# CHAPTER 4

# WAVELET-BASED INITIAL TRIANGULATION

This chapter will present the algorithm used to perform initial mesh triangulation based on detailed information of the original image. It will start with discussion in detail of the initial triangulation proposed by Dr. Sang-Mook Lee [Lee 02], which is also used as the initial triangulation in this thesis. This includes examining the use of wavelet coefficients, the design of basic and variants templates and the solution to the vertex incompatibility problem.

## 4.1 Overview of the Initial Triangulation

Since initial triangulation is the first step in mesh construction, it is considered one of the most important steps because it can largely influence the final reconstructed images. There are many methods to perform initial triangulation. Some of them employ wavelet coefficients and quadtree data structures [Gross, Staadt and Gatti 96] while others use hierarchical triangulation [Bonneau 98]. In this thesis, initial triangulation is based on the algorithm used by Dr. Lee in his dissertation.

The algorithm starts by performing a wavelet transform on the original image at level $M$. The choice of $M$ usually depends on the application and the size of the image. Next the results of the horizontal, vertical and diagonal wavelet coefficients of that level are compared for appropriate templates. If the largest magnitude among the horizontal, vertical and diagonal wavelet coefficients are lower than a flatness threshold value, $\tau_M$ , a flat template is assigned. Otherwise, a vertical, a horizontal or a diagonal template will be chosen.

Next, these horizontal, vertical and diagonal wavelet coefficients are compared for the selection of a non-flat template. If the magnitude of the horizontal coefficient is the largest, the horizontal template is assigned. This also holds for the case of vertical and diagonal coefficients. Another important note before assigning a vertical or horizontal basic template is to check if the coefficient has a positive or negative value. The method will be discussed in

Section 4.2.1. Lastly, Section 4.2.2 will show how inconsistency in template connection can occur and how it can be solved.

## 4.2 Basic and Variant Templates

This algorithm, which uses the magnitude of the wavelet coefficient as well as its directional properties, is found to be efficient in both time and memory to construct the initial triangular meshes. First of all, like many other initial triangulation approaches, the basic template and its variants are predefined. There are four basic templates: flat, vertical, horizontal and diagonal templates, which are shown in Figure 4.1. The nine fundamental forms for variants construction are shown in Figure 4.2. Variant templates $\pi_1$ to $\pi_4$ are used to solve for flat template cases while variant templates $\pi_5$ to $\pi_9$ are used to solve for diagonal template cases. Templates $\pi_1$ and $\pi_8$ are used to smooth three adjacent connections while $\pi_2$ and $\pi_9$ are used to smooth all four adjacent connections. Likewise, $\pi_3$ and $\pi_5$ are similar in that they are used to solve for only one vertex incompatibility while $\pi_4$ and $\pi_7$ are appropriate to solve for smoothness in a vertical and a horizontal connection. Template $\pi_6$ is suitable for smoothing any two opposite connections that lie either in the vertical or horizontal orientation.

## 4.3 SML Initial Triangulation

The first step of the initial triangulation is to apply the discrete wavelet transform to the original image for $M$ iterations. The results are the image approximation and its vertical, horizontal and diagonal wavelet coefficients at different level. The coefficients that are needed for initial triangulation are the vertical $d^1_{M,i,j}$, horizontal $d^2_{M,i,j}$ and diagonal $d^3_{M,i,j}$ wavelet coefficients at level $M$. Their sizes are $\frac{N_r}{2^M}$ and $\frac{N_c}{2^M}$ in height and width respectively, where $N_r$ and $N_c$ are the image height (row) and width (column) respectively. Therefore if the original image size is 128x128 and $M$=4, these wavelet coefficients size will be 8x8. Each of these wavelet coefficients at location at $0 \le i \le \frac{N_r}{M}$ and $0 \le j \le \frac{N_c}{M}$ are compared with the flatness threshold value, $\tau_M$. If it does not exceed the threshold, a flat template will be assigned to that location $(i, j)$. Otherwise, the dominant wavelet coefficients will determine an appropriate basic template to that region. For example if $\left|d^1_{M,i,j}\right| = \max_{1\le k\le 3}\left(\left|d^k_{M,i,j}\right|\right)$ and $\left|d^1_{M,i,j}\right| > \tau_M$, then the vertical basic template is assigned as the dominant coefficient for that rectangular patch. This means that this region mainly contains a large discontinuity in the vertical direction. Figure 4.3 shows the required wavelet coefficients for initial triangulation as well as results obtained by wavelet coefficient comparison.

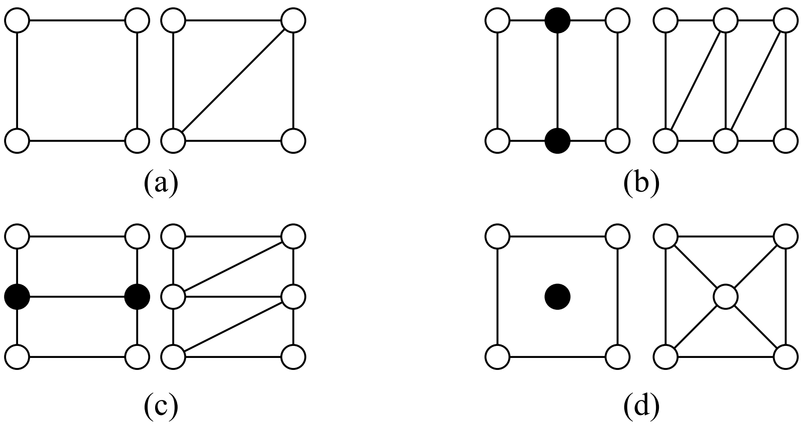



Figure 4.1 - Construction of basic templates. (a) Flat template $\pi_f$. (b) Vertical template $\pi_v$. (c) Horizontal template $\pi_h$. (d) Diagonal template $\pi_d$. The black vertices indicate the additional vertices that need to be added to the vertex list.

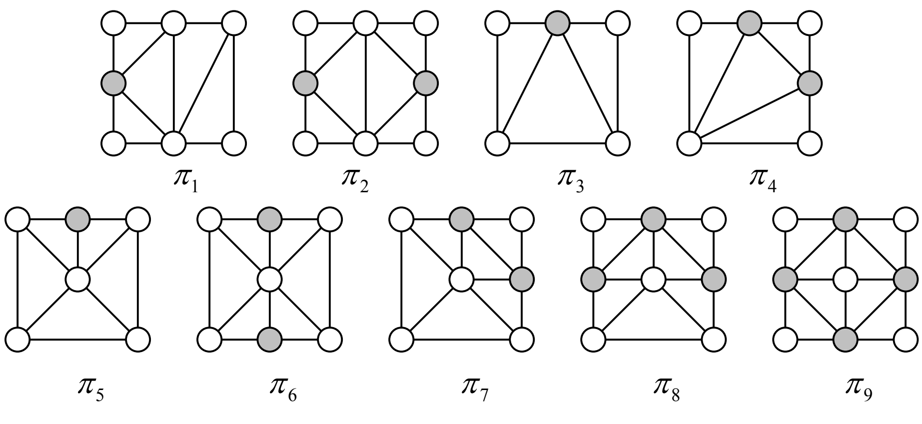
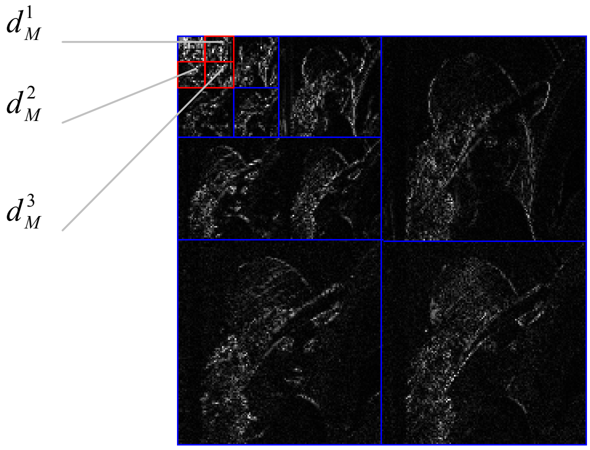



Figure 4.2 – Nine fundamental forms for variant construction.



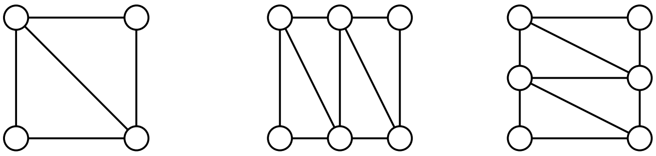

(a)

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| V | V | H | H | H | H | V | H |
| V | V | H | D | D | H | V | V |
| V | V | H | V | D | D | V | H |
| V | V | H | H | V | V | V | V |
| V | V | V | D | H | V | V | V |
| V | V | V | V | V | V | V | F |
| V | V | V | V | F | V | V | V |
| V | V | V | V | V | V | H | F |

(b)

Figure 4.3 – Wavelet coefficients required for initial triangulation and their comparison. (a) Wavelet coefficient requirement for construction of initial triangulation. (b) Result obtained by comparing wavelet coefficients.

In addition to these basic templates, the algorithm should also consider the directional property of the edges. Therefore, the dual templates, as important as the basic templates themselves, must be predefined. These templates are shown in Figure 4.4



(a) (b) (c)

Figure 4.4 – Dual templates. (a) Dual of flat template $\hat{\pi}_f$. (b) Dual of vertical template $\hat{\pi}_v$. (c) Dual of horizontal template $\hat{\pi}_d$.

These dual templates can be expressed as

$$\begin{aligned} g_v(\pi_f) &= \hat{\pi}_f \\ g_v(\pi_v) &= \hat{\pi}_v \\ g_v(\pi_h) &= \hat{\pi}_h \end{aligned} \tag{4.1}$$

The importance of these dual templates can be explained by the directionality property of discontinuities of the region. To illustrate this, the Haar wavelet is used as an example for demonstration. The wavelet coefficients from the Haar wavelet are actually the difference of the signal in either vertical or horizontal directions. These slopes can be represented as

$$d_m^1 \approx \frac{\partial s^{m-1}}{\partial x}, d_m^2 \approx \frac{\partial s^{m-1}}{\partial y} \tag{4.2}$$

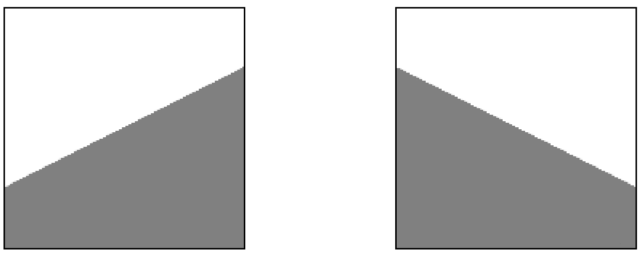
where $s^{m-1}$ is the average signal at level *m-1*. Therefore, the approximation of the orientation of discontinuity can be calculated from

$$\theta^m(x,y) = \tan^{-1}\left(\frac{d_m^2}{d_m^1}\right) \tag{4.3}$$

This concept can be demonstrated by the images in Figure 4.5. Assume that the dark region represents the region with low values while the white region represents region with high values. Both (a) and (b) have horizontal discontinuity as the main discontinuity.

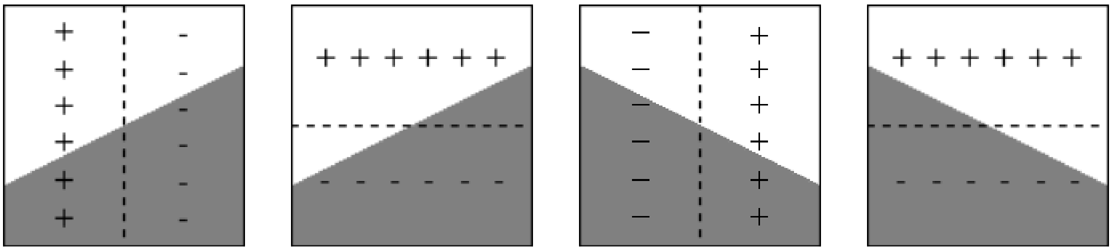
If the algorithm moves from left to right across Figure 4.5 (a), it will go from high values to low values and if it moves downward from top to bottom, it will also go from high values to low values. This movement is shown in Figure 4.6 (a) and (b). However, if it moves horizontally left to right across Figure 4.5 (b), it will be moving from low values to high values while the vertical movement will go from high values to low values. This demonstration is shown in Figure 4.6 (c) and (d).

Since both of the examples have the main discontinuity in the horizontal direction, the magnitudes of their horizontal coefficients, $d_{m,i,j}^1$, are the largest. However they can be positive or negative. In the positive case, which occurs for Figure 4.5 (a), it is more appropriate to assign the basic horizontal template while in the negative case, which occurs for Figure 4.5(b), its dual template is more suitable. Figure 4.7 shows the appropriate assignment for the template with different sign in direction.



(a) (b)

Figure 4.5 – Demonstration for the directionality property for the discontinuity. Both (a-b) have the main horizontal discontinuity but with different direction.




(a) (b) (c) (d)

Figure 4.6 – Directionality and slope of discontinuity. (a) Moving across Figure 4.4(a) in horizontal direction from white region (+) to darker region (-) (b) Moving vertically downward across Figure 4.5(a) (c) Moving horizontal across Figure 4.5(b). (d) Moving vertically across Figure 4.5(b).

(a) (b)

Figure 4.7 – Basic template assignment. Basic templates are assigned according to the directionality property of discontinuities. (a) The appropriate template for Figure 4.4 (a) is basic horizontal template. (b) The appropriate template for Figure 4.4 (b) is the dual for the basic horizontal template.

## 4.4 Vertex Compatibility and Variant Templates

The choice of templates depends on the horizontal, vertical and diagonal wavelet coefficients at level *M*. However with only basic and its dual templates, this does not guarantee the consistency in their connection. Inconsistency in connection can lead to violation in triangulation rules. For example, when a vertical template, $\pi_v$, is placed next to a horizontal template $\pi_h$, this can cause violation to the triangulation in their vertex connection. Figure 4.8 shows the vertex incompatibility of the two templates and one of the possible solutions.

Therefore, in the construction of initial triangulation, it is necessary to introduce variant templates to solve vertex incompatibility problems. Figure 4.9 shows neighboring rectangles of each basic template that need to be checked for vertex compatibility. For example, the flat and diagonal templates need to check all of adjacent rectangles for vertex compatibility. While the vertical template needs to check for its left and right rectangle, the horizontal template requires checking of rectangles above and below it. The rectangles that need to be checked are shown in gray.

An example of using a variant template to solve the vertex incompatibility is shown in Figure 4.10. Assume that the top-left, top-right, bottom-left and bottom-right rectangles are assigned with flat, negative vertical, positive horizontal and diagonal templates respectively. This causes the problem between the vertical connection of negative vertical and diagonal templates and between the horizontal connection of positive horizontal and diagonal templates. To solve this vertex incompatibility, the replacement of basic templates with variant templates is necessary and, unavoidable; two vertices have to be introduced. Variant template $\pi_7$ is chosen because it is appropriate for diagonal case with one vertical and one horizontal inconsistent connection.

(a) (b)

Figure 4.8 - Vertex incompatibility. (a) Two connected basic templates can cause vertex incompatibility. (b) Introducing variant templates can solve the vertex incompatibility.

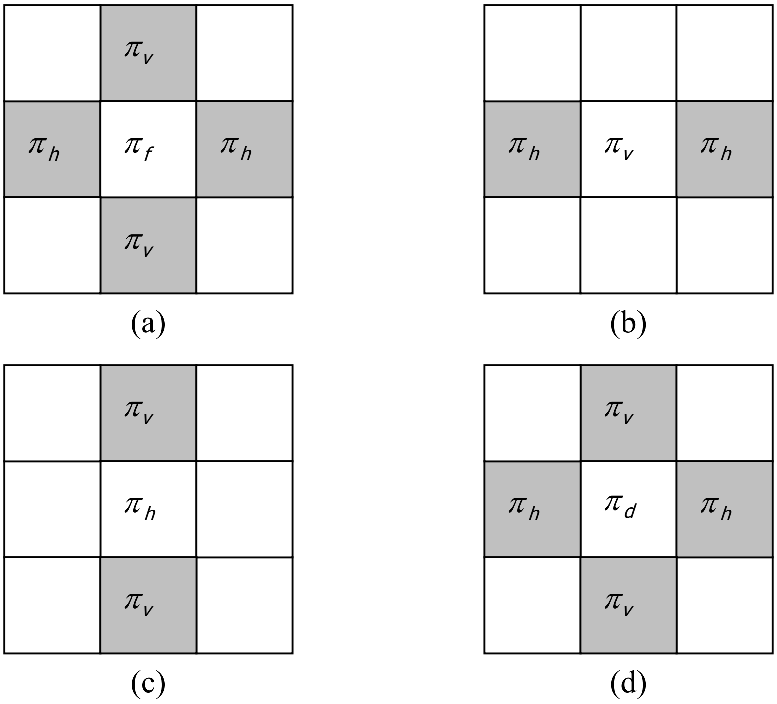



Figure 4.9 – Neighbor checking for vertex compatibility. Necessary neighbors checking (gray blocks) for (a) flat template $\pi_f$, (b) vertical template $\pi_v$, (c) horizontal template $\pi_h$ and (d) diagonal template $\pi_d$.

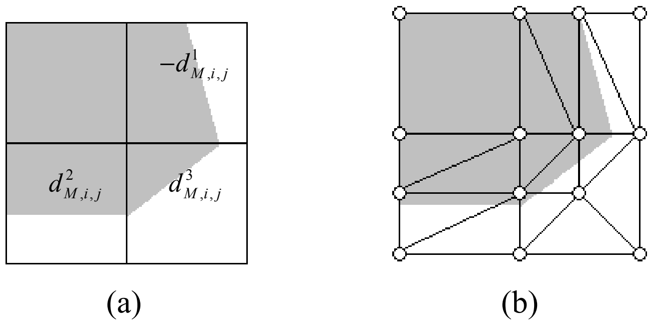



Figure 4.10 – Solution to vertex incompatibility. (a) Dominant coefficients. (b) Initial triangulations from predefined templates.

## 4.5 Complete Set

The basic, dual and variant templates are the fundamental templates. However they are not complete because vertex incompatibility can occur at any location of the rectangle. Therefore isomorphism is required to define a complete set of templates to solve for all situations of vertex compatibility. Let $g_h$, $g_v$ and $g_r$ be isomorphism functions that correspond to horizontal flip, vertical flip and rotation of vertices and edges respectively. The composite isomorphism function can be defined as

$$g_{hvr} = g_h \circ g_v \circ g_r \tag{4.3}$$

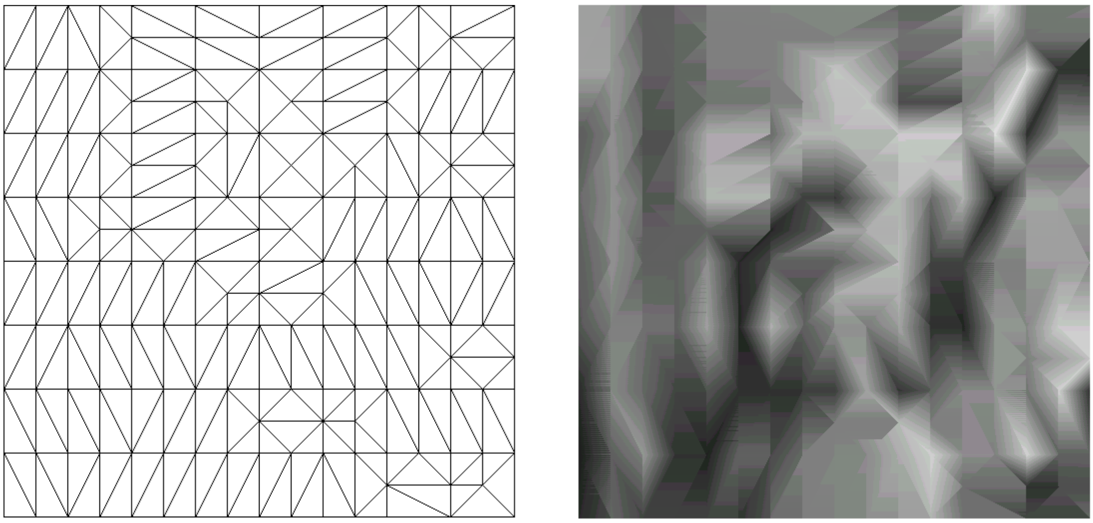
Normally, this isomorphism function is not commutative. Therefore

$$g_{hvr} \neq g_{hrv} \neq g_{rvh} \neq g_{vhr} \neq g_{rvh} \neq g_{rhv} \tag{4.4}$$

Therefore the eight-isomorphism functions can be defined as

$$\begin{array}{cccc} g_{000} = I & g_{001} = g_r & g_{010} = g_v & g_{011} = g_{vr} \\ g_{100} = g_h & g_{101} = g_{hr} & g_{110} = g_{hv} & g_{111} = g_{hvr} \end{array} \tag{4.5}$$

where $I$ represents the identity. Figure 4.11 shows the final initial triangulation of the Lena image performed with wavelet transform at level 4. The complete set of templates based on isomorphism is shown in Figure 4.12.



(a) (b)

Figure 4.11 – Example of final initial triangulation. (a) Initial triangulation from Lena image. (b) Reconstructed the Lena image from the initial triangulation.

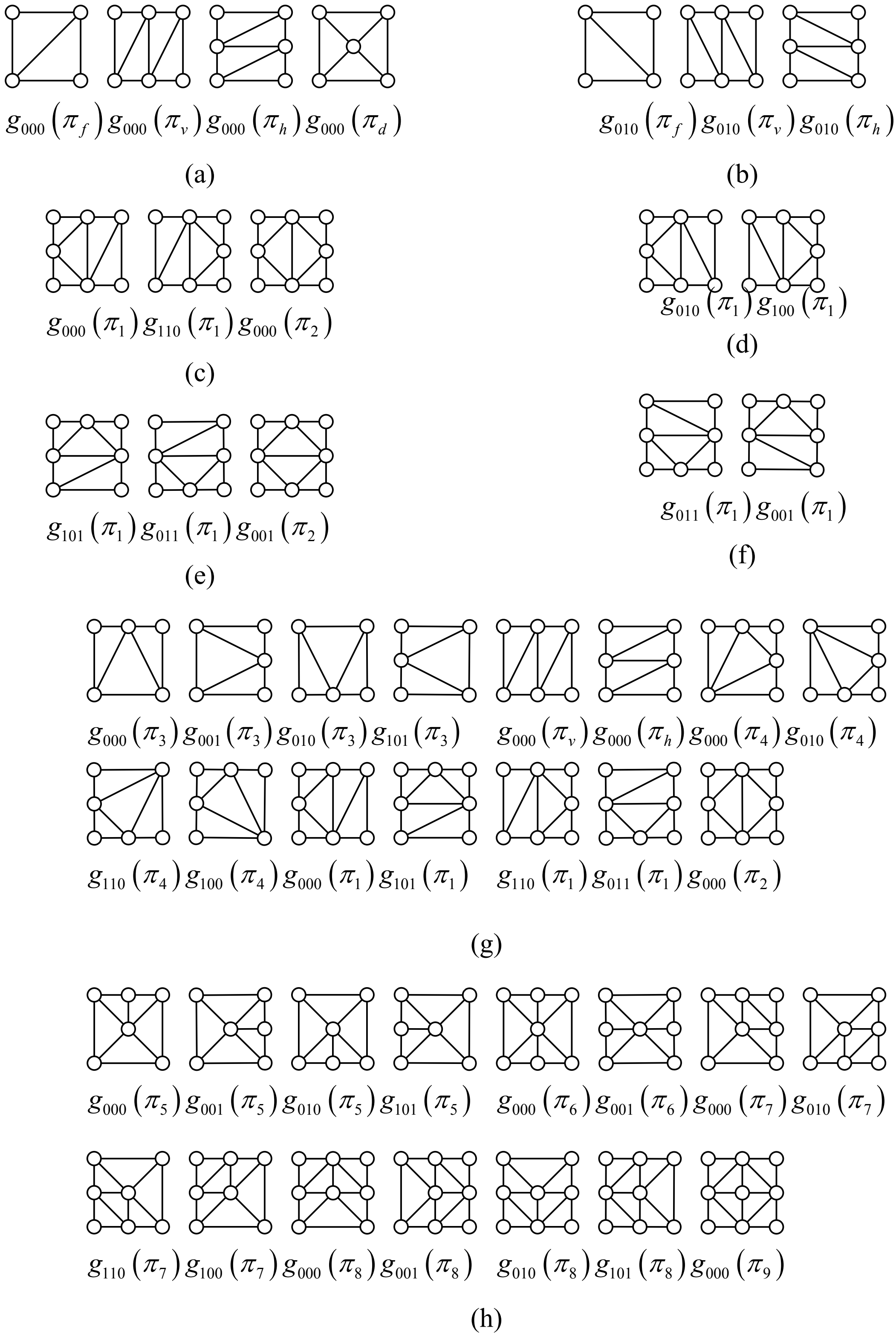



Figure 4.12 – Complete set of the templates and their generating isomorphism functions. (a) Basic templates. (b) Dual for basic templates. (c) Vertical variants. (d) Dual for vertical variants. (e) Horizontal variants. (f) Dual for horizontal variants. (g) Variants for flat template. (h) Variants for diagonal templates