsilon^2 \dfrac{(1+a_1)\rho}{\sin\Theta_o}$, see equation (116-c)

Subscripts:

o = identifies modulation constants, i = 1,2

i = identifies modulation constants, i = 1,2

j = identifies constants of integration, j = 1,2,...

k = identifies special integration functions, k = 1,2,3

c = identifies composite solution

s = initial conditions after drag step function

Superscripts:

(ℓ) = identifies perturbation order, ℓ = 0, 1, 2, ...

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I. INTRODUCTION

During the past ten years there have been numerous articles published concerning approximate solutions for reentry trajectories. These early papers have considered the nonsimilar-type of solutions for both lifting and nonlifting vehicles assuming constant aerodynamic coefficients. In addition, there have been some specialized solutions developed for variable aerodynamic forces assuming conditions such as constant flight path angle, constant acceleration, et cetera. Often, however, these previous solutions could not be matched with a Keplerian flight regime.

More recently, Shen[1] has developed a series solution which is valid for the initial altitude at reentry conditions but, more significantly, he has used aerodynamic coefficients expressed as functions of altitude and speed. Although his solutions match with the Keplerian flight regime, some other solution type must be used for the lower altitude flight regimes; i. e., for those altitudes where the aerodynamic forces dominate.

Just prior to that time Broglio[2] developed a set of governing equations of motion which were expressed in similarity form and were written in terms of arbitrary lift and drag modulation functions. Unfortunately, only a small amount of analytical work has been done using his equations. Of significance, however, was the fact that he showed how the usual isothermal approximation for atmospheric density could be approximated by a power function of altitude rather than the usual exponential function. Furthermore, this power function admits the formulation of a similarity solution, whereas the exponential function requires a solution that depends (though weakly) on certain of the body parameters.

In a recent paper (1967), Willes, et al.[3], applied the method of matched asymptotic expansions to the reentry problem. In this solution, the investigators studied the regimes of applicability and the limits imposed on such well-known solutions as those of Eggers[4], Chapman[5], Lees, et al.[6], Shen[1], Loh[7], and others. Furthermore, they developed a composite solution, applicable for constant aerodynamic force coefficients, which was uniformly valid over three distinct flight regimes--namely, the Keplerian, the initial, and the skip-trajectory. Their solution gives exceptional agreement with the numerically integrated solutions presented in that paper.

The original intent in this thesis was to develop the governing equations of motion, in similarity form, for variable lift and drag; and then to solve these equations by the method of matched asymptotic expansions. It was found, however, that any non-trivial order of magnitude transformation of the variables in these governing equations only reproduced the same terms (and generally in a degenerate form) that could be obtained by a straight-forward perturbation solution of the initial equations. Unfortunately, the non-trivial order of magnitude transformations yielded either coupled simultaneous differential equations, which could not be solved in closed form, or solutions which could not be matched uniformly. However, it was found that by writing the governing equations in a different form than that used by Willes[3], that a straight-forward perturbation solution - from the initial conditions - yielded a solution not unlike that which would be obtained by using Picard's iteration method. In fact, the first few terms in the assumed expansion can be shown, by inspection, to agree with Picard's method. It is implied, therefore, that the solution which follows approaches the exact solution for non-skip trajectories.

II. THE DYNAMIC EQUATIONS FOR PLANAR FLIGHT

The basic equations of motion for non-thrusting planar flight in a non-rotating atmosphere, about a spherical earth, or planet, can be written as follows (see figure 1):

$$\frac{d\tilde{v}}{d\tilde{t}} = -\frac{C_D}{2\tilde{m}}\tilde{\rho}\tilde{v}^2\tilde{A} + \frac{\mu}{\tilde{r}^2}\sin\theta, \tag{1-a}$$

$$-\tilde{v}\frac{d\theta}{d\tilde{t}} = \frac{C_L}{2\tilde{m}}\tilde{\rho}\tilde{v}^2\tilde{A} - \frac{\mu}{\tilde{r}^2}\cos\theta\left(1 - \frac{\tilde{r}\tilde{v}^2}{\mu}\right); \tag{1-b}$$

also, from kinematics,

$$\frac{d\phi}{d\tilde{t}} = \frac{\tilde{v}}{\tilde{r}}\cos\theta, \tag{1-c}$$

and

$$\frac{d\tilde{r}}{d\tilde{t}} = \tilde{v}\sin\theta. \tag{1-d}$$

Let

$$C_D \overset{\Delta}{=} k_1\eta, \tag{2-a}$$

and

$$C_L \overset{\Delta}{=} k_2\lambda, \tag{2-b}$$

where k_1 and k_2 are constants to be specified later; then η is the nondimensional drag modulation function, and λ is the nondimensional lift modulation function.

Now, if the following dimensionless quantities are chosen:

$$r = \frac{\tilde{r}}{\tilde{r}_o}, \quad t = \frac{\tilde{v}_{co}\tilde{t}}{\tilde{r}_o}, \quad v = \frac{\tilde{v}}{\tilde{v}_{co}}, \quad \rho = \frac{k_1\tilde{A}\tilde{r}_o\tilde{\rho}}{\tilde{m}}, \tag{2-c}$$

where

$$\tilde{v}_{co} = \left(\frac{\mu}{\tilde{r}_o}\right)^{\frac{1}{2}},$$

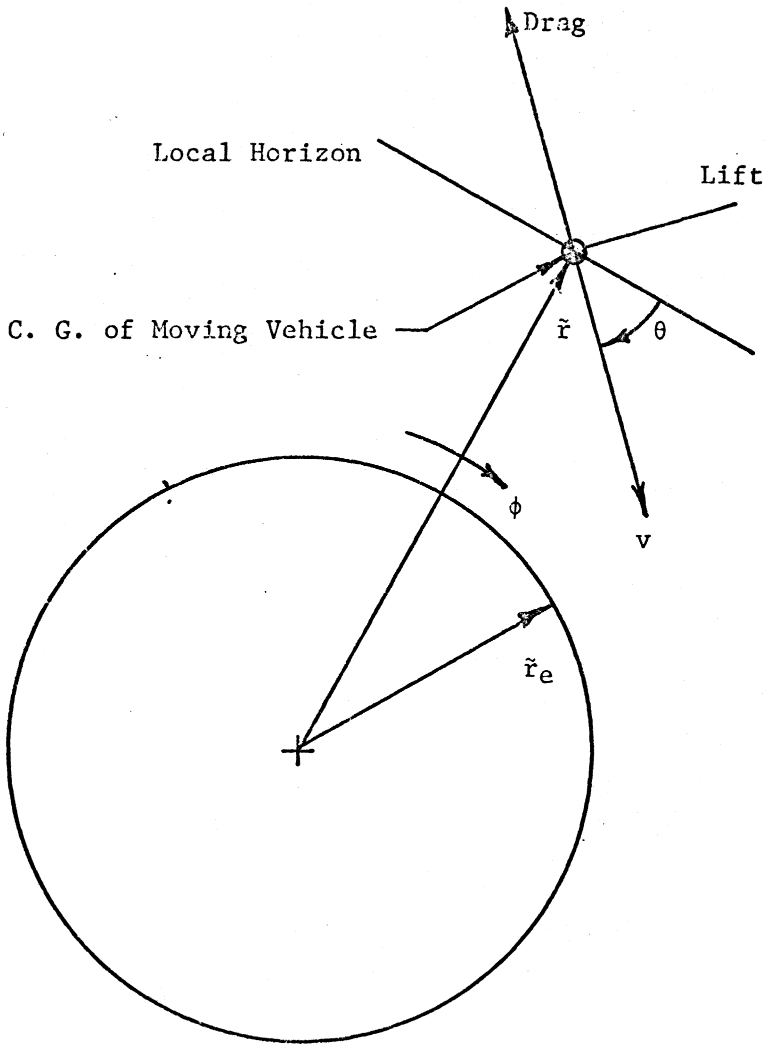

Figure 1. An Inertial Coordinate System showing a particle in a reentry situation.

then equations (1-a) through (1-d) become the following nondimensional expressions:

$$\frac{dv}{dt} = -\frac{\eta \rho v^2}{2} + \frac{\sin\theta}{r^2}, \tag{3-a}$$

$$-v\frac{d\theta}{dt} = \lambda \frac{k_2}{k_1} \rho \frac{v^2}{2} - \frac{\cos\theta}{r^2}(1 - rv^2); \tag{3-b}$$

with
$$\frac{d\phi}{dt} = \frac{v\cos\theta}{r}, \tag{3-c}$$

and
$$\frac{dr}{dt} = -v\sin\theta. \tag{3-d}$$

Assuming an isothermal atmosphere, then the atmospheric density can be expressed by Broglio's power law[2] as:

$$\tilde{\rho} = \tilde{\rho}_o \left(\frac{\tilde{r}_o}{\tilde{r}}\right)^\alpha, \tag{4-a}$$

or
$$\rho = \rho_o \left(\frac{r_o}{r}\right)^\alpha = \rho_o \left(\frac{1}{r}\right)^\alpha, \tag{4-b}$$

since
$$r_o = 1;$$

wherein
$$\alpha = \frac{\tilde{v}_{co}^2}{\tilde{v}_a} = M_{co}^2, \tag{5}$$

and $\tilde{v}_a$ is the speed of sound for the atmosphere, while M_{co} is the Mach number corresponding to circular orbit speed (at $\tilde{r} = \tilde{r}_o$).

Since the time, t, appears only in the various ordered derivatives, it is convenient to transform to a new independent variable, ρ, by means of the following relations:

writing
$$\frac{d}{dt} = \frac{d\rho}{dr}\frac{dr}{dt}\frac{d}{d\rho}, \tag{6-a}$$

where, from equation (3-d)

$$\frac{dr}{dt} = -v\sin\theta,$$

and from equation (4-b)

$$\frac{d\rho}{dr} = - \frac{\alpha\rho}{r} ;$$

then the transformation equation becomes,

$$\frac{d}{dt} = \frac{\alpha\rho}{r} v \sin\Theta \frac{d}{d\rho} . \tag{6-b}$$

Next transforming equations (3), one obtains:

$$\frac{dv^2}{d\rho} = - \varepsilon^2 \frac{\eta v^2 r}{\sin\Theta} + \varepsilon^2 \frac{2}{r\rho} , \tag{7-a}$$

$$\frac{d \cos\Theta}{d\rho} = \varepsilon^2 \frac{\lambda k_2 r}{2k_1} - \varepsilon^2 \frac{\cos\Theta}{\rho}\left(\frac{1}{rv^2} - 1\right) ; \tag{7-b}$$

with $$\frac{d\phi}{d\rho} = \varepsilon^2 \frac{\cos\Theta}{\rho \sin\Theta} . \tag{7-c}$$

Also, from equation (6-b),

$$\frac{dt}{d\rho} = \varepsilon^2 \frac{r}{\rho v \sin\Theta} , \tag{7-d}$$

wherein $\varepsilon^2 = 1/\alpha.$

Equation (4-b) gives the required relation between r and ρ. For convenience, set $\rho_0 = 1$; then, from equation (4-b),

$$r = e^{-\varepsilon^2 \ln\rho} \tag{8-a}$$

which can be expanded to yield a series function in increasing powers of ε as follows:

$$r = 1 - \varepsilon^2 \ln\rho + \frac{\varepsilon^4 (\ln\rho)^2}{2!} - \frac{\varepsilon^6 (\ln\rho)^3}{3!} + \ldots . \tag{8-b}$$

Now, noting that v appears only as a squared term in equations (7-a) and (7-b), let

$$v^2 = x. \tag{9}$$

Next, using equation (8-b) to eliminate r, then equations (7-a) and (7-b) become, respectively:

$$\frac{dx}{d\rho} = - \varepsilon^2 \; \frac{\eta x}{\sin \Theta} \; (1 - \varepsilon^2 \ln\rho + \ldots) + \varepsilon^2 \; \frac{2}{\rho} \; (1 + \varepsilon^2 \ln\rho + \ldots), \tag{10-a}$$

and

$$\frac{d \cos \theta}{d\rho} = \varepsilon^2 \; \frac{\lambda k_2}{2k_1} (1 - \varepsilon^2 \ln\rho + \ldots) - \varepsilon^2 \; \frac{\cos \Theta}{\rho} [\frac{1}{x} \; (1 + \varepsilon^2 \ln\rho + \ldots) - 1], \tag{10-b}$$

with the initial conditions being

$$r_0 = 1, \; \rho_0 = 1, \; x(1) = x_0 = v_0^2 \; , \; \Theta(1) = \Theta_0, \; \phi(1) = 0, \; t(1) = 0. \tag{10-c}$$

Note: Equations (7-c) and (7-d) can be solved using results obtained from the solutuions to equations (10-a) through (10-c).

III. ASSUMED AERODYNAMIC FORCE LAWS

In principle, any arbitrary general drag and lift laws may be employed in equations (10-a) and (10-b). For this problem, however, only very small lift coefficients will be considered; thus, the drag may be considered as independent of the lift. This will be discussed in the following paragraphs.

3.1 Discussion of the Assumed Drag Law

The drag law used in this analysis (and often found in the literature) is a relatively simple parabolic lift-drag polar which, in coefficient form, can be expressed as

$$C_D = C_{Do} + K\, C_L^2, \tag{11-a}$$

wherein C_{Do} is the profile drag coefficient (taken at zero lift); and,

K is a constant relating the contribution of the lift to the drag.

Using the notation in equations (2-a) and (2-b), this lift-drag relation can be rewritten as

$$k_1 \eta = k_o \eta_o + K\, k_2^2\, \lambda^2 , \tag{11-b}$$

wherein

$$C_{Do} \overset{\Delta}{=} k_o \eta_o , \tag{11-c}$$

and where k_o is the reference profile-drag coefficient,

while η_o is the profile-drag modulation function.

Also, here,

k_1 is the reference drag coefficient,

and k_2 is the reference lift coefficient.

The modulation functions can be thought of, respectively, as:

η, an arbitrary drag modulation function depending on η_0 and λ;

η_0, an arbitrary function which represents the drag changes which depend on non-lifting configuration changes;

and λ, an arbitrary function which represents the vehicle attitude and configuration changes affecting the lift.

Without loss of generality, it is convenient to set $k_1 = 1$; thus equation (11-b) becomes

$$\eta = k_0\eta_0 + K\,k_2^2\lambda^2 \; . \qquad (12\text{-a})$$

Now if the reference condition for the aerodynamic force coefficients is chosen to coincide with the initial conditions at an altitude prior to the appearance of a significant density; i.e., $\rho = O(\varepsilon^2)$, and if the modulation functions for small density are chosen such that

$$\eta(\varepsilon^2) = \eta_0(\varepsilon^2) = \lambda(\varepsilon^2) = 1, \qquad (12\text{-b})$$

the relationship between the vehicle parameters is fixed and from equation (12-a) is found to be

$$1 = k_0 + K\,k_2^2 \; . \qquad (12\text{-c})$$

Equations (11) and (12) represent a rather general parabolic drag law in a form corresponding to that which is usually found in the literature. However, if very small lift coefficients are considered, it is apparent that the contribution of the lift to the drag will be quite small. In fact it will be shown, in the solution which follows, that in order to generate a solution for non-skip trajectories one must assume the lift coefficient is of $0(\varepsilon^2)$, or at most of $0(\varepsilon)$. From equation (12-c), then, the contribution of the lift to the drag for this case is small, and, at most,

$$k_0 = 1 - 0(\varepsilon^2) \ . \qquad (13\text{-a})$$

It is reasonable, therefore, to neglect the contribution of lift to drag and set

$$k_0 = 1; \qquad (13\text{-b})$$

thus, from equation (12-a),

$$\eta = \eta_0 \ . \qquad (13\text{-c})$$

3.2 Assumed Modulation Functions

In general the modulation functions can be arbitrary in that they may include the effects of velocity, flight path, and altitude changes. As one might expect, however, the inclusion of anything other than the simplest terms greatly complicates the overall analysis. Furthermore, if the contribution of lift to drag is included, products of these modulation terms will appear in the lift component of the drag term. Because of the rather severe algebraic complications imposed by including numerous terms in a general solution, this analysis will be restricted to a relatively simple but bounded function of altitude only. Several other possible terms are discussed which, along with other expressions that the reader may wish to consider, can - in principle - be handled in the same way as the terms used in this thesis.

The following expression, chosen for this work, it bounded and yields a smoothly varying lift or drag modulation law which should be satisfactory for preliminary design studies.

The basic modulation function, written as the profile-drag modulation function, is

$$\eta_o = 1 + \frac{\varepsilon a_1 \rho}{a_2 + \varepsilon \rho} ; \qquad (14)$$

wherein the constants a_1 and a_2, both or order unity, may be arbitrarily chosen.

A graphical representation of equation (14) is given in Figure 2.

3.3 Other Possible Modulation Functions

Numerous other modulation functions which could be applied in this analysis may be visualized. It is felt, however, that due to the complexity brought about through the addition of terms to equation (14), any added quantities would be warranted only in the case of a special design study. Even though these terms are not included in this solution, several of them are noted in the following paragraphs:

In the event that slender bodies are to be studied, then the aerodynamic force coefficients can be modified using the unified hypersonic-supersonic similarity law as noted below. Mach number dependency can be included through a multiplication of the modulation terms, in equation (14), by the factor

$$\frac{M_o}{(M^2 - 1)^{1/2}} .$$

This can be written in terms of the nondimensional variables of this analysis as

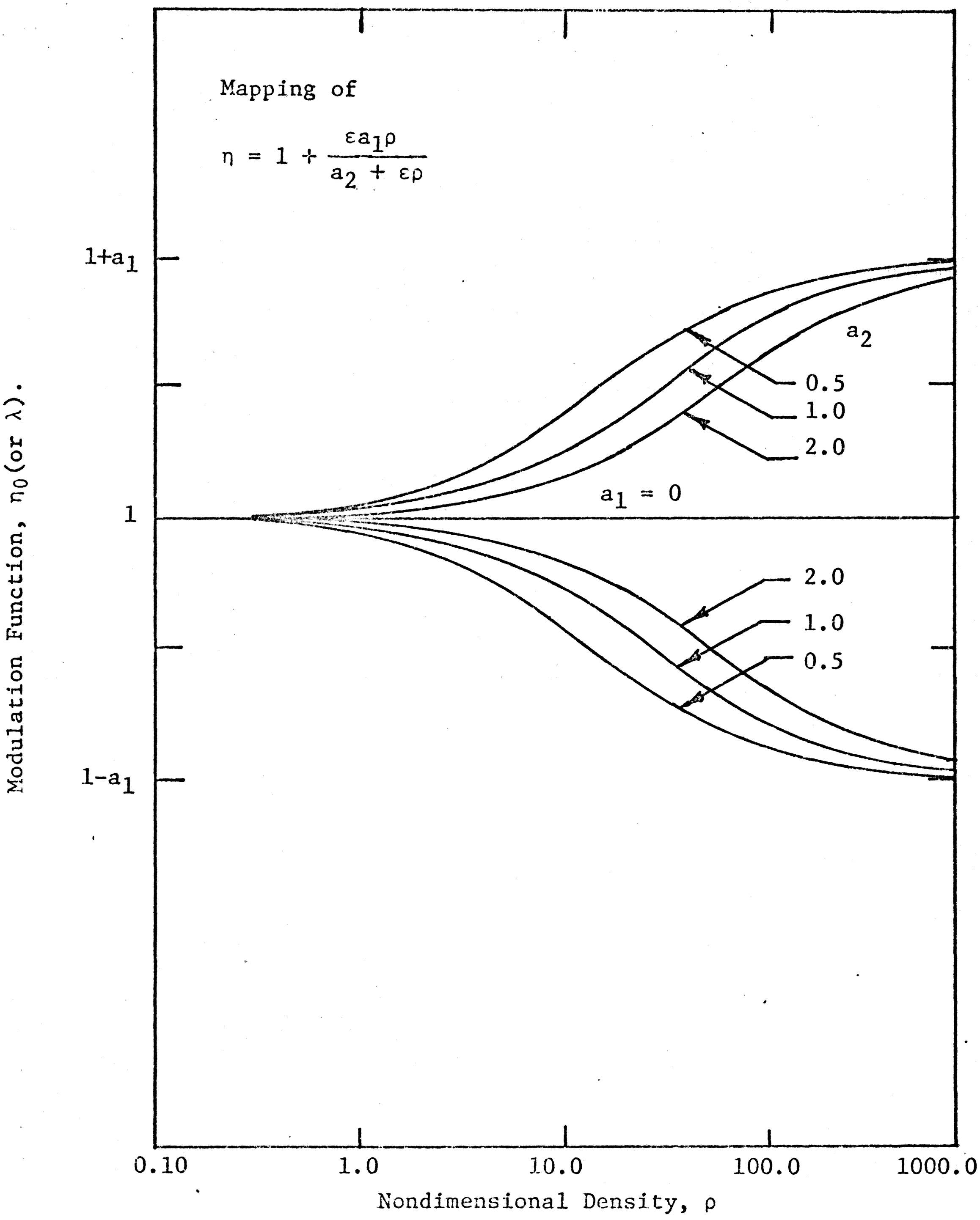

Figure 2. Schematic Depicting Modulation Law.

$$\frac{M_0}{(M^2 - 1)^{1/2}} = \frac{\varepsilon v_0}{(x - \varepsilon^2)^{1/2}} . \qquad (15\text{-a})$$

As a consequence the modulation function would become:

$$\eta_0 \ (x,\rho) = \frac{\varepsilon v_0}{(x - \varepsilon^2)^{1/2}} \eta_o(\rho), \qquad (15\text{-b})$$

where η_0 (ρ) is the modulation function given by equation (14).

Other velocity dependent terms can be considered also; for example, suppose that one assumes

$$\eta_0(x) = a_0 + \varepsilon a_1 x + \varepsilon^2 a_2 x + \ldots \ , \qquad (15\text{-c})$$

where the coefficients may be chosen to fit a given velocity-dependent drag-law.

Another possibility would be to consider a drag law which would be dependent on dynamic pressure. For example:

let $$\eta_o = \frac{a_1}{a_2 + \rho x} \ ; \qquad (15\text{-e})$$

this is a bounded function which yields a decrease in the drag coefficient at maximum dynamic pressure (this coincides approximately with the maximum deceleration).

IV. GENERAL SOLUTION

Originally, the basic technique that was to be used for this solution consisted of a straight-forward perturbation scheme which was to yield asymptotic solutions for any given flight regime. In this sense a regime would be defined wherein all of the variables in the governing equations, and/or the resulting equations obtained by order of magnitude transformations, would be considered as being of order unity. Thus, the resulting asymptotic expansions would be formed into a composite expansion which should be uniformly valid over several regimes. In the course of this study it was found that no nontrivial order of magnitude transformed equations could be (both) solved and uniformly matched to an original regime. It was found, however, that if equations (10) were rearranged into a higher order function of the dependent variables, then a straight-forward perturbation solution would yield (at least for the first few terms) the same results as could be obtained by Picard's iteration method. This implied that the n^{th} order solution in this analysis is equivalent to the significant terms of the n^{th} solution that could be obtained by Picard's method. Furthermore, this implied that in this case the perturbation solution approaches the exact solution for all flight regimes. As will be discussed in the following sections, however, many terms of the assumed expansions are required in order to obtain all of the significant terms when the density, ρ , becomes large.

4.1 Simplification of Differential Equations

The basic differential equations of motion have been developed in Section II. In these expressions the nondimensional radius has been retained; however, in order to reduce the problem to one independent variable the radius was expanded into a power series in $\varepsilon^2 \ln\rho$; e.g., see equation (8-b). Clearly, for the values of ρ expected over the range of reentry it would be possible to write

$$r \cong 1 + 0(\varepsilon^2) \; ; \qquad (16)$$

since it is expected that

$$0(\varepsilon^2) \leq \rho \leq 0(1/\varepsilon^2); \qquad (17)$$

and for this range,

$$\ln\rho = 0(1) , \qquad (18\text{-a})$$

and hence

$$\varepsilon^2 \ln\rho << 1$$

Thus without any significant loss in accuracy, equations (10-a) and (10-b) become (approximately)

$$\frac{dx}{d\rho} = - \varepsilon^2 \frac{\eta x}{\sin\Theta} + \varepsilon^2 \frac{2}{\rho} , \qquad (19\text{-a})$$

and

$$\frac{d\cos\Theta}{d\rho} = \varepsilon^2 \frac{k_2\lambda}{2} - \varepsilon^2 \frac{\cos\Theta}{\rho}\left(\frac{1}{x} - 1 \right) , \qquad (19\text{-b})$$

In addition, equations (7-c) and (7-d) become

$$\frac{d\phi}{d\rho} = \varepsilon^2 \frac{\cos\Theta}{\rho \sin\Theta} \tag{19-c}$$

and

$$\frac{dt}{d\rho} = \varepsilon^2 \frac{1}{\rho v \sin\Theta} , \tag{19-d}$$

with the initial conditions being, now,

$$\begin{aligned} \rho_0 &= 1 , \\ x(1) &= v_0^2 , \\ \Theta(1) &= \Theta_0 , \\ \text{and} \qquad \phi(1) &= t(1) = 0 . \end{aligned} \tag{19-e}$$

4.2 General Technique

In general, if the governing differential equations can be written in terms of a small parameter, ε, then a solution can be generated by assuming a form, which for this case can be written as,

$$x(\rho;\varepsilon) = E(\varepsilon)^{(0)} x(\rho)^{(0)} + E(\varepsilon)^{(1)} x(\rho)^{(1)} + \dots ; \tag{20-a}$$

and, in addition,

$$\cos\Theta(\rho;\varepsilon) = E(\varepsilon)^{(0)} \cos\Theta(\rho)^{(0)} + E(\varepsilon)^{(1)} \cos\Theta(\rho)^{(1)} + \dots ; \tag{20-b}$$

wherein the superscripts represent the order of magnitude solutions obtained by substituting the assumed expansions into the differential equations and equating coefficients of like terms in $E(\varepsilon)^{(n)}$. In this generalized notation the proper values of $E(\varepsilon)^{(n)}$ are determined by requiring that the ordered differential equations, which are obtained by the procedure above, are in the least degenerate form for each order of magnitude of the perturbation. One of the disadvantages (and probably the most significant one) of this technique is that often the higher order terms appear to be more significant for large changes in the independent variable than those of lower order. One method for correcting this disadvantage, as given by Van Dyke,[8] is to transform the variables by an order of magnitude transformations to one or more of the other regimes, and then to generate asymptotic solutions within these new regimes. If these several asymptotic solutions are matched in their limits, then these solutions can be used to form a composite expansion. This last result is generally more accurate, over all regimes, than the individual expansions are over their own respective regimes. Furthermore, this technique generates a composite solution which is uniformly valid over the several regimes, with only a few terms of the expansion being required in any one regime.

This is the basic technique which was used by Willes, et al.[3] However, in their formulation of the governing equations the authors obtained a singular solution, in the initial variables, which yielded the Keplerian solution but which could not be used to indicate aerodynamic effects. Thus, they were forced to use order of magnitude transformations to obtain these latter effects. In contrast, the present formulation is non-singular in the original variables and, consequently, it can be used to generate a uniformly valid solution in the initial altitude regime; one which yields the Keplerian terms also.

Assuming that a solution, in the form of equations (20-a) and (20-b), has been obtained (this result is straightforward); it is desirable now to transform the variables in the governing equations in such a manner that they emphasize some of the other aspect of the problem. For example, Willes[3] transformed the independent variable in a manner which emphasized the effect of the aerodynamic lift. In terms of the variables used in this thesis, this same effect can be accomplished by setting $\rho = 0(1/\varepsilon^2)$. Therefore let

$$\rho = \rho_2 / \varepsilon^2 , \qquad (21\text{-a})$$

wherein

$$\rho_2 = 0(1) \quad . \qquad (21\text{-b})$$

Now, holding all other variables unchanged equations (19-a) and (19-b) become

$$\frac{dx}{d\rho_2} = - \frac{x}{\sin\Theta} + \varepsilon^2 \frac{2}{\rho_2} , \qquad (21\text{-c})$$

and

$$\frac{d\cos\Theta}{d\rho_2} = \frac{k_2\lambda}{2} - \varepsilon^2 \frac{\cos\Theta}{\rho_2} \left(\frac{1}{x} - 1 \right) . \qquad (21\text{-d})$$

The zeroth order solution for these equations was obtained; it was found to yield a result similar to that obtained by Willes and correspond to the solution which he called a "skip-trajectory". It was found, however, that the zeroth order solution in this analysis (when expanded) generated the same lift-terms as those obtained for the initial regime; i.e., the terms containing k_2. However, Willes' solution does not properly yield the effect of velocity decay; and therefore, is restricted to skip-trajectories in which a pull-up occurs before there is a significant velocity decay:i.e., while $x = 0(1)$ remains satisfied. It follows, then, that this skip-solution is trivial in that nothing new is obtained.

The proper scheme to employ in this density regime would be to transform both the density, ρ, and the dimensionless speed ratio, x. That is, define the situation so that as the density approaches $0(1/\varepsilon^2)$, the speed ratio decays and approaches $0(\varepsilon^2)$. Thus one should properly write

$$\rho = \rho_2/\varepsilon^2 , \tag{22-a}$$

and

$$x = \varepsilon^2 x_2 . \tag{22-b}$$

Using this description the governing equations (19-a) and (19-b) become

$$\frac{dx_2}{d\rho_2} = -\frac{\eta x_2}{\sin\Theta} + \frac{2}{\rho_2} , \tag{23-a}$$

and

$$\frac{d\cos\Theta}{d\rho_2} = \frac{k_2\lambda}{2} - \frac{\cos\Theta}{\rho_2 x_2} + \varepsilon^2 \frac{\cos\Theta}{\rho_2} , \tag{23-b}$$

For this case the zeroth order equations are coupled and cannot be solved by elementary methods.

Had it been assumed that x is of $O(\varepsilon)$ rather than $O(\varepsilon^2)$ the zeroth order equations could have been solved analytically; however, the next order of magnitude solution could not be matched uniformly to the solution expressed in terms of the initial variables.

Similar results were found for all other order of magnitude transformations of the initial variables. That is, the transformed equations either duplicated the initial solution, or led to a degenerate form of the initial solution; or the zeroth order differential equations were found to be coupled and could not be solved directly.

As a result of the rather extensive investigation outlined above, it has become apparent that a straightforward perturbation solution, in terms of the original variables, would yield the best solution possible by this method. Also, it was found, that for best results the original equations should be rewritten in a form which would generate a less degenerate solution. For example, it was found that the perturbation solution of the governing expressions - in the form of equation (19-a) and (19-b) - generated the expanded form of exponential functions. Fortunately a less degenerate solution can be indicated for the problem by means of the following considerations:

If it is assumed that the sine function in equation (19-a) can be written in terms of the density, ρ, i.e., as

$$\sin\Theta = G_1(\rho) ; \tag{24-a}$$

and, similarly, the speed term in equation (19-b) can be written as

$$\left(\frac{1}{x} - 1\right) = G_2(\rho) ; \tag{24-b}$$

then a solution to equations (19-a) and (19-b) can be written as

$$x \exp\left\{\int_1^\rho \frac{\varepsilon^2\eta}{\sin\Theta}\, d\rho\right\} = \int_1^\rho \frac{\varepsilon^2 2}{\rho} \exp\left\{\int_1^\rho \frac{\varepsilon^2\eta\, d\rho}{\sin\Theta}\right\} d\rho + v_o^2 , \tag{25-a}$$

and

$$\cos\Theta \exp\left\{\int_1^\rho \varepsilon^2\left(\frac{1}{x} -1\right) \frac{d\rho}{\rho}\right\} = \int_1^\rho \varepsilon^2 \frac{k_2\lambda}{2} \exp\left\{\int_1^\rho \varepsilon^2\left(\frac{1}{x} -1\right) \frac{d\rho}{\rho}\right\} d\rho + \cos\Theta_o ; \tag{25-b}$$

or,

$$\ln x = \ln\left\{\left\{v_o^2 + 2\varepsilon^2 \int_1^\rho \exp\left\{\varepsilon^2 \int_1^\rho \frac{\varepsilon^2\eta}{\sin\Theta}\, d\rho\right\} \frac{d\rho}{\rho}\right\}\right\} - \varepsilon^2 \int_1^\rho \frac{\eta\, d\rho}{\sin\Theta} , \tag{26-a}$$

and

$$\ln\cos\Theta = \ln\left\{\cos\Theta_o + \varepsilon^2 \frac{k_2}{2} \int_1^\rho \lambda \exp\left[\varepsilon^2 \int_1^\rho \left(\frac{1}{x} -1\right)\frac{d\rho}{\rho}\right] d\rho\right. - \varepsilon^2 \int_1^\rho \left(\frac{1}{x} - 1\right)\frac{d\rho}{\rho} . \tag{26-b}$$

Actually not much can be done with these equations other than to employ some iterative means for an evaluation of the integrals. The form of equations (26-a) and (26-b) suggests, however, that fewer terms of a perturbative solution will be required if the dependent variables in the original governing equations are rewritten in terms of logarithmic functions. This is easily accomplished and yields, from equations (19-a) and (19-b),

$$\frac{dZ}{d\rho} = - \frac{\varepsilon^2\eta}{\sqrt{1-\varepsilon^{2Y}}} + \varepsilon^2 \frac{2e^{-Z}}{\rho} \tag{27-a}$$

and

$$\frac{dY}{d\rho} = \varepsilon^2 \frac{k_2\lambda e^{-Y}}{2} - \frac{\varepsilon^2}{\rho}(e^{-Z} - 1) , \tag{27-b}$$

where

$$Z = \ln x \tag{27-c}$$

and

$$Y = \ln\cos\Theta ; \tag{27-d}$$

with the initial conditions being

$$\begin{aligned} \rho_o &= 1 \\ Z(1) &= \ln(v_o^2) \\ Y(1) &= \cos\Theta_o . \end{aligned} \tag{27-e}$$

The expressions for the equations for range angle, ϕ, and the time, t, obtained from equations (19-c) and (19-d), become

$$\frac{d\phi}{d\rho} = \frac{\varepsilon^2 e^Y}{\rho\sqrt{1-e^{2Y}}} \tag{28-a}$$

and

$$\frac{dt}{d\rho} = \frac{\varepsilon^2 e^{-\frac{Z}{2}}}{\rho\sqrt{1-e^{2Y}}} \quad , \qquad (29\text{-b})$$

respectively; with the corresponding initial conditions being

$$\phi(1) = t(1) = 0 \ . \qquad (29\text{-c})$$

It will be assumed now that a uniformly valid solution of these equations can be obtained by means of the following series expansions:

$$Z(\rho;\varepsilon) = Z(\rho)^{(0)} + \varepsilon^2 Z(\rho)^{(1)} + \varepsilon^4 Z(\rho)^{(2)} + \ldots \quad , \qquad (30\text{-a})$$

$$Y(\rho;\varepsilon) = Y(\rho)^{(0)} + \varepsilon^2 Y(\rho)^{(1)} + \varepsilon^4 Y(\rho)^{(2)} + \ldots \quad , \qquad (30\text{-b})$$

$$\phi(\rho;\varepsilon) = \phi(\rho)^{(0)} + \varepsilon^2 \phi(\rho)^{(1)} + \varepsilon^4 \phi(\rho)^{(2)} + \ldots \quad , \qquad (30\text{-c})$$

and

$$t(\rho;\varepsilon) = t(\rho)^{(0)} + \varepsilon^2 t(\rho)^{(1)} + \varepsilon^4 t(\rho)^{(2)} + \ldots \qquad (30\text{-d})$$

where, from the initial conditions,

$$\left.\begin{aligned} Z(1)^{(0)} &= \ln v_o^2 \ , & Z(1)^{(n)} &= 0 \ , \\ Y(1)^{(0)} &= \ln \cos\Theta_0 \ , & Y(1)^{(n)} &= 0 \ , \\ \phi(1)^{(0)} &= 0 \ , & \phi(1)^{(n)} &= 0 \ , \\ \text{and} \\ t(1)^{(0)} &= 0 \ , & t(1)^{(n)} &= 0 \ . \end{aligned}\right\} n > 1 \qquad (30\text{-e})$$

The general technique is to substitute the assumed expansions into the governing equations and generate a sequence of ordered differential equations by equating coefficients of powers of ε. Clearly, then, in this case, the zeroth order equations and the initial conditions yield

$$\begin{aligned} Z(\rho)^{(0)} &= \ln v_o^2 \quad , \\ Y(\rho)^{(0)} &= \ln \cos\Theta_o \quad , \\ \phi(\rho)^{(0)} &= 0 \quad , \end{aligned} \tag{31}$$

and

$$t(\rho)^{(0)} = 0 \quad .$$

Before proceeding with the expansion of the governing equations, the reciprocal of the sine function which appears in equation (27-a), will be considered as follows:

The function of interest here is:

$$\frac{1}{\sin\Theta} = \frac{1}{\sqrt{1-e^{2Y}}} \quad . \tag{32-a}$$

Now, using the zeroth term, equation (31), and equation (30-b), then

$$Y = \ln\cos\Theta_0 + \varepsilon^2 Y^{(1)} + \varepsilon^4 Y^{(2)} + \ldots \tag{32-b}$$

Next, substituting this expansion into equation (32-a) gives

$$\frac{1}{\sin\Theta} = \frac{1}{\sqrt{1-\cos^2\Theta e^{\psi}}} \quad , \tag{33-a}$$

wherein

$$\psi = 2[\varepsilon^2 Y^{(1)} + \varepsilon^4 Y^{(2)} + \dots] \, . \tag{33-b}$$

For real values of this radical it is clear that

$$|\cos\Theta_o e^{\psi}| < 1 \tag{33-c}$$

For this restriction (non-skip trajectories) the radical can be expanded (by the binomial theorem) to yield

$$\frac{1}{\sin\Theta} = 1 + \frac{1}{2} \cos^2\Theta_o e^{\psi} + \frac{3}{2^2 2!} \cos^4\Theta_o \varepsilon^{2\psi} + \frac{3.5}{2^3 3!} \cos^6\Theta_o \varepsilon^{3\psi} + \dots \tag{34}$$

This form of the expansion is not amenable to expressing the series in terms of the perturbation parameter, however. Therefore, it is more convenient to rewrite equation (34) as

$$\frac{1}{\sin\Theta} = 1 + \frac{1}{2} \cos^2\Theta_o [(e^{\psi} - 1) + 1] + \frac{3}{2^2 2!} \cos^4\Theta_o [1 + (e^{\psi}-1)]^2 + \frac{3.5}{2^3 3!} \cos^6\Theta_o [1 + (e^{\psi}-1)]^3 + \dots \tag{35}$$

where each of the bracketed terms can be expanded (by the binomial theorem) into a form

$$[1 + (e^{\psi} - 1)]^n = 1 + (e^{\psi}-1) + \frac{n(n-1)}{2!} (e^{\psi}-1)^2 + \dots + (e^{\psi}-1)^{n-1} + (e^{\psi}-1)^n . \tag{36}$$

Using equation (36) in equation (35), collecting coefficients of the powers of $(e^{\psi}-1)$, and rearranging, yields

$$\frac{1}{\sin\Theta} = \frac{1}{\sin\Theta_0}\left[1 + \frac{1}{2}\cot^2\Theta_0\,(e^{\psi}-1) + \frac{3}{2^2 2!}\cot^4\Theta_0(e^{\psi}-1)^2 + \frac{3.5}{2^3 3!}\cot^6\Theta_0\,(e^{\psi}-1)^3 + \ldots\right] . \tag{37}$$

Interestingly, if the exponential functions in equation (34) are expanded and the coefficients of like powers of ψ are collected, etc., the ultimate result also yields equation (37). It appears that even though the form of equation (34) converges for all values of ψ (which correspond to $0 < \Theta < 90^o$), the collective series which will allow the reciprocal sine function to be written in terms of the perturbation parameter, ε, is convergent only if

$$|\cot^2\Theta_0(e^{\psi}-1)| < 1. \tag{38}$$

The effect of this restriction can be determined by noting that equation (33-a) can be rewritten as

$$\frac{1}{\sin\Theta} = \frac{1}{\sin\Theta_0}\,\frac{1}{\sqrt{1-\cot^2\Theta_0(e^{\psi}-1)}} , \tag{39}$$

which is valid for $0 < \Theta < 90^o$; but, can be expanded to yield equation (37) only for the restriction given by equation (38). Hence, as is readily seen from equation (39), the range of permissible flight path angles is

$$0 < \sin\Theta < 1.414 \sin\Theta_0 . \tag{40}$$

Actually this limitation on the solution is not too severe if small initial angles, Θ_0, are avoided. Should the case arise where very small initial angles must be analyzed, then some other expansion will have to be utilized.

Within the limits specified by equation (40), equations (33-b) and (37) yield

$$\frac{1}{\sin\Theta} = \frac{1}{\sin\Theta_0}\left[1 + \cot^2\Theta_0\left[\varepsilon^2\, Y(1) + \varepsilon^4\,(Y^{(2)} + Y^{(1)^2}) + \ldots\right] + \frac{3}{2}\cot^4\Theta_0\left[\varepsilon^2\, Y^{(1)} + \varepsilon^4\,(Y^{(2)} + Y^{(1)^2}) + \ldots\right]^2 + \ldots\right] \qquad (41)$$

which can be written easily in ascending powers of the perturbation parameter, ε.

Using this last result, equation (41), and substituting the assumed expansions, equations (30-a, -b, -c, -d) into equations (27-a, -b) and (28-a, -b) and arranging the results in powers of ε yields the following expressions:

$$\frac{dZ^{(0)}}{d\rho} + \varepsilon^2\frac{dZ^{(1)}}{d\rho} + \varepsilon^4\frac{dZ^{(2)}}{d\rho} + \ldots = -\frac{\varepsilon^2\eta}{\sin\Theta_0}\left\{1 + \varepsilon^2\cot^2\Theta_0\, Y^{(1)} + \varepsilon^4\left[\cot^2\Theta_0\,(Y^{(2)} + Y^{(1)^2}) + \frac{3}{2}\cot^4\Theta_0 Y^{(1)^2}\right] + \ldots\right\} + \varepsilon^2\,\frac{2}{\rho}\left\{1 - \varepsilon^2 Z^{(1)} - \varepsilon^4(Z^{(2)} - \frac{Z^{(1)2}}{2}) + \ldots\right\}, \qquad (42\text{-a})$$

$$\frac{dY^{(0)}}{d\rho} + \varepsilon^2\frac{dY^{(1)}}{d\rho} + \varepsilon^4\frac{dY^{(2)}}{d\rho} + \ldots = \varepsilon^2\,\frac{k_2\lambda}{2\cos\Theta_0}\left\{1 - \varepsilon^2 Y^{(1)} + \varepsilon^4(Y^{(2)} - \frac{Y^{(1)2}}{2}) + \ldots\right\} - \frac{\varepsilon^2}{\rho}\left\{(\frac{1}{v_0^2} - 1) - \varepsilon^2\,\frac{Z^{(1)}}{v_0^2} - \right.$$

$$\left. \frac{\varepsilon^4}{v_o^2} \left(Z^{(2)} - \frac{Z^{(1)^2}}{2}\right) + \ldots \right\} , \tag{42-b}$$

$$\begin{aligned} \frac{d\phi^{(0)}}{d\rho} + \varepsilon^2 \frac{d\phi^{(1)}}{d\rho} + \varepsilon^4 \frac{d\phi^{(2)}}{d\rho} + \ldots = \varepsilon^2 \frac{\cot\Theta_0}{\rho} \Bigg\{ 1 + \frac{\varepsilon^2 Y^{(1)}}{\sin^2\Theta_0} + \\ \varepsilon^4 \left[\frac{Y^{(2)}}{\sin^2\Theta_0} + \frac{Y^{(1)} \cot^2\Theta_0}{2} (3\cot^2\Theta_0 + 1)\right] + \ldots \Bigg\} , \end{aligned} \tag{42-c}$$

and

$$\begin{aligned} \frac{dt^{(0)}}{d\rho} + \varepsilon^2 \frac{dt^{(1)}}{d\rho} + \varepsilon^4 \frac{dt^{(2)}}{d\rho} + \ldots = \frac{\varepsilon^2}{\rho v_o \sin\Theta_0} \Bigg\{ 1 - \varepsilon^2 \left[\frac{Z^{(1)}}{2} - \cot^2\Theta_0 Y^{(1)}\right] - \\ \varepsilon^4 \left[\frac{Z^{(2)}}{2} - \frac{Z^{(1)^2}}{8} + \frac{\cot^2\Theta_0}{2} Z^{(1)} Y^{(1)} - \cot^2\Theta_0 Y^{(2)} - \frac{3}{2} \cot^4\Theta_0 Y^{(1)^2}\right] + \ldots \Bigg\} . \end{aligned} \tag{42-d}$$

These expressions are the expanded form of the governing differential equations and they can be solved, in principle, after equation coefficients of the powers of ε. Note that the zeroth solutions have already been obtained and have been given previously as equation (31).

Before proceeding with the solution, however, it is instructive to compare these expansions with Picard's iteration method.

4.3 Comparison with Picard's Method

As previously discussed, the advantage of considering matched asymptotic expansions is that the first few terms of assumed expansions can be obtained for several flight regimes. A composite of these several results usually yields a more accurate resultant than any one of the individual expansions. Unfortunately, as was discussed at the beginning of this chapter, no significant transformations could be solved for in this particular problem. It behooves us, then, to consider the validity of the higher order terms which are obtained by solving the order of magnitude equations - represented by equation (42). These appear to become more significant than the lower order expressions as $\rho \to 0(1/\varepsilon^2)$. With this intent in mind, one can consider Picard's iteration method.[9] Given an initial value problem of the form

$$Y' = F(x,y),\ Y(x_o) = Y_o$$

then the existence and uniqueness theorems can be stated as follows:

Existence Theorem - If $F(x,y)$ is continuous at all points (x,y) in some rectangle

$$R:\ |x - x_o| < a,\quad |y - y_o| < b$$

and is bounded in R by, say,

$$|F(x,y)| \leq k, \text{ for all } (x,y) \text{ in } R,$$

then the initial value problem has at least one solution, $y(x)$, which is defined for at least all values of x in the interval $|x - x_o| < \alpha$ where α is the smaller of the two numbers a and b/K.

Uniqueness Theorem - If $F(x,y)$ and $\partial F/\partial Y$ are continuous for all (x,y) in that rectangle R and bounded by, say,

$$(a)\ |F| \leq K,\quad (b)\ |\partial F/\partial Y| \leq M \text{ for all } (x,y) \text{ in } R,$$

then the initial value problem has only one solution, $Y(x)$, which is defined for at least all values of x in that interval $|x - x_o| < \alpha$. This solution can be obtained by Picard's iteration method, that is by the sequence: Y_o, Y_1, Y_2, $\ldots$, Y_n, $\ldots$, wherein

$$Y_n(x) = Y_o + \int_{x_o}^{x} F(t, Y_{n-1}(t))dt, \; n = 1, 2, \ldots,$$

and which converges to that solution, $Y(x)$.

It is assumed that these theorems are extendible to the simultaneous initial value problem,

$$Y' = F(x,y,z(x)), \; Y(x_o,z_o) = y_o, \qquad (a)$$

and

$$Z' = G(x,z,y(x)), \; Z(x_o,y_o) = Z_o. \qquad (b)$$

This is a logical extension, since, under this assumption, equations (a) and (b) can be expressed as

$$Y' = F(x,y) \; ; \quad Y(x_o) = Y_o, \qquad (c)$$

and

$$Z' = G(x,z), \quad Z(x_o) = Z_o. \qquad (d)$$

Equations (c) and (d), now, independently satisfy the existence and uniqueness theorems.

It follows that equations (27) and (28) can be solved by Picard's method provided $\theta > 0$.

The solution of equations (27) by Picard's iteration method now can be written and compared with the perturbation solution in the following manner:

The first terms in the Picard sequences are

$$Z_1 = \ln v_o^2 \tag{43-a}$$

and

$$Y_1 = \ln \cos\Theta_o \; ; \tag{43-b}$$

which, by a comparison with equations (37), yield

$$Z_1 = Z^{(0)} \; , \tag{43-c}$$

and

$$Y_1 = Y^{(0)} \; , \tag{43-d}$$

Next, the second terms in the Picard sequences become

$$Z_2 = \ln v_o^2 + \varepsilon^2 \int_1^\rho \frac{-\eta d\rho}{\sin\Theta_o} + \frac{\varepsilon^2 2}{v_o^2} \int_1^\rho \frac{d\rho}{\rho} \; ; \tag{44-a}$$

and

$$Y_2 = \ln \cos\Theta_o + \frac{\varepsilon^2 k_2}{2\cos\Theta_o} \int_1^\rho \lambda d\rho - \varepsilon^2 \left(\frac{1}{v_o^2} - 1 \right) \int_1^\rho \frac{d\rho}{\rho} \; . \tag{44-b}$$

Comparing this result with equations (32-a) and (32-b) one finds that

$$Z_2 = Z^{(0)} + \varepsilon^2 Z^{(1)} \; , \tag{44-c}$$

and

$$Y_2 = Y^{(0)} + \varepsilon^2 Y^{(1)} \; . \tag{44-d}$$

Continuing, the third terms in the Picard sequences can be written as

$$Z_3 = \ln v_o^2 + \varepsilon^2 \int_1^\rho \frac{-\eta}{\sin\Theta_0} \left\{ 1 - \varepsilon^2 \cot^2\Theta_0 Y^{(1)} + \varepsilon^4 Y^{(1)^2} (\cot^2\Theta_0 + \frac{3}{2}\cot^4\Theta_0) + \ldots \right\} d\rho + \frac{\varepsilon^2}{v_o^2} \int_1^\rho \left\{ 1 - \varepsilon^2 Z^{(1)} + \varepsilon^4 \frac{Z^{(1)^2}}{2} + \ldots \right\} \frac{d\rho}{\rho}, \qquad (45\text{-a})$$

and

$$Y_3 = \ln \cos\Theta_0 + \frac{\varepsilon^2 k_2}{2\cos\Theta_0} \int_1^\rho \lambda \left\{ 1 - \varepsilon^2 Y^{(1)} + \frac{\varepsilon^4 Y^{(1)^2}}{2} + \ldots \right\} d\rho - \varepsilon^2 \int_1^\rho \left\{ (\frac{1}{v_o^2} - 1) - \varepsilon^2 Z^{(1)} + \varepsilon^4 \frac{Z^{(1)^2}}{2} + \ldots \right\} \frac{d\rho}{\rho}. \qquad (45\text{-b})$$

Once again, comparing these results with equations (42-a) and (42-b) yields

$$Z_3 = Z^{(0)} + \varepsilon^2 Z^{(1)} + \varepsilon^4 Z^{(2)} + O(\varepsilon^6), \qquad (45\text{-c})$$

and

$$Y_3 = Y^{(0)} + \varepsilon^2 Y^{(1)} + \varepsilon^4 Y^{(2)} + O(\varepsilon^6). \qquad (45\text{-d})$$

By a continuation of this process it is readily shown that the two methods yield the same solution, in the limit. That is, the nth terms in the Picard sequences become

$$Z_n = Z^{(0)} + \varepsilon^2 Z^{(1)} + \ldots + \varepsilon^{2(n-1)} Z^{(n-1)} + O(\varepsilon^{2n}), \qquad (46\text{-a})$$

and

$$Y = Y^{(0)} + \varepsilon^2 Y^{(1)} + \ldots + \varepsilon^{2(n-1)} Y^{(n-1)} + O(\varepsilon^{2n}), \qquad (46\text{-b})$$

which are exactly the assumed perturbation expansions. Therefore, it follows that since Picard's iteration method converges to a solution and since, in the limit, the perturbation expansion yields Picard's solution, then this perturbation expansion must be uniformly valid in the limit as n becomes large.

Furthermore, if all of the terms in the perturbation expansion can be generated, which are significant for any given flight regime, this solution should be at least as good as Picard's solution taken to the same degree of accuracy. In addition, this technique also yields an order of magnitude estimate of the next term in the ordered expansions.

4.4 Discussion of Ordered Terms

The governing equations of motion have been rewritten in what appears to be a less degenerate form of the dependent variables in section 4.2; and these are presented as equations (27) and (28). Subsequently, it was presumed that a uniformly valid solution could be obtained by assuming the series expansions given in equations (30-a, -b, -c, -d). The resulting order of magnitude terms have been shown to generate in the limit the n^{th} term in the sequences produced by Picard's iteration method (see section 4.3). Furthermore, if the problem is restricted to non-skip trajectories, specifically of $\Theta > 0$, then the governing equations must satisfy Picard's existence and uniqueness theorems. Thus, it follows that if a sufficient number of terms in the assumed expansions can be determined, the expansions will converge to Picard's solution and, therefore, to a unique solution.

On the other hand, if one proceeds to evaluate the ordered terms in the assumed expansions it soon becomes apparent that it is not practical to consider more than a first few terms. Because of this one usually considers solutions (by series expansions) for several flight regimes. As previously noted, a general technique for this type of analysis can be found in Reference 8.

Specifically, by properly transforming the initial variables (by order of magnitude transformations), asymptotic expansions for several flight regimes can be obtained. If these several expansions can be matched in their limits, then they can be used to form a composite expansion which is generally more accurate over all flight regimes than the individual asymptotic expansions for each specific region. In addition, this technique allows one to generate the asymptotic expansions to any given order of magnitude of accuracy for each flight regime of interest.

For the particular problem investigated here, the initial expansion yields an "exact solution" to within a reasonable order of accuracy by means of only a few terms, provided the density, ρ, does not become too large. This affect (of ρ large) is readily apparent, however, from a simple examination of the solved expansions for the affect of a large density (ρ). More specifically, only a first few terms of the expansion are needed to obtain good accuracy through the flight regime where $\rho = 0(1/\varepsilon)$. However, when $\rho = 0(1/\varepsilon^2)$ there are terms appearing which are of the form,

$$\sum_{n=1}^{\infty} A_n(\varepsilon^2 Y(\rho)^{(1)})^n \,, \quad \sum_{n=1}^{\infty} B_n(\varepsilon^2 Z(\rho)^{(1)})^n \,, \quad \text{etc.};$$

these become significant and must be generated for this flight regime. Fortunately, the significant terms for the case of ρ large can be obtained by properly transforming the governing equations and matching the new (assumed) expansions to the original expansions.

Before proceeding, however, it is informative to reexamine the expansion that has been generated for the reciprocal sine function, equation (41); this is repeated here for convenience:

$$\frac{1}{\sin\Theta} = \frac{1}{\sin\Theta_0}\left\{1 + \varepsilon^2 \cot^2\Theta_0 \left[Y^{(1)} + \varepsilon^2 \left(Y^{(2)} + Y^{(1)^2}\right) + \ldots\right] + \frac{3}{2}\, \varepsilon^4 \cot^4\Theta_0 \left[Y^{(1)} + \varepsilon^2 \left(Y^{(2)} + Y^{(1)^2}\right) + \ldots\right]^2 + \ldots\right\} . \tag{41}$$

This series will converge quite rapidly for flight conditions where the sine of the flight path angle does not change by more than a few percent from its initial value. On the other hand, for conditions which require a large change in path angle many terms in the series are required. Since it is not practical to consider more than a few terms it seems necessary to restrict the solution to those flight conditions in which the change in flight path angle is small. Specifically, if even the simplest aerodynamic modulation function is considered (as will become evident from the following solution) it does not seem practical to carry more than the first significant term in equation (41).

The relative importance of these terms in equation (41), compared with the unexpanded form, equation (39), is presented on Figure 3. As is noted on this figure, the first order expansion is within 6% of the actual value for changes as large as 20% of the initial value. Noting that the integration of this function-in the ensuing solution- suppresses this error, then it is reasonable to expect good accuracy in the solution for changes in the sine of the flight path angle as large as 20% of its initial value.

The restriction imposed on the solution of considering only the first order expansion (and possible the second order) requires that the lift force must be quite small. Actually, as will be illustrated later, the solution for non-skip trajectories requires that the lift coefficient, k_2, be of $O(\varepsilon^2)$, or at most of $O(\varepsilon)$, anyway. The effect of velocity decay on the flight path angle, on the other hand, is generally quite small except for very small initial entry angles; thus this condition will not impose any significant additional restrictions on the solution.

Since the lift coefficient, k_2, must be restricted to rather small values, the contribution of lift to the drag will be considered as negligible. Therefore, the drag modulation function, neglecting the lift term is obtained from equations (13-c) and (14) as:

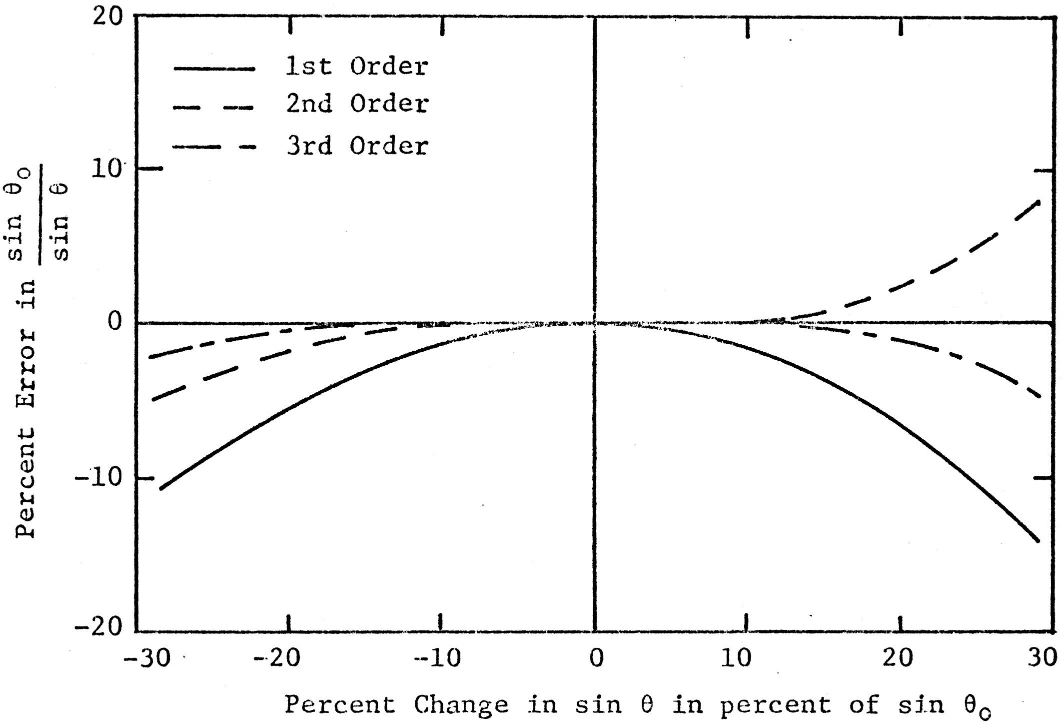

Figure 3. Percent Error in 1st, 2nd, and 3rd Order Expansion for sin θ_0/sin θ, equation (41).

$$\eta = 1 + \frac{\varepsilon a_1 \rho}{a_2 + \varepsilon\rho} . \tag{47}$$

Also, because of the relatively small allowable changes in the flight path angle, the effects of lift modulation become trivial, at least when compared to the added algebraic complexity which they would introduce. Hence, the lift modulation term will be set at unity; i.e.,

$$\lambda = 1. \tag{48}$$

4.5 Solution for the Initial Flight Regime

The zeroth terms of the assumed expansions have been given previously by equations (31); these are repeated here for convenience:

$$Z^{(0)} = \ln v_o^2 , \tag{49-a}$$

$$Y^{(0)} = \ln \cos\Theta_0 , \tag{49-b}$$

$$\phi^{(0)} = 0 , \tag{49-c}$$

and

$$t^{(0)} = 0 . \tag{49-d}$$

The higher order terms can be solved for now in the following fashion:

From equations (42-a) through (42-d) the first order governing differential equations are:

$$\frac{dZ^{(1)}}{d\rho} = - \frac{\eta}{\sin\Theta_0} + \frac{2}{v_o^2 \rho} , \tag{50-a}$$

$$\frac{dY^{(1)}}{d\rho} = \frac{k_2 \lambda}{2 \cos\Theta_0} - \left(\frac{1}{v_o^2} - 1 \right) \frac{1}{\rho} , \tag{50-b}$$

$$\frac{d\phi^{(1)}}{d\rho} = \cot\Theta_0 \frac{1}{\rho} , \tag{50-c}$$

and

$$\frac{dt^{(1)}}{d\rho} = \frac{1}{v_o \sin\Theta_0} \frac{1}{\rho} . \tag{50-d}$$

Next, using equations (47) and (48) for the modulation functions, integrating, and using the initial conditions given by equation (30-e) yield:

$$Z^{(1)} = - \frac{1}{\sin\Theta_0} \left[(1 + a_1)(\rho - 1) - \frac{a_1 a_2}{\varepsilon} \ln \left(\frac{a_2 + \varepsilon\rho}{a_2 + \varepsilon} \right) \right] + \frac{2}{v_o^2} \ln\rho , \tag{51-a}$$

$$Y^{(1)} = \frac{k_2}{2\cos\Theta_0} (\rho - 1) + (1 - \frac{1}{v_o^2}) \ln\rho , \tag{51-b}$$

$$\phi^{(1)} = \cot\Theta_0 \ln\rho , \tag{51-c}$$

and

$$t^{(1)} = \frac{1}{v_o \sin\Theta_0} \ln\rho . \tag{51-d}$$

Equations (51-a) through (51-d) are the required first order terms of the assumed expansions.

The second order equations are obtained by equating the coefficients of ε^4 from equations (32-a) through (32-d); this procedure leads directly to:

$$\frac{dZ^{(2)}}{d\rho} = -\frac{\eta}{\sin\Theta_0}\cot^2\Theta_0\, Y^{(1)} - \frac{2}{v_o^{\,2}}\frac{Z^{(1)}}{\rho}, \tag{52-a}$$

$$\frac{dY^{(2)}}{d\rho} = -\frac{k_2\lambda}{2\cos\Theta_0}Y^{(1)} + \frac{Z^{(1)}}{v_o^2\rho}, \tag{52-b}$$

$$\frac{d\phi^{(2)}}{d\rho} = \cot\Theta_0\frac{Y^{(1)}}{\rho}, \tag{52-c}$$

and

$$\frac{dt^{(2)}}{d\rho} = -\frac{1}{v_o\sin\Theta_0}\left[\frac{Z^{(1)}}{2\rho} - \cot^2\Theta_0\frac{Y^{(1)}}{\rho}\right]. \tag{52-d}$$

The solution to these equations is, for the most part, straightforward, though algebraically it can be complicated. Because of the number of terms involved in this solution, it is now convenient to consider this integration termwise. Hence, one may consider, first, the integral

$$I_1 = \int_1^\rho \eta Y^{(1)}\, d\rho \tag{53}$$

where the limits of integration are consistent with the initial conditions given by equation (30-e).

Substituting equations (47) and (51-b) into this integral leads to the expanded form,

$$I_1 = \int_1^\rho \left(1 + \frac{\varepsilon a_1\rho}{a_2+\varepsilon\rho}\right)\left[\frac{k_2}{2\cos\Theta_0}(\rho-1) + \left(1 - \frac{1}{v_o^{\,2}}\right)\ln\rho\right] d\rho\,. \tag{54}$$

Completing the product and integrating, term by term, and then after rearranging, one finds that

$$I_1 = \frac{k_2}{2\cos\Theta_0}\left\{(1+a_1)\frac{(\rho-1)^2}{2} - \frac{a_1a_2}{\varepsilon}\left[\varepsilon(\rho-1) - (a_2+\varepsilon)\ln\left(\frac{a_2+\varepsilon\rho}{a_2+\varepsilon}\right)\right]\right\} +$$

$$\left(1 - \frac{1}{v_o^2}\right)\left\{(1+a_1)\,[\rho\ln\rho - (\rho-1] - \frac{a_1a_2}{\varepsilon}\left[\ln\rho\,\ln\left(1+\frac{\varepsilon\rho}{a_2}\right) - I_2\right]\right\}, \tag{55}$$

wherein

$$I_2 = \int_1^\rho \ln\left(1 + \frac{\varepsilon\rho}{a_2}\right)\frac{d\rho}{\rho}\ . \tag{56}$$

The integral, I_2, can be integrated successively by parts to yield

$$I_2 = \frac{1}{2}\ln^2\left(1+\frac{\varepsilon\rho}{a_2}\right) - \frac{1}{2}\ln^2\left(1+\frac{\varepsilon}{a_2}\right) +$$

$$\sum_{n=1}^{\infty}\frac{1}{n^2}\left[\left(\frac{\varepsilon\rho}{a_2+\varepsilon\rho}\right)^n - \left(\frac{\varepsilon}{a_2+\varepsilon}\right)^n\right]. \tag{57}$$

Next, consider the integral

$$I_3 = \int_1^\rho Z^{(1)}\frac{d\rho}{\rho}\ . \tag{58}$$

Using equation (51-a) for $Z^{(1)}$ and integrating, one obtains

$$I_3 = -\frac{1}{\sin\Theta_0}\left\{(1+a_1)\,[\rho-1-\ln\rho] - \frac{a_1a_2}{\varepsilon}\,[I_2 - \ln\rho\ln(1+\frac{\varepsilon}{a_2})]\right\}+$$

$$\frac{1}{v_o^2}\ln^2\rho\ , \tag{59}$$

where I_2 is again given as equation (57).

The solution for $Z^{(2)}$, from equation (52-a), can be expressed now as

$$\begin{aligned} Z^{(2)} = -\frac{\cot^2\Theta_o}{\sin\Theta_o}\Biggl\{\Biggl\{\frac{k_2}{2\cos\Theta_o}\Biggl\{(1+a_1)\frac{(\rho-1)^2}{2} - \frac{a_1a_2}{\varepsilon}\times \\ \Biggl[\varepsilon(\rho-1) - (a_2+\varepsilon)\ln\left(\frac{a_2+\varepsilon\rho}{a_2+\varepsilon}\right)\Biggr]\Biggr\} + (1-\frac{1}{{v_o}^2})(1+a_1)\times \\ [\rho\ln\rho - (\rho-1)]\Biggr\}\Biggr\} + \frac{2}{v_o^2\sin\Theta_o}\Biggl\{(1+a_1)[\rho-1-\ln\rho] + \\ \frac{1}{{v_o}^4}\ln^2\rho\Biggr\} - \frac{a_1a_2}{\varepsilon\sin\Theta_o}\Biggl[\frac{2}{{v_o}^2} + \cot^2\Theta_o(1-\frac{1}{{v_o}^2})\Biggr]\Biggl\{\frac{1}{2}\ln^2(1+\frac{\varepsilon\rho}{a_2}) - \\ \frac{1}{2}\ln^2(1+\frac{\varepsilon}{a_2}) + \sum_{n=1}^{\infty}\frac{1}{n^2}\Biggl[\left(\frac{\varepsilon\rho}{a_2+\varepsilon\rho}\right)^n - \left(\frac{\varepsilon}{a_2+\varepsilon}\right)^n\Biggr] - \ln\rho\ln(1+\frac{\varepsilon}{a_2})\Biggr\}\ . \end{aligned} \tag{60}$$

Similarly, the solution for $Y^{(2)}$ can be expressed as

$$\begin{aligned} Y^{(2)} = -\frac{k_2}{2\cos\Theta_o}{}^2\int_1^\rho(\rho-1)d\rho - \frac{k_2}{2\cos\Theta_o}(1-\frac{1}{{v_o}^2}) \\ \int_1^\rho\ln\rho\frac{d\rho}{\rho} + \frac{1}{{v_o}^2}I_3\ . \end{aligned} \tag{61}$$

Completing this integration and using equation (59) for I_3 yield

$$\begin{aligned} Y^{(2)} = -\left(\frac{k_2}{2\cos\Theta_o}\right)^2\frac{(\rho-1)^2}{2} - \frac{k_2}{2\cos\Theta_o}(1-\frac{1}{{v_o}^2})[\rho\ln\rho-(\rho-1)] - \\ \frac{1}{v_o^2\sin\Theta_o}\Biggl\{\Biggl\{(1+a_1)[\rho-1-\ln\rho] - \frac{a_1a_2}{\varepsilon}\Biggl\{\frac{1}{2}\ln^2(1+\frac{\varepsilon\rho}{a_2}) - \frac{1}{2}\ln^2(1+\frac{\varepsilon}{a_2}) + \end{aligned}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2}\left[\left(\frac{\varepsilon\rho}{a_2+\varepsilon\rho}\right)^n - \left(\frac{\varepsilon}{a_2+\varepsilon}\right)^n\right] - \ln\rho\ln(1+\frac{\varepsilon}{a_2}) \Big\}\Big]\Big] + \frac{1}{v_o^4}\ln^2\rho \ . \qquad (62)$$

In a similar manner, the solutions for $\phi^{(2)}$ and $t^{(2)}$ are easily shown to be

$$\phi^{(2)} = \cot\Theta_0 \left\{\frac{k_2}{2\cos\Theta_0}\,[\rho-1-\ln\rho] + (1-\frac{1}{v_o^2})\,\frac{\ln^2\rho}{2}\right\}, \qquad (63)$$

and

$$t^{(2)} = \frac{1}{2v_o\sin^2\Theta_0}\left\{\!\!\left\{(1+a_1)\,[\rho-1-\ln\rho] - \frac{a_1a_2}{\varepsilon}\left\{\frac{1}{2}\ln^2(1+\frac{\varepsilon\rho}{a_2}) - \right.\right.\right.$$

$$\frac{1}{2}\ln^2(1+\frac{\varepsilon}{a_2}) + \sum_{n=1}^{\infty}\frac{1}{n^2}\left[\left(\frac{\varepsilon\rho}{a_2+\varepsilon\rho}\right)^n - \left(\frac{\varepsilon}{a_2+\varepsilon}\right)^n\right] - \ln\rho\ln(1+\frac{\varepsilon}{a_2})\Big\} +$$

$$2\cos\Theta_0\cot\Theta_0\left\{\frac{k_2}{2\cos\Theta_0}\,[\rho-1-\ln\rho] + (1-\frac{1}{v_o^2})\,\frac{\ln^2\rho}{2}\right\}\!\!\Big\}\Big\} . \qquad (64)$$

The second order solution of the governing differential equations, expressed as equations (42-a-b-c-d), can now be written by substituting the appropriate ordered terms above into the assumed expansions, equations (30). Thus, a value for the dependent variable Z becomes

$$Z = \ln v_o^2 - \frac{\varepsilon^2}{\sin\Theta_0}\,[(1+a_1)(\rho-1) - \frac{a_1a_2}{\varepsilon}\ln\left(\frac{a_2+\varepsilon\rho}{a_2+\varepsilon}\right)] + \frac{\varepsilon^2 2}{v_o^2}\ln\rho -$$

$$\varepsilon^4\,\frac{\cot^2\Theta 0}{\sin\Theta_0}\left\{\!\!\left\{\frac{k_2}{2\cos\Theta_0}\left\{(1+a_1)\,\frac{(\rho-1)^2}{2} - \frac{a_1a_2}{\varepsilon^2}\left[\varepsilon(\rho-1) - (a_2+\varepsilon)\times\right.\right.\right.\right.$$

$$\left.\ln\left(\frac{a_2+\varepsilon\rho}{a_2+\varepsilon}\right)\right]\Big\} + (1-\frac{1}{v_o^2})\,(1+a_1)\,[\rho\ln\rho - (\rho-1)]\Big\}\Big\} +$$

$$\frac{\varepsilon^4 2}{v_o^2\sin\Theta_o}\Bigg\{(1+a_1)\,[\rho-1-\ln\rho]+\frac{1}{v_o^2}\ln^2\rho\Bigg\}-\varepsilon^3\frac{a_1a_2}{\sin\Theta_o}\Bigg[\frac{2}{v_o^2}+$$

$$\cot^2\Theta_o\left(1-\frac{1}{v_o^2}\right)\Bigg]\Bigg\{\frac{1}{2}\ln^2\left(1+\frac{\varepsilon\rho}{a_2}\right)-\frac{1}{2}\ln^2\left(1+\frac{\varepsilon}{a_2}\right)+\sum_{n=1}^{\infty}\frac{1}{n^2}\Bigg[\left(\frac{\varepsilon\rho}{a_2+\varepsilon\rho}\right)^n-$$

$$\left(\frac{\varepsilon}{a_2+\varepsilon}\right)^n-\ln\rho\ln\left(1+\frac{\varepsilon}{a_2}\right)\Bigg]\Bigg\}+0(\varepsilon^6)\,,\qquad(65)$$

where Z is the logarithm of the nondimensional speed ratio.

In a like manner the solutions for Y, ϕ and t become, respectively,

$$Y=\ln\cos\Theta_o+\frac{\varepsilon^2k_2}{2\cos\Theta_o}(\rho-1)+\varepsilon^2\left(1-\frac{1}{v_o^2}\right)\ln\rho-\varepsilon^4\left(\frac{k_2}{2\cos\Theta_o}\right)^2\frac{(\rho-1)^2}{2}-$$

$$\frac{\varepsilon^4k_2}{2\cos\Theta_o}\left(1-\frac{1}{v_o^2}\right)[\rho\ln\rho-(\rho-1)]-\frac{\varepsilon^4}{v_o^2\sin\Theta_o}\Bigg\{\Bigg\{(1+a_1)\,[\rho-1-\ln\rho]-$$

$$\frac{a_1a_2}{\varepsilon}\Bigg\{\frac{1}{2}\ln^2\left(1+\frac{\varepsilon\rho}{a_2}\right)-\frac{1}{2}\ln^2\left(1+\frac{\varepsilon}{a_2}\right)+\sum_{n=1}^{\infty}\frac{1}{n^2}\Bigg[\left(\frac{\varepsilon\rho}{a_2+\varepsilon\rho}\right)^n-\left(\frac{\varepsilon}{a_2+\varepsilon}\right)^n\Bigg]-$$

$$\ln\rho\ln\left(1+\frac{\varepsilon}{a_2}\right)\Bigg\}\Bigg\}\Bigg\}+\frac{\varepsilon^4}{v_o^4}\ln^2\rho+0(\varepsilon^6)\,,\qquad(66)$$

$$\phi=\varepsilon^2\cot\Theta_o\ln\rho+\frac{\varepsilon^4\cot\Theta_o}{\sin^2\Theta_o}\Bigg\{\frac{k_2}{2\cos\Theta_o}[\rho-1-\ln\rho]+\left(1-\frac{1}{v_o^2}\right)\frac{\ln^2\rho}{2}\Bigg\}+0(\varepsilon^6)$$

$$(67)$$

$$t=\frac{\varepsilon^2}{v_o\sin\Theta_o}\ln\rho+\frac{\varepsilon^4}{2v_o\sin^2\Theta_o}\Bigg\{\Bigg\{(1+a_1)\,[\rho-1-\ln\rho]-$$

$$\frac{a_1a_2}{\varepsilon}\Bigg\{\frac{1}{2}\ln^2\left(1+\frac{\varepsilon\rho}{a_2}\right)-\frac{1}{2}\ln^2\left(1-\frac{\varepsilon}{a_2}\right)+\sum_{n=1}^{\infty}\frac{1}{n^2}\Bigg[\left(\frac{\varepsilon\rho}{a_2+\varepsilon\rho}\right)^n-\left(\frac{\varepsilon}{a_2+\varepsilon}\right)^n\Bigg]-$$

$$\left. \ln\rho \ln\left(1 + \frac{\varepsilon}{a_2}\right)\right\} + 2\cos\Theta_0 \cot\Theta_0 \left\{ \frac{k_2}{2\cos\Theta_0} [\rho - 1 - \ln\rho] + \left(1 - \frac{1}{v_a^2}\right) \times \right.$$

$$\left. \frac{\ln^2\rho}{2} \right]\Bigg]\Bigg\} + O(\varepsilon^6) ; \tag{68}$$

where Y is the logarithm of $\cos\Theta$, ϕ is the range angle and t is the nondimensional flight time.

The solution generated thus far should yield good results up to the order of magnitude indicated, provided the conditions are chosen so that the sine of the flight path angle does not change by more than (about) 20% from its initial value and also provided the density, ρ, is not too large. For example, as $\rho \to O(\frac{1}{\varepsilon^2})$, the principal $O(\varepsilon^4)$ terms become as significant as the $O(\varepsilon^2)$ terms.

In principle, all higher order terms can be generated until the resulting expansion converges for the case of a large density. This procedure is not very practical, however, due to the large number of terms which would be required. The significant terms for large density, ρ, can be generated in a more orderly manner, however, as will be demonstrated in the next section.

4.6 Aerodominated Flight Regime

In the previous section a straightforward perturbation technique was applied to the governing differential equations to account for terms through the second order. It was noted, however, that as the density became large, i.e., as $\rho \to O(1/\varepsilon^2)$, the higher order solution contained terms which were as significant as those obtained in the first order solution. Fortunately, all of these quantities can be acquired by transforming the variables in the governing equations in a manner which will admit a solution including the significant terms in this flight regime.

Before proceeding with this analysis it is informative to consider the solution obtained by Willes, et al[3]. In that paper, the authors transformed their equations by assuming ρ was of $O(1/\varepsilon^2)$. The resulting zeroth solution gave their so-called skip-trajectory which provided good agreement with computer solutions whenever the pull-up or skip occured before a significant velocity decay developed. Unfortunately, the more significant transformation, which would include the assumption that the velocity was small, i.e., $x = O(\varepsilon^2)$, led to coupled zeroth order equations which could not be successfully manipulated analytically.

Contrary to this work, when non-skip trajectories are considered, it is necessary to use a transformation which reflects the relative importance of both the lift force and the centrifugal term. The most obvious transformation to use here would assume that ρ was of $O(1/\varepsilon^2)$ and x was of $O(\varepsilon^2)$; however this has already been ruled out since it gives the coupled equations which cannot be solved by this technique. An alternate transformation, which reflects the proper relationship between the lift and the effect of velocity decay, is obtained by assuming ρ of $O(1/\varepsilon^2)$ and k_2 of $O(\varepsilon^2)$; however, this is a more degenerate form. Nevertheless this transformation can be matched uniquely with the original solution and can be used to generate the significant terms in this flight regime.

Therefore, let

$$\begin{aligned} \rho &= \rho_2/\varepsilon^2 \;, \\ k_2 &= \varepsilon^2 k_2' \;, \\ Z &= Z_2 \;, \\ Y &= Y_2 \;, \\ \phi &= \phi_2 \;, \end{aligned} \tag{69}$$

and

$$t = t_2 \;;$$

where these new variables are considered to be of order one. The governing equations, (27-a) and (27-b), become

$$\frac{dZ_2}{d\rho_2} = \frac{-\eta}{\sqrt{1 - e^{2Y_2}}} + \frac{\varepsilon^2 2 e^{-Z_2}}{\rho_2} \tag{70-a}$$

and

$$\frac{dY_2}{d\rho_2} = \varepsilon^2 \frac{k_2'\lambda}{2} e^{-Y_2} - \frac{\varepsilon^2}{\rho_2} (e^{-Z_2} - 1) \;. \tag{70-b}$$

The equations for the range angle and the time, (28-a) and (28-b), written in terms of the transformed variables, are

$$\frac{d\phi_2}{d\rho_2} = \frac{\varepsilon^2 e^{Y_2}}{\rho_2\sqrt{1-e^{2Y_2}}} \;, \tag{70-c}$$

and

$$\frac{dt_2}{d\rho_2} = \frac{\varepsilon^2 e^{-\frac{Z_2}{2}}}{\sqrt{1-e^{2Y_2}}} \; . \tag{70-d}$$

The constants of integration for these expressions will be determined by matching the ordered solutions (here) with the corresponding ordered solutions from the initial flight regime.

The modulation functions for this flight regime, obtained from equation (69) used in (47), become

$$\eta = 1 + \frac{a_1\rho_2}{\varepsilon a_2 + \rho_2} \; ; \tag{71-a}$$

and from equation (48),

$$\lambda = 1. \tag{71-b}$$

For the solution in this flight regime it is convenient to expand the modulation function by means of the binomial theorem. This leads directly to the expansion

$$\eta = (1 + a_1) - \frac{\varepsilon a_1 a_2}{\rho_2} + \varepsilon^2 \frac{a_1 a_2^2}{\rho_2^2} + \dots \quad . \tag{72}$$

In this regime it will be assumed that the solution can be expressed as a power series in the parameter ε. Recall that this was the procedure followed in the initial flight regime. Before expanding equations

(70-a) through (70-d), however, it is expedient to note the form of equation (70-b); it is such that the zeroth order equation is

$$\frac{dY_2^{(0)}}{d\rho_2} = 0 \; ; \tag{73}$$

thus,

$$Y_2^{(0)} = C_6 . \tag{74}$$

Comparing this zeroth expansion with the corresponding zeroth expansion for the initial regime, equation (31), it is clear that

$$C_6 = \ln\cos\Theta_0 \; . \tag{75}$$

Taking this into account, along with the expanded form of the modulation function, the proper series expansions become

$$Z_2(\rho_2;\varepsilon) = Z_2(\rho_2)^{(0)} + \varepsilon\, Z_2(\rho_2)^{(1)} + \varepsilon^2\, Z_2(\rho_2)^{(2)} + \ldots$$

$$Y_2(\rho_2;\varepsilon) = \ln\cos\Theta_0 + \varepsilon^2\, Y_2(\rho_2)^{(1)} + \varepsilon^3\, Y_2(\rho_2)^{(2)} + \ldots$$

$$\phi_2(\rho_2;\varepsilon) = \phi_2(\rho_2)^{(0)} + \varepsilon^2\, \phi_2(\rho_2)^{(1)} + \varepsilon^4\, \phi_2(\rho_2)^{(2)} + \ldots \tag{76}$$

and

$$t_2(\rho_2;\varepsilon) = t_2(\rho_2)^{(0)} + \varepsilon^2\, t_2(\rho_2)^{(1)} + \varepsilon^3\, t_3(\rho_2)^{(2)} + \ldots$$

Substituting equations (69), (71-b), (72) and (76) into (70-a) through (70-d) and rearranging yield

$$\frac{dZ_2^{(0)}}{d\rho_2} + \varepsilon \frac{dZ_2^{(1)}}{d\rho_2} + \varepsilon^2 \frac{dZ_2^{(2)}}{d\rho_2} + \ldots = -\frac{1}{\sin\Theta_0}\left\{(1 + a_1) - \frac{\varepsilon a_1 a_2}{\rho_2} + \varepsilon^2\left[\frac{a_1 a_2^2}{\rho_2^2} + (1 + a_1)\cot^2\Theta_0\, Y_2^{(1)}\right] + \ldots\right\} + \varepsilon^2 \frac{2}{\rho_2} e^{-Z_2^{(0)}}\left\{1 - \varepsilon Z_2^{(1)} + \ldots\right\}, \qquad (77\text{-a})$$

and

$$\frac{dY_2^{(0)}}{d\rho_2} + \varepsilon^2 \frac{dY_2^{(1)}}{d\rho_2} + \varepsilon^3 \frac{dY_2^{(2)}}{d\rho_2} + \ldots = \varepsilon^2 \frac{k_2'}{2\cos\Theta_0}\left[1 - \varepsilon^2 Y_2^{(1)} + \ldots\right] + \frac{\varepsilon^2}{\rho_2} - \varepsilon^2 \frac{e^{-Z_2^{(0)}}}{\rho_2}\left[1 - \varepsilon Z_2^{(1)} + \ldots\right]; \qquad (77\text{-b})$$

where the zeroth term, $Y_2^{(0)}$ has already been accounted for previously and is given by equation (74). Also, one obtains here:

$$\frac{d\phi_2^{(0)}}{d\rho_2} + \varepsilon^2 \frac{d\phi_2^{(1)}}{d\rho_2} + \varepsilon^4 \frac{d\phi_2^{(1)}}{d\rho_2} + \ldots = \varepsilon^2 \frac{\cot\Theta_0}{\rho_2}\left[1 + \varepsilon^2 \frac{Y_2^{(1)}}{\sin^2\Theta_0} + \ldots\right], \qquad (77\text{-c})$$

and

$$\frac{dt^{(0)}}{d\rho_2} + \varepsilon^2 \frac{dt^{(1)}}{d\rho_2} + \varepsilon^3 \frac{dt^{(2)}}{d\rho_2} + \ldots = \frac{\varepsilon^2 e^{-\frac{Z_2^{(0)}}{2}}}{\rho_2 \sin\Theta_0}\left[1 - \frac{\varepsilon Z_2^{(1)}}{2} + \ldots\right]. \qquad (77\text{-d})$$

The zeroth ordered equations, obtained by equating the coefficients of ε^{o}, are found to be

$$\frac{dZ_2^{(0)}}{d\rho_2} = -\frac{(1 + a_1)}{\sin\Theta_0}, \tag{78-a}$$

$$\frac{d\phi_2^{(0)}}{d\rho_2} = 0 , \tag{78-b}$$

and

$$\frac{dt_2^{(0)}}{d\rho_2} = 0 \text{ , respectively .} \tag{78-c}$$

Solutions to equations (78-b) and (78-c) when compared with the corresponding zeroth results for the intermediate regime yield

$$\phi_2^{(0)} = 0 \tag{79-a}$$

and

$$t_2^{(0)} = 0 . \tag{79-b}$$

Next, equation (78-a) is integrated to give

$$Z_2^{(0)} = -\frac{(1+a_1)}{\sin\Theta_0}\rho_2 + C_7 , \tag{80}$$

where the constant, C_7 is determined by matching this zeroth order expansion with the corresponding expansion in the initial regime.

Substituting this resultant into the assumed expansion, expressing it in terms of the initial variables, and then after passing to the limit of O(1), one finds that

$$Z_2(\rho,\ k_2) = C_7 \quad . \tag{81}$$

The corresponding zeroth order expansion for the initial regime was then found to be

$$Z = \ln v_o^2 \quad ; \tag{82}$$

thus

$$C_7 = \ln v_o^2 \quad ; \tag{83}$$

and now, in the aerodominated regime,

$$Z_2^{(0)} = \ln v_o^2 - \frac{(1+a_1)\rho_2}{\sin\Theta_0} \quad . \tag{84}$$

The first order equations are now

$$\frac{dZ_2^{(1)}}{d\rho_2} = \frac{a_1 a_2}{\sin\Theta_0}\,\frac{1}{\rho_2} \quad , \tag{85-a}$$

$$\frac{dY_2^{(1)}}{d\rho_2} = \frac{k_2'}{2\cos\Theta_0} + \frac{1}{\rho_2} - \frac{e^{\frac{(1+a_1)\rho}{\sin\Theta_0}}}{v_o^2\,\rho_2} \quad , \tag{85-b}$$

$$\frac{d\phi_2^{(1)}}{d\rho_2} = \frac{\cot\Theta_0}{\rho_2} \quad , \tag{85-c}$$

and

$$\frac{dt_2^{(1)}}{d\rho_2} = \frac{1}{v_o \sin\Theta_0}\,\frac{e^{\frac{(1+a_1)\rho_2}{\sin\Theta_0}}}{\rho_2} \quad , \tag{85-d}$$

where the necessary zeroth order terms have been inserted.

Integrating equation (85-a) yields

$$Z_2^{(1)} = \frac{a_1 a_2}{\sin\Theta_o} \ln\rho_2 + C_8 \; . \qquad (86)$$

When this result is employed in equation (84), the first order expansion is found to be

$$Z_2 = \ln v_o^2 - \frac{(1+a_1)\rho_2}{\sin\Theta_0} + \frac{\varepsilon a_1 a_2}{\sin\Theta_0} \ln\rho_2 + \varepsilon C_8 \quad . \qquad (87)$$

Expressing this in terms of the initial variables and passing to the limit $O(\varepsilon^2)$, it is found that

$$Z_2(\rho, k_2) = \ln v_o^2 - \varepsilon^2 \frac{(1+a_1)\rho}{\sin\Theta_0} + \frac{\varepsilon a_1 a_2}{\sin\Theta_0} \ln(\varepsilon^2\rho) + \varepsilon C_8 \quad . \qquad (88)$$

The corresponding first order expansion in the initial regime from equation (65) has been obtained as

$$Z_1 = \ln v_o^2 - \varepsilon^2 \frac{(1+a_1)}{\sin\Theta_0} (\rho-1) + \frac{\varepsilon a_1 a_2}{\sin\Theta_0} \ln\left(\frac{a_2+\varepsilon\rho}{a_2+\varepsilon}\right) + \varepsilon^2 \frac{2}{v_o^2} \ln\rho \quad .$$

Now, if this expression if written in terms of the aerodominated variables and expanded, then on passing to the limit of $O(\varepsilon)$, one finds that

$$Z(\rho_2, k_2') = \ln v_o^2 - \frac{(1+a_1)\rho_2}{\sin\Theta_0} + \frac{\varepsilon a_1 a_2}{\sin\Theta_0}\left[\ln\rho_2 - \ln(\varepsilon a_2)\right] . \qquad (89)$$

Before these expressions can be matched to the initial regime, it is necessary to write both expressions in terms of the same variables. Thus, after rewriting equation (89) one finds

$$Z(\rho, k_2) = \ln v_o^2 - \frac{\varepsilon^2(1+a_1)\rho}{\sin\Theta_0} + \frac{\varepsilon a_1 a_2}{\sin\Theta_0}\left[\ln(\varepsilon^2\rho) - \ln(\varepsilon a_2)\right] . \qquad (90)$$

Equating equations (88) and (90) determines C_8 as:

$$C_8 = - \frac{a_1 a_2}{\sin\Theta_0} \ln(\varepsilon a_2) \ ; \qquad (91)$$

thus,

$$Z_2^{(1)} = \frac{a_1 a_2}{\sin\Theta_0} \ln\left(\frac{\rho_2}{\varepsilon a_2}\right) . \qquad (92)$$

Integrating equations (85-b), (85-c), and (85-d) and matching in a similar manner yield

$$Y_2^{(1)} = \frac{k_2' \rho_2}{2\cos\Theta_0} + \left(1 - \frac{1}{v_o^2}\right) \ln\left(\frac{\rho_2}{\varepsilon^2}\right) - \sum_{n=1}^{\infty} \frac{1}{n n!}\left[\frac{(1+a_1)\rho_2}{\sin\Theta_0}\right]^n , \qquad (93)$$

$$\phi_2^{(1)} = \cot\Theta_0 \ln(\rho_2/\varepsilon^2) , \qquad (94)$$

and

$$t_2^{(1)} = \frac{1}{v_o \sin\Theta_0}\left\{\ln(\rho_2/\varepsilon^2) + \sum_{n=1}^{\infty}\left[\frac{(1+a_1)\rho_2}{2\sin\Theta_0}\right]^n\right\} . \qquad (95)$$

It should be noted that the first order expansions for 1/2, ϕ_2 and t_2 are all of $O(\varepsilon^2)$; see equation (76).

The second order expressions, from equations (77-a) through (77-d), are

$$\frac{dZ_2^{(2)}}{d\rho_2} = -\frac{1}{\sin\Theta_0}\left[\frac{a_1 a_2^2}{\rho_2^2} + (1+a_1)\cot^2\Theta_0 Y_2^{(1)}\right] + \frac{2}{v_o^2\rho_2}\, e^{\frac{(1+a_1)\rho_2}{\sin\Theta_0}} . \qquad (96\text{-a})$$

$$\frac{dY_2^{(2)}}{d\rho_2} = \frac{e^{\frac{(1+a_1)\rho_2}{\sin\Theta_0}}}{v_o^2\rho_2} Z_2^{(1)} , \qquad (96\text{-b})$$

$$\frac{d\phi_2^{(2)}}{d\rho_2} = \frac{\cot\Theta_0}{\rho_2}\,\frac{Y_2^{(1)}}{\sin^2\Theta_0} , \qquad (96\text{-c})$$

and

$$\frac{dt_2^{(2)}}{d\rho_2} = -\frac{Z_2^{(1)}\, e^{\frac{(1+a_1)\rho_2}{2\sin\Theta_0}}}{2v_o\sin\Theta_0\rho_2} ; \qquad (96\text{-d})$$

where the zeroth terms have been inserted previously, and the first order terms are given by equations (92) and (93).

Using equation (93) for $Y_2^{(1)}$, then equation (96-a) can be integrated to yield

$$Z_2^{(2)} = -\frac{1}{\sin\theta_o}\left\{\left\{ -\frac{a_1 a_2^2}{\rho} + (1+a_1)\cot^2\theta_o \left\{\frac{k_2'}{2\cos\Theta_0}\,\frac{\rho_2^2}{\rho_2} + \right.\right.\right.$$

$$(1 - \frac{1}{v_o^2})\rho_2[\ln(\rho_2/\varepsilon^2) - 1] - \frac{\rho_2}{v_o^2} \sum_{n=1}^{\infty} \frac{1}{n(n+1)!} \left[\frac{(1+a_1)\rho_2}{\sin\Theta_0}\right]^n \Bigg\}\Bigg\} +$$

$$\frac{2}{v_o^2} \left\{ \ln\rho_2 + \sum_{n=1}^{\infty} \frac{1}{nn!} \left[\frac{(1+a_1)\rho_2}{\sin\Theta_0}\right]^n \right\} + C_{12} \ . \tag{97}$$

This resultant is used along with equations (84) and (92) to form this second order expansion. This new resultant then is matched with the corresponding second order expansion for the initial flight regime, equation (65), to yield

$$C_{12} = \frac{(1+a_1)}{\sin\Theta_0} - \frac{2}{v_o^2} \ln(\varepsilon^2) \ . \tag{98}$$

In a similar manner, equations (96-b), (96-c) and (96-d) can be integrated and matched with the corresponding expansions for the initial flight regime to yield

$$Y_2^{(2)} = \frac{a_1a_2}{v_o^2\sin\Theta_0} \Bigg\{\Bigg\{ \frac{1}{2} \ln^2\left(\frac{\rho_2}{\varepsilon a_2}\right) + \sum_{n=1}^{\infty} \frac{1}{nn!} \left[\frac{(1+a_1)\rho_2}{\sin\Theta_0}\right]^n \times$$

$$\left[\ln\left(\frac{\rho_2}{\varepsilon a_2}\right) - \frac{1}{n}\right] \Bigg\}\Bigg\} , \tag{99}$$

$$\phi_2^{(2)} = \frac{\cot\Theta_0}{\sin^2\Theta_0} \left\{ \frac{k_2'}{2\cos\Theta_0} \rho_2 + (1 - \frac{1}{v_o^2}) \frac{\ln^2(\rho_2/\varepsilon^2)}{2} - \right.$$

$$\left. \frac{1}{v_0^2} \sum_{n=1}^{\infty} \frac{1}{n^2n!} \left[\frac{(1+a_1)\rho_2}{\sin\Theta_0}\right]^n \right\} , \tag{100}$$

and

$$t_2^{(2)} = -\frac{a_1a_2}{2v_o\sin^2\Theta_0}\left\{\frac{\ln^2(\rho_2/\varepsilon a_2)}{2} + \sum_{n=1}^{\infty}\frac{1}{nn!}\left[\frac{(1+a_1)\rho_2}{\sin\Theta_0}\right]^n \times\right.$$

$$\left.\left[\ln(\rho_2/\varepsilon\, a_2) - \frac{1}{n}\right]\right\}. \qquad (101)$$

The asymptotic expansion for this flight regime can now be written after substituting the ordered solutions into equations (76); the results are

$$Z_2 = \ln v_o^2 - \frac{(1+a_1)}{\sin\Theta_0}(\rho_2-\varepsilon^2) + \frac{\varepsilon a_1a_2}{\sin\Theta_0}\ln(\rho_2/\varepsilon a_2) - \frac{\varepsilon^2a_1a_2}{\sin\Theta_0}\frac{1}{\rho_2} -$$

$$\varepsilon^2\frac{(1+a_1)\cot^2\Theta_0}{\sin\Theta_0}\left\{\frac{k_2'}{2\cos\Theta_0}\frac{\rho_2^2}{2} + (1-\frac{1}{v_o^2})\,\rho_2[\ln(\rho_2/\varepsilon^2) - 1] -\right.$$

$$\left.\frac{\rho_2}{v_o^2}\sum_{n=1}^{\infty}\frac{1}{n(n+1)!}\left[\frac{(1+a_1)\rho_2}{\sin\Theta_0}\right]^n\right\} + \frac{\varepsilon^2 2}{v_o^2}\left\{\ln(\rho_2/\varepsilon^2) + \right.$$

$$\left.\sum_{n=1}^{\infty}\frac{1}{nn!}\left[\frac{(1+a_1)\rho_2}{\sin\Theta_0}\right]^n\right\} + O(\varepsilon^3)\,, \qquad (102)$$

$$Y_2 = \ln\cos\Theta_0 + \frac{\varepsilon^2k_2'\rho_2}{2\cos\Theta_0} + \varepsilon^2\,(1-\frac{1}{v_o^2})\,\ln^2(\rho_2/\varepsilon^2) -$$

$$\frac{\varepsilon^2}{v_o^2}\sum_{n=1}^{\infty}\frac{1}{nn!}\left[\frac{(1+a_1)\rho_2}{\sin\Theta_0}\right]^n + \frac{\varepsilon^3a_1a_2}{v_o^2\sin\Theta_0}\left\{\frac{\ln^2(\rho_2/\varepsilon^2)}{2} +\right.$$

$$\sum_{n=1}^{\infty} \frac{1}{nn!} \left[\frac{(1+a_1)\rho_2}{\sin\Theta_0} \right]^n \left[\ln(\rho_2/\varepsilon a_2) - \frac{1}{n} \right] \Bigg\} + O(\varepsilon^4) \quad , \tag{103}$$

$$\phi_2 = \varepsilon^2 \cot\Theta_0 \ln(\rho_2/\varepsilon^2) + \frac{\varepsilon^4 \cot\Theta_0}{\sin^2\Theta_0} \left\{ \frac{k_2' \rho_2}{2\cos\Theta_0} + \left(1 - \frac{1}{v_o^2}\right) \frac{\ln^2(\rho_2/\varepsilon^2)}{2} - \right.$$

$$\left. \frac{1}{v_o^2} \sum_{n=1}^{\infty} \frac{1}{nn!} \left[\frac{(1+a_1)\rho_2}{\sin\Theta_0} \right]^n \right\} + O(\varepsilon^5) \quad , \tag{104}$$

and

$$t_2 = \frac{\varepsilon^2}{v_o \sin\Theta_0} \left\{ \ln(\rho_2/\varepsilon^2) + \sum_{n=1}^{\infty} \frac{1}{nn!} \left[\frac{(1+a_1)\rho_2}{2\sin\Theta_0} \right]^n \right\} -$$

$$\frac{\varepsilon^3 a_1 a_2}{2 v_o \sin^2\Theta_0} \left\{ \frac{\ln^2(\rho_2/\varepsilon a_2)}{2} + \sum_{n=1}^{\infty} \frac{1}{nn!} \left[\frac{(1+a_1)\rho_2}{2\sin\Theta_0} \right]^n \left[\ln^2(\rho_2/\varepsilon a_2) - \right. \right.$$

$$\left. \left. \frac{1}{n} \right] \right\} + O(\varepsilon^4) \quad . \tag{105}$$

As indicated above, the next order of magnitude terms in this flight regime are $O(\varepsilon^3)$, $O(\varepsilon^4)$, $O(\varepsilon^5)$ and $O(\varepsilon^4)$, respectively, as noted from equations (102) through (105). In spite of this indicated accuracy, however, it should be noted that neither equation (103) - for the flight path angle - nor equation (105) - for the flight time-reflect an effect from a change in the flight path angle. Although this effect is somewhat suppressed in the integration, a large percentage change in flight path angle has a significant affect on the velocity decay;

this, in turn, affects the centrifugal acceleration term in the equation for the flight path angle.

The significance of changes in the flight path angle can be deduced, somewhat, by noting that only the first significant term of the expanded sine function, equation (41), has been obtained in the solution for the speed. It can be noted, quite easily, from a numerical investigation that at least two significant terms, from equation (41), are required if angular changes of more than approximately 20% are to be allowed. It follows, then, that this solution will not be valid when changes in the flight path angle of more than 20% are to be expected. As a consequence, this restricts the solution to small lift cases, $k_2 \leq O(\varepsilon^2)$, and to initial flight path angles which are not too small; i.e., to where $\sin\Theta_o > O(\varepsilon)$.

V. COMPOSITE EXPANSION

In Chapter IV the matched asymptotic expansions were obtained for the initial and the aerodominated flight regimes. For this problem in particular, the expansion for the initial regime was shown to match, in the limit, Picard's iterative solution, which would be a unique solution. However, it was apparent, as is usually the case, that the algebraic complexity-after obtaining the first few terms - becomes prohibitive. Hence only the first three terms(of zeroth, first, and second order) were generated. The basic differential equations were then appropriately transformed - by order of magnitude transformations - and a new second order expansion was generated which matched uniquely to the initial expansion. In this chapter, these two matched asymptotic expansions will be used to form a uniformly valid composite expansion.

5.1 Definition of A Composite Expansion

A composite expansion is defined as any series which reduces to a first expansion, when expanded asymptotically for small ε in the first variables, and to a second expansion when expanded asymptotically for small ε in the second variables.

5.2 The Additive Composition

The additive composition for matched asymptotic expansions, as given by Van Dyke[8], can be expressed as

$$F_c^{(m,n)} = \begin{array}{l} F_i^{(m)} + F_o^{(n)} - [F_o^{(n)}]_i^{(m)} \\ \\ F_o^{(n)} + F_i^{(m)} - [F_i^{(m)}]_o^{(n)} \end{array} \; ; \qquad (106)$$

where $F_c^{(m,n)}$ is the composite expansion correct to ε^m in the first variables and correct to ε^n in the second;

$F_i^{(m)}$ is the first expansion to ε^m;

$F_o^{(n)}$ is the second expansion to ε^n;

$[F_o^{(n)}]_i^{(m)}$ is the second expansion to ε^n, expanded in the first variables to ε^m;

$[F_i^{(m)}]_o^{(n)}$ is the first expansion to ε^m, expanded in the second variables to ε^n.

The two expressions obtained from equation (106) are equivalent.

5.3 A Composite Expansion for the Velocity

A composite expansion for the velocity can be formed by using equations (65) and (102) in the additive technique as indicated above. It should be noted that the solution in the initial flight regime contains all of the significant terms for ρ --small; thus, this expansion is valid up to $O(\varepsilon^4)$ for the Keplerian flight regime. Therefore, it follows that the forming of a composite expansion, using the result for the initial and the aerodominated regimes, will yield an expansion which is valid from the Keplerian flight regime down through the aerodominated regime.

In the notation of equation (106), the composite expansion becomes

$$Z_c^{(4,3)} = Z_1^{(4)} + Z_2^{(3)} - [Z_2^{(3)}]_1^{(4)} \quad , \qquad (107)$$

where $Z_1^{(4)}$ is given by equation (65), and

$Z_2^{(3)}$ is given by equation (102) .

Substituting the appropriate expressions into equation (107) only serves to eliminate the common terms and, in the original variables, yields

$$\begin{aligned}
Z_c^{(4,3)} = {} & \ln v_o^2 - \frac{\varepsilon^2}{\sin\Theta_0} \Bigg\{ (1 + a_1)(\rho - 1) - \frac{a_1 a_2}{2} \ln \left(\frac{a_2+\varepsilon\rho}{a_2+\varepsilon} \right) \Bigg\} - \\
& \varepsilon^4 \frac{\cot^2\Theta_0}{\sin\Theta_0} \Bigg\{\!\!\Bigg\{ \frac{k_2}{2\cos\Theta_0} \Bigg\{ (1 + a_1) \frac{(\rho-1)^2}{2} - \frac{a_1 a_2}{\varepsilon} \Bigg[\varepsilon(\rho - 1) - \\
& (a_2 + \varepsilon) \ln \left(\frac{a_2+\varepsilon\rho}{a_2+\varepsilon} \right) \Bigg] \Bigg\} + \left(1 - \frac{1}{v_o^2}\right) (1 + a_1) [\rho \ln\rho - (\rho-1)] - \frac{\rho}{v_o^2} \times \\
& \sum_{n=1}^{\infty} \frac{1}{n(n+1)!} \left[\frac{\varepsilon^2 (1+a_1)\rho}{\sin\Theta_0} \right] \Bigg\}\!\!\Bigg\} + \varepsilon^2 \frac{2}{v_o^2} \Bigg\{\!\!\Bigg\{ \ln\rho + \sum_{n=1}^{\infty} \Bigg\{ \frac{1}{nn!} \times \\
& \left[\frac{\varepsilon^2 (1+a_1)\rho}{\sin\Theta_0} \right]^n \Bigg\} - \varepsilon^2 \frac{(1+a_1)}{\sin\Theta_o} (1 + \ln\rho) \Bigg\}\!\!\Bigg\} - \varepsilon^3 \frac{a_1 a_2}{\sin\Theta_0} \left[\frac{2}{v_o^2} + \cot^2\Theta_0 \left(1 - \frac{1}{v_o^2}\right) \right] \times
\end{aligned}$$

$$\left\{\frac{1}{2}\ln^2\left(1+\frac{\varepsilon\rho}{a_2}\right)-\frac{1}{2}\ln^2\left(1+\frac{\varepsilon\rho}{a_2}\right)+\sum_{n=1}^{\infty}\frac{1}{n^2}\left[\left(\frac{\varepsilon\rho}{a_2+\varepsilon\rho}\right)^n-\left(\frac{\varepsilon}{a_2+\varepsilon}\right)^n\right]-\ln\rho\,\ln\left(1+\frac{\varepsilon}{a_2}\right)\right\}+\frac{\varepsilon^4}{v_o{}^4}\ln^2\rho\ . \tag{108}$$

5.4 A Composite Expansion for the Flight Path Angle

The composite expansion for the flight path angle can be formed now, using the results for the initial and aerodominated regimes. In the notation of equation (106) this becomes

$$Y_c^{(4,3)}=Y_1^{(4)}+Y_2^{(3)}-[Y_2^{(3)}]_1^{(4)}\ , \tag{109}$$

where $Y_1^{(4)}$ is given as equation (66), and

$Y_2^{(3)}$ is given as equation (103) .

Substituting the appropriate expansions into equation (109) and rearranging in terms of the original variables yield

$$Y_c^{(4,3)}=\ln\cos\Theta_0+\varepsilon^2\frac{k_2(\rho-1)}{2\cos\Theta_0}-\frac{1}{2}\left(\frac{\varepsilon^2k_2(\rho-1)}{2\cos\Theta_0}\right)^2+\varepsilon^2\left(1-\frac{1}{v_o^2}\right)\ln\rho-$$

$$\frac{\varepsilon^4k_2}{2\cos\Theta_0}\left(1-\frac{1}{v_o^2}\right)[\rho\ln\rho-(\rho-1]-\frac{\varepsilon^2}{v_o^2}\sum_{n=1}^{\infty}\frac{1}{nn!}\left[\frac{\varepsilon^2(1+a_1)\rho}{\sin\Theta_0}\right]^n+$$

$$\varepsilon^4 \frac{(1+a_1)(1+\ln\rho)}{v_o^2 \sin\Theta_0} + \frac{\varepsilon^3 a_1 a_2}{v_o^2 \sin\Theta_0} \left\{ \frac{1}{2} \ln^2(1 + \frac{\varepsilon\rho}{a_2}) - \frac{1}{2} \ln^2(1 + \frac{\varepsilon}{a_2}) + \right.$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \left[\left(\frac{\varepsilon\rho}{a_2+\varepsilon\rho} \right)^n - \left(\frac{\varepsilon}{a_2+\varepsilon} \right)^n \right] - \ln\rho \ln (1 + \frac{\varepsilon}{a_2}) +$$

$$\left. \sum_{n=1}^{\infty} \frac{1}{n n!} \left[\frac{\varepsilon^2(1+a_1)\rho}{\sin\Theta_0} \right]^n [\ln(1 + \frac{\varepsilon\rho}{a_2}) - \frac{1}{n}] \right\} + \frac{\varepsilon^4}{v_o^4} \ln^2\rho; \qquad (110\text{-a})$$

where, in order to preserve the proper sense of the higher order modulation term for the case of small density, the logarithmic term in equation (103) is modified. That is, this term is altered to read

$$\ln(\rho_2/\varepsilon a_2) \simeq \ln(1 + \rho_2/\varepsilon a_2) \; . \qquad (110\text{-b})$$

This approximation was not required in order to satisfy the matching technique used in the previous section. On the other hand, the argument of this logarithmic term was generated as a result of having expanded the drag modulation function in the solution for the aerodominated flight regime; see equation (72). Thus, this term when compared with the other expressions that arise from the modulation function appears to be over emphasized where the nondimensional density is small; i.e., when $\rho_2 \to O(\varepsilon^2)$. The approximation given as equation (110-b) is used to correct this apparent error for ρ small. It is easily shown, however, that the error that arises from this approximation is at most of $O(\varepsilon)$ in the aerodominated flight regime and is, therefore, within the accuracy of this term.

5.5 Composite Expansions for the Range Angle and the Flight Time

The composite expansion for the range angle, obtained by using equation (106), can be expressed as

$$\phi_c^{(4,4)} = \phi_1^{(4)} + \phi_2^{(4)} - [\phi_2^{(4)}]_1^{(4)} \; ; \tag{111}$$

wherein $\phi_1^{(4)}$ and $\phi_2^{(4)}$ are obtained from equations (67) and (104), respectively. Using the appropriate expressions, this composite expansion becomes

$$\phi_c^{(4,4)} = \varepsilon^2 \cot\Theta_0 \ln\rho + \frac{\varepsilon^4 \cot\Theta_0}{\sin^2\Theta_0} \left\{ \frac{k_2}{2\cos\Theta_0} [\rho - 1 - \ln\rho] + \frac{1}{2}(1 - \frac{1}{v_o^2})\ln^2\rho - \frac{1}{v_o^2} \sum_{n=1}^{\infty} \frac{1}{n^2 n!} \left[\frac{\varepsilon^2(1+a_1)\rho}{\sin\Theta_0}\right]^n \right\} . \tag{112}$$

The composite expansion for the flight time, again obtained by using equation (106), can be expressed as

$$t_c^{(4,3)} = t_1^{(4)} + t_2^{(3)} - [\,t_2^{(3)}\,]_1^{(4)} \; ; \tag{113}$$

where $t_1^{(4)}$ and $t_2^{(3)}$ are given by equations (68) and (105), respectively.

Similar to the previous cases, equation (113) leads directly to

$$t_c^{(4,3)} = \frac{\varepsilon^2}{v_o \sin\Theta_o}\left\{\ln\rho + \sum_{n=1}^{\infty}\frac{1}{nn!}\left[\frac{\varepsilon^2(1+a_1)\rho}{2\sin\Theta_0}\right]^n - \varepsilon^2\frac{(1+a_1)}{2\sin\Theta_0}(1+\ln\rho)\right\} - \frac{\varepsilon^3 a_1 a_2}{2v_o\sin\Theta_0}\left\{\frac{1}{2}\ln^2\left(1+\frac{\varepsilon\rho}{a_2}\right) - \frac{1}{2}\ln^2\left(1+\frac{\varepsilon}{a_2}\right) + \sum_{n=1}^{\infty}\frac{1}{n^2}\left[\left(\frac{\varepsilon\rho}{a_2+\varepsilon\rho}\right)^n - \left(\frac{\varepsilon}{a_2+\varepsilon}\right)^n\right] - \ln\rho\ln\left(1+\frac{\varepsilon}{a_2}\right) + \sum_{n=1}^{\infty}\frac{1}{nn!}\left[\frac{\varepsilon^2(1+a_1)\rho}{2\sin\Theta_0}\right]^n\left[\ln\left(\frac{\varepsilon\rho}{a_2}\right) - \frac{1}{n}\right]\right\} + \varepsilon^4\frac{\cot^2\Theta_0}{\sin\Theta_0}\left\{\frac{k_2}{2\cos\Theta_0}[\rho-1-\ln\rho] + \left(1-\frac{1}{v_o^2}\right)\frac{\ln^2\rho}{2}\right\}. \tag{114}$$

5.6 Rearranged Composite Solutions

Equations (108) and (110) can be written now in a somewhat more acceptable form by comparing this result with the form of the solution given as equations (26-a) and (26-b). With this in mind and recalling, from equation (27-c), that

$$Z = \ln x , \tag{27-c}$$

then by comparison, the corresponding terms in equations (108) and (26-a) lead to

$$x = X_1 e^{X_2} \; ; \tag{115-a}$$

where

$$X_1 = v_o^2 + 2\varepsilon^2\left[\ln\rho + \Phi_1(\omega) - \varepsilon^2 \frac{(1-a_1)}{\sin\Theta_0}(1+\ln\rho) - \varepsilon\frac{a_1a_2}{\sin\Theta_0}M_1\right] , \tag{115-b}$$

and

$$\begin{aligned} X_2 = &- \frac{\varepsilon^2}{\sin\Theta_0}\left[(1+a_1)(\rho-1) - \frac{a_1a_2}{\varepsilon}\ln\left(\frac{a_2+\varepsilon\rho}{a_2+\varepsilon}\right)\right] - \\ &\varepsilon^4 \frac{\cot^2\Theta_0}{\sin\Theta_0}\Bigg\{\Bigg[\frac{k_2}{2\cos\Theta_0}\Bigg\{(1+a_1)\frac{(\rho-1)^2}{2} - \frac{a_1a_2}{\varepsilon^2}\Bigg[\varepsilon(\rho-1) - \\ &(a_2+\varepsilon)\ln\left(\frac{a_2+\varepsilon\rho}{a_2+\varepsilon}\right)\Bigg]\Bigg\} + \left(1 - \frac{1}{v_o^2}\right)\Bigg\{(1+a_1)[\rho\ln\rho - (\rho-1)] + \\ &\frac{a_1a_2}{\varepsilon}M_1\Bigg\} - \frac{\rho}{v_o^2}\Phi_2(\omega)\Bigg]\Bigg\} . \end{aligned} \tag{115-c}$$

In this resultant, the following quantities are defined:

$$\Phi_1(\omega) = \sum_{n=1}^{\infty}\frac{\omega^n}{n\,n!} , \tag{116-a}$$

$$\Phi_2(\omega) = \sum_{n=1}^{\infty}\frac{\omega^n}{n(n+1)!} , \tag{116-b}$$

wherein

$$\omega = \varepsilon^2 \frac{(1+a_1)\rho}{\sin\Theta_0} \; ; \qquad (116\text{-c})$$

and

$$M_1 = \frac{1}{2}\ln^2(1 + \frac{\varepsilon\rho}{a_2}) - \frac{1}{2}\ln^2(1 + \frac{\varepsilon}{a_2}) - \ln\rho\ln(1 + \frac{\varepsilon}{a_2}) +$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2}\left[\left(\frac{\varepsilon\rho}{a_2+\varepsilon\rho}\right)^n - \left(\frac{\varepsilon}{a_2+\varepsilon}\right)^n\right] . \qquad (116\text{-d})$$

Similarly, recalling that from equation (27-d)

$$Y = \ln\cos\Theta \; ; \qquad (27\text{-d})$$

then by comparing the corresponding terms in equations (110) and (26-b) leads to

$$\cos\Theta = (\cos\Theta_0 + \frac{k_2}{2} \beta_1)\, e^{\beta_2} \; ; \qquad (117\text{-a})$$

wherein

$$\beta_1 = \varepsilon^2(\rho-1) + \varepsilon^4(1 - \frac{1}{v_o^2})\;[\rho\ln\rho - (\rho-1)\;, \qquad (117\text{-b})$$

and

$$\beta_2 = \varepsilon^2(1 - \frac{1}{v_o^2})\ln\rho - \frac{\varepsilon^2}{v_o^2}\,[1 - \frac{\varepsilon a_1 a_2}{\sin\Theta_0}\ln(1 + \frac{\varepsilon\rho}{a_2})\;]\;\Phi_1(\omega) -$$

$$\varepsilon^3 \frac{a_1 a_2}{v_o^2 \sin\Theta_o} \Phi_3(\omega) + \varepsilon^4 \cdot \frac{(1+a_1)(1+\ln\rho)}{v_o^2 \sin\Theta_0} + \frac{\varepsilon^3 a_1 a_2}{v_o^2 \sin\Theta_0} M_1 \ . \qquad (117\text{-c})$$

For this resultant $\Phi_1(\omega)$, M_1, and ω are given as equations (116-a), (116-b), and (116-c), respectively, and

$$\Phi_3(\omega) = \sum_{n=1}^{\infty} \frac{\omega^n}{n^2 n!} \ . \qquad (118)$$

The summation functions, Φ_1, Φ_2, and Φ_3 are presented on Figure 4. Figure 5 shows a plot of the modulation term M_1.

Equation (117-a), however, is not fully amenable to a numerical solution for small flight path angles. It is convenient, therefore, to rearrange equation (117-a); that is, after squaring and expanding, it can be shown that

$$\sin^2\Theta = \sin^2\Theta_0 - k_2\cos\Theta_0\beta_1 - \left(\frac{k_2\beta_1}{2}\right)^2 -$$

$$(2\beta_2 + \beta_2^2 + \ldots)\left[\cos^2\Theta_0 + k_2\cos\Theta_0\beta_1 + \left(\frac{k_2\beta_1}{2}\right)^2\right] . \qquad (119)$$

In keeping with the original solution, the lift coefficient, k_2, when multiplied by the function β_1, and the function β_2 must be small. Thus the higher order terms here may be neglected; i.e., when considering maximum changes in $\sin\Theta$ of about 20%. Hence, for this situation equation (119) becomes

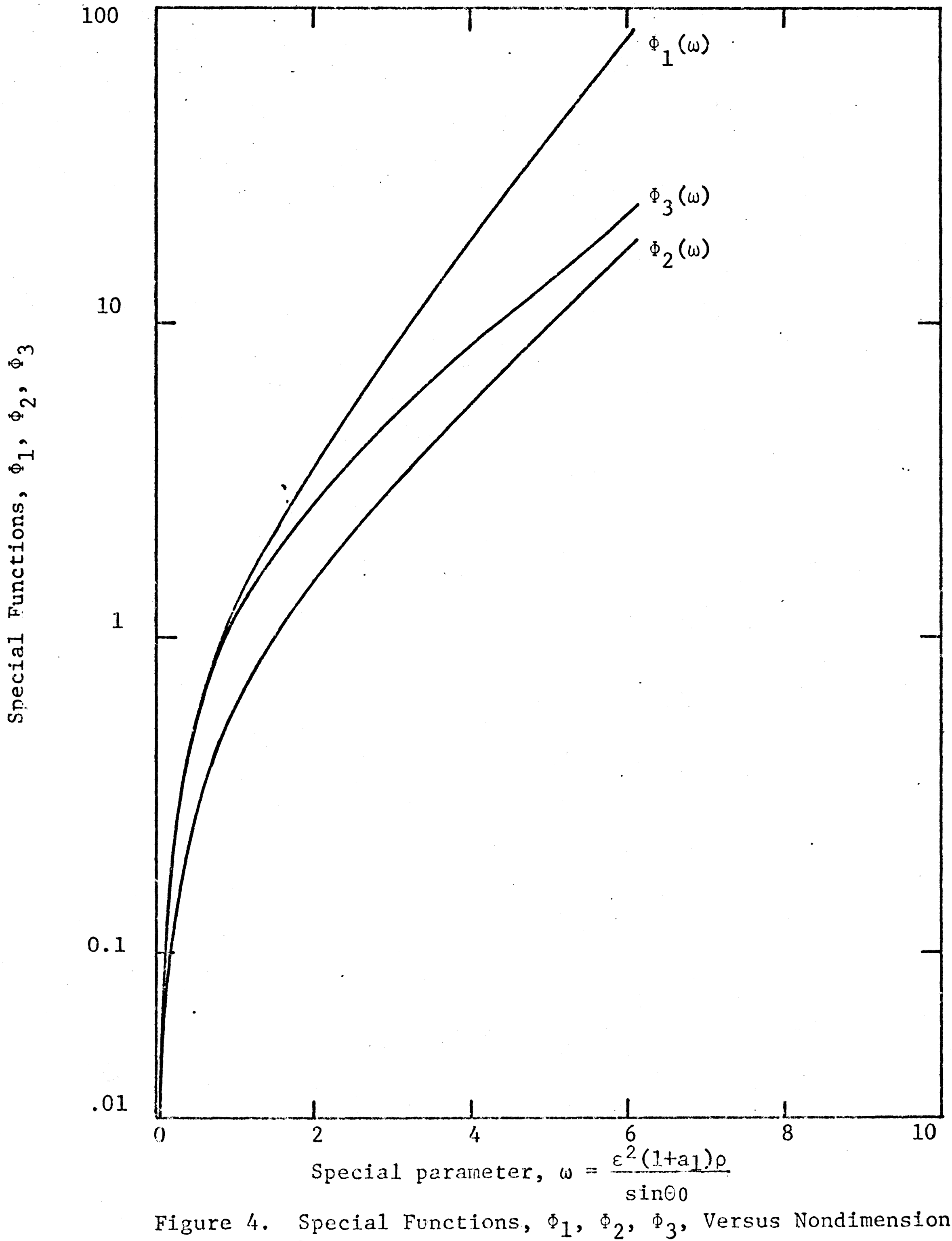

Figure 4. Special Functions, Φ_1, Φ_2, Φ_3, Versus Nondimensional Parameter, ω.

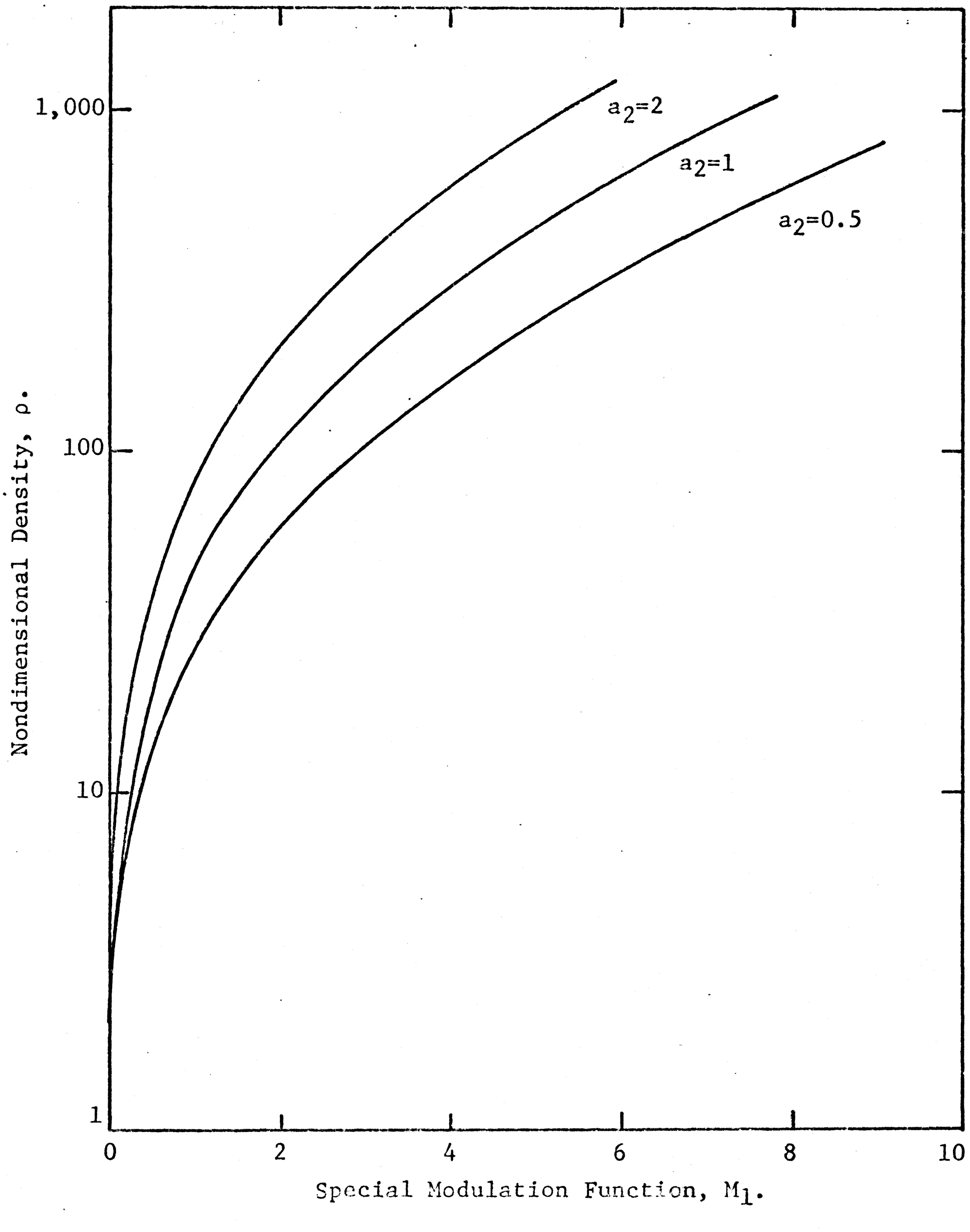

Figure 5. Special Modulation Function, M_1, Versus Nondimensional Density, ρ.

$$\sin^2\Theta = \sin^2\Theta_0 - k_2\cos\Theta_o\beta_1 - 2\cos^2\Theta_o\beta_2 \ ; \qquad (120)$$

where the functions β_1 and β_2 have been given by equations (117-b) and (117-c), respectively.

Equations for the range angle and the flight time cannot be rearranged into a more convenient form, but they can be rewritten in terms which are similar to those used above. In this regard equations (112) and (114) are rewritten as

$$\phi = \varepsilon^2\cot\Theta_0\ln\rho + \varepsilon^4 \frac{\cot\Theta_0}{\sin^2\Theta_0}\left\{\frac{k_2}{2\cos\Theta_0}[\rho-1-\ln\rho] + \frac{1}{2}\left(1 - \frac{1}{v_o^2}\right)\ln^2\rho - \frac{1}{v_o^2}\Phi_3(\omega)\right\}, \qquad (121)$$

and

$$t = \frac{\varepsilon^2}{v_o\sin\Theta_0}\left\{\ln\rho + \Phi_1\left(\frac{\omega}{2}\right) - \varepsilon^2\frac{(1+a_1)}{2\sin\Theta_0}(1+\ln\rho)\right\} - \frac{\varepsilon^3a_1a_2}{2v_o\sin\Theta_0}\left\{M_1 + \ln\left(1+\frac{\varepsilon\rho}{a_2}\right)\Phi_1\left(\frac{\omega}{2}\right) - \Phi_3\left(\frac{\omega}{2}\right)\right\} + \frac{\varepsilon^4\cot^2\Theta_0}{v_o\sin\Theta_0}\left\{\frac{k_2}{2\cos\Theta_0}[\rho-1-\ln\rho] + \frac{1}{2}\left(1-\frac{1}{v_o^2}\right)\ln^2\rho\right\} \ ; \qquad (122)$$

where the functions Φ_1, Φ_3, and M_1 are given as equations (116-a), (118), and (116-d), respectively.

The rearranged composite solutions obtained in this section, in general, can not be considered more significant than the original composite solutions that were generated previously. The resultant expressions are somewhat more convient, however, for numerical computation and for further analysis - such as follows in Chapter VII, "Analysis for the Region of Validity."

VI. ANALYSIS OF RESULTS

In this section the solution obtained in this thesis is compared with computer generated numerical results. In order to retain consistence with other published materials, the density is converted into a corresponding nondimensional altitude. Finally, the numerical results from this analysis are plotted versus this altitude parameter for presentation purposes.

6.1 Nondimensional Altitude

Broglio's power density law[2] has been found to yield essentially the same result as that obtained by means of the usual exponential density law, but as was shown by Broglio, it is more adaptable to the writing of the differential governing equations in similarity form. In order to present the generated data in a form consistent with that of previous investigators it is more convenient to convert the power-law to the usual exponential form, as demonstrated by the following manipulation.

From equation (8-b) one can write

$$r = 1 - \varepsilon^2 \ln \rho + \varepsilon^4 \frac{\ln^2 \rho}{2} + \ldots + , \tag{8-b}$$

or

$$r - 1 = - \varepsilon^2 \ln \rho + \varepsilon^4 \frac{\ln^2 \rho}{2} + \ldots \quad . \tag{123}$$

Over the range of nondimensional density, ρ, used for this reentry study,

$$\varepsilon^2 \ln\rho \ll 1. \tag{124}$$

Thus, to a reasonable and good approximation, the higher order terms may be neglected.

As a consequence of neglecting terms of ε^4 and higher, equation (123) can be written as

$$\rho = \exp\left[\frac{1-r}{\varepsilon^2}\right] , \tag{125}$$

and on using equation (2-c) this becomes

$$\rho = \exp\left[\frac{\tilde{r}_o-\tilde{r}}{\varepsilon^2\tilde{r}_o}\right] . \tag{126}$$

Now, writing

$$\tilde{r} = \tilde{r}_e + \tilde{h} , \tag{127}$$

and

$$\tilde{r}_o = \tilde{r}_e + \tilde{h}_o ; \tag{128}$$

where $\tilde{h}$ is the altitude above the planet surface,

$\tilde{h}_o$ is the altitude above the planet surface at the initial conditions,

and $\tilde{r}_e$ is the radius of the planet;

equation (126) can be replaced by

$$\rho = \exp \frac{\tilde{h}_o-\tilde{h}}{\varepsilon^2\tilde{r}_o} = \exp(H) , \tag{129}$$

where $H = \dfrac{\tilde{h}_o-\tilde{h}}{\varepsilon^2\tilde{r}_o}$ is a nondimensional altitude measured from the initial altitude. This result can be reduced further to a more familiar form as follows:

Consider

$$\frac{1}{\tilde{r}_o} = \frac{1}{\tilde{r}_e+\tilde{h}_o} = \frac{1}{\tilde{r}_e} (1 + \frac{\tilde{h}_o}{\tilde{r}_e})^{-1} ; \tag{130}$$

which, by expanding, becomes

$$\frac{1}{\tilde{r}_o} = \frac{1}{\tilde{r}_e}\left[1 - \frac{\tilde{h}_o}{\tilde{r}_e} + \left(\frac{\tilde{h}_o}{\tilde{r}_e}\right)^2 + \ldots \right]. \tag{131}$$

On substituting this into the expression for H, one finds that

$$H = H'\left[1 - \frac{\tilde{h}_o}{\tilde{r}_e} + \left(\frac{\tilde{h}_o}{\tilde{r}_e}\right)^2 + \ldots\right] , \tag{132}$$

where

$$H' = \frac{\tilde{h}_o-\tilde{h}}{\varepsilon^2\tilde{r}_e} ; \tag{133}$$

this is in the form of the usual nondimensional altitude function used by Loh[7]. Note, however, that this quantity is measured from the initial reentry altitude towards the planet surface; whereas Loh's value is measured upward from the planet surface.

For reasonable initial reentry altitudes,

$$\frac{\tilde{h}_o}{\tilde{r}_e} << 1 ; \tag{134}$$

thus H can be replaced by H', and the density law reverts to the form usual exponential form,

$$\rho = \exp\left[\frac{\tilde{h}_o - \tilde{h}}{\varepsilon^2 \tilde{r}_e}\right] = \exp(H') . \tag{135}$$

Evaluating equation (135) at sea-level conditions yields

$$\rho_{SL} = \exp\left[\frac{\tilde{h}_o}{\varepsilon^2 \tilde{r}_e}\right] = \left(\frac{k_1 \tilde{A}}{\tilde{m}}\right) \tilde{r}_o \tilde{\rho}_{SL} ; \tag{136}$$

while at the initial conditions

$$1 = \left(\frac{k_1 \tilde{A}}{\tilde{m}}\right) \tilde{r}_o \tilde{\rho}_o . \tag{137}$$

From these two results it follows that

$$\tilde{\rho}_o \doteq \tilde{\rho}_{SL} \exp\left[-\frac{\tilde{h}_o}{\varepsilon^2 \tilde{r}_e}\right], \tag{138}$$

where $\tilde{\rho}_o$ can be determined for any given initial altitude. Using the definition of the nondimensional density and evaluating this at the initial conditions yield

$$\frac{\tilde{m}}{k_1 \tilde{A}} = \tilde{r}_o \tilde{\rho}_o \, , \tag{139}$$

or

$$\frac{\tilde{w}}{k_1 \tilde{A}} = \tilde{g} \;\; \tilde{r}_o \tilde{\rho}_o \quad . \tag{140}$$

This resultant is presented on Figure 6, where the ballistic parameter, $\tilde{w}/k_1\tilde{A}$, is plotted as a function of the initial altitude, $\tilde{h}_o$.

6.2 Comparison with Numerical Solution

Numerical solutions for the governing differential equations were generated using a fourth-order Runge-Kutta numerical integrator which was programmed for use on an IBM-7040 computer located at the Auburn University Computer Center. These numerical solutions were used as a basis for comparison by the analytical solutions obtained in this thesis.

Representative results for the comparison between the numerical solutions and these analytical solutions are presented in the next nine figures. These graphs are explained below.

Figures 7 and 8 are comparisons for classical nonlifting bodies

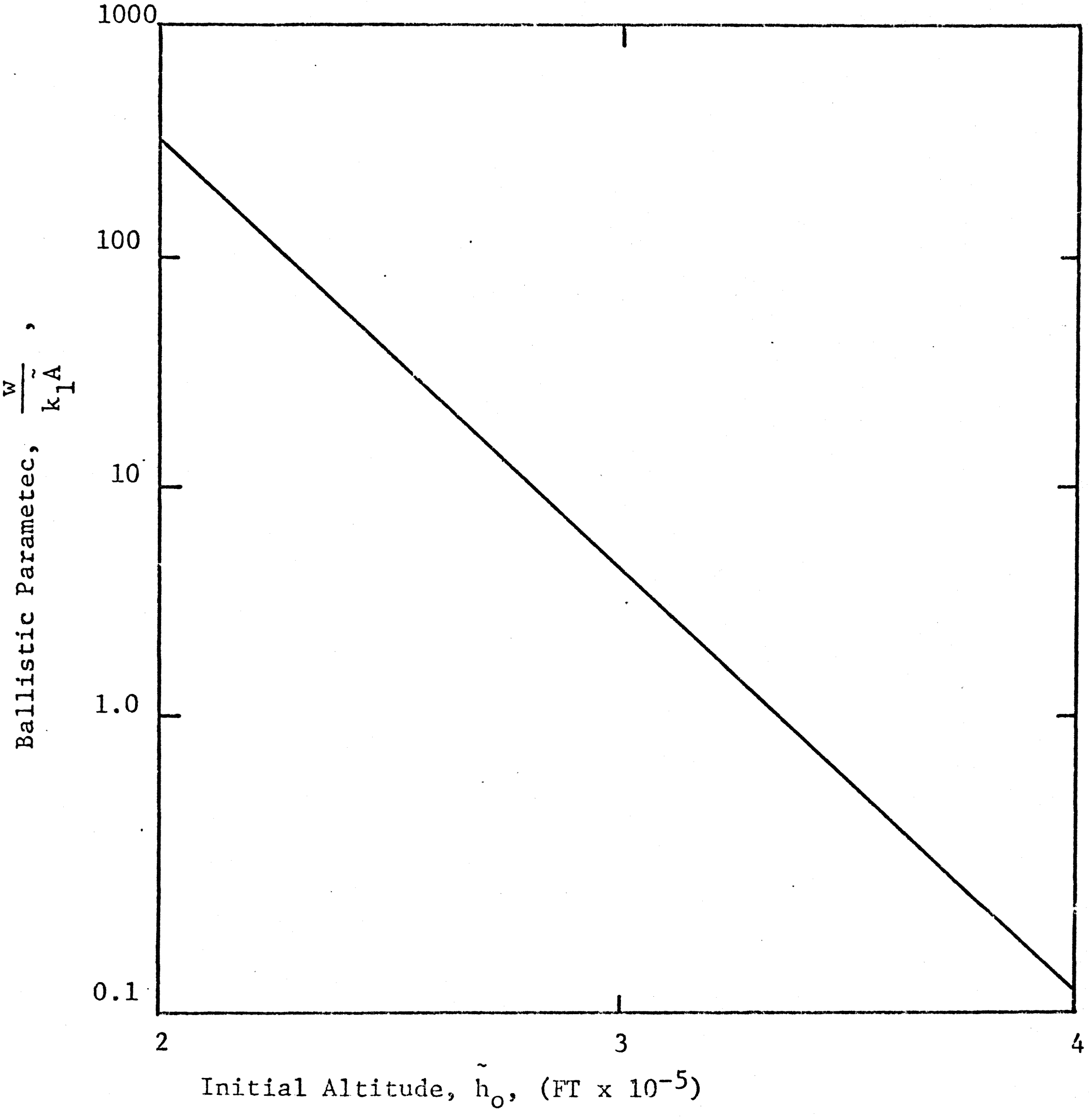

Figure 6. Ballistic Parameter Versus Initial Altitude for Initial Nondimensional Altitude of Unity.

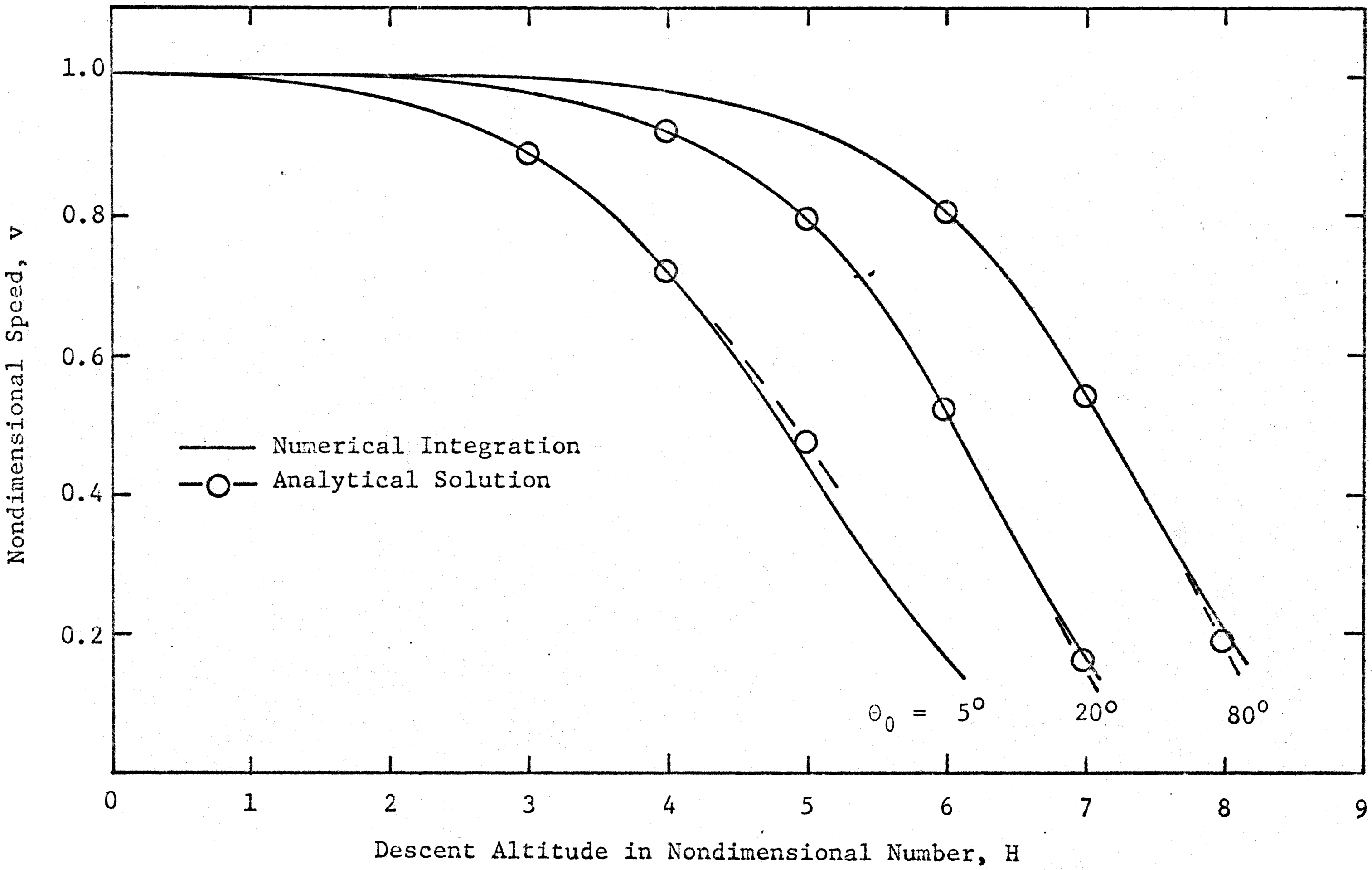

Figure 7. Comparison of Analytical Solution with Numerical Calculation for Non-Lifting Bodies with Constant Drag Coefficient.

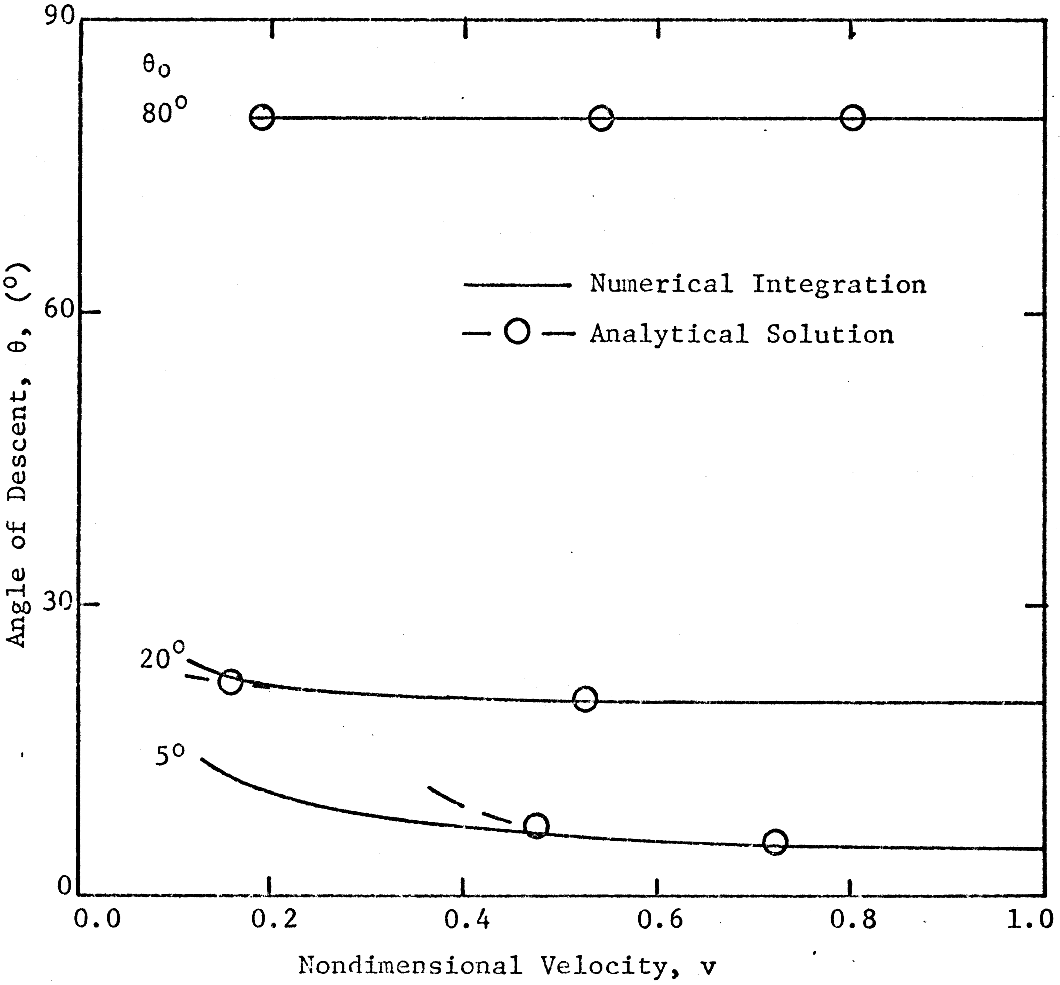

Figure 8. Comparison of Analytical Solution with Numerical Calculations for Non-Lifting Reentry with Constant Drag Coefficient.

assuming a constant drag coefficient; the comparisons show excellent agreement for large initial entry angles, over a range of velocities from the initial value down to rather small values; i.e., for $v_o \geq v \geq 0.2$. For small initial angles, as had been anticipated, the agreement is poor for values of the non-dimensional velocity less than about 0.6.

The next two figures, 9 and 10, show typical results obtained for constant lift and drag coefficients. In addition, these two figures indicate an excellent agreement in the predicting of the Keplerian orbital effects for an initial speed corresponding to excape speed; i.e., v = 1.414 (see Figure 10).

Figures 11, 12, and 13 are representative of the agreement between the exact and analytical solutions for constant lift coefficient with drag modulation. The first two of these figures are similar to those shown for the previous cases, in form and in relative agreement. The third graph, Figure 13, is a plot of the angle of descent versus the non-dimensional altitude. This figure shows the excellent agreement that is obtained for flight path angles which do not change by more than a few percent from the initial value.

Figures 14 and 15 present the relative agreement for the range and flight time during the reentry.

As had been anticipated in developing this analytical solution, the agreement of this analytic solution with the numerically integrated values is excellent so long as the sine of the flight path

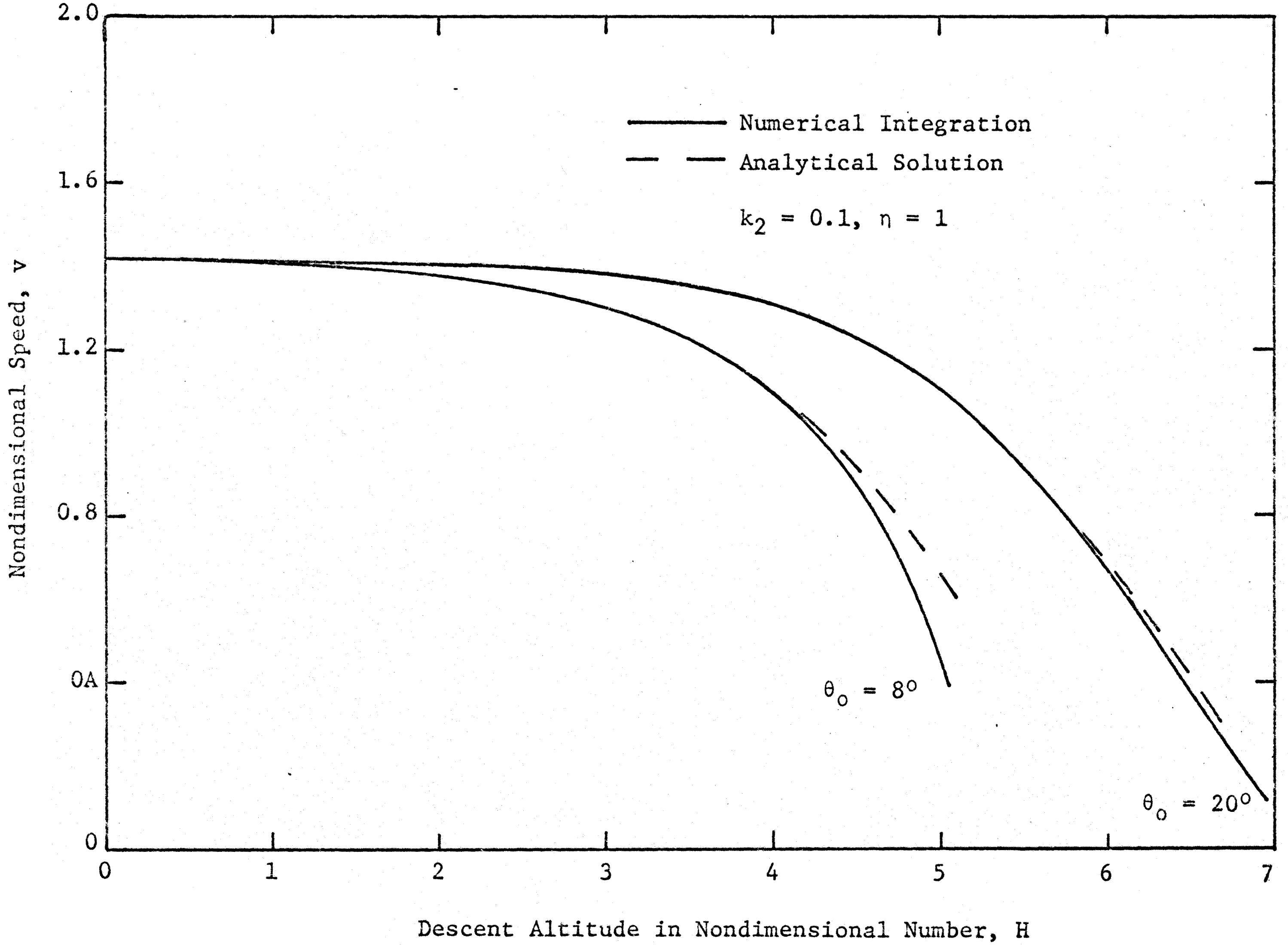

Figure 9. Comparison of Analytical Solution with Numerical Calculations.

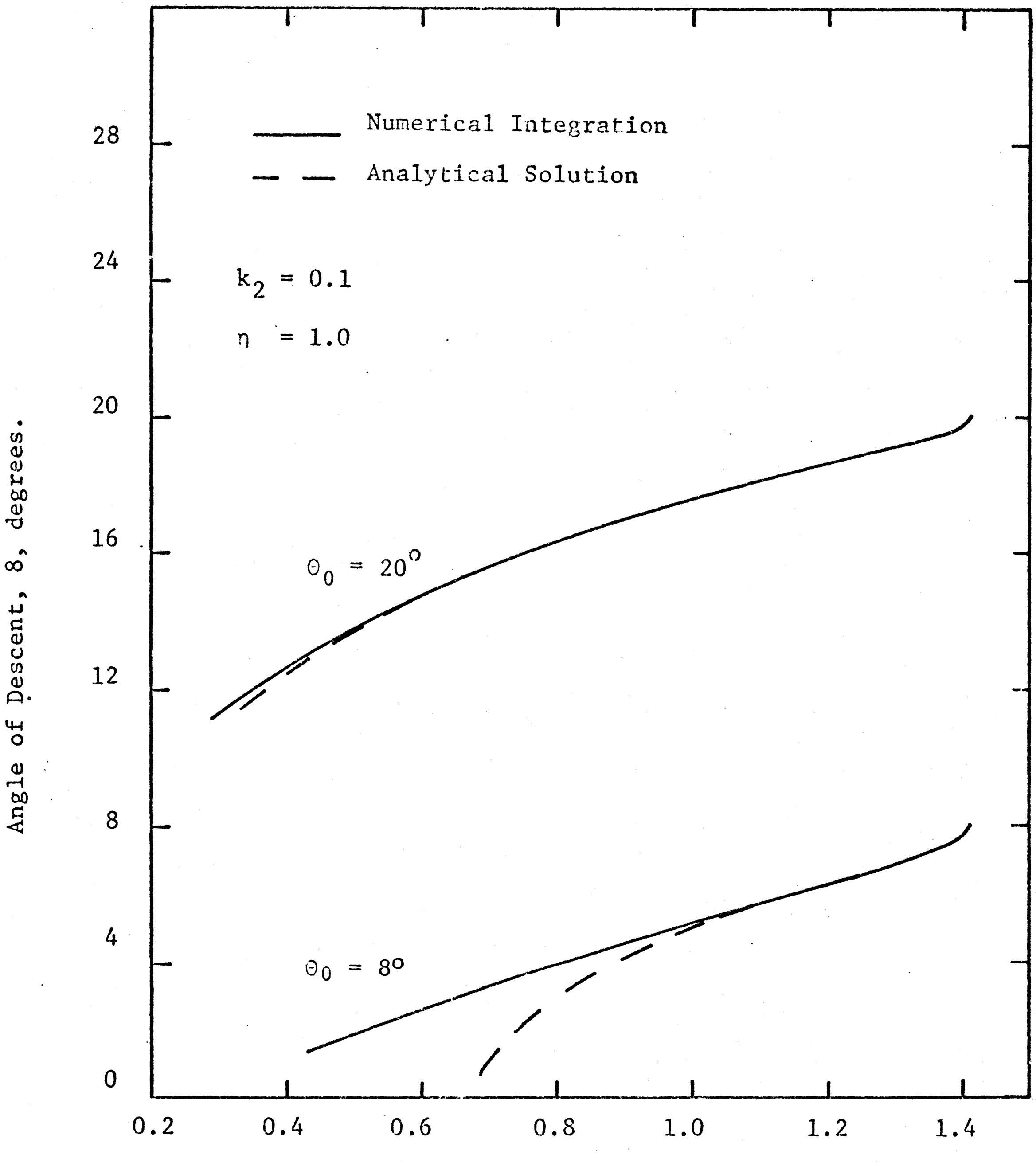

Figure 10. Comparison of Analytical Solution with Numerical Calculations.

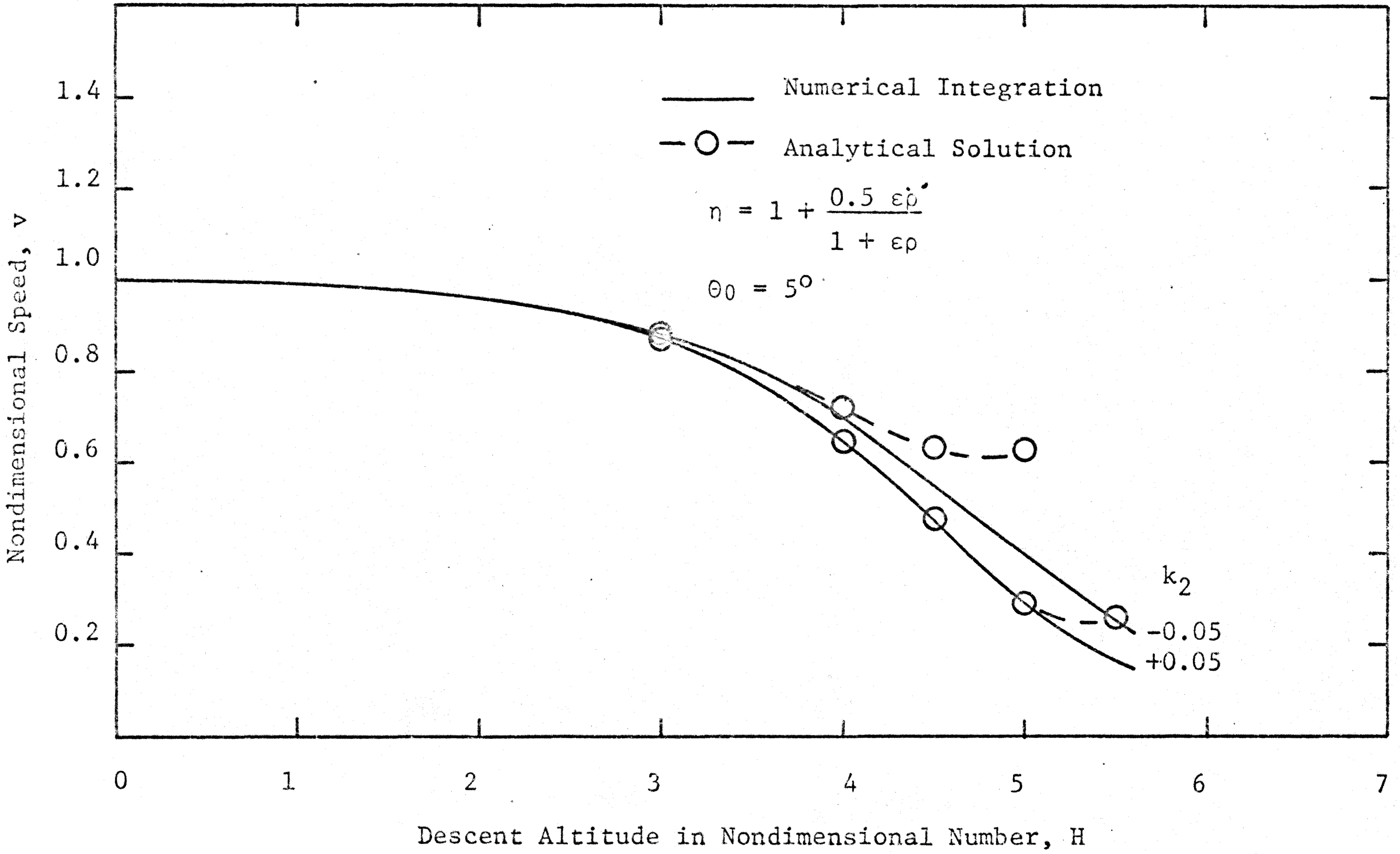

Figure 11. Comparison of Analytical Solution with Numerical Calculations for Modulated Drag.

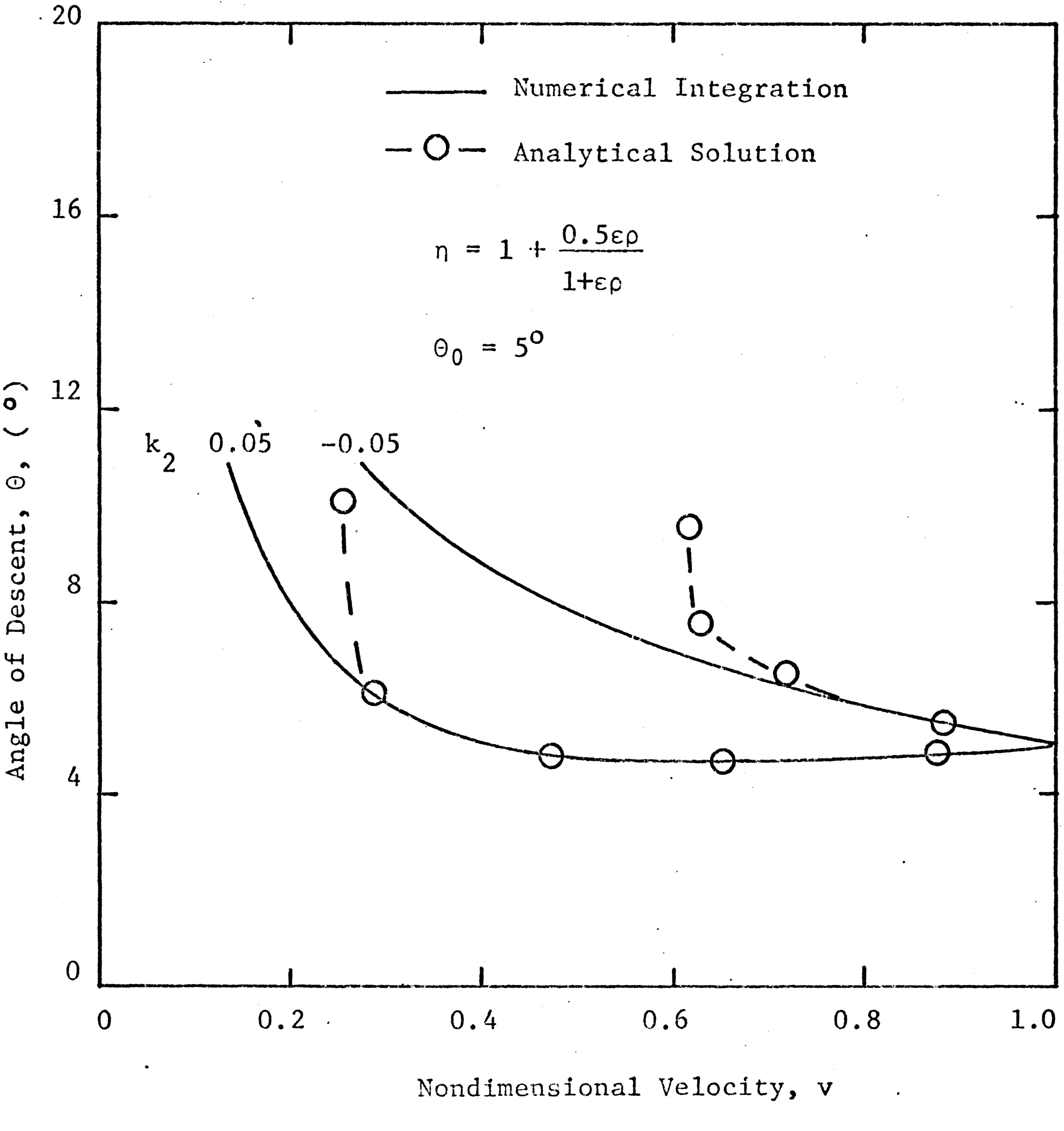

Figure 12. Comparison of Analytical Solution with Numerical Calculation for Modulated Drag.

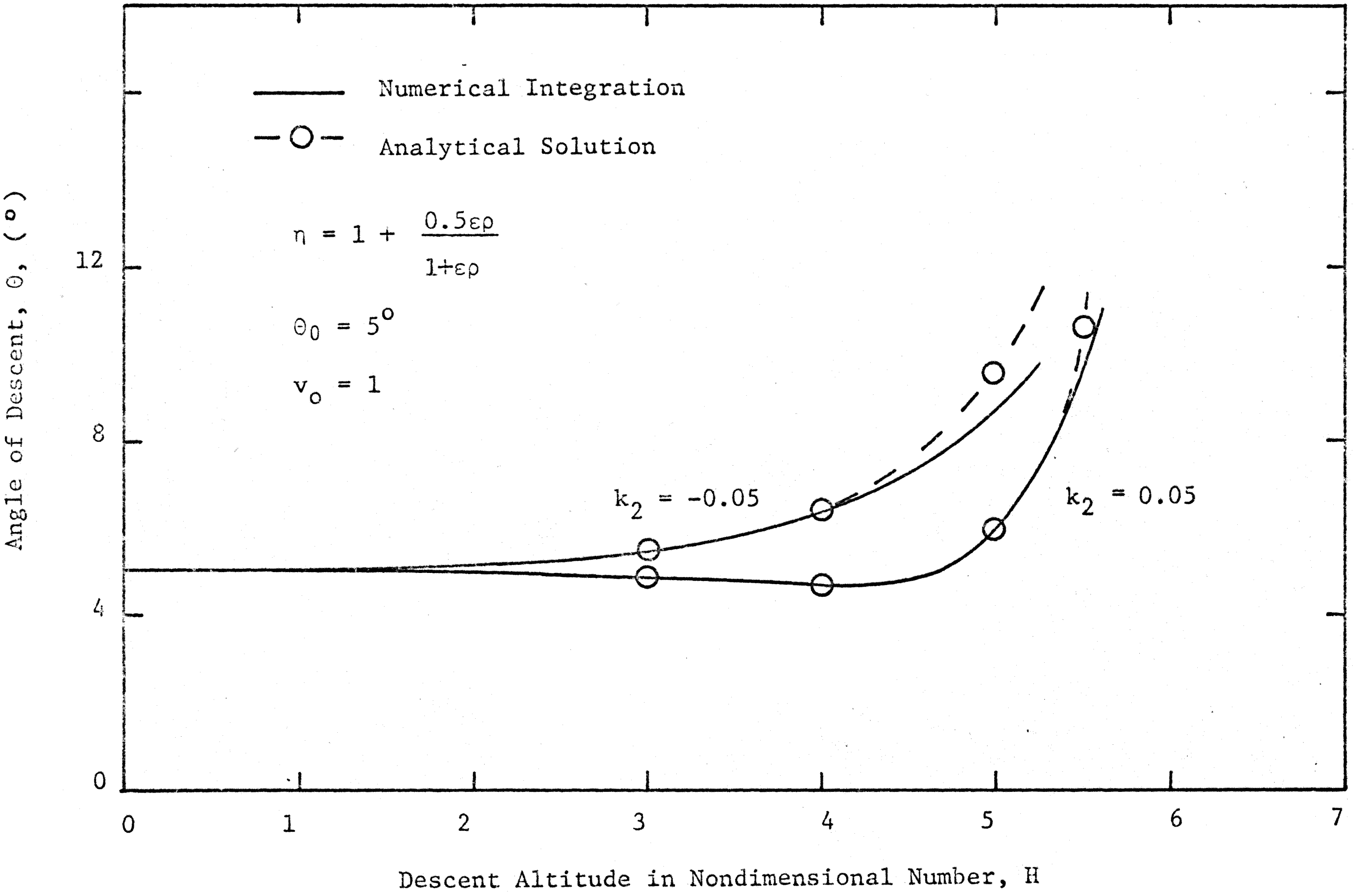

Figure 13. Comparison of Analytical Solution with Numerical Calculations for Modulated Drag.

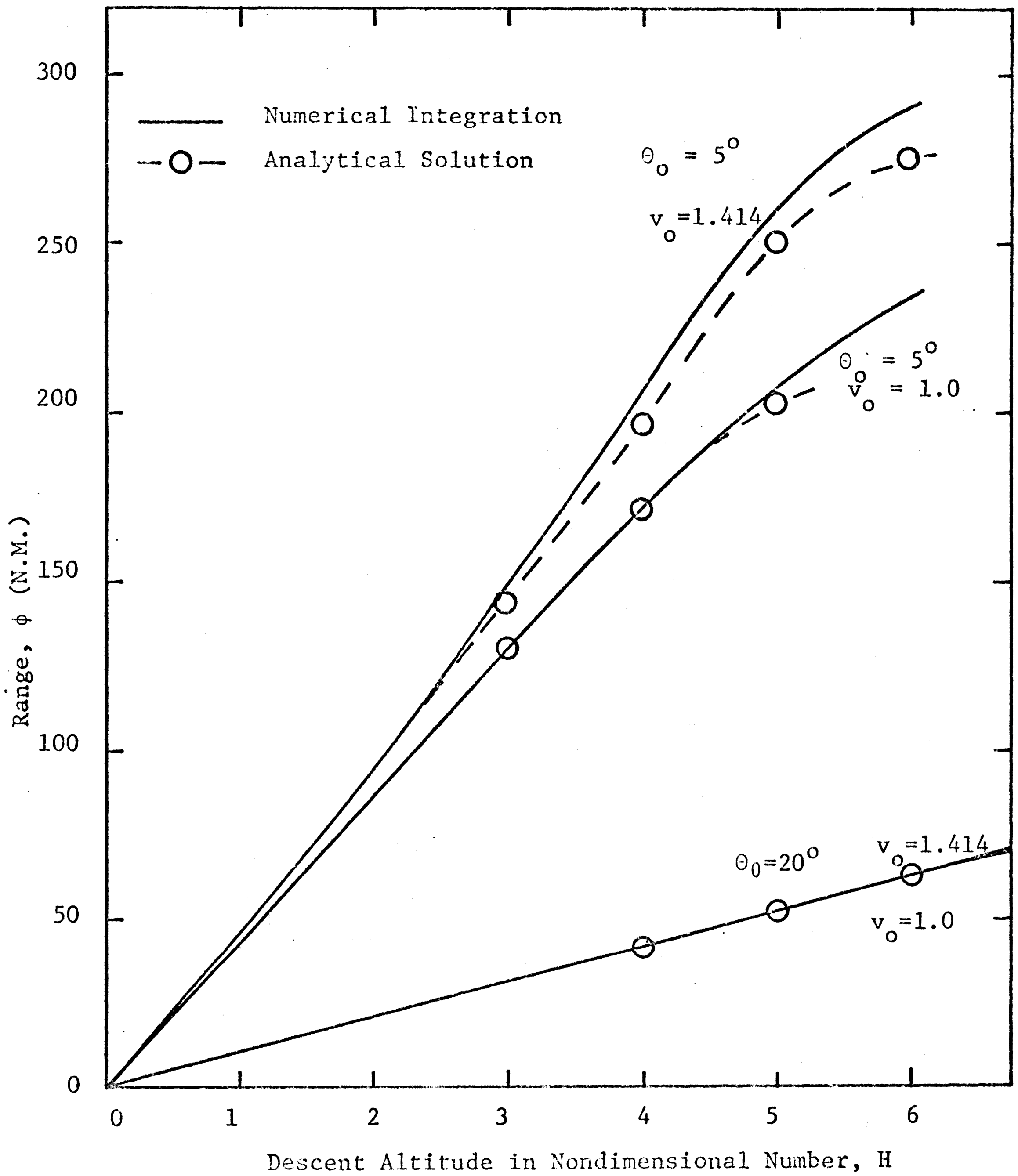

Figure 14. Comparison of Analytical Solution for Range with Numerical Solution for Range for Non-lifting Bodies with Constant Drag Coefficient.

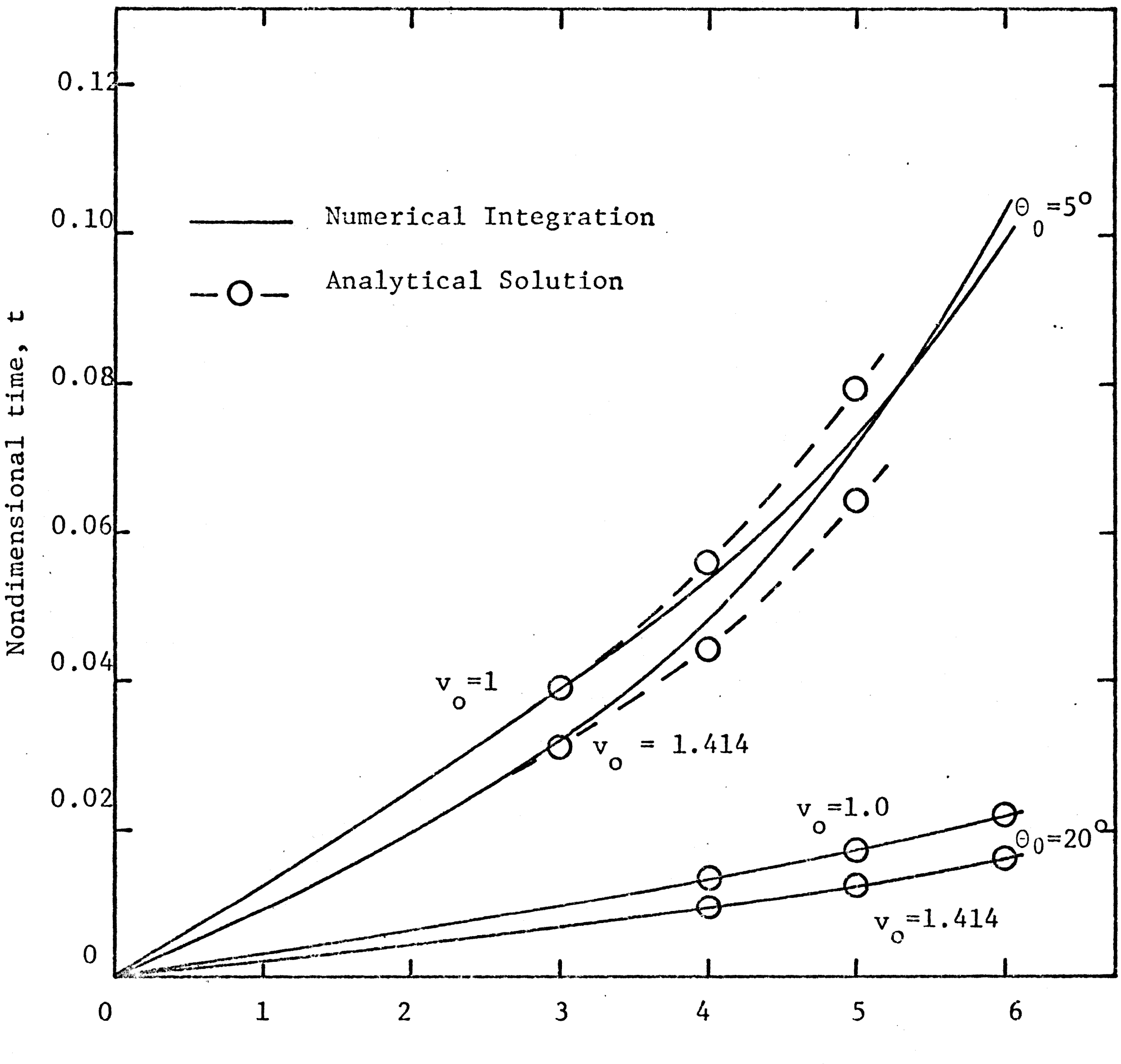

Figure 15. Comparison of Analytical Solution with Numerical Solution for Time for Non-lifting Bodies with Constant Drag Coefficient.

angle does not change by more than 20% from its initial value and when small velocities are avoided, i.e., for

$$0.8 \sin\Theta_0 \leq \sin\Theta \leq 1.2 \sin\Theta_0 ; \qquad (141)$$

and

$$v \geq 0.2 . \qquad (142)$$

The limit imposed by a small velocity, equation (142), presents no difficulty in that it only requires that the solution should be terminated when this condition is met. On the other hand, the limit on the flight path angle is more significant in that certain combinations of the initial conditions and the lift coefficients should be avoided. A reasonable description for these conditions can be obtained; this is presented in the next chapter.

VII. ANALYSIS FOR THE REGION OF VALIDITY

Throughout the previous analysis it was noted that the accuracy of this analytic solution depends, in a large part, on ones ability to generate the effects of changes in the flight path angle. Specifically, the expansion of the reciprocal sine function, equation (41), was required in equation (27-a) for the velocity solution. Although, as discussed in Section (5.5), the effects of the higher order terms of this expansion are somewhat suppressed in the integration, they are significant for changes in flight path angle greater than approximately 20%.

Assuming, then, that the limiting condition on this solution corresponds to

$$\sin\Theta = (1 \pm 0.2)\ \sin\Theta_o; \tag{143}$$

then the required limit boundaries can be determined as follows:

Substituting equation (143) into (120) and rearranging yield

$$(\pm\ 0.2 + 0.02)\ \tan^2\Theta_o = -\ \frac{k_2}{2\cos\Theta_o}\ \ \beta_1 - \beta_2\ , \tag{144}$$

where β_1 and β_2 are given by equations (117-b) and (117-c), respectively. This resultant has been solved, by iteration, to determine the limiting conditions on the solution. Some care must be used in evaluating equation (144), however, since the exact solution may be double valued. Specifically, for lifting trajectories it is possible that the flight path angle may change by amounts approaching 100% of the initial value without skipping. That is, the flight path angle decreases to a minimum and then increases again. Consequently, in this case, the first time the limiting conditions are reached corresponds to a description on the limits to the solution. Fortunately this limit is characterized by a small velocity decay. Hence, the proper solution can be obtained by subordinating the terms in equation (117-c) which arise as a consequence of the aerodynamic drag.

Typical boundaries, showing the limiting conditions as obtained from equation (144), are presented as Figures 14, 15, and 16. It is apparent from these figures that the solution is generally not of much value for small initial entry angles. On the other hand, for initial angles greater than about six degrees and for small lift, this solution should give good accuracy over most of the reentry trajectory; i.e., for $v \geq 0.2$.

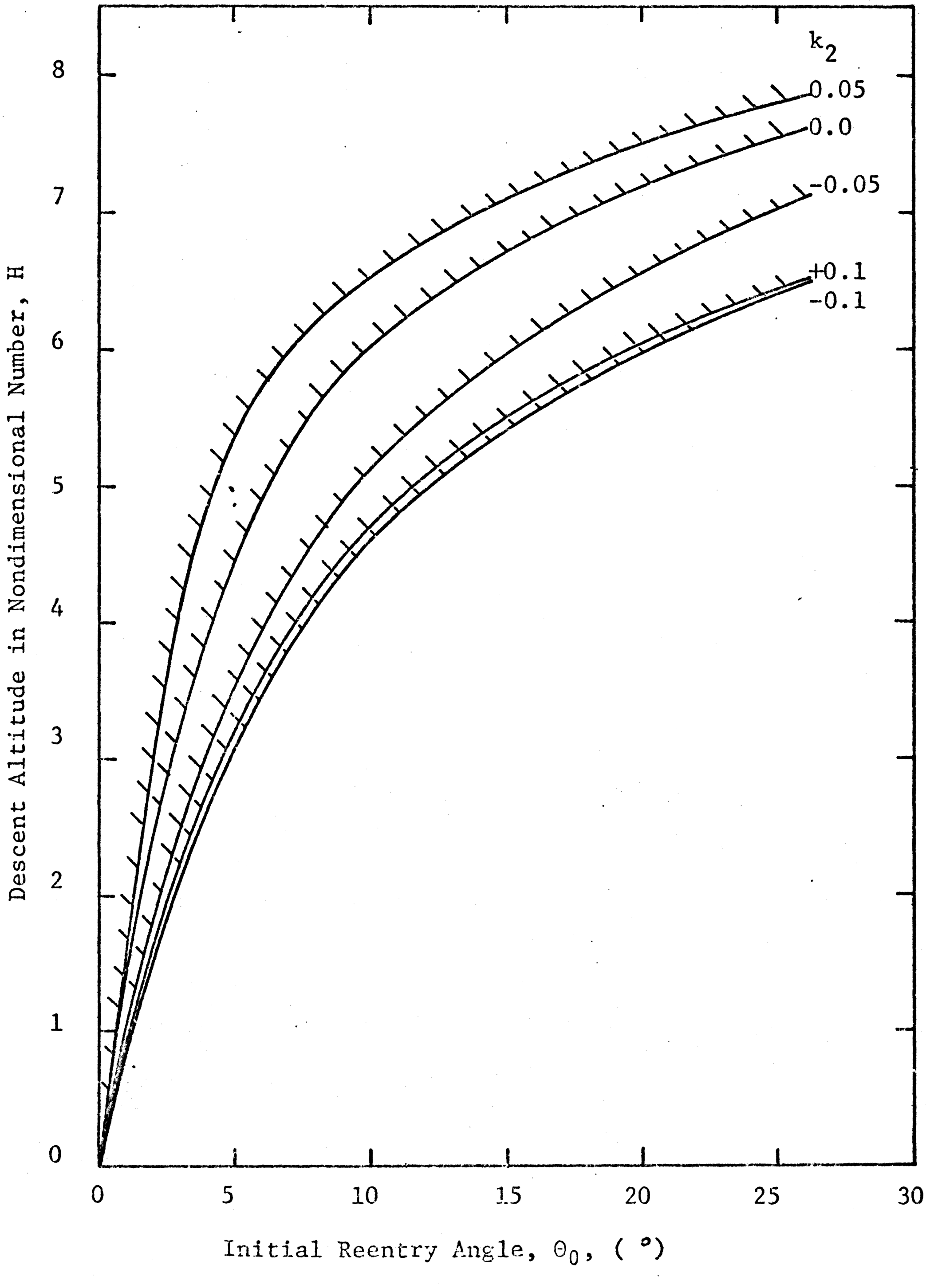

Figure 16. Limiting Conditions for Solution Based on ± 20% Change in $\sin\Theta$ for Constant Drag and Lift Coefficients and Initial Nondimensional Speed, $v_o = 1$.

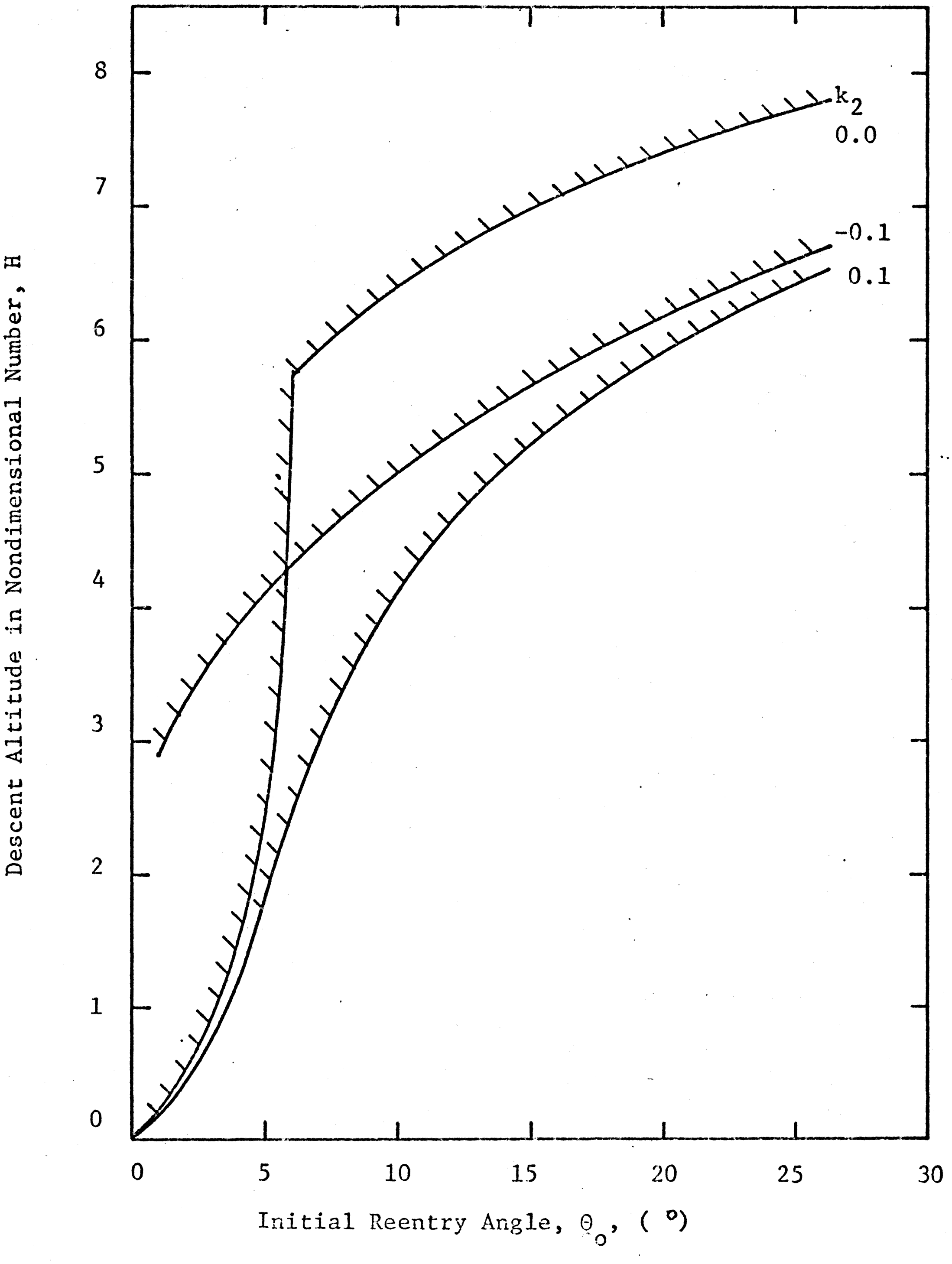

Figure 17. Limiting Conditions Based on ± 20% Change in sinΘ for Constant Drag and Lift Coefficients and Initial Nondimensional Speed, $v_o = 1.414$.

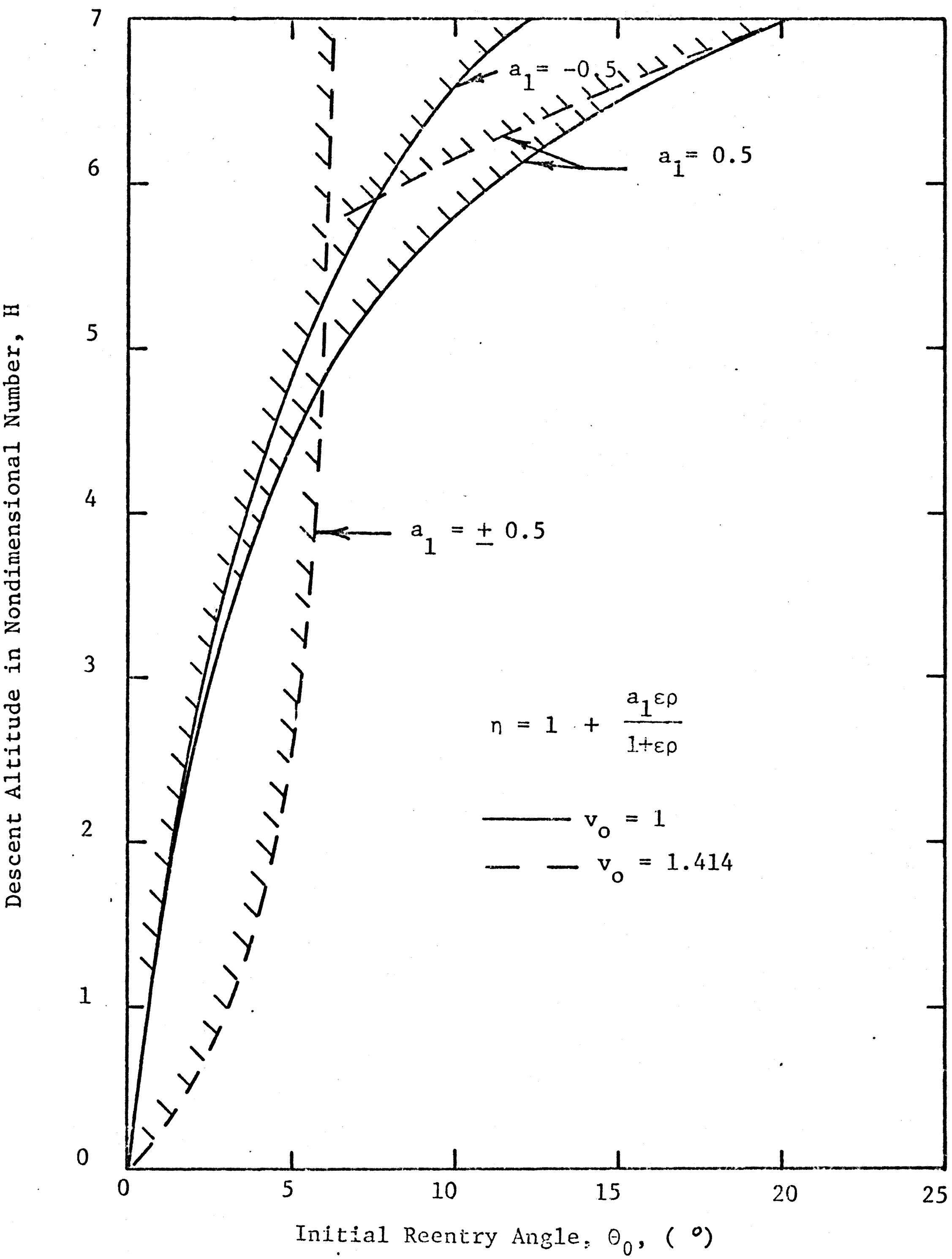

Figure 18. Limiting Conditions for Solution Based on ± 20% Change in sinΘ For Nonlifting Bodies with Drag Modulation.

VIII. ANALYSIS FOR DRAG STEP FUNCTION

The special drag modulation effects, such as those associated with drag-chutes, can be investigated from the basic solutions developed in this thesis. In principle, the effect of deploying a drag device can be treated as a step function. Fortunately, the form of the developed equations is amenable to a consideration of this case by means of a simple transformation of the ballistic parameter, $\frac{k_1\tilde{A}}{\tilde{m}}$, which appears in the nondimensional density. This will be accomplished as follows:

The nondimensional density, from equation (2-c), is

$$\rho = \frac{k_1\tilde{A}}{\tilde{m}} \tilde{r}_o\tilde{\rho} \; ; \qquad (2\text{-}c)$$

this is set equal to unity at the initial conditions and becomes

$$1 = \frac{k_1\tilde{A}\tilde{r}_o\tilde{\rho}_o}{\tilde{m}} \; . \qquad (145)$$

Suppose that at some arbitrary altitude, during reentry, a drag device is either deployed or released. This would correspond to an abrupt change in the ballistic parameter; thus, using the subscript s to represent the density at this occurrence, then equation (2-c) can be written as

$$\rho_S = \left(\frac{k_1 \tilde{A}}{\tilde{m}}\right) \tilde{r}_o \, \tilde{\rho}_S \quad ; \tag{146}$$

and a new initial density can be defined as

$$\rho_{os} = \left(\frac{k_1 \tilde{A}}{\tilde{m}}\right)_s \tilde{r}_{os} \, \tilde{\rho}_{os} \quad . \tag{147}$$

Taking the ratio of these two expressions one obtains

$$\rho_{os} = \left[\frac{\left(\frac{k_1 \tilde{A}}{\tilde{m}}\right)_s}{\left(\frac{k_1 \tilde{A}}{\tilde{m}}\right)} \; r_{os}\right] \rho_s \; , \tag{148}$$

since

$$\tilde{\rho}_{os} = \tilde{\rho}_s \; , \tag{149}$$

and

$$\frac{\tilde{r}_{os}}{\tilde{r}_o} = r_{os} \; . \tag{150}$$

The nondimensional position radius, r_{os}, from equation (8-b), can be written as

$$r_{os} = 1 - \varepsilon^2 \ln\rho_s + \frac{\varepsilon^4}{2} \ln^2\rho_s + \dots \quad . \tag{151}$$

Consequently, within the accuracy of the ordered solutions, equation (148) becomes

$$\rho_{os} = \frac{\left(\frac{k_1\tilde{A}}{\tilde{m}}\right)_s}{\left(\frac{k_1\tilde{A}}{\tilde{m}}\right)} \rho_s \; . \tag{152}$$

Equation (152) defines a new initial density not necessarily equal to unity but consistent with the ordered solution already developed.

The initial flight regime now will have the new initial conditions:

$$\begin{aligned} x(F_o) &= v_s^2 \; , \\ \Theta(F_o) &= \Theta_s ; \end{aligned} \tag{153}$$

which correspond to the new density function,

$$F = \left(\frac{k_1\tilde{A}}{\tilde{m}}\right) \tilde{r}_{os} \, \tilde{\rho} \quad , \tag{154}$$

where F is the density based on the new ballistic parameter, and

$$F_o = \rho_{os} \quad . \tag{155}$$

Taking into account the new initial conditions, then equations (115-a) and (120) for the special case of nonlifting bodies with constant drag coefficient become

$$x = X_1 \, e^{X_2} \quad , \tag{156}$$

and

$$\sin^2\Theta = \sin^2\Theta_s - 2\cos^2\Theta_s \, \beta_2 \; ; \tag{157}$$

where

$$X_1 = v_s^2 + 2\varepsilon^2 \left[\ln(F/F_o) + \Phi_1(\omega) - \frac{\varepsilon^2 F_o}{\sin\Theta_s} [1 + \ln(F/F_o)] \right] , \qquad (158)$$

$$X_2 = - \frac{\varepsilon^2}{\sin\Theta_s} (F - F_o) - \varepsilon^4 \frac{\cot^2\Theta_s}{\sin\Theta_s} \left[(1 - \frac{1}{v_s^2}) [F \ln(F/F_o) - \right.$$

$$\left. (F-F_o)] - \frac{F}{v_s^2} \Phi_2(\omega) \right] , \qquad (159)$$

$$\beta_2 = \varepsilon^2 (1 - \frac{1}{v_s^2}) \ln(F/F_o) - \frac{\varepsilon^2}{v_s^2} \Phi_1(\omega) + \varepsilon^4 \frac{F_o[1+\ln(F/F_o)]}{v_s^2 \sin\Theta_s} , \qquad (160)$$

and

$$\omega = \frac{\varepsilon^2 F}{\sin\Theta_s} . \qquad (161)$$

Herein, $\Phi_1(\omega)$ and $\Phi_2(\omega)$ are the functions given as equations (116-a) and (116-b), respectively.

The corresponding expressions for range and flight time, obtained from equations (121) and (122), become

$$\phi = \varepsilon^2 \cot\Theta_s \ln(F/F_o) + \varepsilon^4 \frac{\cot\Theta_s}{\sin^2\Theta_s} \left\{ \frac{1}{2} (1 - \frac{1}{v_s^2}) \ln^2(F/F_o) - \right.$$

$$\left. \frac{1}{v_s^2} \Phi_3(\omega) \right\} , \qquad (162)$$

and

$$t = \frac{\varepsilon^2}{v_s \sin\Theta_s} \left\{ \ln(F/F_o) + \Phi_1(\frac{\omega}{2}) - \frac{\varepsilon^2 F_o}{2 \sin\Theta_s} [1 + \ln(F/F_o)] \right\} +$$

$$\frac{\varepsilon^4 \cot^2\Theta_s}{2v_s \sin\Theta_s} \left(1 - \frac{1}{v_s^{\,2}} \right) \ln^2 (F/F_o) \; ; \tag{163}$$

wherein $\Phi_3(\omega)$ is obtained from equation (118).

The nondimensional altitude can be described as follows:

First, the general relationship between the original density, ρ, and the new density, F, is acquired from equations (2-C) and (154); that is,

$$\rho = \frac{\left(\frac{k_1\tilde{A}}{\tilde{m}}\right)}{\left(\frac{k_1\tilde{A}}{\tilde{m}}\right)_s} F \; , \tag{164}$$

where the higher order terms in the nondimensional radius have been neglected [see equation (151)].

Next, from equation (129),

$$H = \ln\rho \quad ; \tag{165}$$

hence, equation (164) can be rewritten as

$$H = \ln \left[\left(\frac{k_1\tilde{A}}{\tilde{m}}\right) \Big/ \left(\frac{k_1\tilde{A}}{\tilde{m}}\right)_s \right] + \ln F \; . \tag{166}$$

Now, in terms of the initial conditions for the step function, this last result may be expressed as

$$H = H_s + \ln(F/F_o) \; . \tag{167}$$

Typical results obtained from this analysis are given as Figures 19 and 20. The first of these figures shows the nondimensional speed versus nondimensional altitude for initial conditions corresponding to parabolic escape speed, $v = 1.414$, and to a reentry angle, $\Theta_o = 8^o$. At point "1" on this curve it is assumed that a drag device is released which abruptly decreases the ballistic parameter to 1/20.1 of the original value (this yields a new initial density of $F_o = 1$). The corresponding new velocity curve is then plotted showing a comparison with the constant drag curve. In order to show the relative effect of this given drag device it is assumed now that the device was released at point "2" on the speed curve corresponding to ($F_o = 1.65$). This second resulting speed curve is also shown on Figure 19.

Figure 20 is indicative of the relative deceleration effects brought about by the two cases discussed above.

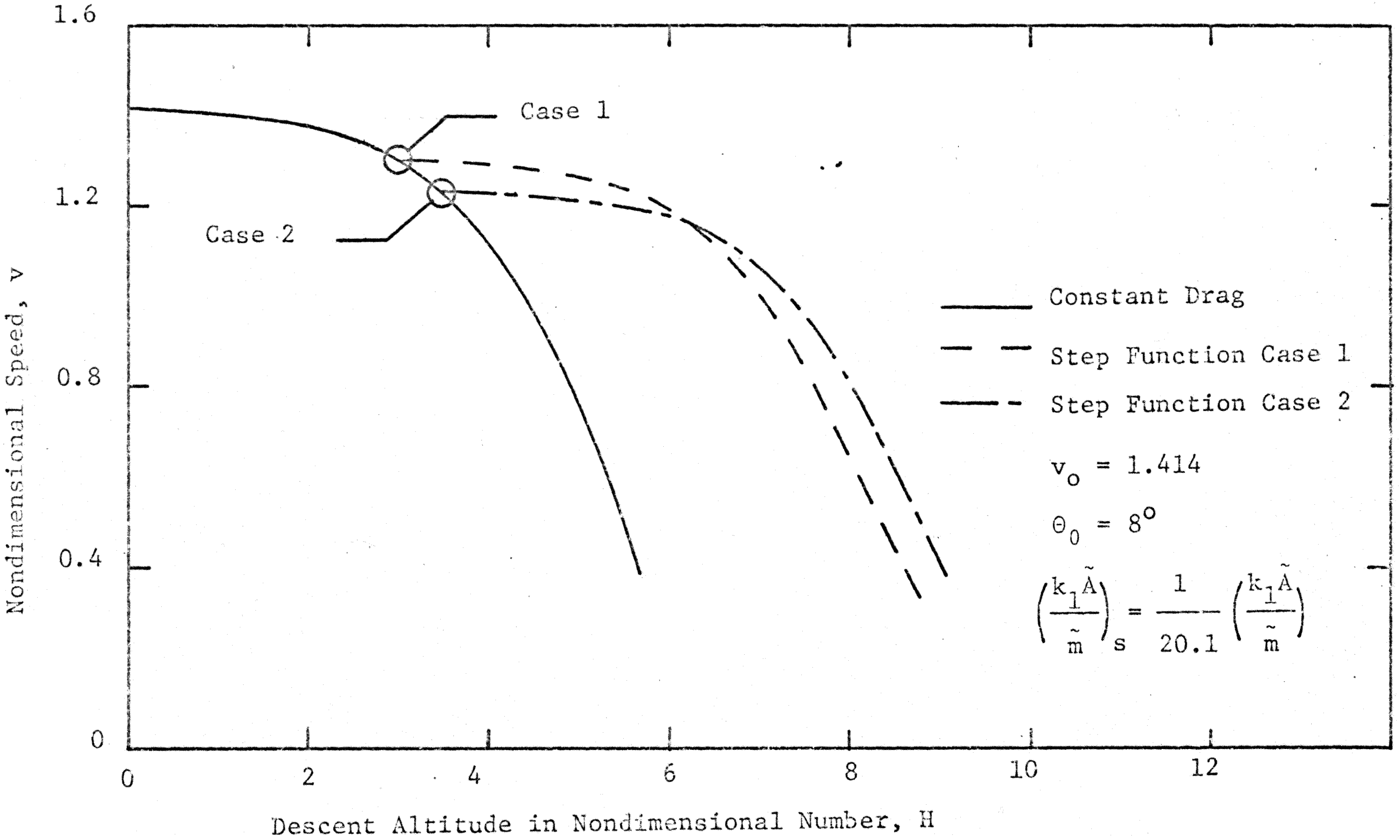

Figure 19. Speed Curve Showing Effects of A Drag Step Function.

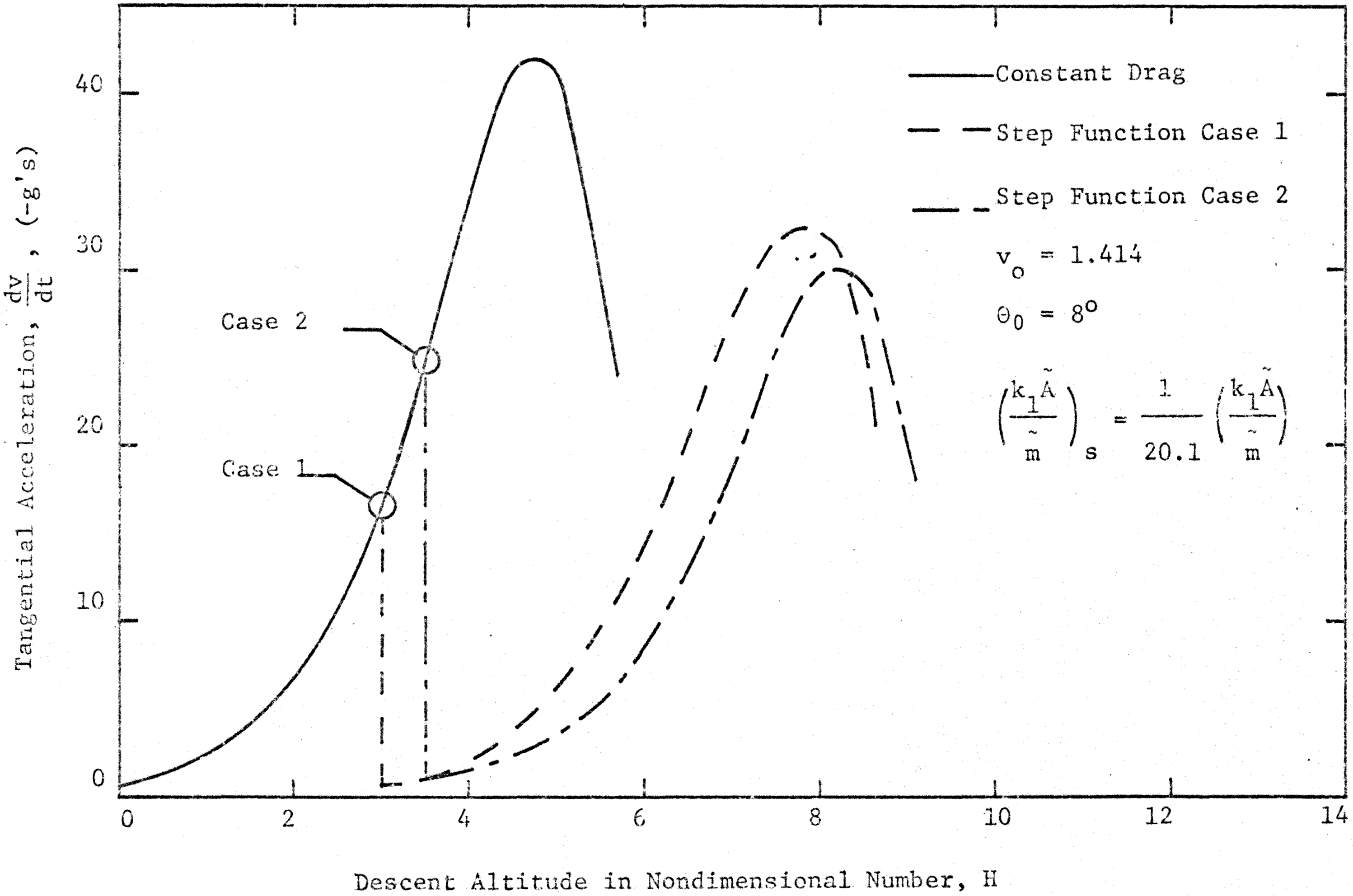

Figure 20. Tangential Acceleration Curve Showing Effects of A Drag Step Function.

VIX. SUMMARY OF ANALYSIS

In this thesis the reentry equations of motion for planar flight, written in the tangent and normal directions relative to the flight path, were nondimensionalized and presented in a form that admits to similarity solutions. The independent variable for these equations was changed from the time, which appeared only in the derivatives, to the density. This transformation yielded a set of equations suitable for nonsingular perturbation solutions.

It was noted that the transformed equations could be assumed to be in a form of linear differential equations with variable coefficients. The solutions implied by the assumed linear equations led to a less degenerate form of the dependent variables; that is, the speed and the cosine function of the flight path angle were rewritten as logarithmic functions. This transformation led, ultimately, to a perturbation solution which generated the significant terms of the argument of an exponential function for the speed rather than the expanded form of the exponential.

A straightforward perturbation solution in the form of power series was assumed next. Subsequently, it was shown that this perburation solution led, in the limit, to Picard's iterative solution. From a practical point of view, however, it was necessary to generate the significant terms for a large density by using first an order of magnitude transformation on the differential equations. This operation yielded a new series expansion which was matched with the original expansion at their limits. The two expansions were then used to form a composite expansion which was uniformly valid for both density regimes.

It is of interest to note that in this particular problem it was seen that the terms generated for the large density regime (except for the initial constants) were, for the most part, the higher order terms which would have been obtained by the second iteration of Picard's method. The significance of the perturbation technique used herein, over Picard's method, however, is twofold. First, in Picard's method, all terms are carried in each successive iteration; whereas, in the current technique only the successively higher order terms were required. Second, the present technique yields an order of magnitude estimate of successively higher order terms.

In the solution only the first significant term of the expanded sine function, equation (41), was obtained. The effect of the higher order terms was discussed, however, and the approximate conditions limiting the resulting solution were obtained and are presented on Figures 16, 17, and 18. As can be seen from these figures, this solution is limited, in general, to initial reentry angles greater than six degrees and to small lift coefficients; i.e., $k_2 \leq 0(0.05)$. These conditions depend in part however on the initial reentry speed.

However, it should be noted that had it been assumed that the initial reentry angle was such that $\sin\Theta_0 = O(\varepsilon)$, then equation (41) would imply that all terms in that expansion become significant for all values of density. On the other hand, if small flight path angles are assumed (i.e., $\sin\Theta = O(\varepsilon)$), then the zeroth order equations become coupled as the density approaches order $1/\varepsilon$; and a solution cannot be generated by this technique. On the other hand, from a comparison with Picard's method it is implied that the inclusion of higher order terms in the expanded sine function, equation (41), would significantly extend the validity of the solution for small initial reentry angles.

As also pointed out in the body of this thesis, the effect of the centrifugal acceleration term in the governing equation for the flight path angle is not properly accounted for as the speed becomes small. Attempts to account for this effect by an order of magnitude transformation led to coupled equations similar to those obtained for small flight path angles. Basically, however, the more degenerate solution - which was obtained - fails for small speeds because the higher order terms in the expanded exponential function for the speed become significant. It follows, again based on the comparison with Picard's method, that a higher order solution would pick up these higher order terms along with those higher order terms of the sine function which was discussed above. The algebraic complexity of this becomes formidable, however.

Finally, the resulting solution was specialized to the case of abrupt changes in the ballistic parameter -- represented by a drag step function. This special case should yield the same degree of accuracy and have the same limiting conditions as the basic solution so long as the transformed initial density is of order unity. This presents no problem, however, for those cases where a drag device is released and thereby reduces the ballistic parameter. On the other hand, if a drag device having a large ballistic parameter is deployed so that the transformed density is of order $1/\varepsilon^2$, then the summation functions Φ_1, Φ_2, and Φ_3 will be large and not properly accounted for at the new initial conditions. In general, however, whenever conditions are such that these functions are negligible at the initial conditions for the step function, this analysis should yield consistently accurate results.

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A SIMILARITY SOLUTION FOR PLANAR ATMOSPHERIC REENTRY WITH AERODYNAMIC FORCE MODULATION

by

Fred William Martin

Abstract

The differential equations for planar motion describing reentry into a planetary atmosphere, with arbitrary lift and drag modulation functions, are written in the tangential and normal direction for the flight path. Broglio's power law approximation for the density, in an isothermal atmosphere, is used in order to write these equations in a nondimensional form that admits to self similar solutions with respect to body parameters. The independent variable, time, is transformed into a nondimensional density so that the governing equations can be expressed in terms of a small parameter. This operation yields a set of equations which are suitable for a non-singular perturbation solution.

Next, the governing equations of motion are written in a form which, as shown in this thesis, yields a less degenerate perturbation solution than the original form. A straightforward perturbation solution of these equations is shown to yield, in the limit, Picard's

iteration solution which for non-skip trajectories leads to the unique solution.

A second order perturbation solution, using a bounded drag modulation function and a constant lift coefficient, is generated in terms of the original nondimensional variables (called "the expansion in the initial flight regime"). In order to generate the significant terms in the solution for large density the initial variables are appropriately transformed by certain order of magnitude transformations; thus, a new second order perturbation solution is generated (called "the expansion in the aero-dominated flight regime").

These two expansions are matched, in their limits, and used to form a composite solution which is uniformly valid for both flight regimes. The nondimensional density, employed in this analysis, is correlated to the usual exponential density for isothermal atmospheres; and a nondimensional altitude, similar to the usual altitude number, is defined.

The solutions obtained here are compared with computer generated numerical solutions; these results show excellent agreement for those cases where the initial flight conditions are within the analytically-predicted limits.

Finally, the basic solution from this thesis is applied to the special case of a step-type drag function.