A STUDY OF THE LAW OF PLASTIC FLOW OF METALS

A THESIS

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Introduction

In order to make a study of the laws of plastic flow of metals by employing data on thick-walled cylinders, cold worked by internal pressures, it will be necessary to select the best theory of combined stress for the elastic material; to determine the stress in the elastic portion of the wall when the outer fiber of the cylinder has not been stressed beyond its elastic limit.

There are a number of theories advanced as to the cause of rupture of materials under combined stresses and are as follows:

1. Rankin's or maximum stress theory.

The yielding starts when the maximum stress in any direction exceeds the elastic value obtained by simple tension or compression test (ABCD Fig. 1).

$$S_x = S_{yp}$$

S_x, S_y, S_z = stresses along the three principal axes.

Assume $S_x > S_y > S_z$

S_{yp} = stress at yield point.

2. St. Venant's or maximum strain theory.

Failure begins when the maximum strain is greater than the elastic strain in simple tension or compression test (EFGH Fig. 1).

$$\frac{S_x}{E} = \frac{S_{zp}}{E} + \frac{1}{mE}(S_y + S_z)$$

E = Young's Modulus

$\frac{1}{m}$ = Poisson's Ratio

3. Coulomb's or maximum shear theory.

The maximum shear causes yielding if the shear is greater than that of simple tension or compression test (DJKLM Fig. 1).

$$\tfrac{1}{2}\,(S_x + S_y) = \tfrac{1}{2}S_{yp}$$

4. The resilience or strain energy theory.

The quantity of strain energy is constant for all failures, and a material fails if this value is exceeded. (Ellipse Fig. 1). The constant for a material may be determined by the simple tension test.

$$W = \frac{1}{2E}\,(S_x^{\,2} + S_y^{\,2} + S_z^{\,2}) - \frac{1}{mE}\,(S_xS_z + S_{xy} + S_{yz}) = \frac{S_{yp}}{2E}$$

5. Mohr's circles.

This is a special development of the shear theory and gives the same values as shear theory for materials of equal tensile and compressive elastic limits.

$$S_v = \tfrac{1}{2}\sqrt{S_nS_m}$$

S_v = shearing yielding stress

S_m = tension yielding stress

S_n = compression yielding stress

S_w = compression and tension (equal) which together produce shearing

$$S_w = \frac{S_nS_m}{S_m + S_n}$$

if $S_m = S_n$ $\qquad S_w = \frac{S_m}{2} = S_v$

6. Hancock's Theory.

This is an elliptical function

$$\frac{S_s^2}{S_v^2} + \frac{S_g^2}{S_{yp}} = 1$$

S_s = shearing stress

Hancock suggests the shear theory be changed to

$$S_g = \sqrt{S_z^2 + \frac{S_{y^2p}}{S_v^2} S_b^2}$$

S_g = maximum shearing stress

if $S_m = S_n$, the above coincides with the shear theory.

7. Becker's theory.

This is a combination of the strain and shear theories. The shear of ductile materials is taken as 0.6 the allowable tension. The figure OPQRSTUVWX (Fig. 1) represent Becker's values. P to Q, R to S, U to V, and W to X are the lines of the strain theory. Lines QR and VW are parallel to the lines of the shear theory and at such a distance that $\frac{1}{2}(S_x + S_y) = .6S_{yp}$
Lines ST, TU, OP, and OX are also parallel to the lines of the shear diagram but S_x (max.) = $1.2\ S_{y \cdot p}$.

The internal friction theory and Wehage's theory at present are not sufficiently developed to be accepted or used. Still other theories of minor importance have been proposed, but are not supported by sufficient experimental data.

The wide variation of materials, the condition of the test, the value used for Poisson's ratio and other details not being available for the tests supporting the various theories make it difficult to

secure a definite comparison between the results as reported.

The work of Guest on thin-walled cylinders loaded axially and radially, is the principal support for the shear theory. The resilience theory checks closely with results of Cook and Robertson's thick-walled cylinder test. Haigh states that the resilience ellipse fits Guest's work as near as the shear polygon. Becker's work supports his theory, but the work of Hancock is given as a close check. From the test at the Watertown Arsenal on thick-walled cylinders under internal radial pressure, Norris has found that St. Venant's theory checks with the applied pressure better than Rankin's or Coulomb's theories. The gun formula (equation 6) was used in this work.

The properties of the cylinders used by the Arsenal are available and the values are the results of very careful and accurate work, so these values are compared with the principal theories of combined stress. The values of the required tensile stress at R_o and P_o are calculated from equation (2), and these results are checked against the allowable tensile stress for P_o by the four leading theories — Table 1 (a fifth theory is plotted in Fig. 1). It is clearly seen that the strain theory fits these values much better than any of the other theories.

From the work of Lamé the following equations have been developed for thick-walled cylinders under radial pressures with lateral loading.

$$p = \frac{R_o^2 P_o - R_1^2 P_1}{R_1^2 - R_o^2} - \frac{(P_o - P_1) R_1^2 R_o^2}{r^2 (R_1^2 - R_o^2)} \qquad (1)$$

$$t = \frac{R_o^2 P_o - R_1^2 P_1}{R_1^2 - R_o^2} + \frac{(P_o - P_1) R_1^2 R_o^2}{r^2 (R_1^2 - R_o^2)} \qquad (2)$$

$$S_p = Ev_p = (1 - \frac{1}{m}) \frac{R_o^2 P_o - R_1^2 P_1}{R_1^2 - R_o^2} - (1 + \frac{1}{m}) \frac{(P_o - P_1) R_1^2 R_o^2}{r^2 (R_1^2 - R_o^2)} \qquad (3)$$

$$S_t = Ev_t = (1 - \frac{1}{m}) \frac{R_o^2 P_o - R_1^2 P_1}{R_1^2 - R_o^2} + (1 + \frac{1}{m}) \frac{(P_o - P_1) R_1^2 R_o^2}{r^2 (R_1^2 - R_o^2)} \qquad (4)$$

p = radial stress in pounds per square inch.

t = tangential stress in pounds per square inch.

S_p = equivalent radial stress in pounds per square inch.

S_t = equivalent tangential stress in pounds per square inch.

P_o = inside pressure in pounds per square inch.

P_1 = outside pressure in pounds per square inch.

E = modulus of elasticity in pounds per square inch.

$\frac{1}{m}$ = Poisson's ratio.

U_p, U_t = unit radial, unit tangential strain.

$R_o R_1$ = inside, outside radius in inches.

r = any radius in inches.

Poisson's ratio for gun steel has been proven to be $\frac{1}{3}$.

If $P_1 = 0$ and $m = 3$

$$S_p = \frac{2}{3} P_o \frac{R_o^2}{R_1^2 - R_o^2} \left(1 - \frac{2R_1^2}{r^2}\right) \tag{5}$$

$$S_t = \frac{2}{3} P_o \frac{R_o^2}{R_1^2 - R_o^2} \left(1 + \frac{2R_1^2}{r^2}\right) \tag{6}$$

Equation (6) is commonly called the gun formula.

If the stress of the outer fiber is held constant for cylinders of the same material, but of variable ratio of R_o to R_1, it will be seen that the S_t value for $\frac{R_1}{r}$ will be the same for all cylinders. That is p at $\frac{R_1}{r} = 2$ is equal to P_o for $\frac{R_1}{R_o} = 2$ when S_{to} is the same.

$$P_n = \frac{S_{to}}{2} \frac{(R_1^2 - R_n^2)}{R_n^2} \tag{7}$$

p_n = radial pressure at radius r.

R_n = radius n.

S_{to} = stress in outer fiber.

These results check closely to values of $R_o = R_n$ in equation (6) if S_t at $R_n = S_{y.p.}$

Take the radius R_a to be such that

$$S_{ta} = S_{y.p}$$

By using S_{ta} and R_a in (6) and eliminate Ra by substitution from (7).

$$R_a^2 = \frac{2S_{to}R_1^2}{3S_{yp} - S_{to}} \tag{8}$$

This equation is used in determining the division of the elastic and plastic material on a thick-walled cylinder, cold worked partly through the wall. These values are found to check with the values of elastic strain limit from Fig. 3 or Fig. 7.

If a material is under greater strains than its elastic value, it is said to be in the plastic state and its properties to resist strain are changed. For simple stress the stress-strain diagram is used to determine the stress by the corresponding strain. Norris has developed an empirical equation for the plastic portion of a diagram of true stress to reduction of diameter (reduction of area is about the same) in simple tension test.

$$\text{True stress} = \frac{\text{load}}{\text{resultant area}}$$

$$S_b = \frac{T^{(1 - .0018Tx)} - T}{T^{.0018Tx}} + .2\sqrt{Tx}$$

S_b = stress in thousands of pounds per square inch.

T = ultimate tensile stress in thousands of pounds per square inch

$\left(\frac{\text{load}}{\text{original area}} \right)$

x = number of percent reduction of diameter. In this the apparent ultimate strength is used as a base.

This equation fits very well for the values above 2% reduction of diameter, but the values from the elastic limit to 2% are shown by test to be erratic.

Some studies of crystal slippage have been made by the use of the x-ray, but the methods of test and interpretation of results have not been developed sufficiently to know just what might be learned from these tests. Nadai has presented work on the plastic state of metal along the theory advanced by Lévy. This work is based on Mohr's diagram representing the strain (in place of stress) and assuming the material has a constant volume. The equations are somewhat involved and become extremely complicated for general problems, especially if the stress strain relation is a function of the higher order. For plastic flow Haigh has suggested the use of the resilience theory using $m = 2$. The strain theory is also a function of $\frac{1}{m}$ and as it is the best for elastic relation, the plastic properties should be more nearly represented by a strain relation with $m = 2$ than by the other theories.

The residual stresses in cold worked material have also been studied, but with even less success. The change in the elastic limit, the hysteresis loops, thermal changes, temperature effects, time effects and other undetermined causes have left results rather indefinite. Laws of plastic flow might aid in solving these internal stresses after unloading or these residual stresses probably would be useful in determining plastic laws.

Norris, in a study on auto fretted cylinders, has advanced the idea that the recovery from the strain under load is elastic. This theory gives results that are a fair check to experimental values, but the variation increases with the load. The increase might be explained by the change of yield point resulting from cold working.

Method of Test

These tests were conducted at the Watertown Arsenal by the Army Ordnance Department in a series of tests on thick-walled cylinders. To develop the required hydraulic pressures an 18:1 intensifier was used in series with a high pressure water pump. A Crosby test gage was used on the low side of the intensifier and readings were corrected from calibration of the intensifier. Pressure was also checked by the electrical resistance of a manganin coil in the high pressure line. The bore of the cylinder was sealed near each end by rubber packing which was carried on a spool, thus putting no longitudinal load on the cylinder. Figure (6) represents the method used in getting the enlargements. The gage ring (A) around the cylinder (B) carried three Ames dials (C) and an adjustment screw (D). The small holes (E) in the cylinder are drilled to the desired depth and the points of the dial set in them and the increase in diameter is taken by the sum of the readings of the dials at 180° apart. These dials are calibrated to .0001 inch.

Table I gives the eleven cylinders of report No. 258 which is a test of the limiting pressure of a cylinder without causing permanent strain. Two 3"x9" cylinders Nos. 1A and 2A of report No. 251 and three cylinders Nos. II, III, IV of Bridgeman's work report No. 238 are also included in Table I. Bridgeman used an electrical measuring device for inside diameters.

Data of a 6"x18" cylinder from test 6 of report No. 258 in which the following readings were taken: enlargement under load and after load for the 7", 9", 12", 15", and 18" diameters for various pressures, the 40000, 48000, 56000, 60000, 64000, 72000, 76000, 80000, 84000, and 88000 pounds per square inch are values available. The steel was 64900 pounds per inch elastic limit, 101000 pounds per square inch ultimate strength, and 30000000 pounds per square inch modulus of elasticity.

Hypotheses of Plastic Flow

The volume of steel during plastic flow is not absolutely constant, which is shown by the change in density of cold worked material, but these variations are not enough to materially affect the results. Therefore, the volume will be assumed constant.

Let V = Unit volume change

$e_x e_y e_z$ = Unit extension in the three principal directions

Now $V = e_x - e_y - e_z = 0$

Also $V = (1 - \frac{2}{m})(e_x - e_y - e_z) = 0$

Then $m = 2$

Therefore, Poisson's ratio is $\frac{1}{2}$

For elastic material the relation of P_x and e_x is a constant.

$$\frac{P_x}{e_x} = E$$

P_x = Unit load pounds

In the plastic state this relation of P_x and e_x is a variable and the values are influenced by analysis and the heat treatment of the material. The value of P_x for given e_x may be taken from the equation or from the true stress-strain diagram. The commonly used stress-strain diagram may be used for small extensions without causing noticeable error.

From the strain theory:

$$S_x = E e_x = P_x - \frac{1}{m} P_y \qquad (9)$$

$$S_y = Ee_y = P_y - \frac{1}{m} P_x \qquad (9')$$

Now if the strain theory is used for the plastic flow with $m = 2$ these equations become:

$$S_x = P_x - \tfrac{1}{2}P_y \qquad (10)$$

$$S_y = P_y - \tfrac{1}{2}P_x \qquad (10')$$

If the compression and tension maintain an equal elastic limit, equation (10) can be used for values of P_x up to $0.666S_{y.p.}$ and equation (10") for higher stresses.

$$S_x = P_y - \frac{P_x}{2} \qquad (10")$$

In plastic material S_x is equal the yielding stress for the given amount of cold working.

The unit deformation for radial and circumferential direction are equal.

$$C = 2\pi r$$

$$C + \Delta C = 2\pi(r + \Delta r)$$

$$C = 2\pi \Delta r$$

$$\frac{\Delta C}{C} = \frac{\Delta r}{r}$$

This relation of strains makes it possible to use the unit radial deformations in the gun formula without corrections.

The values of R_a as determined by equation (8) are found to check very closely with the results obtained from the intersection (B) of the strain curves and the line of the elastic limit strain. Values from the true stress-strain diagrams are used in the checking of these values. The local effect of elastic and plastic material at the radius R_a is probably small as the steel does not have a very

sharp change, so this effect is neglected. The elastic cylinder of radii R_1 and R_a is worked according to the gun formula and the plastic cylinder of radii R_a and R_o is worked by the equation developed in the following paragraph.

If Fig. 5 is an annulus element from the plastic cylinder, the forces shown must be in equilibrium because without a lateral load no other forces can act on this element

$$2 P_1 r_1 + t_p(r_1 - r_o) = 2 P_o r_o$$

$$r_1 - r_o = \Delta r$$

$$P_o = \frac{P_1 r_1}{r_o} + \frac{t_p(\Delta r)}{r_o} \qquad (11)$$

The smaller the values of r are taken, the greater the accuracy of the results. When values of $\frac{1}{2}(P_1 - P_a)$ approaches $0.666 S_{y.p.}$ and if too large a Δr is used the results may be appreciably affected. The values allowable of t_p is determined by the plastic strains theory for the new elastic limit according to the stress-strain diagram.

As an illustration of the application of this method the 56000 pound test will be used.

P_o = pounds per square inch

S_{to} = 15270 pounds per square inch

S_{yp} = 64900 pounds per square inch

R_a = 3.73 inches

P_a = 36300 pounds per inch (Eq. 7)

Let $r_a = R_a$

r_b = 3.5 inches

r_c = 3.01 inches (.01 inches estimated enlargement of bore)

By equation (11)

$$P_b = \frac{36300 \times 3.73}{3.5} + 45900\frac{(3.73 - 3.5)}{3.5}$$

$$= 43290 \text{ pounds per square inch}$$

t_p was determined by trial from $P_r = \frac{P_1 - P_o}{2}$ and according to equation (10)

$$= 66500 - \tfrac{1}{2}\frac{(36300 + 43290)}{2}$$

$$= 45900 \text{ pounds per inch.}$$

By the same process for r_b and r_c, P_o = 55400 pounds per square inch. This gives a result 6 pounds per square inch pressure or almost 1 percent low according to the observed applied pressure.

The combined stresses are taken for corresponding strain from the diagram Fig. 4 and these values are plotted in Fig. 7 for the different pressure of the test.

Residual Stresses

A cylinder cold worked by internal pressure has a greater permanent unit enlargement of the inside fiber than of the outer fiber. Unequal strains cause internal stress in an elastic body. Cold worked steel has elastic properties similar to annealed steel, but the values of these properties are changed according to the amount of straining. The variation of the strain in a thick-walled cylinder causes the allowable stresses to be different for each radial distance.

In a cylinder radial pressures cause tangential stresses; therefore, the residual strains are produced by combined internal stresses. To determine the stresses of these a true stress-strain curve as in Fig. (13) will be used for the following equation

$$\delta_L - \frac{S_{YP}}{E_x} = \delta_o \qquad (12)$$

δ_L = Unit strain under load

S_{yp} = Yielding stress (after cold working) pounds per square inch

E_h' = Average modulus from D' to H' pounds per square inch

δ_R = Unit residual strain

δ_S = Unit strain from

δ_o = Unit permanent strain that causes no stress

H' = δ_o in Fig. (13)

E' = S_{yp} in Fig. (13)

(Note: δ_o = 0 in material not strained beyond the elastic limit)

Fig. (14) is an illustration of this method curve ABC from the unit strain at 72000 pound pressure from Fig. (3), curve IJK, the residual unit strain after releasing above pressure, from Fig. Z (12) LM is calculated by equation (12), and curve JO'N is calculated from equation (13) and then multiplied by E.

The curves of Fig. (15) are the JO'N curves for the various pressures used in the tests.

TABLE I

Cyl.No.	Size	Elas. Limit	P_o used	t(eq.1)	Allowable t by Rankin's	Coulomb's	St.Venant	Resilience
1	3x4.5	75000	25000-	65000	75000	50000	66270	62870
2	3x6	75000	37000	63200	75000	37000	63300	53400
3	3x9	75000	51000-	63700	75000	24000	58000	40600
4	6x9	75000	25000	65000	75000	50000	66270	62870
5	6x12	75000	37000	63200	75000	37000	63300	53400
6	6x18	65000	47000	58700	65000	18000	49300	46300
78	x18x12	77100	24000-	62400	77100	53100	69100	65700
8	8x16	73750	36500-	60900	73750	37250	61500	52800
9	8x24	70750	43200-	54000	70750	27450	56150	43400
10	9.5x14.25	71100	23500-	61000	71100	47600	63260	59700
11	9.5x19	64700	35000+	58300	64700	28700	58810	44000
1A	3x9	54250	38000-	47500	54250	16250	41590	28040
2A	3x9	52500	35100-	43900	52500	17400	40800	29200
		Estimated(not plotted in Fig.1)						
II	1.5x3	65000	31880	53200	65000	33120	49060m	43000
III	1.5x4.5	65000	45190	56300	65000	19810	42410	30000
IV	1.5x6.75	65000	49800	54000	65000	15200	40100	24500

Data and Computed Results

40000 pounds per square inch pressure.
R_a = below 3 inches
P_a = x

Radius	3	3.5	4	4.5	5	5.5	6	6.5	7	7.5	8	8.5	9
Enlargement at 40000 lbs		.010775		.009425			.007325			.006425			.00605
Enlargement at 0 lbs		.000025		.00005			.0001			.000025			.000075
δ_L		.00154		.001047			.000621			.000428			.000334
δ_R		.000003		.000006			.00001			.000022			.000004
$E\delta_L$	63000	46180		31420			18310			12850			10080
$E\delta_R$		-107		-167			-250			-50			-125

(Note: all dimensions given in inches.)

48000 pounds per square inch pressure.
R_a=3.34 inches
P_a=39400 pounds per square inch

Radius inches	3	3.5	4	4.5	5	5.5	6	6.5	7	7.5	8	8.5	9
Enlargement at 48000Lbs	.	.0133		.1165			.0089			.0079			.007425
Enlargement at 0 lbs		.0004		.0004			.000075			.0001			.00005
δ_L	.0025	.0019		.00129			.00074			.00053			.00041
δ_R	.00007	.00006		.000045			.000006			.000006			.000003
$E\delta_L$		57000		38830			22250			15800			12375
$E\delta_R$		1700		1320			190			200			80
St	66500	64900 max.											
t_p	44500												
P_r	44500	(Note: heavy line divides elastic and plastic values.)											
P_o	47850												
Applied P	48000												
Error	- 250	.5%											
$S_{max} \div E$													
δ_o		—											
δ_s		.00006		.000045			.000006			.000006			.000003
$E\delta_s$		1700		1320			190			200			80

56000 pounds per square inch. pressure.
R_a=3.73 inches
P_a=36300 pounds per square inch

Radius	3	3.5	4	4.5	5	5.5	6	6.5	7	7.5	8	8.5	9
Enlargement at 56000 lbs		.0175		.015075			.01165			.00975			.009225
Enlargement at 0 lbs.		.002125		.0020			.00085			.0006			.0006
δ_L	.00328	.0025		.00166			.00093			.00065			.00051
δ_R	.0008	.0003		.00022			.00007			.00004			.00005
$E\delta_L$				50250			28130			19500			15270
$E\delta_R$				6670			2130			1200			1000
S_t	67000	66500	64900 max.										
t_p	33000	46000											
P_r	50000	41150											
P_o	55400	43290											
Applied P	56000												
Error	-600	1%											
$S_{max} \div E$		.0022											
δ_o		.00028											
δ_S		.000023		.00022			.00007			.0004			.00003
$E\delta_S$		690		6670			2130			1200			1000

60000 pounds per square inch pressure.
R_a=3.9 inches
P_a=36400 pounds per square inch

Radius	3	3.5	4	4.5	5	5.5	6	6.5	7	7.5	8	8.5	9
Enlargement at 60000 lbs		.019575		.01655			.0121			.010525			.00995
Enlargement at 0 lbs.		.00315		.0029			.00105			.000775			.0008
δ_L	.0042	.00279		.00184			.00101			.0007			.00055
δ_R		.00048		.000332			.000088			.000052			.000045
$E\delta_L$				55200			30400			21000			16600
$E\delta_R$		14400		9670			2620			1560			1480
S_t	67100	66000	64900 Max.										
t_p	28100	44750											
P_r	52500	42500											
Po	59200	45950											
Applied P	60000												
Error	-800	1.3%											
$S_{max} \div E$	.00223	.0022											
δ_o		.00059											
δ_s		-.00011		.000332			.000088			.000052			.000045
$E\delta_s$		-3300		9670			2620			1560			1480

64000 pounds per square inch pressure.
R_a=4.18 inches
P_a=35600 pounds per square inch

Radius	3	3.5	4	4.5	5	5.5	6	6.5	7	7.5	8	8.5	9
Enlargement at 64000 lbs		.02265		.018925			.01375			.01195			.011325
Enlargement at 0 lbs.		.00543		.0047			.001875			.001375			.0015
d_L	.0045	.0032	.00267	.0021			.00116			.0008			.00063
d_R	.002	.00078.		.000523			.000156			.000091			.000083
$E d_L$				.63800			34800			23900			18800
$E d_R$				15670			4690			2750			2500
S_t	67500	66600	66000	64900 Max.									
t_p	22200	44350	47450										
P_r	56400	44300	37100										
P_o	62500	50380	38320										
Applied P	64000												
Error	-1500	2.3%											
$S_{max} \div E$		.00225											
d_o		.00098											
d_s		-.00021		.000523			.000156			.000091			.000083
$E d_s$		- 6200		15670			4690			2750			2500

72000 pounds per square inch pressure.
R_a=5.58 inches
P_a=24600 pounds per square inch

Radius	3	3.5	4	4.5	5	5.5	6	6.5	7	7.5	8	8.5	9
Enlargement at 72000 lbs		.042125		.033825			.022725			.019025			.018725
Enlargement at 0 lbs.		.020025		.016225			.008375			.0063			.00655
δ_L	.008	.00602	.0048	.00374	.0030	.0024	.001894			.00127			.00104
δ_R		.00284		.0018			.000695			.00042			.00036
$E\delta_L$							56810			38500			31210
$E\delta_R$							20940			12600			10920
S_t	68300	68100	66900	66600	66400	66000	64900 Max.						
t_p	1300	21000	41000	47500	52100	53000							
P_r	67830	57720	46680	38260	28950	25500							
P_o	72820	62860	52580	42780	33800	26100							
Applied P	72000												
Error	+820	1.1%											
$S_{max} \div E$		.00228		.00225									
δ_o		.00374		.00149									
δ_s		-.00089		.00029			.000695			.00042			.00036
$E\delta_s$		-25800		8700			20940			12600			10920

76000 pounds per square inch pressure.
R_a=6.23 inches
R_a=20800 pounds per square inch

Radius	3	3.5	4	4.5	5	5.5	6	6.5	7	7.5	8	8.5	9
Enlargement at 76000 lbs.		.05225		.04135			.028875			.02265			.0225
Enlargement at 0 lbs.		.02875		.02245			.01245			.0090			.00955
	.0110	.00746	.00595	.00459	.00366	.0028	.00241			.00151			.00125
	.0127	.00411		.00249			.00104			.0006			.00053
E										45300			37500
E										18000			15920
S_t	69200	68700	68300	68000	67500	67000	66200	64900 Max.					
t_p	-6400	12100	31000	46500	51000	53400	55100						
P_r	72080	62720	52790	43460	34330	26830	22000						
Po	76730	66700	58000	47580	38350	30320	23350						
Applied P	76000												
Error	+730	.9%											
S_{max}+E		.00228		.00227			.0022						
		-.00566		-.00232			.0022						
		-.00122		-.00013			.00084			.0006			.00053
E		-36600		- 3900			25200			18000			15920

80000 pounds per square inch pressure.
R_a=7.3 inches
P_a=12200 pounds per square inch

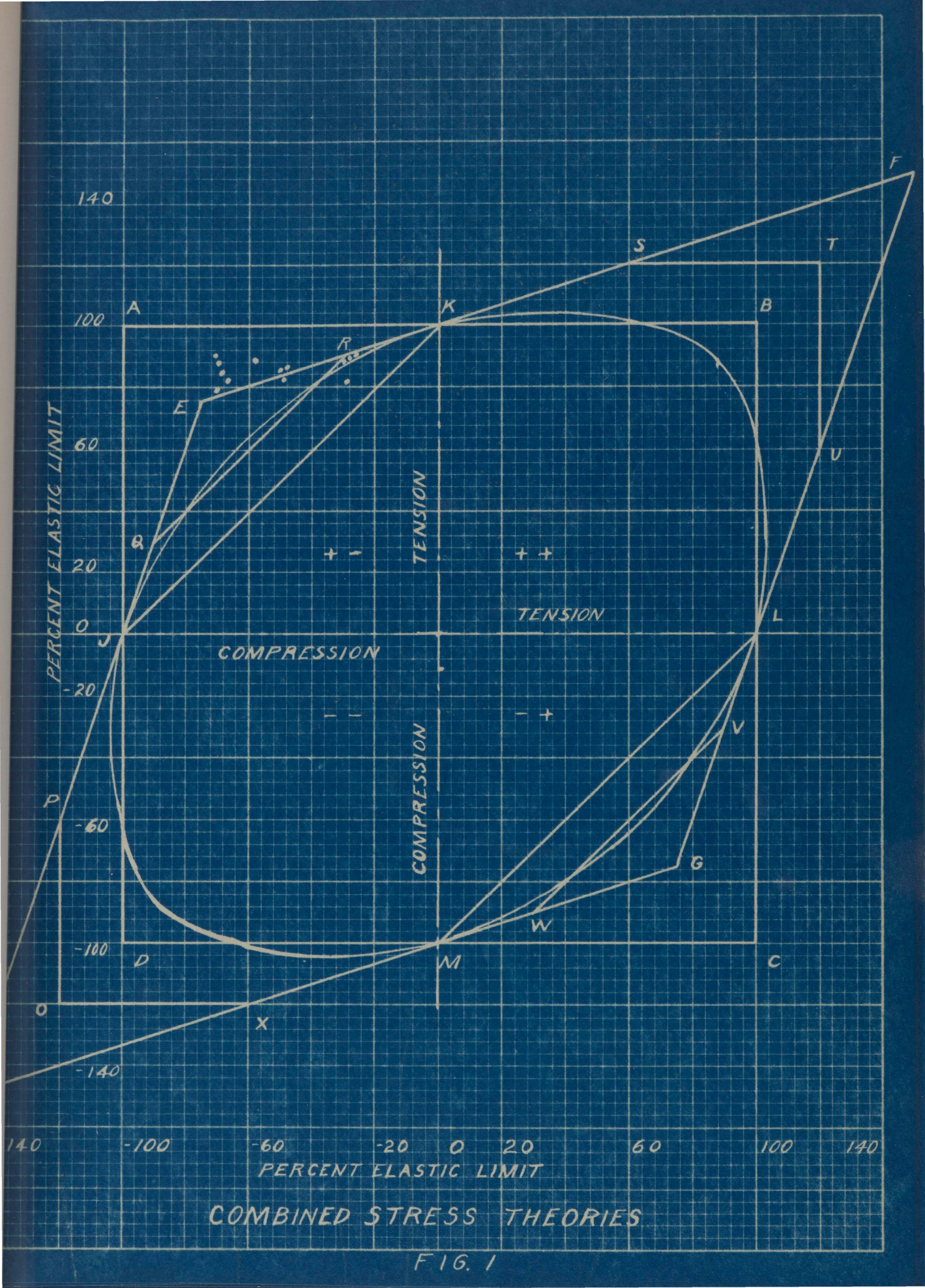
Radius	3	3.5	4	4.5	5	5.5	6	6.5	7	7.5	8	8.5	9
Enlargement at 80000 lbs		.0654		.051925			.034325			.027625			.027675
Enlargement at 0 lbs.		.04005		.03145			.017675			.013025			.013675
δ_L	.0137	.00834	.00734	.00577	.0044	.00347	.00286	.0026	.00216	.00184			.00154
δ_R	.0131	.00572		.00349			.00147			.00087			.00076
$E\delta_L$										54725			46120
$E\delta_R$										26050			22790
S_t	69200	68800	68300	68000	67900	67500	67100	66200	65000	64900 Max.			
t_p	-12300	6500	23500	41500	48300	52100	56500	57400					
P_r	75400	65800	56540	47290	38440	30670	23870	17830	15200				
P_o	80190	70710	60900	52180	42600	34280	27070	20660	15000				
Applied P	80000												
Error	+ 190	.2%											
$S_{max} \div E$		.00231		.00226			.0022						
δ_o		.00703		.00351			.00066						
δ_s		-.00127		-.00001			.00080			.00087			.00076
$E\delta_s$		-38200		-3000			24000			26050			22790

84000 pounds per square inch pressure.
R_a=8.25 inches
P_a=5500 pounds per square inch

Radius	3	3.5	4	4.5	5	5.5	6	6.5	7	7.5	8	8.5	9
Enlargement at 84000 lbs		.080675		.0647			.0427			.03465			.03435
Enlargement at 0 lbs.		.052925		.042175			.02475			.0186			.019075
d_L	.0157	.0124	.00905	.0072	.00565	.0044	.00353	.0031	.00286	.00231	.00217		.00191
d_R		.00765		.00468			.00206			.00124			.00106
$E d_L$													57250
$E d_R$													31790
S_t	70500	69500	68800	68300	68000	67900	67500	67100	66600	66000	65000		
t_p	-12000	1200	19500	38000	47800	51000	54200	57000	58800	60800	61700		
P_r	77560	68560	59280	49970	41270	33500	26660	20590	15160	10300	6600		
P_o	82000	73150	63920	54550	45400	37150	29850	23480	17710	12400	7650		
Applied P	84000												
Error	-2000	2.4%											
$S_{max} \div E$		.00232		.00228			.00223			.00220			
d_0		.00920		.00492			.00133			.00017			
d_s		-.00164		-.00025			.00073			.00107			.00106
$E d_s$		-49200		-3000			21900			32000			31790

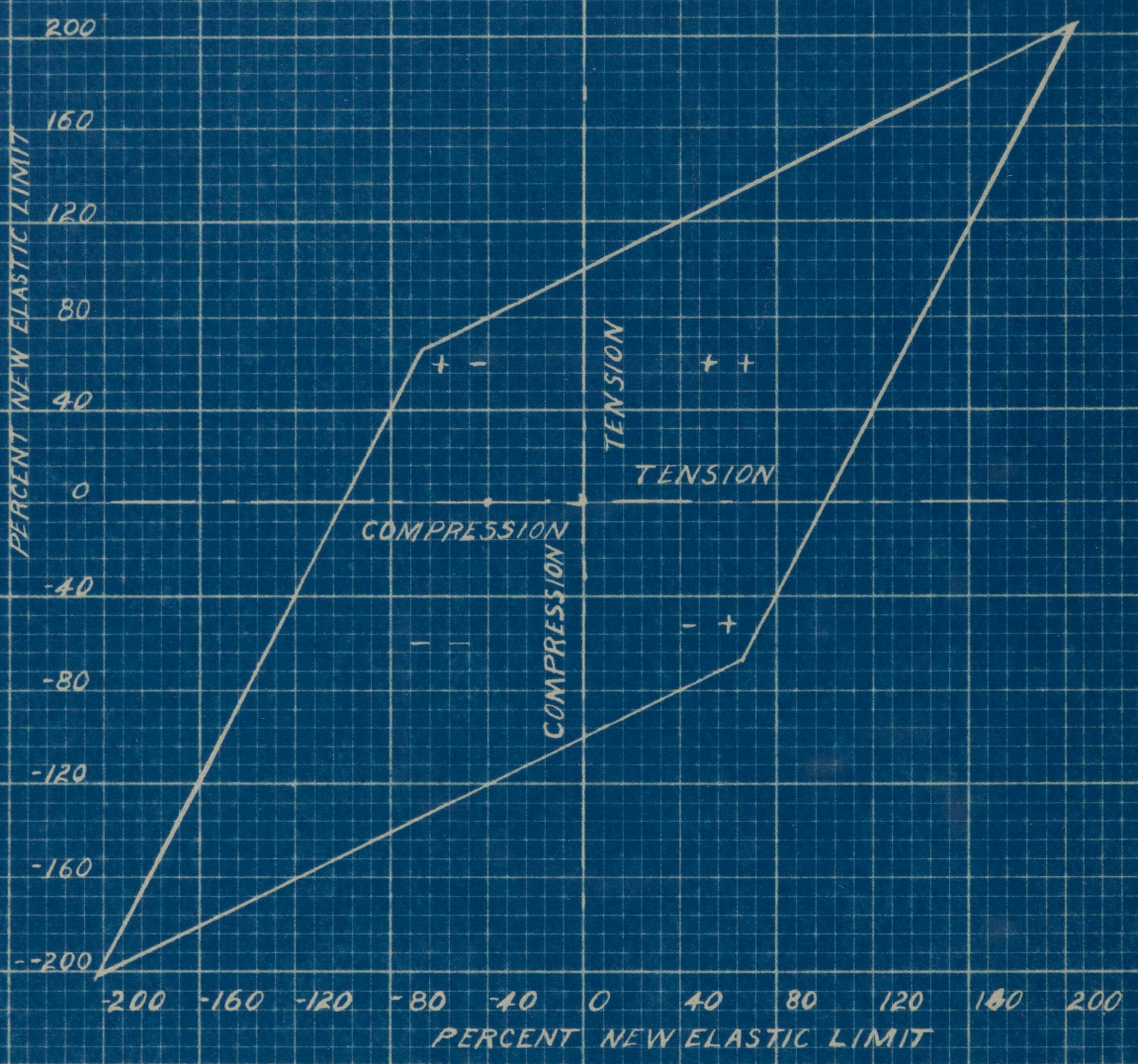
88000 pounds per square inch pressure.
R_a=9+ inches
P_a=0 pounds per square inch

Radius	3	3.5	4	4.5	5	5.5	6	6.5	7	7.5	8	8.5	9
Enlargement at 88000 lbs		.107125		.084825			.05605			.04475			.04555
Enlargement at 0 lbs		.0771		.06175			.03695			.028025			.028825
d_L	.0250	.0153	.0122	.00943	.0070	.00577	.00467	.0040	.0034	.00299	.00273	.00265	.00253
d_R		.01101		.00686			.003079			.00187			.0016
E d_L		xxxxx											
E d_R		xxxxx											
S_t	80000	70500	69500	68600	68200	68000	67900	67800	67500	67000	66700	66400	66000
t_p	-100	2000	20000	36700	47500	51200	54600	57500	59800	62000	63500	65500	
P_r	80350	69940	59770	50590	41720	33770	26800	20720	15280	10420	6020	1860	
P_o	85200	75500	64380	55170	46010	37440	30100	23600	17850	12720	8130	3910	0
Applied P	88000												
Error	-2800	3.2%											
$S_{max} \div E$		.02313		.00229			.00227			.00224			.00223
d_o		.01299		.00714			.00240			.00074			.00030
d_s		-.00199		-.00028			.0007			.00126			.0013
E d_s		-59700		-8400			21000			37800			39000



COMBINED STRESS THEORIES

FIG. 1



PLASTIC STRAIN THEORY

FIG. 2

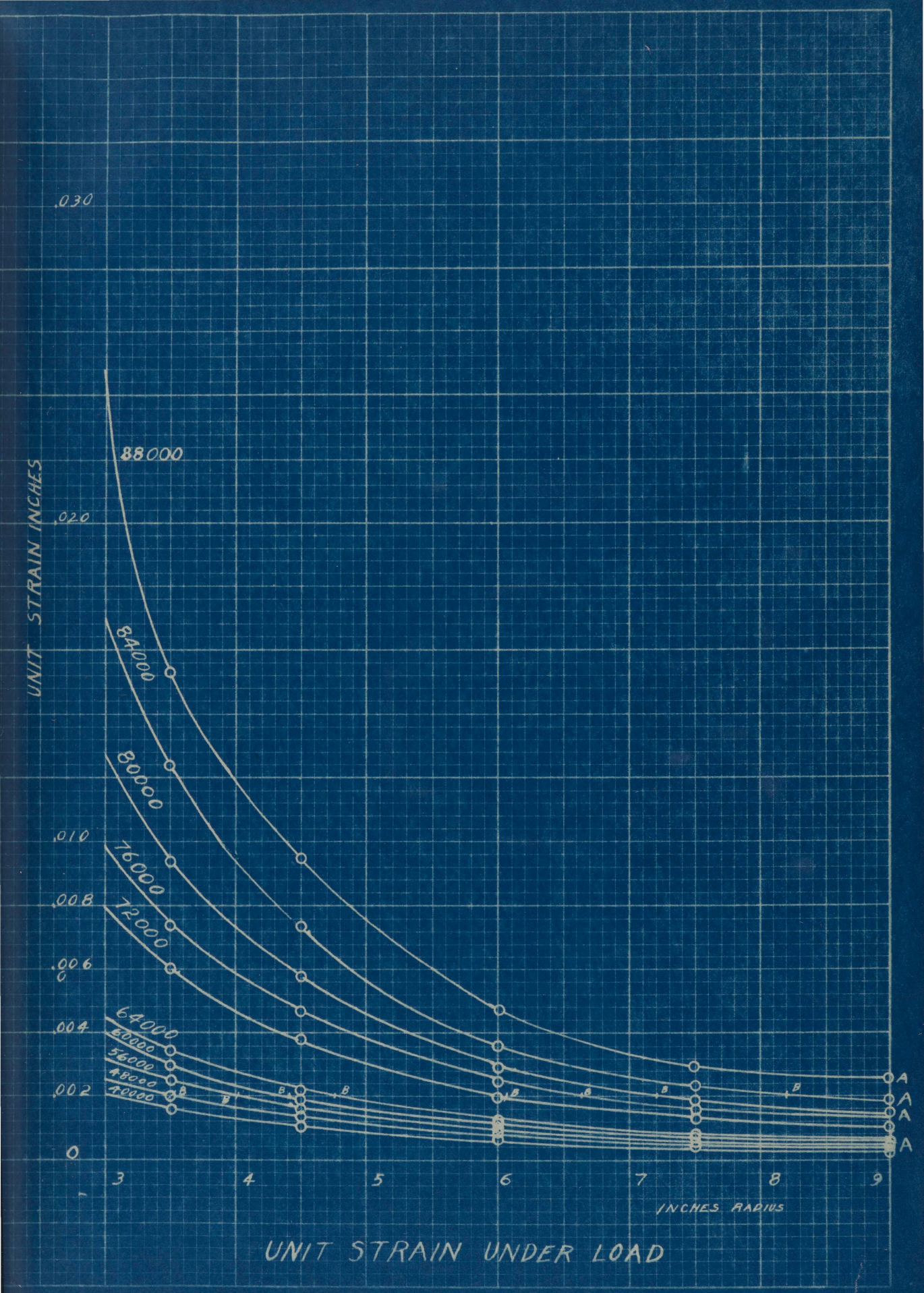
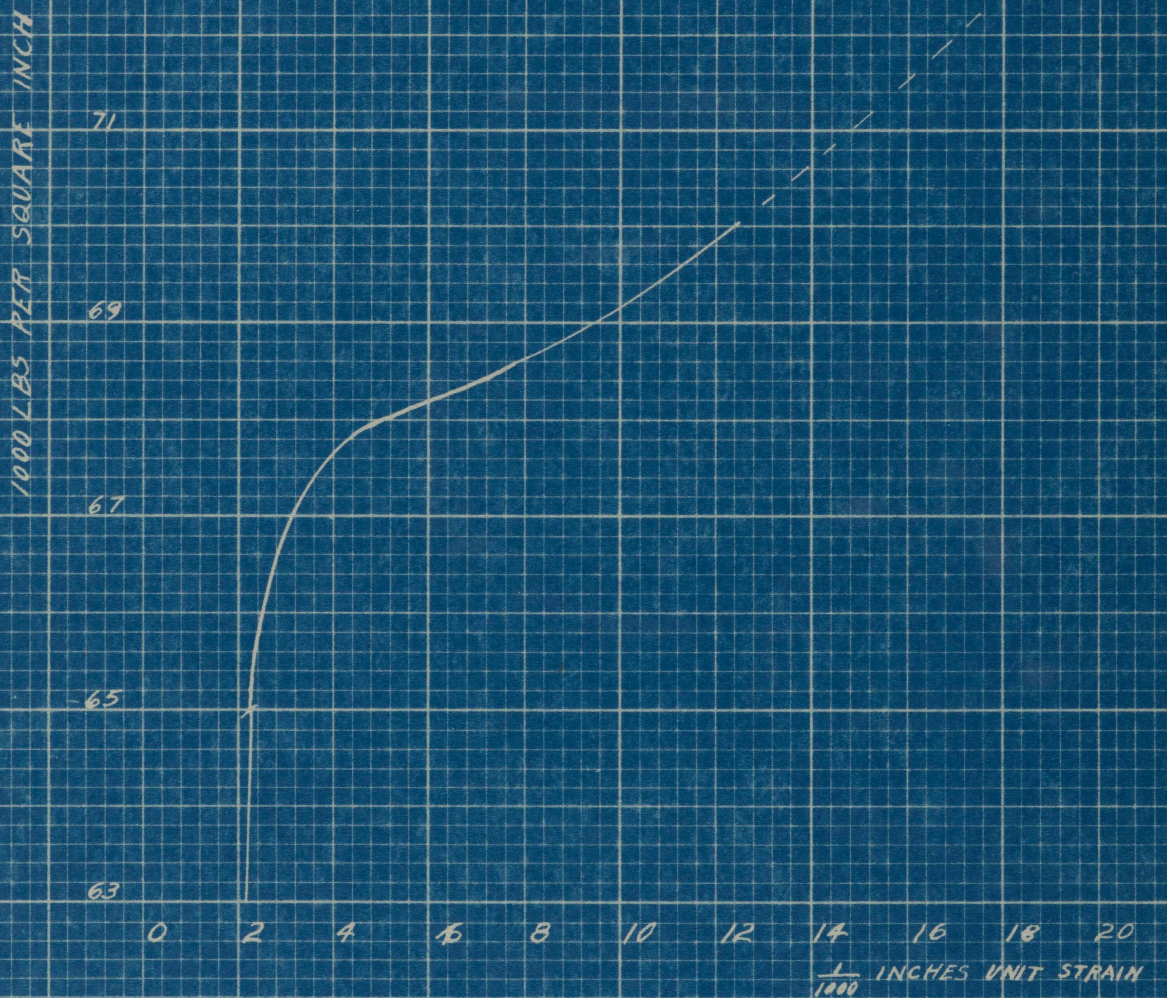

.030
.020
.010
.008
.006
.004
.002
0
UNIT STRAIN INCHES
88000
84000
80000
76000
72000
64000
60000
56000
48000
40000
A
B
3
4
5
6
7
8
9
INCHES RADIUS
UNIT STRAIN UNDER LOAD

FIG. 3



TRUE STRESS - STRAIN DIAGRAM

FIG. 4

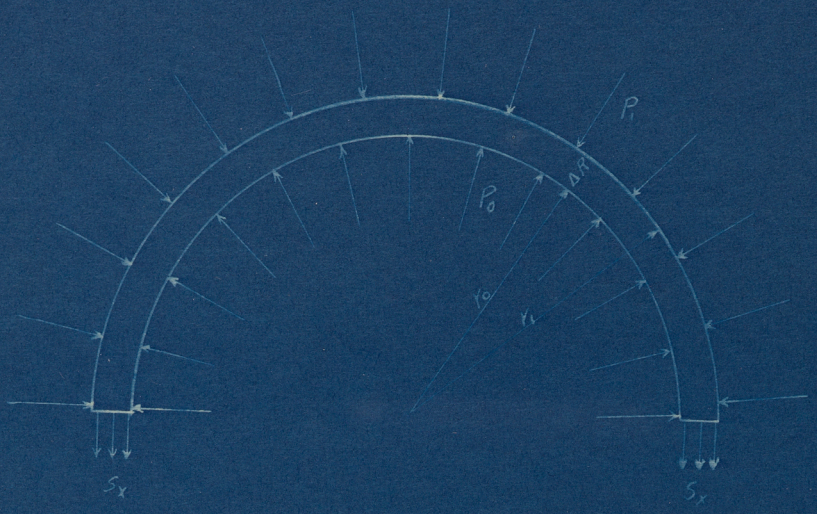


FIG. 5

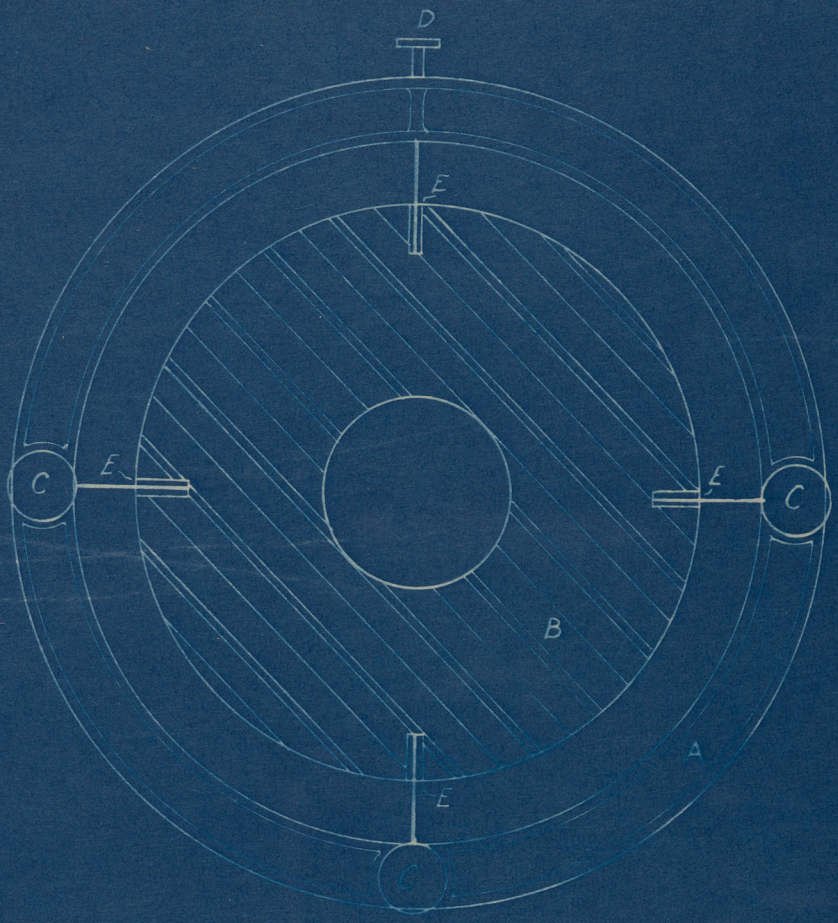
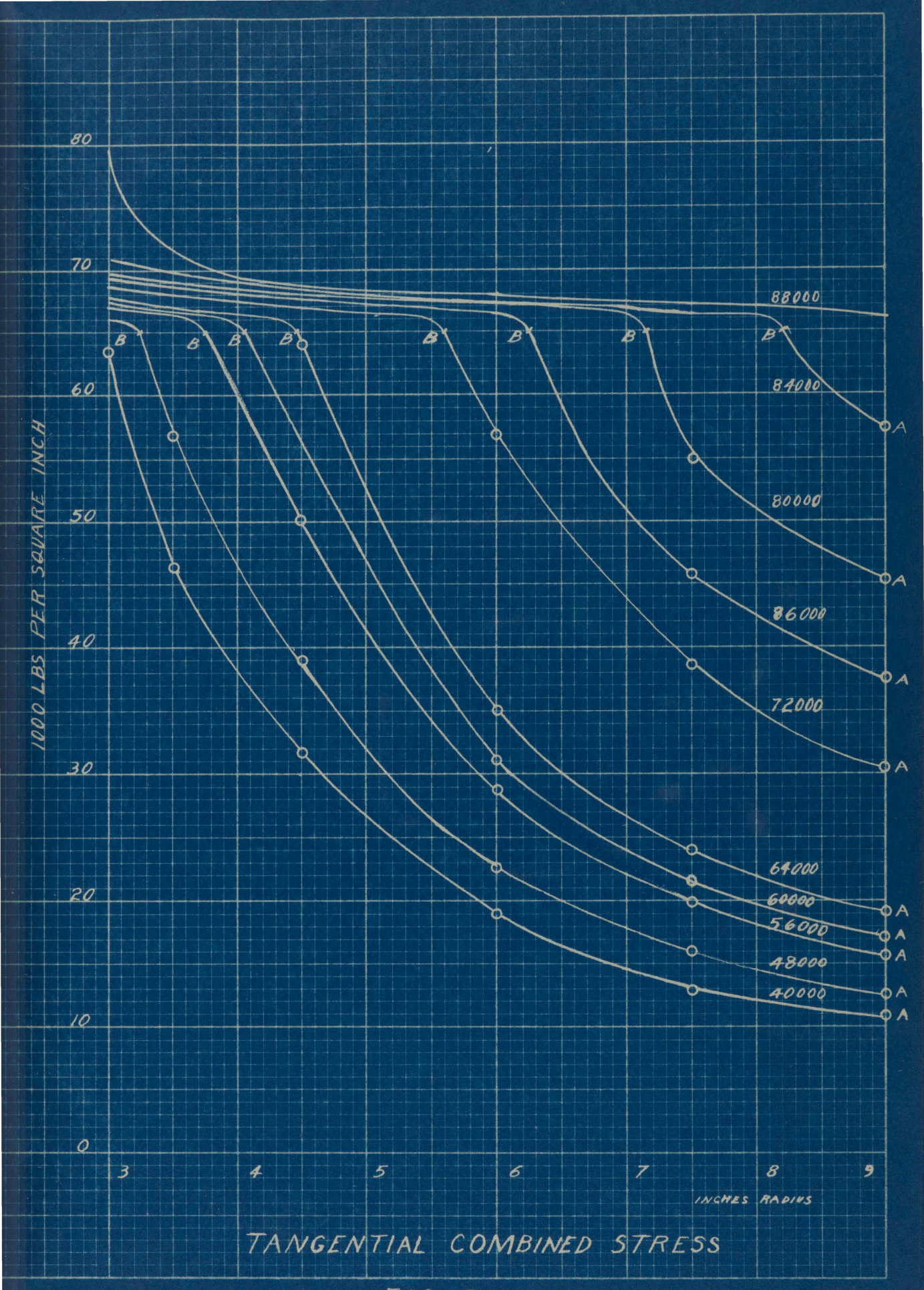


FIG. 6



TANGENTIAL COMBINED STRESS

FIG. 7

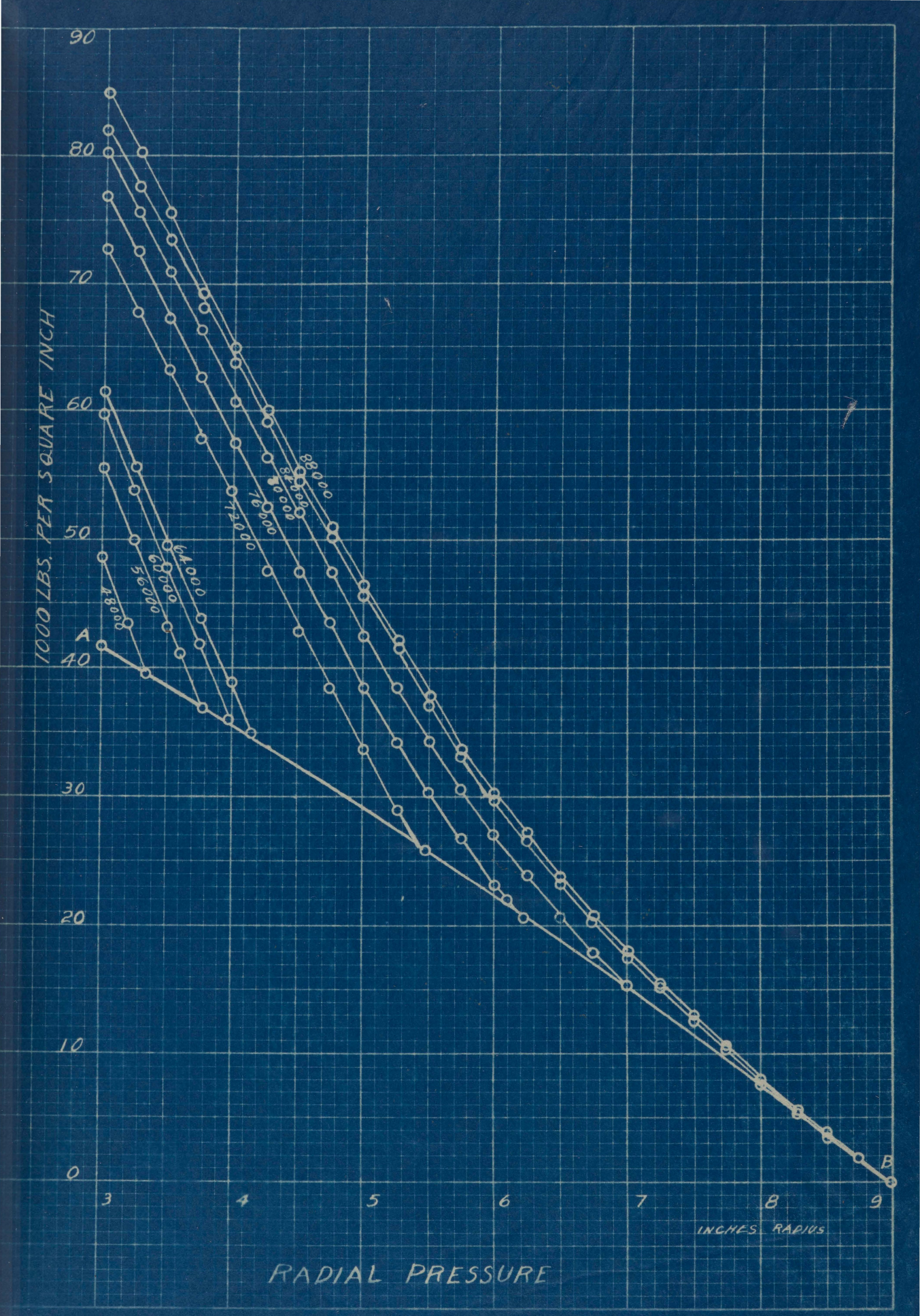

90
80
70
60
50
40
30
20
10
0
1000 LBS. PER SQUARE INCH
A
B
88000
80000
72000
64000
56000
48000
3
4
5
6
7
8
9
INCHES RADIUS
RADIAL PRESSURE

FIG. 8

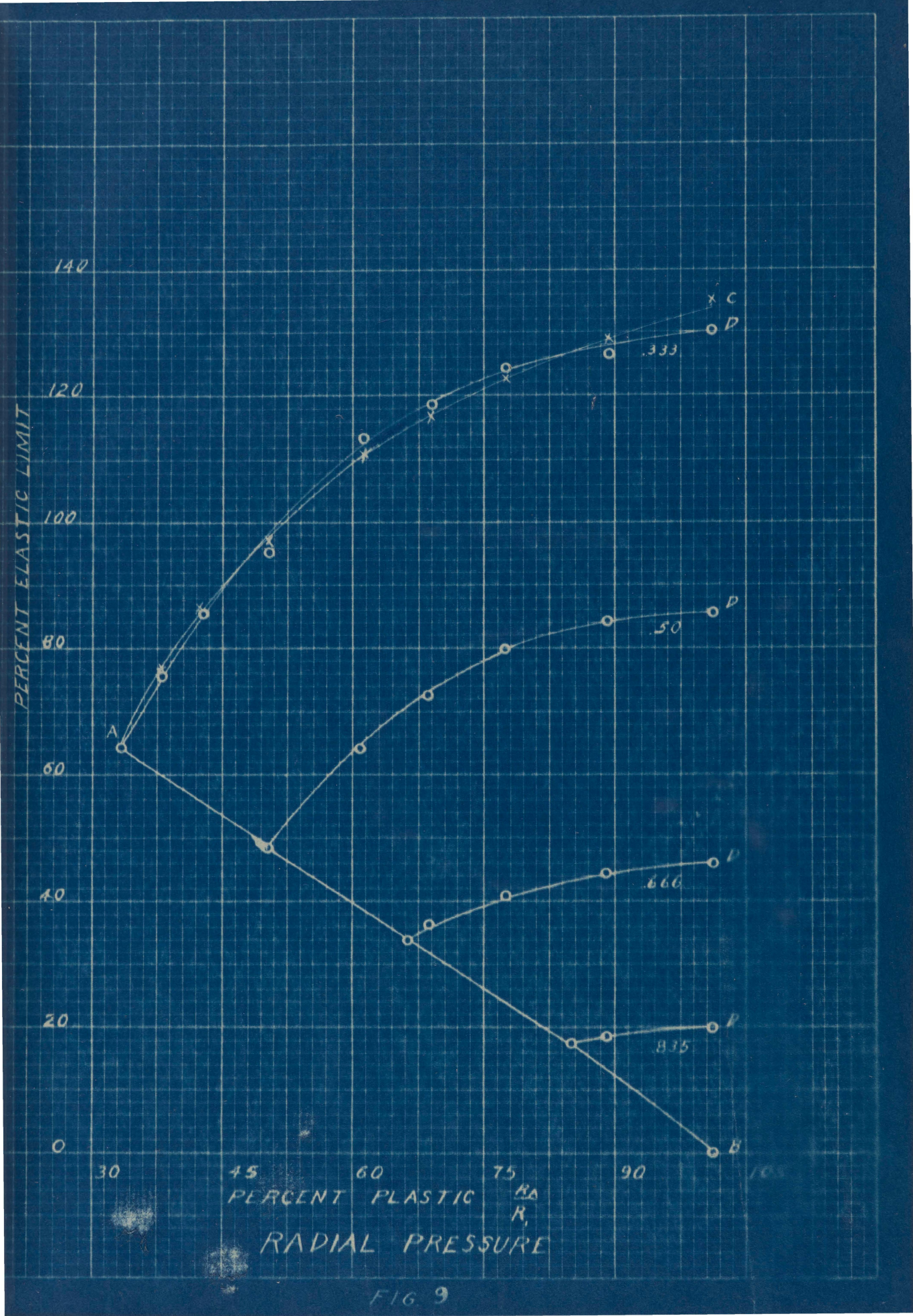

PERCENT ELASTIC LIMIT
140
120
100
80
60
40
20
0
30
45
60
75
90
PERCENT PLASTIC $\frac{R_A}{R_i}$
RADIAL PRESSURE
A
C
D
.333
D
.50
D
.666
P
.835
b

FIG 9

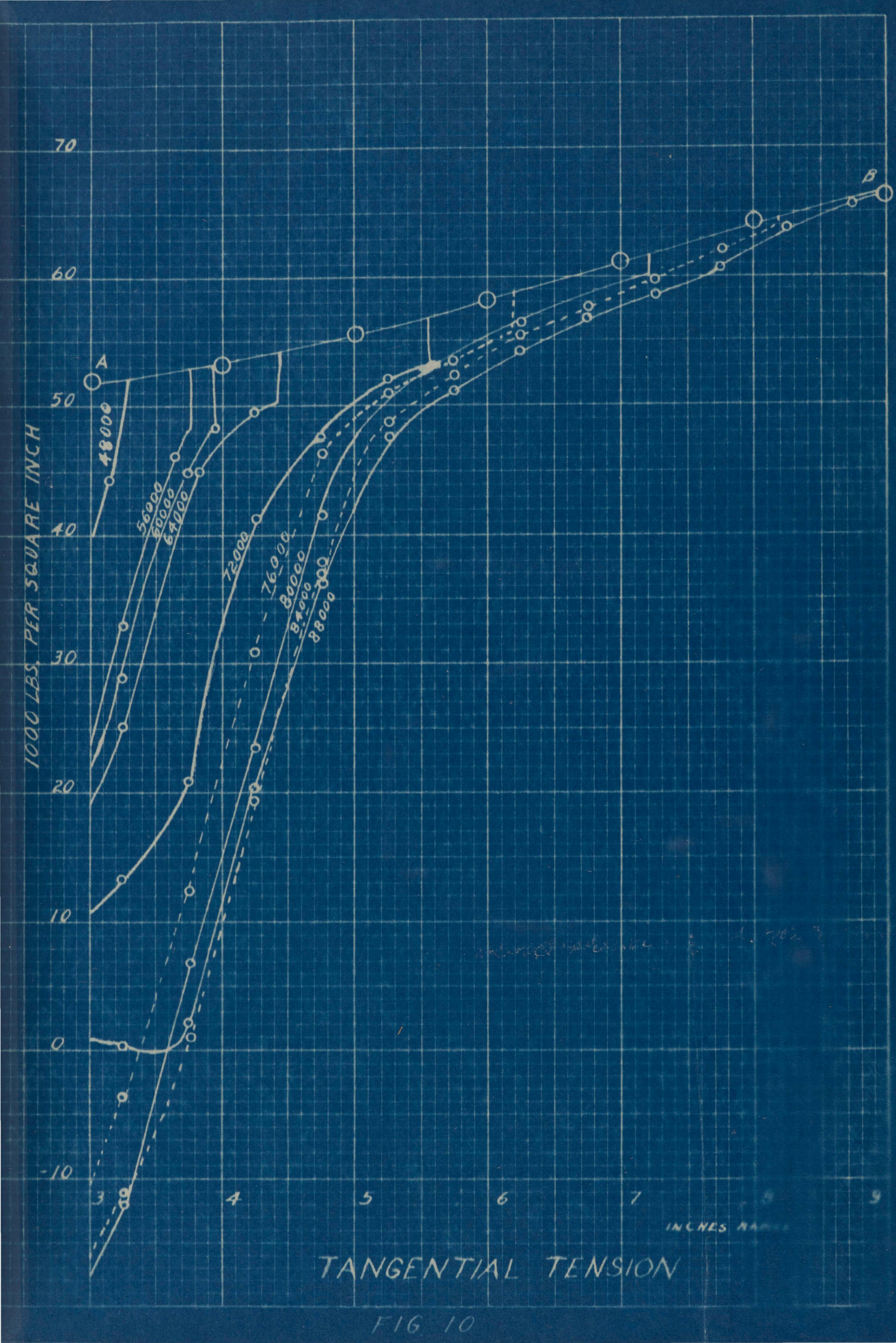

70
60
50
40
30
20
10
0
-10
1000 LBS PER SQUARE INCH
A
B
48000
56000
60000
64000
72000
76000
80000
84000
88000
3
4
5
6
7
INCHES
TANGENTIAL TENSION

FIG 10

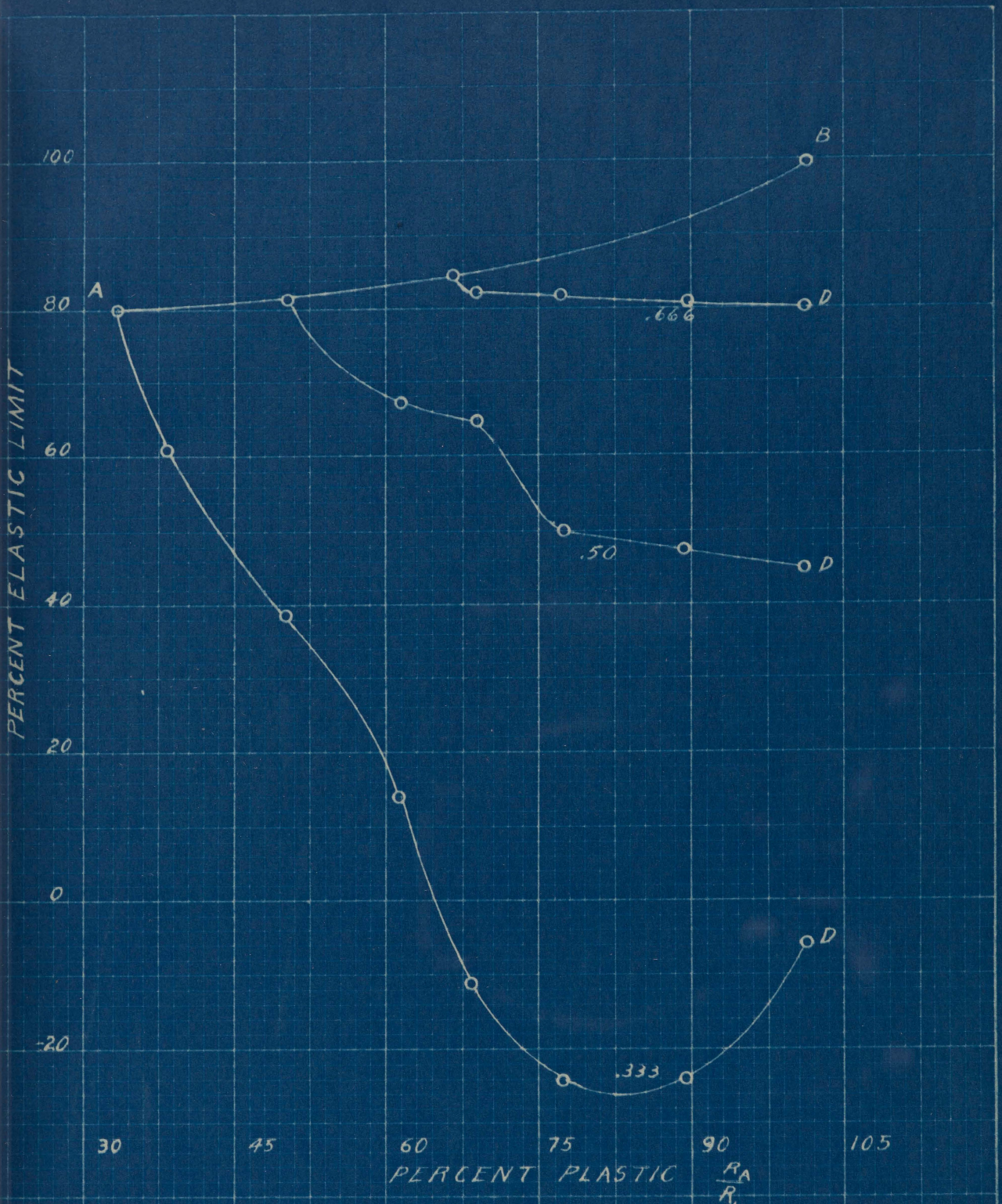
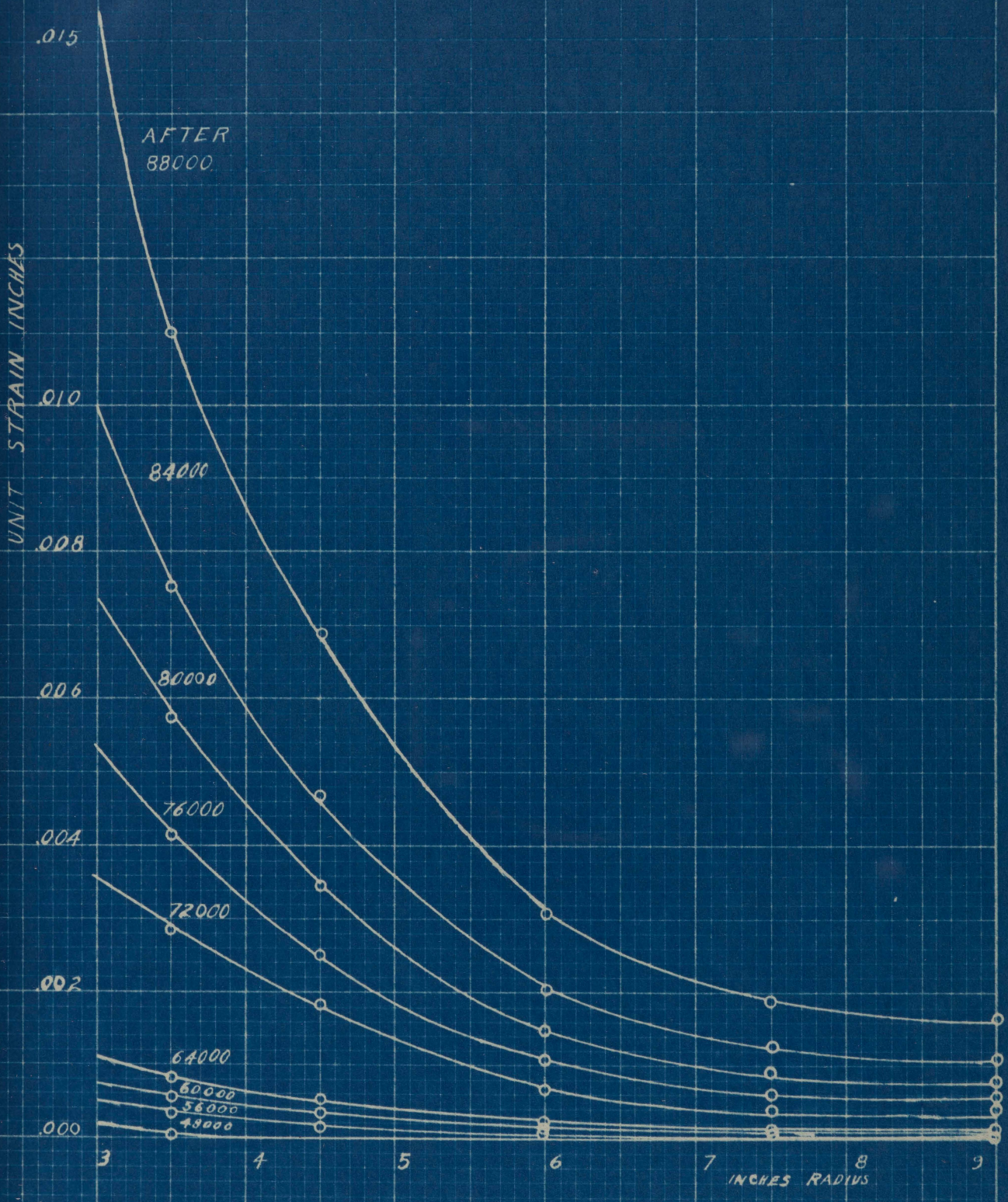
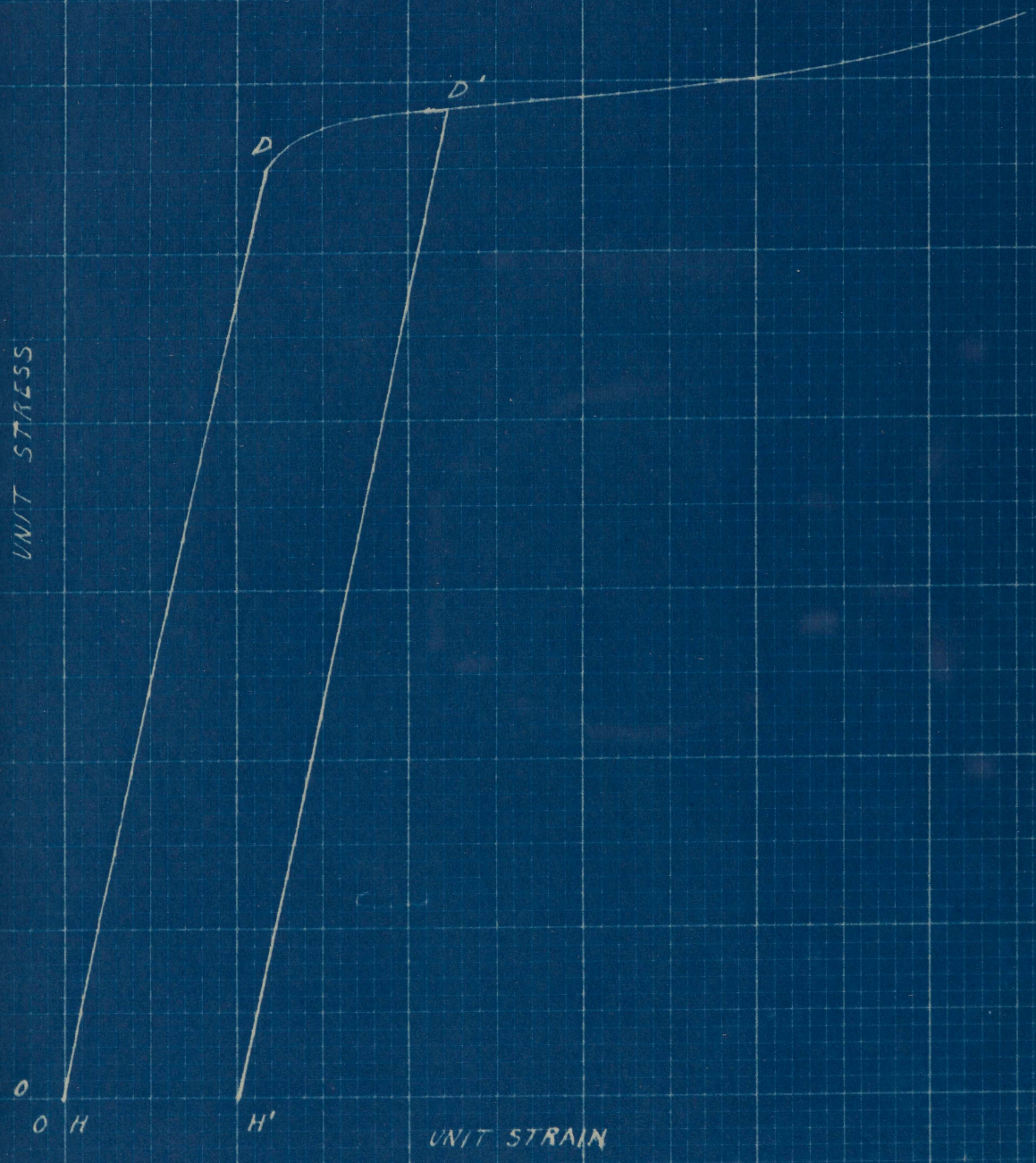

100
80
60
40
20
0
-20
A
B
D
D
D
.666
.50
.333
30
45
60
75
90
105
PERCENT ELASTIC LIMIT
PERCENT PLASTIC $\frac{R_A}{R_1}$
TANGENTIAL TENSION

FIG. 11



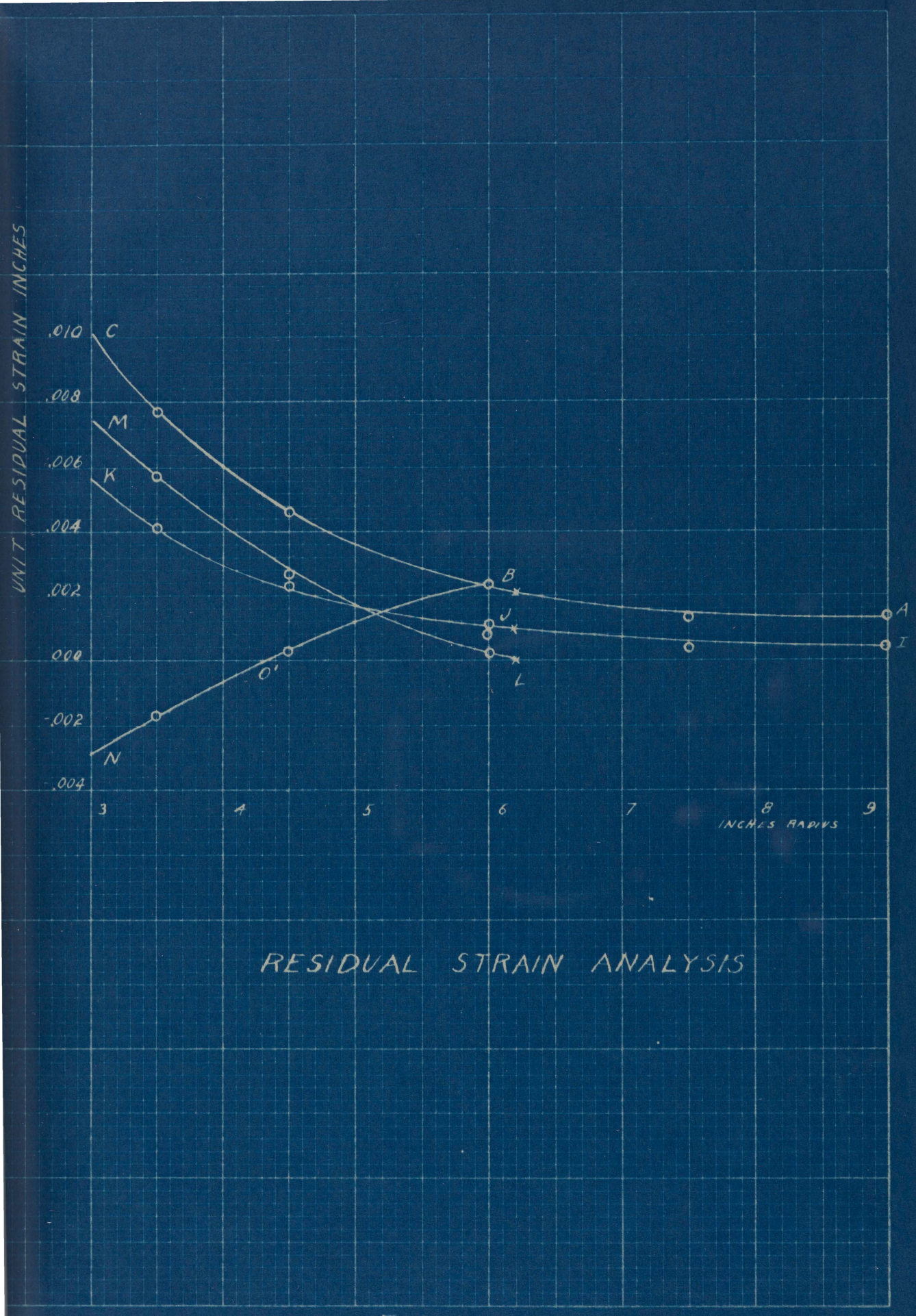
UNIT RESIDUAL STRAIN

FIG. 12



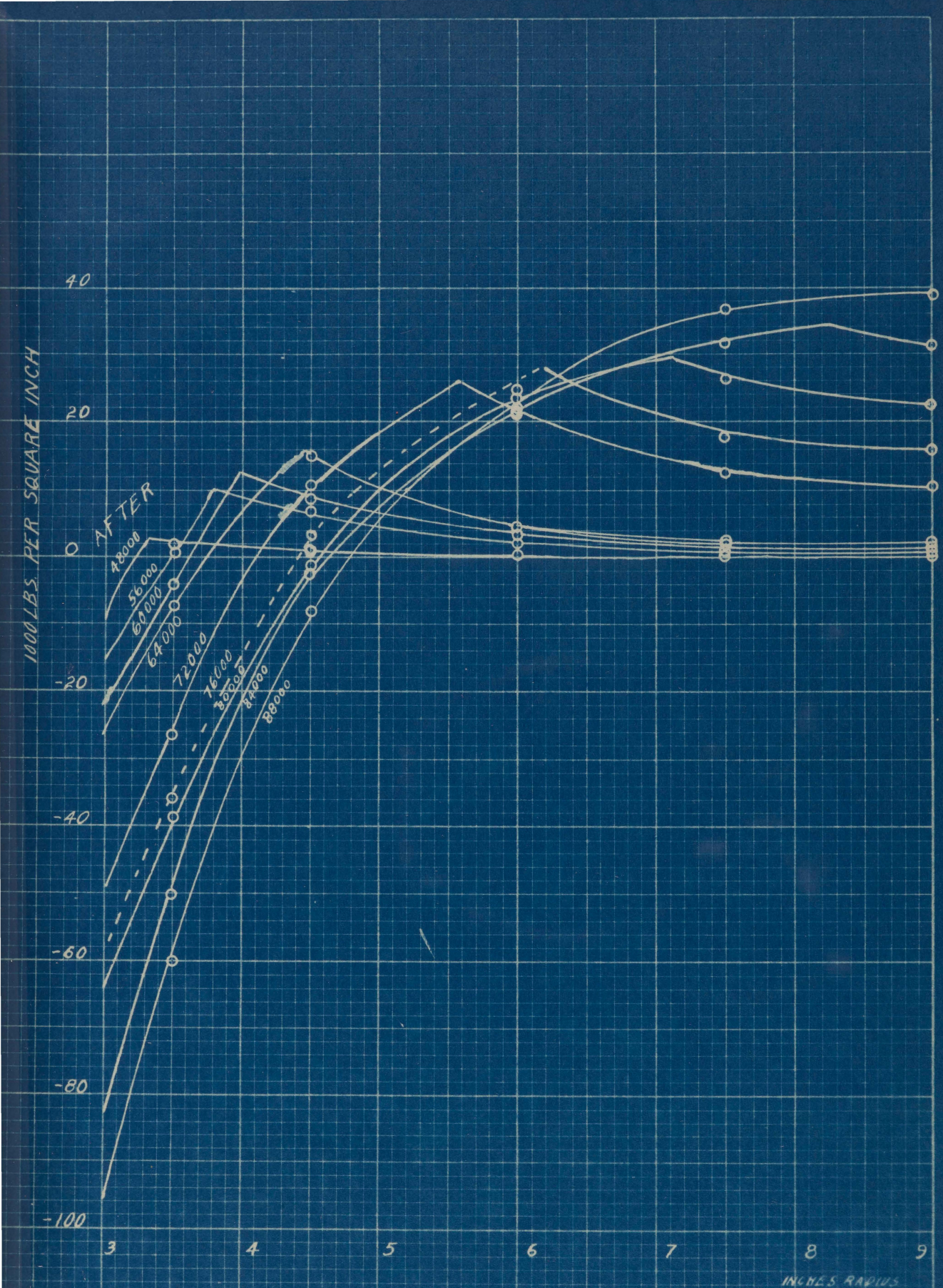
TRUE-STRESS STRAIN DIAGRAM

FIG. 13



RESIDUAL STRAIN ANALYSIS

FIG. 14



TANGENTIAL RESIDUAL COMBINED STRESS

FIG. 15

Discussion

The comparison of the combined stress theories is represented in Fig. 1. The thirteen points each represent values of tension of the inner fiber (by the well-known equation 2) according to an applied pressure which produces a very small permanent enlargement (.0001 in.) of outside diameter according to test at Watertown Arsenal. Only absolutely reliable data has been shown, although all the tests rejected were a good fit to the strain theory, some of the details of methods used or the properties of the materials were not positively known to be correct.

All available literature was searched and all theories of which sufficient information could be obtained are represented. (Mohr's and Hancock's are the same as Coulomb's for a symmetrical material like steel.) The much used strain theory is clearly seen to be the best representation of the results.

The strain theory being the best for elastic properties, naturally would be expected to be the best to convert into a plastic law. No complicated equations have been developed to create a theory of plastic flow, but just a logical conversion of the elastic law to conform to the known plastic properties. The strain theory of plastic material is supported by the check from the test of cylinder No. 6. The results are shown as calculated to be the methods previously discussed. The curve of Fig. 4 is not absolutely known, but by careful search it is found that a like grade of steel of similar treatment is a curve of the nature shown. The curve was drawn and then the values calculated. Under these conditions it would be a

very rare chance of values checking if the stress-strain curve was not approximately correct.

Graphic values show very close agreement to formula results; also there is a good check between data of a point and values calculated for this point from data of another point. A comparison of the values of P_o can be seen on Fig. 8 at the three-inch radius or perhaps better on the .333 ratio line AD and applied pressures line AC in Fig. 9. The variation of AC and AD from the 80% to 100% plastic radius might be credited to the stress-strain curve deviating from the true curve at these higher strains where the curve is not very definitely known.

Fig. 9 is a percent stress and percent plastic radius curve which can be used for all cylinders of material of proportional stress strain diagrams. If, for instance, a 2 to 1 ratio cylinder is used the 0.5 AD line will be used as inside radius; cylinder No. II checks very nicely by this diagram. Fig. 10 and Fig. 11 are tangential tension curves for pressure as shown in Figs. 8 and 9. Fig. 7 is the combined stress curves of tangential stress, a sum of radial and tangential effect for a 6" by 18" cylinder, but may be used for 3 to 1 cylinders.

The fact that the tangential tension is very low at the bore is the result of the high radial pressure producing so much strain that there is little left for the tension according to the measured distortions. These forces are combined according to the converted strain theory and cases are shown where the tangential force has become compression to support the radial resistance against

the applied pressures that are above the new elastic limit of the material for the given amount of cold working.

The general shape of these curves approximately agrees with work of others along this line, who have assumed the plastic strength as equal the maximum elastic stress which makes a smooth set of curves. The stresses as in Fig. 10 and Fig. 11 are necessarily irregular as the radial pressure must decrease outward, the maximum strains increase inward, and both are rather smooth curves which leave the tension to absorb all the irregularities of the stress-strain diagram and the effect of crossing the 0.666 percent of corresponding elastic limit, which changes the equation of combination.

The limiting pressure that a cylinder of an infinite outside radius could withstand (end condition being sufficiently strong) is not known, but twice the true breaking load is believed to be a maximum value for steel according to the indication from the stress distribution in the walls. Some tests indicate that there is no limit under these conditions, but the effect is not well understood. More tests of materials under high hydraulic pressure may bring some light on this subject.

Some questions have arisen as to where failure begins in thick-walled cylinders, and both inside and outside failures have been reported from test. From this study of the stress distribution, it appears that both might be correct, but each is limited depending upon the properties of the material and the diameter ratio. A brittle material should show the first failure at the inside radius because the material lacks the ductility to stand the strain necessary

to put the outside under a high stress. A very ductile material will flow almost indefinitely under compression. Therefore, the inside fiber will decrease in tension (might get into compression) when under high pressures, but the fiber will stay in contact even though it is severely strained. The outside is under tension; therefore, it may show the first break, but this is because the inside has ceased to carry any tension. To get outside failures it seems necessary to have high pressures which require large radii ratios.

With the general conclusion that a thick-walled cylinder will split laterally, this study of stresses is in agreement.

The plastic flow of steel from about 10 percent reduction of area up to the point of necking is practically a straight line (true stress - true strain diagram) which is very easy to use in the equations. The plastic strength just above the elastic limit is a very irregular value, but at between 1 and 2 percent reduction of area it settled to a smooth curve up to the straight line relation. It is these irregular values that give the trouble because there is no way to get partial cold-working without being in this range. It is clear that a straight line seldom fits these conditions and in general the equation of the curve is very difficult to use.

Some interesting indications supporting Norris' relation of plastic strength to the apparent ultimate strength have been found from work with data from Bridgeman's test. Although the steel is different from those used by Norris and the values are too close to the elastic limit to be very definite, but a check within a $\pm$ 7% maximum was obtained by use of equations (10) and (11). These

checks were on both a 3 to 1 and a 2 to 1 cylinder of the same material. More data would make a very interesting and valuable study in cold worked steel.

The break in laws from the elastic to the plastic line is probably related to the internal friction change of the material and the change of deformation in the crystals to slippage. Apparently there is a range where both conditions exist; however, such a relation has not yet been presented in the form of a definite equation. More x-ray work and comparison to true stress-true strain relations have possibilities of giving an analysis of plastic flows at this point. The value of the laws of plastic flow to industry is in the application of these principles to the masses of materials formed by various amounts of cold working and subjected to all kinds of loading.

The residual stress as shown in Fig. 15 gives a very good representation of how the internal stresses are believed to be distributed in the material. The limited accuracy of instruments and the previously listed difficulties in the material have made it impossible to get a good check on these results; however, there are good indications that this method is in the right direction.

Cold worked cylinders have been found to act in approximate accordance to elastic laws as long as the original cold working pressure is not exceeded. This is very good evidence that there is a (σ_0) zero stress line; also that a compression and a tension force exist in the wall. Further investigation supporting this is given by a study of strain rings cut from the cylinder after a test. The

rings from near the bore expanded and those from the outer portion contracted.

The hypothesis as developed here needs more data to test its reliability, but very good agreement has been found from the available data.

Conclusion

Elastic material:

The relation of combined stress in elastic material is in accordance with the strain theory as proposed by St. Venant.

Plastic material:

1. The plastic (or converted) strain theory with $m = 2$ is the correct combined stress relation for plastic materials.

2. The true stress-strain diagram or true stress-true elongation (reduction of area) diagram should be used.

Residual stresses:

Residual stresses are caused by the differences of the permanent stretch given a fiber and the residual strain remaining in this fiber after removing the load.

Failures of cylinders:

The properties of the material and dimensions of the cylinder determine the radius where failure begins.

List of Symbols Used

A = original area - sq. in.

A_o = resultant area - sq. in.

p = radial stress - pounds per sq. in.

t = tangential stress - pounds per sq. in.

S_p = equivalent radial stress - pounds per sq. in.

S_t = equivalent tangential stress - pounds per sq. in.

U_p = unit radial distortion.

U_t = unit tangential distortion.

$\frac{1}{m}$ = Poisson's ratio.

R_o = inside radius - inches.

R_1 = outside radius - inches.

R_a = plastic-elastic division - inches.

r = any radius - inches

P_o = internal pressure - pounds per sq. in.

P_1 = external pressure - pounds per sq. in.

P_a = pressure at R_a - pounds per sq. in.

E = Young's modulus.

C_x, C_y, C_z = unit elongation.

P_x, P_y, P_z = pressure - pounds per sq. in.

P_r = residual pressure in plastic material - pounds per sq. in.

r_o, r_1, r_a = radii of annulus - inches

δ_L = unit strain under load.

δ_R = unit residual strain.

δ_o = unit residual strain of zero stress.

δ_S = unit effective residual strain.

S, S_x, S_y = stress - pounds per sq. in.

t_p = tangential stress in plastic material - pounds per sq. in.

$S_{y.p.}$ = yield point stress - pounds per sq. in.

$S_{e.l.}$ = elastic limit stress - pounds per sq. in.

S_v = shearing yield point - pounds per sq. in.

S_n, S_m = tension, compression yielding stress - pounds per sq. in.

S_w = compression and tension (equal) which together cause shearing = S_v

S_s = shearing stress - pounds per sq. in.

p_n = p at radius n - pounds per sq. in.

R_n = radius at n - inches.

S_{to} = stress in outer fiber - pounds per sq. in.

S_b = stress in thousands of pounds per sq. in.

T = ultimate tensile stress - thousands of pounds per sq. in.

x = number of percent reduction of diameter.

C = circumference - inches.

Bibliography

Norris, E. B. - Army Ordnance Department Experimental Report No. 288.

Norris, E. B. - Combined Stresses in Thick-Walled Cylinders. A. S. M. E. Transactions 1929.

Norris, E. B. - Plastic Flow and the Strength of Cold-Worked Steel. Engineering News Record, October 6, 1927.

Timoshenko and Lessell - Applied Elasticity, Chap. XVII, p. 497, p. 249, p. 11 to 17, p. 54, p. 406.

Timoshenko - Strength of Materials II, p. 705, p. 658, p. 654.

Tschappat - Ordnance and Gunnery, Chap. IV, p. 221, 181-182, 183.

Love - Mathematical Theory of Elasticity, p.99, p. 114, 116, 121, 144, 620 (1927).

Westergard, H. M. - Condition of Yield. Journal of Franklin Institute, 1920.

Becker, A. J. - University of Illinois Bulletin. 85.

Turner, L. B. - Eng., London, July 28, 1911.

Haigh, B. P. - Maximum Resilience Theory. Eng., London, Jan. 30, 1920.

Cook and Robertson - The Strength of Thick-Hollow Cylinders. Eng., London, Dec. 15, 1911.

Hancock - Result of Test of Materials Subjected to Combined Stresses. A. S. T. M. 1905-6-7-8.

Nadai, A. - On the Mechanics of Plastic State of Metals. A. S. M. E. Trans., 1929.

DeForest - Some Experiments on the Plastic Elongation of Wire. A. S. T. M. Proc. 1926.

Norton and Warren - Plastic Deformation of Metals. A. I. M. M. E. 1926.

Kimble and Lovell - Internal Friction in Solids. A. S. M. E. 1926.

Wegel - Viscosity in Solids. Bell Lab. Record. Nov. 1929.

Lester and Aborn - The Behavior Under Stress of the Iron Crystals in Steel. Journal Army Ordnance Association, Vol. VI, No. 32, Sept.-Oct.; No. 33, Nov.-Dec. 1925; No. 34, Jan.-Feb.; No. 35, Mar.-Apr. 1926.

Clark - X-ray Examination of Materials in Industries. A. S. T. M. 1927.

Seely and Putnam - University of Illinois Bulletin No. 115.

Editorial - Eng., London, Nov. 29, 1912.