

Chapter 2: Formulation

2.1 Assumptions

A few assumptions have been made regarding the general behavior of the geosynthetic tubes. First, to allow for a closed-form integral solution, friction is neglected. This includes friction between the tube and ground, and friction between two tubes. Next, the tubes are assumed to be inextensible and have no bending stiffness. Because they have no bending stiffness, it is assumed that they are able to conform to sharp corners. The tubes are also assumed to lay flat on rigid surfaces. They will also stay in contact with deformable foundations. The weight of the geosynthetic material is neglected. The tubes are very long, and a two-dimensional analysis of a cross section is performed.

2.2 Basic Formulation

The equations used to determine the shape and tension of the tubes is based on just a few formulas. Consider the condition shown in Figure 2.1, where a single tube rests on a rigid foundation. This represents the cross section of a long, slender tube. As referenced in Plaut and Suherman (1998), the horizontal coordinate is X , the vertical coordinate is Y , and the arc length of the tube from a given starting point is S . The variable θ represents the angle of the tangent to the tube from the horizontal, and L represents the total perimeter of the tube. W is the base length when the tube rests on a rigid foundation. X , Y , and θ are each a function of S .

It is assumed that the tube is filled with an incompressible fluid with specific weight γ . The specific weight of the fill is typically 1.5 – 2.0 times the specific weight of water. The pressure in the tube is a linear function of height relative to the external air pressure, and its value at the bottom of the tube is P . The circumferential tension per width perpendicular to the cross section is T . The tension remains constant in the tube because only normal forces are acting on the tube (Plaut and Suherman 1998).

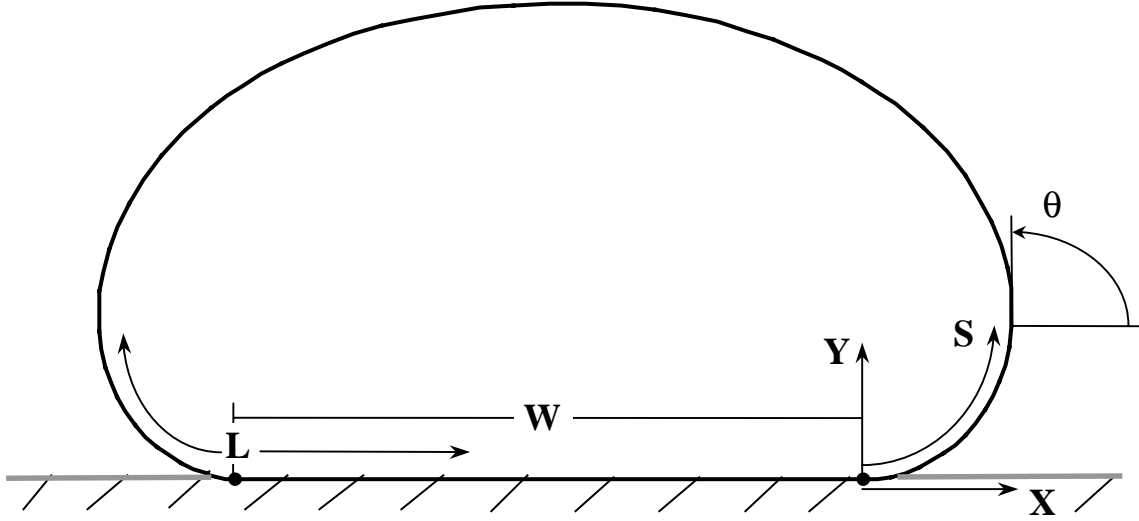


Figure 2.1 Single tube

To alleviate the problem of units, the following nondimensional quantities are used:

$$\begin{aligned}
 x &= \frac{X}{L}, & y &= \frac{Y}{L}, & s &= \frac{S}{L}, \\
 p &= \frac{P}{\gamma L}, & t &= \frac{T}{\gamma L^2}, & w &= \frac{W}{L},
 \end{aligned} \tag{2.1}$$

From Plaut and Suhrman (1998), the controlling nondimensional equations are:

$$\begin{aligned}
 \frac{d\theta}{ds} &= \frac{p-y}{t}, & \frac{dx}{ds} &= \cos\theta, & \frac{dy}{ds} &= \sin\theta
 \end{aligned} \tag{2.2, 2.3, 2.4}$$

2.2.1 Formulation for y

By multiplying by t and using the chain rule in equation (2.2), the equation becomes:

$$t \frac{d\theta}{dy} \frac{dy}{ds} = p - y \tag{2.5}$$

Substituting (2.4) into (2.5) and integrating,

$$t \int \sin\theta d\theta = \int (p - y) dy \tag{2.6}$$

Integration gives:

$$-t \cos\theta = c_0 + py - \frac{y^2}{2} \tag{2.7}$$

Solving for the constant using $\theta = 0$ when $y = 0$ gives $c_0 = -t$. Substitution yields:

$$\frac{y^2}{2} - py - t(\cos\theta - 1) = 0 \quad (2.8)$$

Using the quadratic formula,

$$y = p - \sqrt{p^2 + 2t(\cos\theta - 1)} \quad (2.9)$$

where the negative sign is chosen since $y = 0$ when $\theta = 0$.

2.2.2 Formulation for s

To solve for s, substitute equation (2.9) into (2.2):

$$\frac{d\theta}{ds} = \frac{1}{t} \sqrt{p^2 + 2t(\cos\theta - 1)} \quad (2.10)$$

Integration gives

$$s = t \int_0^\theta \frac{d\theta}{\sqrt{p^2 + 2t(\cos\theta - 1)}} \quad (2.11)$$

with $s = 0$ when $\theta = 0$.

2.2.3 Formulation for x

Equation (2.3) can be written as:

$$\frac{dx}{d\theta} \frac{d\theta}{ds} = \cos\theta \quad (2.12)$$

Substituting (2.2) into (2.12) gives:

$$\frac{dx}{d\theta} \frac{p-y}{t} = \cos\theta \quad (2.13)$$

or,

$$\frac{dx}{d\theta} = \frac{t}{p-y} \cos\theta \quad (2.14)$$

Integration yields:

$$x = t \int_0^{\theta} \frac{\cos\theta d\theta}{\sqrt{p^2 + 2t(\cos\theta - 1)}} \quad (2.15)$$

with $x = 0$ when $\theta = 0$.

2.2.4 General formulas

Considering that p is a linear function of height, it must be adjusted for segments that start at a height above $y = 0$. This can be accounted for by replacing p with G , where $G = p$ at $y = 0$. Another constant, J , is introduced as a multiplier of y in equation (2.2). This changes equation (2.2) to:

$$\frac{d\theta}{ds} = \frac{G - Jy}{t} \quad (2.16)$$

In the same way as above, y is derived here again:

$$t \int \sin\theta d\theta = \int (G - Jy) dy \quad (2.17)$$

$$-t \cos\theta = c_0 + Gy - \frac{Jy^2}{2} \quad (2.18)$$

$$\frac{Jy^2}{2} - Gy - t(\cos\theta - 1) = 0 \quad (2.19)$$

$$y = \frac{1}{J} (G - \sqrt{G^2 + 2Jt(\cos\theta - 1)}) \quad (2.20)$$

Due to the complexity of the tubes to be analyzed, most formulations will use multiple segments. A subscript i will be added to all parameters to represent the segment number. The three basic equations can be used for most segments by substituting the proper values for each variable. Summarizing the three equations:

$$y_i = \frac{1}{J_i} \left[G_i - \sqrt{G_i^2 - 2J_i t_i (\cos\theta_{i\alpha} - \cos\theta_i)} \right] \quad (2.21)$$

$$x_i = t_i \int_{\theta_{i\alpha}}^{\theta_i} \frac{\cos\theta_i d\theta_i}{\sqrt{G_i^2 - 2J_i t_i (\cos\theta_{i\alpha} - \cos\theta_i)}} \quad (2.22)$$

$$s_i = t_i \int_{\theta_{i\alpha}}^{\theta_i} \frac{d\theta_i}{\sqrt{G_i^2 - 2J_i t_i (\cos\theta_{i\alpha} - \cos\theta_i)}} \quad (2.23)$$

where $\theta_{i\alpha}$ is the initial angle of the tangent of the segment of the tube to the horizontal (at point α). The integrals for x_i and s_i can be solved for exactly with the use of elliptic integrals. These integrations are found by using a computer program, discussed in section 2.5.

The integrands of equations (2.21) - (2.23) are even functions of θ_i . This will result in a symmetric shape about a local maximum or minimum in the profile of a segment.

2.2.5 Formulas for specific points

The values of x_i , y_i , and s_i at the end of a segment (at point ω), denoted $x_{i\omega}$, $y_{i\omega}$, and $s_{i\omega}$, respectively can be determined from equations (2.21) - (2.23) as follows.

$$y_{i\omega} = \frac{1}{J_i} \left[G_i - \sqrt{G_i^2 - 2J_i t_i (\cos\theta_{i\alpha} - \cos\theta_{i\omega})} \right] \quad (2.24)$$

$$x_{i\omega} = t_i \int_{\theta_{i\alpha}}^{\theta_{i\omega}} \frac{\cos\theta_i d\theta_i}{\sqrt{G_i^2 - 2J_i t_i (\cos\theta_{i\alpha} - \cos\theta_i)}} \quad (2.25)$$

$$s_{i\omega} = t_i \int_{\theta_{i\alpha}}^{\theta_{i\omega}} \frac{d\theta_i}{\sqrt{G_i^2 - 2J_i t_i (\cos\theta_{i\alpha} - \cos\theta_i)}} \quad (2.26)$$

However, an exception must be made when the formula for G_i contains the variable $y_{i\omega}$. The computer program, Mathematica, cannot accept this form because it attempts to solve for $y_{i\omega}$ in terms of $y_{i\omega}$. Mathematica will repeatedly input equation (2.24) for G_i . Equation (2.24) can be rearranged as shown below:

$$G_i - J_i y_{i\omega} = \sqrt{G_i^2 - 2J_i t_i (\cos\theta_{i\alpha} - \cos\theta_{i\omega})} \quad (2.27)$$

Squaring both sides yields:

$$G_i^2 - 2G_i J_i y_{i\omega} + J_i^2 y_{i\omega}^2 = G_i^2 - 2J_i t_i (\cos\theta_{i\alpha} - \cos\theta_{i\omega}) \quad (2.28)$$

Dividing by J_i gives:

$$J_i y_{i\omega}^2 - 2G_i y_{i\omega} + 2t_i (\cos\theta_{i\alpha} - \cos\theta_{i\omega}) = 0 \quad (2.29)$$

If G_i has the form

$$G_i = Q_i + R_i y_{i\omega}, \quad (2.30)$$

then equation (2.29) becomes

$$J_i y_{i\omega}^2 - 2Q_i y_{i\omega} - 2R_i y_{i\omega}^2 + 2t_i (\cos\theta_{i\alpha} - \cos\theta_{i\omega}) = 0 \quad (2.31)$$

or,

$$(J_i - 2R_i) y_{i\omega}^2 - 2Q_i y_{i\omega} + 2t_i (\cos\theta_{i\alpha} - \cos\theta_{i\omega}) = 0 \quad (2.32)$$

Using the quadratic formula, the solutions are:

$$y_{i\omega} = \frac{2Q_i \pm \sqrt{4Q_i^2 - 8t_i (J_i - 2R_i) (\cos\theta_{i\alpha} - \cos\theta_{i\omega})}}{2(J_i - 2R_i)} \quad (2.33)$$

or,

$$y_{i\omega} = \frac{Q_i \pm \sqrt{Q_i^2 - 2t_i (J_i - 2R_i) (\cos\theta_{i\alpha} - \cos\theta_{i\omega})}}{(J_i - 2R_i)} \quad (2.34)$$

For cases in which $R_i = J_i$, equation (2.34) reduces to:

$$y_{i\omega} = \frac{-Q_i \pm \sqrt{Q_i^2 + 2t_i J_i (\cos\theta_{i\alpha} - \cos\theta_{i\omega})}}{J_i} \quad (2.35)$$

The sign of the radical must agree with the sign of Q_i (before it is input into equation (2.35)), so that $y_{i\omega} = 0$ when $\theta_{i\omega} = \theta_{i\alpha}$. Though the sign of Q_i may be initially unknown, trial and error will lead to a correct assumption.

2.2.6 Formulas for decreasing θ values

When G_i is negative, θ_i will decrease in value from its initial starting point. The sign of the radical must be changed in equations (2.21) and (2.24) to satisfy the condition that $y = 0$ when $\theta = 0$:

$$y_i = \frac{1}{J_i} \left[G_i + \sqrt{G_i^2 - 2J_i t_i (\cos\theta_{i\alpha} - \cos\theta_i)} \right] \quad (2.36)$$

$$y_{i\omega} = \frac{1}{J_i} \left[G_i + \sqrt{G_i^2 - 2J_i t_i (\cos\theta_{i\alpha} - \cos\theta_{i\omega})} \right] \quad (2.37)$$

The bounds of the integration in equations (2.22, 2.23, 2.25, and 2.26) must also be switched to satisfy the rules of integration:

$$x_i = t_i \int_{\theta_i}^{\theta_{i\alpha}} \frac{\cos\theta_i d\theta_i}{\sqrt{G_i^2 - 2J_i t_i (\cos\theta_{i\alpha} - \cos\theta_i)}} \quad (2.38)$$

$$s_i = t_i \int_{\theta_i}^{\theta_{i\alpha}} \frac{d\theta_i}{\sqrt{G_i^2 - 2J_i t_i (\cos\theta_{i\alpha} - \cos\theta_i)}} \quad (2.39)$$

$$x_{i\omega} = t_i \int_{\theta_{i\omega}}^{\theta_{i\alpha}} \frac{\cos\theta_i d\theta_i}{\sqrt{G_i^2 - 2J_i t_i (\cos\theta_{i\alpha} - \cos\theta_i)}} \quad (2.40)$$

$$s_{i\omega} = t_i \int_{\theta_{i\omega}}^{\theta_{i\alpha}} \frac{d\theta_i}{\sqrt{G_i^2 - 2J_i t_i (\cos\theta_{i\alpha} - \cos\theta_i)}} \quad (2.41)$$

2.3 Constraints

In order to solve for the unknown variables in the equations, a number of constraints must be defined. Three constraints can be based on geometric considerations of the shape:

$$\sum y_i = 0 \quad (2.42)$$

$$\sum x_i = 0 \quad (2.43)$$

$$\sum s_i = L_i \quad (2.44)$$

where the summation is over all segments of the tube and L_i is the perimeter length. Each segment used in constructing the shape must be included. However, for the example in Figure 2.1 and for all problems involving rigid foundations, the base length is unknown. As in the example above:

$$x_i + w_i = 0 \quad (2.45)$$

$$s_i + w_i = L_i \quad (2.46)$$

By simple algebra, w_i can be eliminated:

$$s_i - x_i - L_i = 0 \quad (2.47)$$

Each analysis will use the three basic constraints, equations (2.42), (2.43), and (2.44). However, some more complicated problems will need more constraints, which will be defined later.

2.4 Computer Analysis

For complex integration and solving, the technical computing program Mathematica from Wolfram Research was utilized (Wolfram 1996). Mathematica allowed for easy analysis of the tubes that could not have been performed as simply by conventional techniques.

Appendix A contains a sample Mathematica program and its results. After careful input of all equations, constants, and constraints, Mathematica would perform the analysis. Using the “FindRoot” command in Mathematica, the program could solve for any number of unknowns using an equal number of equations. The only requirement from Mathematica was to provide an initial guess for each variable.

Depending on the complexity of the problem, many solutions for a problem may exist. Mathematica often would not find the expected solution. However, rationally changing the initial guesses can result in a physical solution. Though many solutions may exist, Mathematica would return only one solution for most cases that would satisfy the assumptions and conditions of the problem.

From the solutions Mathematica found, a number of data points could be generated. An example Mathematica program used and its results can be found in Appendix B. Note that the subscripted form is omitted in the appendices. These data points are then transferred to Microsoft Excel, where high-quality graphs could be created.

2.5 Deformable foundation

In many applications these geosynthetic tubes will be placed atop muddy riverbanks or soft soils. The tube will settle due to the weight of the internal slurry mix on the soft deformable foundation. This will affect the shape, circumferential tension, and height of the tube as will be discussed in Chapter 3.

Geotechnical engineers sometimes model a deformable foundation as an infinite number of elastic springs, known as a Winkler foundation. The distributed force from the springs, proportional to the downward deflection of the ground, is assumed here to act normal to the contact region of the inextensible tube. The normal force exerted on the tube by the tensionless Winkler foundation, shown in Figure 2.2, is unlike the vertical force assumed by Plaut and Suherman (1998). In this case, the assumed normal force will allow for constant tension throughout the tube's perimeter. The resulting shape of the part of the tube's perimeter in contact with the ground will be symmetric about its center where the greatest ground deflection occurs.

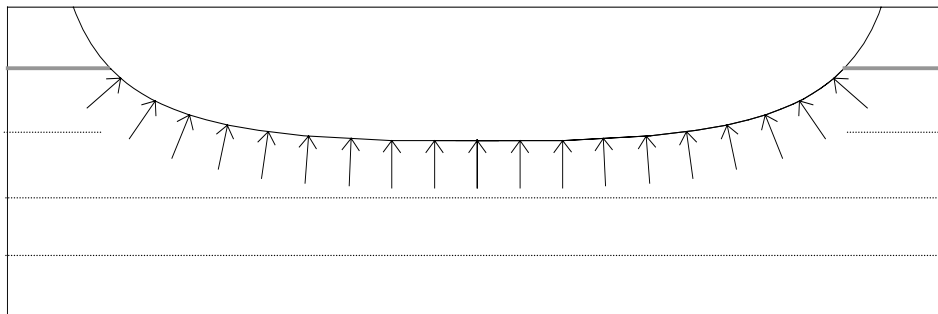


Figure 2.2 Normal force from a Winkler foundation with stiffness coefficient k .

The elastic constant for these springs is referred to as the stiffness coefficient, K . This coefficient depends on a number of factors and is not only dependent on a given soil. These factors include the length and width of the foundation. Though not discussed here, Das (1990) and Scott (1981) give a detailed description of the procedure. The stiffness coefficient, or force over volume, can also be defined as a dimensionless quantity, k :

$$k = \frac{K}{\gamma_1} \tag{2.48}$$

Structures on very stiff foundations possess k values near ∞ . These structures can be analyzed approximately by the procedure for rigid foundations. To determine the dimensional stiffness coefficient for a particular soil and foundation, please refer to the previously mentioned books. The examples shown in the following chapters use relatively soft soils so that the deformation of the foundation can be easily visualized.