

Appendix B-2

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**INTEGRATION OF A TIME-INDEPENDENT AND TIME-DEPENDENT
CONSTITUTIVE MODEL FOR SOFT ROCK**

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ABSTRACT

A framework to describe the time- and rate-dependent behavior observed in soft rocks is presented. This constitutive model is a rate-type model formulated in the framework of bounding surface plasticity, where the position of the bounding surface relative to the stress point is related to a quantity called the volumetric age of the material. In this model, the rate of inelastic deformations is proportional to the volumetric age. In addition to the constitutive model, a new integration algorithm for cap plasticity models is developed for both the rate-independent and rate-dependent cases. This integration algorithm reduces the number of equations and unknowns to be solved for a loading step. A numerical example illustrating the use of this constitutive model and integration algorithm are presented.

Keywords: Geomaterials, integration, time-dependence, constitutive model

INTRODUCTION

Time- and rate-dependent deformations are commonly observed in soft rocks during laboratory testing programs. Two laboratory test types which illustrate the phenomenon of time-dependent deformations are creep (*i.e.*, constant stress) tests and stress-relaxation (*i.e.*, constant deformation) tests.

Rate-dependent behavior is also observed in the deformation of soft rocks. Such behavior is presented by Hayano et al. (2001) for sedimentary silt-sandstones and mudstones. In these results, the effects of rate are described using an “isotach” or constant-rate model. The isotach model is based on the conclusion that irreversible strain rate cannot be linearly decomposed into plastic and time-dependent components. Therefore, two components of strain must be included in a model to describe rate-dependent behavior, instead of three as has been previously assumed (Tatsuoka et al., 2000).

Elasto-viscoplastic models are most commonly used to describe rate-dependent deformations in soft rocks. In this formulation, irreversible deformations occur only when the stress point lies outside or on the yield surface. However, it is notable that irreversible deformations occur in soft rocks at stresses which lie within the “elastic” zone; this behavior is not predicted by standard

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elasto-viscoplastic models. Rate-type models, as those of Gutierrez (1999) and deWaal (1986), allow and predict this type of behavior. Therefore, rate-type models are preferred.

A rate-type cap plasticity constitutive model which is used to describe the behavior of soft rocks is presented briefly in this article. In addition, a new integration method for cap plasticity models of both rate-independent and rate-dependent types is proposed and described herein. Finally, a numerical example illustrating use of the rate-type model and new integration algorithm is presented.

DESCRIPTION OF CONSTITUTIVE MODEL

For soft rocks, shearing and pore collapse yielding mechanisms occur independently. A schematic of the composite yield surface for the model is shown in Fig. 1. This article will focus on the pore collapse mechanism. The pore collapse model described here is an elliptical cap plasticity model, similar to the Modified Cam-Clay model (Roscoe and Burland, 1968). As for Modified Cam-Clay, this model undergoes isotropic volumetric hardening which manifests itself as an increase in isotropic preconsolidation stress p_c . Rate-dependent and time-dependent effects also appear in the model. In addition to hardening by mechanical loading, hardening may also occur due only to passage of time.

To account for rate-dependence, the model is formulated within the framework of bounding surface plasticity. In this formulation, the yield surface is replaced by an isotropic hardening bounding surface with its size determined by the isotropic preconsolidation stress p_c and its aspect ratio in p - q space determined by the slope M (Fig. 1). In the general case, the stress point does not lie on the bounding surface; instead, the stress point lies on an elliptical loading surface, similar to the bounding surface, with its size determined by an equivalent preconsolidation stress p_{eq} (M is the same for the bounding surface and the loading surface).

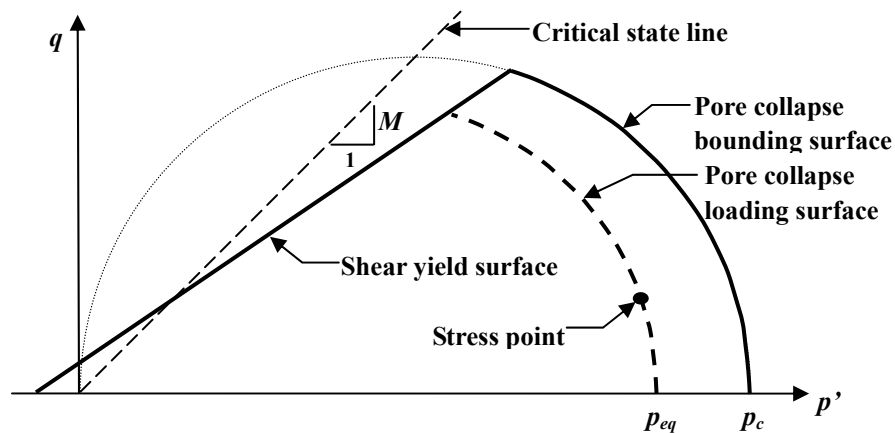


FIG. 1. Multiple yield mechanisms for constitutive model in p - q space.

The relationship between the loading surface and bounding surface is defined with respect to time and a quantity called the volumetric age, t_v . The volumetric age is calculated using the logarithmic creep formulation of Bjerrum (1967). This quantity was later used in the time-dependent model of Borja and Kavazanjian (1985), and is proportional to the distance in

stress space between the loading surface and bounding surface:

$$t_v = t_{vo} \left(\frac{p_c}{p_{eq}} \right)^{\frac{\lambda-\kappa}{\psi}} \quad (1)$$

where ψ is the creep parameter, λ and κ are the slopes of the virgin compression curve and recompression curve in e - $\ln p$ space, and t_{vo} is the minimum volumetric age allowed in the model (*i.e.*, the volumetric age for a stress point on the bounding surface).

Time-dependent deformation is proportional to the volumetric age. Irreversible time-dependent deformation is calculated using the logarithmic creep formulation of Bjerrum (1967):

$$d\varepsilon_v^{ir} = \frac{\psi}{1 + e} \frac{dt}{t_v} \quad (2)$$

NEW INTEGRATION METHOD

A new integration method is proposed here which does not use “plastic consistency” as its basis for solution. This section contains a brief description of plastic consistency and its use in integration of constitutive equations, a description of the new method for integration of rate-independent constitutive models, and a description of how the method may be extended to rate-dependent models. It should be noted that this new method is applicable to all elliptical cap plasticity models.

Conventional Integration for Elasto(visco)plasticity Using Plastic Consistency

Plastic consistency is inherent in the solution procedures commonly used to integrate constitutive relations. The elasto(visco)plastic response of most materials is characterized by the following incremental constitutive relations:

$$d\varepsilon_{ij} = d\varepsilon_{ij}^e + d\varepsilon_{ij}^{ir} \quad (3)$$

$$d\sigma_{ij} = D_{ijkl}^e d\varepsilon_{kl}^e \quad (4)$$

$$d\varepsilon_{ij}^{ir} = \phi \frac{\partial g}{\partial \sigma_{ij}}(\sigma_{ij}, q_\alpha) \quad (5)$$

$$dq_\alpha = \phi h_\alpha(\sigma_{ij}, q_\alpha) \quad (6)$$

In these equations, $d\varepsilon_{ij}$, $d\varepsilon_{ij}^e$, and $d\varepsilon_{ij}^{ir}$ are increments of the total, elastic, and inelastic or irreversible strain tensors, $d\sigma_{ij}$ is the increment of the Cauchy stress tensor, D_{ijkl}^e is the elasticity constitutive tensor, ϕ is a plastic multiplier, $\partial g / \partial \sigma_{ij}$ is the plastic flow direction, q_α is a set of plastic hardening variables, and h_α is the plastic hardening function. Eq. (3)-(6)

represent the properties of strain additivity, incremental elasticity, plastic flow rule, and plastic hardening rule. For rate-independent elastoplasticity, the quantity $d\varepsilon_{ij}^{ir}$ becomes the plastic strain increment $d\varepsilon_{ij}^p$. For points below the yield surface ($f < 0$), $\varphi = 0$, while for points above the yield surface ($f > 0$), $\varphi = \varphi(f)$.

Both the stress point and yield surface change due to plastic or viscoplastic processes. Since the current integration methods assume that the plastic processes which affect the stress point and yield surface are determined as functions of the same plastic multiplier φ , the plastic multiplier is also known as the plastic consistency parameter.

It is possible to combine Eq. (3)-(5) into a form which determines the elastoviscoplastic stress increment $d\sigma_{ij}$ as a function of the plastic consistency parameter:

$$d\sigma_{ij} = D_{ijkl}^e d\varepsilon_{kl} - \varphi D_{ijkl}^e \frac{\partial g}{\partial \sigma_{kl}} \quad (7)$$

while the plastic variable increment dq_α may be solved, also as a function of φ , as in Eq. (6). In Eq. (7), the first term represents the elastic predictor and the second term represents the plastic corrector, which are the key components to a return algorithm method.

When using the return algorithm method for rate-independent elastoplasticity, it is presumed that a converged final solution is obtained when the consistency condition satisfies the yield function at the final stress point and final yield surface:

$$f(\sigma_{ij,f}, q_{\alpha,f}) = f(\sigma_{ij,0} + d\sigma_{ij}, q_{\alpha,0} + dq_\alpha) = 0 \quad (8)$$

To satisfy the loading-unloading criterion, the final value of the yield function for an elastoplastic loading step should equal zero for a non-negative value of φ . It is possible to express the yield function in terms of only φ by substituting Eq. (6)-(7) into Eq. (8):

$$f(\varphi) = f\left(\sigma_{ij,0} + D_{ijkl}^e \left(d\varepsilon_{kl} - \varphi \frac{\partial g}{\partial \sigma_{kl}}\right), q_{\alpha,0} + \varphi h_\alpha\right) \quad (9)$$

Solution for φ using implicit integration requires simultaneous solution of several equations.

Plastic consistency is also inherent in the explicit substepping method of Sloan (1987). This method has its basis in Prager's consistency condition, which is used to form the elastoplastic constitutive matrix D^{ep} and the increment to the plastic hardening parameters. See Sloan (1987) for details regarding this method.

Description of New Integration Method for Rate-Independent Model

The basis for the new integration method is that the virgin isotropic compression behavior of isotropic cap plasticity models is fully defined by the isotropic compression line in e - $\ln p$ space. The preconsolidation pressure and void ratio of an isotropically normally consolidated geomaterial are defined by a one-to-one correspondence. For an isotropically normally

consolidated geomaterial:

$$p = p_c = \exp\left(\frac{N - e}{\lambda}\right) \quad (10)$$

where e is the void ratio, $p = \frac{1}{3}\sigma_{ii}$ is the mean stress, N is the void ratio that anchors the virgin compression line at $p = 1$, λ is the slope of the virgin compression line in e - $\ln p$ space, and p_c is the isotropic preconsolidation pressure. In triaxial compression, $p = \frac{1}{3}(\sigma_1 + 2\sigma_3)$.

Eq. (10) gives an exact solution for virgin isotropic compression of an isotropically normally consolidated geomaterial. Eq. (10) applies to the strain-controlled loading condition, where the void ratio is known and the mean stress must be found. Since void ratio is the input of Eq. (10), the following relationship is useful to convert between volumetric strain and void ratio:

$$e_f = \frac{1 + e_0}{\exp(\Delta\varepsilon_v)} - 1, \text{ where } \Delta\varepsilon_v = \Delta\varepsilon_{ii} \quad (11)$$

where the subscripts 0 and f refer to states at the beginning and end, respectively, of a given loading step. Compressive stresses and strains are positive in Eq. (11).

For general loading conditions, it is necessary to generalize Eq. (10) for non-isotropic conditions. Eq. (10) can be modified to describe anisotropically normally consolidated geomaterials. For anisotropically normally consolidated geomaterials:

$$p_c = \exp\left[\frac{N - e_f + \kappa \ln(p_c/p)}{\lambda}\right] \quad (12)$$

$$p = (p/p_c)p_c \quad (13)$$

$$q = \eta p = \eta(p/p_c)p_c \quad (14)$$

In Eq. (12)-(14), κ is the slope of the isotropic recompression line in e - $\ln p$ space, $q = \sqrt{\frac{3}{2}s_{ij}s_{ij}}$ is the stress invariant representing the deviatoric stress (where $s_{ij} = \sigma_{ij} - \delta_{ij}p$ and δ_{ij} is the Kronecker delta), and $\eta = q/p$ is the shear stress ratio. In triaxial compression, $q = \sigma_1 - \sigma_3$.

To use Eq. (12)-(14), it is necessary to first solve for the ratio (p_c/p) . Fortunately, it is possible to solve for this ratio using many cap plasticity models. For example, the yield surface for the Modified Cam-Clay model is an ellipse defined in p - q space by the following function:

$$f = g = q^2 - M^2 p(p_c - p) \quad (15)$$

It is possible to find the ratio between the preconsolidation pressure and the mean stress for a stress point on the yield surface by setting $f = 0$ and rearranging Eq. (15). For Modified Cam-Clay, as shown by Wood (1990):

$$\frac{p_c}{p} = \frac{\eta^2 + M^2}{M^2} \quad (16)$$

By substituting Eq. (16) into Eq. (12)-(14), it is possible to solve for the stress invariants p and q and plastic hardening parameter (preconsolidation pressure p_c) for the Modified Cam-Clay model. The stress components for the triaxial compression case can then be derived from the stress invariants p and q for an isotropic geomaterial by making appropriate assumptions. We see that *all stress components and plastic hardening parameters may be determined in an internally consistent manner for a normally consolidated geomaterial by solving for only the shear stress ratio η .*

For Modified Cam-Clay:

$$p_c = \exp \left[\frac{N - e_f + \kappa \ln \left(\frac{\eta^2 + M^2}{M^2} \right)}{\lambda} \right] \quad (17)$$

$$p = \frac{M^2}{\eta^2 + M^2} p_c \quad (18)$$

$$q = \frac{\eta M^2}{\eta^2 + M^2} p_c \quad (19)$$

Therefore, when using a cap plasticity model like the rate-independent Modified Cam-Clay model, the only unknown which must be determined to define the state of stress for a geomaterial which lies on the yield surface and has a known void ratio is the shear stress ratio η .

A converged value of shear stress ratio η may be found by maintaining consistency between the trial stress point, final stress point, and plastic flow direction. Only 1 equation must be solved to find the converged value, as opposed to the system of 4 equations which must be solved using current implicit integration procedures. The value may be obtained by iteration using the Newton-Raphson procedure. Details of the solution procedure are omitted here.

Extension of New Integration Method to Rate-Dependent Model

For the rate-dependent model, a second unknown (volumetric age) exists. Due to the similarity between the loading surface and bounding surface, the same relationship exists between mean stress p and equivalent preconsolidation stress p_{eq} as does between p and p_c for the rate-independent model:

$$\frac{p_{eq}}{p} = \frac{\eta^2 + M^2}{M^2} \quad (20)$$

In addition, we may rearrange Eq. (1) to show that a well-defined relationship, which depends on volumetric age, between p_c and p_{eq} exists:

$$\frac{p_c}{p_{eq}} = \left(\frac{t_v}{t_{vo}} \right)^{\frac{\psi}{\lambda - \kappa}} \quad (21)$$

It is possible to substitute Eq. (20)-(21) into Eq. (12)-(14) to solve for the stress invariants and isotropic preconsolidation stress, as for the rate-independent case. In the rate-dependent case, *all stress components and plastic hardening parameters may be determined in an internally consistent manner for a normally consolidated geomaterial by solving for only the shear stress ratio η and volumetric age t_v .*

For Modified Cam-Clay:

$$p_c = \exp \left[\frac{N - e_f + \kappa \ln \left(\frac{\eta^2 + M^2}{M^2} \right) + \frac{\kappa \psi}{\lambda - \kappa} \ln \left(\frac{t_v}{t_{vo}} \right)}{\lambda} \right] \quad (22)$$

$$p = \frac{M^2}{\eta^2 + M^2} \left(\frac{t_{vo}}{t_v} \right)^{\frac{\psi}{\lambda - \kappa}} p_c \quad (23)$$

$$q = \frac{\eta M^2}{\eta^2 + M^2} \left(\frac{t_{vo}}{t_v} \right)^{\frac{\psi}{\lambda - \kappa}} p_c \quad (24)$$

Converged values of shear stress ratio η and volumetric age t_v may be found using a similar procedure as is used for the rate-independent model. Again, details of the solution procedure are omitted here.

NUMERICAL EXAMPLE

One example illustrating the behavior of a water-saturated chalk sample under K_0 compression with creep stages is presented here. The chalk sample is from the Stevens Klint outcrop and serves as an analog to the petroleum reservoir chinks of the North Sea.

The laboratory loading program for the chalk sample consisted of 4 stages and is summarized as follows: Stage 1 was a K_0 compression loading stage (69 hours in length); stage 2 was a K_0 creep stage (208 hours); stage 3 was a K_0 compression loading stage (17 hours); and stage 4 was a K_0 creep stage (222 hours).

Results of the laboratory test and its numerical simulation are shown in Fig. 2. Fig. 2(a) shows the stress-strain behavior of the chalk, while Fig. 2(b) shows the creep behavior of the chalk. It is apparent that the model is able to closely simulate the observed behavior of the chalk.

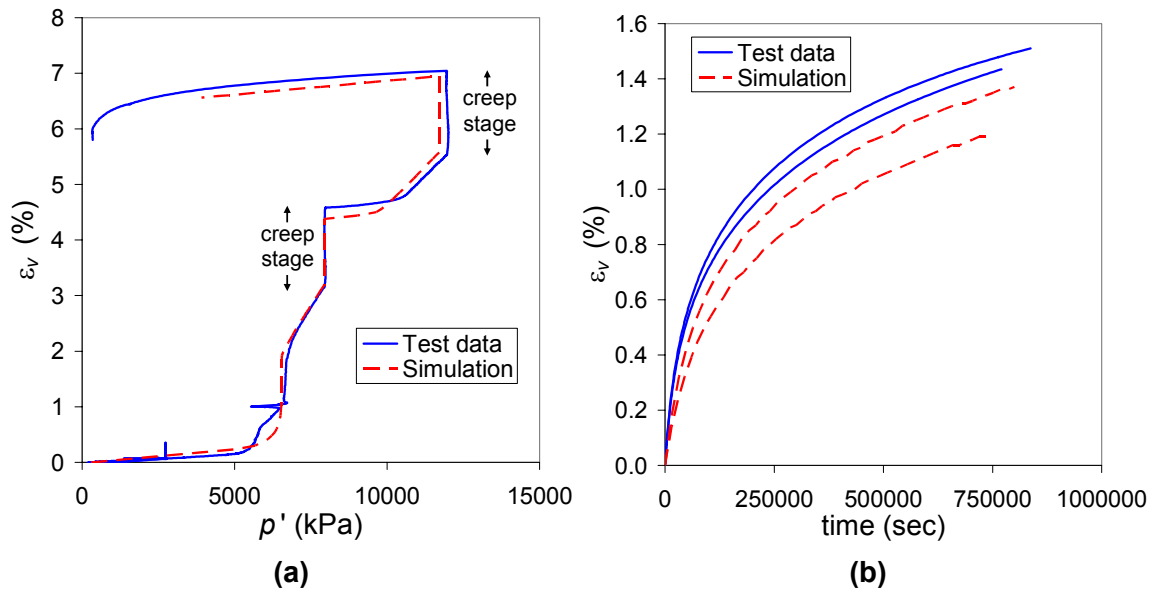


FIG. 2. Observed and simulated material behavior curves for the Stevns Klint chalk: (a) stress-strain behavior; (b) time-strain creep behavior.

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