EFFECT OF CORRELATION BETWEEN SHADOWING AND SHADOWED POINTS IN ROUGH SURFACE SCATTERING

by

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Thesis submitted to the Faculty of the
Virginia Polytechnic Institute and State University
in partial fulfillment of the requirements for the degree of

MASTER OF SCIENCE
in
Electrical Engineering

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May, 1993
Blacksburg, Virginia

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(ABSTRACT)

Shadowing of random surfaces is accounted for by multiplying the Kirchhoff electric current density by a shadowing function. The shadow-corrected incoherent intensity is computed in the backscatter direction and is found to be proportional to the probability of a specular point being illuminated from the source. This probability is computed numerically using an infinite series of integrals, developed by Ricciardi and Sato, and by Monte Carlo computer simulations. The results obtained are compared to the analytic approximations of Wagner and Smith, which neglect correlation between the shadowing points and the shadowed point. Assumptions made by Wagner are explained using the infinite series of integrals. Furthermore, comparison is made to Wagner's results which include correlation between the shadowed point and the shadowing point.

ACKNOWLEDGEMENTS

I would like to thank Dr. Gary Brown for his support and guidance throughout this thesis. Thanks also for your patience, thoroughness and high standards. Thanks for providing me with resources (ECF) and funding, without which would make this thesis and my education impossible.

I would like to thank Dr. Richard Claus for giving me the opportunity to work at the Fiber and Electro-Optics Research Center during my first year of graduate school. This experience was one of the best I've had. Thanks for serving on my committee and the many others things you have done for me.

I owe much gratitude to the first professor I had in electromagnetics, Dr. I. Besieris. Your incredible gift as a teacher has inspired many students throughout the years, especially me. Thanks for serving on my committee and the many discussions we have had.

I would like to thank Keith Tyeryar for his friendship, honesty, determination and help throughout the past three years. I don't know how many times I have said to myself how glad I am to be working with you.

I would like to thank Mike Newkirk for his help over the years. It is not an overestimation to say that this thesis would have taken at least six months longer without your help (if you can imagine such a thing). It is truly appreciated.

I would like to thank the many people who worked at FEORC in 1988-89, especially Kim Bennett, Kent Murphy, and Bernd Zimmermann (U.S. PATENT #5,189,299 !).

I would like to thank Dr. W. A. Davis for making EM understandable to me as an undergraduate. If it wasn't for you I doubt I would be in EM today.

I would like to thank Argy Chatzipetros for his friendship and for reading much of this thesis.

I would finally like to thank Mrs. Marion Bradley Via, who passed away on January, 3, 1993. It was through her generosity and kindness that made my education possible.

This thesis was supported, in part, by the Army Research Office.

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1.0 INTRODUCTION AND SUMMARY

1.1 Overview

The calculation of electromagnetic and acoustic waves scattered from rough surfaces has been a challenging problem for researchers over the last four decades [Ishimaru and Chen, 1991; DeSanto and Brown, 1986; Eckart, 1953; Rice, 1951; Blake, 1950]. During this time, much progress in understanding the mechanisms responsible for the behavior of these scattered fields has been made. However, despite the large effort, the exact solution to this problem has not been obtained. Approximate solutions have been provided in the literature [Barrick, 1968b; Beckmann, 1965b] that result from assumptions which are made for mathematical convenience, and do not always represent the wide variety of surface and operating conditions found in practice [Chan, 1990; Helmken, 1990; Moore, 1985].

The classical methods for solving the rough surface problem include the Kirchhoff or tangent plane approximation [Beckmann, 1965b] and the small perturbation method [Rice, 1951]. These approaches have served as useful check cases from which to compare other results. However, their range of validity, in the past, has been unclear due to the lack of superior results to compare them to. Recently, computer simulations of scattering from one dimensional surfaces has allowed comparison of these theories with known results, for a wide range of surface conditions [Chen and Bai, 1990; Thorsos, 1988; Tran and Celli, 1988; Nieto-Vesperinas, 1987; Axline and Fung, 1978]. From these studies, it is clear that there is a need to develop more accurate methods of prediction.

Although computer simulations of scattering from random surfaces in one dimension have been performed, scattering from surfaces rough in two dimensions is presently beyond the storage and speed capabilities of today's computers. Furthermore, there is not only a need to find numerical results, but also a need to understand the physics behind the scattering process. For these reasons, a theoretical approach may be fruitful and provides insights into the scattering problem.

Unfortunately, the most promising theoretical methods have remained largely formal [Brown, 1984a; Watson and Keller, 1984], mainly due to the fact that the mathematics appear intractable in closed form. This poses its own problems from a computational point of view, even for simple one dimensionally rough surfaces. However, it is easier to extract the physics associated with the problem using a theoretical approach, as opposed to computer simulations, and thus the results are more amenable to interpretation.

One approach used to simplify the solution process is to consider the incident field as a high frequency plane wave [Fuks, 1969; Hagfors, 1966; Lynch and Wagner, 1970b]. The high frequency nature allows a dramatic reduction in computational complexity at the expense of ignoring diffraction effects, which can become extremely important [Teokharov, 1987; McDaniel, 1986]. Despite this limitation, a high frequency approximation can offer a possible case against which more rigorous theories can be checked [Brown, 1990]. The technique may also be used to solve some practical problems in optical sensing [Lements and Fuks, 1978; Garnaker'yan and Sosunov, 1976].

A first step toward improving upon the classical methods would be to use a high frequency approximation in conjunction with a shadow-corrected Kirchhoff current. This has the potential of increasing the range over which the incident angle can increase (toward grazing), and thus improve upon the range of validity of the solution. However, doing so requires computing shadowing functions which, in turn, correct for the fact that real rays cannot penetrate the surface, as in the case of a perfect electric conductor (PEC), or pass through the surface without being attenuated, as in the case of a lossy dielectric [Jin, 1988; Wagner, 1967; Smith, 1967b; Beckmann, 1965b; Bass and Fuks, 1964].

Shadowing theory was introduced in the United States by Beckmann in 1965 [Beckmann, 1965b] and in the former Soviet Union by Bass and Fuks in 1964 [Bass and Fuks, 1964]. Although Beckmann's paper was proven to be erroneous

[Brockelman and Hagfors, 1966; Shaw, 1966; McCoy, 1989], the fundamental development of the integral relationship, which expresses shadowing in terms of the random surface structure and the angle of incoming electromagnetic radiation, remains as the basis of shadowing theory found in the literature. A more rigorous version of the shadowing function first suggested by Beckmann was derived by Wagner in 1967 [Wagner, 1967] and by Smith in 1967 [Smith, 1967b]. Shadowing functions were not defined in relation to the surface current until Sancer's paper in 1969 [Sancer, 1969], which incorporated these functions directly into the diffraction integral for the scattered electric field.

One particular method used to solve for the scattered fields is the vector potential approach. This approach is a two step process which allows for the calculation of the scattered fields by first calculating vector potentials. These potentials are derived from equivalent surface currents which produce the proper fields inside and outside the scattering body. The difficulty in the approach lies in the calculation of the surface currents. The method offers the advantage over the direct approach in that only knowledge of the current and not derivatives of the current are required [Brown, 1988].

In the case of a PEC surface, the electric surface current, $\vec{J}_s$, can be found from the boundary condition on the magnetic field. $\vec{J}_s$ allows for the discontinuous nature of the tangential component of the magnetic field across the boundary. In general, dielectric surfaces must contain not only the equivalent electric current, $\vec{J}_s$, but also the equivalent magnetic current, $\vec{M}_s$. In the case of the PEC, however, $\vec{M}_s=0$, since the tangential component of the electric field is continuous across its boundary, and the result simplifies considerably.

This thesis assumes a PEC surface illuminated by a plane wave, and we therefore replace it by the equivalent electric surface current which satisfies an integral equation of the second kind, derived from the boundary condition for the magnetic field, known as the magnetic field integral equation (MFIE). This equation provides the distinct advantage over integral equations of the first kind in that solutions may be obtained by iteration. Once a solution has been obtained, the magnetic vector potential may be calculated and then the scattered electric and magnetic fields. The zeroth order solution to the MFIE is the Kirchhoff approximation and higher order iterates account for multiple scattering and diffraction. In the high frequency limit, iterates produce mathematical rays which can penetrate the surface (even in the case of a PEC), thus shadowing functions must be included "ad hoc" when computing the scattered field. The role of the shadowing function is to convert mathematical rays which can penetrate the surface to real rays which bounce about on the surface [Brown, 1984].

When *single* scattered rays are considered, there are two kinds of shadowing functions discussed in the literature: (1) incident shadowing function, and (2) scatter shadowing function. The role of the incident shadowing function is to null those elements of current on the surface which are not visible from the source and do not contribute to the *single* scattering process. Likewise, the scatter shadowing function nulls those current elements which produce rays that are blocked by other points on the surface prior to reaching the observer, and are assumed not to contribute to *single* scattering. Multiple scattering further complicates matters since shadowing functions need to be developed for rays which bounce from one point on the surface to another [Jin, 1990; Jin, 1988, Lynch and Wagner, 1970b].

The incident shadowing function is defined as the probability that a point with a specular slope will be illuminated by the source. This is due to the fact that the main contribution to the scattering process in the high frequency limit comes from specular slopes, or slopes which give rise to reflection in the direction of the observer. Thus, the shadowing function is the ratio of illuminated reflection points to the total number of reflection points. The scatter shadowing function is defined in a similar manner relative to the observer. We note that the probability of a reflection point being shadowed is equivalent to having the surface cross the ray path between the reflection point and the source (Figure 1). Thus, shadowing is inherently related to surface upcrossings, where an upcrossing is defined to be the event that the surface crosses the incoming ray from below. The probability density of the distance from the last point in which the incoming ray intersects the surface, as one moves from the source to the surface, and the next to last crossing of the ray by the surface is called the first passage in time function [Blake and Linsey, 1973]. The integral of this density is the probability that a reflection point will be shadowed. The shadowing function is defined to be one minus this integral, and thus is the probability of a reflection point being illuminated.

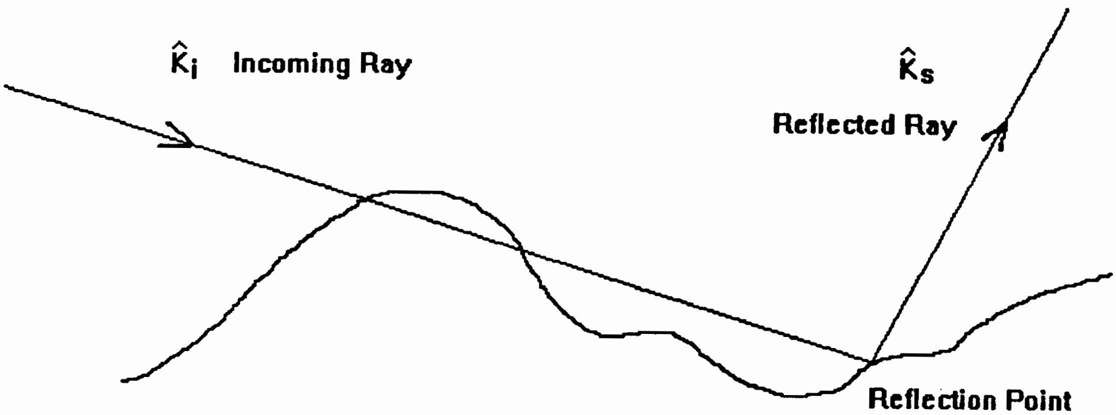

Figure 1: Shadowing of a Reflection Point

There is an extensive body of literature concerning the first passage in time problem, as it has found its way into many areas of physics and engineering [Abrahams, 1986; Blake and Linsey, 1973; Slepian, 1962; Siegert, 1951]. A brief review of some of the important articles is presented in Chapter 3. Closed form solutions to this problem have only been found in a few special cases [Abrahams, 1984; Slepian, 1961]. Recently, however, a infinite series of integrals has been derived by Ricciardi and Sato [Ricciardi and Sato, 1983], based on Rice's work in random noise [Rice, 1945], which allows us to express the unknown function in terms of integrals with known kernels. The focus of this thesis is to use this series to compute the shadowing function "exactly" using numerical integration and to compare our results to those found in the literature and to Monte Carlo computer simulations.

1.2 Description of Thesis

Chapter 2 provides the background for calculating the scattered fields from a random perfect electric conducting (PEC) surface illuminated by a plane wave. The PEC is replaced by an equivalent surface current which results from the discontinuity of the tangential component of the magnetic field across the surface. The MFIE is used to solve for the surface current, where the total current is approximated by its zeroth order solution, known as the Kirchhoff approximation. A shadow-corrected Kirchhoff surface current is developed and substituted into the far-field integral for the scattered electric field. The second moment for the scattered field, the scattered intensity, is formed in terms of the shadowing function.

Chapter 3 provides a literature review of scattering from random rough surfaces, shadowing theory, and the first passage in time function as it applies to shadowing theory. The primary focus of this chapter will be a review of shadowing theory, particularly Wagner's [Wagner, 1967] conditional shadowing function and his assumptions and approximations.

Chapter 4 develops the theory used to calculate the shadowing functions found in Chapter 2. The unknown shadowing functions are expressed in terms of an infinite Rice-like series of integrals with known integrands, developed by Ricciardi and Sato [Ricciardi and Sato, 1986].

Chapter 5 computes the shadowing function in the case when correlation between the shadowing points and the shadowed point is ignored. This result turns out to correspond to the one developed by Wagner [Wagner, 1967], and thus provides us with the physical interpretation of his assumptions.

Chapter 6 presents the derivations leading up to the numerical integration of the first three terms in the Rice-like series. Here, correlation between all shadowing points and shadowed point is included. Monte Carlo simulations are performed, similar to those of Brockelman and Hagfors [Brockelman and Hagfors, 1966], so as to provide a check with our numerical results. The results are also compared to Wagner's [Wagner, 1967] and Smith's work [Smith, 1967b].

Chapter 7 extends Chapter 2 to include doubly scattered rays. The final result is expressed in terms of shadowing functions, but no attempt is made to evaluate these functions.

Chapter 8 presents a summary of our work and conclusions.

An extensive list of references is presented following the conclusion of this thesis. Since the body of literature in rough surface scattering is so vast, it was our intention to provide a list of major papers in this area, particularly as related to shadowing theory. It was also our hope to tie together two vast bodies of literature, that is, the literature pertaining to shadowing theory as it applies to electromagnetic and acoustic scattering, and the literature concerning the first passage in time problem. The reference list makes the connection between the two areas in the hope that a much more well directed effort can take place in the event that the issues in this thesis are pursued further.

1.3 Summary

Several significant results pertaining to shadowing are presented in this thesis. First, we have properly defined scatter shadowing and developed the bistatic shadowing function. Previous definitions of the scatter shadowing function in the literature have led to the unphysical situation of surface currents which depend on the observation angle. We have defined the function such that the currents only depend on the incoming electromagnetic wave and the surface structure and not on the direction of observation. Secondly, the analytical results of Wagner [Wagner, 1967] and Smith [Smith, 1967b] have been checked using an infinite series of integrals for incident angles greater than previously possible with

Monte Carlo simulations. Thirdly, the approximations made by Wagner [Wagner, 1967] in developing the "uncorrelated" shadowing function have been explained based on the physics of the problem. Wagner [Wagner, 1967] developed closed form solutions for shadowing by making approximations. However, these approximations were not made based on physical reasoning, rather for mathematical convenience. The fourth main result is that an error has been identified in Wagner's work concerning the "correlated" shadowing function [Wagner, 1967]. Wagner's analytic results have been considered the authoritative work on shadowing theory; however, we have found that this result is actually poorer than that of Wagner's "uncorrelated" shadowing function, which is supposedly less accurate. Finally, we have solved the first passage in time problem for a Gaussian process, with a Gaussian correlation function, crossing a ramp (i.e. incoming ray of incident field). This problem has been solved by making the approximation that the intersection points of the ramp with the surface are uncorrelated (i.e. the shadowed point and the shadowing points are uncorrelated) and also solved numerically making no approximations.

2.0 MATHEMATICAL PRELIMINARIES

2.1 Introduction

In dealing with random surface scattering, the corresponding scattering field is a random variable. Characterizing the scattered field is achieved by computing the statistical moments. In general, the moments do not necessarily exist, and the distribution of the scattered field is unknown. In order to solve for the scattered field, we assume that the surface statistics are known. Thus, we are solving the direct problem rather than the inverse scattering problem. Inverse problems predict the statistics of the surface from the known scattered field, which is measured, while direct problems predict the scattered field from the known surface statistics. Direct problems provide more physical insight into the mechanisms which affects the scattered field, while inverse problems are more useful in remote sensing applications when the surface structure is to be determined.

In this chapter we motivate the development of the shadowing functions by calculating the second moment of the scattered electric field, the scattered intensity, in the high frequency limit (the first moment is zero in the high frequency limit). This result was first derived by Barrick in 1968 [Barrick, 1968b; Barrick and Bahar, 1981] using the stationary phase method and later by Sancer in 1969 [Sancer, 1969] using the modern approach with shadowing. We present Sancer's results here with some modification and explanation to show the need for further study of shadowing functions.

2.2 Approaches to Finding the Scattered Intensity

There are two approaches for computing the scattered intensity $\langle \vec{E}_s(\vec{r}) \cdot \vec{E}_s^*(\vec{r}) \rangle$. The first approach involves the stationary phase approximation. A high frequency limit is taken on the surface integration used to find the scattered field, $\vec{E}_s(\vec{r})$, which is the sum of the contributions from all the specular points. The intensity is found by multiplying $\vec{E}_s(\vec{r})$ by its complex conjugate. Then, the ensemble average of the random variables is taken. This procedure, used by Kodis in 1966 [Kodis, 1966] and completed by Barrick and Bahar in 1981 [Barrick and Bahar, 1981] is called the classical approach. An alternate method, proposed by Sancer in 1969 [Sancer, 1969], takes the ensemble average of the random variables prior to integration over the surface. After averaging, the surface integration is performed using a high frequency asymptotic technique. His method is called the modern approach. Both methods lead to the same high frequency result, but Kodis' result provides more physical insight into the scattering process, while Sancer's is more straightforward. We shall describe Sancer's approach here.

2.2.1 Sancer's Derivation of the Scattered Intensity in the High Frequency Limit

Let us consider the case of an infinite random perfect electric conducting (PEC) surface, in free space, illuminated by a time harmonic high frequency plane wave. These simplifications offer the possibility to gain some physical insight into the problem. Our intent is to provide a known check case against which others may compare their results and also to examine problems in the optical regime.

There are two approaches in solving for the scattered fields: (1) the direct field approach, and (2) the vector potential approach. The direct field approach requires knowledge of $\vec{J}(\vec{r}_s)$, $\nabla_s \times \vec{J}(\vec{r}_s)$, and $\nabla_s \rho_s$, while the vector potential approach requires knowledge only of $\vec{J}(\vec{r}_s)$. The sources, the volume charge distribution, ρ_s, and the electric current density, $\vec{J}(\vec{r}_s)$, can be viewed as equivalent charges and currents as in the case when scattering objects are replaced using equivalence. However, the derivatives of these quantities may not exist, as in the case when discontinuities in the current arise [Brown, 1988].

We take the vector potential approach to solve the scattering problem and replace the PEC surface (closed at infinity) by an equivalent electric surface current, induced by the discontinuity of the tangential component of the magnetic field across the boundary. This equivalent current produces the correct fields inside and outside the body. With the scattered body replaced by the equivalent currents, the medium is free space everywhere, including the region in which the PEC once occupied. The original and equivalent problems are given in Figures 2 and 3, respectively.

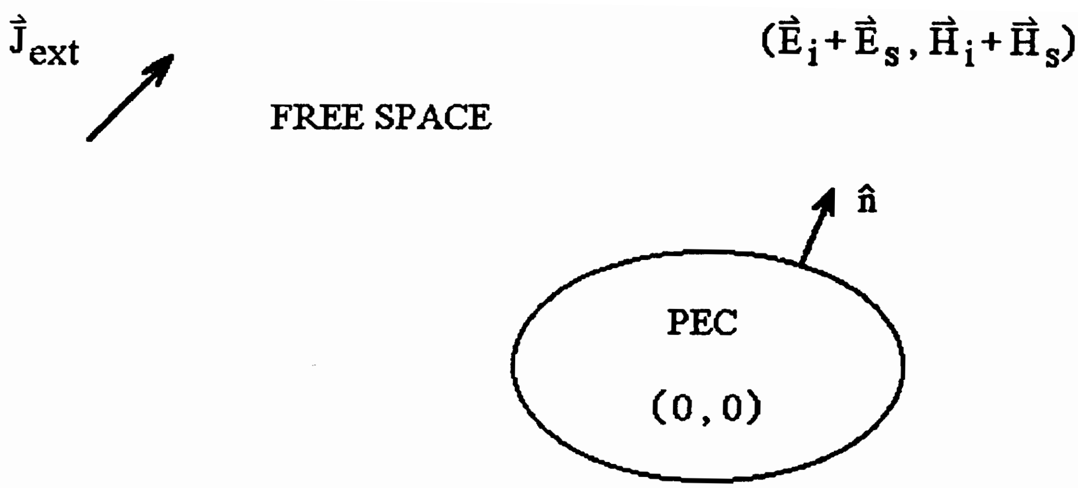

Figure 2: Original Problem

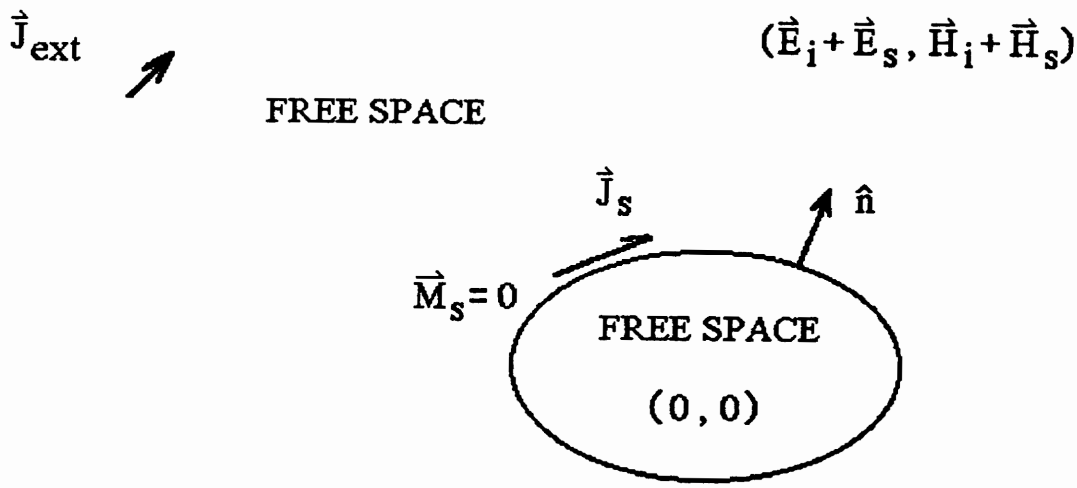

Figure 3: Equivalent Problem

The equivalent surface currents (denoted by the subscript "$_s$") are as follows:

$$\vec{J}_s(\vec{r}_s) \;\; = \hat{n} \times (\vec{H}_i(\vec{r}) + \vec{H}_s(\vec{r}))\Big|_{\vec{r} = \vec{r}_s} \tag{2.1a}$$

$$\vec{M}_s(\vec{r}_s) \;\; = \hat{n} \times (\vec{E}_i(\vec{r}) + \vec{E}_s(\vec{r}))\Big|_{\vec{r} = \vec{r}_s} = 0, \tag{2.1b}$$

where $\vec{M}_s(\vec{r}_s) = 0$ reflects the fact that the tangential components of the electric field are continuous across a PEC boundary. Upon computing the unknown surface current we can calculate the magnetic vector potential using the free space Green's function approach. The calculation of the scattered fields then follows from Maxwell's equations. Thus [Brown, 1988],

$$\vec{A}_s(\vec{r}) = \int_S \vec{J}_s(\vec{r}_s)\, G(\,|\,\vec{r} - \vec{r}_s\,|\,)\, d\vec{r}_s, \tag{2.2}$$

where $\vec{A}_s(\vec{r})$ is the scattered magnetic vector potential and

$$G(\,|\,\vec{r} - \vec{r}_s\,|\,) = \frac{exp\{\,-\,jk_o\,|\,\vec{r} - \vec{r}_s\,|\,\}}{4\pi\,|\,\vec{r} - \vec{r}_s\,|} \tag{2.3}$$

is the 3-dimensional free space Green's function. We note here that $\vec{J}_s(\vec{r}_s)$ only produces scattered fields and not the total fields in this region. The incident field is defined as the field present in the absence of the object. The scattered field is zero when $\vec{J}_s(\vec{r}_s)$ is zero. We further note that the Green's function is the response of the magnetic vector potential (not the fields) to an ideal dipole. To find the scattered electric and magnetic fields we now use one of the defining equations for the magnetic vector potential, namely

$$\vec{H}_s(\vec{r}) = \nabla\times\ \vec{A}_s(\vec{r}). \tag{2.4}$$

From Ampere's law,

$$\vec{E}_s(\vec{r}) = \frac{1}{j\omega\epsilon}\nabla\times\ \vec{H}_s(\vec{r})$$

$$= \frac{1}{j\omega\epsilon}\nabla\times\ \nabla\times\ \vec{A}_s(\vec{r}), \tag{2.5}$$

where $\vec{r}$ is any point not in the source region.

The diffraction integral for the scattered electric field, $\vec{E}_s(\vec{r})$, becomes;

$$\vec{E}_s(\vec{r}) = \frac{1}{j\omega\epsilon}\nabla\times\ \nabla\times\ \int_S \vec{J}_s(\vec{r}_s)\, G(\,|\,\vec{r}-\vec{r}_s\,|\,)\, d\vec{r}_s. \tag{2.6}$$

In the far field this simplifies to [Brown, 1988]

$$\vec{E}_s(\vec{r}) = jk_o\eta_o\frac{exp\{-jk_or\}}{4\pi r}\,\hat{k}\times\hat{k}\times\int_S d\vec{r}_s\ \vec{J}_s(\vec{r}_s)exp\Big\{j\vec{k}\cdot\vec{r}_s\Big\}, \tag{2.7}$$

where k_o is the free space wavenumber (k_o=$2\pi/\lambda_o$,) η_o is the characteristic impedance, r is the magnitude of the vector $\vec{r}$ which extends from the origin to the point of observation,

$$\vec{r}_s = x_o\hat{x}\ + y_o\hat{y} + \zeta(x_o,y_o)\hat{z} = \vec{r}_{op}\ + \zeta(x_o,y_o)\hat{z}$$

is a vector extending from the origin to a point on the surface, $d\vec{r}_s$ is an elemental area on the surface, $\hat{k}$ is the direction of observation, $\vec{J}_s(\vec{r}_s)$ is the surface current, and S is the illuminated surface area (Figure 4).

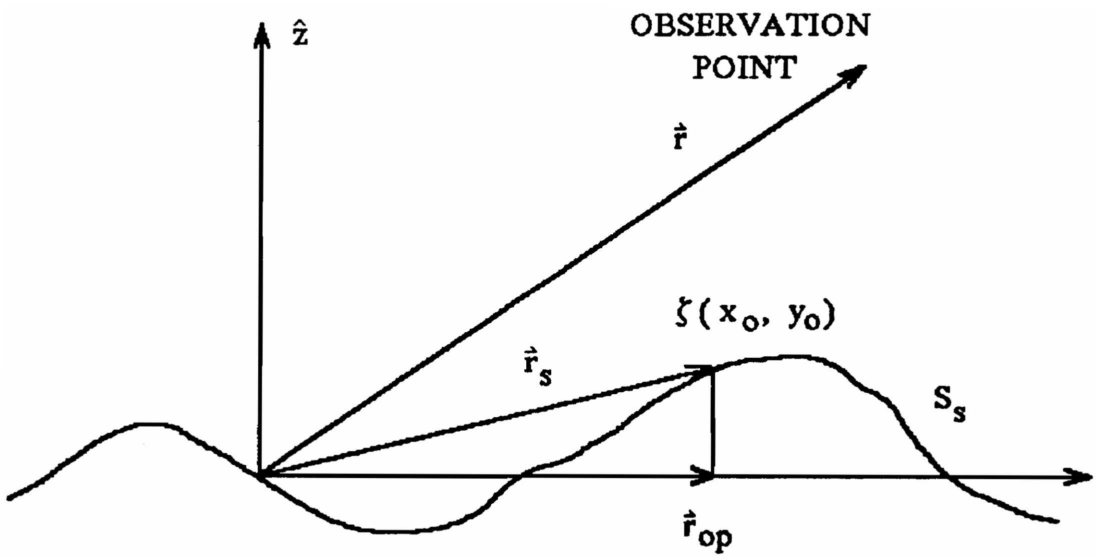

Figure 4: Scattering Coordinate System

We have taken the surface area, S, to be finite and will eventually allow it to go to infinity so as to be consistent with our assumption of an infinite surface. In deriving (2.7) we have assumed that r $\gg \mathrm{D}_{max}$ and r $\gg \pi \dfrac{\mathrm{D}^2_{max}}{\lambda_o}$, where D_{max} is the maximum dimension of the illuminated surface area. We can see here that as $\lambda_o \to 0$ and $S \to \infty$ ($\mathrm{D}_{max} \to \infty$) we must let r increase in such a way that these relationships hold [Brown, 1983b]. The scattered electric field is now proportional to the Fourier transform of the surface current and the field contains no vector component in the direction of observation since $\hat{k} \times \hat{k} \times \vec{J}_s(\vec{r}_s) = \hat{k}(\hat{k} \cdot \vec{J}_s(\vec{r}_s)) - \vec{J}_s(\vec{r}_s)$.

Introducing the incident and scatter shadowing functions into the diffraction integral we have

$$\vec{E}_s(\vec{r}) = jk_o\eta_o \frac{exp\{-jk_o r\}}{4\pi r}\, \hat{k} \times \hat{k} \times \int_S d\vec{r}_s\; S_i(\vec{r}_s;\hat{k}_i) S_s(\vec{r}_s;\hat{k}_i;\hat{k}_s) \vec{J}_s(\vec{r}_s) * exp\left\{j\vec{k} \cdot \vec{r}_s\right\}, \tag{2.8}$$

where S_i is the incident shadowing function defined as

$$S_i(\vec{r}_s;\hat{k}_i) = \begin{cases} 1 \text{ if the point at } \vec{r}_s \text{ is illuminated by } \hat{k}_i \\ 0 \text{ if the point at } \vec{r}_s \text{ is shadowed from } \hat{k}_i, \end{cases} \tag{2.9}$$

where $\hat{k}_i$ is the direction of incoming electromagnetic radiation. S_s is the scatter shadowing function defined as

$S_s(\vec{r}_s;\, \hat{k}_i;\, \hat{k}_s) =$ 1 if the ray, $\hat{k}_s$, resulting from the reflection of $\hat{k}_i$ at the point at $\vec{r}_s$ leaves the surface without intersecting another point on the surface.

and

$$S_s(\vec{r}_s;\, \hat{k}_i;\, \hat{k}_s) = 0 \text{ if the ray, } \hat{k}_s\text{, resulting from the reflection of } \hat{k}_i \text{ at the point at } \vec{r}_s \text{ intersects another point on the surface.} \tag{2.10}$$

We note that any multiply bouncing ray on the surface is explicitly neglected since $S_s(\vec{r}_s;\hat{k}_i;\hat{k}_s) = 0$ at any point beyond the first intersection of the incident ray with the surface. Thus, other multiple scattering effects have to also be accounted for when this approach is taken. Both shadowing functions operate on the surface current $\vec{J}_s(\vec{r}_s)$. The incident shadowing function converts the incident mathematical rays to real rays and by eliminating the current at those points which are not visible from the source. In doing so, it makes the surface current discontinuous since there are only abrupt changes from lit to dark regions and there are no transition regions due to diffraction. The scatter shadowing function eliminates all currents which give rise to multiply scattered rays and converts mathematical (scattering) rays to real rays. We make an important distinction here between our definition of the scatter shadowing function and the one typically encountered in the literature. Our function does not depend on observation angle as objected to by Brown [Brown, 1984b], but only on incident angle and surface structure. Each current element which gives rise to a ray intersecting the surface for a second time is eliminated. Whether or not a ray intersects the surface a second time depends only on the surface structure and incident angle, and has nothing to do with the observation angle. Thus, we avoid the unphysical situation of requiring the surface current to be a function of observation angle by simply re-defining it. This definition makes more sense than eliminating currents which are not visible from the observer, since points which are visible do not necessarily contribute to the scattering process and points that are not visible are not necessarily specular points.

We now face the more difficult problem of determining the unknown surface current, $\vec{J}_s(\vec{r}_s)$. Substituting (2.2) and (2.4) into (2.1a), we obtain

$$\vec{J}_s(\vec{r}_s)= \hat{n}\times\vec{H}_i(\vec{r}_s)+\hat{n}\times\ \nabla_s\times\int_S \vec{J}_s(\vec{r}\,'_s)\ G(\,|\,\vec{r}_s-\vec{r}\,'_s\,|\,)\ d\vec{r}\,'_s, \qquad (2.11)$$

where ∇_s operates on the observation coordinates. Using the vector identity $\nabla_s\times(\vec{J}G) = \nabla_s G\times\vec{J} + G\ \nabla_s\times\vec{J}$, where $\nabla_s\times\vec{J} = 0$ (since $\vec{J}$ is a function of source coordinates) and $\nabla_s G = -\nabla'_s G$, we get

$$\vec{J}_s(\vec{r}_s)= \hat{n}\times\vec{H}_i(\vec{r}_s)+\hat{n}\times\ \int_S \vec{J}_s(\vec{r}\,'_s)\times\nabla'_s\ G(\,|\,\vec{r}_s-\vec{r}\,'_s\,|\,)\ d\vec{r}\,'_s. \qquad (2.12)$$

We see here that the Green's function has a removable singularity at the point $\vec{r}_s = \vec{r}\,'_s$. If we break the integral up into two parts, such that

$$\int_S(\cdot) = \int_{S-A_\epsilon}(\cdot) + \int_{A_\epsilon}(\cdot) \qquad (2.13)$$

where A_ϵ is a small circular area of radius ϵ centered about $\vec{r}_s = \vec{r}\,'_s$, and then let $\epsilon \to 0$, we find that

$$\lim_{\epsilon\to 0}\int_{A_\epsilon}(\cdot) = \frac{\vec{J}_s(\vec{r}_s)}{2} \qquad (2.14)$$

and

$$\lim_{\epsilon \to 0} \int_{S - A_\epsilon} (\cdot) = \oint_S (\cdot) \tag{2.15}$$

where $\oint$ is a Cauchy integral.

Inserting (2.13), (2.14), and (2.15) into (2.12) we end up with

$$\vec{J}_s(\vec{r}_s) = 2\hat{n} \times \vec{H}_i(\vec{r}_s) + 2\hat{n} \times \oint_S \vec{J}_s(\vec{r}'_s) \times \nabla'_s G(|\vec{r}_s - \vec{r}'_s|)\, d\vec{r}'_s. \tag{2.16}$$

This equation is called the magnetic field integral equation (MFIE). It is a Fredholm integral equation of the second kind since the unknown current, $\vec{J}_s(\vec{r}_s)$, is both outside and inside the integral.

The first term on the right hand side of this equation is called the Kirchhoff current or the tangent plane approximation. We see that this term results from the doubling of the $\vec{H}$ field at the "self-point" or the point of reflection (as is the case when a plane wave strikes an infinite planar boundary). The integral term clearly relates to contributions from other points on the surface and thus appears to contain information associated with multiple scattering and diffraction.

For the case in which diffraction and multiple scattering effects can be ignored, we take the surface current to equal the Kirchhoff current, namely,

$$\vec{J}_K(\vec{r}_s) = 2\hat{n} \times \vec{H}_i(\vec{r}_s)$$

$$= 2\hat{n} \times H_o \hat{h}_o exp\left\{ -jk_o \hat{k}_i \cdot \vec{r}_s \right\}, \tag{2.17}$$

where we have set the incident magnetic field to be a plane wave, i.e. $\vec{H}_i(\vec{r}) = H_o\widehat{h}_o exp\left\{-jk_o\widehat{k}_i\cdot\vec{r}\right\}$ with amplitude H_o and polarization $\widehat{h}_o$. $\widehat{n}$ is the upward directed unit vector normal to the surface given by

$$\widehat{n} = -\frac{\nabla\zeta(x_o,y_o)}{|\nabla\zeta(x_o,y_o)|} = \frac{-\zeta_{x_o}\widehat{x} - \zeta_{y_o}\widehat{y} + \widehat{z}}{\sqrt{\zeta_{x_o}^2+\zeta_{y_o}^2 + 1}}, \tag{2.18}$$

where $\zeta_{x_o} = \zeta_{x_o}(\vec{r}_o) = \dfrac{\partial\zeta(x_o,y_o)}{\partial x_o}$ is the slope of the surface at $\vec{r}_s$ in the $\widehat{x}$ direction and $\zeta_{y_o} = \zeta_{y_o}(\vec{r}_o) = \dfrac{\partial\zeta(x_o,y_o)}{\partial y_o}$ is the slope at $\vec{r}_s$ in the $\widehat{y}$ direction.

As suggested by Brown [Brown, 1984b], we project an elemental area on the surface to the plane below. From elementary geometry we find that $d\vec{r}_s = \sqrt{\zeta_{x_o}^2+\zeta_{y_o}^2+ 1}\; d\vec{r}_o$, where $d\vec{r}_o$ is the projected area of $d\vec{r}_s$ on the plane below. Substituting, these expressions into (2.8) and integrating over the plane below, S_o, we have

$$\vec{E}_s(\vec{r}) = 2jk_o\eta_oH_o\frac{exp\{-jk_or\}}{4\pi r}\int_{S_o} d\vec{r}_o\, S_i(\vec{r}_o;\widehat{k}_i)S_s(\vec{r}_o;\widehat{k}_i;\widehat{k}_s)$$
$$*\left(\widehat{k}\times\widehat{k}\times\left\{\vec{N}(\vec{r}_o)\times\widehat{h}_o\right\}\right) exp\left\{jk_o\left(-\widehat{k}_i\cdot\vec{r}_s+\widehat{r}\cdot\vec{r}_s\right)\right\}, \tag{2.19}$$

where $\vec{N}(\vec{r}_o) = -\zeta_{x_o}\widehat{x} - \zeta_{y_o}\widehat{y} + \widehat{z}$ and $\vec{k} = k_o\widehat{r}$. We have replaced $\vec{r}_s$ by $\vec{r}_o$ ($\vec{r}_o$=$x_o\widehat{x}$ + $y_o\widehat{y}$) in the shadowing functions since they are a function of surface structure which is a function of $\vec{r}_o$.

We now form the intensity, $\vec{E}_s(\vec{r})\cdot\vec{E}_s^*(\vec{r})$, bringing the $\widehat{k}\times\widehat{k}\times$ operator, which is independent of the surface coordinate system, inside the integration;

$$\vec{E}_s(\vec{r})\cdot\vec{E}_s^*(\vec{r}) = \left(\frac{k_o\eta_o H_o}{2\pi r}\right)^2 \int_{S_o}\int_{S_o'} d\vec{r}_o\, d\vec{r}_o'\, S_i(\vec{r}_o;\hat{k}_i)S_S(\vec{r}_o;\hat{k}_i;\hat{k}_s)$$

$$* \; S_i(\vec{r}_o';\hat{k}_i)S_S(\vec{r}_o';\hat{k}_i;\hat{k}_s')\Big(\hat{k}\times\hat{k}\times\Big\{\vec{N}(\vec{r}_o)\times\hat{h}_o\Big\}\Big)$$

$$* \Big(\hat{k}\times\hat{k}\times\Big\{\vec{N}(\vec{r}_o')\times\hat{h}_o\Big\}\Big)\, exp\Big\{jk_o\Big(-\hat{k}_i+\hat{r}\Big)\cdot\Big(\vec{r}_s-\vec{r}_s'\Big)\Big\}, \qquad (2.20)$$

where

$$\vec{r}_s = x_o\hat{x} + y_o\hat{y} + \zeta(x_o,y_o)\hat{z} \qquad (2.21a)$$

$$\vec{r}_s' = x_o'\hat{x} + y_o'\hat{y} + \zeta(x_o',y_o')\hat{z}, \qquad (2.21b)$$

and

$$\vec{N}(\vec{r}_o) = -\zeta_{x_o}\hat{x} \;\; -\zeta_{y_o}\hat{y} \;\; +\hat{z} \qquad (2.22a)$$

$$\vec{N}(\vec{r}_o') = -\zeta_{x_o'}\hat{x} \;\; -\zeta_{y_o'}\hat{y} \;\; +\hat{z}. \qquad (2.22b)$$

$\zeta_o = \zeta(x_o,y_o)$ and $\zeta_o' = \zeta(x_o', y_o')$ are the heights of the surface in the unprimed and primed coordinate system, respectively, and ζ_{x_o}, ζ_{y_o}, $\zeta_{x_o'}$, and $\zeta_{x_o'}$ are the respective slopes. As $k_o \to \infty$, the major contributions to each of these integrals is in the neighborhoods around the specular points or the points of stationary phase which satisfy the following equations

$$\nabla\Big\{jk_o(-\hat{k}_i+\hat{r})\cdot\vec{r}_s\Big\} = 0 \qquad (2.23a)$$

$$\nabla'\Big\{jk_o(-\hat{k}_i+\hat{r})\cdot\vec{r}_s'\Big\} = 0, \qquad (2.23b)$$

where ∇ operates on the unprimed coordinates and ∇' operates on the primed coordinates. Thus, the only portions of the surface which contribute to the scattering process occur when $\vec{r}_s$ and $\vec{r}_s'$ are in a neighborhood of the stationary phase points. If $\vec{r}_s$ and $\vec{r}_s'$ are located at different stationary phase points, one can show that in the high frequency limit (or for sufficiently rough surfaces away from grazing incidence) the interference term goes to zero (this is more obvious in the work of Kodis [Kodis, 1966]). Therefore, we can now expand the height $\zeta_o(x_o,y_o)$ about the point (x'_o, y'_o) to get

$$\begin{aligned}\zeta(x_o,y_o) = \; & \zeta(x'_o,y'_o) + \zeta_{x_o}(x'_o,y'_o)\,(x_o - x'_o) + \zeta_{y_o}(x'_o,y'_o)\,(y_o - y'_o) + \\ & \frac{1}{2}\,\zeta_{x_o x_o}(x'_o,y'_o)\,(x_o - x'_o)^2 + \frac{1}{2}\,\zeta_{y_o y_o}(x'_o,y'_o)\,(y_o - y'_o)^2 + \\ & \zeta_{x_o y_o}(x'_o,y'_o)\,(x_o - x'_o)\,(y_o - y'_o) + \text{H.O.T.},\end{aligned} \tag{2.24}$$

where

$$\zeta_{x_o}(x'_o,y'_o) = \left.\frac{\partial\zeta(x_o,y_o)}{\partial x_o}\right|_{(x_o,y_o)=(x'_o,y'_o)} \tag{2.25a}$$

$$\zeta_{y_o}(x'_o,y'_o) = \left.\frac{\partial\zeta(x_o,y_o)}{\partial y_o}\right|_{(x_o,y_o)=(x'_o,y'_o)} \tag{2.25b}$$

$$\zeta_{x_o x_o}(x'_o,y'_o) = \left.\frac{\partial^2\zeta(x_o,y_o)}{\partial x_o^2}\right|_{(x_o,y_o)=(x'_o,y'_o)} \tag{2.25c}$$

$$\zeta_{y_o y_o}(x'_o,y'_o) = \left.\frac{\partial^2\zeta(x_o,y_o)}{\partial y_o^2}\right|_{(x_o,y_o)=(x'_o,y'_o)} \tag{2.25d}$$

$$\zeta_{x_o y_o}(x'_o,y'_o) = \left.\frac{\partial^2\zeta(x_o,y_o)}{\partial x_o \partial y_o}\right|_{(x_o,y_o)=(x'_o,y'_o)}. \tag{2.25e}$$

Then,

$$\zeta(x_o,y_o) - \zeta(x'_o,y'_o) = \zeta_{x_o}(x'_o,y'_o)\,(x_o - x'_o) + \zeta_{y_o}(x'_o,y'_o)\,(y_o - y'_o) +$$

$$\frac{1}{2}\,\zeta_{x_ox_o}(x'_o,y'_o)\,(x_o - x'_o)^2 + \frac{1}{2}\,\zeta_{y_oy_o}(x'_o,y'_o)\,(y_o - y'_o)^2 +$$

$$\zeta_{x_oy_o}(x'_o,y'_o)\,(x_o - x'_o)\,(y_o - y'_o) + \text{H.O.T.} \qquad (2.26)$$

If we let
$$\vec{q} = -\hat{k}_i + \hat{r} = q_1\hat{x} + q_2\hat{y} + q_3\hat{z} \qquad (2.27)$$

(similar to Sancer's notation [Sancer, 1969]) so that

$$q_1 = -k_{ix} + \frac{x}{r} \qquad (2.28a)$$
$$q_2 = -k_{iy} + \frac{y}{r} \qquad (2.28b)$$
$$q_3 = -k_{iz} + \frac{z}{r}, \qquad (2.28c)$$

we have

$$exp\Big\{jk_o\Big(-\hat{k}_i + \hat{r}\Big)\cdot\Big(\vec{r}_s - \vec{r}'_s\Big)\Big\} =$$

$$exp\Big\{jk_o\Big[q_1(x_o - x'_o) + q_2\,(y_o - y'_o) + q_3\,\zeta_{x_o}(x'_o,y'_o)\,(x_o - x'_o) +$$

$$q_3\zeta_{y_o}(x'_o,y'_o)\,(y_o - y'_o) + \frac{q_3}{2}\,\zeta_{x_ox_o}(x'_o,y'_o)\,(x_o - x'_o)^2 +$$

$$\frac{q_3}{2}\,\zeta_{y_oy_o}(x'_o,y'_o)\,(y_o - y'_o)^2 + q_3\zeta_{x_oy_o}(x'_o,y'_o)\,(x_o - x'_o)\,(y_o - y'_o) +$$

$$\text{H.O.T.}\Big]\Big\}. \qquad (2.29)$$

If we let $u = k_o(x_o - x'_o)$ and $v = k_o(y_o - y'_o)$, the exponential becomes

$$exp\Big\{jk_o(-\hat{k}_i+\hat{r})\cdot(\vec{r}_s-\vec{r}_s')\Big\} = exp\Big\{jq_1u+jq_2v+jq_3\zeta_{x_o}u + jq_3\zeta_{y_o}v+$$

$$\frac{q_3}{2}\zeta_{x_ox_o}\frac{u^2}{k_o} + \frac{q_3}{2}\zeta_{y_oy_o}\frac{v^2}{k_o} + q_3\zeta_{x_oy_o}\frac{uv}{k_o} + O\Big(\frac{1}{k_o^2}\Big)\Big\}$$

$$= exp\Big\{j(q_1+q_3\zeta_{x_o})u + (q_2+q_3\zeta_{y_o})v + O\Big(\frac{1}{k_o}\Big)\Big\}. \qquad (2.30)$$

We now have, with $\frac{1}{k_o^2}\, dudv = d\vec{r}\,'_o$,

$$\vec{E}_s(\vec{r})\cdot\vec{E}_s^*(\vec{r}) = \Big(\frac{\eta_o H_o}{2\pi r}\Big)^2 \int_{S_{ox}} dx_o \int_{S_{oy}} dy_o \int_{k_o(x_o-S_{oxmin})}^{k_o(x_o-S_{oxmax})} du \int_{k_o(y_o-S_{oymin})}^{k_o(y_o-S_{oymax})} dv$$

$$* \; S_i(\vec{r}_o;\hat{k}_i)S_s(\vec{r}_o;\hat{k}_i;\hat{k}_s)\; S_i(x_o-\frac{u}{k_o},\, y_o-\frac{v}{k_o};\hat{k}_i)S_s(x_o-\frac{u}{k_o},\, y_o-\frac{v}{k_o};\hat{k}_i;\hat{k}_s')$$

$$*\Big(\hat{k}\times\hat{k}\times\Big\{\vec{N}(\vec{r}_o)\times\hat{h}_o\Big\}\Big)\Big(\hat{k}\times\hat{k}\times\Big\{\vec{N}(\vec{r}_o)\times\hat{h}_o\Big\}\Big)$$

$$* \; exp\Big\{j(q_1+q_3\zeta_{x_o})u + (q_2+q_3\zeta_{y_o})v + O\Big(\frac{1}{k_o}\Big)\Big\}. \qquad (2.31)$$

We now take the integration to be over a rectangular region with dimensions S_{oxmax} and S_{oymax} (Figure 5) denoting the largest value for x_o' and y_o', i.e., $x_o' \leq S_{oxmax}$ and $y_o' \leq S_{oymax}$, and S_{oxmin} and S_{oymin} denoting the smallest values for x_o' and y_o', i.e., $x_o' \geq S_{oxmin}$ and $y_o' \geq S_{oymin}$. The geometry of the integration region is immaterial although the variables will not be independent (as shown) unless the region is rectangular. Nevertheless, as the frequency approaches infinity, the limits of the integrals approach $\pm\infty$ regardless of the integration geometry.

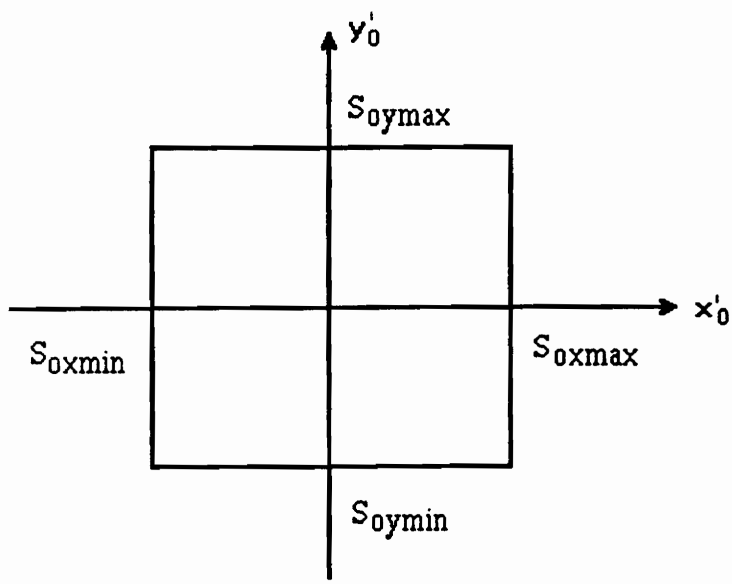

Figure 5: Domain of Surface Integration

Sancer [Sancer, 1969] gives the case when the incident beam is circular and hence the integration region over the surface is elliptical.

We note that the normal vector to the surface at $\vec{r}'_o$, $\vec{N}(\vec{r}'_o)$, is equal to the normal vector to the surface at $\vec{r}_o$, i.e. $\vec{N}(\vec{r}'_o) = \vec{N}(\vec{r}_o)$, and is not effected by the change of variables since it depends only on the specular slopes and not their location. As $k_o \to \infty$, $O\left(\frac{1}{k_o}\right) \to 0$, and the shadowing functions in the primed and unprimed coordinate systems become approximately the same due to the rapid decorrelation of the electromagnetic field as $\vec{r}_o$ and $\vec{r}'_o$ move apart. Likewise, the slopes in the neighborhood around the specular points, where the integral is non-zero, are also approximately the same. The limits on the u and v become

$$
\begin{aligned}
k_o(x_o - S_{oxmax}) &\to -\infty \\
k_o(x_o - S_{oxmin}) &\to \infty \\
k_o(y_o - S_{oymax}) &\to -\infty \\
k_o(x_o - S_{oymin}) &\to \infty,
\end{aligned}
$$

where we have allowed the area in the unprimed coordinate system to be slightly less than the area in the primed coordinate system so that these relationships hold everywhere, including the outermost edges of the source region. Therefore, we have

$$\lim_{k_o\to\infty}\left(\vec{E}_s(\vec{r})\cdot\vec{E}_s^*(\vec{r})\right)=\left(\frac{\eta_o H_o}{2\pi r}\right)^2\int_{S_{ox}} dx_o \int_{S_{oy}} dy_o \int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} dv$$

$$* \; S_i^2(\vec{r}_o;\hat{k}_i)S_s^2(\vec{r}_o;\hat{k}_i;\hat{k}_s)\left(\hat{k}\times\hat{k}\times\left\{\vec{N}(\vec{r}_o)\times\hat{h}_o\right\}\right)^2$$

$$* \; exp\left\{j(q_1+q_3\zeta_{x_o})u \; + j(q_2+q_3\zeta_{y_o})v\right\},$$

$$\lim_{k_o\to\infty}\left(\vec{E}_s(\vec{r})\cdot\vec{E}_s^*(\vec{r})\right)=\left(\frac{\eta_o H_o}{2\pi r}\right)^2\int_{S_{ox}} dx_o \int_{S_{oy}} dy_o \; S_i^2(\vec{r}_o;\hat{k}_i)S_s^2(\vec{r}_o;\hat{k}_i;\hat{k}_s)$$

$$*\left(\hat{k}\times\hat{k}\times\left\{\vec{N}(\vec{r}_o)\times\hat{h}_o\right\}\right)^2\int_{-\infty}^{\infty} du \; exp\left\{j(q_1+q_3\zeta_{x_o})u\right\}$$

$$*\int_{-\infty}^{\infty} dv \; exp\left\{j(q_2+q_3\zeta_{y_o})v\right\}. \tag{2.32}$$

But since

$$\int_{-\infty}^{\infty} du \; exp\left\{j(q_1+q_3\zeta_{x_o})u\right\} = 2\pi\delta(q_1+q_3\zeta_{x_o}) \tag{2.33a}$$

and

$$\int_{-\infty}^{\infty} dv \; exp\left\{j(q_2+q_3\zeta_{y_o})v\right\} = 2\pi\delta(q_2+q_3\zeta_{y_o}), \tag{2.33b}$$

where we have taken the Fourier transform and inverse Fourier transform to be

$$H(\omega) = \Im\{h(x)\} = \int_{-\infty}^{\infty} h(x)exp\{j\omega x\}\,dx \tag{2.34a}$$

$$h(x) = \Im^{-1}\{H(f)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega)exp\{-j\omega x\}d\omega, \tag{2.34b}$$

respectively, we have

$$\lim_{k_o \to \infty}\left(\vec{E}_s(\vec{r}) \cdot \vec{E}_s^*(\vec{r})\right) = \int_{S_{ox}} dx_o \int_{S_{oy}} dy_o \;\; S_i^2(\vec{r}_o;\widehat{k}_i) S_s^2(\vec{r}_o;\widehat{k}_i;\widehat{k}_s)$$

$$* \left(\widehat{k} \times \widehat{k} \times \left\{\vec{N}(\vec{r}_o) \times \widehat{h}_o\right\}\right)^2 \delta(q_1 + q_3\zeta_{x_o})\delta(q_2 + q_3\zeta_{y_o}). \qquad (2.35)$$

Now taking the ensemble average to form the second moment and dropping the limit, we obtain

$$\langle \vec{E}_s(\vec{r}) \cdot \vec{E}_s^*(\vec{r}) \rangle = \left\langle \left(\frac{\eta_o H_o}{r}\right)^2 \int_{S_{ox}} dx_o \int_{S_{oy}} dy_o \; S_i^2(\vec{r}_o;\widehat{k}_i) S_s^2(\vec{r}_o;\widehat{k}_i;\widehat{k}_s)\right.$$

$$\left. * \left(\widehat{k} \times \widehat{k} \times \left\{\vec{N}(\vec{r}_o) \times \widehat{h}_o\right\}\right)^2 \delta(q_1 + q_3\zeta_{x_o})\delta(q_2 + q_3\zeta_{y_o}) \right\rangle. \qquad (2.36)$$

Taking the ensemble average inside the surface integrals yields

$$\langle \vec{E}_s(\vec{r}) \cdot \vec{E}_s^*(\vec{r}) \rangle = \left(\frac{\eta_o H_o}{r}\right)^2 \int_{S_{ox}} dx_o \int_{S_{oy}} dy_o \left\langle S_i^2(\vec{r}_o;\widehat{k}_i) S_s^2(\vec{r}_o;\widehat{k}_i;\widehat{k}_s)\right.$$

$$\left. * \left(\widehat{k} \times \widehat{k} \times \left\{\vec{N}(\vec{r}_o) \times \widehat{h}_o\right\}\right)^2 \delta(q_1 + q_3\zeta_{x_o})\delta(q_2 + q_3\zeta_{y_o}) \right\rangle. \qquad (2.37)$$

Now, the only random variables are S_i, S_s, ζ_{x_o}, and ζ_{y_o} so

$$\langle \cdot \rangle = \int_{-\infty}^{\infty} dS_i \int_{-\infty}^{\infty} dS_s \int_{-\infty}^{\infty} d\zeta_{x_o} \int_{-\infty}^{\infty} d\zeta_{y_o} \; (\;\cdot\;) \; \mathrm{p}\; (S_i, S_s, \zeta_{x_o}, \zeta_{y_o}), \qquad (2.38)$$

where $\mathrm{p}(S_i, S_s, \zeta_{x_o}, \zeta_{y_o})$ is a fourth order joint probability density function (PDF). This PDF can be written as

$$\mathrm{p}(S_i, S_s, \zeta_{x_o}, \zeta_{y_o}) = \mathrm{P}(\widehat{k}_i, \widehat{k}, \zeta_{x_o}, \zeta_{y_o}) \; \delta(S_i - 1) \; \delta(S_s - 1), \qquad (2.39)$$

since the integrand is only non-zero when $S_i = 1$ and $S_s = 1$ and $P(\widehat{k}_i, \widehat{k}, \zeta_{x_o}, \zeta_{y_o})$ is the probability at that point in the density $p(S_i, S_s, \zeta_{x_o}, \zeta_{y_o})$. In other words

$$P(\widehat{k}_i, \widehat{k}, \zeta_{x_o}, \zeta_{y_o}) = \int\int p(S_i, S_s, \zeta_{x_o}, \zeta_{y_o}) dS_i dS_s.$$

$P(\widehat{k}_i, \widehat{k}, \zeta_{x_o}, \zeta_{y_o})$ can also be written as

$$P(\widehat{k}_i, \widehat{k}, \zeta_{x_o}, \zeta_{y_o}) = P(\widehat{k}_i, \widehat{k} \mid \zeta_{x_o}, \zeta_{y_o})\, p(\zeta_{x_o}, \zeta_{y_o}), \tag{2.40}$$

using Baye's theorem, where $P(\widehat{k}_i, \widehat{k} \mid \zeta_{x_o}, \zeta_{y_o})$ is the conditional probability that a point on the surface will be illuminated from $\widehat{k}_i$ and scatter in the direction of observation, $\widehat{k}$, given the specular slopes at that point. $p(\zeta_{x_o}, \zeta_{y_o})$ is the joint PDF of the slopes evaluated at the specular slopes.

If we make a change of variables, and let $\alpha = q_1 + q_3\zeta_{x_o}$, $d\alpha = q_3\, d\zeta_{x_o}$ and $\beta = q_2 + q_3\zeta_{y_o}$, $d\beta = q_3 d\zeta_{y_o}$, we get

$$\begin{aligned}
\langle \vec{E}_s(\vec{r}) \cdot \vec{E}_s^*(\vec{r}) \rangle = \frac{1}{q_3^2}\left(\frac{\eta_o H_o}{r}\right)^2 \int_{S_{ox}} dx_o \int_{S_{oy}} dy_o \\
* \int_{-\infty}^{\infty} dS_i \int_{-\infty}^{\infty} dS_s \int_{-\infty}^{\infty} d\alpha \int_{-\infty}^{\infty} d\beta \cdot S_i^2(\vec{r}_o; \widehat{k}_i) S_s^2(\vec{r}_o; \widehat{k}_i; \widehat{k}_s) \cdot \\
* \left\{ \widehat{k} \times \widehat{k} \times \vec{N}\left(\frac{\alpha - q_1}{q_3}, \frac{\beta - q_2}{q_3}\right) \times \widehat{h}_o \right\}^2 \cdot \delta(\alpha)\delta(\beta)\, \delta(S_i - 1)\, \delta(S_s - 1) \cdot \\
* P\left(\widehat{k}_i, \widehat{k} \mid \frac{\alpha - q_1}{q_3}, \frac{\beta - q_2}{q_3}\right) P\left(\frac{\alpha - q_1}{q_3}, \frac{\beta - q_2}{q_3}\right).
\end{aligned} \tag{2.41}$$

This reduces to

$$\langle \vec{E}_s(\vec{r}) \cdot \vec{E}_s^*(\vec{r}) \rangle = \frac{1}{q_3^2}\left(\frac{\eta_o H_o}{r}\right)^2 \int_{S_{ox}} dx_o \int_{S_{oy}} dy_o \left\{ \hat{k} \times \hat{k} \times \vec{N}(-\tfrac{q_1}{q_3}, -\tfrac{q_2}{q_3}) \times \hat{h}_o \right\}^2$$

$$* \ \mathrm{P}(\hat{k}_i, \hat{k} \mid -\tfrac{q_1}{q_3}, -\tfrac{q_2}{q_3}) \ \mathrm{P}(-\tfrac{q_1}{q_3}, -\tfrac{q_2}{q_3}). \quad (2.42)$$

Note that the scatter shadowing function $S_s(\vec{r}_o; \hat{k}_i; \hat{k}_s)$ which depends on the slopes at each point on the surface reduces to $S_s(\vec{r}_o; \hat{k}_i; \hat{k})$, where $\hat{k}$ is the observation vector. This of due to the fact that $\hat{k}_s$ depends on ζ_{x_o} and ζ_{y_o} and hence on α and β, so as the frequency increases only the slopes which give rise to scattering in the direction of observation become important.

Since the integrand is independent of location, we have

$$\langle \vec{E}_s(\vec{r}) \cdot \vec{E}_s^*(\vec{r}) \rangle = \frac{1}{q_3^2}\left(\frac{\eta_o H_o}{r}\right)^2 \left\{ \hat{k} \times \hat{k} \times \vec{N}(-\tfrac{q_1}{q_3}, -\tfrac{q_2}{q_3}) \times \hat{h}_o \right\}^2$$

$$* \ \mathrm{P}(\hat{k}_i, \hat{k} \mid -\tfrac{q_1}{q_3}, -\tfrac{q_2}{q_3}) \ \mathrm{P}(-\tfrac{q_1}{q_3}, -\tfrac{q_2}{q_3}) \ A_{illum}, \quad (2.43)$$

where $A_{illum} = S_{ox} S_{oy}$ is the projected area of illumination.

We are also interested in the radar cross section (RCS) per unit area of the surface, since the scattered intensity goes to infinity as $A_{illum} \to \infty$. Therefore, we compute the normalized radar cross section, σ^o, given as:

$$\sigma^o = \lim_{A_{illum} \to \infty} \lim_{r \to \infty} \frac{4\pi r^2 \langle \vec{E}_s(\vec{r}) \cdot \vec{E}_s^*(\vec{r}) \rangle}{A_{illum} E_o^2}. \quad (2.44)$$

Here we have let the illuminated area go to infinity to be consistent with our assumption of an infinite surface. The normalized RCS becomes

$$\sigma^o = \frac{4\pi}{q_3^2}\left\{\widehat{k}\times\widehat{k}\times\vec{N}(-\tfrac{q_1}{q_3},-\tfrac{q_2}{q_3})\times\widehat{h}_o\right\}^2$$

$$* \; \mathrm{P}(\widehat{k}_i,\widehat{k}\mid -\tfrac{q_1}{q_3},-\tfrac{q_2}{q_3})\;\mathrm{P}(-\tfrac{q_1}{q_3},-\tfrac{q_2}{q_3}), \qquad (2.45)$$

where $\mathrm{P}(\widehat{k}_i,\widehat{k}\mid -\frac{q_1}{q_3},-\frac{q_2}{q_3})$ is the bistatic shadowing function. Using Baye's theorem, this function can be written more conveniently as

$$\mathrm{P}(\widehat{k}_i,\widehat{k}\mid -\tfrac{q_1}{q_3},-\tfrac{q_2}{q_3}) = \frac{\mathrm{P}(\widehat{k}_i,\widehat{k}\,,-\frac{q_1}{q_3},-\frac{q_2}{q_3})}{\mathrm{P}(-\frac{q_1}{q_3},-\frac{q_2}{q_3})}$$

$$= \mathrm{P}(\widehat{k}\mid\widehat{k}_i,-\tfrac{q_1}{q_3},-\tfrac{q_2}{q_3})\;\mathrm{P}(\widehat{k}_i\mid -\tfrac{q_1}{q_3},-\tfrac{q_2}{q_3}), \qquad (2.46)$$

where $\mathrm{P}(\widehat{k}_i\mid -\frac{q_1}{q_3},-\frac{q_2}{q_3})$ is the incident shadowing function and $\mathrm{P}(\widehat{k}\mid\widehat{k}_i,-\frac{q_1}{q_3},-\frac{q_2}{q_3})$ is the scatter shadowing function. Specifically, $\mathrm{P}(\widehat{k}_i\mid -\frac{q_1}{q_3},-\frac{q_2}{q_3})$ is the probability that a point on the surface will be illuminated by the incident ray given the specular slopes at that point. $\mathrm{P}(\widehat{k}\mid\widehat{k}_i,-\frac{q_1}{q_3},-\frac{q_2}{q_3})$ is the probability that given the incident ray illuminates a given point on the surface and that its slope is specular that the scattered ray will leave the surface without being blocked by another point on the surface.

The scatter shadowing function defined *before* the integration and averaging processes (eq. (2.10)) is only a function of surface structure and incident angle. It is only *after* the integration and averaging process that the dependence on the observation angle becomes apparent. This is due to the fact that the only contribution to the diffraction integral will come from those points whose slopes are specular, and not merely arbitrary slopes. Hence, the joint probability of a ray incident on the surface reaching the observer is conditioned on the specular slopes (and hence, the direction of observation), as we expect. The same reasoning follows for the dependence of the incident shadowing function on q_1, q_2, and q_3. This generalizes and clarifies the work of Sancer [Sancer, 1969] who modified the surface current using bistatic shadowing, in which it appeared the surface current became of function of observation angle, a property objected to by Brown [Brown, 1984b].

When $\hat{k} = -\hat{k}_i$, as in the case for backscatter, $\mathrm{P}(\hat{k}\ |\hat{k}_i, -\frac{q_1}{q_3}, -\frac{q_2}{q_3}) = 1$, since we are guaranteed that a ray traveling in the $\hat{k}$ direction will exit, given it was illuminated by a ray traveling in the $\hat{k}_i$ direction, which takes the same path but travels in the opposite direction (Figure 6). So, this leads to

$$\mathrm{P}(\hat{k}_i, \hat{k}\ | -\tfrac{q_1}{q_3}, -\tfrac{q_2}{q_3}) = \mathrm{P}(\hat{k}_i\ | -\tfrac{q_1}{q_3}, -\tfrac{q_2}{q_3}) \tag{2.47}$$

for backscatter. If we consider the case where the incident ray and the scattered ray both lie in a vertical plane, we have for $\hat{k}_i$ and $\hat{k}$ well separated [Sancer, 1969] (Figure 7),

$$\mathrm{P}(\hat{k}_i, \hat{k}\ | -\tfrac{q_1}{q_3}, -\tfrac{q_2}{q_3}) = \mathrm{P}(\hat{k}\ | -\tfrac{q_1}{q_3}, -\tfrac{q_2}{q_3})\ \mathrm{P}(\hat{k}_i\ | -\tfrac{q_1}{q_3}, -\tfrac{q_2}{q_3}). \tag{2.48}$$

Also, for the case when $\hat{k}$ is below $\hat{k}_i$ in the plane of incidence (and $\hat{k}_i$ is below the normal to the z=0 plane, Figure 8), we can write

$$\mathrm{P}(\hat{k}_i, \hat{k}\ | -\tfrac{q_1}{q_3}, -\tfrac{q_2}{q_3}) = \mathrm{P}(\hat{k}_i\ |\ \hat{k}, -\tfrac{q_1}{q_3}, -\tfrac{q_2}{q_3})\ \mathrm{P}(\hat{k}\ | -\tfrac{q_1}{q_3}, -\tfrac{q_2}{q_3}). \tag{2.49}$$

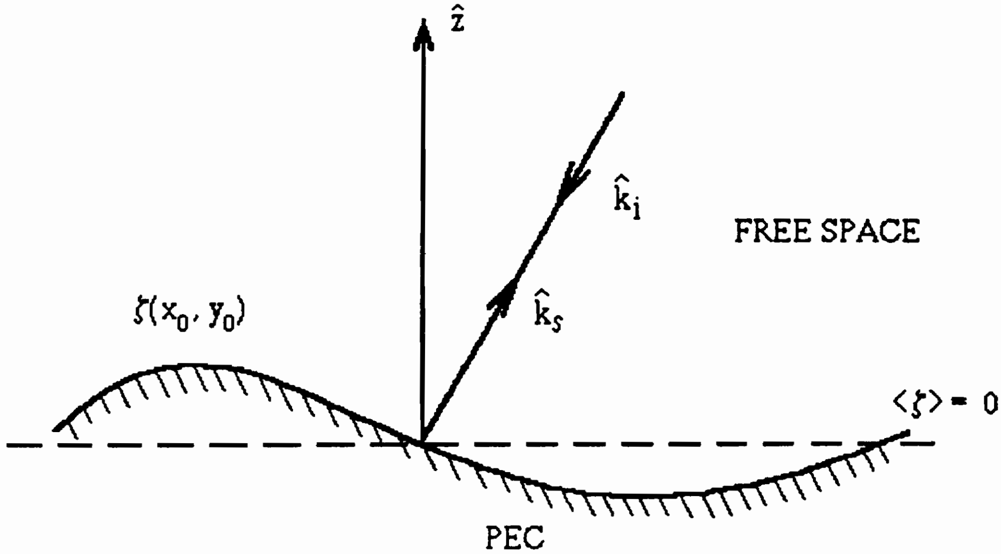

Figure 6: Backscattering

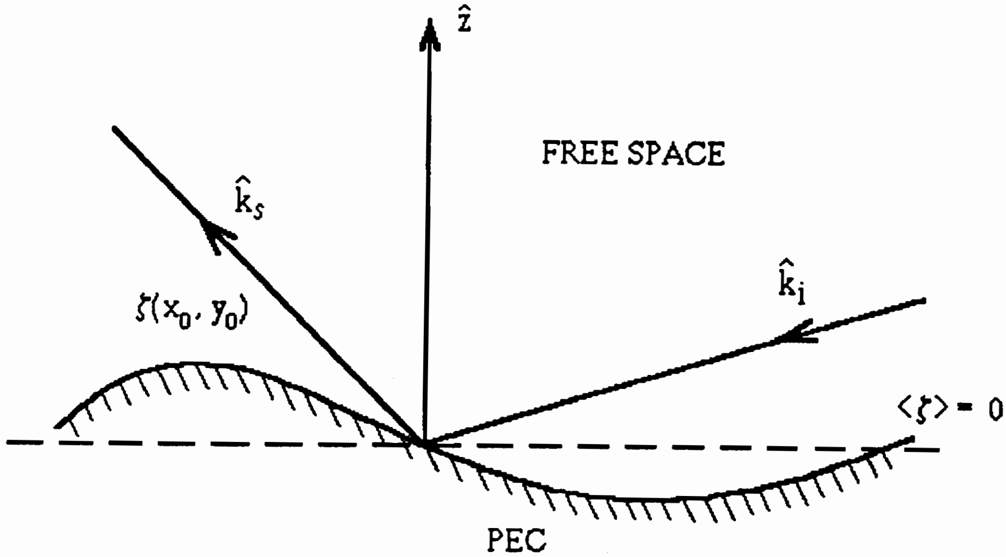

Figure 7: Forward Scattering

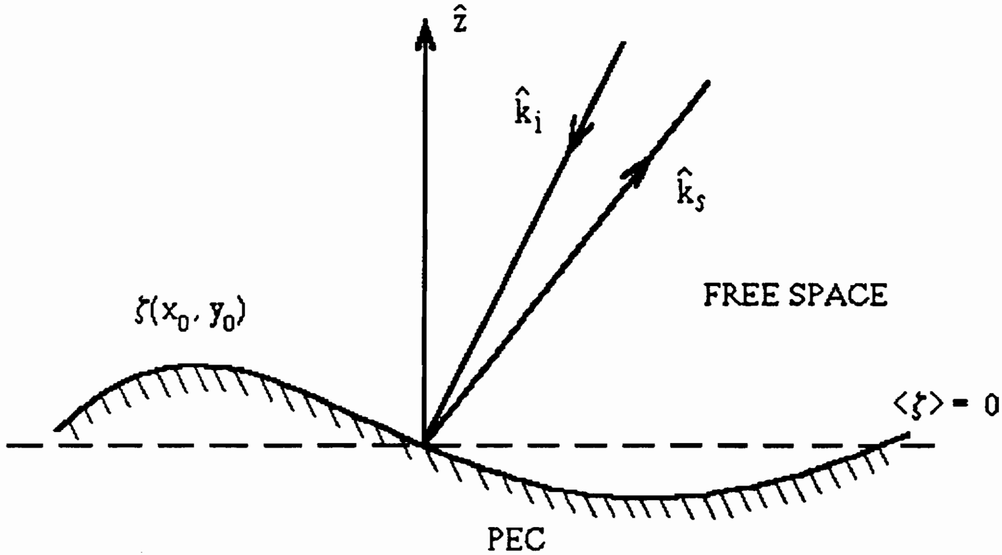

Figure 8: Scattering with the Scattered Ray Below the Incident Ray

The first probability on the right-hand side equals 1, since we are guaranteed illumination under this condition. Thus:

$$P(\hat{k}_i, \hat{k} \mid -\frac{q_1}{q_3}, -\frac{q_2}{q_3}) = P(\hat{k} \mid -\frac{q_1}{q_3}, -\frac{q_2}{q_3}). \tag{2.50}$$

Likewise, for the case when $\hat{k}_i$ is below $\hat{k}$ (and $\hat{k}$ is below the normal to the z=0 plane. Figure 9):

$$P(\hat{k}_i, \hat{k} \mid -\frac{q_1}{q_3}, -\frac{q_2}{q_3}) = P(\hat{k}_i \mid -\frac{q_1}{q_3}, -\frac{q_2}{q_3}), \tag{2.51}$$

since

$$P(\hat{k} \mid -\frac{q_1}{q_3}, -\frac{q_2}{q_3}) = 1.$$

This concludes the modern approach proposed by Sancer [Sancer, 1969]. This formalism provides us with the motivation for calculating the incident shadowing function (since we will be assuming backscatter), which this thesis focuses on. Before a more detailed development of shadowing theory is discussed, it is important to discuss the results of others presented in the literature. This is done in the next chapter.

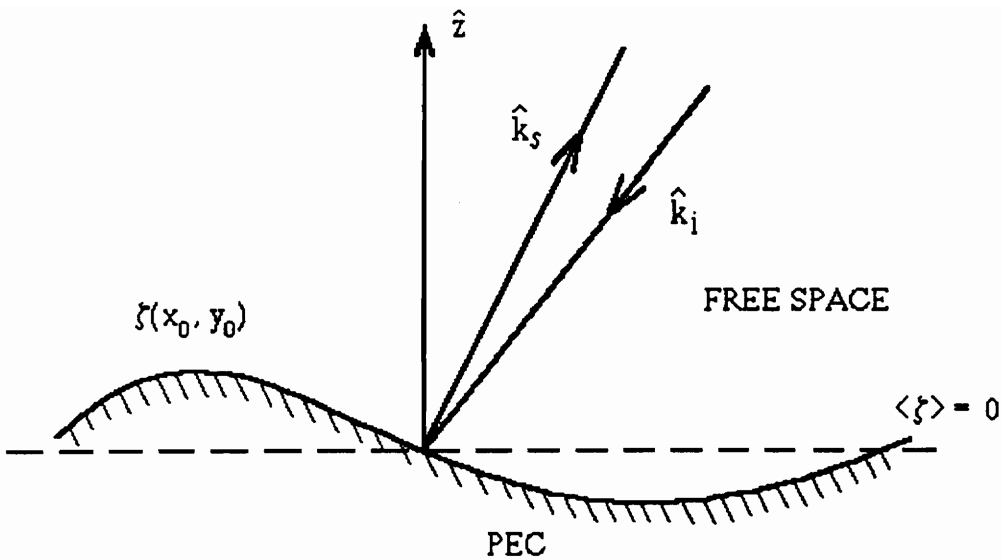

Figure 9: Scattering with the Incident Ray Below the Scattered Ray

3.0 LITERATURE REVIEW

3.1 Introduction

Over the past four decades a large amount of literature has accumulated pertaining to scattering of acoustic and electromagnetic waves from random surfaces. Understandably, most literature reviews are broad based and thus, are not very useful for one wishing to explore a topic in more detail. It is therefore our purpose here to give a complete review of the articles pertaining directly to shadowing theory and present a brief discussion of the major research in this area.

Shadowing theory has been studied for the past three decades, since Beckmann published the first paper (in the United States) in 1965. Shadowing functions have been corrected [Wagner, 1967] and extended [Jin, 1988] to account for the blockage of rays on the surface (propagation shadowing) when multiple scattering effects are considered [Jin, 1990]. The main articles on shadowing are given in the following references [Jin, 1990; McCoy, 1989; Brown, 1984b, 1980a; Welton, 1973; Hardin, 1972; Fuks, 1969; Wagner, 1967; Smith, 1967b; Brockelman and Hagfors, 1966; Shaw, 1966; Beckmann, 1965b, 1966; Hamilton, 1966; Bass and Fuks, 1964].

The mechanisms behind "shadowing theory" have been investigated for almost a century. The relationship of shadowing theory to random surfaces was first introduced by Beckmann in 1965 [Beckmann, 1965b]. It became clear in Beckmann's paper that the solution to the shadowing problem and the solution to what is called the "first passage in time" (FPIT) problem were really the same. In fact, the shadowing function was shown to be simply the integral of the first passage function over all space. It was not until Wagner's paper in 1967 [Wagner, 1967] , however, that the name "first passage" first appeared in a footnote in a scattering paper. It is interesting to note that, besides this footnote, no other references were made in other articles, prior to 1988. The enormous body of literature related to the first passage problem was apparently overlooked.

We make no attempt here to provide a complete list of relevant references on the FPIT problem, as this would take many months. However, we believe that we have accumulated some of the major references in this area and are as follows: [Rainal, 1990, 1988, 1987, 1968; Perez, 1987, Abrahams, 1986, 1984; Ricciardi and Sato, 1986, 1983; Durbin, 1985; Ricciardi, 1977; Blake and Linsey, 1973; Mehr and McFadden, 1965; Slepian, 1962; 1961; Siegert, 1951; Rice, 1945]. We shall concentrate here on shadowing theory as it is known in the electromagnetic and acoustic literature with some references to the FPIT problem. This thesis uses and extends the results of Ricciardi and Sato [Ricciardi and Sato, 1986], whose work was based on Rice's celebrated paper on random noise [Rice, 1945] published in 1945, 20 years before Beckmann's paper on shadowing theory in 1965.

A description of the main work on shadowing theory such as Beckmann's paper, as well as other major contributions, is provided here.

3.2 Shadowing Due to Beckmann

Shadowing of random rough surfaces was first introduced in the United States by Beckmann 1965 [Beckmann, 1965b] and in the Soviet Union in 1964 by Bass and Fuks [Bass and Fuks, 1964]. The impetus at the time was the study of the moon's surface using radar returns [Fuks, 1984; Smith, 1967a; Hagfors, 1966; Beckmann, 1965a; Hagfors, 1964]. While shadowing had been predicted for some time, no one had accounted for this affect until Beckmann.

To this end, Beckmann proposed a correction factor to the coherent and incoherent intensity. According to Beckmann, the incoherent intensity is proportional to the illuminated area (2.43), and the coherent intensity is proportional to the illuminated area squared. Shadowing then can be taken into account, to a first approximation, by simply multiplying the uncorrected incoherent intensity by $S(\theta_i)$ and the uncorrected coherent part by $S^2(\theta_i)$. The shadowing function, $S(\theta_i)$, is the ratio of the illuminated to total area projected on a plane below the surface with θ_i defined as the grazing angle relative to the mean surface height (taken to be the z=0 plane). Thus,

$$\langle \vec{E}_s(\vec{r}) \rangle^2_{corr} = S^2(\theta_i) \langle \vec{E}_s(\vec{r}) \rangle^2_{uncorr} \tag{3.1}$$

is the shadow corrected coherent intensity, and

$$\langle \vec{E}_s^2(\vec{r}) \rangle_{corr} = S(\theta_i) \, \langle \vec{E}_s^2(\vec{r}) \rangle_{uncorr} \tag{3.2}$$

is the shadow corrected scattered intensity. Beckmann stated that a function, $S(x,y)$, equal to 1 on the illuminated parts of the surface and equal to zero otherwise, would multiply the kernel of the diffraction integral. Integration would take place over $S(x,y)XY$ rather than XY, where X and Y are the lengths of the surface in the $\hat{x}$ and $\hat{y}$ directions. The end result could then be approximated by equations (3.1) and (3.2), with $S(\theta_i) = \langle S(x,y) \rangle$.

Beckmann's analysis was met with criticism from various authors, notably from Brockelman and Hagfors [Brockelman and Hagfors, 1966], Shaw [Shaw, 1966], and more recently by McCoy [McCoy, 1989]. Experimental evidence by Hamilton [Hamilton, 1966], one of Beckmann's students, also showed discrepancies between Brockelman and Hagfors results and the theoretically predicted shadowing function proposed by Beckmann.

The criticism centered around two major points in Beckmann's derivation. The first, pointed out by Brockelman and Hagfors, was that the major contribution to scattering will be from specular points on the surface, that is, those points whose slopes allow the incoming ray to be reflected in the direction of the observer (for backscatter, this would be points whose slopes are perpendicular to the incoming ray). Thus, "the effect of shadowing would not be a question of the fraction of illuminated area, but a question of the fraction of illuminated, favorably oriented, surface elements." While Beckmann's shadowing function, $S(\theta_i)$, is the ratio of illuminated points to the total number of points, Brockelman and Hagfors function, which they defined to be $R(\theta_i)$, is the ratio of illuminated specular points to the total number of specular points.

Brockelman and Hagfors performed an experiment using correlated receiver noise and calculated $S(\theta_i)$ and $R(\theta_i)$ using a digital computer. A comparison was made between $S(\theta_i)$ obtained theoretically by Beckmann, and $S(\theta_i)$ obtained "experimentally." They found that Beckmann's theory agreed, at best, "qualitatively" to their experiments, and the "experimental" shadowing function, based on Beckmann's definition, overestimated the effects of shadowing for all tested values of surface statistics and incident angles when compared with $R(\theta_i)$. This focused attention on a mathematical error made in Beckmann's paper and on his original definition for the shadowing function and its role in scattering theory. Brockelman and Hagfors results are generally accepted as proof that Beckmann's function is incorrect.

The next issue concerning Beckmann's result was formally addressed some years later in 1989 by McCoy [McCoy, 1989]. McCoy challenged Beckmann's assumption that shadowing affects could be taken into account by a simple multiplication of $S(\theta_i)$. According to Beckmann

$$\begin{aligned} \langle \vec{E}_s(\vec{r}) \rangle_{corr} &= \langle S(x,y) \rangle \langle \vec{E}_s(\vec{r}) \rangle_{uncorr} \\ &= S(\theta_i) \langle \vec{E}_s(\vec{r}) \rangle_{uncorr}. \end{aligned} \tag{3.3}$$

McCoy explained that in order for an ensemble average of the product of the unaveraged shadowing function and the uncorrected component of the field to equal the product of the averages the shadowing function, $S(x,y)$, and the uncorrected scattered field must be uncorrelated. However, both the shadowing function (which is a function of location) and the scattered field depend on the same stochastic process. Hence, the only way to justify the use of Beckmann's $S(\theta_i)$ is to assume statistical decorrelation which cannot be true.

It should be noted that Beckmann's shadowing theory does not conserve energy. This is due to the fact that the coherent and incoherent power are both attenuated by the introduction of the shadowing function. Clearly energy must be conserved as we move from normal incidence closer to grazing angles. We recognize this to be a failure of the Kirchhoff approximation to account for multiply scattered rays.

Despite the shortcomings of Beckmann's paper, it remains fundamental to most theories which incorporate shadowing into their analysis. More importantly, he made the connection of the shadowing function to the first passage in time function. For further discussions of Beckmann's paper, we refer the reader to two noteworthy comments in reference [Shaw, 1966] and the private communication by J. P. M. Thomas [Beckmann, 1966].

As pointed out by Brockelman and Hagfors, in the high frequency limit where shadowing plays a major role, contributions from specular points play a major role to the scattering process. It is logical, therefore, at this point to present a brief discussion on specular point theory.

3.3 Specular Point Theory

An attempt to compute the shadow-corrected scattered field from a perfect electric conducting (PEC) random rough surface was made by Kodis [Kodis, 1966] and Barrick [Barrick, 1968b]. In Kodis' work, an infinite PEC is illuminated by a plane wave and the equivalent surface current is the Kirchhoff current, i.e.

$$\vec{J}_s(\vec{r}_s) = \begin{cases} 2\hat{n} \times \vec{H}_i(\vec{r}_s) & \text{in the illuminated regions} \\ 0 & \text{in the shadowed regions.} \end{cases} \tag{3.4}$$

All multiple scattering effects are ignored as well as effects due to diffraction. The method of stationary phase is applied to the far-field equation for the scattered electric field to produce a result which depends on the specular slopes as well as the curvature at each specular point. The stationary phase (specular) points are simply those points whose slopes give rise to reflection in the direction of observation. These slopes are perpendicular to the incident ray. The ensemble average of $\vec{E}_s \cdot \vec{E}_s^*$ is taken by first summing the contributions from all specular points and then taking the average. A expression for the radar cross section results which is proportional to the product of the average of the number of specular points per unit area and the average of the product of the principle radii of curvature at the specular points.

The correct analysis, provided by Barrick and Bahar [Barrick and Bahar, 1981] involves computing the average of the product of the number of specular points and the principal radii of curvature and not the product of the average of these two quantities. This result was contrary to that previously published by Barrick [Barrick, 1968b], whose result was based on Kodis' work [Kodis, 1966].

Barrick obtained

$$\sigma(\widehat{k}_i,\ \widehat{k}_s) = \pi\ sec^4\gamma\ \ \mathrm{p}(\mathrm{h_{xsp}},\mathrm{h_{ysp}})\ \mathrm{R}^2\ \mathrm{P}_2(\widehat{k}_i,\ \widehat{k}_s\ |\mathrm{h_{xsp}},\mathrm{h_{ysp}}), \tag{3.5}$$

where $\sigma(\widehat{k}_i,\ \widehat{k}_s)$ is the radar cross section in the $\widehat{k}_s$ direction when the surface is illuminated by a plane wave traveling in the $\widehat{k}_i$ direction, γ is the angle between the normal to the mean surface height and the normal to the specular points, $\mathrm{h_{xsp}}$ and $\mathrm{h_{ysp}}$ are the specular slopes in the $\widehat{x}$ and $\widehat{y}$ directions, R is the reflection coefficient, and $\mathrm{P}_2(\widehat{k}_i,\ \widehat{k}_s \mid \mathrm{h_{xsp}},\mathrm{h_{ysp}})$ is what is commonly referred to as the bistatic shadowing function. $\mathrm{P}_2(\widehat{k}_i,\ \widehat{k}_s \mid \mathrm{h_{xsp}},\mathrm{h_{ysp}})$ is equal to the joint probability that a point on the surface will be illuminated by the incident ray, $\widehat{k}_i$, and the reflected ray, $\widehat{k}_s$, will leave the surface without striking the surface a second time, given the point has a specular slope. Barrick made no attempt at computing the joint probability function $\mathrm{P}_2(\widehat{k}_i,\ \widehat{k}_s\ |\mathrm{h_{xsp}},\mathrm{h_{ysp}})$. An attempt at computing this joint probability function was made by Wagner in 1967 [Wagner, 1967], ironically before the work of Barrick, who later justified its incorporation into the scattering model. Wagner's result is probably the most used and quoted work concerning shadowing in the literature and thus warrants a brief discussion. His work along with that of Smith [Smith, 1967b] and Hardin [Hardin, 1972] will be discussed next.

3.4 Shadowing Due to Wagner, Smith, and Hardin

A year after the work of Kodis [Kodis, 1966], another investigator, R.J. Wagner [Wagner, 1967], developed what is now the accepted version of the shadowing function. Wagner's shadowing function differed from that of

Beckmann's in that it is a conditional shadowing function. It was defined such that

$S(\theta \mid \zeta_1, \zeta_1')$ = Probability that a point, (ζ_1, ζ_1'), on the surface is illuminated by a ray at θ (grazing) given the height, ζ_1, and slope, ζ_1', at that point. (3.6)

Wagner developed a differential equation for the shadowing function, similar to Beckmann's, and was able to solve it to give

$$S(\theta \mid \zeta_1, \zeta_1') = u(\eta_o - \zeta') \; exp\left\{ - \int_0^{\infty} q(\tau) d\tau \right\}, \tag{3.7}$$

where η_o is the slope of the incident ray, $u(\eta_o - \zeta_1')$ is the Heaviside step function, defined as

$$u(\eta_o - \zeta_1') = \begin{cases} 1 & \eta_o > \zeta_1' \\ 0 & \eta_o < \zeta_1' \end{cases} \tag{3.8}$$

and $q(\tau)d\tau$ = Pr[the surface crosses the incident ray in $(\tau, \tau + d\tau)$ | the surface does not cross in $(0, \tau)$; ζ_1, ζ_1']. $u(\eta_o - \zeta_1')$ accounts for the so called "self-shadowing" of a point by itself, since any point greater than the slope of the incoming ray will be shadowed.

The function $q(\tau)$ deserves some special attention. $q(\tau)$ is related to the first passage in time problem to which much time and energy has been devoted. Unfortunately, despite numerous attempts by various authors, no solution has been found except for a very few special cases, and approximations have to be made to obtain closed form solutions.

Wagner went on to make approximations in order to evaluate the shadowing function; however, it is difficult to see under what conditions these approximations are valid. Additionally, Wagner considered the case in which the shadowed point and the shadowing point are independent, an assumption which becomes valid as the two points move sufficiently far away from each other. In doing so, he was able to obtain a closed form solution to $S(\theta \mid \zeta_1, \zeta_1')$. His result seems to compare well with Brockelman and Hagfors for the range of data which they provided (above 20 deg. grazing). Wagner then attempted to remove this restriction by calculating $q(\tau)$ as $\tau \rightarrow 0$ and as $\tau \rightarrow \infty$ and adding the two asymptotic expressions together. Since the height and slope at the shadowing point and the shadowed point (the origin) are uncorrelated, an analytic expression was possible in the limit as $\tau \rightarrow \infty$. As $\tau \rightarrow 0$, the integral was obtained by expanding the autocorrelation function in a Taylor series around the point η_o, which supposedly maximizes the function $q(\tau)$, and performing term by term integration. The calculations were straightforward but very tedious, and he was forced to expand to the seventh term in the series in order to get a leading order result.

A similar analysis to Wagner's uncorrelated shadowing function (we will call the shadowing function uncorrelated if it neglects correlation between the shadowing point and the shadowed point) was undertaken by Smith a short time later in 1967 [Smith, 1967b]. The same integral equation which defined $S(\theta \mid \zeta_1, \zeta_1')$ in Wagner's paper was used here, only the manner in which $q(\tau)$ was evaluated changed. Smith, like Wagner, neglected correlation between the shadowed point and the shadowing point in order to get closed form results.

Smith carried out his results for Gaussian statistics, an assumption later removed by Brown [Brown, 1980a], and which agree with the numerical simulations of Brockelman and Hagfors.

Along the same lines as Wagner, Hardin [Hardin, 1972] considered the case of a point source fixed at a finite height, h, above a point, r_o, on the surface. Two important points are worth noting for this case. The first is that for increasingly shorter distances between the shadowed point and the shadowing point, surface correlation plays a greater role, and neglect of this correlation, according to Hardin, leads to a "significant overestimation of the probability of illumination." In terms of Wagner's work

$$S(\theta \mid \zeta_1, \zeta_1') = \lim_{\substack{h \to \infty \\ r_o \to \infty}} S(\theta(r_o, h) \mid \zeta_1, \zeta_1'), \tag{3.9}$$

where $S(\theta(r_o, h) \mid \zeta_1, \zeta_1')$ is the probability that a point on the surface will be illuminated by a ray at a grazing angle of $\theta(r_o, h)$ given the height, ζ_1, and slope, ζ_1' at that point.

Hardin claimed he obtained a slightly better agreement with Brockelman and Hagfors results, under the no correlation case, than Wagner since he retained more information about the shadowed point. However, as he points out, when Wagner then attempted to remove the no correlation restriction, by the methods mentioned above, he obtained excellent results.

We end this section on the theoretically predicted shadowing functions by a discussion of a more detailed analysis of shadowing theory developed by Bass and

Fuks [Bass and Fuks, 1979]. General expressions before and after the no correlation assumption are given in order to more clearly understand the problem. Calculations are performed in order to try to understand when the no correlation assumption breaksdown.

3.5 Distribution of Illuminated Heights and Slopes

A detailed analysis and generalization of shadowing functions was provided by Bass and Fuks in 1979 [Bass and Fuks, 1979]. An integral for the density of the heights and slopes of the illuminated part of the surface at a given angle of incidence was given to be

$$W_e(\zeta,\gamma;\psi) = W(\zeta,\gamma)\; u(tan\psi - \gamma)\; exp\left\{ - \int_0^{\infty} P_\tau(A \mid B) d\tau \right\}, \qquad (3.10)$$

where $W(\zeta,\gamma)$ is the joint PDF for the heights, ζ, and slopes, γ; $tan\psi$ is the slope of the incoming ray, and $P_\tau(A \mid B)$ = Pr[at a distance τ away, the incoming ray intersects the surface given there are no intersections in (0,τ)]. We note again that $P_\tau(A \mid B)$ is related to the first passage of time problem to be discussed in detail later in this thesis. Notice the similarly between equations (3.7) and (3.10). $P_\tau(A \mid B)$ is then rewritten as

$$P_\tau(A \mid B) = \frac{P_\tau(A,B)}{P_\tau(B)}, \qquad (3.11)$$

using Baye's theorem, where $P_\tau(B)$ can be expressed as

$$P_\tau(B) = \lim_{\substack{\Delta\to 0\\ N\to\infty}} \prod_{n=1}^{N=\tau/\Delta\tau} \left(\int_{-\infty}^{\zeta + n\Delta\tau tan\psi} d\zeta_n \int_{-\infty}^{\infty} d\gamma_n \right) W_{2N+2}(\zeta,\gamma;\zeta_1;\gamma_1;...;\zeta_N,\gamma_N), \tag{3.12}$$

where $W_{2N+2}(\zeta,\gamma;\zeta_1;\gamma_1;...;\zeta_N,\gamma_N)$ is the joint probability density function of heights $(\zeta,\zeta_1,...,\zeta_N)$ and slopes $(\gamma,\gamma_1,...,\gamma_N)$ at the points $\tau = 0, \Delta\tau, 2\Delta\tau, ...,N\Delta\tau$. $P_\tau(A,B)$ is given as

$$P_\tau(A,B) = \lim_{\substack{\Delta\to 0\\ N\to\infty}} \prod_{n=1}^{N} \left(\int_{-\infty}^{\zeta + n\Delta\tau tan\psi} d\zeta_n \int_{-\infty}^{\infty} d\gamma_n \right)$$
$$* \int_{tan\psi}^{\infty} d\gamma''(\gamma'' - tan\psi) W_{2N+4}(\zeta,\gamma;\zeta_1;\gamma_1;...;\zeta_N,\gamma_N;\zeta'' = \zeta + \tau\ tan\psi, \gamma''), \tag{3.13}$$

where ζ'' and γ'' are the height and slope at a distance τ from the shadowed point. Substituting these two quantities into the relation for $W_e(\zeta,\gamma;\psi)$ we obtain

$$W_e(\zeta,\gamma;\psi) = W(\zeta,\gamma)\ u(tan\psi - \gamma) \tag{3.14}$$

$$* \ exp\left\{ - \int_0^\infty d\tau \frac{\lim_{\substack{\Delta\to 0\\ N\to\infty}} \prod_{n=1}^{N} \left(\int_{-\infty}^{\zeta + n\Delta\tau tan\psi} d\zeta_n \int_{-\infty}^{\infty} d\gamma_n \right) \int_{tan\psi}^{\infty} d\gamma''(\gamma'' - tan\psi) W_{2N+4}}{\lim_{\substack{\Delta\to 0\\ N\to\infty}} \prod_{n=1}^{N=\tau/\Delta\tau} \left(\int_{-\infty}^{\zeta + n\Delta\tau tan\psi} d\zeta_n \int_{-\infty}^{\infty} d\gamma_n \right) W_{2N+2}} \right\},$$

which is the general form for the PDF of illuminated heights and slopes for a given angle of incidence.

Bass and Fuks go on to evaluate these integrals for the cases of intense surface shadowing ($tan\psi \ll \gamma_o$) and weak shadowing ($tan\psi \gg \gamma_o$), where γ_o denotes the RMS slope of the surface. Here they assume, like Smith [Smith, 1967b] and Wagner [Wagner, 1967], that the correlation between the heights and slopes at the shadowed and shadowing point is weak and thus can be neglected. An extremely simplified form results

$$W_e(\zeta,\gamma;\psi) = = W(\zeta,\gamma)\ u(tan\psi - \gamma)\ exp\left\{ -\int_0^\infty d\tau \int_{tan\psi}^\infty d\gamma''\ W(\zeta'' = \zeta + \tau\ tan\psi, \gamma'') \right\}. \tag{3.15}$$

Interestingly, Bass and Fuks perform an area defect calculation, which measures the projected area of the surface which is illuminated, to determine the range of validity of this assumption. They computed the difference between the projection of the area illuminated by the incident beam on a plane below, calculated by simple geometry (i.e. $S_o sin\psi$, S_o = the projection of the beam width on the plane below), and by using $W_e(\zeta,\gamma;\psi)$. They found that at both intense and weak shadowing the two agreed well, but as the angle of incidence approached intermediate angles (i.e. $tan\psi \simeq \gamma_o$) the area defect rose to as much as 0.3. Thus, about 30% of the surface that was considered to be illuminated was in fact located in shadow [Bass and Fuks, 1979].

This is perhaps explained by the fact that as the angle of incidence approaches grazing angles, the average distances between the shadowed points and the shadowing points increases, thus they become decorrelated. If the angle of incidence is very high, there is a high degree of correlation between these two points, but the shadowing is so weak as to not make a difference. Thus, intuitively, it appears that in the range of intermediate angles is where we see the no correlation assumption break down. We note that Bass and Fuks compare the illuminated area computed from the distribution of illuminated heights and slopes to a known result easily obtained from geometry and not the illuminated specular points. Thus, we still have no analytic expression which compares Wagner's conditional shadowing function to a known result. Bass and Fuks have basically disproven Beckmann's result for intermediate shadowing.

In addition to the numerous theoretical derivations for computing shadowing, we have encountered one paper in which shadowing measurements based on a true experiment (as opposed to a computer simulation) were performed. The single paper is perhaps testimony to the difficulty in carrying out the necessary experiments.

3.6 Experimental Shadowing Measurements

Experimental evidence by Welton, et al., [Welton, 1973] using a high contrast photographic technique shows qualitative agreement with the theoretically derived unconditional shadowing function, $S(\theta)$, given by Wagner. Welton illuminated various rough surfaces at relatively low grazing angles and measured the ratio of illuminated to total area using a camera with high contrast film and a planimeter. We note that although qualitative agreement was obtained here, Welton's result and Wagner's shadowing function are strictly not the same. Wagner's shadowing function is a ratio of the illuminated number of specular points to the total number of specular points and Welton's results were simply the ratio of illuminated to total area (as Beckmann's function was). Thus, the conclusions here can only be taken to mean that Wagner's function generally agrees with the experiment. To our knowledge no experiment has ever been performed to experimentally calculate Wagner's function exactly.

We now turn toward the work of Sancer [Sancer, 1969] which was given in the previous chapter.

3.7 Sancer's Approach to Calculating the Scattered Fields from Random Surfaces

The method of Kodis [Kodis, 1966] involved computing the single scatter (bounce) incoherent power by first using the stationary phase approximation on the far-field integral equation for the scattered field, summing the contribution from all specular points, forming the product of the summation with its complex conjugate and then computing the ensemble average. While this approach lead to physical insight into the scattering mechanism it is cumbersome and difficult to evaluate, particularly if one is considering doubly scattered rays.

Another approach to this problem was proposed by Sancer in 1969 (see Chapter 2 for a complete derivation along the same lines). Sancer reversed the order of the ensemble average with the integration over the surface. He considered the more general case of a bounded plane wave scattering from a dielectric/air interface. The Kirchhoff approximation for the current was modified by the shadowing functions and thus the joint probability density function in the ensemble average necessarily includes them. A high frequency approximation is then made which allows him to evaluate the joint PDF at the specular slopes. We note that the PDF contains only the slopes and the shadowing functions as random variables, whereas Kodis' work necessarily involves surface curvature terms as well. Sancer's result is equivalent to the work of Barrick and Bahar [Barrick and Bahar, 1981] but extends his work to include the case of bistatic shadowing for the cases of (1) backscatter, and (2) forward scattering, when the angle of incident and angle of observation are well separated.

We note here that the shadowing function Sancer used is conditional, unlike that of Beckmann's [Beckmann, 1965b], and unlike the $S(\theta)$ used by Wagner [Wagner, 1967], but like that of $S(\theta \mid \zeta_1')$ developed by Wagner as an intermediate step toward computing $S(\theta)$. Sancer proceeds to calculate $p\left(-\frac{q_1}{q_3}, -\frac{q_2}{q_3}\right)$ and $P\left(\hat{k}_i, \hat{k}_s \middle| -\frac{q_1}{q_3}, -\frac{q_2}{q_3}\right)$, where $-\frac{q_1}{q_3}$ and $-\frac{q_2}{q_3}$ are the specular slopes (see eq. 2.27), assuming that the PDF of slopes is Gaussian and relying on the results due to Smith [Smith, 1967b], who neglects correlation between the shadowing point and the shadowed point. $p\left(-\frac{q_1}{q_3}, -\frac{q_2}{q_3}\right)$ is the joint PDF of the slopes evaluated at the specular slopes and $P\left(\hat{k}_i, \hat{k}_s \middle| -\frac{q_1}{q_3}, -\frac{q_2}{q_3}\right)$ is the joint probability that a point on the surface will be illuminated by a ray, $\hat{k}_i$, and the subsequent reflecting ray, $\hat{k}_s$, will reach the observer without intersecting the surface again given that the point is a specular point.

Sancer's work is the first to directly and correctly incorporate the shadowing functions into the diffraction integral for the scattered field. An extension of Sancer's paper which includes scattering up to a double bouncing ray was carried out by Jin in 1988 [Jin, 1988].

3.8 Correlation Between the Incident and Scatter Shadowing Function

We stated in the preceding paragraphs that Sancer's bistatic shadowing function was only valid when (1) we were in the backscatter direction or (2) the incident and reflected rays are not too close together. This is due to the fact that we do not know, at this point, how the incident shadowing function and the scatter shadowing function are correlated. We simply know that $S(\theta_s \mid \zeta_1') = 1$ in the backscatter direction, and presumably, although still unclear, that $S(\theta_i, \theta_s \mid \zeta_1') = S(\theta_i \mid \zeta_1') S(\theta_s \mid \zeta_1')$ if the probability of the scattered ray leaving the surface conditioned on the probability of the incident ray illuminating that point with a specular slope is equal to the probability that a point is visible from the observer, along $\hat{k}_s$, given the point is specular, i.e. $S(\theta_s | \theta_i, \zeta_1') = S(\theta_s | \zeta_1')$.

The scatter shadowing function is equal to unity in the backscatter direction since we are given that the point is illuminated by the incident ray and the slope is specular. Therefore, we are guaranteed, with probability equal to one, that the reflected ray will leave the surface in the opposite direction from which it came. We use this fact to simplify the results, since $\mathrm{S}(\theta_i, \theta_s \mid \zeta_1') = \mathrm{S}(\theta_i \mid \zeta_1')$ in this case. Furthermore, we know that we can write

$$
\begin{aligned}
S(\theta_i, \theta_s \mid \zeta_1') &= \frac{S(\theta_i, \theta_s, \zeta_1')}{P(\zeta_1')} \\
&= \frac{S(\theta_s | \theta_i, \zeta_1') \, S(\theta_i \mid \zeta_1') \, P(\zeta_1')}{P(\zeta_1')} \\
&= S(\theta_s | \theta_i, \zeta_1') \, \mathrm{S}(\theta_i \mid \zeta_1'). \qquad (3.16)
\end{aligned}
$$

Thus, not only does the scatter shadowing function depend on the specular slope, as does the incident shadowing function, we also see a dependence on the incident shadowing event. This effect is seen in Sancer's paper [Sancer, 1969] in equations (54), (55), and (56). Fuks [Fuks, 1976] concludes, after analyzing this effect, that the incident and scattered shadowing functions become independent rapidly as the incident and observation angles move apart. Fuks further states that the scattering cross-section can increase by as much as a factor of two in the backscatter direction and terms this effect backscattering reinforcement, not to be confused with backscatter enhancement caused by multiple scattering effects.

To attempt to explain this more clearly, as Fuks' paper is difficult to follow, consider first the backscatter case. It is clear that $S(\theta_i, \theta_s \mid \zeta_1')=S(\theta_i \mid \zeta_1')$ since $S(\theta_s \mid \theta_i, \zeta_1')=1$. Thus, knowing that a point on the surface is illuminated by a ray traveling with an incident angle θ_i provides us with a great deal of information when assessing the probability of that same ray reflecting off the surface in the direction of observation. We say in this case that the incident shadowing function and the scatter shadowing function are perfectly correlated. It is important to mention here that since we are only interested in the case in which the slopes are specular, that is, we are given a point on the surface that has a specular slope, that $S(\theta_i, \theta_s \mid \zeta_1')$ is equivalent to the joint probability that a point will be illuminated from both the source direction and the observation direction given it has a specular slope. It is clear that $S(\theta_s \mid \theta_i, \zeta_1') \neq S(\theta_s \mid \zeta_1')$ since $S(\theta_s \mid \theta_i, \zeta_1')=1$ and $S(\theta_s|\zeta_1') \neq 1$, where $S(\theta_s|\zeta_1')$ is the probability that a point is visible from the observer given it has a specular slope. Thus, we see for the backscatter case, at least, that the shadowings are not independent. However, as the incident and scattering angles move farther apart the dependence, or lack thereof, changes.

The question of correlation between shadowings boils down to the difference between $S(\theta_s\ |\theta_i,\ \zeta_1')$ and $S(\theta_s\ |\zeta_1')$, and we must now ask ourselves what *additional* information is provided to $S(\theta_s\ |\theta_i,\ \zeta_1')$ by knowing that a point is visible from the source? It is also important to note what we are not saying. It is clear that the angle of incidence affects $S(\theta_s|\ \zeta_1')$ since it also directly affects the specular slopes upon which this function is conditioned. However, it is not a question of dependence between the incident and scattered rays, as they are perfectly correlated, but a question of dependence between shadowings. $S(\theta_s\ |\zeta_1')$ is still affected by the incident angle through the specular slopes, and its affect will be felt regardless of the dependence of incident shadowing. Another way to look at this is to ask what effect does knowing that a point is illuminated by some angle, θ_i, have on the probability of visibility from the observer? In the backscatter case, it is clear that knowing the point was illuminated by the source has a profound affect upon our prediction of whether this ray will also be visible from the observer. However, it is unclear what effects incident shadowing will have on scatter shadowing when the incident and scattering angles grow farther apart. We can see when θ_i approaches 90 deg. $S(\theta_s\ |\theta_i,\zeta_1') = S(\theta_s\ |\zeta_1')$ since knowing a point was illuminated from above provides no *additional* information in determining $S(\theta_s|\ \theta_i,\ \zeta_1')$ (because all points are visible from above). Thus, the shadowings are said to be independent. Sancer states that as the incident and observer angles move farther apart the shadowings become independent. However, there appear to be certain transition regions in which we move from dependent shadowings to independent shadowings, and it does not seem, as Fuks states, that the incident and scattered shadowing functions become independent rapidly as the incident and observation angles move apart. This is easily proven by examining the case in which the incident angle is near grazing and the observation angle is at away from grazing but less than 90 degrees in the same plane. Thus, we conclude that Sancer is probably correct in his assumption about dependence on shadowing relying on the separation distance between incident and observation angles but clearly more work needs to be done, as this is only an intuitive hypothesis yet to be proven. We still are uncertain how far apart the incident and observation angles need to be before the independence assumption can be made.

3.9 The Validity of the Shadowing Functions

Before any discussion of shadowing functions takes place, one must, out of necessity, justify the incorporation of those functions into the analysis for calculating the currents induced on the surface and the fields resulting from those currents. One common method of calculating the scattered fields from a surface is to use the vector potential approach. This technique replaces the surface by equivalent currents which satisfy the boundary conditions for the fields. These currents are located in free space, and once known, the scattered field can, in principle, be calculated using the standard diffraction integral. The difficulty, of course, is in computing the surface currents. For a PEC surface, the equivalent magnetic current is zero, due to the requirement of the continuity of the tangential component of the electric field across the interface. Thus, we are left with the electric current density, $\vec{J}_S$, which is defined by the following equation

$$\vec{J}_S(\vec{r}) = 2\hat{n}(\vec{r}) \times \vec{H}^i(\vec{r}) + 2\hat{n}(\vec{r}) \times \oint_{S_o} \vec{J}_S(\vec{r}_o) \times \nabla_o G(|\vec{r} - \vec{r}_o|) dS_o. \quad (3.17)$$

This equation is called the magnetic field integral equation (MFIE). We note that this equation has the benefit of being able to be iterated, to successively improve upon the leading term in the series usually taken to be $2\hat{n}(\vec{r}) \times \vec{H}^i(\vec{r})$. The first term $[2\hat{n}(\vec{r}) \times \vec{H}^i(\vec{r})]$ leads to what is called the Kirchhoff approximation, which neglects shadowing and multiple scattering effects, and becomes valid when each and every point on the surface has a curvature which is large compared to a electromagnetic wavelength (tangential plane approximation), and hence is often called "gently undulating." The second term, an integral over the surface, is recognized to be associated which multiple scattering and effects due to diffraction.

Our purpose here is to review the papers which justify both the incident and scatter shadowing functions associated with correcting the Kirchhoff approximation. The shadowing function, as previously mentioned, converts a single mathematical ray which can penetrate the surface to a real ray which bounces on the surface.

The Kirchhoff approximation is often used as a starting point to iterate the MFIE, where the equation is then solved using the stationary phase approximation ($k_o \rightarrow \infty$). This analysis was carried out for incident shadowing originally by Liska and McCoy [Liska and McCoy, 1982] for the one dimensional surfaces and later by Brown [Brown, 1984b] for the two dimensional case. They successfully showed that the current located in shadow from the direct incident field was equal to zero, thus validating the use of the incident shadowing function. It is interesting to note a remark made by Liska and McCoy regarding a reflected ray off the surface. " . . . if a reflected signal intersects a surface once in passing from the reflection point to the field point ; . . . the reflected signal; will be canceled by a contribution from a second iteration; i.e. a reflection followed by a shadowing." However, in a detailed analysis performed by Brown [Brown, 1984b], it was found that this simple iterating process, using the stationary phase method, did not automatically produce the correct currents on the surface and likewise the scattered field was not zero in the specular direction of a scattered ray when that ray was blocked by some other portion of the surface. Thus, it appeared that the scatter shadowing function could not be rigorously derived from the MFIE.

3.10 The First Passage in Time Density Function

The first passage in time density function is closely related to the shadowing of a random rough surface, and thus, it is important that we recognize the great body of literature surrounding this function and methods to compute it. Its relationship was first provided by Beckmann in 1965 with his paper on shadowing theory, although the name first passage did not appear in the engineering literature until Wagner's paper in 1967.

According to Siegert [Siegert, 1951], the first passage in time problem has been studied for many years, dating back as far as 1915 with the work of Schrödinger and Smoluchowski. "Schrödinger and Smoluchowski found the probability that a free particle in Brownian Motion [Wang, 1945] in a medium of high viscosity reaches a marker at a for the first time in $(t, t + dt)$ after starting from y_o at $t = 0$." Other applications include Rice's work on signals corrupted by random noise and Rainal's paper, based on Rice's work, which computes the first and second passage times of a sine wave plus noise crossing a constant level [Rainal, 1988; Rice, 1945] . "This random process is of interest since it serves as a realistic model for the output of the IF amplifier of a typical radio or radar receiver during the reception of a sinusoidal signal immersed in Gaussian noise" [Rainal, 1968]. "The first passage problem in random vibration is concerned with the determination of the probability that a random process will exceed some critical level within a specified time interval. In practical situations the random process might represent the stress in some critical component of a structure which is subjected to random environmental forces, while the critical level might be yield stress or buckling load" [Langley, 1988]. The first passage problem also occurs in biology with the modeling of neuron activity [Kostyukov, 1981].

We begin by reviewing some important papers and provided the terminology used in this area. No attempt will be made at providing a comprehensive list, although, we believe we have cited some of the more well known articles and also some of the more useful articles pertaining to the shadowing problem.

Two literature surveys are worth mentioning. The first, by Blake and Linsey, published in 1973 [Blake and Linsey, 1973], and more recently Abrahams published in 1986 [Abrahams, 1986]. Much attention is given to Markov [Tuchwell, 1984; Lindenburg, 1975; Mehr and McFadden, 1965; Darling and Siegert, 1953] and Wiener [Shepp and Slepian, 1976] processes and various statistics surrounding the level crossing problem [Sato and Tanabe, 1979; Longuet-Higgins, 1956]. Since we are concerned with non-Markov processes (i.e. a Gaussian PDF with a Gaussian autocorrelation function) a detailed look at these papers will not be given. It is informative, however, at this point to classify the different types of level crossing problems to facilitate researchers starting out in this area who face a vast compilation of literature.

The three broad areas include (1) the one-sided barrier problem [Slepian, 1962], which involves the probability that process "x(t) first crosses a constant level, a, or a curve $a(t)$, during an interval of length τ; the probability that x(t) does not cross the fixed level during an interval of length τ, and; the probability distribution of the maximum of x(t) over an interval of length τ" [Abrahams, 1986] (2) The two-sided barrier problem, which "concerns the probability that x(t) first exits a region (a,b) during an interval $(\tau, \tau + d\tau)$, and; the probability distribution of the difference between the maximum of x(t) over an interval and its global minimum" [Abrahams, 1986]. (3) The zero-crossing problem, which "concerns the probability distribution of the time between zero crossings and the number of zero crossings in an interval" [Abrahams, 1986] Our interest lies in the one-sided barrier problem since we are concerned with the probability that surface x(t) crosses a ray, a(t), in an interval (0,τ).

Our problem can be though of as Gaussian noise crossing a ramp, where the noise may be the surface of the sea, and the ramp the incident or exit ray. To our knowledge, this problem has never been completely solved. In fact, in only a few special cases has a closed form result been obtained to the FPIT problem [Abrahams, 1984; Slepian, 1961], and usually some sort of approximation is necessary to get an answer [Rainal, 1990, 1968; Ricciardi and Sato, 1983]. Unfortunately, the approximations have to do with mathematical simplicity or necessity and cannot be made based on physical reasoning.

One paper, by Ricciardi and Sato [Ricciardi and Sato, 1983], based on Rice's work [Rice, 1945], examines the case of Gaussian noise crossing a time varying boundary. The first passage density is expressed as an infinite series of integrals with known integrands. The integrands themselves, however, are integrals of a conditional PDF of heights and slopes over a constrained domain. Thus, only the first term in the series can be obtained in closed form and one must resort to numerical methods to obtain results. The terms in the series provided improved lower and upper bounds to the first passage density and under certain conditions the first term in the series may provide an accurate approximation to the FPIT density function. To the best of our knowledge there is no other way to compare the approximations proposed by Wagner [Wagner, 1967] without making other approximations to the exact FPIT density function which are also unjustifiable. It is regrettable that this is the case, since computation of higher order terms in the series requires multidimensional integration, and thus we are limited by the speed of our computer as to how many terms we can carry out. But it is unclear at this point what effect these higher order terms have on the result and so we are not left with much choice.

Ricciardi and Sato [Ricciardi and Sato, 1986] compute the second term in the infinite series for the case in which a Gaussian process with Butterworth covariance crosses a constant level. It is important to note that their densities are conditioned on the height (amplitude) and thus they are forced to assume that the process begins either above or below the constant level at t=0. This is necessary since they concern themselves with computing the density of the upcrossing (or downcrossing) times. Since they have no knowledge of the slope at the initial time, at t=0, as with random noise, this must be the case. Fortunately for us, we do have knowledge of the slope, that is, we know the initial slope is specular, and so we are able to extend their work to include the important case in which the surface (or noise) intersects the ray (ramp) at the initial point (t=0). It follows then that our conditional densities will depend not only on the initial height but also on the initial slope. Furthermore, we include the case in which the correlation function is Gaussian and are able to compute three terms in the series. This, to our knowledge, has never been done before.

An extensive examination and derivation of the infinite series, similar to that given by Ricciardi and Sato [Ricciardi and Sato, 1986], will be presented in Chapter 4. Despite its extensive use, we were unable to find a clear and complete derivation of Rice's result (the closest being given in [Bass an Fuks, 1979]) from which Ricciardi and Sato based their work. This is the topic of our next chapter.

4.0 SHADOWING THEORY

4.1 Introduction

In this chapter we provide a mechanism for calculating the shadowing functions discussed in Chapter 2. We specialize the result here to backscatter and calculate the incident shadowing function. As previously mentioned, the scatter shadowing function is equal to one (1) in the backscatter direction. The analysis extends to bistatic shadowing when the two shadowings are independent of each other (2.48). Simply substituting the incident angle with the observation angle allows us to obtain the scatter shadowing function.

We are interested in computing the probability that a point on the surface will be directly illuminated by the source, given that the point has a specular slope. This need is a direct result of equations (2.43) and (2.45). We assume that a high frequency plane wave is incident on a conducting surface. We have made no mention so far of the distribution of the surface, correlation, or surface statistics, except to say that the surface is isotropic and stationary. "A rough surface is isotropic if the statistics of the surface are independent of direction along the surface" [Ogilvy, 1991]. Thus, the correlation on the surface is independent of direction, and is a function only of the distance between two points. Stationarity implies "that the probability of a surface point being at height $h(\boldsymbol{r})$ is independent of $\boldsymbol{r}$. Similarly, any statistical properties dependent on two or more surface points, such as height correlations, depend only on the vector joining these points and not their absolute positions" [Ogilvy, 1991]. The stationarity assumption has already been made in going from equation (2.42) to (2.43), where we assumed that the bistatic shadowing function, as well as the distribution of the slopes, is independent of location. As a result of these assumptions, shadowing becomes independent of the direction of illumination and the analysis simplifies considerably.

For the special case of backscatter, the specular points are those points whose slopes are perpendicular to the direction of the incident ray. It is well known that in the high frequency limit ($k_o \rightarrow \infty$) the major contribution to the scattering process comes from neighborhoods around the specular points (i.e. those points which satisfy Snell's law of reflection). This helps us, not only in simplifying the result to (2.43), but also in computing the shadowing. As we mentioned, in this chapter we will extend the results of Ricciardi and Sato [Ricciardi and Sato, 1986] to include the case of a random process (surface) crossing a level (ray) when the process and the level intersect at the origin.

Previously, work in this area [Ricciardi and Sato, 1986] specialized the level crossing problem to the case in which the process would be either above or below the level at the origin (Figure 10). Since we know that the last point which intersects the incoming ray, as one moves along the ray toward the surface, is given to be specular, the slope of the random process must be perpendicular to the ray and hence heading downward as one moves in the direction of the source (Figure 11). This fact can be utilized to extend Ricciardi and Sato's work to help us solve our problem.

4.2 Derivation of the Incident Shadowing Function in terms of the First Passage in Time Density Function.

Our goal in this section is to compute the probability that a point on the surface is illuminated by an incoming ray, $\widehat{k}_i$, given that the point has a specular slope [Sancer, 1969, Smith, 1967]. We will denote the specular slope as ζ'_{osp} and the height at that point as ζ_{osp}. The subscript "o" denotes the fact that the height and slope are at the origin and the subscript "sp" the fact that the point has a specular slope. We do not break the slope up into partial derivatives of the height in the $\widehat{x}$ and $\widehat{y}$ directions at this point, as in (2.36), rather we imply by ζ'_{osp} a derivative with respect to the variable t. Of course, this derivative can be expressed in terms of the slope in the $\widehat{x}$ and $\widehat{y}$ directions by applying the chain rule. We define a coordinate system such that our axis, t, is the projection of the incoming ray on the plane below (z=0 plane). The origin of the t axis is at the last intersection of the incoming ray with the surface as one moves from the source to the surface.

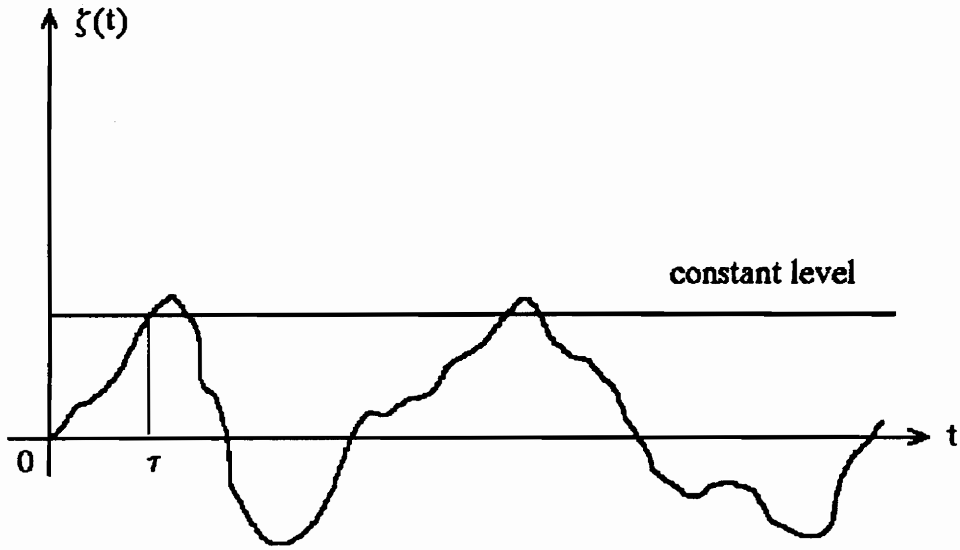

Figure 10: A Random Process Crossing a Constant Level Starting from Below the Level at t=0.

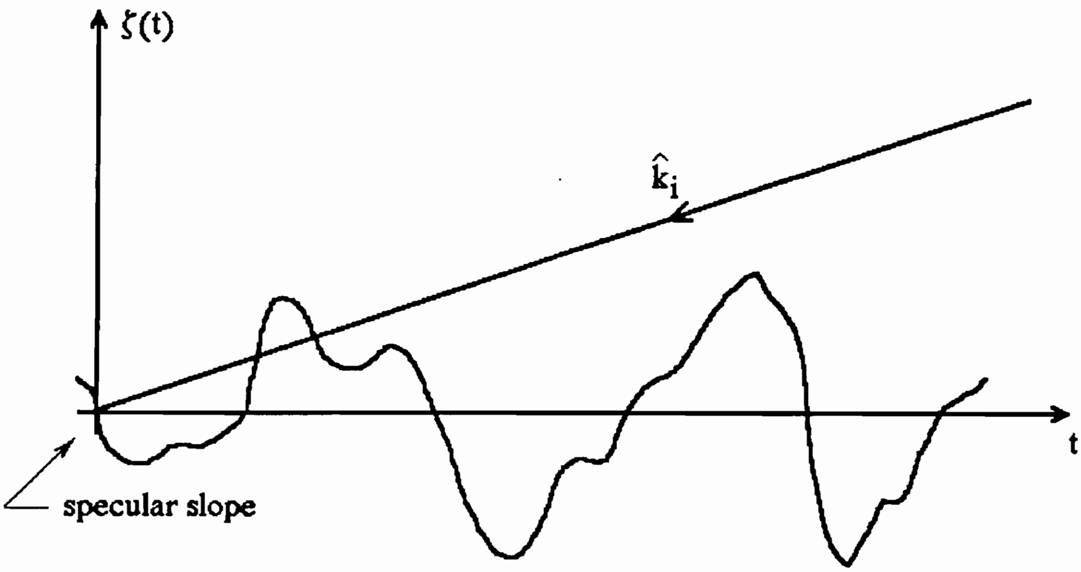

Figure 11: A Random Process Crossing a Ray (Ramp)

Although we desire the probability of illumination for a point on the surface, given that the point has a specular slope, in carrying out the actual calculations we must first compute the intermediate step of computing this probability conditioned not only on the slope but on the height as well. This point then, defined to be at the origin, uniquely defines this point as the last one which intersects the incident ray. After this probability is computed, the height then takes on values commensurate with its distribution and we arrive at

$$P(\hat{k}_i \,|\zeta'_{osp}) = \int_{-\infty}^{\infty} d\zeta_{osp}\, P(\hat{k}_i \,|\zeta_{osp},\zeta'_{osp})\; p(\zeta_{osp}), \tag{4.1}$$

where $p(\zeta_{osp})$ is the PDF the the specular heights. The integrand can then be written as [Wagner, 1967]

$$P(\hat{k}_i \,|\zeta_{osp},\zeta'_{osp}) = \lim_{\tau\to\infty} P(\hat{k}_i \,|\zeta_{osp},\zeta'_{osp};\, \tau), \tag{4.2}$$

where $P(\hat{k}_i \,|\zeta_{osp},\zeta'_{osp};\, \tau)$ is equal to the probability that $\zeta(t)$ does not cross the ray $\hat{k}_i$ in $(0,\tau)$. This is understood to be a local shadowing function valid in $(0,\tau)$, where τ is an arbitrary point on the t-axis. Equivalently, $P(\hat{k}_i \,|\zeta_{osp},\zeta'_{osp})$ can be written as

$$P(\hat{k}_i \,|\zeta_{osp},\zeta'_{osp}) = \lim_{\tau\to\infty}[1\text{-Probability } \zeta(t) \text{ does cross the ray, } \hat{k}_i\text{, in } (0,\tau)], \tag{4.3}$$

using the law of total probability. In order to derive this probability in terms of the first passage density function we now segment the interval, $(0,\tau)$ [Bass and Fuks, 1979] into n equally spaced sections such that

$$(0,\tau) = (0,\tau_1) \cup (\tau_1,\tau_2) \cup (\tau_2,\tau_3) \cup \ldots \cup (\tau_{n-1},\tau_n)$$

$$= (0,\ \tau_1) \bigcup_{i=1}^{n-1} (\tau_i,\ \tau_{i+1}), \qquad (4.4)$$

where $\tau_i - \tau_{i-1} = \Delta\tau_i$ and $\tau_n = \tau$. As we divide the fixed interval into an increasingly large number of equally spaced subdivisions (the points do not have to be equally spaced but it makes the explanation easier to follow) we see that as $n \to \infty$ $\Delta\tau_i = \Delta\tau = \tau_i - \tau_{i-1} \to d\tau$, an infinitesimal length on the t-axis, and we move from a discretized interval to a continuous one.

Since the surface crosses the ray from above at the specular point at t=0, as we move from left to right on the t axis, we are assured that next time the random surface crosses the ray, if it does, it will be from below (i.e. an upcrossing). If we denote the event that the surface crosses the ray from above in $\Delta\tau$ at τ_{i-1} as $\delta_i=1$, and as $\delta_i=0$ if no upcrossing occurs (an example is given in Figure 12) [Bass and Fuks, 1979], then the probability of a point at t=0 being shadowed can be expressed as:

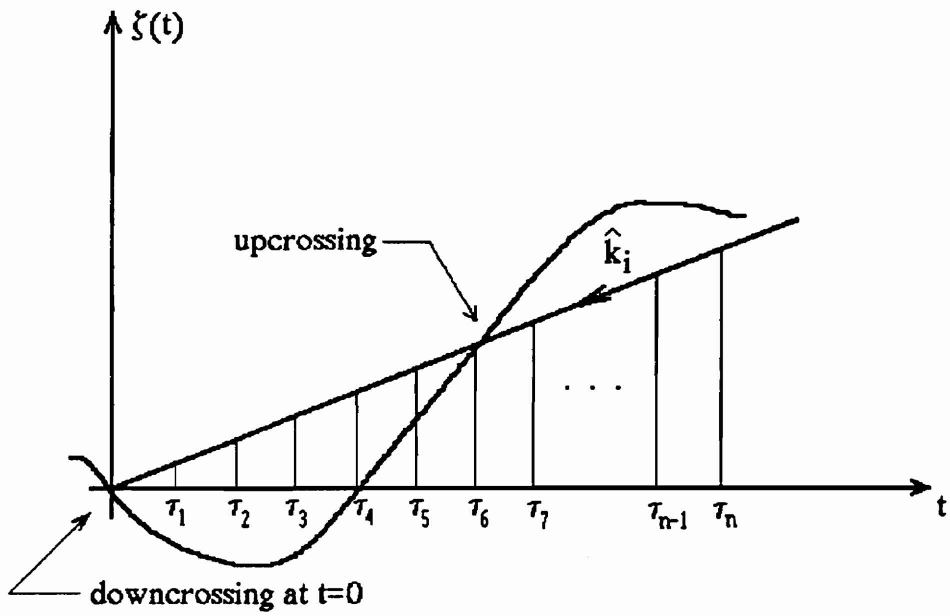

Figure 12: The Event (δ=0, . . ., δ_6=0, δ_7=1, δ_8=0, . . ., δ_n=0)

$$1\text{-}P(\widehat{k}_i\,|\zeta_{osp}, \zeta'_{osp}) = \lim_{\tau\to\infty}\lim_{n\to\infty}[\,P(\delta_1{=}0,\ \delta_2{=}0,\ldots,\delta_{n\text{-}1}{=}0,\delta_n{=}1\,|\zeta_{osp}, \zeta'_{osp})$$

$$+\,P(\delta_1{=}0,\ \delta_2{=}0,\ldots,\ \delta_{n\text{-}2}{=}0,\ \delta_{n\text{-}1}{=}1,\delta_n{=}0\,|\zeta_{osp}, \zeta'_{osp})$$

$$+\,P(\delta_1{=}0,\ \delta_2{=}0,\ldots,\ \delta_{n\text{-}2}{=}0,\ \delta_{n\text{-}1}{=}1,\delta_n{=}1\,|\zeta_{osp}, \zeta'_{osp})$$

$$+\ldots+\ldots$$

$$+\,P(\delta_1{=}1,\ \delta_2{=}1,\ldots,\delta_{n\text{-}1}{=}1,\delta_n{=}1\,|\zeta_{osp}, \zeta'_{osp})]. \qquad (4.5)$$

We note that the probability of a downcrossing is an assured event if an upcrossing occurs, and therefore it is sufficient to consider only the probability of of having an upcrossing in each of the intervals. We have also assumed that only one upcrossing can occur in each of the n intervals, but since we can make the interval spacing as small as we like, this poses no problem in the analysis.

The limit as $n\to\infty$ allows us to pass from a discretized interval to a continuous one. We choose the method of segmenting the interval to more clearly demonstrate the role of the first passage density in shadowing theory. To put this into more compact form, we let,

A_{τ_i} = the event that the surface crosses the ray, $\widehat{k}_i$, in $(\tau_i,\ \tau_i + \Delta\tau)$, and
B_{τ_i} = the event that the surface does not cross the ray, $\widehat{k}_i$, in $(0\ ,\ \tau_i)$.

We can express (4.5) as

$$1\text{-}P(\widehat{k}_i \,|\zeta_{osp}, \zeta'_{osp}) = \lim_{\tau\to\infty} \lim_{n\to\infty} \sum_{t=\tau_o}^{t=\tau_{n-1}} P(A_t \cap B_t \,|\zeta_{osp}, \zeta'_{osp}), \qquad (4.6)$$

where we have used the law of total probability. To be more explicit,

$$P(A_{\tau_{n-2}} \cap B_{\tau_{n-2}} \,|\zeta_{osp}, \zeta'_{osp}) = P(\delta_1{=}0,\ \delta_2{=}0,\ldots,\ \delta_{n-2}{=}0,\ \delta_{n-1}{=}1,\ \delta_n{=}0 \,|\zeta_{osp}, \zeta'_{osp})$$
$$+\ P(\delta_1{=}0,\ \delta_2{=}0,\ldots,\delta_{n-2}{=}0,\ \delta_{n-1}{=}1, \delta_n{=}1 \,|\zeta_{osp}, \zeta'_{osp}),$$

for example.

Taking the limit as $n\to\infty$ and $\Delta\tau_i\to 0$, where we have $\tau_o\to 0$ (t=0) , and $\tau_n\to\tau$, we arrive at

$$1\text{-}P(\widehat{k}_i \,|\zeta_{osp}, \zeta'_{osp}) = \lim_{\tau\to\infty} \int_0^{\tau} g[S(t), t \,|\zeta_{osp}, \zeta'_{osp})]\, dt \qquad (4.7)$$

and

$$P(\widehat{k}_i \,|\zeta_{osp}, \zeta'_{osp}) = 1 - \int_0^{\infty} g[S(t), t \,|\zeta_{osp}, \zeta'_{osp})]\, dt, \qquad (4.8)$$

and finally

$$P(\widehat{k}_i \,|\zeta'_{osp}) = 1 - \int_{-\infty}^{\infty} d\zeta_{osp} \left\{ \int_0^{\infty} g[S(t), t \,|\zeta_{osp}, \zeta'_{osp})]\, dt \right\} p(\zeta_{osp}), \qquad (4.9)$$

where $g[S(t), t|\, \zeta_{osp}, \zeta'_{osp})]\, dt$ equals $P(A_t \cap B_t \,|\zeta_{osp}, \zeta'_{osp})$ as $\Delta t \to dt$ and is the probability that the surface crosses the ray, $\widehat{k}_i$, in dt <u>and</u> that no crossing has occurred prior (i.e. from 0 to t) <u>given</u> the point at the origin is specular with height ζ_{osp} and slope ζ'_{osp}. $S(t)=\zeta_{osp}+\eta_o t$ is the equation of the height of the ray, $\widehat{k}_i$, where η_o is the slope of the ray.

The function $g[S(t), t| \zeta_{osp},\zeta'_{osp})]$ is called the first passage in time (FPIT) density function, and has been studied for many years by various authors [Abrahams, 1986; Blake and Linsey, 1973]. We note the distinction between this function and the one used by Wagner. His conditional first passage density function is defined to be the probability density of an upcrossing occurring in dt given that no crossings occurred prior and the point at the origin has a specular slope and height. We further note that

$$1 = P(\widehat{k}_i \,|\zeta_{osp}, \zeta'_{osp}) + \int_0^\infty g[S(t), t\,|\zeta_{osp}, \zeta'_{osp})]\, dt \qquad (4.10)$$

and, in general, for a non zero probability of illumination

$$\int_0^\infty g[S(t), t|\, \zeta_{osp}, \zeta'_{osp})]\, dt \neq 1. \qquad (4.11)$$

This point is noted since we use the term "density" in discussing the FPIT problem, when it is clear that this "density" is not a true probability density function whose total area equals one. The reason for this is simple; the outcome in which an arbitrary point on the surface is illuminated is not included upon integration. It is only when the probability of every possible outcome is summed does the integral of a density sum to 1. Of course, this is intentional, because the whole point of the derivation is to compute either the probability of a point being shadowed or the probability of a point being illuminated and not to compute the probability of a point being shadowed or illuminated (which is equal to 1).

While much effort has been put forth in attempting to find this density function, in only a few cases have closed form solutions been found. The most interesting and practical cases (at least to us) have been given little attention. Generally, one must make assumptions which cannot be justified, such as the process is Markov, in order to use the results presented in the literature.

In 1945, Steven O. Rice developed in his "Mathematical Theory of Random Noise" an infinite series which expresses an unknown function, similar to g[S(t), t $|\zeta_{osp}, \zeta'_{osp})$], in terms of known functions. His expression is now well known in the literature and of fundamental importance in the study of the first passage density. We will depart here from Wagner's approximations to the conditional density function, which he makes to get closed form results, in order to test their validity. In chapters 5 and 6, we will attempt to compare Wagner's theory with our theory using an infinite series similar to that of Rice, Ricciardi and Sato. We will further attempt to investigate when the approximations made by Wagner become invalid, what they mean, and when we must look for a more exact solution.

4.3 First Passage Density Expressed as a Infinite Series

The first passage of time density function is expressed in terms of an infinite series of integrals, similar to that of Rice [Rice, 1945] and Ricciardi and Sato [Ricciardi and Sato, 1986, 1983]. The integrals are of known functions which provide us, for the first time, the ability to compute g[S(t), t $|\zeta_{osp}, \zeta'_{osp}]$. The method is based upon the 'inclusion-exclusion' principle, and the series is developed in such a way as to give us improving upper and lower bounds to the desired function [Ricciardi and Sato, 1986]. The first term gives the very crudest approximation, while the second and third terms provide a lower and upper limit to the first passage density function, respectively. We will find that in the case of no correlation between the shadowing points and the shadowed point, the infinite series appears to reduce to the same shadowing function given by Wagner. However, when correlation is included we are left with complicated expressions which cannot be solved in closed form and therefore must be completed using numerical integration procedures.

We begin by returning to the segmented interval $(0,\tau)$ for which we can write

$$\sum_{t=\tau_o}^{t=\tau_{n-1}} \mathrm{P}(\mathrm{A}_t \cap \mathrm{B}_t \,|\zeta_{\mathrm{osp}}, \zeta'_{\mathrm{osp}}) = \sum_{t=\tau_o}^{t=\tau_{n-1}} \mathrm{P}(\mathrm{A}_t \,|\zeta_{\mathrm{osp}}, \zeta'_{\mathrm{osp}})$$

$$- \sum_{t=\tau_1}^{t=\tau_{n-1}} \sum_{t_1=\tau_o}^{t_1=t-\Delta\tau} \mathrm{P}(\mathrm{A}_t \cap \mathrm{A}_{t_1} \,|\zeta_{\mathrm{osp}}, \zeta'_{\mathrm{osp}})$$

$$+ \sum_{t=\tau_2}^{t=\tau_{n-1}} \sum_{t_1=\tau_1}^{t_1=t-2\Delta\tau} \sum_{t_2=t_1+\Delta\tau}^{t_2=t-\Delta\tau} \mathrm{P}(\mathrm{A}_t \cap \mathrm{A}_{t_1} \cap \mathrm{A}_{t_2} \,|\zeta_{\mathrm{osp}}, \zeta'_{\mathrm{osp}})$$

$$- \ \ldots \qquad (4.12)$$

As we will see, in doing this we have now expressed an unknown result in terms of probabilities which can be computed. $P(A_{\tau_i} \,|\zeta_{osp}, \zeta'_{osp})$ is the probability of an upcrossing occurring in an interval $\Delta\tau_i$ at a distance $t=\tau_i$. One notices that the probabilities within the first summation on the right-hand side contain one entry for those events which have one upcrossing, two entries for those that contain two upcrossings, etc. This is due to the fact that during the summation process over the entire interval for $P(A_t \,|\zeta_{osp}, \zeta'_{osp})$, the counting of one event does not exclude that event from being counted again, if the surface crosses the ray again. For example, the event that an upcrossing occurs in $\Delta\tau_1$, included in $P(A_{\tau_1}|\zeta_{osp}, \zeta'_{osp})$, does not exclude that event from being counted again, if the surface crosses the ray again, say in $\Delta\tau_4$, which is also included in $P(A_{\tau_4}|\, \zeta_{osp}, \zeta'_{osp})$. Clearly, both $P(A_{\tau_1} \,|\zeta_{osp}, \zeta'_{osp})$ and $P(A_{\tau_4} \,|\zeta_{osp}, \zeta'_{osp})$ contain the probability of the event $(\delta_1=0,\ \delta_2=1,\ \delta_3=0,\ \delta_4=0,\ \delta_5=1,\ \delta_6=0,\ \ldots,\ \delta_n=0 \,|\zeta_{osp}, \zeta'_{osp})$. The key point is that $P(A_{\tau_i} \,|\zeta_{osp}, \zeta'_{osp})$ is only concerned with the probability of having an upcrossing in $\Delta\tau_i$ at a distance τ_i and it does not care what occurs elsewhere in the interval. Thus, we imply by $P(A_{\tau_i} \,|\zeta_{osp}, \zeta'_{osp})$ a *summation* of probabilities. This summation is the sum of probabilities of each event which has an upcrossing in $\Delta\tau_i$ (and possibly elsewhere), and not just a single event.

This logic continues to the double summation where we have multiple entries for probabilities which contain three or more upcrossings. It is easy to see, for example, that $P(A_{\tau_1}, A_{\tau_3} \,|\zeta_{osp}, \zeta'_{osp})$, $P(A_{\tau_1}, A_{\tau_4} \,|\zeta_{osp}, \zeta'_{osp})$, and $P(A_{\tau_3}, A_{\tau_4} \,|\zeta_{osp}, \zeta'_{osp})$, all contain the event $(\delta_1=0,\ \delta_2=1,\ \delta_3=0,\ \delta_4=1,\ \delta_5=1,\ \delta_6,\ \ldots,\ \delta_n=0|\, \zeta_{osp}, \zeta'_{osp})$. So when these probabilities are summed $P(\delta_1=0,\ \delta_2=1,\ \delta_3=0,\ \delta_4=1,\ \delta_5=1,\ \delta_6=0,\ \ldots,\ \delta_n=0 \,|\zeta_{osp}, \zeta'_{osp})$ is counted three times.

Likewise, the triple summation contains multiple entries for probabilities with four or more upcrossings, and so on. Thus, where $P(\delta_1=0, \delta_2=1, \delta_3=0, \delta_4=1, \delta_5=1, \delta_6=0, ..., \delta_n=0 \mid \zeta_{osp}, \zeta'_{osp})$ was counted three times in the first summation, it was subtracted off three times in the second summation, and will be re-added back once in the third summation. This process is known as the 'inclusion-exclusion' principle. It is important to note that each event is counted only once when the entire infinite series is considered, as it should.

One can now see when the higher order terms in the series might play a role and contribute significantly in determining $P(A_t \cap B_t \mid \zeta_{osp}, \zeta'_{osp})$. If the probability of having multiple crossings is reduced then we would expect that the higher order terms, which sum these probabilities, to become less important. To relate this back to the shadowing problem, one would thus expect that as the grazing angle increases the probability of multiple crossings would diminish and the first summation might become a good approximation to the shadowing function.

If we now take the limit as $n \rightarrow \infty$, $\Delta\tau_i \rightarrow d\tau$, $\tau_0 \rightarrow 0$, $\tau_1 \rightarrow 0$, $\tau_2 \rightarrow 0$, etc., and $\tau_n \rightarrow \tau$, the summations become integrals, where

$$\lim_{n\rightarrow\infty} \sum_{t=\tau_o}^{t=\tau_n} P(A_t \cap B_t \mid \zeta_{osp}, \zeta'_{osp}) = \int_0^{\tau} g[S(t), t \mid \zeta_{osp}, \zeta'_{osp}]\, dt, \quad (4.13)$$

and where

$$\int_0^{\tau} g[S(t), t|\, \zeta_{osp}, \zeta'_{osp}]\, dt = \int_0^{\tau} dt\, W_1(t\, |\zeta_{osp}, \zeta'_{osp})$$

$$- \int_0^{\tau} dt \int_0^{t} dt_1\, W_2(t, t_1\, |\zeta_{osp}, \zeta'_{osp})$$

$$+ \int_0^{\tau} dt \int_0^{t} dt_1 \int_{t_1}^{t} dt_2\, W_3(t, t_1, t_2\, |\zeta_{osp}, \zeta'_{osp})$$

$$- \ . \ . \ . \qquad (4.14)$$

The W_n's are the $P(A_t, A_{t_1}, \ldots, A_{t_{n-1}}\, |\zeta_{osp}, \zeta'_{osp})$'s divided by Δt, Δt_1, ... Δt_{n-1}, where we have let $t_0=0$, $t_n=t$ (to conform to Ricciardi and Sato's notation), $n \to \infty$, and $\Delta t_i \to dt_i$. The W_n 's are known functions to be derived in the next section. This series is very similar to that of Ricciardi and Sato based on Rice's work, except that ours is conditioned on the slope as well as the height at the origin.

In the limit as $\tau \to \infty$ we have

$$\int_0^{\infty} g[S(t), t\, |\zeta_{osp}, \zeta'_{osp}]\, dt = \int_0^{\infty} dt\, W_1(t\, |\zeta_{osp}, \zeta'_{osp})$$

$$- \int_0^{\infty} dt \int_0^{t} dt_1\, W_2(t, t_1\, |\zeta_{osp}, \zeta'_{osp})$$

$$+ \int_0^{\infty} dt \int_0^{t} dt_1 \int_{t_1}^{t} dt_2\, W_3(t, t_1, t_2\, |\zeta_{osp}, \zeta'_{osp})$$

$$- \ \ldots \qquad (4.15)$$

In words, $W_1(t\, |\zeta_{osp}, \zeta'_{osp})dt$ is the probability of a single upcrossing occurring in dt at a distance t given the height and slope of the specular point at the origin; $W_2(t, t_1 | \zeta_{osp}, \zeta'_{osp}) dt dt_1$ is the joint probability of a single upcrossing occurring in dt at a distance t and another in dt_1 at a distance t_1 given the height and slope of the specular point, etc.

We are now in a position to derive the W_n's in terms of the probability density function of the heights and slopes on the surface. For simplicity of notation (and to be consistent with Ricciardi and Sato's notation), at this point let us make a change of variables such that

$$W_1(t | \zeta_{osp}, \zeta'_{osp})\, dt \rightarrow W_1(t_1 | \zeta_{osp}, \zeta'_{osp})\, dt_1 \tag{4.16a}$$

$$W_2(t, t_1 | \zeta_{osp}, \zeta'_{osp})\, dt\, dt_1 \rightarrow W_2(t_2, t_1 | \zeta_{osp}, \zeta'_{osp})\, dt_2\, dt_1 \tag{4.16b}$$

$$W_3(t, t_1, t_2 | \zeta_{osp}, \zeta'_{osp})\, dt\, dt_1 dt_2 \rightarrow W_3(t_3, t_1, t_2 | \zeta_{osp}, \zeta'_{osp})\, dt_3\, dt_1\, dt_2 \tag{4.16c}$$

$$\vdots\ ,$$

where after the W's are calculated, $t_1 \rightarrow t$ in W_1, $t_2 \rightarrow t$ in W_2, $t_3 \rightarrow t$ in W_3, etc. The convenience of doing this will become obvious in a moment. We begin by considering $W_1(t_1 | \zeta_{osp}, \zeta'_{osp}) dt_1$. From Wagner [Wagner, 1967] we see that there are two restrictions for an upcrossing to occur in dt_1 at a distance t_1. One is on the slope and the other on the height. Clearly, for an upcrossing to occur in dt_1 at t_1 [Wagner, 1967]

$$\zeta'(t_1) \; > \; \eta_0, \tag{4.17}$$

where η_0 is the slope of the incident ray, $\widehat{k}_i$, along the t axis. This is illustrated in Figure 13. The second condition is on the height at t_1. At the crossing point, τ_c, we have [Wagner, 1967]

$$\zeta_{osp} \; + \; \eta_0\tau_c = \zeta(t_1) \; + \; \zeta'(t_1)(\tau_c - t_1), \tag{4.18}$$

since dt_1 is assumed infinitesimally small. We also have

$$t_1 < \tau_c < t_1 + dt_1. \tag{4.19}$$

Solving for τ_c we have

$$\tau_c = \frac{[\zeta(t_1) \; - \; \zeta_{osp} \; - \zeta'(t_1) \; t_1]}{\eta_0 - \zeta'(t_1)}, \tag{4.20}$$

and substituting (4.20) into (4.19) we get

$$t_1 < \frac{[\zeta(t_1) \; - \zeta_{osp} \; - \zeta'(t_1) \; t_1]}{\eta_0 - \zeta'(t_1)} \; < \; t_1 + dt_1. \tag{4.21}$$

This reduces to

$$t_1\eta_0 \; + \zeta_{osp} > \zeta(t_1) > \zeta_{osp} \; + \; \eta_0 t_1 \; - \; (\zeta'(t_1) \; - \; \eta_0) \; dt_1. \tag{4.22}$$

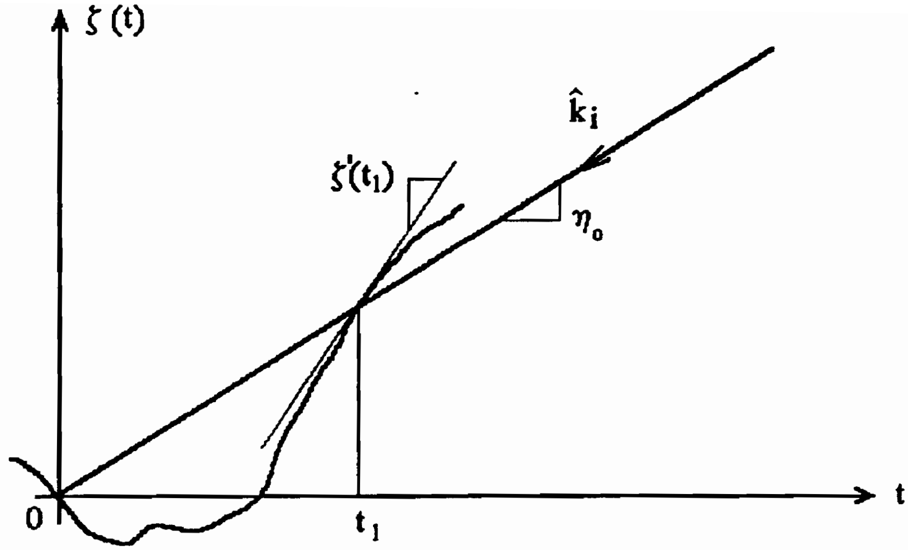

Figure 13: $\zeta'(t_1) > \eta_o$ at the Shadowing Point

Therefore, the probability of an upcrossing occurring in dt_1 is

$$W_1(t_1\ |\zeta_{osp},\zeta'_{osp})dt_1 = \int_{\eta_0}^{\infty} d\zeta' \int_{\zeta_{osp}+\eta_o t_1-(\zeta'(t_1)-\eta_o)\,dt_1}^{\zeta_{osp}+\eta_o t_1} d\zeta\ p(\zeta(t_1),\zeta'(t_1)\ |\zeta_{osp},\zeta'_{osp}), \quad (4.23)$$

where $p(\zeta(t_1),\zeta'(t_1)\ |\zeta_{osp},\zeta'_{osp})$ is the joint probability density function for the height and slope at t_1 conditioned on the height and slope of the specular point at $t = 0$.

For a small dt_1, we can approximate the integral to produce the probability of an upcrossing occurring in dt_1,

$$W_1(t_1\ |\zeta_{osp},\ \zeta'_{osp})dt_1 = dt_1 \int_{\eta_0}^{\infty} d\zeta'\ (\zeta'(t_1)\text{-}\eta_0)\ p(\zeta_{osp}+\eta_0 t_1,\ \zeta'(t_1)\ |\zeta_{osp},\zeta'_{osp}).$$

Thus,

$$W_1(t_1\ |\zeta_{osp},\zeta'_{osp}) = \int_{\eta_0}^{\infty} d\zeta'\ (\zeta'(t_1)\text{-}\eta_0)\ p(\zeta_{osp}+\eta_0 t_1,\ \zeta'(t_1)\ |\zeta_{osp},\ \zeta'_{osp}). \quad (4.24)$$

Likewise, the joint probability of having an upcrossing in dt_1 and dt_2 is

$$W_2(t_2,\ t_1\ |\zeta_{osp},\ \zeta'_{osp})dt_1 dt_2 = dt_1 dt_2 \int_{\eta_0}^{\infty} d\zeta'_1 \int_{\eta_0}^{\infty} d\zeta'_2\ (\zeta'(t_1)\text{-}\eta_0)\ (\zeta'(t_2)\text{-}\eta_0)$$

$$*\ p(\zeta_{osp}+\eta_0 t_1,\ \zeta'(t_1),\ \zeta_{osp}+\eta_0 t_2,\ \zeta'(t_2)\ |\zeta_{osp},\ \zeta'_{osp}). \quad (4.25)$$

Thus,

$$W_2(t_2,\, t_1\, |\zeta_{osp}, \zeta'_{osp}) = \int_{\eta_0}^{\infty} d\zeta'_1 \int_{\eta_0}^{\infty} d\zeta'_2\, (\zeta'(t_1)\text{-}\eta_0)\, (\zeta'(t_2)\text{-}\eta_0)$$

$$*p(\zeta_{osp} + \eta_0 t_1,\, \zeta'(t_1),\, \zeta_{osp} + \eta_0 t_2,\, \zeta'(t_2)\, |\zeta_{osp}, \zeta'_{osp}). \quad (4.26)$$

Continuing in this manner, we see that

$$W_n(t_1,\, t_2,\, \ldots,\, t_n\, |\zeta_{osp}, \zeta'_{osp}) = \int_{\eta_0}^{\infty} d\zeta'_1 \int_{\eta_0}^{\infty} d\zeta'_2 \ldots \int_{\eta_0}^{\infty} d\zeta'_n \prod_{i=1}^{n} (\zeta'(t_i)\text{-}\eta_0)$$

$$*\; p(\zeta_{osp} + \eta_0 t_1,\, \zeta'(t_1),\, \zeta_{osp} + \eta_0 t_2,\, \zeta'(t_2),\, \ldots,\, \zeta_{osp} + \eta_0 t_n,\, \zeta'(t_n)\, |\zeta_{osp}, \zeta'_{osp}). \quad (4.27)$$

For convenience we define $S_i = S(t_i) = \zeta_{osp} + \eta_0 t_i$, $\zeta'(t_i)$, and $\zeta'_i = \zeta'(t_i)$, for $i=1, \ldots, n$. This gives us

$$W_n(t_1,\, t_2, \ldots,\, t_n\, |\zeta_{osp},\, \zeta'_{osp}) = \int_{\eta_0}^{\infty} d\zeta'_1 \int_{\eta_0}^{\infty} d\zeta'_2 \ldots \int_{\eta_0}^{\infty} d\zeta'_n \prod_{i=1}^{n} (\zeta'_i\text{-}\eta_o)$$

$$*\; p(S(t_1),\, \zeta'_1,\, S(t_2),\, \zeta'_2,\, \ldots,\, S(t_n),\, \zeta'_n\, |\zeta_{osp},\, \zeta'_{osp}), \quad (4.28)$$

which is similar to equation A.6 of Ricciardi and Sato [Ricciardi and Sato, 1986].

Our next task is to actually compute the W_n's for a given joint probability density function with known surface statistics, correlation function, and a known angle of incidence. Up until this point our results hold for any statistically isotropic stationary surface. Analytic results based on these probabilities are presented in the next chapter.

5.0 ANALYTICAL RESULTS

5.1 Introduction

A theoretical approach to calculating the incident shadowing function was presented in Chapter 4. The shadowing function was expressed in terms of an infinite series of integrals [Ricciardi and Sato, 1986]. One disadvantage with this approach is that multiple integrations are necessary to obtain a result, and the dimensionality of the integrations increase with higher order terms in the series. One possible solution to this problem is to carry out the integrations analytically, as far as possible, and then use numerical techniques to complete the analysis. Another possibility is to make an approximation that will result in some simplification of the infinite series.

From a physical point of view, one can imagine that if the surface is rough enough and/or the grazing angle low enough in such a way that the shadowed points and the shadowing points become separated by several correlation lengths, they become decorrelated of one another. If we make use of this assumption, great simplicity is achieved, and it provides us with a case in which to compared "exact" numerical results.

We begin this chapter by neglecting the correlation between the shadowed points and shadowing points. We also neglect the correlation between the shadowing points. It is important to note that although we refer to this assumption as the "no correlation" assumption, we are not assuming the surface has no correlation (i.e. a delta function correlation); rather, we are assuming that the shadowing points and the shadowed point are so far away from each other that they become uncorrelated (likewise, as the shadowing points become separated they become uncorrelated). We further assume that any contribution to shadowing when these points are close together is negligible. Thus, we are using the no correlation assumption where it is clearly not valid in the hope that this portion of the integrals does not contribute much to the overall shadowing function.

5.2 Calculation of the W_n's

In the previous chapter the incident shadowing function was expressed in terms of the integral of the first passage in time density function. This unknown density function was further expressed as an infinite series of integrals whose integrands could be expressed in terms of known functions.

To restate the problem mathematically, we found in Chapter 4 that

$$P(\hat{k}_i \,|\zeta'_{osp}) = 1 - \int_{-\infty}^{\infty} d\zeta_{osp} \left\{ \int_0^{\infty} g[S(t), t \,|\zeta_{osp}, \zeta'_{osp})] \, dt \right\} p(\zeta_{osp}), \quad (5.1)$$

where

$$\begin{aligned} \int_0^{\infty} g[S(t), t \,|\zeta_{osp}, \zeta'_{osp})] \, dt = & \int_0^{\infty} dt \; W_1(t \,|\zeta_{osp}, \zeta'_{osp}) \\ & - \int_0^{\infty} dt \int_0^{t} dt_1 \; W_2(t, t_1 \,|\zeta_{osp}, \zeta'_{osp}) \\ & + \int_0^{\infty} dt \int_0^{t} dt_1 \int_{t_1}^{t} dt_2 \; W_3(t, t_1, t_2 \,|\zeta_{osp}, \zeta'_{osp}) \\ & - \ldots \end{aligned} \quad (5.2)$$

and

$$\begin{aligned} W_n(t_1, t_2, \ldots, t_n \,|\zeta_{osp}, \zeta'_{osp}) = & \int_{\eta_0}^{\infty} d\zeta'_1 \int_{\eta_0}^{\infty} d\zeta'_2 \ldots \int_{\eta_0}^{\infty} d\zeta'_n \prod_{i=1}^{n} (\zeta'_i - \eta_0) \\ & * \, p(S(t_1), \zeta'_1, S(t_2), \zeta'_2, \ldots, S(t_n), \zeta'_n \,|\zeta_{osp}, \zeta'_{osp}). \end{aligned} \quad (5.3)$$

To reiterate the physical meaning of each of these terms, we have that $P(\hat{k}_i \,|\zeta'_{osp})$ is the probability that a point on the surface will be illuminated by a ray, $\hat{k}_i$, given that the point has a specular slope ζ'_{osp}; $g[S(t), t \,|\zeta_{osp}, \zeta'_{osp})]\, dt$ is the probability that the surface will cross the ray, $\hat{k}_i$ (with equation $S(t)=\zeta_{osp}+\eta_0 t$), from below and that no crossing has occurred prior, given the specular slope and height at the origin. We saw that the probability of a point being shadowed was simply the total area under this first passage density, where we then averaged over the specular heights on the surface (5.1). This unknown density, we found, could be formulated in terms of an infinite series of integrals (5.2) with known integrands (meaning they are integrals of known functions) (5.3). These integrands, written as $W_n(t_1, t_2, ..., t_n \,|\zeta_{osp}, \zeta'_{osp})$, are the probability densities of having upcrossings at $t_1, t_2, ..., t_n$ given ζ_{osp} and ζ'_{osp}.

At this point we must now specify the problem in more detail to obtain results. We assume the following:

- Gaussian statistics for the heights and slopes (each with zero mean).
- Gaussian height correlation for the surface.
- variances of height and slopes are known.

From Chapter 2, we assumed that the random process is both stationary and isotropic. Thus, the shadowing function is independent of the direction in which the source illuminated the surface. The stationarity assumption allows us to obtain a result which is independent of location on the surface. Both of these assumptions simplify matters considerably.

The justification for using Gaussian statistics is that many physical processes can be approximated by this process, particularly when the central limit theorem can be used. For example, under certain conditions, the ocean may be approximated by a Gaussian process [Longuet-Higgins, 1956]. The correlation function is assumed to be Gaussian in order to facilitate comparison with other results on shadowing theory, particularly Wagner's [Wagner's, 1967]. The actual correlation function for many realistic surfaces, such as the ocean, is Gaussian only about the origin, but this limitation does not defeat our approach since one could substitute any correlation function in place of the Gaussian and proceed in a similar manner, with a few additional constraints.

We begin by attempting to compute $W_1(t \mid \zeta_{osp}, \zeta'_{osp})$ in (5.3) where we have set $t_1 = t$ in anticipation of (5.2). We will first make the aforementioned approximation that the height and slope at the shadowing point (at a distance t) and the height and slope at the shadowed point (t=0) are uncorrelated. We expect that this approximation will be valid when the two points are sufficiently far away, in terms of correlation lengths, and will breakdown when the two points come closer together. In terms of rough surface scattering, this appears to happen when the specular points are low relative to the mean of the heights on the surface [Hardin, 1972].

5.3 Calculation of $W_1(t \mid \zeta_{osp}, \zeta'_{osp})$ Assuming No Correlation Between the Shadowing Point and the Shadowed Point.

As a first order approximation to $g[S(t), t \mid \zeta_{osp}, \zeta'_{osp}]$, we compute the first term $W_1(t \mid \zeta_{osp}, \zeta'_{osp})$. We know that W_1 will become a better approximation to $g[S(t), t \mid \zeta_{osp}, \zeta'_{osp}]$ for the case in which multiple crossings are minimized. As previously mentioned, there is much less probability of having multiple crossings for higher grazing angles, thus reducing the affect of the higher order terms in (5.2).

In order to obtain closed form solutions for W_1, and hence obtain a first approximation to $g[S(t), t \,|\zeta_{osp}, \zeta'_{osp}]$, we further assume no correlation between the height and slope $(\zeta_{osp}, \zeta'_{osp})$ at the origin (shadowed point) and the height and slope $(\zeta(t), \zeta'(t))$ at t (shadowing point). We then have from (5.3) that

$$W_1(t \,|\zeta_{osp}, \zeta'_{osp}) = \int_{\eta_0}^{\infty} d\zeta' \, (\zeta'(t) - \eta_0) \, p(S(t), \zeta'(t) \,|\zeta_{osp}, \zeta'_{osp}), \quad (5.4)$$

where t_1 was set equal to t and $S(t) = \zeta_{osp} + \eta_0 t$. Since the heights and slopes at t and at the origin are assumed to be uncorrelated (and hence statistically independent, since the density function is Gaussian), we can break the density function up such that

$$p(S(t), \zeta'(t) \,|\zeta_{osp}, \zeta'_{osp}) = p(S(t), \zeta'(t)). \quad (5.5)$$

Further simplification results by considering the Gaussian autocorrelation function

$$R(\tau) = \sigma_1^2 \, exp\left\{ - \frac{\tau^2}{l^2} \right\}, \quad (5.6)$$

where σ_1^2 is the variance of the heights, l is the correlation length (defined to be the $1/e$ point of the autocorrelation function), and τ is the horizontal distance between two points on the surface. We can see that because

$$R(\tau) \doteq E[\zeta(t+\tau)\zeta(t)],$$

where $E[\cdot]$ is the expected value operator, $R(0) = E[\zeta^2(t)] = \sigma_1^2$. We also know that the variance for the slopes is defined to be $\sigma_2^2 \doteq E[\zeta_{osp}'^2]$ where

$$\sigma_2^2 \doteq E[\zeta_{osp}'^2] = -\frac{R''(\tau)}{d\tau}\bigg|_{\tau=0} = 2 \cdot \frac{\sigma_1^2}{l^2}, \tag{5.7}$$

thus we have

$$l = \sqrt{\frac{2\sigma_1^2}{\sigma_2^2}}, \tag{5.8}$$

where we have expressed the correlation length in terms of the height and slope second moments. The joint expected value for a height and a slope at a particular point is zero, which implies that they are uncorrelated. This is due to the fact that the correlation function has zero slope at the origin, i.e.

$$E[\zeta(t)\zeta'(t)] = -\frac{R(\tau)}{d\tau}\bigg|_{\tau=0} = 2\,\frac{\tau}{l^2}\,\sigma_1^2\,exp\left\{-\frac{\tau^2}{l^2}\right\}\bigg|_{\tau=0} = 0. \tag{5.9}$$

We can use this fact to further break up the density function in (5.5) (since decorrelation implies statistical independence for Gaussian random variables) such that

$$p(S(t), \zeta'(t)) = p(S(t))\; p(\zeta'(t)). \tag{5.10}$$

Since S(t) is independent of slope, we can bring p(S(t)) outside the integral in (5.4). Substituting (5.10) and (5.5) into (5.4) we get

$$W_1(t\;|\zeta_{osp}, \zeta_{osp}') = p(S(t)) \int_{\eta_0}^{\infty} d\zeta'\; (\zeta'(t) - \eta_0)\; p(\zeta'(t)), \tag{5.11}$$

where

$$p(S(t)) = \frac{exp\left\{-\frac{[S(t)]^2}{2\sigma_1^2}\right\}}{\sqrt{2\pi\sigma_1^2}} \tag{5.12}$$

is the probability density function of the heights at t, $S(t)=\zeta_{osp}+\eta_0 t$, and

$$p(\zeta'(t)) = \frac{exp\left\{-\frac{[\zeta'(t)]^2}{2\sigma_2^2}\right\}}{\sqrt{2\pi\sigma_2^2}} \tag{5.13}$$

is the probability density function of the slope at t. To perform the integration in (5.11), we can break the integrand into two parts, writing $p(\zeta'(t))$ explicitly, to get

$$W_1(t\;|\zeta_{osp},\zeta'_{osp}) = \frac{p(S(t))}{\sqrt{2\pi\sigma_2^2}}\left\{\int_{\eta_0}^{\infty} d\zeta'\;\zeta'(t)\;\; exp\left\{-\frac{[\zeta'(t)]^2}{2\sigma_2^2}\right\} - \right.$$

$$\left. \eta_0\int_{\eta_0}^{\infty} d\zeta'\;\; exp\left\{-\frac{[\zeta'(t)]^2}{2\sigma_2^2}\right\}\right\}. \tag{5.14}$$

If we make a change of variables in the first integral such that $\beta = \frac{[\zeta'(t)]^2}{2\sigma_2^2}$ and $d\beta = \frac{\zeta'(t)}{\sigma_2^2}\,d\zeta$ we get

$$\int_{\eta_0}^{\infty} d\zeta'\;\zeta'(t)\;\; exp\left\{-\frac{[\zeta'(t)]^2}{2\sigma_2^2}\right\} = \sigma_2^2\int_{\frac{\eta_0^2}{2\sigma_2^2}}^{\infty} d\beta\; exp\{-\beta\} = \sigma_2^2\; exp\left\{-\frac{\eta_0^2}{2\sigma_2^2}\right\}. \tag{5.15}$$

If in the second integral we let $\mu = \frac{\zeta'(t)}{\sqrt{2\sigma_2^2}}$ and $d\mu = \frac{1}{\sqrt{2\sigma_2^2}}\,d\zeta'$ we get

$$\int_{\eta_0}^{\infty} d\zeta' \; exp\left\{ -\frac{[\zeta'(t)]^2}{2\sigma_2^2} \right\} = \sqrt{2\sigma_2^2} \int_{\frac{\eta_0}{\sqrt{2\sigma_2^2}}}^{\infty} d\mu \; exp\{ -\mu^2 \}$$

$$= \sqrt{\frac{\pi\sigma_2^2}{2}} \; erfc(\eta_0 / \sqrt{2\sigma_2^2}), \qquad (5.16)$$

where we have used the following definition of the complementary error function

$$erfc(x) \doteq \frac{2}{\sqrt{\pi}} \int_{x}^{\infty} dy \; exp\{ -y^2 \}. \qquad (5.17)$$

Finally, we end up with

$$W_1(t \; | \zeta_{osp}, \zeta'_{osp}) = \frac{exp\left\{ -\frac{[\zeta_{osp} + \eta_0 t]^2}{2\sigma_1^2} \right\}}{2\pi\sigma_1\sigma_2}$$

$$* \left\{ \sigma_2^2 \; exp\left\{ -\frac{\eta_0^2}{2\sigma_2^2} \right\} - \eta_0 \sqrt{\frac{\pi\sigma_2^2}{2}} \; erfc(\eta_0 / \sqrt{2\sigma_2^2}) \right\} \qquad (5.18)$$

$$\doteq g^{(1)}[S(t), t \; | \zeta_{osp}, \zeta'_{osp}], \qquad (5.19)$$

where $g^{(1)}[S(t), t \; | \zeta_{osp}, \zeta'_{osp}]$ is a first order (no correlation) approximation to $g[S(t), t \; | \zeta_{osp}, \zeta'_{osp}]$.

A first order (no correlation) approximation to the shadowing function $P(\widehat{k}_i \; | \zeta_{osp}, \zeta'_{osp})$ is given as

$$P^{(1)}(\widehat{k}_i \; | \zeta_{osp}, \zeta'_{osp}) = 1 - \int_0^{\infty} g^{(1)} \, [S(t), t \; | \zeta_{osp}, \zeta'_{osp})] \; dt. \qquad (5.20)$$

Integrating over the heights (see 5.1), we get

$$P^{(1)}(\widehat{k}_i \,|\zeta'_{osp}) = \int_{-\infty}^{\infty} d\zeta_{osp}\, P^{(1)}(\widehat{k}_i \,|\zeta_{osp}, \zeta'_{osp})\, p(\zeta_{osp}), \qquad (5.21)$$

where $p(\zeta_{osp}) = \dfrac{exp\left\{-\dfrac{\zeta^2_{osp}}{2\sigma_1^2}\right\}}{\sqrt{2\pi\sigma_1^2}}$ is the pdf of the specular heights at the origin.

Since we wish to generalize the problem as much as possible, we perform the final integration in (5.2) over t from 0 to τ, and then let $\tau \rightarrow \infty$. This facilitates comparison with Hardin [Hardin, 1972]. Carrying out the appropriate integration of W_1 in (5.2) using (5.18), we find

$$\int_0^{\tau} dt\; W_1(t\,|\zeta_{osp},\zeta'_{osp}) \;=\; \frac{\left\{\sigma_2^2\, exp\left\{-\dfrac{\eta_0^2}{2\sigma_2^2}\right\} - \eta_0\sqrt{\dfrac{\pi\sigma_2^2}{2}}\; erfc\,(\eta_0/\sqrt{2\sigma_2^2})\right\}}{2\pi\sigma_1\sigma_2}$$

$$* \frac{\sigma_1}{\eta_o}\sqrt{\frac{\pi}{2}}\left\{erfc(\zeta_{osp}/\sqrt{2}\sigma_1) \;-\; erfc((\zeta_{osp}+\eta_0\tau)/\sqrt{2}\sigma_1)\right\}. \qquad (5.22)$$

Now taking the limit as $\tau \rightarrow \infty$ and using the fact that $erfc((\zeta_{osp}+\eta_0\tau)/\sqrt{2}\sigma_1) \rightarrow erfc(\infty) = 0$ as $\tau \rightarrow \infty$ $(\eta_0 > 0)$, we arrive at

$$\int_0^{\infty} dt\; W_1(t\,|\zeta_{osp},\zeta'_{osp}) = \frac{\left\{\sigma_2^2\, exp\left\{-\dfrac{\eta_0^2}{2\sigma_2^2}\right\} - \eta_0\sqrt{\dfrac{\pi\sigma_2^2}{2}}\; erfc\,(\eta_0/\sqrt{2\sigma_2^2})\right\}}{2\sqrt{2\pi}\sigma_2\eta_0}$$

$$* \; erfc(\zeta_{osp}/\sqrt{2}\sigma_1)$$

Substituting $\nu = \frac{\eta_0}{\sqrt{2\sigma_2^2}}$ we get

$$\int_0^\infty dt\ W_1(t\ |\zeta_{osp},\zeta'_{osp}) = \frac{1}{4\sqrt{\pi}\nu}\left\{ exp\{-\nu^2\} - \sqrt{\pi}\nu\ erfc(\nu)\right\}\ erfc(\zeta_{osp}/\sqrt{2\sigma_1}). \quad (5.23)$$

In summary, we have approximated $g[S(t), t\ |\zeta_{osp}, \zeta'_{osp})]$ by $W_1(t\ |\zeta_{osp}, \zeta'_{osp}) \doteq g^{(1)}[S(t), t\ |\zeta_{osp}, \zeta'_{osp})]$ after truncating the infinite series after the first term and we have assumed no correlation between the shadowing point and the shadowed point. Therefore,

$$\begin{aligned} P(\widehat{k}_i\ |\zeta_{osp}, \zeta'_{osp}) &= 1 - \int_0^\infty g[S(t), t\ |\zeta_{osp}, \zeta'_{osp})]\ dt \\ &\approx 1 - \int_0^\infty g^{(1)}\ [S(t), t|\ \zeta_{osp}, \zeta'_{osp})]\ dt \\ &= 1 - \int_0^\infty dt\ W_1(t\ |\zeta_{osp},\zeta'_{osp}) \end{aligned} \quad (5.24)$$

or

$$P^{(1)}(\widehat{k}_i\ |\zeta_{osp}, \zeta'_{osp}) = 1\ -\ \frac{1}{4\sqrt{\pi}\nu}\left\{ exp\{-\nu^2\} - \sqrt{\pi}\nu\ erfc(\nu)\right\}\ erfc(\zeta_{osp}/\sqrt{2\sigma_1}), \quad (5.25)$$

where the second term on the right-hand side is like Wagner's result [Wagner, 1967] in the exponential (eq. 17 of [Wagner, 1967]). We can see here that

$$exp\left\{ -\frac{1}{4\sqrt{\pi}\nu}\left\{ exp\{-\nu^2\} - \sqrt{\pi}\nu\ erfc(\nu)\right\} erfc(\zeta_{osp}/\sqrt{2\sigma_1})\right\}$$

$$\approx 1 - \frac{1}{4\sqrt{\pi}\nu}\left\{ exp\{-\nu^2\} - \sqrt{\pi}\nu\ erfc(\nu)\right\} erfc(\zeta_{osp}/\sqrt{2\sigma_1})$$

when $\frac{1}{4\sqrt{\pi}\nu}\left\{ exp\{-\nu^2\} - \sqrt{\pi}\nu\ erfc(\nu)\right\} erfc(\zeta_{osp}/\sqrt{2\sigma_1})$ is small or when ν is large. This implies that our results should agree when the shadowing is weak, i.e. when η_0, the slope of the incoming ray, is large or σ_2 is small.

Before we proceed to calculate higher order terms in the series, we shall finish the derivation by calculating the actual shadowing function that will go into our analysis, namely $P^{(1)}(\widehat{k}_i\ |\zeta'_{osp})$. To do this we substitute (5.25) into (5.21) to get

$$P^{(1)}(\widehat{k}_i\ |\zeta'_{osp}) = 1 - \frac{1}{4\sqrt{\pi}\nu}\left\{ exp\{-\nu^2\} - \sqrt{\pi}\nu\ erfc(\nu)\right\}, \qquad (5.26)$$

with $\nu = \dfrac{\eta_0}{\sqrt{2\sigma_2^2}}$ and $\eta_0 > 0$. This is a first order approximation to the shadowing function.

We note that for small η_0 (low grazing angles) or large σ_2 the probability of a point on the surface being shadowed appears to become large and greater than 1. This is to be expected for the following reasons. It is understood by the very nature of how $g[S(t), t|\ \zeta_{osp}, \zeta'_{osp})]$ is computed, that the terms

$$\int_0^\infty dt\ W_1(t\ |\zeta_{osp}, \zeta'_{osp}), \quad \int_0^\infty dt \int_0^t dt_1\ W_2(t, t_1\ |\zeta_{osp}, \zeta'_{osp}), \ldots, \text{etc.}$$

can become greater than one. We can see this most clearly by examining the integral of $W_1(t \,|\zeta_{osp}, \zeta'_{osp})$. The sum of probabilities which make up this integral are not the probabilities of mutually exclusive events. One can have an upcrossing occur in more than one location within the interval $(0,\tau)$. We can see that as the grazing angle becomes small (or the variance of the slopes becomes large) we increase the probability of those events which have more than one upcrossing occurring. But these are precisely the events which are counted more than once in the integral. Hence, there is no reason to expect that their sum is less than one. So, the individual terms in the series should only be viewed as approximations to the actual probabilities desired. Clearly, one cannot have probabilities greater than one, however, we must keep in mind that we are effectively approximating a density function, whose integral is less than one, by a series of functions whose integrals can be greater than one. We can see that under certain conditions, the functions which correct the first term in the series will have a minimal effect, such as at large grazing angles. Of course, as we move down in grazing angle, more and more terms in the series must be kept in order to get a good approximation to the actual shadowing function.

We will now proceed to calculate the second term in the series under the same assumption of no correlation between the shadowing points and the shadowed point. Here we neglect, not only the correlation between the shadowed point and each of the shadowing points, but also the correlation between the shadowing points as well. The idea is again to use the property of Gaussian random variables, i.e. that decorrelation implies statistical independence, to simplify matters in (5.3).

5.4 Calculation of $W_2(t, t_1 | \zeta_{osp}, \zeta'_{osp})$ Assuming No Correlation Between the Shadowed Point and the Shadowing Points.

If we consider the next term in the integral series, in anticipation that this will correct the first approximation, we have from (5.3)

$$W_2(t, t_1 | \zeta_{osp}, \zeta'_{osp}) = \int_{\eta_0}^{\infty} d\zeta'_1 \int_{\eta_0}^{\infty} d\zeta' \, (\zeta'(t_1) - \eta_0)\,(\zeta'(t) - \eta_0)$$

$$* \; p(\zeta_{osp} + \eta_0 t_1, \zeta'(t_1), \zeta_{osp} + \eta_0 t, \zeta'(t) \,|\zeta_{osp}, \zeta'_{osp}). \qquad (5.27)$$

Assuming that the points at $(\zeta_{osp}, \zeta'_{osp})$, $(\zeta_{osp} + \eta_0 t_1, \zeta'(t_1))$, and $(\zeta_{osp} + \eta_0 t, \zeta'(t))$ are uncorrelated, we have

$$W_2(t, t_1 | \zeta_{osp}, \zeta'_{osp}) = \int_{\eta_0}^{\infty} d\zeta'_1 \, p(\zeta_{osp} + \eta_0 t_1, \zeta'(t_1))(\zeta'(t_1) - \eta_0)$$

$$* \int_{\eta_0}^{\infty} d\zeta' \, p(\zeta_{osp} + \eta_0 t, \zeta'(t))(\zeta'(t) - \eta_0). \qquad (5.28)$$

Since the height and slope at a given point are uncorrelated, from (5.9) and (5.10) we have

$$p(\zeta_{osp} + \eta_0 t_1, \zeta'(t_1)) \;=\; p(\zeta_{osp} + \eta_0 t_1)\, p(\zeta'(t_1)), \qquad (5.29a)$$

$$p(\zeta_{osp} + \eta_0 t, \zeta'(t)) \;=\; p(\zeta_{osp} + \eta_0 t) p(\zeta'(t)). \qquad (5.29b)$$

This simplifies W_2 to

$$W_2(t, t_1 \,|\zeta_{osp},\zeta'_{osp}) = p(\zeta_{osp}+\eta_0 t_1) \int_{\eta_0}^{\infty} d\zeta'_1 \,(\zeta'(t_1) - \eta_0) p(\zeta'(t_1))$$

$$* \; p(\zeta_{osp}+\eta_0 t) \int_{\eta_0}^{\infty} d\zeta' \,(\zeta'(t) - \eta_0) p(\zeta'(t)). \qquad (5.30)$$

Each of these integrals are in the same form as $W_1(t\,|\zeta_{osp}, \zeta'_{osp})$ (5.11). Therefore, we have, using (5.18),

$$W_2(t, t_1 \,|\zeta_{osp}, \zeta'_{osp}) = \frac{exp\left\{-\dfrac{[\zeta_{osp}+\eta_0 t]^2}{2\sigma_1^2}\right\} \, exp\left\{-\dfrac{[\zeta_{osp}+\eta_0 t_1]^2}{2\sigma_1^2}\right\}}{(2\pi\sigma_1\sigma_2)^2}$$

$$* \left\{\sigma_2^2 \; exp\left\{-\frac{\eta_0^2}{2\sigma_2^2}\right\} - \eta_0 \sqrt{\frac{\pi\sigma_2^2}{2}} \; erfc(\eta_0/\sqrt{2\sigma_2^2})\right\}^2. \qquad (5.31)$$

We now proceed to calculate the second order approximation to the first passage in time density function and consequently the second order approximation to the overall shadowing function. We will start to notice a pattern in the terms of the series in (5.2). In the sections to follow, we sum the series, after convincing ourselves that a pattern, in fact, does exist, indicated by the third order term containing W_3.

Calculating the first of the double integrals in (5.2) (which integrate W_2) we obtain

$$\int_0^t dt_1\ W_2(t,\ t_1\ |\zeta_{osp},\ \zeta'_{osp}) = \frac{exp\left\{-\dfrac{[\zeta_{osp}+\eta_0 t]^2}{2\sigma_1^2}\right\}}{(2\pi\sigma_1\sigma_2)^2}$$

$$* \left\{\sigma_2^2\ exp\left\{-\frac{\eta_0^2}{2\sigma_2^2}\right\} - \eta_0\sqrt{\frac{\pi\sigma_2^2}{2}}\ erfc\ (\eta_0/\sqrt{2\sigma_2^2})\right\}^2$$

$$* \int_0^t dt_1\ exp\left\{-\frac{[\zeta_{osp}+\eta_o t_1]^2}{2\sigma_1^2}\right\}, \tag{5.32}$$

where the integral is

$$\int_0^t dt_1\ exp\left\{\frac{-[\zeta_{osp}+\eta_o t_1]^2}{2\sigma_1^2}\right\} = \frac{\sigma_1}{\eta_0}\sqrt{\frac{\pi}{2}}\left\{erfc(\zeta_{osp}/\sqrt{2\sigma_1}) - erfc((\zeta_{osp}+\eta_0 t)/\sqrt{2\sigma_1})\right\},$$

using (5.22). We are then left with

$$\int_0^t dt_1\ W_2(t,\ t_1\ |\zeta_{osp},\ \zeta'_{osp}) = \frac{exp\left\{-\dfrac{[\zeta_{osp}+\eta_0 t]^2}{2\sigma_1^2}\right\}}{(2\pi\sigma_1\sigma_2)^2}$$

$$* \left\{\sigma_2^2\ exp\left\{-\frac{\eta_0^2}{2\sigma_2^2}\right\} - \eta_0\sqrt{\frac{\pi\sigma_2^2}{2}}\ erfc(\eta_0/\sqrt{2\sigma_2^2})\right\}^2 \cdot$$

$$* \frac{\sigma_1}{\eta_o}\sqrt{\frac{\pi}{2}}\left\{erfc(\zeta_{osp}/\sqrt{2\sigma_1}) - erfc((\zeta_{osp}+\eta_0 t)/\sqrt{2\sigma_1})\right\}$$

$$\doteq g^{(2)}\ [S(t),\ t\ |\zeta_{osp},\ \zeta'_{osp})], \tag{5.33}$$

where $g^{(2)}\ [S(t),\ t\ |\zeta_{osp},\ \zeta'_{osp})]$ is the second order approximation to $g[S(t),\ t\ |\zeta_{osp},\ \zeta'_{osp})]$. To obtain the probability of a point on the surface being shadowed we must perform another integration over the t variable (see 5.2). We obtain:

$$\int_0^{\tau} dt \int_0^{t} dt_1 \; W_2(t, t_1 \,|\zeta_{osp}, \zeta'{}_{osp}) =$$

$$* \; \frac{1}{(2\pi\sigma_1\sigma_2)^2} \left\{ \sigma_2^2 \; exp\left\{ -\frac{\eta_0^2}{2\sigma_2^2} \right\} - \eta_0 \sqrt{\frac{\pi\sigma_2^2}{2}} \;\; erfc(\eta_0/\sqrt{2\sigma_2^2}) \right\}^2$$

$$* \; \frac{\sigma_1}{\eta_0}\sqrt{\frac{\pi}{2}}[erfc(\zeta_{osp}/\sqrt{2\sigma_1^2})\frac{\sigma_1}{\eta_0}\sqrt{\frac{\pi}{2}}\left\{ erfc(\zeta_{osp}/\sqrt{2\sigma_1^2}) \; - \; erfc((\zeta_{osp} + \eta_0\tau)/\sqrt{2\sigma_1^2}) \right\} \; +$$

$$\frac{\sqrt{2\sigma_1^2}}{\eta_0}\,\frac{\sqrt{\pi}}{4} \; \left\{ erfc^2((\zeta_{osp} + \eta_0\tau)/\sqrt{2\sigma_1^2}) \; - erfc^2(\zeta_{osp}/\sqrt{2\sigma_1^2}) \right\}]. \qquad (5.34)$$

If we now take the limit as $\tau \rightarrow \infty$ and obtain

$$\int_0^{\infty} dt \int_0^{t} dt_1 \; W_2(t, t_1 \,|\zeta_{osp}, \zeta'_{osp}) =$$

$$\frac{\sigma_2^2}{16\pi{\eta_0}^2}\left\{ \sigma_2^2 \; exp\left\{ -\frac{\eta_0^2}{2\sigma_2^2} \right\} - \eta_0 \sqrt{\frac{\pi\sigma_2^2}{2}} \;\; erfc\;(\eta_o/\sqrt{2\sigma_2^2}) \right\}^2$$

$$* \; erfc^2(\zeta_{osp}/\sqrt{2\sigma_1^2}). \qquad (5.35)$$

If we now let $\nu = \dfrac{\eta_0}{\sqrt{2\sigma_2^2}}$ as before we get

$$\int_0^{\infty} dt \int_0^{t} dt_1 \; W_2(t, t_1 \,|\zeta_{osp}, \zeta'_{osp}) = \frac{1}{2}\left(\frac{1}{4\sqrt{\pi}\nu}\right)^2 \left\{ exp\{-\nu^2\} - \sqrt{\pi}\nu \;\; erfc(\nu) \right\}^2$$

$$* \; erfc^2(\zeta_{osp}/\sqrt{2\sigma_1}). \qquad (5.36)$$

But this is exactly $\int_0^{\infty} dt \; W_1(t \,|\zeta_{osp}, \zeta'_{osp})$ squared over 2 (see 5.23).

Performing a final integration over the heights yields

$$\int_{-\infty}^{\infty} d\zeta_{osp} \int_0^{\infty} dt \int_0^{t} dt_1 \; W_2(t, t_1 \,|\zeta_{osp}, \zeta'_{osp}) \; p(\zeta_{osp}) =$$

$$\frac{2}{3}\left(\frac{1}{4\sqrt{\pi}\nu}\right)^2 \left\{ exp\{-\nu^2\} - \sqrt{\pi}\nu \; erfc(\nu) \right\}^2, \quad (5.37)$$

so that, thus far, we have:

$$\begin{aligned}
\int_0^{\infty} g[S(t), t \,|\zeta_{osp}, \zeta'_{osp})] \, dt \quad &\approx \int_0^{\infty} g^{(1)}[S(t), t \,|\zeta_{osp}, \zeta'_{osp})] \, dt \; + \\
&\quad \int_0^{\infty} g^{(2)}[S(t), t \,|\zeta_{osp}, \zeta'_{osp})] \, dt \\
&= \int_0^{\infty} dt \; W_1(t \,|\zeta_{osp}, \zeta'_{osp}) \\
&\quad - \int_0^{\infty} dt \int_0^{t} dt_1 \; W_2(t, t_1 \,|\zeta_{osp}, \zeta'_{osp}),
\end{aligned} \quad (5.38)$$

and so

$$\begin{aligned}
\int_0^{\infty} g[S(t), t \,|\zeta_{osp}, \zeta'_{osp})] \, dt \approx & \frac{1}{4\sqrt{\pi}\nu}\left\{ exp\{-\nu^2\} - \sqrt{\pi}\nu \; erfc(\nu) \right\} erfc(\zeta_{osp}/\sqrt{2\sigma_1}) \\
& -\frac{1}{2}\left(\frac{1}{4\sqrt{\pi}\nu}\right)^2 \left\{ exp\{-\nu^2\} - \sqrt{\pi}\nu \, erfc(\nu) \right\}^2 erfc^2(\zeta_{osp}/\sqrt{2\sigma_1}).
\end{aligned} \quad (5.39)$$

We also have, from (5.1), that

$$P(\hat{k}_i|\zeta'_{osp}) \approx 1 - \frac{1}{4\sqrt{\pi}\nu}\left\{exp\{-\nu^2\} - \sqrt{\pi}\nu\ erfc(\nu)\right\} +$$

$$\frac{2}{3}\left(\frac{1}{4\sqrt{\pi}\nu}\right)^2\left\{exp\{-\nu^2\} - \sqrt{\pi}\nu\ erfc(\nu)\right\}^2. \qquad (5.40)$$

At this point one might wonder whether the form for $\int_0^\infty g[S(t),\ t\ |\zeta_{osp},\ \zeta'_{osp})]\ dt$ might continue in a Taylor series such that

$$\int_0^\infty g[S(t),\ t\ |\zeta_{osp},\ \zeta'_{osp})]\ dt = X - \frac{X^2}{2!} + \frac{X^3}{3!} -, \qquad (5.41)$$

where $X = \frac{1}{4\sqrt{\pi}\nu}\left\{exp\{-\nu^2\} - \sqrt{\pi}\nu\ erfc(\nu)\right\} erfc(\zeta_{osp}/\sqrt{2\sigma_1})$. We try to convince ourselves of this by computing yet another term in the series.

5.5 Calculation of $W_3(t,\ t_1,\ t_2\ |\zeta_{osp},\ \zeta'_{osp})$ Assuming No Correlation Between the Shadowed Point and the Shadowing Points.

We proceed in the same manner as before starting with

$$W_3(t,\ t_1,\ t_2\ |\zeta_{osp},\ \zeta'_{osp}) = \int_{\eta_0}^\infty d\zeta' \int_{\eta_0}^\infty d\zeta'_1 \int_{\eta_0}^\infty d\zeta'_2\ (\zeta'(t) - \eta_0)\ (\zeta'(t_1) - \eta_0)(\zeta'(t_2) - \eta_0)$$

$$* \ p(\zeta_{osp} + \eta_0 t,\ \zeta'(t),\ \zeta_{osp} + \eta_0 t_1,\ \zeta'(t_1),\ \zeta_{osp} + \eta_o t_2,\ \zeta'(t_2)\ |\zeta_{osp},\ \zeta'_{osp}). \qquad (5.42)$$

Assuming the points $(\zeta_{osp},\ \zeta'_{osp})$, $(\zeta_{osp} + \eta_0 t,\ \zeta'(t))$, $(\zeta_{osp} + \eta_0 t_1,\ \zeta'(t_1))$, and $(\zeta_{osp} + \eta_0 t_2,\ \zeta'(t_2))$ are uncorrelated, and setting $S(t) = \zeta_{osp} + \eta_0 t$, $S(t_1) = \zeta_{osp} + \eta_0 t_1$, and $S(t_2) = \zeta_{osp} + \eta_0 t_2$, where $t_3 \rightarrow t$ for (5.2), we have

$$p(S(t),\ \zeta'(t),\ S(t_1),\ \zeta'(t_1),\ S(t_2),\ \zeta'(t_2)\ |\zeta_{osp},\zeta'_{osp}) = \\ p(S(t),\ \zeta'(t))\ p(S(t_1),\ \zeta'(t_1))\ p(S(t_2),\zeta'(t_2)). \qquad (5.43)$$

Using (5.9) and (5.10) we further simplify matters such that

$$p(S(t),\ \zeta'(t))\ p(S(t_1),\ \zeta'(t_1))\ p(S(t_2),\zeta'(t_2)) = \\ p(S(t))p(\zeta'(t))p(S(t_1))p(\zeta'(t_1))p(S(t_2))p(\zeta'(t_2)). \qquad (5.44)$$

So, W_3 now becomes, substituting (5.44) and (5.43) into (5.42)

$$W_3(t,\ t_1,\ t_2\ |\zeta_{osp},\ \zeta'_{osp}) = p(S(t)) \int_{\eta_0}^{\infty} d\zeta\ (\zeta'(t) - \eta_0)\ p(\zeta'(t))$$
$$* \ p(S(t_1)) \int_{\eta_0}^{\infty} d\zeta'_1\ (\zeta'(t_1) - \eta_0) p(\zeta'_1(t_1))$$
$$* \ p(S(t_2)) \int_{\eta_0}^{\infty} d\zeta'_2\ (\zeta'(t_2) - \eta_o)\ p(\zeta'(t_2)). \qquad (5.45)$$

Each of the integrals are in the same form as W_1 (5.11). Therefore, we have, using (5.18)

$$W_3(t,\ t_1,\ t_2\ |\zeta_{osp},\ \zeta'_{osp}) =$$

$$\frac{exp\left\{-\frac{[\zeta_{osp} + \eta_0 t]^2}{2\sigma_1^2}\right\}\ exp\left\{-\frac{[\zeta_{osp} + \eta_0 t_1]^2}{2\sigma_1^2}\right\}\ exp\left\{-\frac{[\zeta_{osp} + \eta_0 t_2]^2}{2\sigma_1^2}\right\}}{(2\pi\sigma_1\sigma_2)^3}$$

$$* \left\{\sigma_2^2\ exp\left\{-\frac{\eta_0^2}{2\sigma_2^2}\right\} - \eta_0\sqrt{\frac{\pi\sigma_2^2}{2}}\ erfc(\eta_0/\sqrt{2\sigma_2^2})\right\}^3. \qquad (5.46)$$

We proceed next to calculate the third order approximation of the first passage density and also the third order term to the shadowing function. This will give us a clear pattern, concerning the behavior of the first passage density under the no correlation assumption.

We define

$$g^{(3)}[S(t),\, t\; |\zeta_{osp},\, \zeta'_{osp})] \doteq \int_0^t dt_1 \int_{t_1}^t dt_2\; W_3(t,\, t_1,\, t_2\; |\zeta_{osp},\, \zeta'_{osp}) \tag{5.47}$$

from (5.2) and perform the inner integral to find that

$$\int_{t_1}^t dt_2\; W_3(t,\, t_1,\, t_2\; |\zeta_{osp},\, \zeta'_{osp}) = \frac{exp\left\{-\frac{[\zeta_{osp}+\eta_0 t]^2}{2\sigma_1^2}\right\}\; exp\left\{-\frac{[\zeta_{osp}+\eta_0 t_1]^2}{2\sigma_1^2}\right\}}{(2\pi\sigma_1\sigma_2)^3}$$

$$* \left\{\sigma_2^2\; exp\left\{-\frac{\eta_0^2}{2\sigma_2^2}\right\} - \eta_0\sqrt{\frac{\pi\sigma_2^2}{2}}\; erfc(\eta_0/\sqrt{2\sigma_2^2})\right\}^3$$

$$* \frac{\sigma_1}{\eta_0}\sqrt{\frac{\pi}{2}}\left\{erfc[(\zeta_{osp}+\eta_0 t_1)/\sqrt{2\sigma_1^2}\,] - erfc[(\zeta_{osp}+\eta_0 t)/\sqrt{2\sigma_1^2}\,]\right\}. \tag{5.48}$$

We then have

$$\int_0^t dt_1 \int_{t_1}^t dt_2 \; W_3(t, t_1, t_2 \,|\zeta_{osp}, \zeta'_{osp}) = \frac{exp\left\{ -\frac{[\zeta_{osp} + \eta_0 t]^2}{2\sigma_1^2} \right\}}{(2\pi\sigma_1\sigma_2)^3}$$

$$* \left\{ \sigma_2^2 \; exp\left\{ -\frac{\eta_0^2}{2\sigma_2^2} \right\} - \eta_0 \sqrt{\frac{\pi\sigma_2^2}{2}} \; erfc(\eta_0/\sqrt{2\sigma_2^2}) \right\}^3$$

$$* \frac{\sigma_1}{\eta_0} \sqrt{\frac{\pi}{2}} \left\{ \int_0^t dt_1 \, exp\left\{ -\frac{[\zeta_{osp} + \eta_0 t_1]^2}{2\sigma_1^2} \right\} erfc[(\zeta_{osp} + \eta_0 t_1)/\sqrt{2\sigma_1^2}\,] - \right.$$

$$\left. erfc[(\zeta_{osp} + \eta_0 t)/\sqrt{2\sigma_1^2}\,] \int_0^t dt_1 \, exp\left\{ -\frac{[\zeta_{osp} + \eta_0 t_1]^2}{2\sigma_1^2} \right\} \right\}.$$

Both of these integrals are known, and are:

$$\int_0^t dt_1 \; exp\left\{ -\frac{[\zeta_{osp} + \eta_0 t_1]^2}{2\sigma_1^2} \right\} erfc[(\zeta_{osp} + \eta_0 t_1)/\sqrt{2\sigma_1^2}\,] =$$

$$-\frac{\sigma_1\sqrt{2\pi}}{\eta_o \;\; 4} \left\{ erfc^2((\zeta_{osp} + \eta_0 t)/\sqrt{2\sigma_1}) \; - erfc^2(\zeta_{osp}/\sqrt{2\sigma_1}) \right\}$$

and

$$\int_0^t dt_1 \, exp\left\{ \frac{-[\zeta_{osp} + \eta_0 t_1]^2}{2\sigma_1^2} \right\} = \frac{\sigma_1}{\eta_0}\sqrt{\frac{\pi}{2}} \left\{ erfc(\zeta_{osp}/\sqrt{2\sigma_1}) \text{ - } erfc((\zeta_{osp} + \eta_0 t)/\sqrt{2\sigma_1}) \right\}.$$

So finally, we have

$$g^{(3)}[S(t), t\ |\zeta_{osp}, \zeta'_{osp})] = \int_0^t dt_1 \int_{t_1}^t dt_2\ W_3(t, t_1, t_2\ |\zeta_{osp}, \zeta'_{osp})$$

$$= \frac{exp\left\{-\frac{[\zeta_{osp}+\eta_0 t]^2}{2\sigma_1^2}\right\}}{(2\pi\sigma_1\sigma_2)^3}\ \frac{\sigma_1}{\eta_0}\sqrt{\frac{\pi}{2}}$$

$$* \left\{\sigma_2^2\ exp\left\{-\frac{\eta_0^2}{2\sigma_2^2}\right\} - \eta_0\sqrt{\frac{\pi\sigma_2^2}{2}}\ erfc(\eta_0/\sqrt{2\sigma_2^2})\right\}^3$$

$$* \left\{-\frac{\sigma_1\sqrt{2\pi}}{\eta_0\ 4}\left\{erfc^2((\zeta_{osp}+\eta_0 t)/\sqrt{2\sigma_1}) - \right.\right.$$

$$\left.\left. erfc^2(\zeta_{osp}/\sqrt{2\sigma_1})\right\}\right) - erfc[(\zeta_{osp}+\eta_0 t)/\sqrt{2\sigma_1^2}]$$

$$* \frac{\sigma_1}{\eta_0}\sqrt{\frac{\pi}{2}}\left\{erfc(\zeta_{osp}/\sqrt{2\sigma_1}) - erfc((\zeta_{osp}+\eta_0 t)/\sqrt{2\sigma_1})\right\}\Big\}. \qquad (5.49)$$

Performing the final integral $\int_0^T dt\ g^{(3)}[S(t), t\ |\zeta_{osp}, \zeta'_{osp})]$ we end up with

$$\int_0^{\tau} dt\ g^{(3)}[S(t),\ t\ |\zeta_{osp},\ \zeta'_{osp})] = \frac{1}{(2\pi\sigma_1\sigma_2)^3}\ \frac{\sigma_1}{\eta_0}\sqrt{\frac{\pi}{2}}$$

$$* \left\{\sigma_2^2\ exp\left\{-\frac{\eta_0^2}{2\sigma_2^2}\right\} - \eta_0\sqrt{\frac{\pi\sigma_2^2}{2}}\ erfc(\eta_0/\sqrt{2\sigma_2^2})\right\}^3$$

$$* \left\{-\frac{\sigma_1\sqrt{2\pi}}{\eta_0\ \ 4}\left(-\frac{1}{3}\frac{\sigma_1}{\eta_0}\sqrt{\frac{\pi}{2}}\left\{erfc^3((\zeta_{osp}+\eta_0\tau)/\sqrt{2\sigma_1}) - erfc^3(\zeta_{osp}/\sqrt{2\sigma_1})\right\}\right.\right.$$

$$\left. - erfc^2(\zeta_{osp}/\sqrt{2\sigma_1})\ \frac{\sigma_1}{\eta_0}\sqrt{\frac{\pi}{2}}\left\{erfc(\zeta_{osp}/\sqrt{2\sigma_1}) - erfc((\zeta_{osp}+\eta_0\tau)/\sqrt{2\sigma_1})\right\}\right) -$$

$$\frac{\sigma_1}{\eta_0}\sqrt{\frac{\pi}{2}}\left\{erfc(\zeta_{osp}/\sqrt{2\sigma_1})\left(-\frac{\sigma_1\sqrt{2\pi}}{\eta_0\ \ 4}\left\{erfc^2((\zeta_{osp}+\eta_0\tau)/\sqrt{2\sigma_1}) - erfc^2(\zeta_{osp}/\sqrt{2\sigma_1})\right\}\right)\right.$$

$$\left.\left. + \frac{1}{3}\frac{\sigma_1}{\eta_0}\sqrt{\frac{\pi}{2}}\left\{erfc^3((\zeta_{osp}+\eta_0\tau)/\sqrt{2\sigma_1}) - erfc^3(\zeta_{osp}/\sqrt{2\sigma_1})\right\}\right\}\right\}. \qquad (5.50)$$

Taking the limit as $\tau\rightarrow\infty$ we have

$$\int_0^{\infty} dt\ g^{(3)}[S(t),\ t\ |\zeta_{osp},\ \zeta'_{osp})] = \left\{exp\left\{-\frac{\eta_0^2}{2\sigma_2^2}\right\} - \frac{\eta_0}{\sigma_2}\sqrt{\frac{\pi}{2}}\ erfc(\eta_0/\sqrt{2\sigma_2^2})\right\}^3$$

$$* \left(\frac{\sigma_2}{2\pi\eta_0}\right)^3\sqrt{\frac{\pi}{2}}\ \frac{\pi}{12}\ erfc^3(\zeta_{osp}/\sqrt{2\sigma_1}). \qquad (5.51)$$

Now, letting $\nu = \dfrac{\eta_0}{\sqrt{2\sigma_2^2}}$ we get

$$\int_0^{\tau} dt\ g^{(3)}[S(t),\ t\ |\zeta_{osp},\ \zeta'_{osp})] = \frac{1}{3!}\left(\frac{1}{4\sqrt{\pi}\nu}\right)^3\left\{exp\{-\nu^2\} - \sqrt{\pi}\nu\ erfc(\nu)\right\}^3$$

$$* \ erfc^3(\zeta_{osp}/\sqrt{2\sigma_1}). \qquad (5.52)$$

But this is just $\int_0^\infty dt\ W_1(t\ |\zeta_{osp}, \zeta'_{osp})$ cubed divided by 3 ! (see 5.23). This gives us, so far,

$$\int_0^\infty g[S(t), t|\ \zeta_{osp}, \zeta'_{osp})]\ dt = X - \frac{X^2}{2!} + \frac{X^3}{3!} - \text{H.O.T.}, \tag{5.53}$$

where $X = \frac{1}{4\sqrt{\pi}\nu}\left\{ exp\{-\nu^2\} - \sqrt{\pi}\nu\ erfc(\nu)\right\} erfc(\zeta_{osp}/\sqrt{2\sigma_1})$.

Assuming the series continues is this form, we'll have

$$\int_0^\infty g[S(t), t\ |\zeta_{osp}, \zeta'_{osp})]\ dt = X - \frac{X^2}{2!} + \frac{X^3}{3!} - \frac{X^4}{4!} + \ldots \tag{5.54}$$

Using the Taylor series expansion for the exponential

$$exp\{-X\} = 1 - X + \frac{X^2}{2!} - \frac{X^3}{3!} + \frac{X^4}{4!} - \ldots,$$

gives us

$$-exp\{-X\} = -1 + X - \frac{X^2}{2!} + \frac{X^3}{3!} - \frac{X^4}{4!} + \ldots,$$

and so

$$1 - exp\{-X\} = X - \frac{X^2}{2!} + \frac{X^3}{3!} - \frac{X^4}{4!} + \ldots,$$

hence

$$\int_0^\infty g[S(t),\, t|\, \zeta_{osp},\, \zeta'_{osp})]\; dt = 1 - exp\{-X\}$$

$$= 1 - exp\Big\{ - \frac{1}{4\sqrt{\pi}\nu}\Big\{exp\{-\nu^2\} - \sqrt{\pi}\nu\ erfc(\nu)\Big\}\ erfc(\zeta_{osp}/\sqrt{2\sigma_1})\Big\}. \tag{5.55}$$

Thus, the shadowing function, conditional on the specular height and slope, appears to equal

$$P(\widehat{k}_i\ |\zeta_{osp}, \zeta'_{osp}) = 1 - \int_0^\infty g[S(t),\, t\ |\zeta_{osp},\, \zeta'_{osp})]\; dt$$

$$= exp\Big\{ - \frac{1}{4\sqrt{\pi}\nu}\Big\{exp\{-\nu^2\} - \sqrt{\pi}\nu\ erfc(\nu)\Big\}\ erfc(\zeta_{osp}/\sqrt{2\sigma_1})\Big\}. \tag{5.56}$$

We recognize this to be eq. (17) of Wagner [Wagner, 1967]. Wagner arrives at the same answer by making two assumptions. The first is that the probability that the surface crosses the ray in $d\tau$ *given* that it does not cross prior to $t=\tau$ may be approximated by the probability that the surface crosses the ray in $d\tau$ (both probabilities are also conditioned on the height and slope of the specular point at $t=0$). In order words, as his first assumption, the conditional first passage of time density function may be approximated by our W_1 (with correlation included). His second assumption is to neglect correlation in W_1, as we have done.

Thus, we conclude that the neglect of correlation between all shadowing points and between the shadowing points and the shadowed point is equivalent to

Wagner's assumptions stated above. We note here that the function is independent of the specular slope, which is an apparent result of the no correlation assumption.

In the same manner as Wagner, we can compute the shadowing function conditional only on the specular slope, by integrating over the heights. Using

$$P(\hat{k}_i \,|\zeta'_{osp}) = \int_{-\infty}^{\infty} d\zeta \; P(\hat{k}_i \,|\zeta_{osp}, \zeta'_{osp}) \; p(\zeta_{osp}),$$

where $p(\zeta_{osp}) = \dfrac{exp\left\{ -\dfrac{\zeta_{osp}^2}{2\sigma_1^2}\right\}}{\sqrt{2\pi\sigma_1^2}}$, we get

$$P(\hat{k}_i \,|\zeta'_{osp}) = \frac{1}{\sqrt{2\pi\sigma_1^2}} \int_{-\infty}^{\infty} d\zeta_{osp}$$

$$* \; exp\left\{ - \frac{1}{4\sqrt{\pi}\nu}\left\{ exp\{-\nu^2\} - \sqrt{\pi}\nu \; erfc(\nu)\right\} erfc(\zeta_{osp}/\sqrt{2}\sigma_1) \right\} exp\left\{ -\frac{\zeta_{osp}^2}{2\sigma_1^2}\right\}.$$

Using Wagner's notation, we let $B = \frac{1}{4\sqrt{\pi}\nu}\left\{ exp\{-\nu^2\} - \sqrt{\pi}\nu \; erfc(\nu)\right\}$ so

$$P(\hat{k}_i \,|\zeta'_{osp}) = \frac{1}{\sqrt{2\pi\sigma_1^2}} \int_{-\infty}^{\infty} d\zeta_{osp} \;\; exp\left\{ -\; B \; erfc(\zeta_{osp}/\sqrt{2}\sigma_1)\right\} \;\; exp\left\{ -\frac{\zeta_{osp}^2}{2\sigma_1^2}\right\}$$

Letting $\kappa = B \; erfc(\zeta_{osp}/\sqrt{2}\sigma_1)$ and $d\kappa/d\zeta_{osp} = -B\sqrt{\frac{2}{\sigma_1^2\pi}} \;\; exp\left\{ -\frac{\zeta_{osp}^2}{2\sigma_1^2}\right\}$, we get

$$P(\widehat{k}_i \,|\zeta'_{osp}) = \frac{1}{2B}\int_0^{2B} d\kappa \; exp\{-\kappa\} = \frac{1}{2B}[1 - exp\{-2B\}], \qquad (5.57)$$

where $B = \frac{1}{4\sqrt{\pi}\nu}\left\{exp\{-\nu^2\} - \sqrt{\pi}\nu \; erfc(\nu)\right\}$, following Wagner's notation.

We notice that the shadowing function is independent of surface height as well as specular slope. We expect that the independence of height is due to the high frequency assumption and the independence of specular slope due to the no correlation assumption.

A plot of this function (5.57), which is the same as Wagner's uncorrelated shadowing function [Wagner, 1967], is given in Figure 14 for incident angles varying from 0 to 90 degrees with σ_2^2 (S2)= 0.05, 0.1, 0.3, 0.5 and 2.0. We can see that for a given angle of incidence, as the variance of the slopes increase, the shadowing increases and the shadowing function (which is a probability of illumination) decreases. Also, as we expected, for a given σ_2^2, as the incident angle goes to 90 degrees the shadowing function also goes to zero; and as the incident angle goes to 0 degrees (illumination from directly overhead) the shadowing function goes to 1.

Before we proceed to investigate the affects of the no correlation assumption, we would like to compute the normalized radar cross section (NRCS) (2.45) derived in Chapter 2. We will consider the case in which the electric field is perpendicular to the plane of incidence (defined by the $\widehat{z}$ and $\widehat{x}$ axis), were the incident ray, $\widehat{k}_i$, has no component in the $\widehat{y}$ direction.

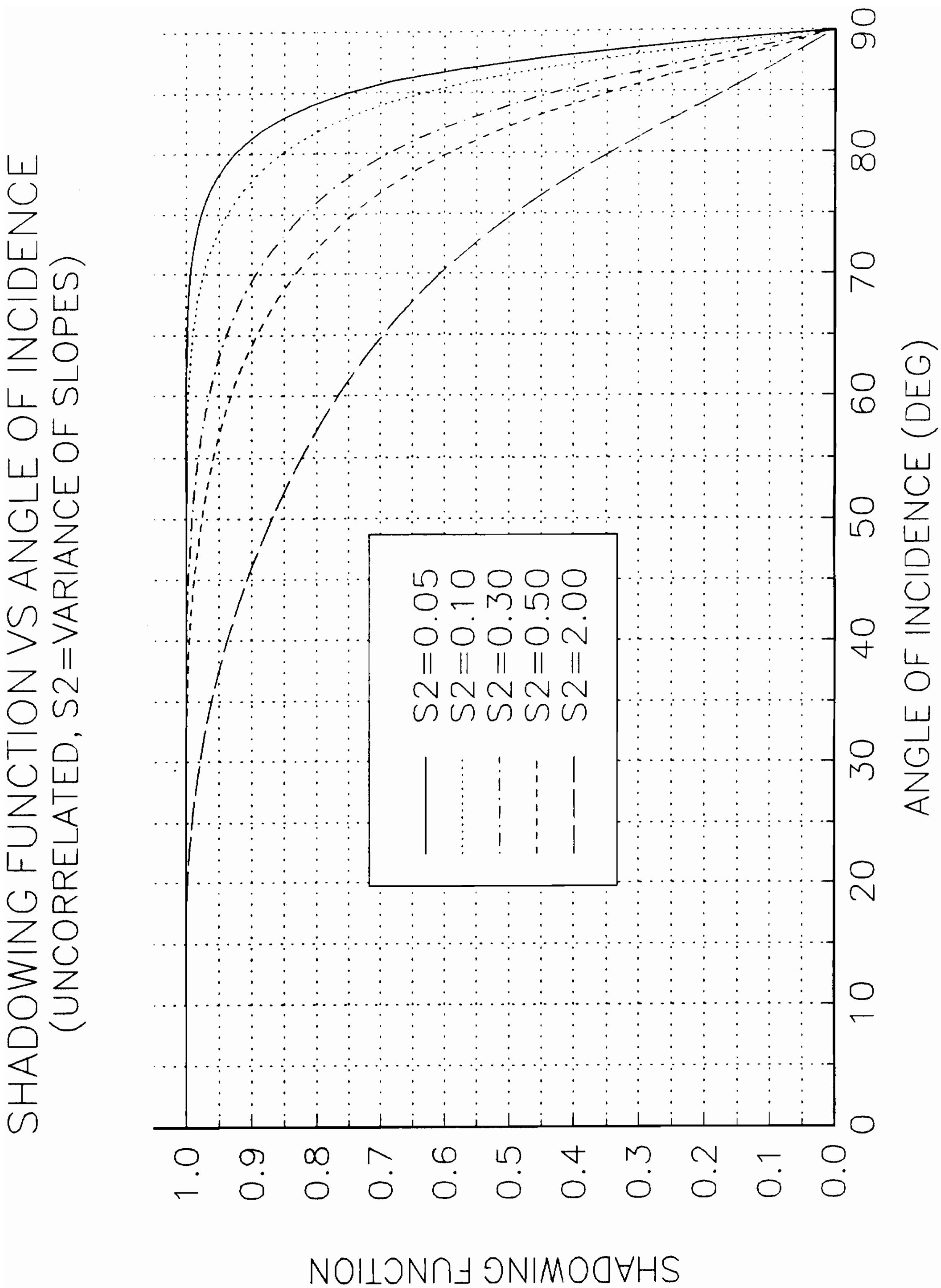

Figure 14: Shadowing Function vs Angle of Incidence

This case, called TE for transverse electric, turns out to produce the same result as when the magnetic field is perpendicular to the plane of incidence (TM) because we are dealing with backscatter.

Restated for convenience, the radar cross section from Chapter 2 is

$$\sigma^\circ = \frac{4\pi}{q_3^2}\left\{\hat{k}\times\hat{k}\times\vec{N}\left(-\frac{q_1}{q_3}, -\frac{q_2}{q_3}\right)\times\hat{h}_0\right\}^2 p\left(-\frac{q_1}{q_3}, -\frac{q_2}{q_3}\right)P\left(\hat{k}_i \,\middle|\, -\frac{q_1}{q_3}, -\frac{q_2}{q_3}\right), \tag{5.58}$$

for $\hat{k}=-\hat{k}_i$ (backscatter) and for TE polarization

$$\hat{h}_0 = -\cos(\theta_i)\hat{x} + \sin(\theta_i)\hat{z}. \tag{5.59}$$

From (2.27) we find

$$q_1 = 2\sin(\theta_i), \tag{5.60a}$$

$$q_2 = 0, \tag{5.60b}$$

$$q_3 = 2\cos(\theta_i). \tag{5.60c}$$

Using (5.60), we find

$$\vec{N}\left(-\frac{q_1}{q_3}, -\frac{q_2}{q_3}\right) = \tan(\theta_i)\hat{x} + \hat{z} \tag{5.61}$$

and since $\hat{k}_i = -\sin(\theta_i)\hat{x} - \cos(\theta_i)\hat{z}$ and $\hat{k}=-\hat{k}_i$ we find

$$\left\{\hat{k} \times \hat{k} \times \vec{N}(-\tfrac{q_1}{q_3}, -\tfrac{q_2}{q_3}) \times \hat{h}_0\right\}^2 = [\sin(\theta_i)\tan(\theta_i) + \cos(\theta_i)]^2 \tag{5.62}$$

and

$$p(-\tfrac{q_1}{q_3}, -\tfrac{q_2}{q_3}) = \frac{1}{2\pi\sigma_2^2}\, exp\left\{-\frac{\left(\frac{q_1}{q_3}\right)^2}{2\sigma_2^2}\right\} \tag{5.63}$$

Substituting (5.59)-(5.63) into (5.58) we find

$$\sigma^\circ = \frac{\pi}{\cos^2(\theta_i)}[\sin(\theta_i)\tan(\theta_i) + \cos(\theta_i)]^2 P(\hat{k}_i \mid -\tfrac{q_1}{q_3}, -\tfrac{q_2}{q_3})$$
$$* \frac{1}{2\pi\sigma_2^2}\, exp\left\{-\frac{\left(\frac{q_1}{q_3}\right)^2}{2\sigma_2^2}\right\}. \tag{5.64}$$

A plot of (5.64) vs incident angle, with and without the above shadowing function is given in Figure 15-19. We note three major points about these graphs. The first is that as the variance of the slopes increase the peak radar cross section shifts away from 0° incidence. This is apparently due to the fact that the Kirchhoff approximation is breaking down as the slopes become large, since we expect the greatest return to always be at normal incidence. We also notice that the plots are symmetric about 0° since the radar cross section and shadowing are the same regardless of which direction the source illuminates the surface (isotropic assumption).

The second major point is that when shadowing is included we see a reduction in the radar cross section for slopes with a variance, σ_2^2 (S2), equal to 0.5 and greater.

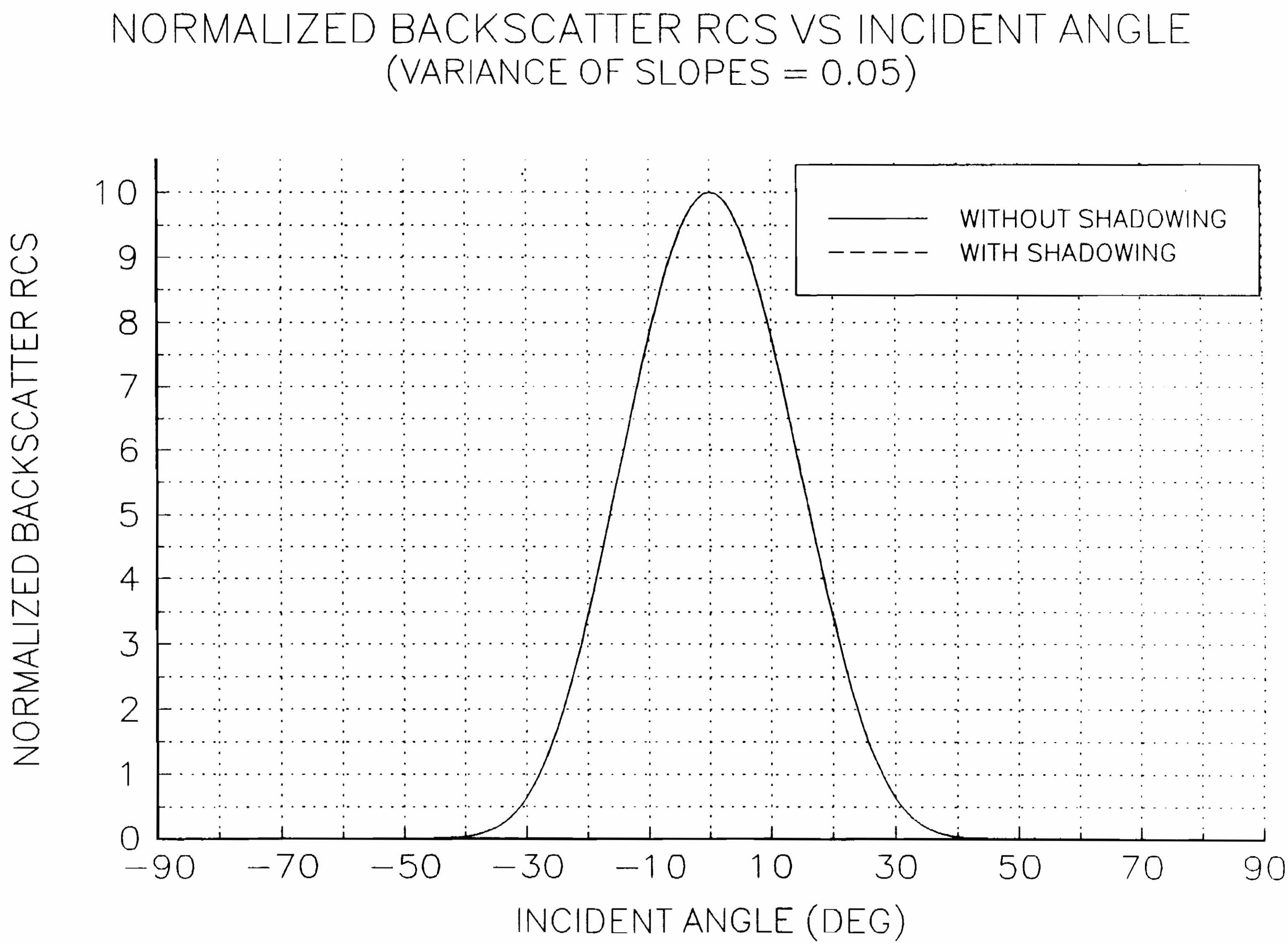

Figure 15: Normalized Backscatter RCS vs Incident Angle (Variance of Slopes=0.05)

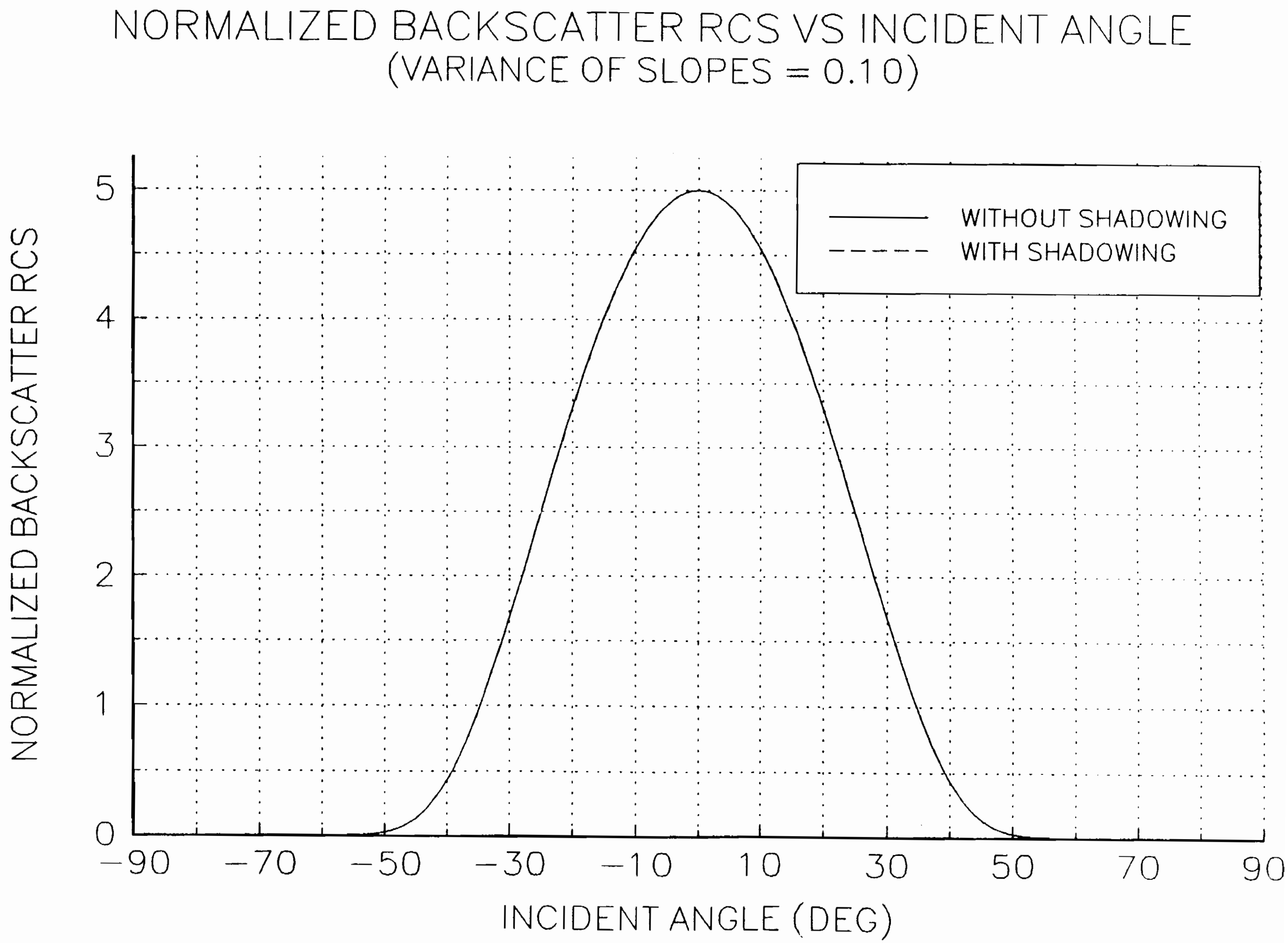

Figure 16: Normalized Backscatter RCS vs Incident Angle (Variance of Slopes=0.10)

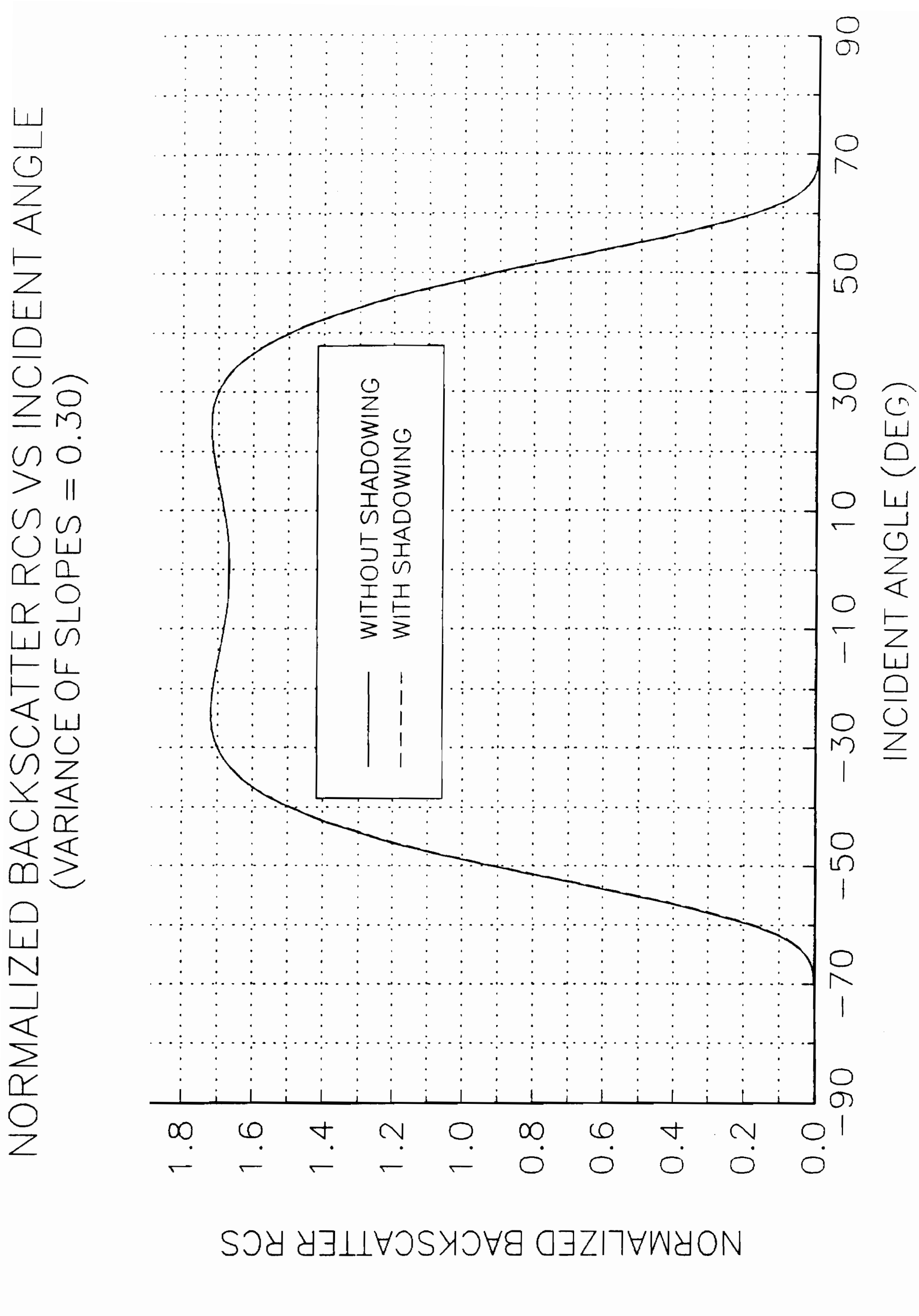

Figure 17: Normalized Backscatter RCS vs Incident Angle (Variance of Slopes=0.30)

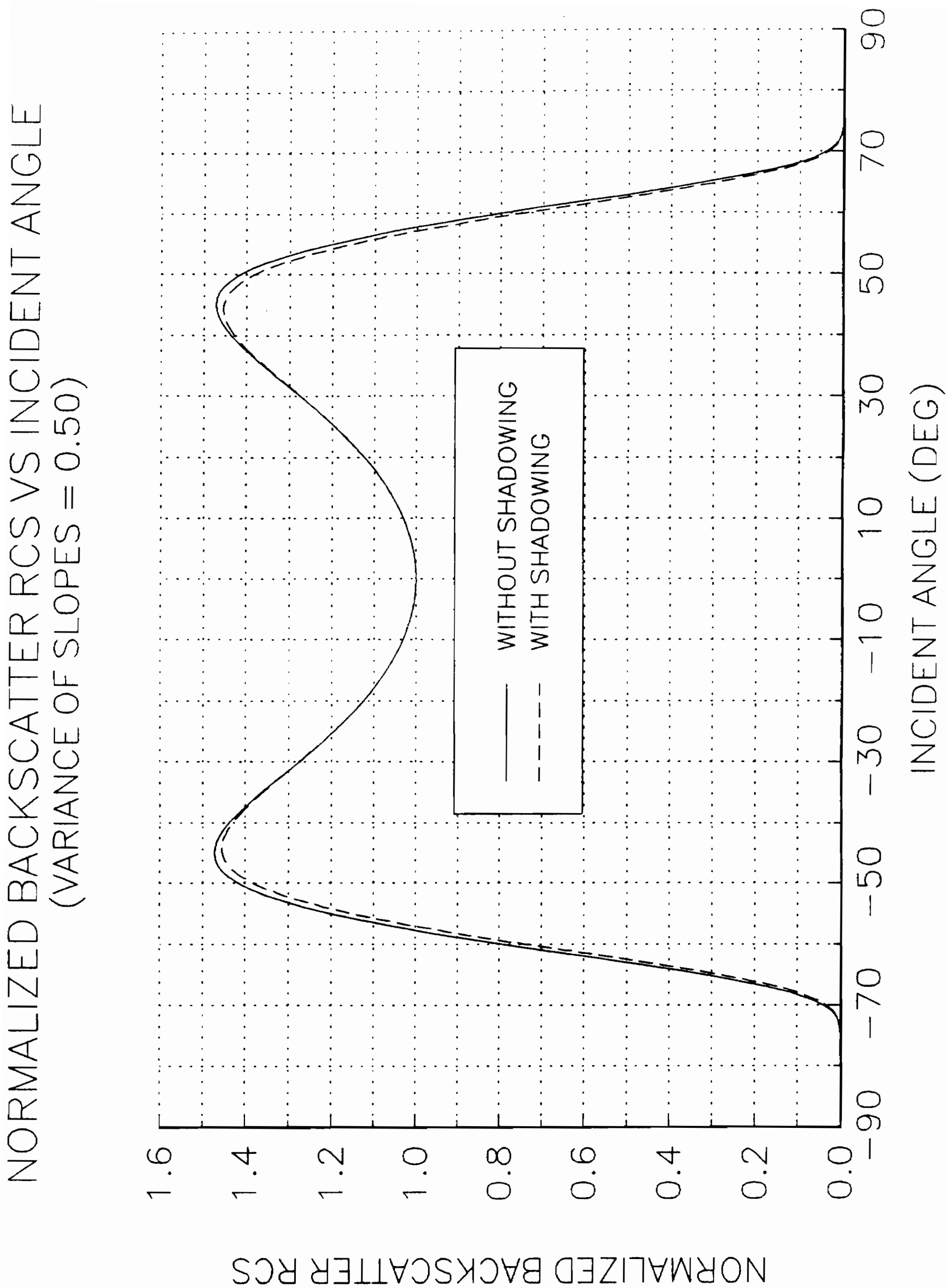

Figure 18: Normalized Backscatter RCS vs Incident Angle (Variance of Slopes=0.50)

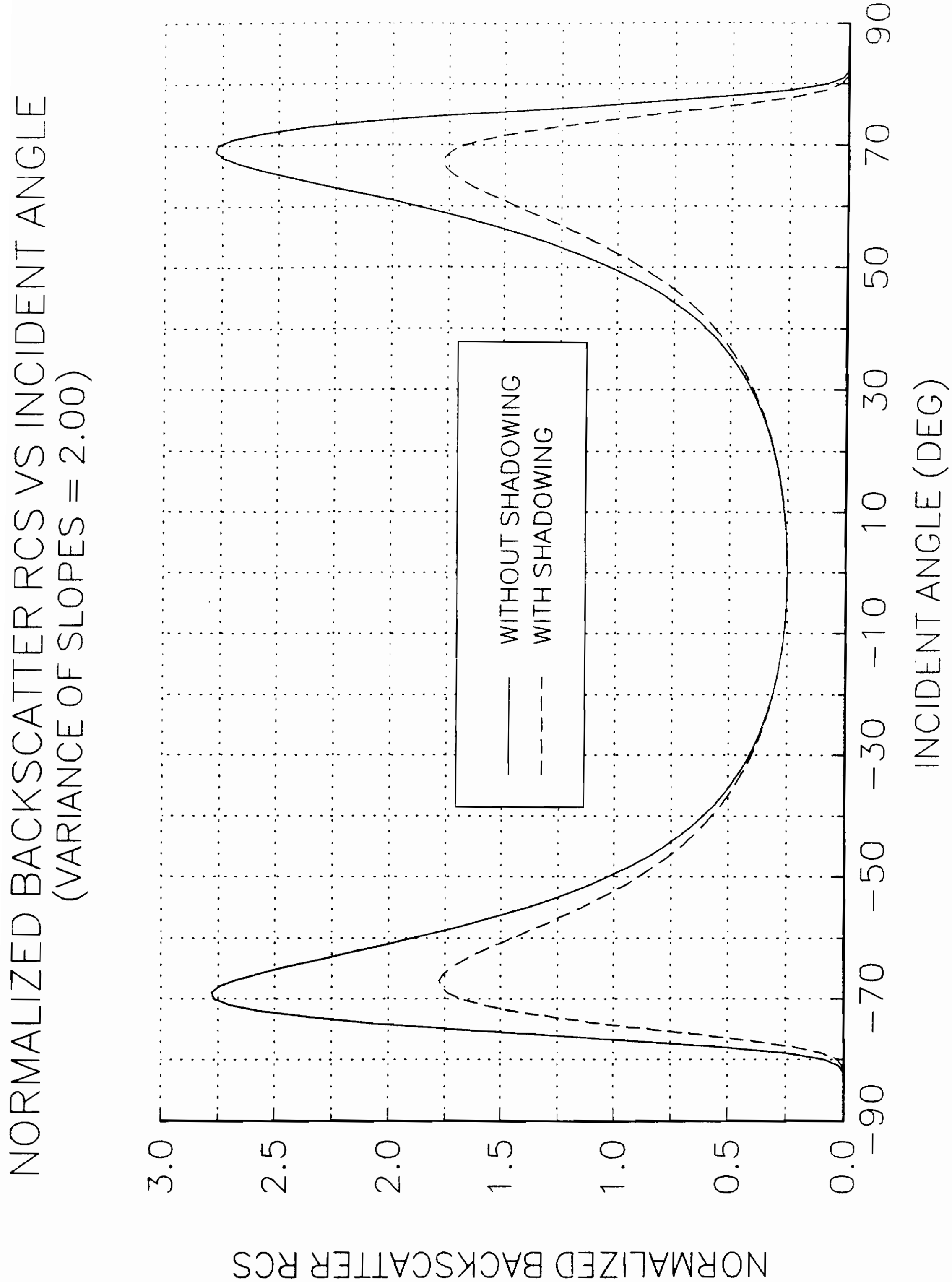

Figure 19: Normalized Backscatter RCS vs Incident Angle (Variance of Slopes=2.00)

Notice the shadowing begins to take affect around an incident angle of 40°. As the variance of slope increases to 2.0, we see a profound difference between the two results (Figure 19). The radar cross section suffers a drop of approximately 36% at its highest point. However, it is doubtful that shadow- corrected Kirchhoff theory is valid for these large slopes due to multiple scattering effects.

The third point to notice is that as the slope variance increases from 0.05 to 2.0 the peak level of the radar cross section decreases and then increases. As shadowing is introduced this effect diminishes but is still apparent. We see from Figure 19 that even when the variance of the slopes is 2.0 (a very rough surface), the peak of the radar cross section without shadowing is still 2.76, a relatively large increase from 1.47 at a variance of slope equal to 0.5. However, for the shadow-corrected Kirchhoff result, the peak NRCS increases from 1.45 to only 1.76, a improvement of about 2 dB for σ_2^2=2.0. Thus, we are apparently noticing the breakdown of the Kirchhoff approximation as correlation length of the surface becomes smaller.

As the incident angle becomes close to 90° the discrepancy between the radar cross section with and without shadowing becomes pronounced. This is shown in Figure 20.

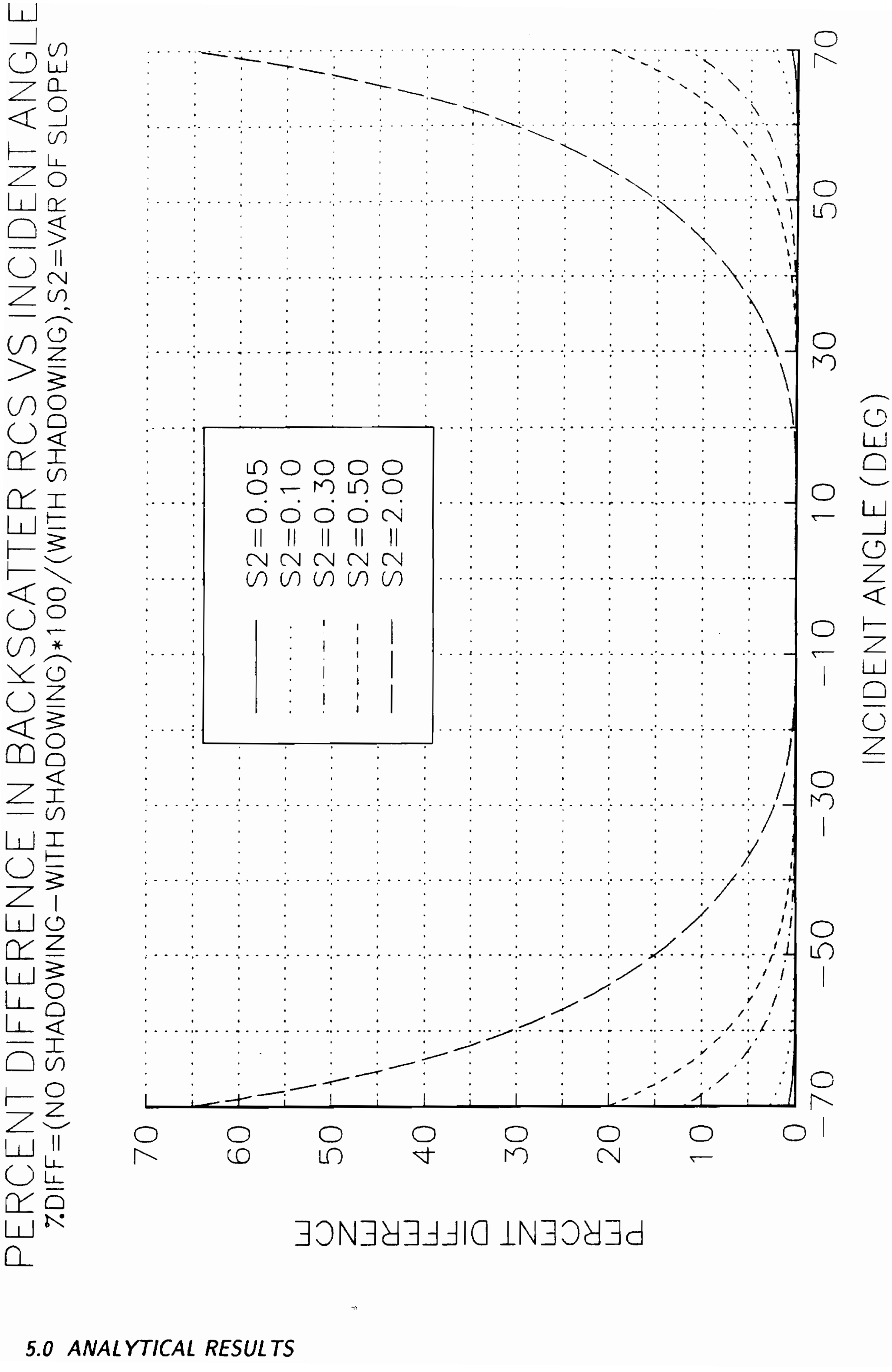

Figure 20: Percent Difference in Backscatter RCS vs Incident Angle

For incident angles greater than 70° and a variance of the slopes greater than 0.5, the percent difference, defined to be

$$\% \text{ Diff} = \frac{\text{RCS w/o Shadowing - RCS w/Shadowing}}{\text{RCS w/Shadowing}} \times 100 \tag{5.65}$$

approaches 100% very rapidly, and the inclusion of shadowing affects becomes mandatory. We see a dramatic increase in the percent (%) difference as the incident angle further increases. One can easily see that since the NRCS with shadowing equals the RCS without shadowing multiplied by the shadowing function $S(\theta_i)$ that the

$$\% \text{ Diff} = \frac{1 - S(\theta_i)}{S(\theta_i)} \times 100, \tag{5.66}$$

as $S(\theta_i) \rightarrow 0$, approaches infinity.

We now wish to investigate further the assumption of no correlation between points which intersect the mathematical ray impinging on the surface. We will proceed analytically as far as possible, but we find that the terms become too complicated when correlation is included to obtain closed form results and we are forced to resort to numerical integration techniques to carry through to a result.

6.0 NUMERICAL RESULTS

6.1 Introduction

Calculation of the shadowing function in Chapter 5 was simplified considerably by neglecting correlation between the shadowing points and the shadowed point. The physical meaning of the assumptions made by Wagner [Wagner, 1967] in developing his shadowing functions have been interpreted by calculating the infinite series (5.2) in this limit. However, inclusion of correlation in our analysis and its effect remains to be completed. In an attempt to resolve this matter we now proceed to determine the shadowing function, with the inclusion of correlation, by numerical integration of the first three terms in (5.2) and compare these results to those obtained in Chapter 5 (which correspond to Wagner's shadowing function in which correlation was neglected), and also to Wagner's correlated shadowing function [Wagner, 1967] and Smith's work [Smith, 1967b].

An alternative to calculating the integrals numerically is to expand the autocorrelation function in a Taylor series around those points in which correlation becomes important, such as near the shadowed point. Along these lines, an analysis was performed by Wagner [Wagner, 1967] for a one dimensional integral related to the shadowing function. In that paper, Wagner was forced to compute the Taylor series out to the seventh term in order to get a leading order result for this integral. This was due to cancellation of terms in the determinant of the covariance matrix located in the kernel. Our case is much more complicated than that of Wagner's due to the multi-dimensional integration procedure and the order of the covariance matrix in the third term in the series (5.2) containing W_3. Given the length of derivations necessary when Wagner's result was repeated by us, we have decided that this approach would not be fruitful, and numerical techniques would be less error prone.

The solution to the first passage in time function has in only a few special cases been found in closed form. Unfortunately, our problem does not fall into any of these categories and we must calculate the first passage function and the shadowing function numerically to obtain an exact result. We begin this chapter by re-introducing the series in (5.2). Our approach is to carry through the analysis analytically until we cannot proceed any further, at which point we will use numerical integration in order to obtain a result. We proceed in a similar manner as in Chapter 5; the first three terms in the series (5.2) are computed. The first term provides us with a leading order approximation to the overall shadowing function while the second and third terms provide us with a upper and lower bound to the shadowing function, respectively, provided that the terms decrease in magnitude. Each term in (5.2) was computed individually to the level of the shadowing function so that the effects of the higher order terms on shadowing could be easily seen. The results were then summed according to (5.2).

6.2 Calculation of $W_1(t\,|\zeta_{osp}, \zeta'_{osp})$ with Correlation Between the Shadowing Point and the Shadowed Point.

The first term in (5.2) will provide the lowest order approximation to the first passage function and hence the shadowing function. We expect that this approximation will provide a good result for gently sloping surfaces away from grazing angles where multiple ray crossings by the surface have a low probability. We wish to perform as many integrations analytically as possible prior to numerical computation. We begin from (5.3) with

$$W_1(t\,|\zeta_{osp}, \zeta'_{osp}) = \int_{\eta_0}^{\infty} d\zeta' \,(\zeta'(t) - \eta_0)\, p(S(t), \zeta'(t)\,|\zeta_{osp}, \zeta'_{osp}), \tag{6.1}$$

were we have let the equation of the incoming ray $\zeta_{osp} + \eta_0 t = S(t)$. In order to isolate the variable of integration, $\zeta'(t)$, we use Baye's theorem to write the conditional probability density function in the integrand of (6.1) such that

$$p(S(t), \zeta'(t)\,|\zeta_{osp}, \zeta'_{osp}) = \frac{p(S(t), \zeta'(t), \zeta_{osp}, \zeta'_{osp})}{p(\zeta_{osp}, \zeta'_{osp})}. \tag{6.2}$$

From this point we can see more easily that

$$p(S(t), \zeta'(t)\,|\zeta_{osp}, \zeta'_{osp}) = p(\zeta'(t)\,|S(t), \zeta_{osp}, \zeta'_{osp})\, p(S(t)\,|\zeta_{osp}, \zeta'_{osp}), \tag{6.3}$$

where we have twice applied Baye's theorem to the numerator of (6.2) and canceled $p(\zeta_{osp}, \zeta'_{osp})$ in the denominator with the same term in the numerator. Therefore, we have that

$$W_1(t \,|\zeta_{osp}, \zeta'_{osp}) = p(S(t) \,|\zeta_{osp}, \zeta'_{osp}) \int_{\eta_0}^{\infty} d\zeta' \, (\zeta'(t) - \eta_0) p(\zeta'(t) \,|S(t), \zeta_{osp}, \zeta'_{osp}), \tag{6.4}$$

where $p(S(t)| \, \zeta_{osp}, \zeta'_{osp})$, independent of $\zeta'(t)$, is brought outside the integral. To perform this integration we break the integrand into two parts so that

$$W_1(t \,|\zeta_{osp}, \zeta'_{osp}) = p(S(t) \,|\zeta_{osp}, \zeta'_{osp}) \int_{\eta_0}^{\infty} d\zeta' \, \zeta'(t) p(\zeta'(t) \,|S(t), \zeta_{osp}, \zeta'_{osp})$$
$$- \eta_0 \, p(S(t) \,|\zeta_{osp}, \zeta'_{osp}) \int_{\eta_0}^{\infty} d\zeta' \, p(\zeta'(t) \,|S(t), \zeta_{osp}, \zeta'_{osp}). \tag{6.5}$$

For notational simplicity we note that

$$p(\zeta'(t) \,|S(t), \zeta_{osp}, \zeta'_{osp}) = N(m_1, k_1) \tag{6.6}$$

and

$$p(S(t) \,|\zeta_{osp}, \zeta'_{osp}) = N(m_2, k_2), \tag{6.7}$$

where $N(m_1, k_1)$ and $N(m_2, k_2)$ signify Normal (Gaussian) random probability density functions with conditional means m_1 and m_2, and condition variances k_1 and k_2. We note that m_1, m_2, k_1, and k_2 are all scalars and are equal to (see Appendix A)

$$m_1 = -\frac{1}{\sigma_1^2\rho_{24}^2 - \sigma_1^4\sigma_2^2 + \sigma_2^2\rho_{23}^2}\Big\{S(t)(\sigma_1^2\rho_{14}\rho_{24} - \sigma_2^2\rho_{23}\rho_{24}) + \zeta_{osp}[\rho_{24}(\sigma_1^2\sigma_2^2 - \rho_{24}^2) -$$

$$\rho_{14}\rho_{23}\rho_{24}] + \zeta'_{osp}[\rho_{23}\rho_{24}^2 - \rho_{14}(\sigma_1^4 - \rho_{23}^2)]\Big\} \tag{6.8}$$

$$m_2 = \frac{\zeta_{osp}\rho_{23}}{\sigma_1^2} + \frac{\zeta'_{osp}\rho_{24}}{\sigma_2^2} \tag{6.9}$$

$$k_1 = \sigma_2^2 + \frac{\rho_{24}^2(\sigma_1^2\sigma_2^2) - \rho_{14}\rho_{23}\rho_{24}^2 - \rho_{14}\rho_{23}\rho_{24}^2 + \rho_{14}^2(\sigma_1^4 - \rho_{23}^2)}{\sigma_1^2\rho_{24}^2 + (\rho_{23}^2 - \sigma_1^4)\sigma_2^2} \tag{6.10}$$

$$k_2 = \sigma_1^2 - \frac{\rho_{23}^2}{\sigma_1^2} - \frac{\rho_{24}^2}{\sigma_2^2}, \tag{6.11}$$

where

$$\rho_{23} \doteq E[S(t)\zeta_{osp}] = E[\zeta_{osp}S(t)] = R(t) = \sigma_1^2\ exp\Big\{-\frac{t^2}{l^2}\Big\}, \tag{6.12a}$$

$$\rho_{24} \doteq E[S(t)\zeta'_{osp}] = E[\zeta'_{osp}S(t)] = -E[\zeta'(t)\zeta_{osp}] = -E[\zeta_{osp}\zeta'(t)] \tag{6.12b}$$

$$= -\frac{dR(t)}{dt} = t\sigma_2^2\ exp\Big\{-\frac{t^2}{l^2}\Big\},$$

and

$$\rho_{14} \doteq E[\zeta'(t)\zeta'_{osp}] = -\frac{d^2R(t)}{dt^2} = (\sigma_2^2 - \frac{\sigma_2^4}{\sigma_1^2}\,t^2)\ exp\Big\{-\frac{t^2}{l^2}\Big\}. \tag{6.12c}$$

Writing $p(\zeta'(t)\ |S(t),\ \zeta_{osp},\ \zeta'_{osp})$ explicitly we have

$$p(\zeta'(t)\ |S(t),\ \zeta_{osp},\ \zeta'_{osp}) = \frac{1}{\sqrt{2\pi k_1}}\ exp\Big\{-\frac{(\zeta'(t) - m_1)^2}{2k_1}\Big\}. \tag{6.13}$$

Substituting this into the first integral in (6.5) we find

$$\int_{\eta_0}^{\infty} d\zeta' \; \zeta'(\mathrm{t}) \; \mathrm{p}(\zeta'(\mathrm{t}) \; |\mathrm{S}(\mathrm{t}), \zeta_{\mathrm{osp}}, \zeta'_{\mathrm{osp}}) = \sqrt{\frac{\mathrm{k}_1}{2\pi}} \; exp\left\{ -\frac{(\eta_0 - \mathrm{m}_1)^2}{2 \; \mathrm{k}_1} \right\} + \frac{\mathrm{m}_1}{2} \; erfc\left(\frac{\eta_0 - \mathrm{m}_1}{\sqrt{2 \; \mathrm{k}_1}}\right). \tag{6.14}$$

The second integral in (6.5) is easily evaluated by making a change of variables; doing so we find

$$\int_{\eta_0}^{\infty} d\zeta' \; \mathrm{p}(\zeta'(\mathrm{t}) \; |\mathrm{S}(\mathrm{t}), \zeta_{\mathrm{osp}}, \zeta'_{\mathrm{osp}}) = \frac{1}{2} \; erfc\left(\frac{\eta_0 - \mathrm{m}_1}{\sqrt{2 \; \mathrm{k}_1}}\right). \tag{6.15}$$

Combining these two results, we obtain

$$\mathrm{W}_1(\mathrm{t} \; |\zeta_{\mathrm{osp}}, \zeta'_{\mathrm{osp}}) = \mathrm{p}(\mathrm{S}(\mathrm{t}) \; |\zeta_{\mathrm{osp}}, \zeta'_{\mathrm{osp}}) \left\{ \sqrt{\frac{\mathrm{k}_1}{2\pi}} \; exp\left\{ -\frac{(\eta_0 - \mathrm{m}_1)^2}{2 \; \mathrm{k}_1} \right\} + \right.$$

$$\left. \frac{\mathrm{m}_1}{2} \; erfc\left(\frac{\eta_0 - \mathrm{m}_1}{\sqrt{2 \; \mathrm{k}_1}}\right) - \frac{\eta_0}{2} \; erfc\left(\frac{\eta_0 - \mathrm{m}_1}{\sqrt{2 \; \mathrm{k}_1}}\right) \right\}, \tag{6.16}$$

where $\mathrm{p}(\mathrm{S}(\mathrm{t}) \; |\zeta_{\mathrm{osp}}, \zeta'_{\mathrm{osp}}) = \frac{1}{\sqrt{2\pi k_2}} \; exp\left\{ -\frac{(\mathrm{S}(\mathrm{t}) \; - \; \mathrm{m}_2)^2}{2\mathrm{k}_2} \right\}$ is the PDF of the heights at a distance t.

Finally, for completeness we now relate this function to the shadowing function. Following (5.1), we have

$$
\begin{aligned}
P^{(1)}(\hat{k}_i \,|\, \zeta'_{osp}) &= 1 - \int_{-\infty}^{\infty} d\zeta'_{osp}\; p(\zeta'_{osp}) \int_0^{\infty} dt\; W_1(t \,|\zeta_{osp}, \zeta'_{osp}) \\
&= 1 - \int_{-\infty}^{\infty} d\zeta'_{osp}\; p(\zeta'_{osp}) \int_0^{\infty} dt\; p(S(t) \,|\zeta_{osp}, \zeta'_{osp}) \\
&\quad * \left\{ \sqrt{\frac{k_1}{2\pi}}\; exp\left\{ -\frac{(\eta_0 - m_1)^2}{2\,k_1} \right\} + \right. \\
&\quad \left. \frac{m_1}{2}\; erfc\left(\frac{\eta_0 - m_1}{\sqrt{2\,k_1}} \right) - \frac{\eta_0}{2}\; erfc\left(\frac{\eta_0 - m_1}{\sqrt{2\,k_1}} \right) \right\},
\end{aligned}
\tag{6.17}
$$

where $P^{(1)}(\hat{k}_i \,|\, \zeta'_{osp})$ is a first order approximation to $P(\hat{k}_i \,|\, \zeta'_{osp})$. Here we must keep in mind that m_1 and k_1 are functions of the autocorrelation function, which is a function of distance t and that m_1 is also a function of the slope ζ'_{osp}.

Following an unsuccessful effort to integrate this function analytically, we were forced to resort to numerical techniques to obtain results. These results are presented at the end of this chapter. We now continue in a similar manner on the second term in (5.2).

6.3 Calculation of $W_2(t_2, t_1 \,|\zeta_{osp}, \zeta'_{osp})$ with Correlation Between the Shadowing Point and the Shadowed Point.

The second term in (5.2) involves integration over W_2, as defined in (5.3). This is the first correction term to the first passage in time density function and forms the lower bound for this function (since, for the cases we consider, the terms in the series decrease in magnitude). Hence, it is also the first order correction term to the shadowing function and forms its upper bound. Restated for convenience, we have

$$W_2(t_2,\, t_1\, |\zeta_{osp},\, \zeta'_{osp}) = \int_{\eta_0}^{\infty} d\zeta'_1 \int_{\eta_0}^{\infty} d\zeta'_2\, (\zeta'(t_2)-\eta_0)(\zeta'(t_1)-\eta_0)$$

$$* \; p(S(t_2),\, \zeta'(t_2),\, S(t_1),\, \zeta'(t_1)\, |\zeta_{osp},\, \zeta'_{osp}), \quad (6.18)$$

where we have set $S(t_2) = \zeta_{osp} + \eta_0 t_2$ and $S(t_1) = \zeta_{osp} + \eta_0\, t_1$ for simplicity. Thus, the first correction term to the shadowing function, defined as $C^{(1)}$, becomes

$$C^{(1)} \doteq \int_{-\infty}^{\infty} d\zeta_{osp}\;\; p(\zeta_{osp}) \int_0^{\infty} dt \int_0^{t} dt_1 W_2(t,\, t_1\, |\zeta_{osp},\, \zeta'_{osp}), \qquad (6.19)$$

where we have set t_2=t.

We can see here that this term is actually a five-fold integration and we must attempt to reduce the result analytically as much as possible before attempting numerical solution. An additional integration in (6.18) and (6.19) can be performed if the last integral, the integration over of heights, ζ_{osp}, is calculated first; to do this we formally reverse the order of integration to give

$$C^{(1)} \doteq \int_0^{\infty} dt \int_0^{t} dt_1 \int_{\eta_0}^{\infty} d\zeta'_1 \int_{\eta_0}^{\infty} d\zeta'\; (\zeta'(t)-\eta_0)(\zeta'(t_1)-\eta_0) \int_{-\infty}^{\infty} d\zeta_{osp}\;\; p(\zeta_{osp})$$

$$* \; p(S(t),\, \zeta'(t),\, S(t_1),\, \zeta'(t_1)\, |\zeta_{osp},\, \zeta'_{osp}). \quad (6.20)$$

From Baye's theorem, we have

$$p(\zeta_{osp})\; p(S(t),\, \zeta'(t),\, S(t_1),\, \zeta'(t_1)\, |\zeta_{osp},\, \zeta'_{osp}) =$$

$$p(S(t_1),\, S(t),\, \zeta_{osp}\, |\zeta'(t_1),\, \zeta'(t),\, \zeta'_{osp})\; p(\zeta'(t_1),\, \zeta'(t)\, |\zeta'_{osp}), \qquad (6.21)$$

where we have used the fact that $p(\zeta_{osp}, \zeta'_{osp}) = p(\zeta_{osp})\, p(\zeta'_{osp})$. We end up with

$$C^{(1)} \doteq \int_0^\infty dt \int_0^t dt_1 \int_{\eta_0}^\infty d\zeta'_1 \int_{\eta_0}^\infty d\zeta' \, (\zeta'(t)-\eta_0)(\zeta'(t_1)-\eta_0)\; p(\zeta'(t_1), \zeta'(t)\,|\zeta'_{osp}) \cdot$$

$$\int_{-\infty}^{\infty} d\zeta_{osp}\; p(S(t_1), S(t), \zeta_{osp}\,|\zeta'(t_1), \zeta'(t), \zeta'_{osp}). \qquad (6.22)$$

We are now left with the task of computing the conditional means and conditional covariance matrix for $p(S(t_1), S(t_2), \zeta_{osp}\,|\zeta'(t_1), \zeta'(t_2), \zeta'_{osp})$ so as to perform the inner integral. We find that (see Appendix B)

$$p(S(t_1), S(t_2), \zeta_{osp}\,|\zeta'(t_1), \zeta'(t_2), \zeta'_{osp}) = \frac{1}{(2\pi)^{\frac{3}{2}}\,[det(\widehat{\underline{c}})]^{\frac{1}{2}}} *$$

$$exp\left\{ -\frac{1}{2}\begin{bmatrix} (\zeta_{osp}+\eta_0 t_1-\widehat{\widehat{m}}_1) & (\zeta_{osp}+\eta_0 t-\widehat{\widehat{m}}_2) & (\zeta_{osp}-\widehat{\widehat{m}}_3) \end{bmatrix} \begin{bmatrix} \tilde{c}_{11} & \tilde{c}_{12} & \tilde{c}_{13} \\ \tilde{c}_{21} & \tilde{c}_{22} & \tilde{c}_{13} \\ \tilde{c}_{31} & \tilde{c}_{32} & \tilde{c}_{33} \end{bmatrix} \cdot \right.$$

$$\left. \begin{bmatrix} (\zeta_{osp}+\eta_0 t_1-\widehat{\widehat{m}}_1) \\ (\zeta_{osp}+\eta_0 t-\widehat{\widehat{m}}_2) \\ (\zeta_{osp}-\widehat{\widehat{m}}_3) \end{bmatrix} \right\}. \qquad (6.23)$$

where

$$\underline{\hat{\hat{m}}} = \hat{\hat{\underline{c}}}_{12}\hat{\hat{\underline{c}}}_{22}^{-1}\hat{\hat{\underline{x}}}_2, \quad \hat{\hat{\underline{x}}}_2 = \begin{bmatrix} \zeta'(t_1) \\ \zeta'(t) \\ \zeta'_{osp} \end{bmatrix}$$

$$= \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} \zeta'(t_1) \\ \zeta'(t) \\ \zeta'_{osp} \end{bmatrix}$$

$$= \begin{bmatrix} \hat{\hat{m}}_1 \\ \hat{\hat{m}}_2 \\ \hat{\hat{m}}_3 \end{bmatrix}. \tag{6.24}$$

and $\hat{\hat{\underline{c}}} = \hat{\hat{\underline{c}}}_{11} - \hat{\hat{\underline{c}}}_{12}\hat{\hat{\underline{c}}}_{22}^{-1}\hat{\hat{\underline{c}}}_{21}$, where

$$\hat{\hat{\underline{c}}}_{11} = \begin{bmatrix} E[S^2(t_1)] & E[S(t_1)S(t)] & E[S(t_1)\zeta_{osp}] \\ E[S(t_1)S(t)] & E[S^2(t)] & E[\zeta_{osp}S(t)] \\ E[S(t_1)\zeta_{osp}] & E[\zeta_{osp}S(t)] & E[\zeta^2_{osp}] \end{bmatrix}, \tag{6.25a}$$

$$\hat{\hat{\underline{c}}}_{12} = \begin{bmatrix} E[S(t_1)\zeta'(t_1)] & E[S(t_1)\zeta'(t)] & E[S(t_1)\zeta'_{osp}] \\ E[\zeta(t)\zeta'(t_1)] & E[\zeta(t)\zeta'(t)] & E[\zeta(t)\zeta'_{osp}] \\ E[\zeta_{osp}\zeta'(t_1)] & E[\zeta_{osp}\zeta'(t)] & E[\zeta_{osp}\zeta'_{osp}] \end{bmatrix}, \tag{6.25b}$$

$$\hat{\hat{\underline{c}}}_{21} = \hat{\hat{\underline{c}}}^{T}_{12}, \tag{6.25c}$$

$$\text{and} \qquad \hat{\underline{c}}_{22} = \begin{bmatrix} E[\zeta'^2(t_1)] & E[\zeta'(t_1)\zeta'(t)] & E[\zeta'(t_1)\zeta'_{osp}] \\ E[\zeta'(t)\zeta'(t_1)] & E[\zeta'^2(t)] & E[\zeta'(t)\zeta'_{osp}] \\ E[\zeta'_{osp}\zeta'(t_1)] & E[\zeta'_{osp}\zeta'(t)] & E[\zeta'^2_{osp}] \end{bmatrix}. \qquad (6.25d)$$

We note that $E[\zeta'^2(t_1)] = E[\zeta'^2(t)] = E[\zeta'^2_{osp}] = \sigma_2^2$, $E[\zeta_{osp}\zeta'_{osp}] = E[\zeta'_{osp}\zeta_{osp}] = E[S(t_1)\zeta'(t_1)] = E[\zeta'(t_1)S(t_1)] = 0$, and we define

$$\hat{\underline{c}}^{-1} \doteq \begin{bmatrix} \tilde{c}_{11} & \tilde{c}_{12} & \tilde{c}_{13} \\ \tilde{c}_{21} & \tilde{c}_{22} & \tilde{c}_{23} \\ \tilde{c}_{31} & \tilde{c}_{32} & \tilde{c}_{33} \end{bmatrix}. \qquad (6.26)$$

After much algebraic manipulation we find

$$C^{(1)} \doteq \int_0^\infty dt \int_0^t dt_1 \int_{\eta_0}^\infty d\zeta'_1 \int_{\eta_0}^\infty d\zeta'_2 \; (\zeta'(t)-\eta_0)(\zeta'(t_1)-\eta_0) \; p(\zeta'(t_1), \zeta'(t) \,|\zeta'_{osp})$$

$$* \frac{1}{(2\pi)^{\frac{3}{2}} \, [det(\hat{\underline{c}})]^{\frac{1}{2}}} \; exp\Big\{ -\frac{1}{2}\Big[(\eta_0 t_1 - \hat{m}_1)^2 \, \tilde{c}_{11} + (\eta_0 t - \hat{m}_2)^2 \, \tilde{c}_{22} + \hat{m}_3^2 \, \tilde{c}_{33} +$$

$$2(\eta_0 t_1 - \hat{m}_1)(\eta_0 t - \hat{m}_2)\tilde{c}_{12} - 2\,\hat{m}_3(\eta_0 t_1 - \hat{m}_1)\tilde{c}_{13} - 2\,\hat{m}_3(\eta_0 t - \hat{m}_2)\tilde{c}_{23}\Big]\Big\}$$

$$* \int_{-\infty}^{\infty} d\zeta_{osp} \;\; exp\Big\{ -\frac{1}{2}\Big[a\zeta^2_{osp} + b\zeta_{osp}\Big]\Big\} \qquad (6.27)$$

where

$$a = \sum_{i=1}^{3} \; \sum_{j=1}^{3} \tilde{c}_{ij} \qquad (6.28)$$

and

$$b = 2\Big\{(\eta_0 t_1 - \widehat{\widehat{m}}_1)(\widetilde{c}_{11} + \widetilde{c}_{12} + \widetilde{c}_{13}) + (\eta_0 t - \widehat{\widehat{m}}_2)(\widetilde{c}_{21} + \widetilde{c}_{22} + \widetilde{c}_{23}) -$$

$$\widehat{\widehat{m}}_3(\widetilde{c}_{31} + \widetilde{c}_{32} + \widetilde{c}_{33})\Big\}. \tag{6.29}$$

Making a change of variables we find

$$\int_{-\infty}^{\infty} d\zeta_{osp} \;\; exp\Big\{ -\frac{1}{2}\left[a\zeta_{osp}^2 + b\zeta_{osp}\right]\Big\} = \sqrt{\frac{2\pi}{a}}\; exp\Big\{\frac{b^2}{8a}\Big\}, \tag{6.30}$$

and so we have

$$C^{(1)} \doteq \int_0^{\infty} dt \int_0^{t} dt_1 \int_{\eta_0}^{\infty} d\zeta_1' \int_{\eta_0}^{\infty} d\zeta' \; (\zeta'(t) - \eta_0)(\zeta'(t_1) - \eta_0)\; p(\zeta'(t_1),\, \zeta'(t)\, |\zeta'_{osp})$$

$$* \frac{1}{(2\pi)^{\frac{3}{2}}\,[det(\widehat{\underline{c}})]^{\frac{1}{2}}} \;\; exp\Big\{ -\frac{1}{2}\Big[(\eta_0 t_1 - \widehat{\widehat{m}}_1)^2\, \widetilde{c}_{11} + (\eta_0 t - \widehat{\widehat{m}}_2)^2\, \widetilde{c}_{22} + \widehat{\widehat{m}}_3^2\, \widetilde{c}_{33} +$$

$$2(\eta_0 t_1 - \widehat{\widehat{m}}_1)(\eta_0 t - \widehat{\widehat{m}}_2)\widetilde{c}_{12} \;\; - 2\, \widehat{\widehat{m}}_3(\eta_0 t_1 - \widehat{\widehat{m}}_1)\widetilde{c}_{13} - 2\, \widehat{\widehat{m}}_3(\eta_0 t - \widehat{\widehat{m}}_2)\widetilde{c}_{23}\Big]\Big\}$$

$$* \sqrt{\frac{2\pi}{a}}\; exp\Big\{\frac{b^2}{8a}\Big\}. \tag{6.31}$$

To further reduce this term we use Baye's theorem and express

$$p(\zeta'(t_1),\, \zeta'(t)\, |\zeta'_{osp}) = p(\zeta'(t)\, |\zeta'(t_1),\, \zeta'_{osp})\; p(\zeta'(t_1)|\, \zeta'_{osp}), \tag{6.32}$$

where

$$p(\zeta'(t)\,|\zeta'(t_1),\,\zeta'_{osp}) = \frac{1}{\sqrt{2\pi k}}\,exp\left\{-\frac{1}{2}\frac{(\zeta'(t)-\mu)^2}{k}\right\} \tag{6.33}$$

is a Gaussian density, with mean

$$\mu = \frac{1}{\sigma_2^4 - R_1''^2}\Big[\dot{x}_1\Big(-R_2''\sigma_2^2 - R_1''R_3''\Big) + \zeta'_{osp}\Big(-R_1''R_2'' - R_3''\sigma_2^2\Big)\Big] \tag{6.34}$$

and variance

$$k = \sigma_2^2 + \frac{\sigma_2^2(-R_2''^2 - R_3''^2) - 2R_1''\,R_2''\,R_3''}{\sigma_2^4 - R_1''^2}, \tag{6.35}$$

where

$$R_1'' = \frac{d^2R(t_1)}{dt_1^2} = -\ (\sigma_2^2 - \frac{\sigma_2^4}{\sigma_1^2}\,t_1^2)\ exp\left\{-\frac{t_1^2}{l^2}\right\}, \tag{6.36a}$$

$$R_2'' = \frac{d^2R(t-t_1)}{d(t-t_1)^2} = -(\sigma_2^2 - \frac{\sigma_2^4}{\sigma_1^2}\,(t-t_1)^2)\ exp\left\{-\frac{(t-t_1)^2}{l^2}\right\}, \tag{6.36b}$$

and

$$R_3'' = \frac{d^2R(t)}{dt^2} = -(\sigma_2^2 - \frac{\sigma_2^4}{\sigma_1^2}\,t^2)\ exp\left\{-\frac{t^2}{l^2}\right\}, \tag{6.36c}$$

with $l \doteq \sqrt{\frac{2\sigma_1^2}{\sigma_2^2}}$. Also,

$$p(\zeta'(t_1)|\ \zeta'_{osp}) = \frac{1}{\sqrt{2\pi d}}\,exp\left\{-\frac{1}{2}\frac{(\zeta'(t_1)-m)^2}{d}\right\}, \tag{6.37}$$

where $m = \frac{-R_1''^2}{\sigma_2^2}$ and $d = \sigma_2^2 - \frac{R_1''^2}{\sigma_2^2}$.

After much algebraic manipulation we are able to reduce the integrals in (6.31) by one yielding

$$C^{(1)} \doteq \int_0^\infty dt \int_0^t dt_1 \int_{\eta_0}^\infty d\zeta_1' \; f\; (\zeta'(t_1) - \eta_0) exp\Big\{a_1 + b_1\zeta'(t_1) + c_1\zeta'^2(t_1)\Big\}$$

$$* \; exp\Big\{\frac{\beta^2}{4\alpha}\Big\}\Bigg\{\frac{1}{2\,|\,\alpha\,|}\; exp\Big\{-\,|\,\alpha\,|\Big[\eta_0 + \frac{\beta}{2\alpha}\Big]^2\Big\} + \Bigg\{-\frac{\beta}{2\sqrt{|\,\alpha\,|}\,\alpha} - \frac{\eta_0}{\sqrt{|\,\alpha\,|}}\Bigg\}\frac{\sqrt{\pi}}{2}$$

$$* \; erfc\Big(\sqrt{|\,\alpha\,|}\Big[\eta_0 + \frac{\beta}{2\alpha}\Big]\Big)\Bigg\}, \tag{6.38}$$

where

$$f = \frac{1}{4\pi^2}\,\frac{1}{[det(\hat{\underline{c}})]^{\frac{1}{2}}}\,\frac{1}{\sqrt{adk}} \tag{6.39}$$

$$a_1 = \; -\frac{\mu^2}{2k} + \frac{\alpha_1^2}{8a} - \frac{m}{2d} - \frac{1}{2}\Big\{\tilde{c}_{11}(\eta_0^2\, t_1^2 - 2\eta_0 t_1 A_{13}\zeta'_{osp} + A_{13}^2\, \zeta'^2_{osp})$$

$$+ \tilde{c}_{22}\Big(\eta_0^2 t^2 - 2\eta_0 t A_{23}\zeta^2_{osp} + A_{23}^2\zeta'^2_{osp}\Big) + \tilde{c}_{33}A_{33}^2\zeta'^2_{osp}\Big\} - \tilde{c}_{12}\Big(\eta_0^2 t_1 t - \eta_0 t_1 A_{23}\zeta'_{osp}$$

$$- \eta_0 t A_{13}\zeta'_{osp} + A_{13}A_{23}\zeta'^2_{osp} - \eta_0 t_1 A_{22}\Big) + \tilde{c}_{13}A_{33}\zeta'_{osp}(\eta_0 t_1 - A_{13}\zeta'_{osp}) +$$

$$\tilde{c}_{23}A_{33}\zeta'_{osp}(\eta_0 t - A_{23}\zeta'_{osp}) \tag{6.40}$$

$$b_1 = -\frac{1}{4a}\alpha_1\alpha_2 + \frac{m}{d} - \frac{1}{2}\Big\{ \tilde{c}_{11}\Big(-2\eta_0 \mathrm{t}_1 A_{11} + 2A_{11}A_{13}\zeta'_{\mathrm{osp}}\Big) +$$

$$\tilde{c}_{22}\Big(-2\eta_0 \mathrm{t} A_{21} + 2A_{21}A_{23}\zeta'_{\mathrm{osp}}\Big) + \tilde{c}_{33} 2A_{31}A_{33}\zeta'_{\mathrm{osp}}\Big\} - \tilde{c}_{12}(-\eta_0 \mathrm{t}_1 A_{21} -$$

$$\eta_0 \mathrm{t} A_{11} + A_{11}A_{23}\zeta'_{\mathrm{osp}} + A_{13}A_{21}\zeta'_{\mathrm{osp}}) + \tilde{c}_{13}[A_{31}(\eta_0 \mathrm{t}_1 - A_{13}\zeta'_{\mathrm{osp}})$$

$$- A_{33}A_{11}] + \tilde{c}_{23}[\mathrm{A}_{31}(\eta_0 \mathrm{t} - A_{23}\zeta'_{\mathrm{osp}}) - A_{33}A_{21}\zeta'_{\mathrm{osp}}] \qquad (6.41)$$

$$c_1 = \frac{\alpha_2^2}{8a} - \frac{1}{2d} - \frac{1}{2}\tilde{c}_{11}A_{11}^2 - \frac{1}{2}\tilde{c}_{22}A_{21}^2 - \frac{1}{2}\tilde{c}_{33}A_{31}^2 - \tilde{c}_{12}A_{11}A_{21} - \tilde{c}_{13}A_{31}A_{11} -$$

$$\tilde{c}_{23}A_{31}A_{21} \qquad (6.42)$$

$$\beta = \frac{\mu}{k} + \frac{1}{8a}(-2\alpha_1\alpha_3 + 2\alpha_2\alpha_3\zeta'(\mathrm{t}_1)) - \frac{1}{2}\tilde{c}_{11}(-2\eta_0 \mathrm{t}_1 A_{12} + 2A_{11}A_{12}\zeta'(\mathrm{t}_1) +$$

$$2A_{12}A_{13}\zeta'_{\mathrm{osp}}) - \frac{1}{2}\tilde{c}_{22}(-2\eta_0 \mathrm{t} A_{22} + 2A_{21}A_{22}\zeta'(\mathrm{t}_1) + 2A_{22}A_{23}\zeta'_{\mathrm{osp}}) -$$

$$\frac{1}{2}\tilde{c}_{33}(2a_{31}A_{32}\zeta'(\mathrm{t}_1) + 2A_{32}A_{33}\zeta'_{\mathrm{osp}}) - \tilde{c}_{12}[-2\eta_0 \mathrm{t}_1 A_{22} - \eta_0 \mathrm{t} A_{12} +$$

$$A_{22}(A_{11}\zeta'(\mathrm{t}_1) + A_{13}\zeta'_{\mathrm{osp}}) + A_{12}(A_{21}\zeta'(\mathrm{t}_1) + A_{23}\zeta'_{\mathrm{osp}})] +$$

$$\tilde{c}_{13}[A_{32}(\eta_0 \mathrm{t}_1 - A_{11}\zeta'(\mathrm{t}_1) - A_{13}\zeta'_{\mathrm{osp}}) - A_{12}(A_{31}\zeta'(\mathrm{t}_1) + A_{33}\zeta'_{\mathrm{osp}})] +$$

$$\tilde{c}_{23}[A_{32}(\eta_0 \mathrm{t} - A_{21}\zeta'(\mathrm{t}_1) - A_{23}\zeta'_{\mathrm{osp}}) - A_{22}(A_{31}\zeta'(\mathrm{t}_1) + A_{33}\zeta'_{\mathrm{osp}})] \qquad (6.43)$$

$$\alpha = -\frac{1}{2k} + \frac{\alpha_3^2}{8a} - \frac{A_{12}^2\tilde{c}_{11}}{2} - \frac{A_{22}^2\tilde{c}_{22}}{2} - \frac{A_{32}^2\tilde{c}_{33}}{2} - \tilde{c}_{12}A_{12}A_{22} - \tilde{c}_{13}A_{32}A_{12} -$$

$$\tilde{c}_{23}A_{32}A_{22} \tag{6.44}$$

$$\alpha_1 = 2\left[\eta_0 t_1(\tilde{c}_{11}+\tilde{c}_{12}+\tilde{c}_{13}) + \eta_0 t(\tilde{c}_{21}+\tilde{c}_{22}+\tilde{c}_{23})\right] - 2\Big[\mathrm{A}_{13}\zeta'_{\mathrm{osp}}(\tilde{c}_{11}+\tilde{c}_{12}+\tilde{c}_{13})$$

$$+\mathrm{A}_{23}\zeta'_{\mathrm{osp}}(\tilde{c}_{21}+\tilde{c}_{22}+\tilde{c}_{23}) + \mathrm{A}_{33}\zeta'_{\mathrm{osp}}(\tilde{c}_{31}+\tilde{c}_{32}+\tilde{c}_{33})\Big] \tag{6.45}$$

$$\alpha_2 = 2\left[\mathrm{A}_{11}(\tilde{c}_{11}+\tilde{c}_{12}+\tilde{c}_{13}) + \mathrm{A}_{21}(\tilde{c}_{21}+\tilde{c}_{22}+\tilde{c}_{23}) + \mathrm{A}_{31}(\tilde{c}_{31}+\tilde{c}_{32}+\tilde{c}_{33})\right] \tag{6.46}$$

$$\alpha_3 = 2\left[\mathrm{A}_{12}(\tilde{c}_{11}+\tilde{c}_{12}+\tilde{c}_{13}) + \mathrm{A}_{22}(\tilde{c}_{21}+\tilde{c}_{22}+\tilde{c}_{23}) + \mathrm{A}_{32}(\tilde{c}_{31}+\tilde{c}_{32}+\tilde{c}\,\mathrm{y}_{33})\right] \tag{6.47}$$

This completes the derivation. Adding the first order correction term consistent with (5.2) we get

$$\mathrm{P}^{(2)}(\widehat{k}_i \mid \zeta'_{\mathrm{osp}}) = \mathrm{P}^{(1)}(\widehat{k}_i \mid \zeta'_{\mathrm{osp}}) + C^{(1)}, \tag{6.48}$$

where $\mathrm{P}^{(2)}(\widehat{k}_i \mid \zeta'_{\mathrm{osp}})$ is a upper bound for the shadowing function.

We next proceed to calculate the third term in the series. This term provides the upper bound for the first passage in time density function, and thus gives us an lower bound on the shadowing function. In this case we are able to reduce the dimensionality of the integrations from seven to five, which is about the maximum number of numerical integrations possible on our 486 33MHz computer.

6.4 Calculation of $W_3(t_3,\ t_2,\ t_1\ |\zeta_{osp},\ \zeta'_{osp})$ with Correlation Between the Shadowing Point and the Shadowed Point.

The third term in the infinite series in (5.2) provides us with with a lower bound for the shadowing function (since, for the cases we consider, the terms in the series decrease in magnitude). It becomes important at this point to reduce the dimensionality of the integration in order to be able to use a numerical procedure such as Romberg integration. We are faced with a seven fold integration at this point, due to the fact that two additional integrations are needed for each additional term in the series. This third term is probably the last term which can feasibly be calculated without resorting to more efficient special purpose integration procedures such as Monte Carlo integration. We restate the expression for W_3 again here for convenience:

$$W_3(t_3,\ t_2,\ t_1|\ \zeta_{osp},\ \zeta'_{osp}) = \int_{\eta_0}^{\infty} d\zeta'_1 \int_{\eta_0}^{\infty} d\zeta'_2 \int_{\eta_0}^{\infty} d\zeta'_3\ (\zeta'(t_3)-\eta_0\ )(\zeta'(t_2)-\eta_0\ )(\zeta'(t_1)-\eta_0\ )$$

$$* \ p(\zeta_{osp}+\eta_0 t_3,\ \zeta'(t_3),\ \zeta_{osp}+\eta_0 t_2,\ \zeta'(t_2),\ \zeta_{osp}+\eta_0 t_1,\ \zeta'(t_1)\ |\zeta_{osp},\ \zeta'_{osp}). \quad (6.49)$$

The second correction term to the shadowing function becomes, using (5.1),

$$C^{(2)} \doteq \int_{-\infty}^{\infty} d\zeta_{osp}\ \ p(\zeta_{osp}) \int_{0}^{\infty} dt\ \int_{0}^{t} dt_1 \int_{t_1}^{t} dt_2\ W_3(t,\ t_1,t_2\ |\zeta_{osp},\ \zeta'_{osp}), \quad (6.50)$$

where we have set t_3=t. If we bring the integration over the heights, ζ_{osp}, inside the seven-fold integration as before we get

$$C^{(2)} \doteq \int_0^\infty dt \int_0^t dt_1 \int_{t_1}^t dt_2 \int_{\eta_0}^\infty d\zeta_1' \int_{\eta_0}^\infty d\zeta_2' \int_{\eta_0}^\infty d\zeta' \; (\zeta'(t)-\eta_0)(\zeta'(t_2)-\eta_0)(\zeta'(t_1)-\eta_0)$$

$$*\int_{-\infty}^\infty d\zeta_{osp} \;\; p(\zeta_{osp})p(\zeta_{osp}+\eta_0 t,\, \zeta'(t),\, \zeta_{osp}+\eta_0 t_2,\, \zeta'(t_2),\, \zeta_{osp}+\eta_0 t_1, \zeta'(t_1)|\zeta_{osp},\, \zeta_{osp}'). \tag{6.51}$$

Using Baye's theorem

$$p(\zeta_{osp})\; p(\zeta_{osp}+\eta_0 t,\, \zeta'(t),\, \zeta_{osp}+\eta_0 t_2,\, \zeta'(t_2),\, \zeta_{osp}+\eta_0 t_1,\, \zeta'(t_1)\; |\zeta_{osp},\, \zeta_{osp}')$$

can be written more conveniently as

$$p(\zeta'(t_1),\zeta'(t_2),\zeta'(t)\; |\zeta_{osp}')\; p(S(t_1),S(t_2),S(t),\zeta_{osp}\; |\zeta'(t_1),\zeta'(t_2),\zeta'(t),\zeta_{osp}'), \tag{6.52}$$

where we have let $S(t_1) = \zeta_{osp}+\eta_0 t_1$, $S(t_2) = \zeta_{osp}+\eta_0 t_2$, and $S(t) = \zeta_{osp}+\eta_0 t$ to simplify notation. (6.51) can now be written as

$$C^{(2)} \doteq \int_0^\infty dt \int_0^t dt_1 \int_{t_1}^t dt_2 \int_{\eta_0}^\infty d\zeta_1' \int_{\eta_0}^\infty d\zeta_2' \int_{\eta_0}^\infty d\zeta_3' \; (\zeta'(t)-\eta_0)(\zeta'(t_2)-\eta_0)$$

$$*\; (\zeta'(t_1)-\eta_0)p(\zeta'(t_1),\zeta'(t_2),\zeta'(t)|\zeta_{osp}')$$

$$*\int_{-\infty}^\infty d\zeta_{osp}\; p(S(t_1),S(t_2),S(t),\zeta_{osp}\; |\zeta'(t_1),\zeta'(t_2),\zeta'(t),\zeta_{osp}'), \tag{6.53}$$

where $p(\zeta'(t_1),\zeta'(t_2),\zeta'(t)|\zeta_{osp}')$ and $p(S(t_1),S(t_2),S(t),\zeta_{osp}\; |\zeta'(t_1),\zeta'(t_2),\zeta'(t),\zeta_{osp}')$ are conditional probability density functions of Gaussian random variables, and hence are also Gaussian density functions. Performing the inner two integrals, similar to what was done if the previous section we find (see Appendix C)

$$C^{(2)} \doteq \int_0^{\infty} \mathrm{dt} \int_0^{t} \mathrm{dt}_1 \int_{t_1}^{t} \mathrm{dt}_2 \int_{\eta_0}^{\infty} d\zeta_1' \int_{\eta_0}^{\infty} d\zeta_2' \; (\zeta'(\mathrm{t}_2) - \eta_0 \,)$$

$$* \; (\zeta'(\mathrm{t}_1) - \eta_0) \frac{1}{\sqrt{2\pi \tilde{\tilde{c}}_3}} \mathrm{p}(\zeta'(\mathrm{t}_1), \zeta'(\mathrm{t}_2) | \zeta'_{\mathrm{osp}}) \; \frac{1}{(2\pi)^2 (det \hat{\tilde{\underline{c}}}\,)^{\frac{1}{2}}} \; \sqrt{\frac{2\pi}{c}} \; exp\left\{ -\frac{\tilde{\tilde{\mathrm{m}}}_3^{\,2}}{2\tilde{\tilde{c}}_3} - \frac{Q}{2} + \frac{E^2}{8c} \right\}$$

$$* \; exp\left\{ \frac{\beta^2}{4\gamma} \right\} \left\{ \frac{1}{2\gamma} exp\left\{ -\gamma \left[\eta_0 - \frac{\beta}{2\gamma} \right]^2 \right\} + \left(\frac{\beta}{2\gamma} - \eta_0 \right) \sqrt{\frac{\pi}{\gamma}} \frac{1}{2} \; erfc\left[\sqrt{\gamma} \left(\eta_0 - \frac{\beta}{2\gamma} \right) \right] \right\} \qquad (6.54)$$

where

$$\gamma = -\left[-\frac{1}{2\tilde{\tilde{c}}_3} - \frac{R}{2} + \frac{F^2}{8c} \right] \text{ and } \beta = \frac{\tilde{\tilde{\mathrm{m}}}_3}{\tilde{\tilde{c}}_3} - \frac{P}{2} + \frac{2EF}{8c}. \qquad (6.55)$$

$$c = \sum_{i,j} \hat{\tilde{c}}_{ij} \,, \qquad (6.56)$$

$$E = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \hat{\tilde{\underline{c}}}^{\,-1} \underline{b} + \underline{b}^{\mathrm{T}} \hat{\tilde{\underline{c}}}^{\,-1} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \qquad (6.57)$$

$$F = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \hat{\tilde{\underline{c}}}^{\,-1} \underline{d} + \underline{d}^{\mathrm{T}} \hat{\tilde{\underline{c}}}^{\,-1} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \qquad (6.58)$$

$$P = \left[\underline{d}^{\mathrm{T}} \, \hat{\tilde{\underline{c}}}^{\,-1} \underline{b} + \underline{b}^{\mathrm{T}} \, \hat{\tilde{\underline{c}}}^{\,-1} \underline{d} \right] \qquad (6.59)$$

$$R = \underline{d}^{\mathrm{T}} \, \hat{\tilde{\underline{c}}}^{\,-1} \, \underline{d} \qquad (6.60)$$

$$\underline{\widehat{c}}^{\,-1} = \underline{\widehat{c}}_{11} - \underline{\widehat{c}}_{12}\underline{\widehat{c}}_{22}^{\,-1}\underline{\widehat{c}}_{21} \doteq \begin{bmatrix} \tilde{\tilde{c}}_{11} & \tilde{\tilde{c}}_{12} & \tilde{\tilde{c}}_{13} & \tilde{\tilde{c}}_{14} \\ \tilde{\tilde{c}}_{21} & \tilde{\tilde{c}}_{22} & \tilde{\tilde{c}}_{23} & \tilde{\tilde{c}}_{24} \\ \tilde{\tilde{c}}_{31} & \tilde{\tilde{c}}_{32} & \tilde{\tilde{c}}_{33} & \tilde{\tilde{c}}_{34} \\ \tilde{\tilde{c}}_{41} & \tilde{\tilde{c}}_{42} & \tilde{\tilde{c}}_{43} & \tilde{\tilde{c}}_{44} \end{bmatrix}. \tag{6.61}$$

$$\underline{b} + \underline{d}\zeta'(\mathrm{t}) = \begin{bmatrix} \eta_0 \mathrm{t}_1 - \tilde{m}_1 \\ \eta_0 \mathrm{t}_2 - \tilde{m}_2 \\ \eta_0 \mathrm{t} - \tilde{m}_3 \\ -\ \tilde{m}_4 \end{bmatrix} \tag{6.62}$$

$$\underline{\tilde{m}} = \begin{bmatrix} \tilde{m}_1 \\ \tilde{m}_2 \\ \tilde{m}_3 \\ \tilde{m}_4 \end{bmatrix} = \underline{\tilde{\mu}}_1 + \underline{\widehat{c}}_{12}\underline{\widehat{c}}_{22}^{\,-1}(\underline{\tilde{x}}_2 - \underline{\mu}_2), \tag{6.63}$$

$$\underline{\tilde{x}}_2 = \begin{bmatrix} \zeta'(\mathrm{t}_1) \\ \zeta'(\mathrm{t}_2) \\ \zeta'(\mathrm{t}) \\ \zeta'_{\mathrm{osp}} \end{bmatrix}, \tag{6.64}$$

$$\underline{\widehat{c}}_{11} = \begin{bmatrix} \sigma_1^2 & R_2 & R_4 & R_1 \\ R_2 & \sigma_1^2 & R_5 & R_3 \\ R_4 & R_5 & \sigma_1^2 & R_6 \\ R_1 & R_3 & R_6 & \sigma_1^2 \end{bmatrix}, \tag{6.65a}$$

$$\hat{\tilde{\underline{c}}}_{12} = \begin{bmatrix} 0 & R'_2 & R'_4 & -R'_1 \\ -R_2^2 & 0 & R'_5 & -R'_3 \\ -R'_4 & -R'_5 & 0 & -R'_6 \\ R'_1 & R'_3 & R'_6 & 0 \end{bmatrix}, \tag{6.65b}$$

$$\hat{\tilde{\underline{c}}}_{21} = \hat{\tilde{\underline{c}}}_{12}^{\mathrm{T}} \tag{6.65c}$$

$$\hat{\tilde{\underline{c}}}_{22} = \begin{bmatrix} \sigma_2^2 & -R''_2 & -R''_4 & -R''_1 \\ -R''_2 & \sigma_2^2 & -R''_5 & -R''_3 \\ -R''_4 & -R''_5 & \sigma_2^2 & -R''_6 \\ -R''_1 & -R''_3 & -R''_6 & \sigma_2^2 \end{bmatrix}, \tag{6.65d}$$

$$\tilde{\tilde{m}}_3 = \begin{bmatrix} -R''_4 & -R''_5 & -R''_6 \end{bmatrix} \begin{bmatrix} \sigma_2^2 & -R''_2 & -R''_1 \\ -R''_2 & \sigma_2^2 & -R''_3 \\ -R''_1 & -R''_3 & \sigma_2^2 \end{bmatrix}^{-1} \tag{6.66}$$

$$\tilde{\tilde{c}}_3 = \sigma_2^2 + \begin{bmatrix} -R''_4 & -R''_5 & -R''_6 \end{bmatrix} \begin{bmatrix} \sigma_2^2 & -R''_2 & -R''_1 \\ -R''_2 & \sigma_2^2 & -R''_3 \\ -R''_1 & -R''_3 & \sigma_2^2 \end{bmatrix}^{-1} \begin{bmatrix} -R''_4 \\ -R''_5 \\ R''_6 \end{bmatrix}, \tag{6.67}$$

where

$$R_1 = \mathrm{E}[\zeta_{\mathrm{osp}}\ \mathrm{S}(\mathrm{t}_1)] = \sigma_1^2\ exp\left\{-\frac{\mathrm{t}_1^2}{l^2}\right\}, \tag{6.68a}$$

$$R_2 = \mathrm{E}[\mathrm{S}(\mathrm{t}_1)\ \mathrm{S}(\mathrm{t}_2)] = \sigma_1^2\ exp\left\{-\frac{(\mathrm{t}_2-\mathrm{t}_1)^2}{l^2}\right\}, \tag{6.68b}$$

$$R_3 = \mathrm{E}[\zeta_{\mathrm{osp}}\ \mathrm{S}(\mathrm{t}_2)] = \sigma_1^2\ exp\left\{-\frac{\mathrm{t}_2^2}{l^2}\right\}, \tag{6.68c}$$

$$R_4 = \mathrm{E}[\mathrm{S}(\mathrm{t}_1)\ \mathrm{S}(\mathrm{t})] = \sigma_1^2\ exp\left\{-\frac{(\mathrm{t}-\mathrm{t}_1)^2}{l^2}\right\}, \tag{6.68d}$$

$$R_5 = \mathrm{E}[\mathrm{S}(\mathrm{t}_2)\ \mathrm{S}(\mathrm{t})] = \sigma_1^2\ exp\left\{-\frac{(\mathrm{t}-\mathrm{t}_2)^2}{l^2}\right\}, \tag{6.68e}$$

$$R_6 = \mathrm{E}[\zeta_{\mathrm{osp}}\ \mathrm{S}(\mathrm{t})] = \sigma_1^2\ exp\left\{-\frac{\mathrm{t}^2}{l^2}\right\}, \tag{6.68f}$$

$$R_1' = \frac{dR_1(\mathrm{t}_1)}{d\mathrm{t}_1} = -\mathrm{t}_1\sigma_2^2\ exp\left\{-\frac{\mathrm{t}_1^2}{l^2}\right\}, \tag{6.68g}$$

$$R_2' = \frac{dR_2(\mathrm{t}_2-\mathrm{t}_1)}{d(t_2-t_1)} = -(\mathrm{t}_2-\mathrm{t}_1)\sigma_2^2\ exp\left\{-\frac{(\mathrm{t}_2-\mathrm{t}_1)^2}{l^2}\right\}, \tag{6.68h}$$

$$R_3' = \frac{dR_3(\mathrm{t}_2)}{d\mathrm{t}_2} = -\mathrm{t}_2\sigma_2^2\ exp\left\{-\frac{\mathrm{t}_2^2}{l^2}\right\}, \tag{6.68i}$$

$$R_4' = \frac{dR_4(\mathrm{t}-\mathrm{t}_1)}{d(\mathrm{t}-t_1)} = -(\mathrm{t}-\mathrm{t}_1)\sigma_2^2\ exp\left\{-\frac{(\mathrm{t}-\mathrm{t}_1)^2}{l^2}\right\}, \tag{6.68j}$$

$$R_5' = \frac{dR_5(\mathrm{t}-\mathrm{t}_2)}{d(\mathrm{t}-\mathrm{t}_2)} = -(\mathrm{t}-\mathrm{t}_2)\sigma_2^2\ exp\left\{-\frac{(\mathrm{t}-\mathrm{t}_2)^2}{l^2}\right\}, \tag{6.68k}$$

$$R_6' = \frac{dR_3(\mathrm{t})}{dt} = -\mathrm{t}\sigma_2^2\ exp\left\{-\frac{\mathrm{t}^2}{l^2}\right\}, \tag{6.68l}$$

$$R_1'' = \frac{d^2R(\mathrm{t}_1)}{d\mathrm{t}_1^2} = -\ (\sigma_2^2-\frac{\sigma_2^4}{\sigma_1^2}\,\mathrm{t}_1^2)\ exp\left\{-\frac{\mathrm{t}_1^2}{l^2}\right\}, \tag{6.68m}$$

$$R_2'' = \frac{d^2R(\mathrm{t}_2-\mathrm{t}_1)}{d(\mathrm{t}_2-\mathrm{t}_1)^2} = -\ (\sigma_2^2-\frac{\sigma_2^4}{\sigma_1^2}\,(\mathrm{t}_2-\mathrm{t}_1)^2)\ exp\left\{-\frac{(\mathrm{t}_2-\mathrm{t}_1)^2}{l^2}\right\}, \tag{6.68n}$$

$$R_3'' = \frac{d^2R(t_2)}{dt_2^2} = -\ (\sigma_2^2 - \frac{\sigma_2^4}{\sigma_1^2}\,t_2^2)\ exp\left\{-\frac{t_2^2}{l^2}\right\}, \tag{6.68o}$$

$$R_4'' = \frac{d^2R(t-t_1)}{d(t-t_1)^2} = -\ (\sigma_2^2 - \frac{\sigma_2^4}{\sigma_1^2}\,(t-t_1)^2)\ exp\left\{-\frac{(t-t_1)^2}{l^2}\right\}, \tag{6.68p}$$

$$R_5'' = \frac{d^2R(t-t_2)}{d(t-t_2)^2} = -\ (\sigma_2^2 - \frac{\sigma_2^4}{\sigma_1^2}\,(t-t_2)^2)\ exp\left\{-\frac{(t-t_2)^2}{l^2}\right\}, \tag{6.68q}$$

$$R_6'' = \frac{d^2R(t)}{dt^2} = -\ (\sigma_2^2 - \frac{\sigma_2^4}{\sigma_1^2}\,t^2)\ exp\left\{-\frac{t^2}{l^2}\right\}. \tag{6.68r}$$

and $\tilde{\underline{\mu}}_1 = \tilde{\underline{\mu}}_2 = 0$.

The shadowing function now becomes

$$\mathrm{P}^{(2)}(\widehat{k}_i \mid \zeta'_{\mathrm{osp}}) = \mathrm{P}^{(1)}(\widehat{k}_i \mid \zeta'_{\mathrm{osp}}) + C^{(1)} - C^{(2)}. \tag{6.69}$$

The second correction term provides us not only with an upper bound on the exact shadowing function, but it also gives us an indication as to how accurate we are with the second term. Since the W functions in (5.3) contain information about the number of upcrossings a surface makes with a ray, we expect that the $(i{+}1)^{th}$ term in (5.2) will decrease in magnitude from the i^{th} term provided there is a decrease from the $(i-1)^{th}$ to the i^{th} term. Thus, if the second correction term is small, we expect that shadowing can adequately be accounted for by using the first two terms in the infinite series.

6.5 Monte Carlo Simulations

Due to the complex nature of the multi-dimensional integrals it is necessary to provide a check against which our results can be verified. Perhaps the most rigorous analysis of the shadowing problem was done by Wagner [Wagner, 1967] in 1967. As previously mentioned, Wagner's conditional shadowing function, in which correlation between the shadowing point and the shadowed point was neglected, corresponds to our analytic result which neglected correlation between all shadowing points and correlation between the those points and the shadowed point. Wagner proceeds to eliminate the no correlation assumption by including correlation between the shadowing point and the shadowed point. This was done be expanding the autocorrelation function around the point $\tau = \eta_0$, for which Wagner claimed q(t) (the first passage in time function) was maximized. Unfortunately, Wagner is forced to expand the autocorrelation function to the seventh term to get a leading order result. We feel that due to the complex nature of the derivations and the subsequent approximations that need to be made to carry through to a final result that a further check is warranted, and, as we will show, justified. We proceed along the same lines as Brockelman and Hagfors [Brockelman and Hagfors, 1966] and compute the shadowing function by generating a random surface with Gaussian statistics and a Gaussian correlation function. The ratio of illuminated specular points to the total number of specular points was computed for various angles and surface conditions and compared with our numerical integration of the series solution as well as with the work of Wagner [Wagner, 1967] and Smith [Smith, 1967b].

A simple and useful method for generating a random surface is the moving average technique [Ogilvy, 1991]. This method computes a sequence of random numbers by a weighted sum of independent random variables. The weights are chosen to produce the proper correlation function.

Following the procedure outlined in Ogilvy [Ogilvy, 1991], and consistent with our previous work, we produce a sequence of correlated Gaussian random heights with a Gaussian correlation function. We thus have

$$\zeta(i) = \sum_j W(j)\,U(i-j), \quad i = 1, 2, 3, ..., \tag{6.70}$$

where $\zeta(i)$ are the heights, $W(j)$ are the weights (to be determined), and $U(i-j)$ are independent zero mean, σ_1^2 variance, Gaussian random variables. We immediately see that

$$\mathrm{E}[\zeta(i)] = 0 \tag{6.71}$$

and

$$\mathrm{Var}[\zeta(i)] = \sigma_1^2 \sum_j W^2(j). \tag{6.72}$$

We therefore normalize the weights by $\dfrac{1}{\sqrt{\sum_j W^2(j)}}$ so that $\mathrm{Var}[\zeta(i)] = \sigma_1^2$. If we form the function

$$\zeta(i)\zeta(i+\tau) = \sum_j \sum_k W(j)\,W(k)\,U(i-j)\,U(i+\tau-k)$$

and then take the expected value, we arrive at

$$\mathrm{E}[\zeta(i)\zeta(i+\tau)] = \sum_j \sum_k W(j)\,W(k)\mathrm{E}[\,U(i-j)\,U(i+\tau-k)]. \tag{6.73}$$

However, since the *U's* are independent

$$\mathrm{E}[U(i-j)U(i+\tau-k)] = \sigma_1^2 \text{ if } k = \tau + j \tag{6.74}$$

and $= 0$, otherwise.

Thus,

$$\mathrm{E}[\zeta(i)\zeta(i+\tau)] = \sigma_1^2 \sum_j W(j)\,W(\tau+j). \tag{6.75}$$

We recognize this to be proportional to the convolution of the weights. To determine the weights we take the Fourier transform (*FT)* of both sides to get

$$FT\{\mathrm{E}[\zeta(i)\zeta(i+\tau)]\} = \sigma_1^2 \; W^2(f), \tag{6.76}$$

where $W(f)$ is the Fourier transform of the weights. Solving for $W(j)$, we get

$$W(j) = FT^{-1}\sqrt{\left\{\frac{FT\{\mathrm{E}[\zeta(i)\zeta(i+\tau)]\}}{\sigma_1^2}\right\}}. \tag{6.77}$$

For a Gaussian correlation function $\mathrm{E}[\zeta(i)\zeta(i+\tau)] = \sigma_1^2 \; exp\left\{-\tau^2/l^2\right\}$, therefore [Ogilvy, 1991]

$$W(j) = \frac{exp\left\{-2(j\Delta x/l)^2\right\}}{\sqrt{\sum_j W^2(j)}}, \tag{6.78}$$

where we have normalized the sum of the squares of the weights to one. Δx is the spacing between random numbers and is chosen such that the results to not change as Δx is decreased. We also wish to chose this spacing such that the summation in (6.70) in which j varies from $-\infty$ to $+\infty$ may be bounded by some reasonably small number, M, to reduce computation time. Thus, we'll have

$$\zeta(i) = \sum_{j=-M}^{j=+M} \frac{exp\left\{-2(j\Delta x/l)^2\right\}}{\sqrt{\sum_j W^2(j)}} U(i-j), \quad i = 1,2,3,... \tag{6.79}$$

M should be chosen such that the weights, $W(j)$, have decayed sufficiently at M and are near zero (i.e. $W(M) \approx 0$). However, if a large surface is desired, as in our case, the size of M greatly effects computation speed and a trade off is involved. We have chosen to generate a sequence of $2(10)^6$ correlated Gaussian random heights with M=100. This provided us with about 31 points per correlation length and a surface extent of over $64(10)^3$ correlation lengths. This large extent was necessary in order to provide results close to grazing angles where the probability of specular points is very low.

To verify our simulation we computed the height and slope distribution along the realization for 20000 points (we were limited to some extent in the number of points we could plot by our plotting routine). The results are presented in Figures 21 and 22. Both appear near Gaussian in shape, and the distributions would no doubt improve had we used a larger number of points. The correlation function was also computed and compared with the exact Gaussian autocorrelation function and was found to be in excellent agreement (Figures 23-27). The tails of the function were raised slightly since we did not pass to the infinite summation needed in computing this function.

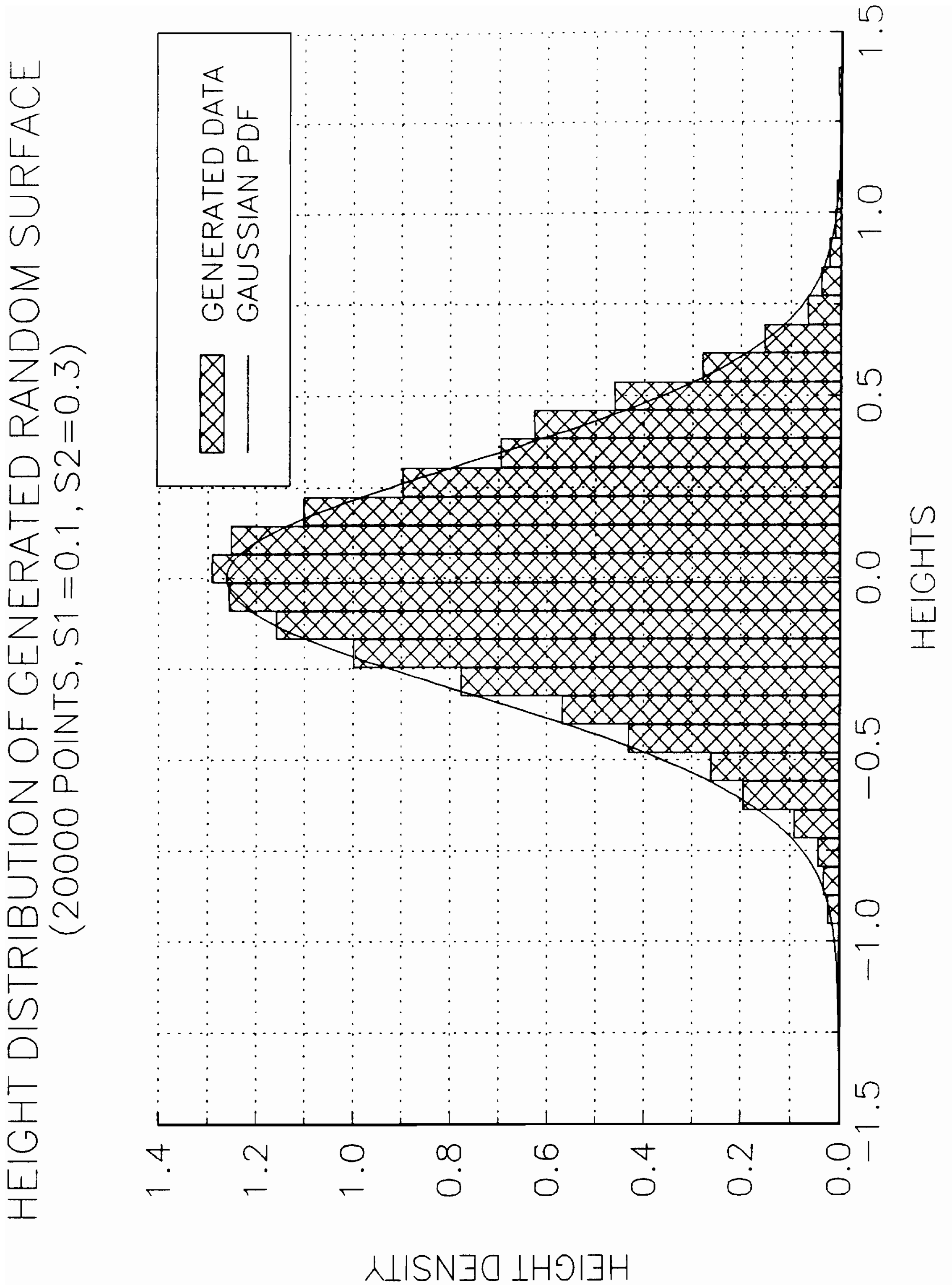

Figure 21: Height Distribution of Generated Random Surface

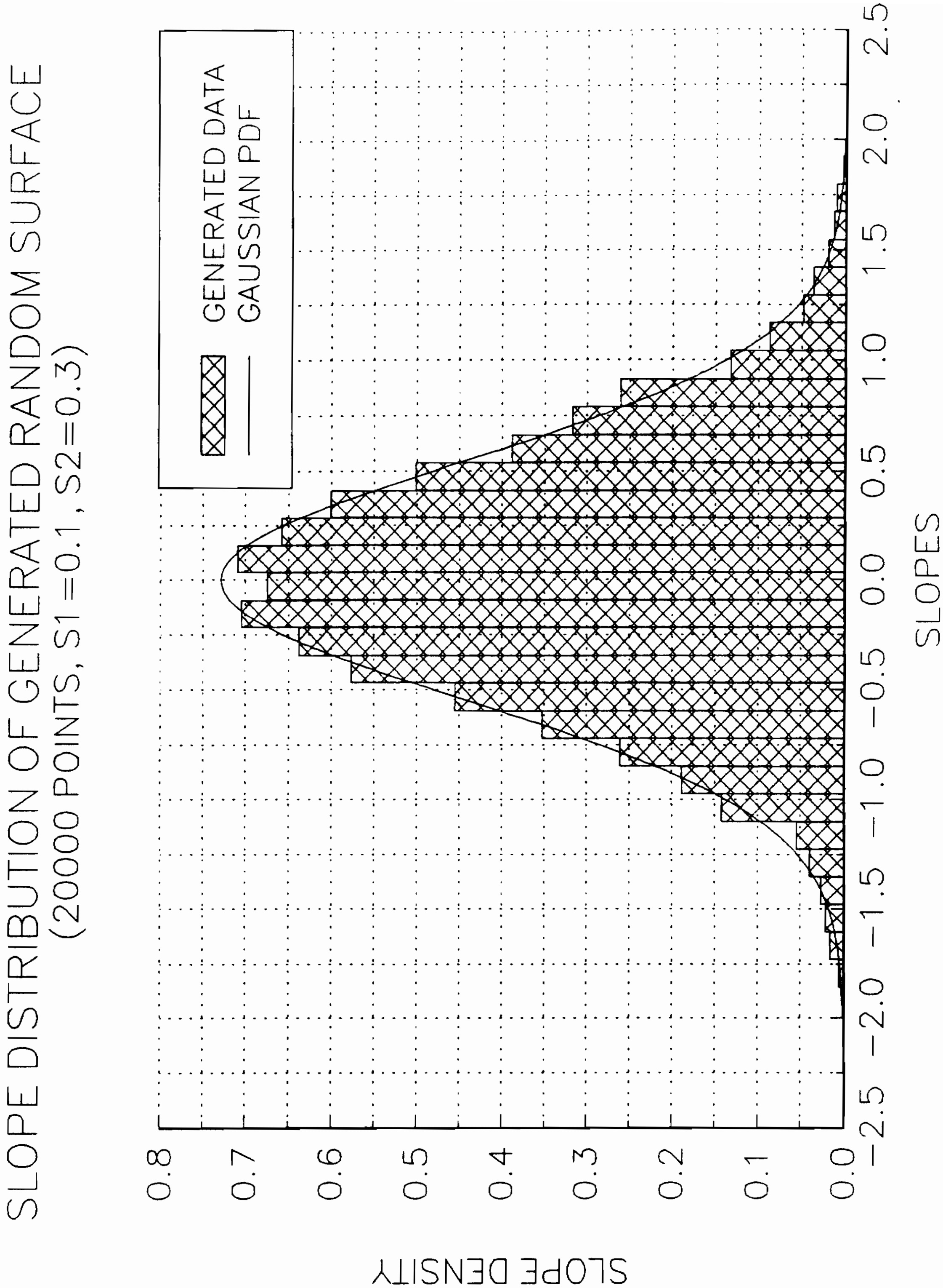

Figure 22: Slope Distribution of Generated Random Surface

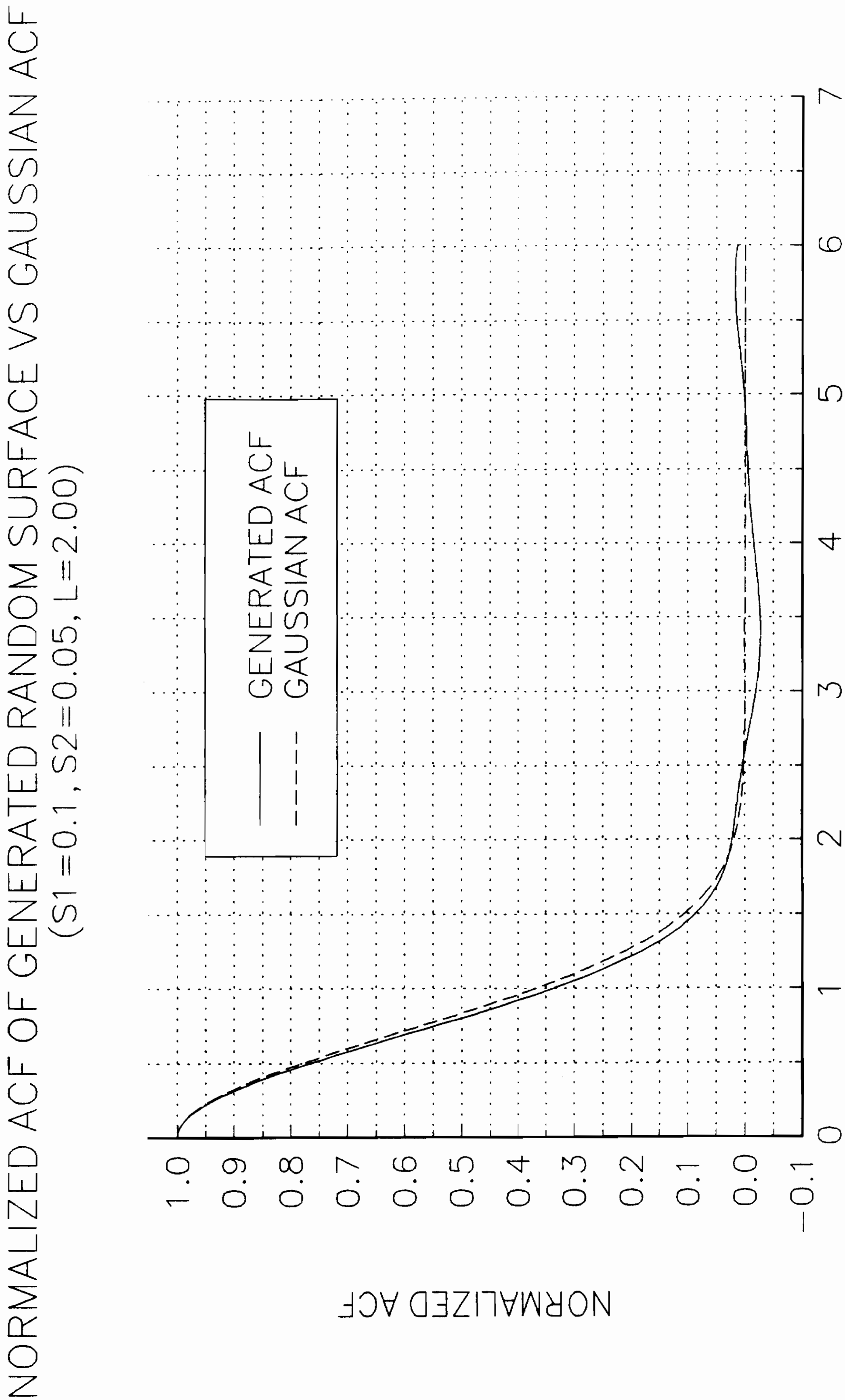

Figure 23: Normalized ACF of Generated Random Surface (Var. of heights=0.1, Var. of Slopes=0.05)

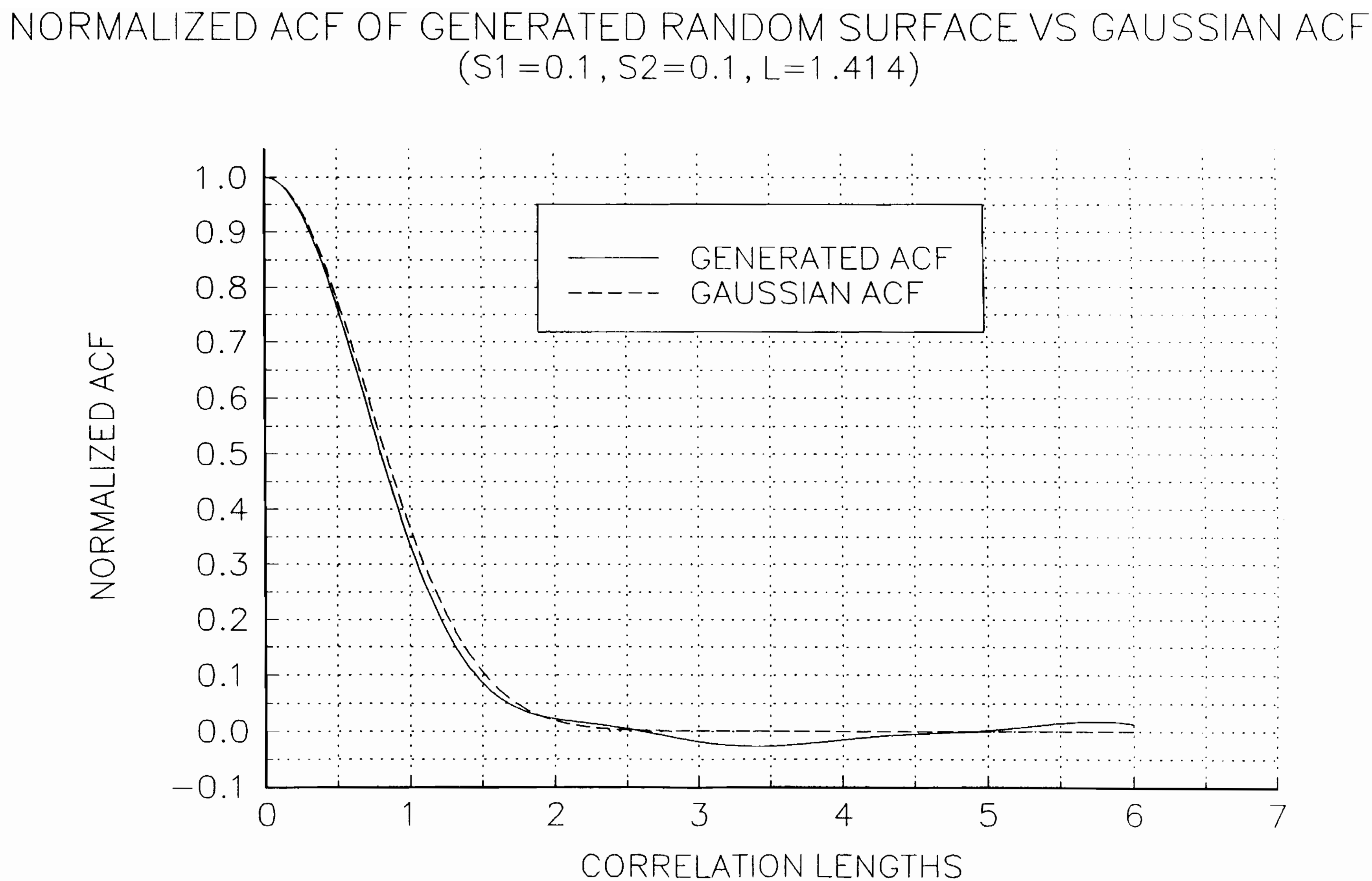

Figure 24: Normalized ACF of Generated Random Surface (Var. of heights=0.1, Var. of Slopes=0.10)

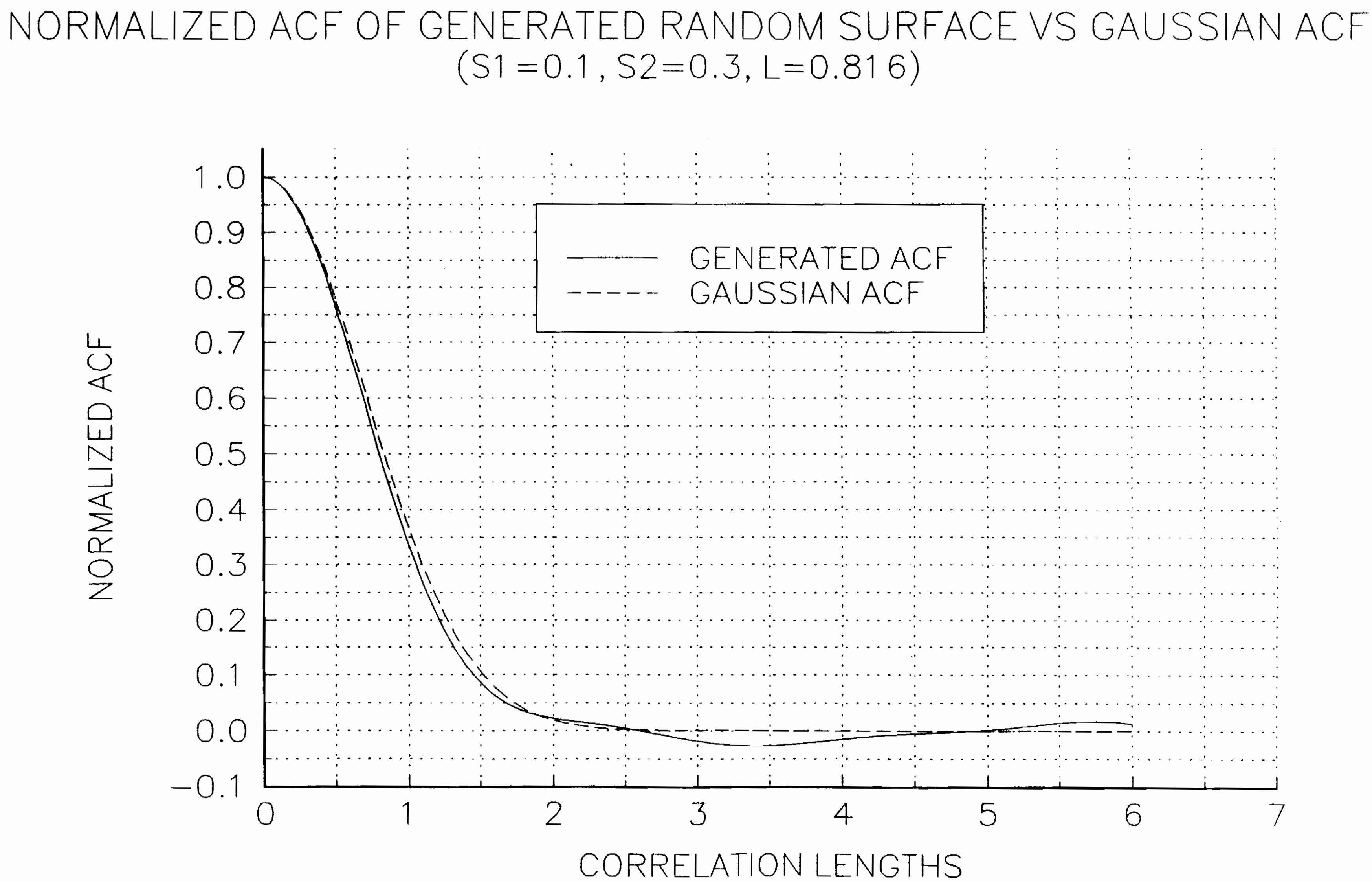

Figure 25: Normalized ACF of Generated Random Surface (Var. of heights=0.1, Var. of Slopes=0.30)

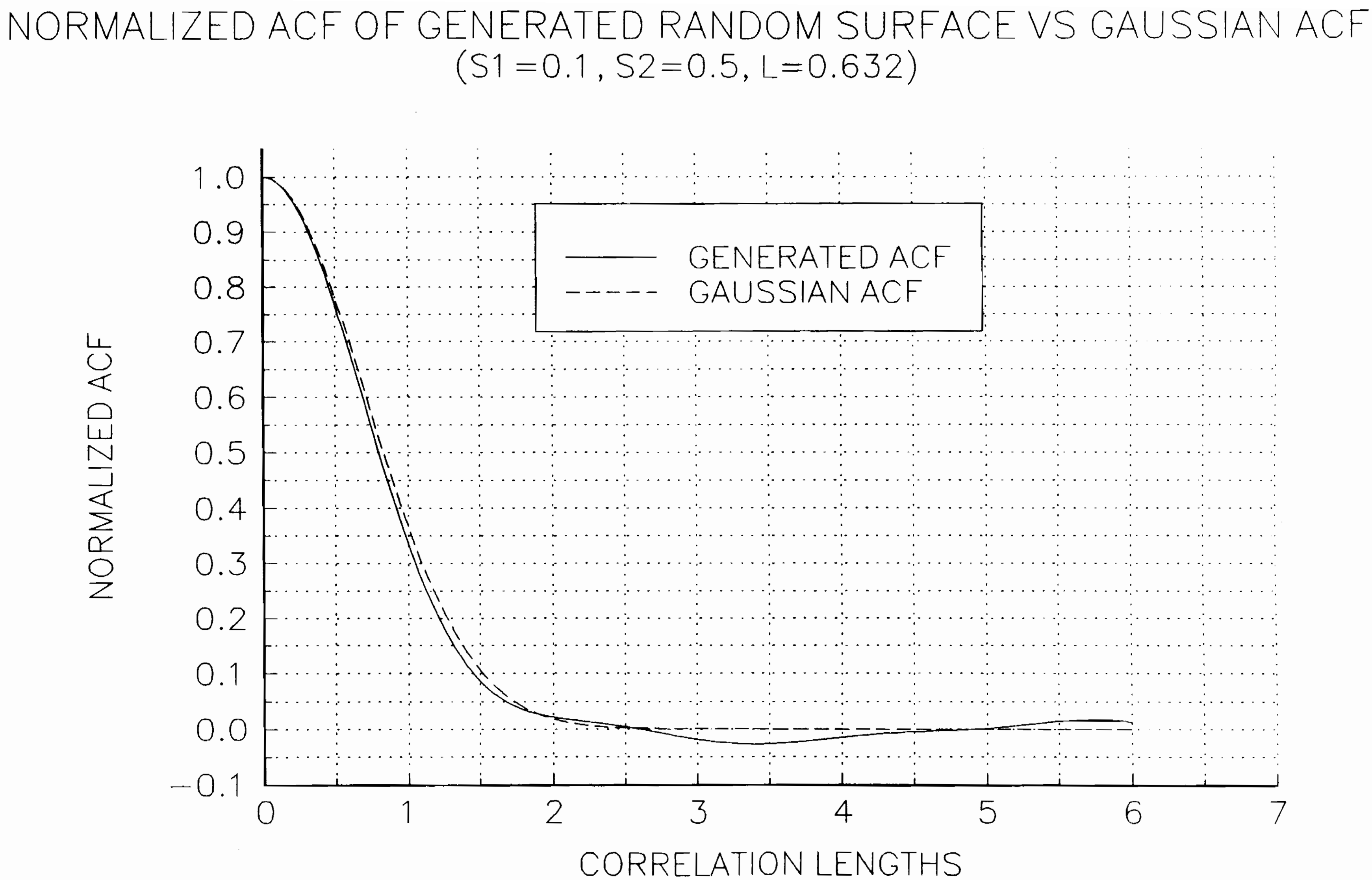

Figure 26: Normalized ACF of Generated Random Surface (Var. of heights=0.1, Var. of Slopes=0.50)

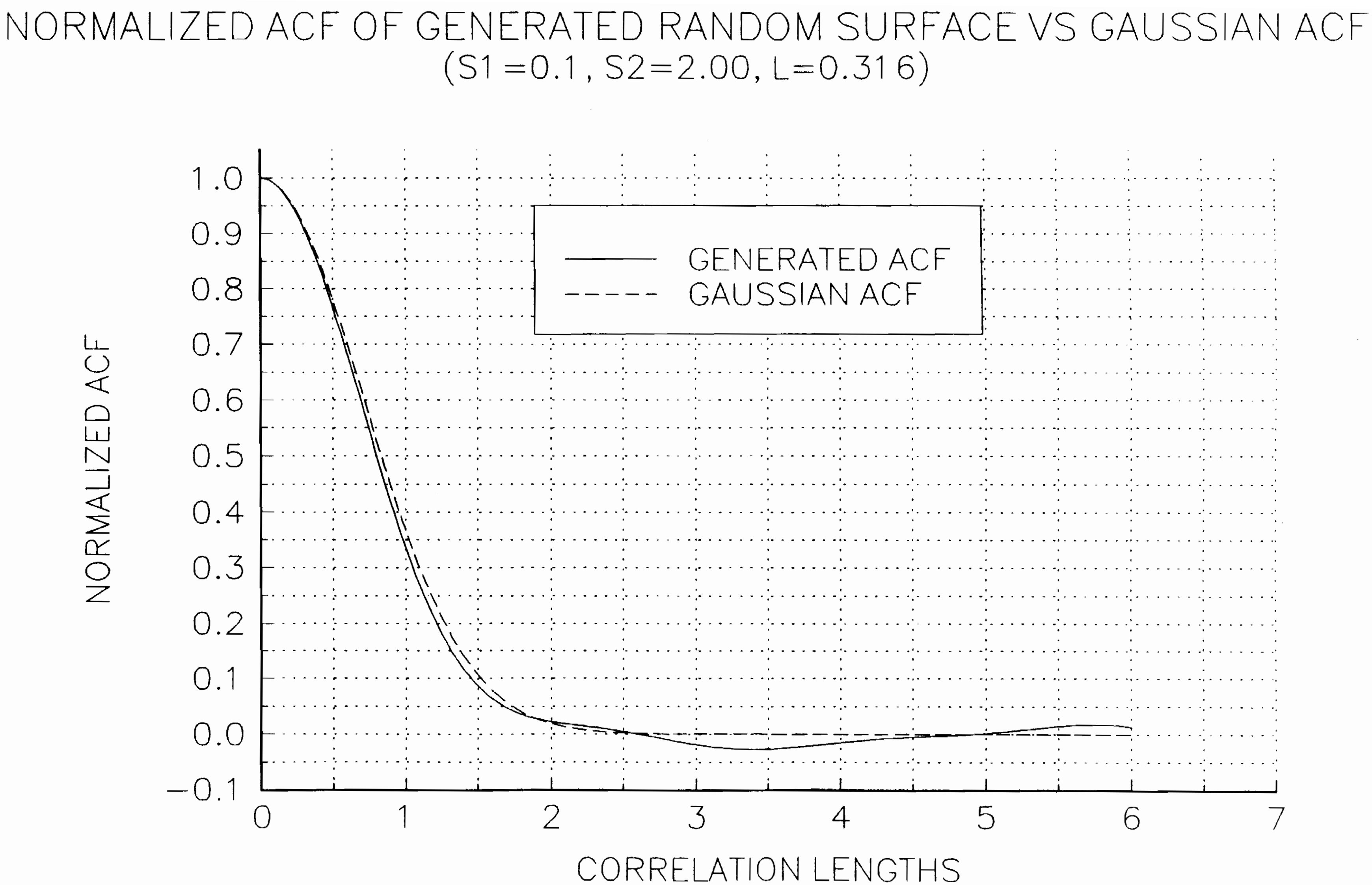

Figure 27: Normalized ACF of Generated Random Surface (Var. of heights=0.1, Var. of Slopes=2.00)

Following Brockelman and Hagfors [Brockelman and Hagfors, 1966], we then computed the number of reflection points, N_R, using the following criterion;

$$\left(tan\theta_i - \frac{d\zeta_i}{d\tau}\right)\left(tan\theta_i - \frac{d\zeta_{i-1}}{d\tau}\right) < 0, \qquad (6.80)$$

where θ_i is the angle of incidence and $\frac{d\zeta_i}{d\tau}$ and $\frac{d\zeta_{i-1}}{d\tau}$ are the slopes of the surface at the points i and $i-1$, respectively, computed by

$$\frac{d\zeta_i}{d\tau} = (\zeta_{i+1} - \zeta_{i-1})/2\Delta x \qquad (6.81a)$$

$$\frac{d\zeta_{i-1}}{d\tau} = (\zeta_i - \zeta_{i-2})/2\Delta x. \qquad (6.81b)$$

We simultaneously computed the number of shadowed points, N_S. The start of the shadow region is detected when the following formula is satisfied

$$\left(cot\theta_i + \frac{d\zeta_i}{d\tau}\right)\left(cot\theta_i + \frac{d\zeta_{i-1}}{d\tau}\right) < 0. \qquad (6.82)$$

If this equation holds then subsequent points which fall below the incident ray's path are shadowed points. Once a point is detected above the ray path then the procedure of searching for the start of the next shadowed region, using (6.82), begins again. The number of reflection points which are also in shadow, N_{RS}, is also computed and the ratio of illuminated reflection points, $N_R - N_S$, to total reflection points, N_R, is computed by

$$R(\theta_i) = \frac{N_R - N_S}{N_R}. \qquad (6.83)$$

This is the shadowing function using the same notation and procedure as Brockelman and Hagfors [Brockelman and Hagfors, 1966] and is the same as our shadowing function. These results were compared against our numerical results and the work of Wagner [Wagner, 1967] and Smith [Smith, 1967b] where possible.

6.6 Discussion of Results

Plots of the shadowing function for the variance of the heights (S1) $\sigma_1^2 = 0.1$, and the variance of the slopes (S2) $\sigma_2^2 = 0.05$, 0.1, 0.3, 0.5, and 2.0 are presented in Figures 28-32, where L is defined to be the correlation length. We see that for a gently sloping surface, where $\sigma_2^2 = 0.05$, that our numerical results differ from both Wagner's correlated shadowing function and Smith's result. A closer agreement is obtained with Wagner's uncorrelated shadowing function (Monte Carlo results for this graph are not presented since they produced a shadowing function of 1 until a 50 degree angle of incidence at which point no reflection points were detected). We see similar results in Figure 29 as the variance of the slopes increases to $\sigma_2^2 = 0.1$. Notice the excellent agreement between our results and Wagner's no correlation shadowing function in both Figure 28 and Figure 29, even near grazing angles. As we increase the slope variance still further to $\sigma_2^2 = 0.3$ we see that the region for which our results are valid is in excellent agreement to Wagner's uncorrelated shadowing function until about 75°. However, our results are beginning to move closer to Wagner's correlated function and Smith's work for σ_2^2=0.5 and 2.0. Notice also the excellent agreement of the Monte Carlo simulation to our results in Figure 30 until about 61 degrees incidence at which point the simulation starts to breakdown due to the fact that there are so few reflection points.

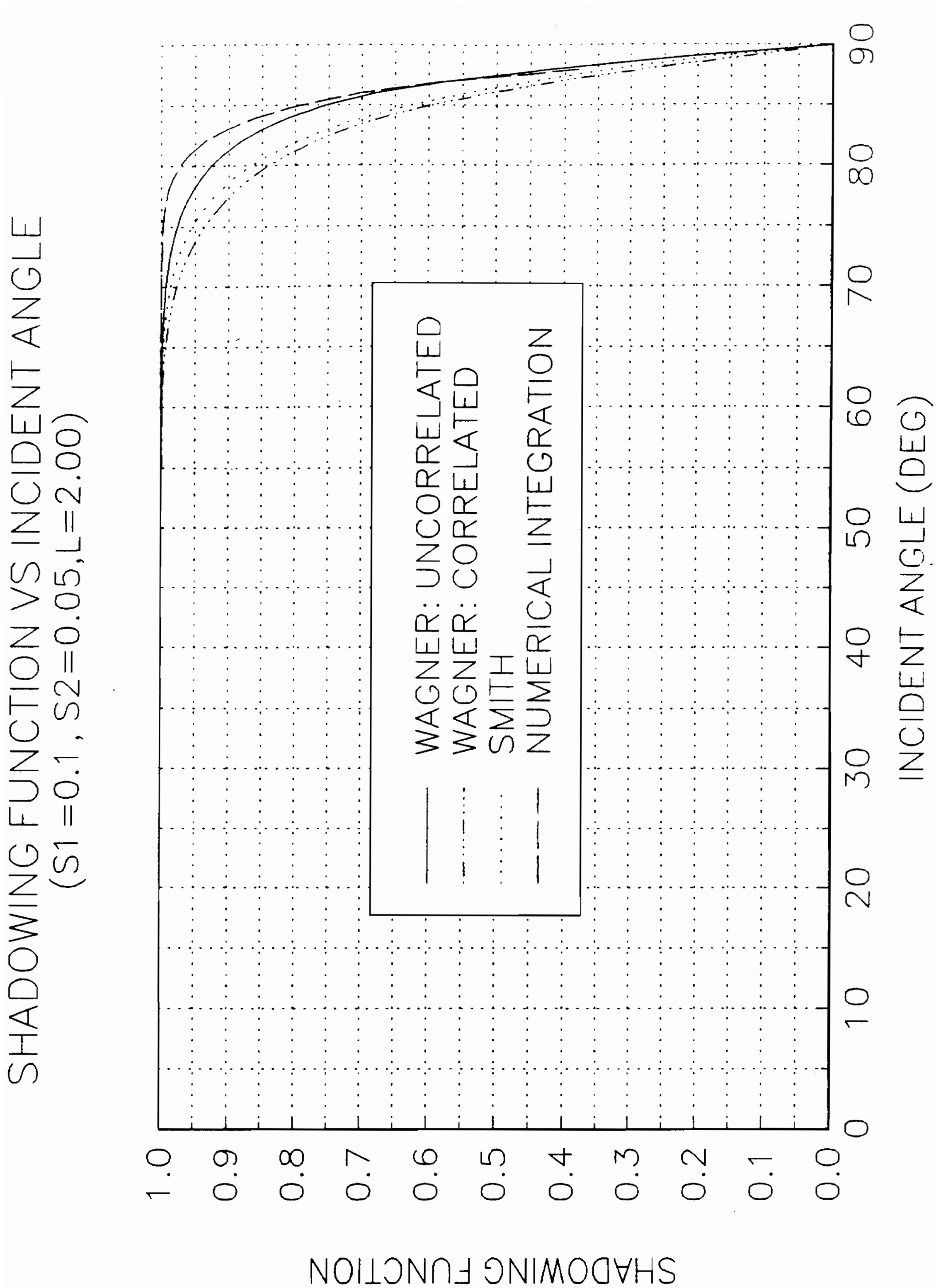

Figure 28: Shadowing Function vs Incident Angle (Var. of Heights=0.1, Var. of Slopes=0.05)

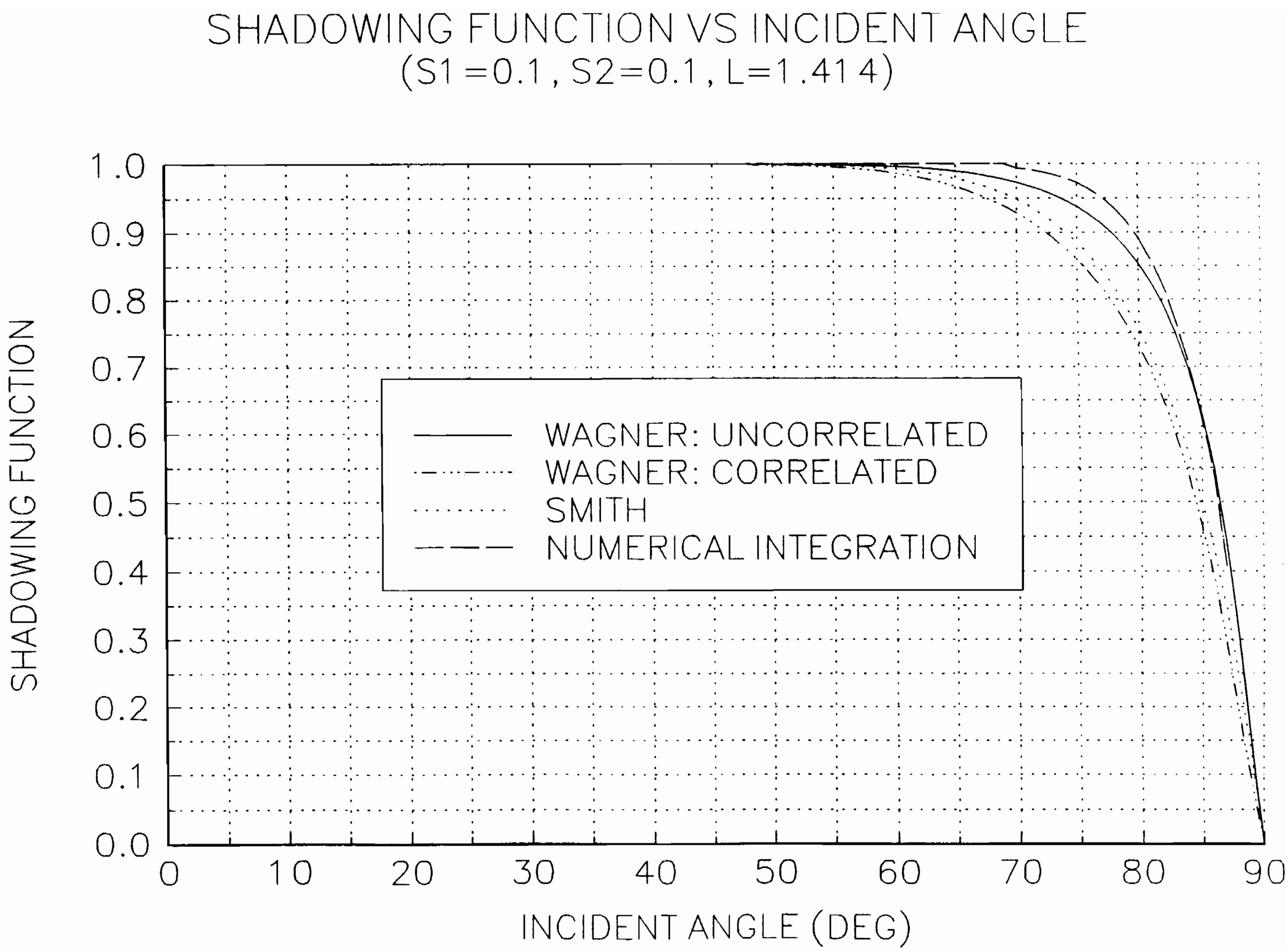

Figure 29: Shadowing Function vs Incident Angle (Var. of Heights=0.1, Var. of Slopes=0.10)

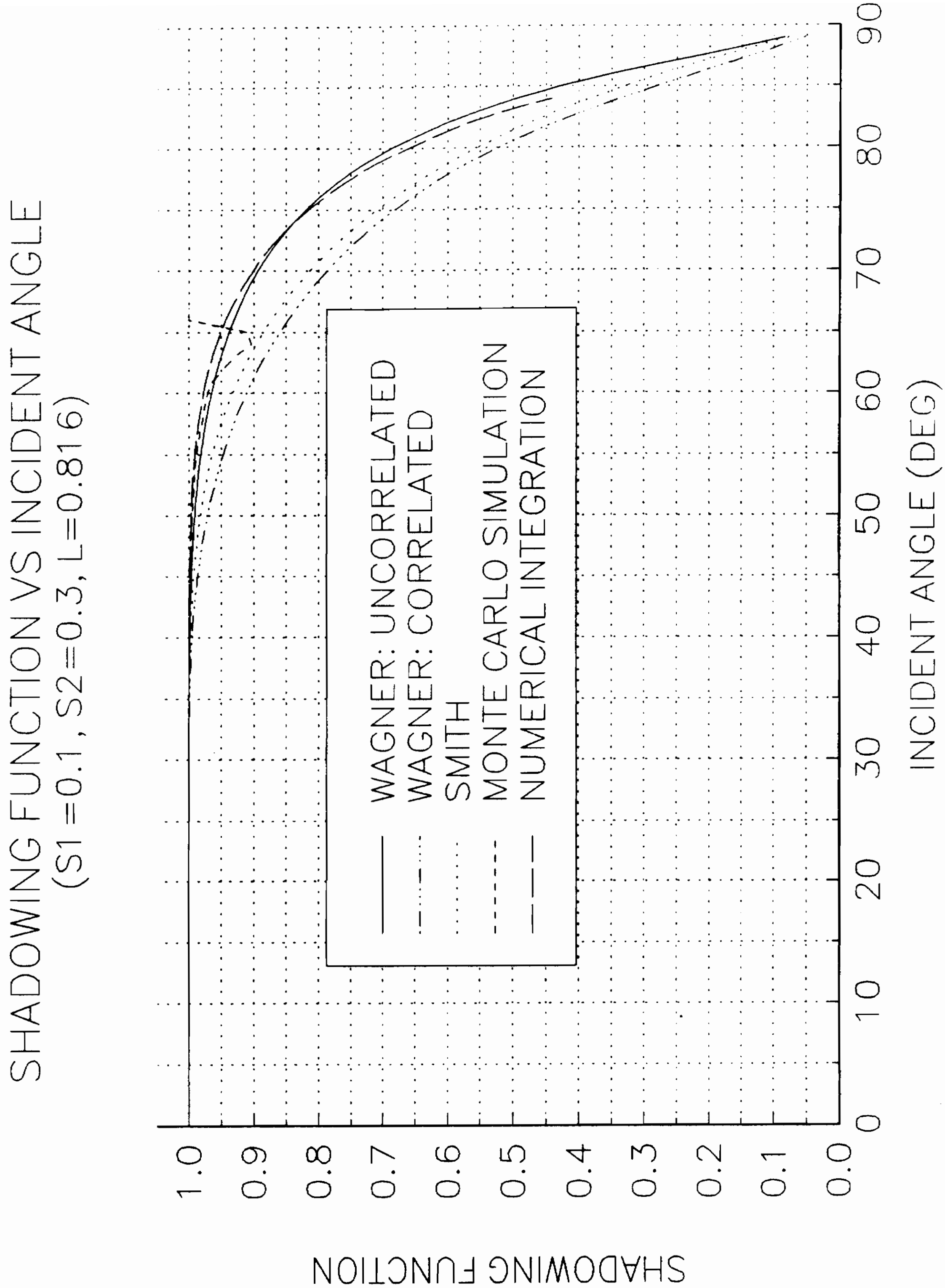

Figure 30: Shadowing Function vs Incident Angle (Var. of Heights=0.1, Var. of Slopes=0.30)

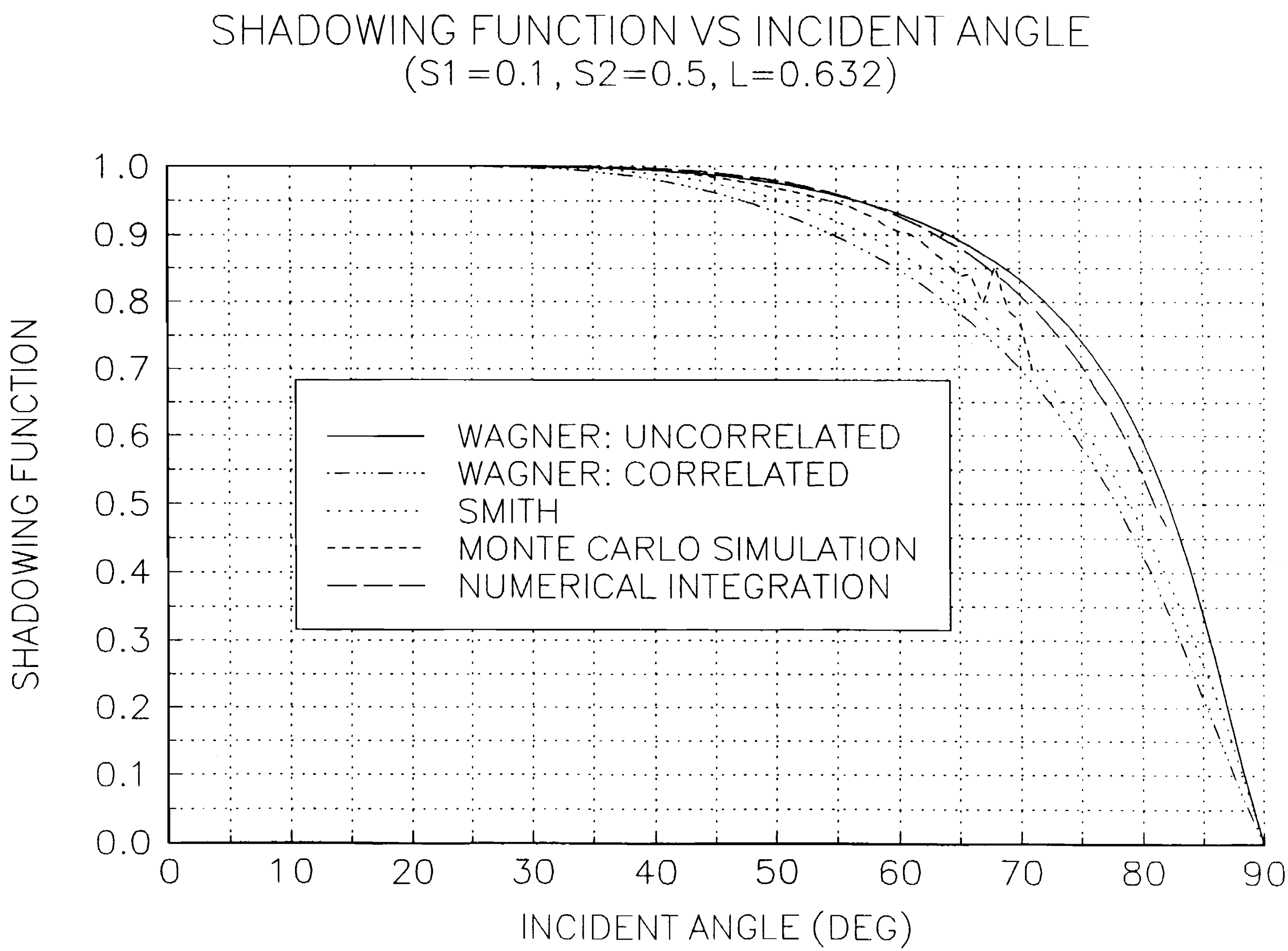

Figure 31: Shadowing Function vs Incident Angle (Var. of Heights=0.1, Var. of Slopes=0.50)

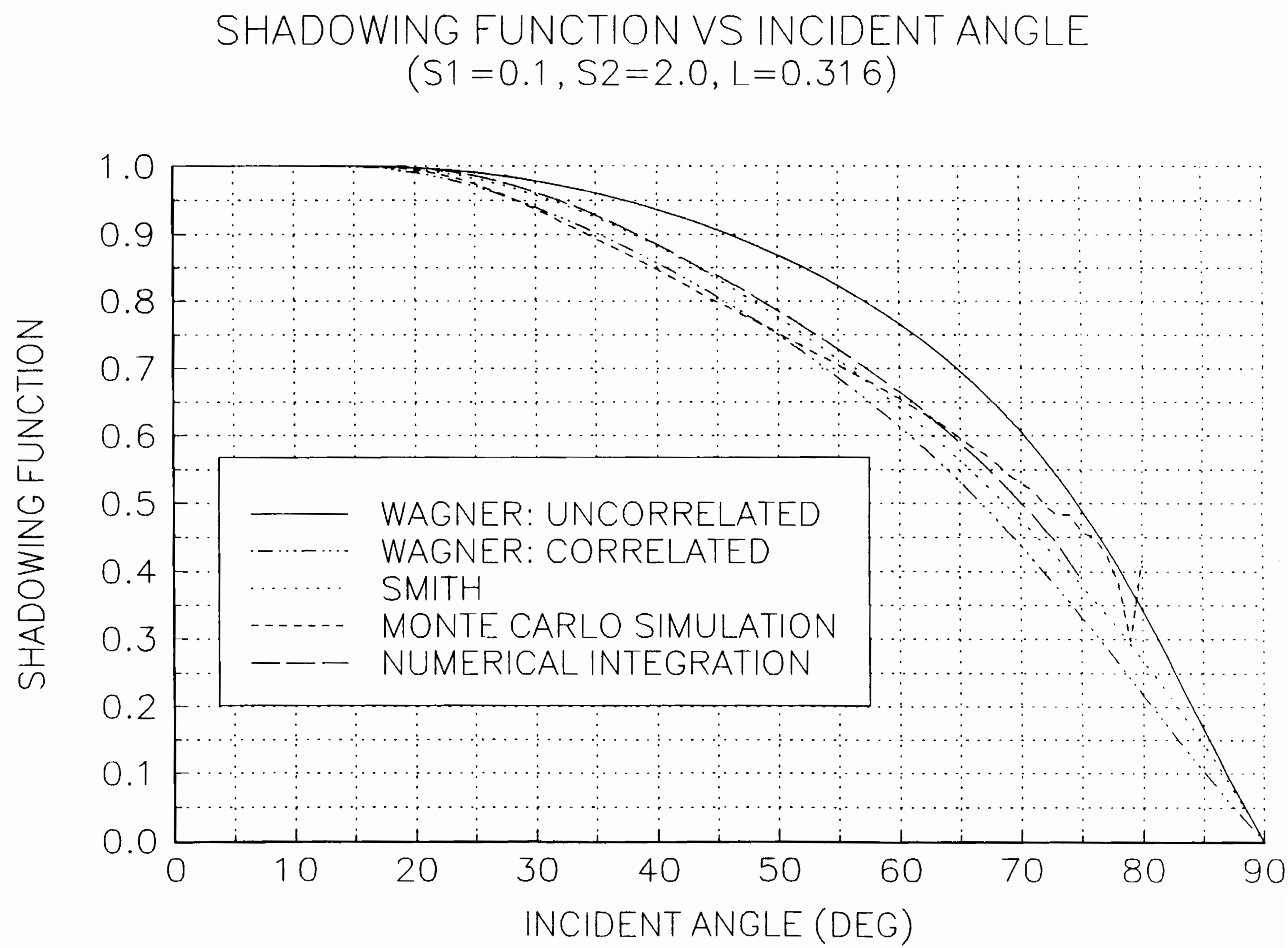

Figure 32: Shadowing Function vs Incident Angle (Var. of Heights=0.1, Var. of Slopes=2.00)

Notice the fairly large discrepancy between our work and that of Wagner's for which correlation is included, particularly at intermediate incident angles. For σ_2^2 = 2.0 we see a much better agreement with the work of Smith, a maximum margin of error of only a few percent. Correspondence with Wagner's correlated shadowing function is also improving.

Plots of percent error in Wagner's and Smith's shadowing function compared to our numerical results are given in Figures 33-37. We see that Wagner's uncorrelated shadowing, for $\sigma_2^2 = 0.05$ and 0.10, compares very well with our numerical results except very near grazing. As the surface slopes are increased to $\sigma_2^2 = 0.3$, we see the analytical results of both Wagner and Smith are breaking down near grazing angles. Wagner's correlated shadowing function and Smith result both produce fairly large errors even for intermediate angles. Wagner's uncorrelated shadowing function, however, is producing very good results until about 75°. As the variance of slopes increases to 0.5, Wagner's uncorrelated shadowing function is still doing better than the other two results. For very large slopes ($\sigma_2^2 = 2.0$), however, we see that Wagner's uncorrelated shadowing function is completely breaking down. Smith's result is doing a good job for all angles. Wagner's correlated shadowing is still doing a poor job close to grazing incidence.

Although good comparisons were generally obtained with Wagner's uncorrelated shadowing function the discrepancy between our work and his more rigorous shadowing function for which correlation was included is worth investigating.

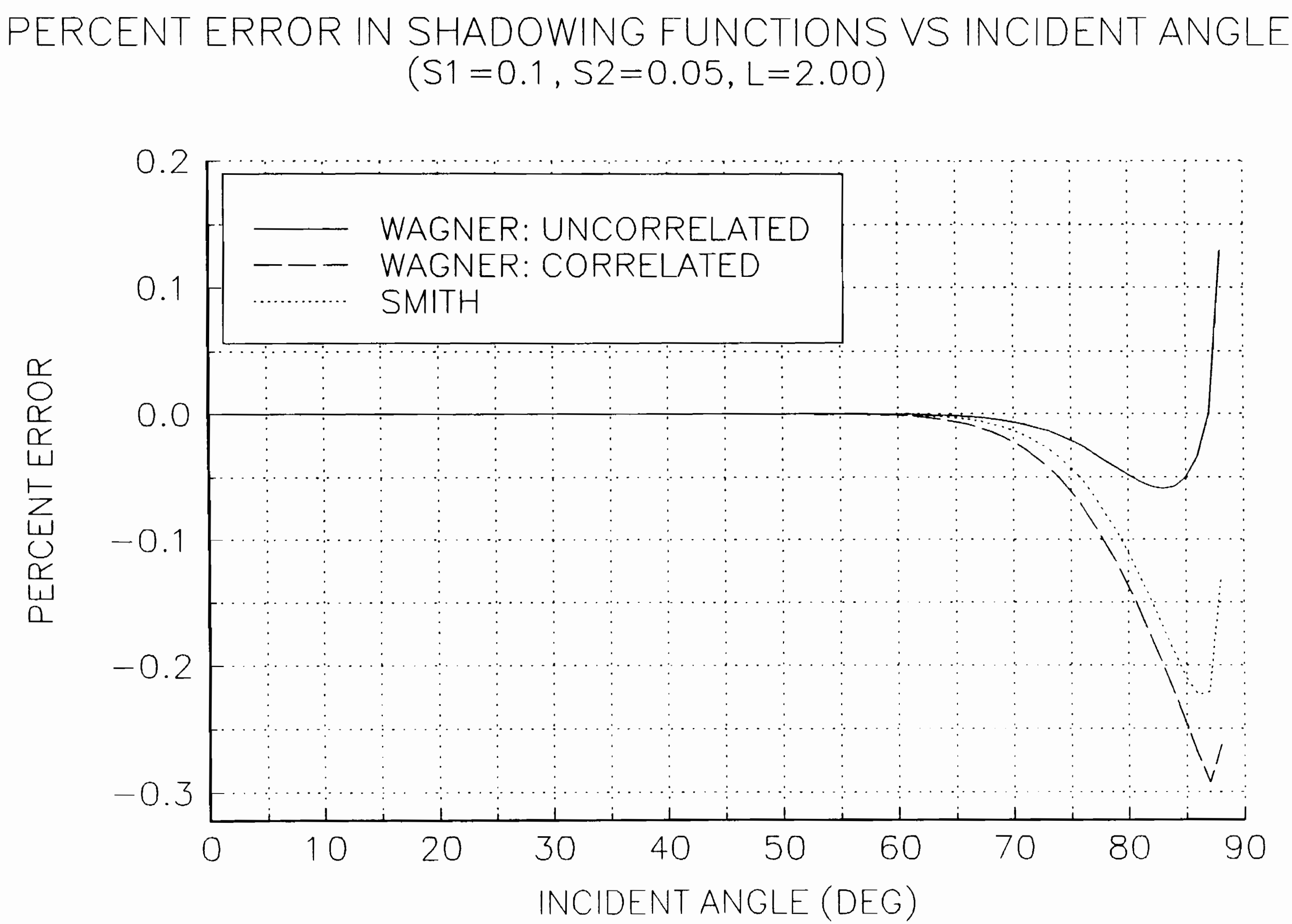

Figure 33: Percent Error in Shadowing Functions vs Incident Angle (Var. of Heights =0.1, Var. of Slopes=0.05)

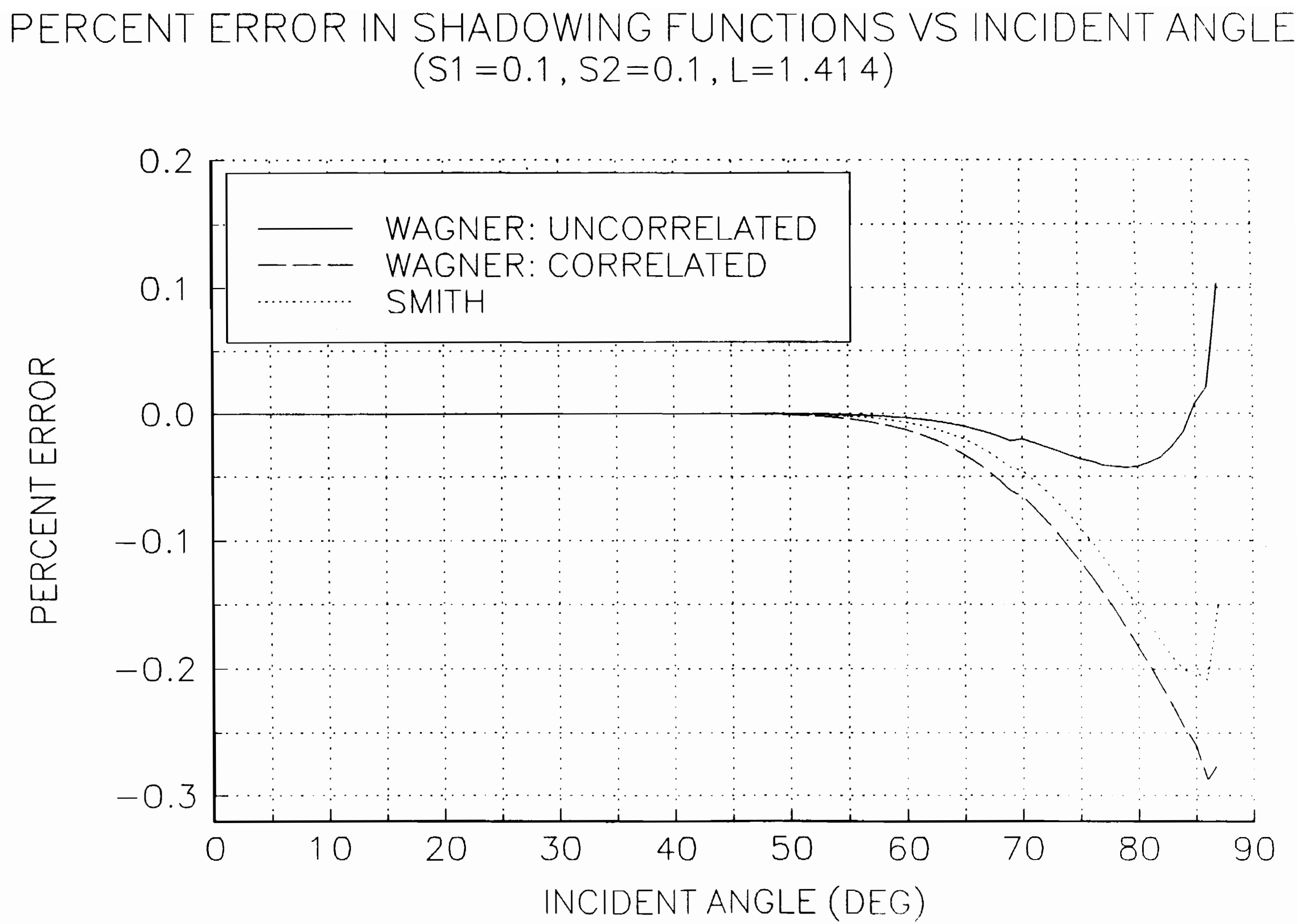

Figure 34: Percent Error in Shadowing Functions vs Incident Angle (Var. of Heights =0.1, Var. of Slopes=0.10)

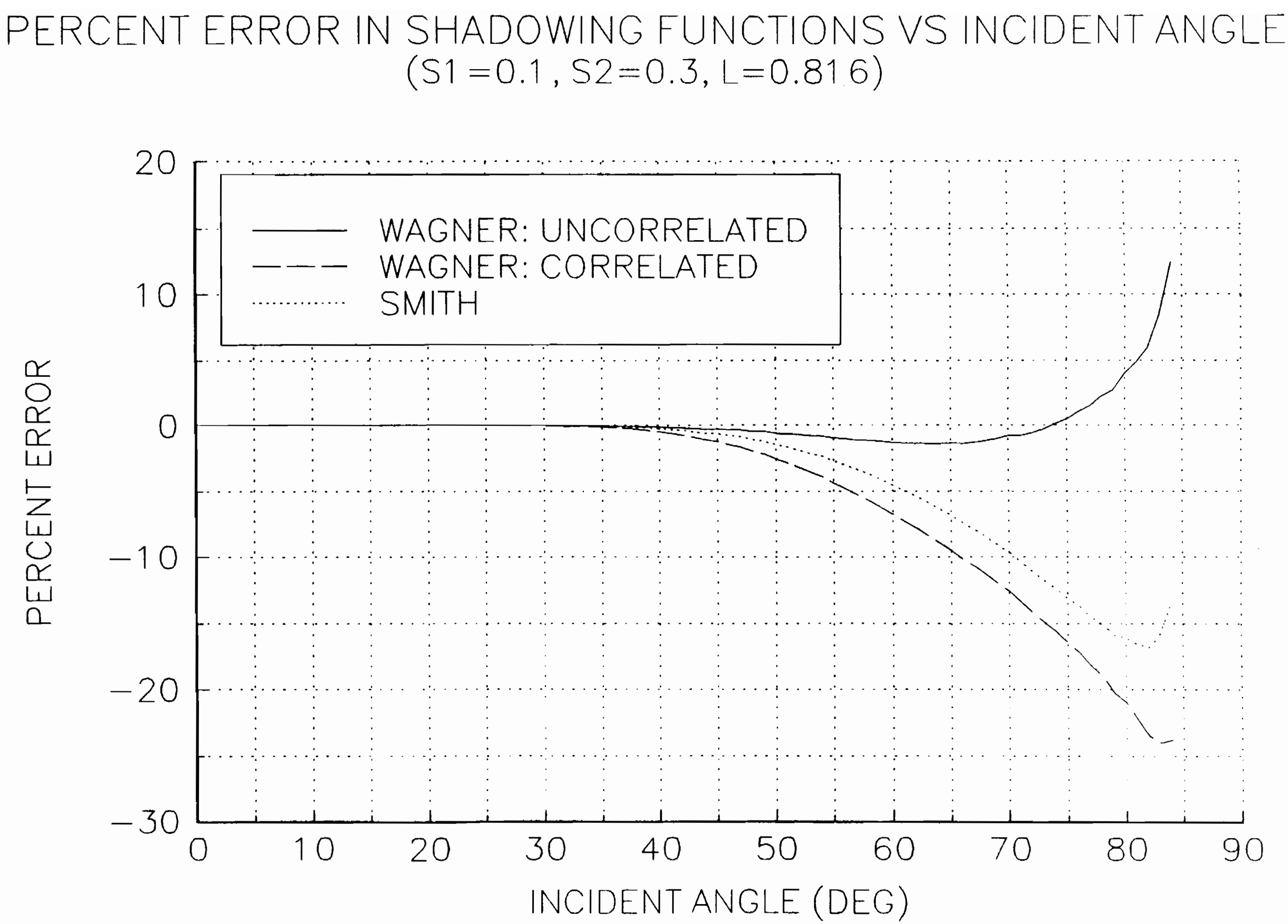

Figure 35: Percent Error in Shadowing Functions vs Incident Angle (Var. of Heights =0.1, Var. of Slopes=0.30)

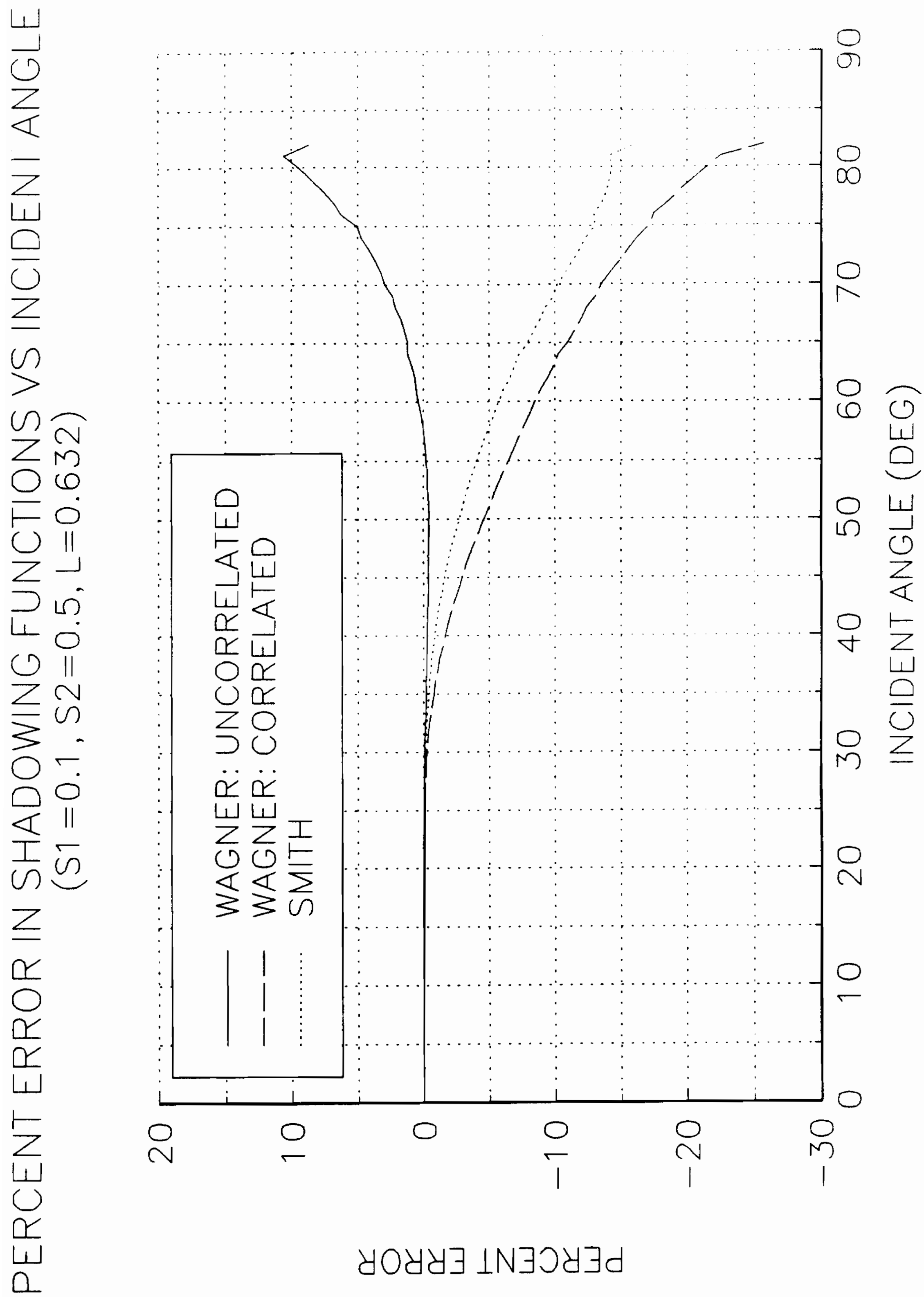

Figure 36: Percent Error in Shadowing Functions vs Incident Angle (Var. of Heights =0.1, Var. of Slopes=0.50)

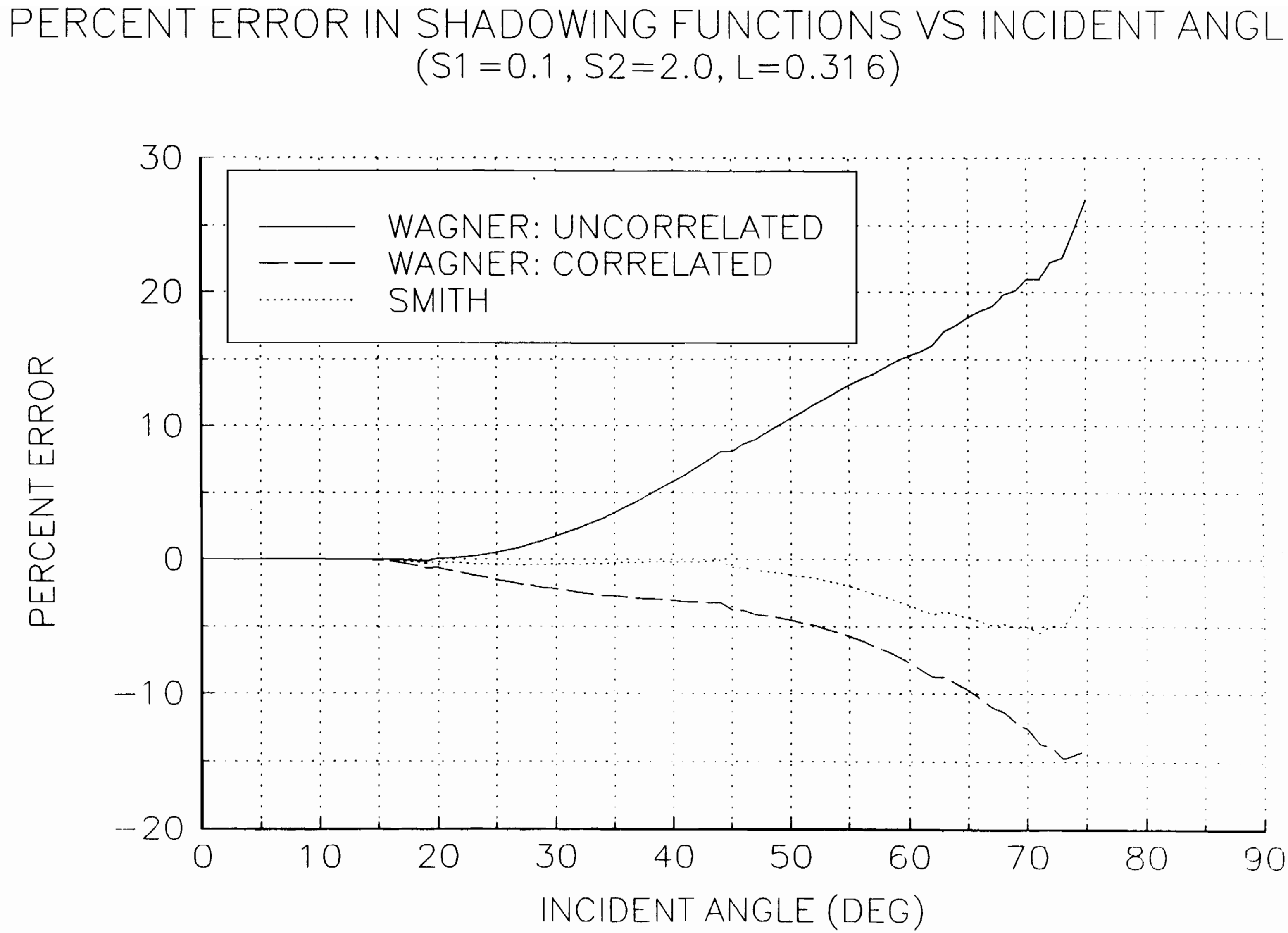

Figure 37: Percent Error in Shadowing Functions vs Incident Angle (Var. of Heights =0.1, Var. of Slopes=2.00)

To resolve this discrepancy we look at the first passage in time function of Wagner's work, q(t), which is conditional on both the height, x_0 ($\zeta_{\rm osp}$), and slope, $\dot{x}_0$ ($\zeta'_{\rm osp}$). This is equivalent to our W_1 function. For σ_1^2=0.1 and σ_2^2=0.05 at a angle of 80 degrees incidence we plotted our result, which is the exact function, against Wagner's uncorrelated shadowing function as well as Wagner's correlated shadowing function for different values of x_0 ($\zeta_{\rm osp}$). A comparison with Smith's work was also made. These plots are shown in Figures 38-40 where we have normalized the distance t to a correlation length (L). There are several points to note here. The first is that none of the approximations to the first passage function, q(t), adequately describe the actual function. From these graphs overestimation of the integral of q(t) would be expected which would result in an over estimation of the shadowing on the surface or an underestimation of the shadowing function. One point to notice is that both Smith's work and Wagner's uncorrelated shadowing function agree very closely with our results far away from the origin, but Wagner's correlated function does not appear to be decaying rapidly enough, probably due to the truncated Taylor series in Wagner's derivation. An important point to note also is that none of the approximations produce the correct result as $t \rightarrow 0^+$, which is that the function q(t) should decay to zero. Our work, however, does produce the correct result in this limit. Physically we expect the function to decay to zero as $t \rightarrow 0^+$ due to the fact that the slope at the origin is perpendicular to the incident ray and we expect that the probability of a first crossing occurring immediately after this point to be zero due to the correlation of the surface.

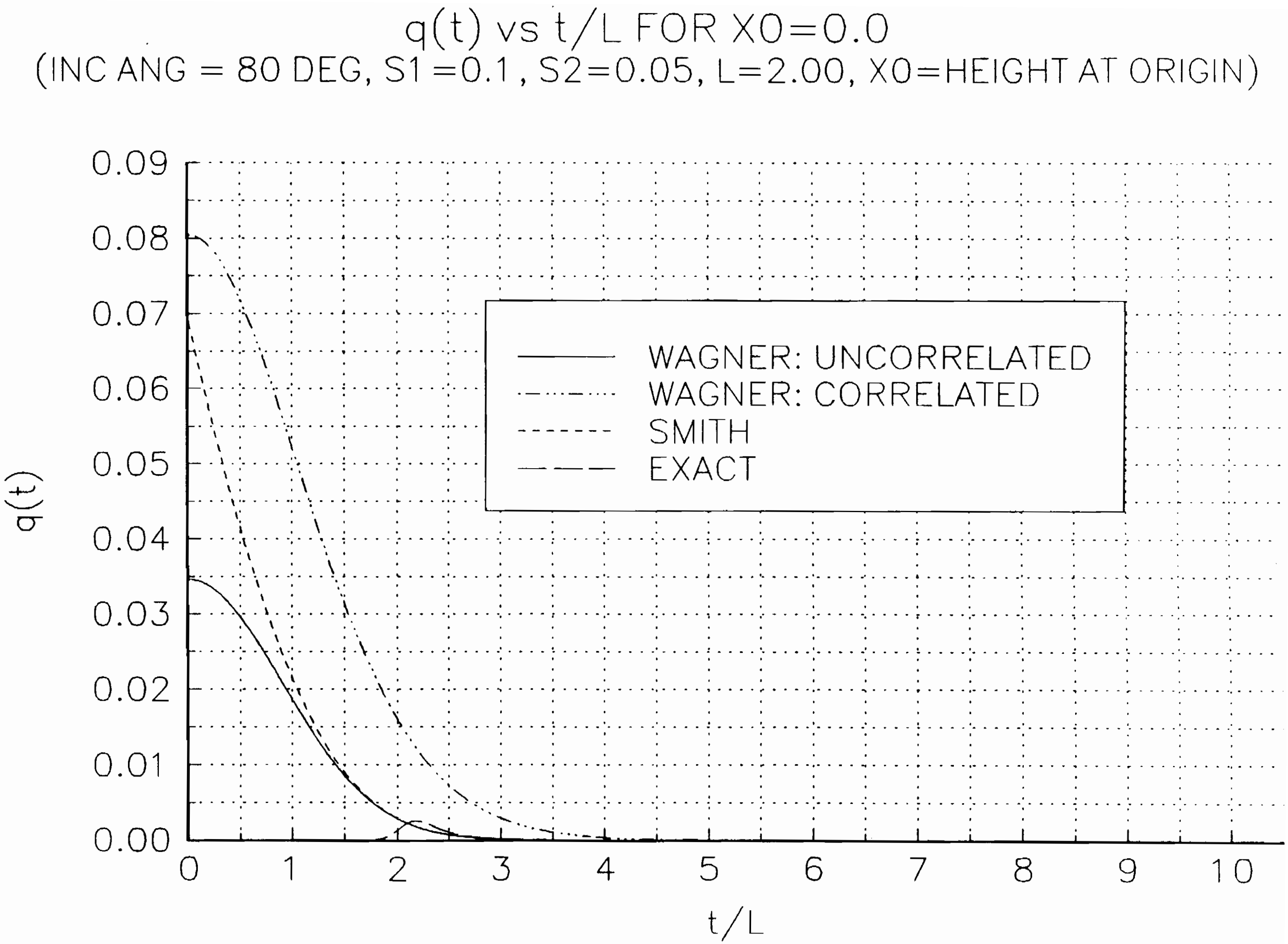

Figure 38: q(t) vs t/L for X0 =0.0, S1=0.1, S2=0.05, L=2.00, θ_i=80°

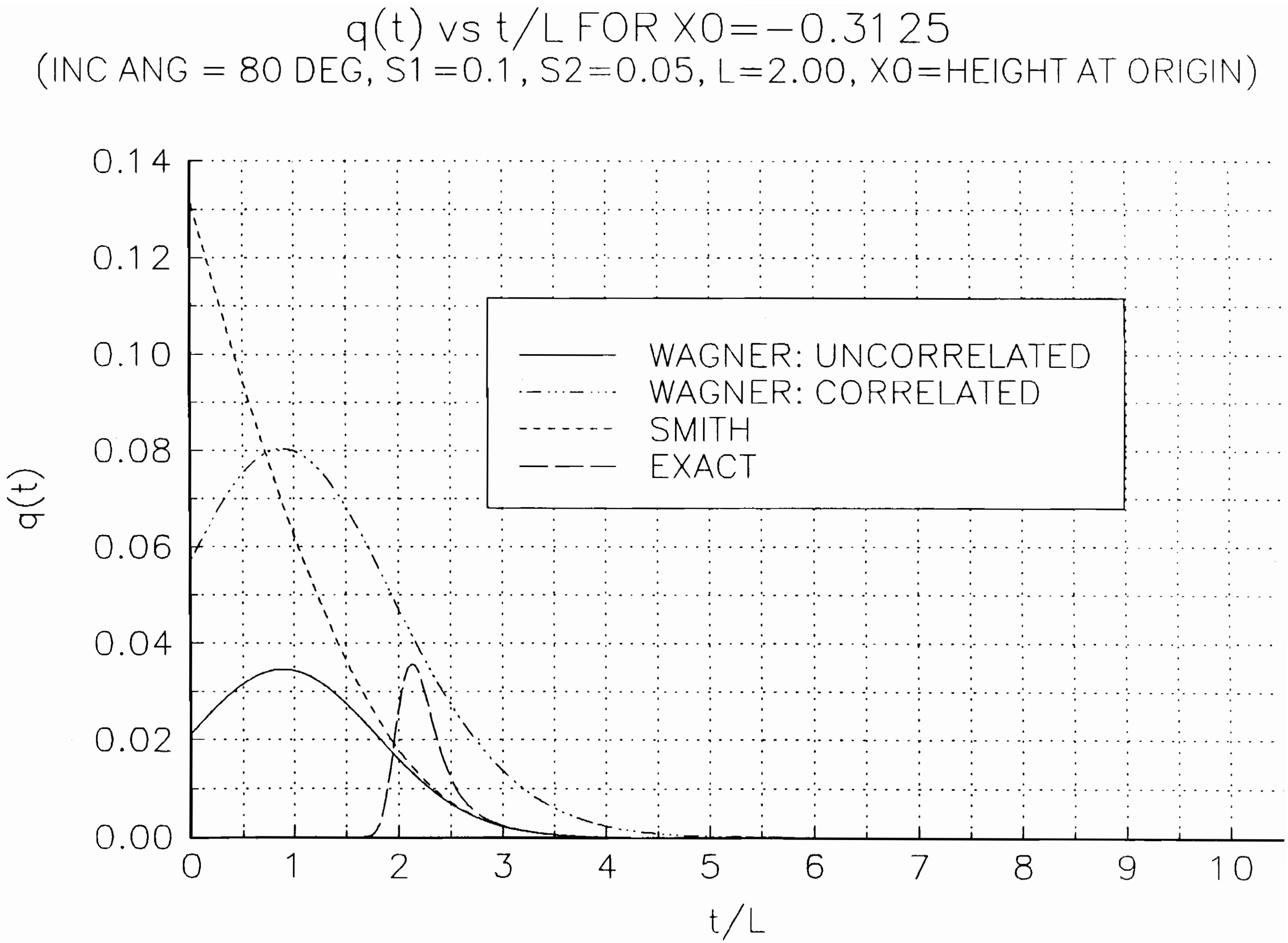

Figure 39: q(t) vs t/L for X0 =-0.3125, S1=0.1, S2=0.05, L=2.00, θ_i=80°

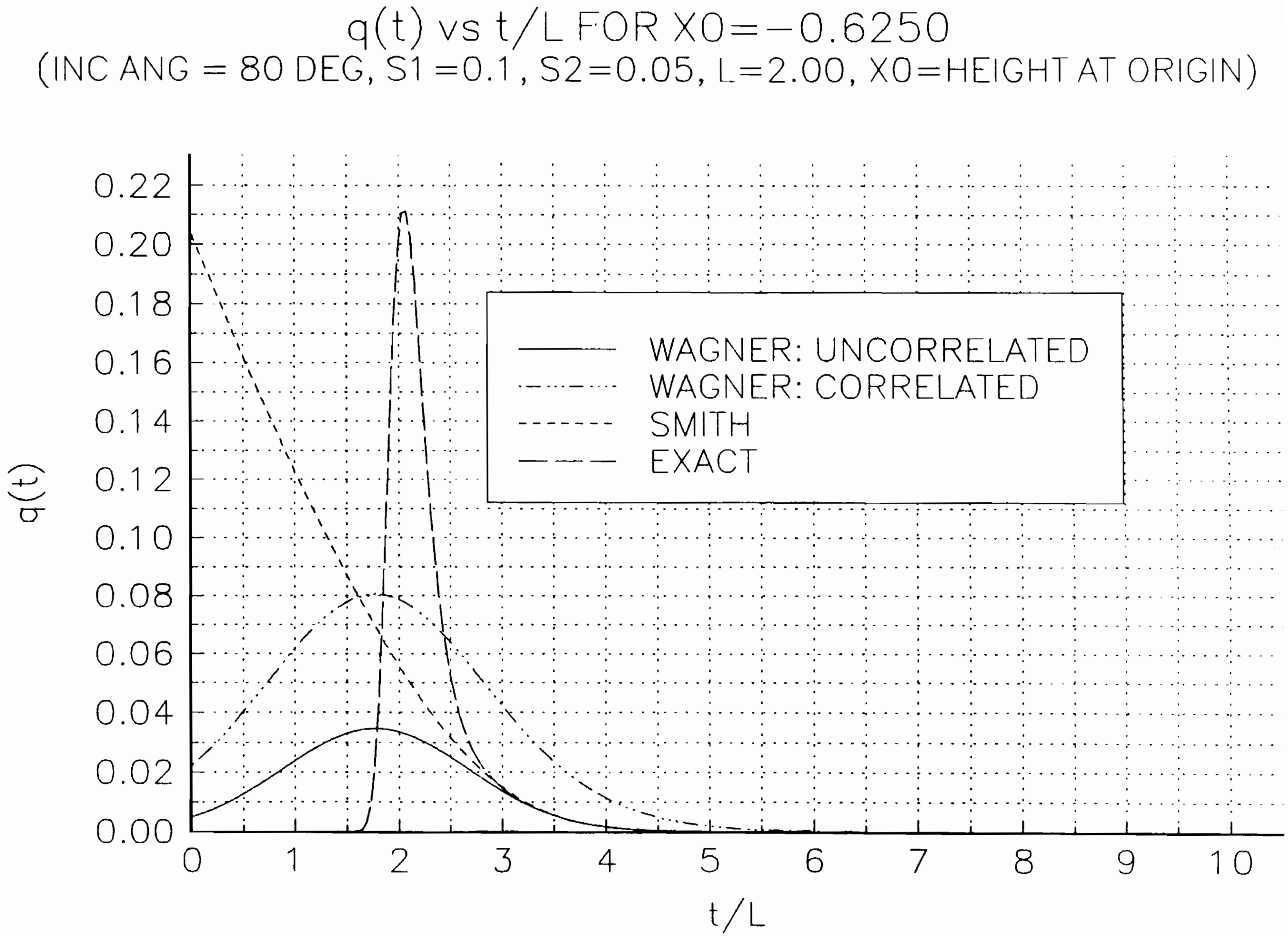

Figure 40: q(t) vs t/L for X0 =-0.6250, S1=0.1, S2=0.05, L=2.00, θ_i=80°

Of course we expect the results of Smith and Wagner which neglect correlation to produce results which do not represent the actual function properly near the origin. However, the inclusion of correlation by Wagner purportedly to improve the solution has apparently made things worse. We have concluded there are probably two reasons for this. The first is that Wagner's result of expanding the autocorrelation function in a Taylor series only produced a leading order result, and had he been able to expand further an improvement would have been obtained. The second reason for the discrepancy is due to what we believe to be a error in logic in Wagner's paper. Wagner obtains two approximations to the q(t) function. The first is $q_1(t)$ which is valid as $t \to \infty$, and is the same as the assumption of no correlation between the shadowing point and the shadowed point. The second is $q_2(t)$ which is due to the Taylor's expansion and is supposedly valid as $t \to 0$. Wagner proceeds to approximate the actual function q(t) by adding $q_1(t)$ and $q_2(t)$. His reasoning according to our interpretation is as follows: Since $q_2(t)$ decays to zero as $t \to \infty$ it will not contribute appreciably to the integral in this region unless grazing angles are approached (if grazing angles are approached then more terms in the Taylor series need to be taken). Also, since $q_1(t)$ "explicitly neglected, for all t, the effect on the crossing probability of the conditioning on ζ_{osp}, and ζ'_{osp} at t=0," this would not contribute significantly at the origin. Although it was not stated in Wagner's paper that he believed $q_1(t)$ to decay at the origin it certainly was implied and it seems to us that this must be true if his approximation is to be a valid one. We have found, however, that this approximation is not a good one since the actual function has decayed to zero while the no correlation function, $q_1(t)$, is still appreciable. We believe it is this that has caused the overestimation of surface shadowing by Wagner. Likewise, Smith's result is invalid at the origin due to both the no correlation assumption and an additional factor which amplifies the error. However, while Smith's work is invalid at the origin, it decay's properly away from the origin and so it provides a more accurate result than that of Wagner's correlated shadowing function.

As the surface becomes extremely rough we see things start to improve slightly. For σ_1^2=0.1 and σ_2^2=2.0 (Figures 41-44) we see that the approximations still do a poor job in accurately representing the exact function. At this point, however, we can see that the area under these curves, particularly Smith's result and our work for $\zeta_{osp} \doteq x_0$ =-0.3125, become closer. So, although the exact function (integrand) is poorly approximated, its integral, which is the shadowing, is fairly close to the exact result.

For completeness, we have plotted the normalized radar cross section of equation (5.64) for the various surface conditions mentioned above. These are presented in Figures 45-49. The graphs behave similarly to those plotted in Chapter 5. As the variance of the slopes increase the peak moves away from zero degrees incidence. This is apparently due to the breakdown of the shadow corrected Kirchhoff approximation. The symmetry of the graphs reflect the isotropic nature of the surface. We see excellent agreement with our results and Wagner's no correlation shadowing function up until $\sigma_2^2 = 0.5$. All the results are apparently so close as to be unnoticeable on the graphs except as the surface becomes very rough ($\sigma_2^2 > 0.5$). We note however, this may change if an attempt is made to use these functions in the context of multiple scattering for rays which bounce from one point on the surface to another. In this case, there is no telling the effects of correlation between the shadowing point and the shadowed point on the final result.

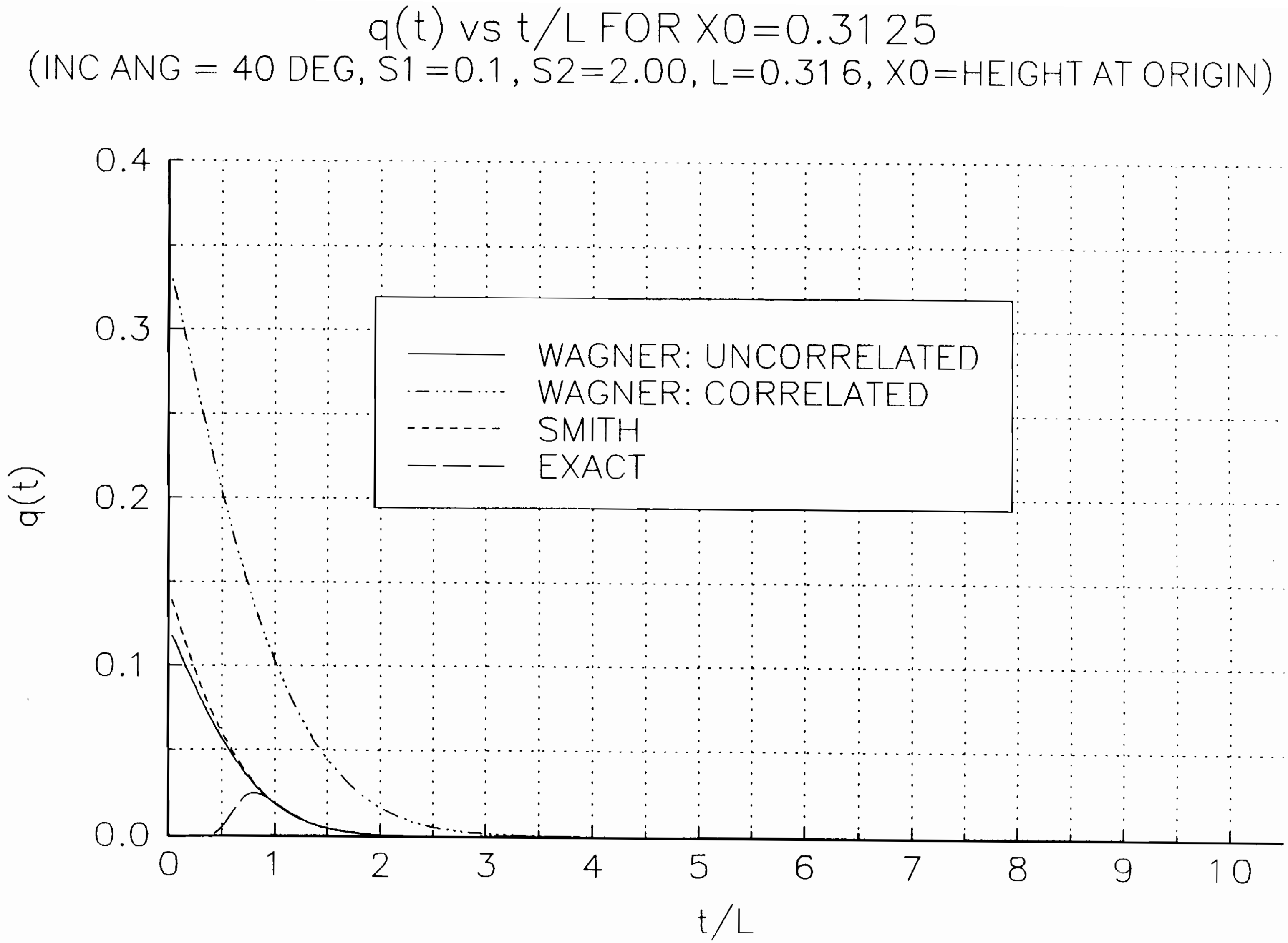

Figure 41: q(t) vs t/L for X0 =0.3125, S1=0.1, S2=2.00, L=0.316, θ_i=40°

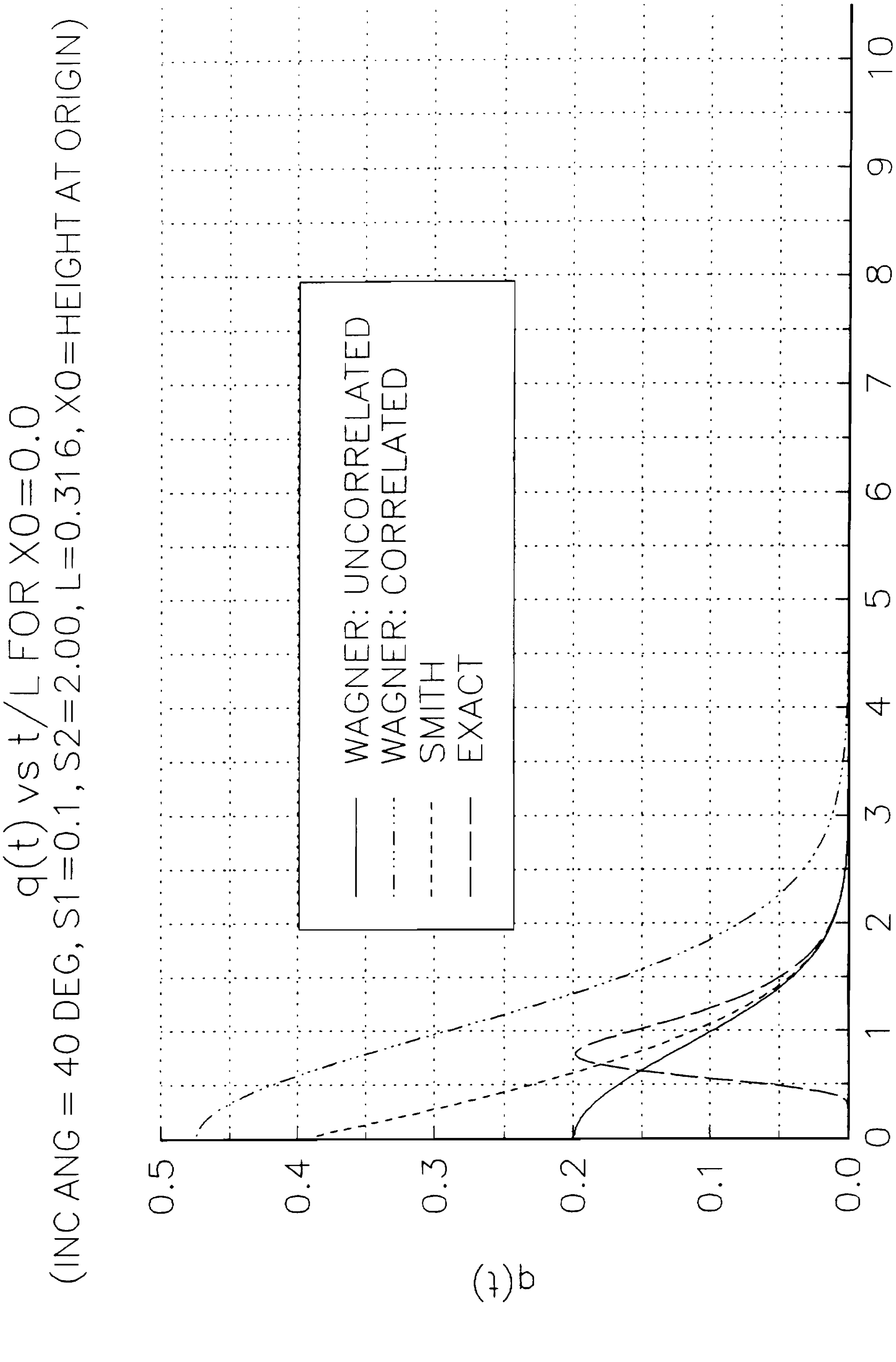

Figure 42: q(t) vs t/L for X0 =0.0, S1=0.1, S2=2.00, L=0.316, θ_i=40°

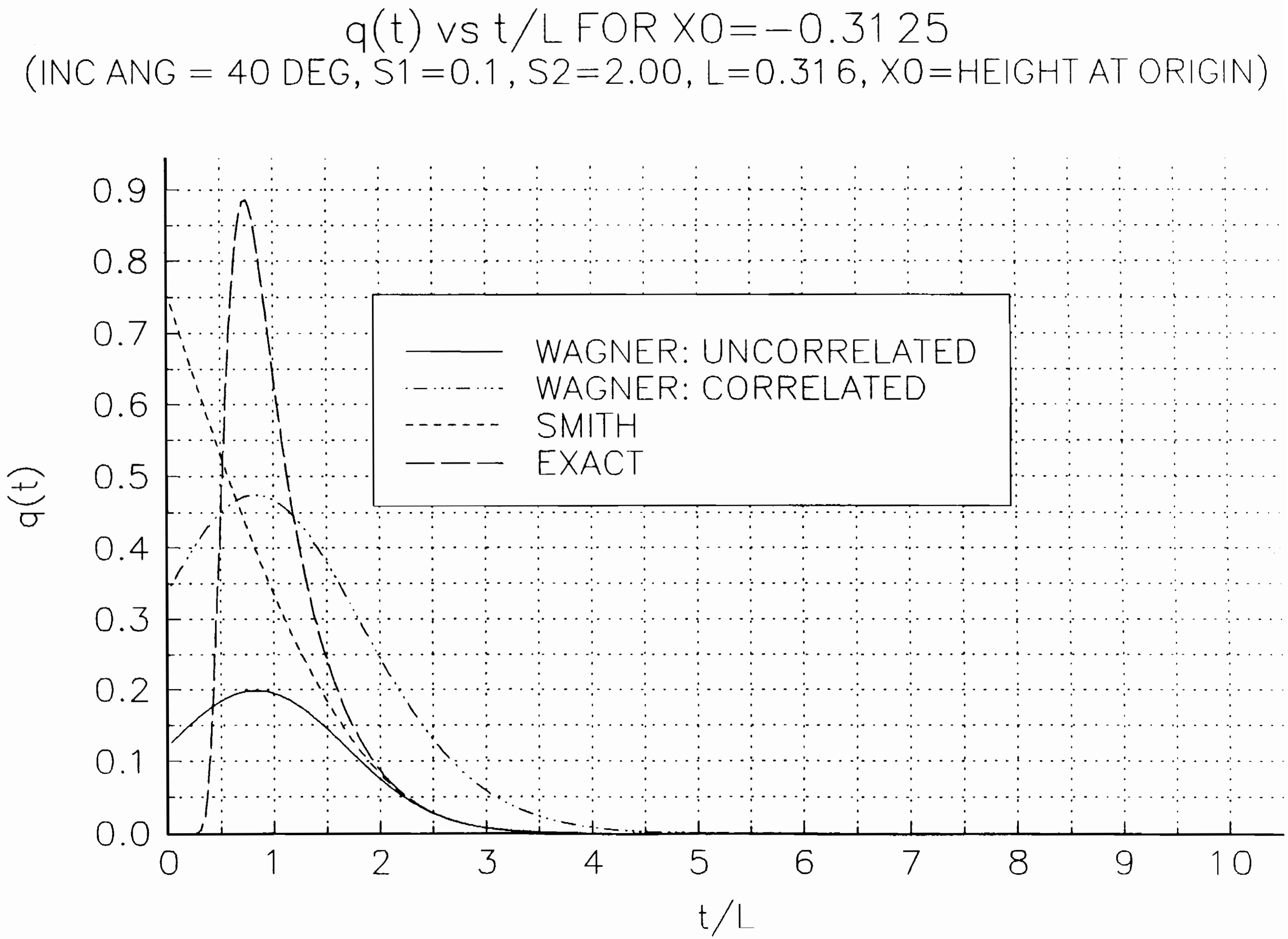

Figure 43: q(t) vs t/L for X0 =-0.3125, S1=0.1, S2=2.00, L=0.316, θ_i=40°

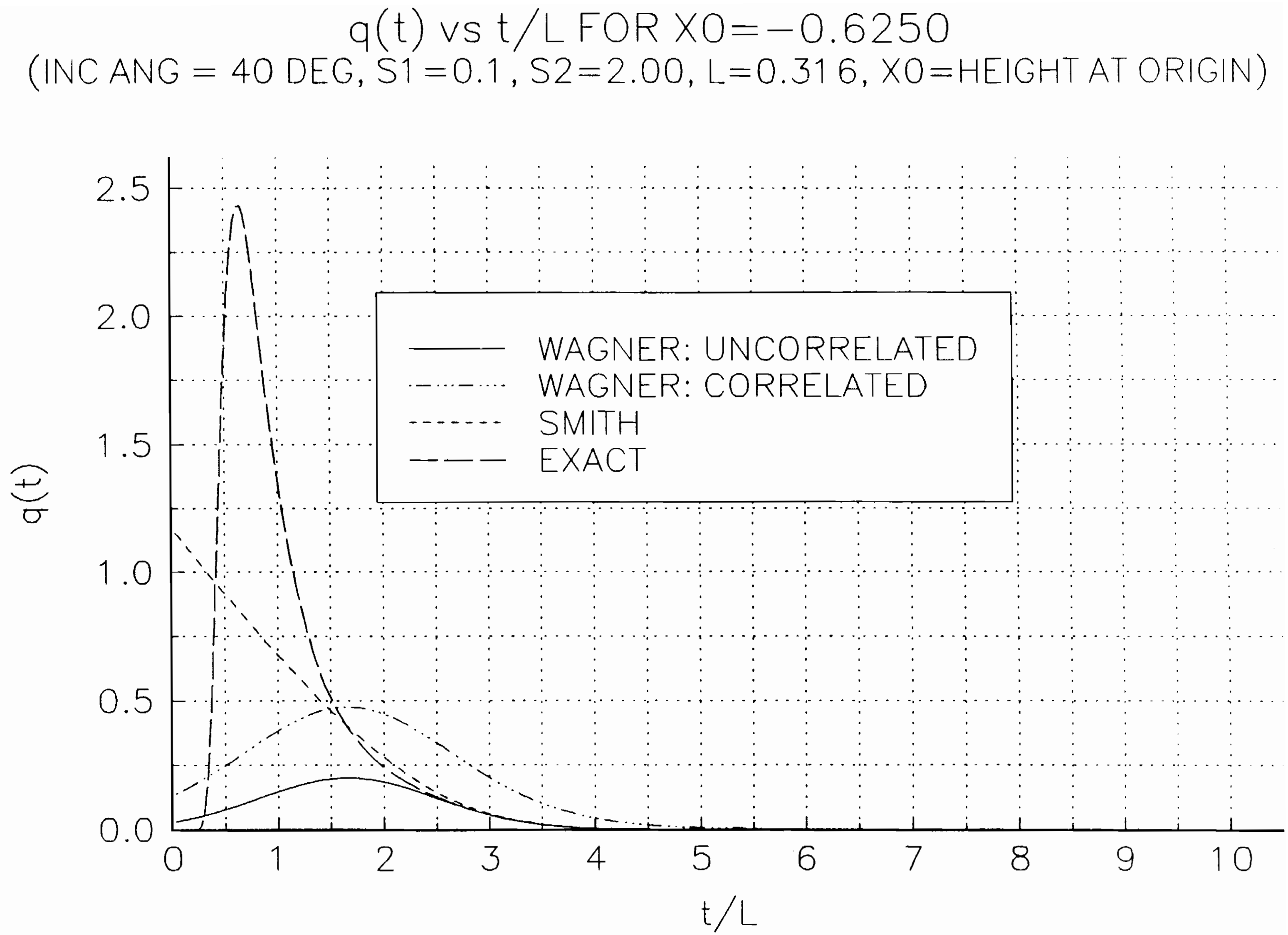

Figure 44: q(t) vs t/L for X0 =-0.6250, S1=0.1, S2=2.00, L=0.316, θ_i=40°

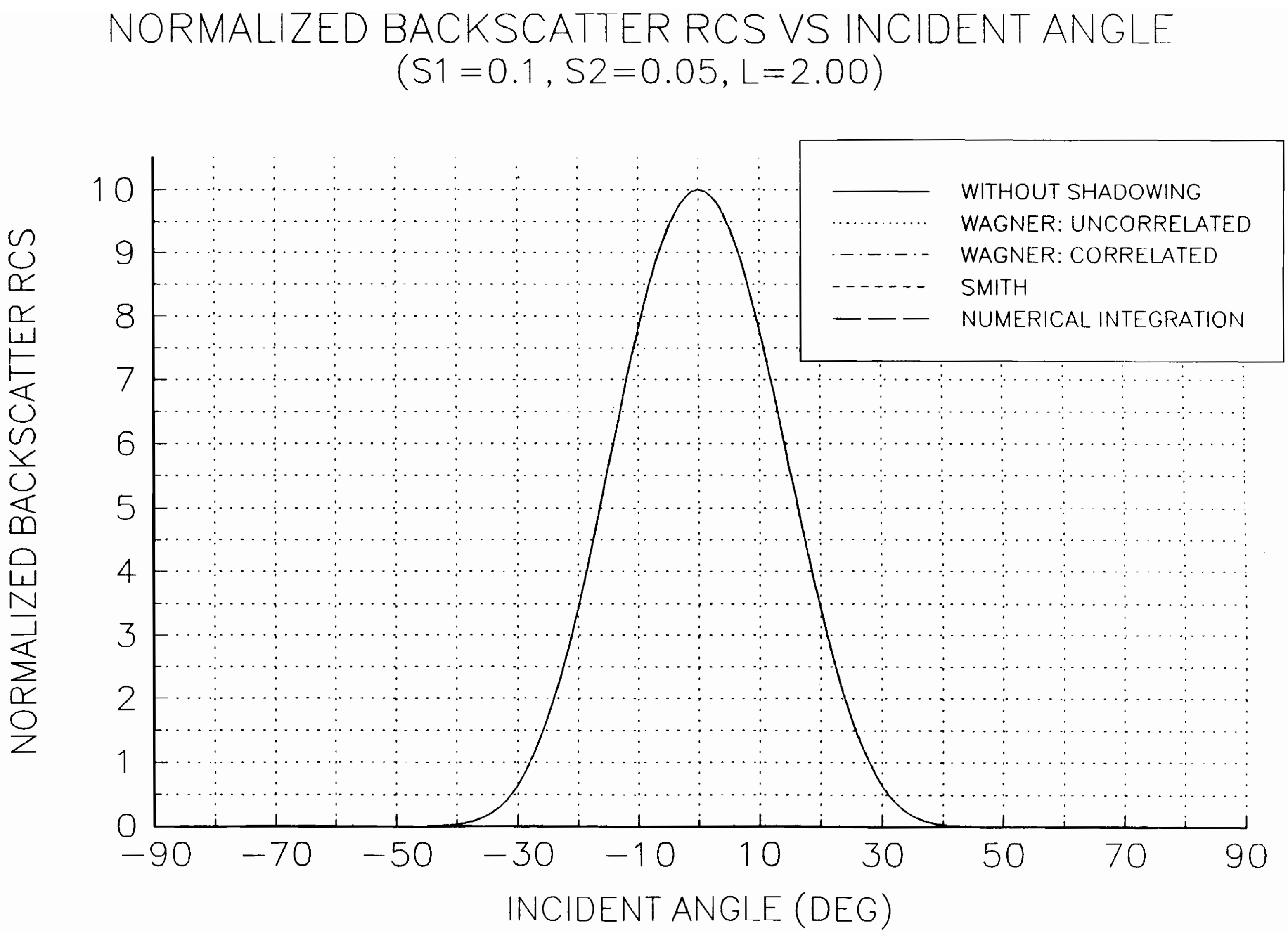

Figure 45: Normalized Backscatter RCS vs Incident Angle (Var. of Heights=0.1, Var. of Slopes=0.05)

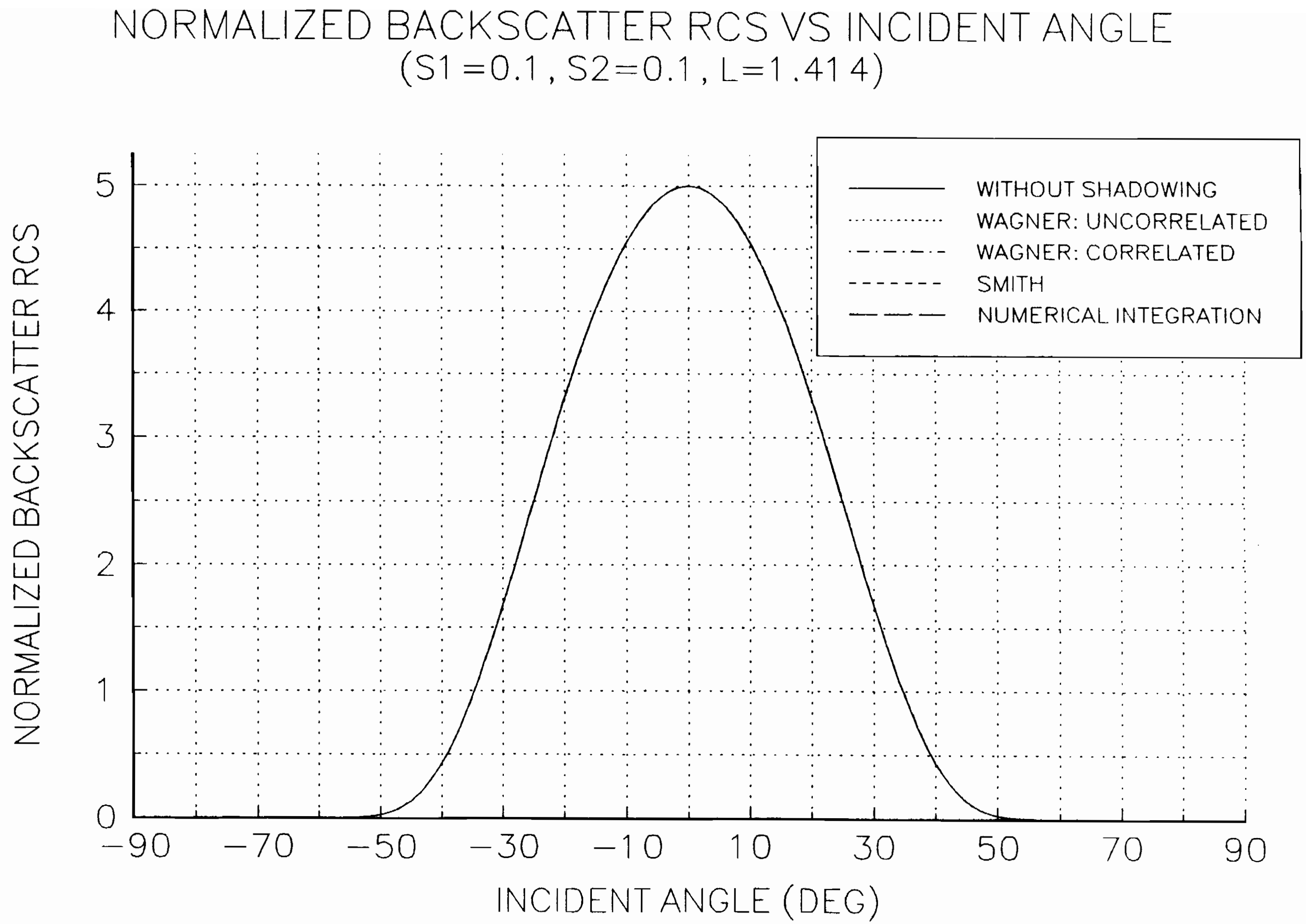

Figure 46: Normalized Backscatter RCS vs Incident Angle (Var. of Heights=0.1, Var. of Slopes=0.10)

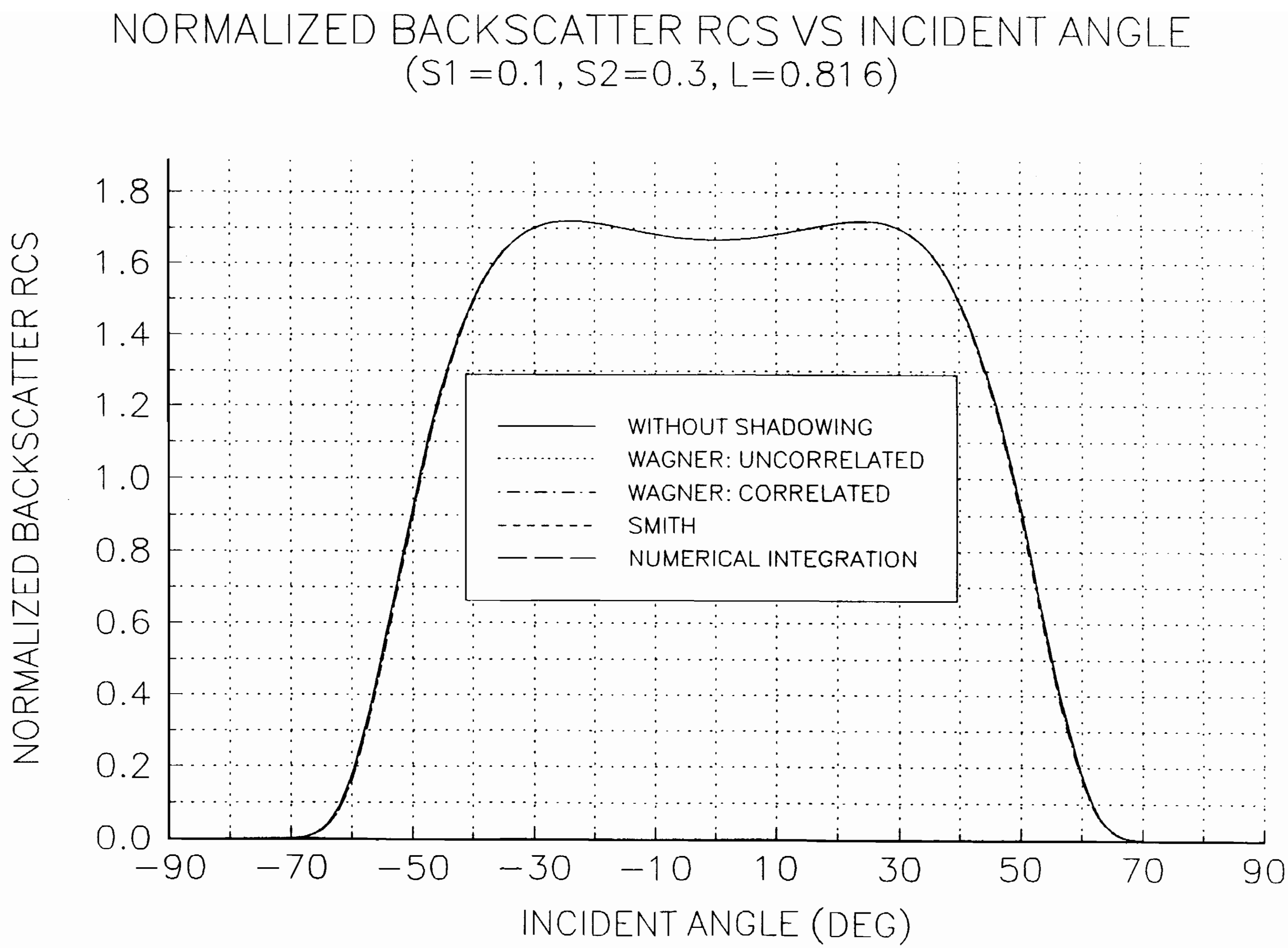

Figure 47: Normalized Backscatter RCS vs Incident Angle (Var. of Heights=0.1, Var. of Slopes=0.30)

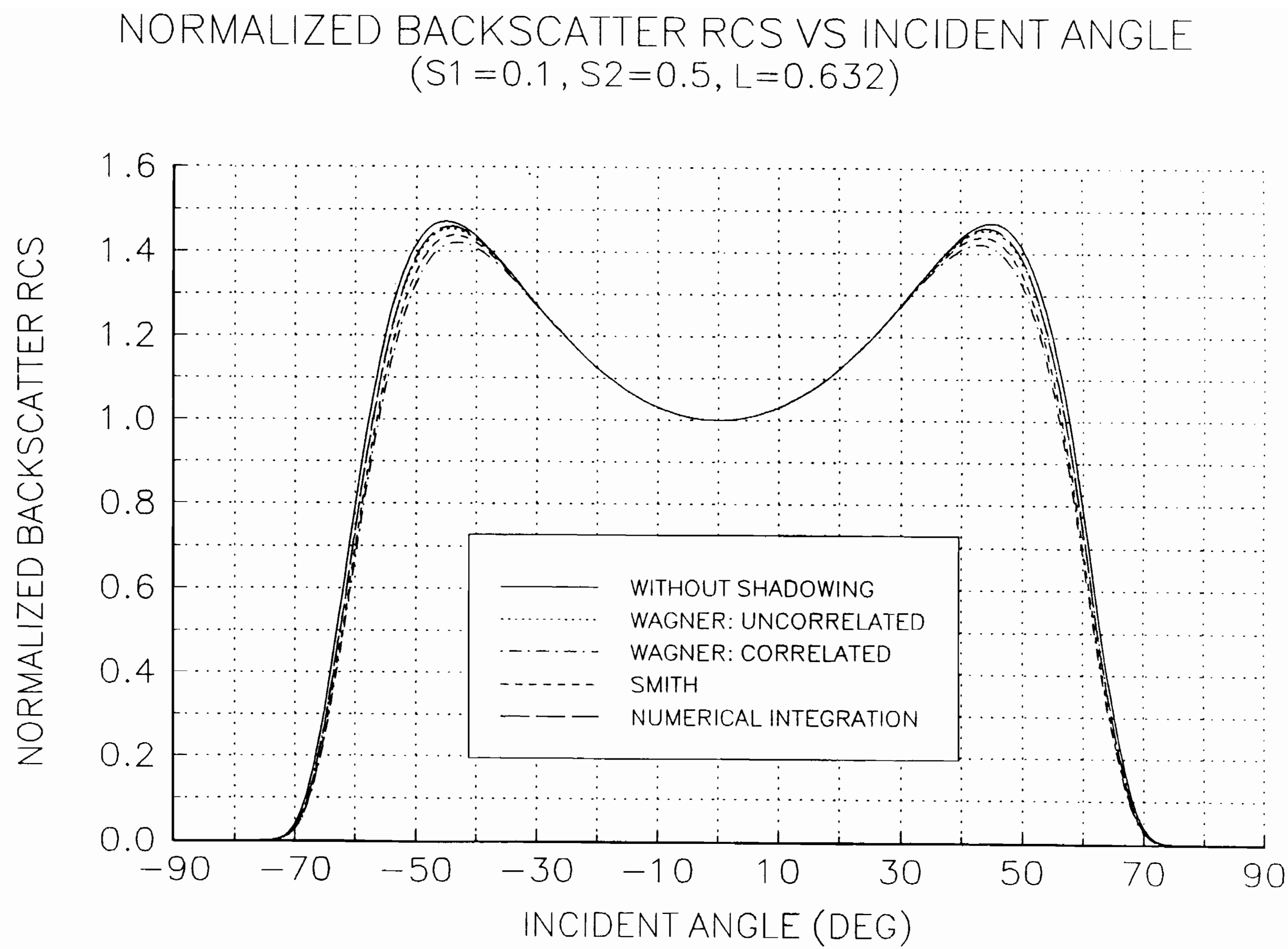

Figure 48: Normalized Backscatter RCS vs Incident Angle (Var. of Heights=0.1, Var. of Slopes=0.50)

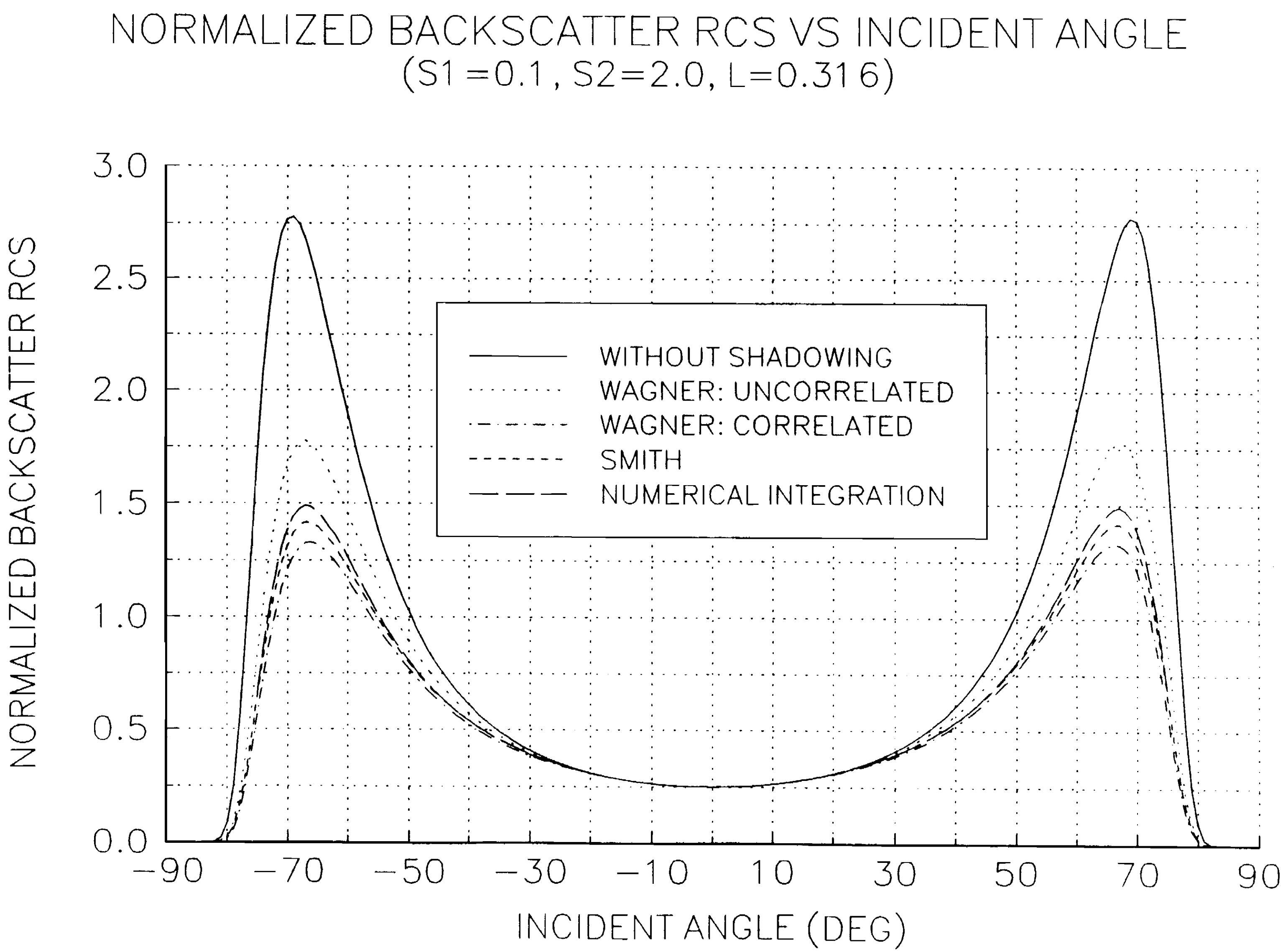

Figure 49: Normalized Backscatter RCS vs Incident Angle (Var. of Heights=0.1, Var. of Slopes=2.00)

7.0 FUTURE RESEARCH

This thesis studied shadowing of electromagnetic fields from randomly rough surfaces. Due to the difficulty of the problem, several assumptions and restrictions were made at the onset. Multiple scattering and diffraction effects were both ignored, leading to a very crude estimate for the scattered field. The lifting of these restrictions are currently being investigated in the literature.

This chapter formally extends Sancer's work [Sancer, 1969] and derives an expression for the doubly scattered electric field in the high frequency limit. We arrive at the incoherent scattered intensity in terms of complicated shadowing functions, but make no attempt to derive these functions explicitly. Our approach is similar to that of Chapter 2, which follows the work of Sancer [Sancer, 1969].

The shadow-corrected Kirchhoff approximation derived in the previous chapters provides a good result for surfaces that are gently undulating when the incident field is above grazing angles. This is due to the fact that for very rough surfaces, or in the case of grazing incidence, there is a great deal of multiple scattering on the surface which cannot be ignored. Since our goal is to determine the radar cross section over as broad a range of angles and surface conditions as possible, a discussion of multiple scattering effects is necessary.

7.1 Multiple Scattering Effects

In this section we consider doubly scattered rays. This is done by performing one iteration of the magnetic field integral equation (MFIE) using appropriately shadowed currents. The result is expected to provide an improvement over the shadow corrected Kirchhoff approximation of the previous chapters, particularly as the surface correlation length becomes smaller or as we approach grazing incidence.

The far-field integral for the electric field and the MFIE are re-stated here for convenience:

$$\vec{E}_s(\vec{r}) = jk_o\eta_o \frac{exp\{-jk_o r\}}{4\pi r}\,\widehat{k}\times\widehat{k}\times\int_{S_o} d\vec{r}_o\; \vec{J}_s(\vec{r}_s) exp\{j\vec{k}\cdot\vec{r}_s\} \tag{7.1}$$

$$\vec{J}(\vec{r}_s) = \vec{J}_K(\vec{r}_s) + \frac{jk_o}{2\pi}\,\vec{N}(\vec{r}_o)\times\int_{S_o} d\vec{r}\,'_o\; \vec{J}_s(\vec{r}\,'_s)\times\frac{\widehat{u}(\vec{r}_s-\vec{r}\,'_s)}{|\vec{r}_s-\vec{r}\,'_s|}\; exp\{-jk_o|\vec{r}_s-\vec{r}\,'_s|\} \tag{7.2}$$

where

$$\vec{r}_s = x_o\hat{x} + y_o\hat{y} + \zeta(x_o, y_o)\hat{z}$$
$$= \vec{r}_o + \zeta(x_o, y_o)\hat{z}$$

and

$$\vec{r}'_s = x'_o\hat{x} + y'_o\hat{y} + \zeta(x'_o, y'_o)\hat{z}$$
$$= \vec{r}'_o + \zeta(x'_o, y'_o)\hat{z}.$$

To compute the doubly scattered field, we set $\vec{J}(\vec{r}'_s) = \vec{J}_k(\vec{r}'_s) = 2\vec{N}(\vec{r}'_o) \times H_o\hat{h}_o exp\left\{ -jk_o\hat{k}_i \cdot \vec{r}'_s \right\}$ under the integral in (7.2) to get

$$\vec{J}(\vec{r}_s) = \vec{J}_K(\vec{r}_s) + \frac{jk_oH_o}{\pi} \vec{N}(\vec{r}_o) \times \int_{S_o} d\vec{r}'_o \, [\vec{N}(\vec{r}'_o) \times \hat{h}_o] \times \frac{\hat{u}(\vec{r}_s - \vec{r}'_s)}{|\vec{r}_s - \vec{r}'_s|}$$
$$* \; exp\left\{ jk_o\left(-\hat{k}_i \cdot \vec{r}'_s - |\vec{r}_s - \vec{r}'_s| \right)\right\} \qquad (7.3)$$

We can write the exponential of the Green's function as

$$exp\left\{ -jk_o|\vec{r}_s - \vec{r}'_s| \right\} = exp\left\{ -jk_o\hat{u}(\vec{r}_s - \vec{r}'_s) \cdot (\vec{r}_s - \vec{r}'_s)\right\} = exp\left\{ -jk_o\hat{k}_1 \cdot (\vec{r}_s - \vec{r}'_s)\right\}, \qquad (7.4)$$

where $\hat{k}_1 = \hat{u}(\vec{r}_s - \vec{r}'_s)$. $\vec{J}(\vec{r}_s)$ now becomes

$$\vec{J}(\vec{r}_s) = \vec{J}_K(\vec{r}_s) + \frac{jk_oH_o}{\pi} \vec{N}(\vec{r}_o) \times \int_{S_o} d\vec{r}'_o[\vec{N}(\vec{r}'_o) \times \hat{h}_o] \times \frac{\hat{k}_1}{|\vec{r}_s - \vec{r}'_s|}$$
$$* \; exp\left\{ jk_o\left(-\hat{k}_i \cdot \vec{r}'_s - \hat{k}_1 \cdot (\vec{r}_s - \vec{r}'_s)\right)\right\}. \qquad (7.5)$$

$\vec{E}_s(\vec{r})$, with this current, is therefore

$$\vec{E}_s(\vec{r}) = jk_o\eta_o\frac{exp\{-jk_or\}}{4\pi r}\,\hat{k}\times\hat{k}\times\int_{S_o} d\vec{r}_o\;\vec{J}_K(\vec{r}_s)exp\Big\{j\vec{k}\cdot\vec{r}_s\Big\}\;+$$

$$jk_o\eta_o\frac{exp\{-jk_or\}}{4\pi r}\,\hat{k}\times\hat{k}\times\int_{S_o} d\vec{r}_o\Bigg\{\frac{jk_oH_o}{\pi}\;\vec{N}(\vec{r}_o)\;\times\int_{S_o} d\vec{r}'_o[\vec{N}(\vec{r}'_o)\times\hat{h}_o]\times\frac{\hat{k}_1}{|\vec{r}_s-\vec{r}'_s|}$$

$$*\;exp\Big\{jk_o\Big[-\hat{k}_i\cdot\vec{r}'_s-\hat{k}_1\cdot(\vec{r}_s-\vec{r}'_s)\Big]\Big\}\Bigg\}exp\Big\{j\vec{k}\cdot\vec{r}_s\Big\}. \tag{7.6}$$

We now modify the result by including the shadowing functions in the kernels of the integrals. They are defined as follows:

$$S_i(\vec{r}'_s;\hat{k}_i) \quad = \begin{cases} 1 \text{ if the point at } \vec{r}'_s \text{ is illuminated by } \hat{k}_i \\ 0 \text{ if the point at } \vec{r}'_s \text{ is shadowed from } \hat{k}_i, \end{cases} \tag{7.7}$$

is the incident shadowing function (2.9),

$S_p(\vec{r}'_s;\hat{k}_i;\hat{k}_1) = 1$ if the ray $\hat{k}_1$ resulting from the reflection of $\hat{k}_i$ at the point $\vec{r}'_s$ strikes the surface at the point $\vec{r}_s$ without first being blocked by another point on the surface (7.8)

$= 0$ if the ray $\hat{k}_1$ resulting from the reflection of $\hat{k}_i$ at the point $\vec{r}'_s$ leaves the surface without being blocked by another point on the surface, is called the propagation shadowing function, and

$$S_s(\vec{r}_s;\ \hat{k}_1;\hat{k}) \quad = 1 \text{ if the ray, } \hat{k}\text{, resulting from the reflection of } \hat{k}_1 \text{ at the point at } \vec{r}_s \text{ leaves the surface without being blocked by another point on the surface.} \tag{7.9}$$

$$= 0 \text{ if the ray, } \hat{k}\text{, resulting from the reflection of } \hat{k}_1 \text{ at the point at } \vec{r}_s \text{ is blocked by another point on the surface,}$$

is the scatter shadowing function (2.10).

We now modify equation (7.6) to get

$$\vec{E}_s(\vec{r}) \quad = jk_o\eta_o\frac{exp\{-jk_or\}}{4\pi r}\,\hat{k}\times\hat{k}\times\int_{S_o} d\vec{r}_o\ S_i(\vec{r}_o;\hat{k}_i)S_s(\vec{r}_o;\hat{k}_i;\hat{k}_s)\vec{J}_K(\vec{r}_s)exp\left\{j\vec{k}\cdot\vec{r}_s\right\}$$

$$-k_o^2\left(\frac{\eta_oH_o}{4\pi^2r}\right)exp\{-jk_or\}\,\hat{k}\times\hat{k}\times\int_{S_o} d\vec{r}_o\int_{S_o} d\vec{r}'_o\ S_i(\vec{r}'_o;\hat{k}_i)\ S_p(\vec{r}'_o;\hat{k}_i;\hat{k}_1)S_s(\vec{r}_o;\hat{k}_1;\hat{k})$$

$$\vec{N}(\vec{r}_o)\ \times\left\{[\vec{N}(\vec{r}'_o)\times\hat{h}_o]\times\frac{\hat{k}_1}{|\vec{r}_s-\vec{r}'_s|}\right\}exp\left\{jk_o(\hat{k}_1-\hat{k}_i)\cdot\vec{r}'_s\right\}\ exp\left\{jk_o(\hat{k}-\hat{k}_1)\cdot\vec{r}_s\right\}.$$

$$= \vec{E}_s^{(1)}(\vec{r}) + \vec{E}_s^{(2)}(\vec{r}). \tag{7.10}$$

If we form the incoherent intensity as before, we obtain

$$\langle \vec{E}_s(\vec{r})\, \vec{E}_s^*(\vec{r})\rangle = \langle \vec{E}_s^{(1)}(\vec{r})\cdot \vec{E}_s^{*(1)}(\vec{r})\rangle + \langle \vec{E}_s^{(1)}(\vec{r})\cdot \vec{E}_s^{*(2)}(\vec{r})\rangle + \langle \vec{E}_s^{*(1)}(\vec{r})\cdot \vec{E}_s^{(2)}(\vec{r})\rangle + \langle \vec{E}_s^{(2)}(\vec{r})\cdot \vec{E}_s^{*(2)}(\vec{r})\rangle,$$

where $\langle \vec{E}_s^{(1)}(\vec{r})\cdot \vec{E}_s^{*(1)}(\vec{r})\rangle$ was solved in the previous chapters using Sancer's approach. We now examine $\langle \vec{E}_s^{(2)}(\vec{r})\cdot \vec{E}_s^{*(2)}(\vec{r})\rangle$ (ignoring the cross terms).

$$\vec{E}_S^{(2)}(\vec{r})\cdot \vec{E}_S^{*(2)}(\vec{r}) = k_o^4 \left(\frac{\eta_o H_o}{4\pi^2 r}\right)^2 \int\limits_{S_o} d\vec{r}_o \int\limits_{S_o} d\vec{r}\,'_o \int\limits_{S_o} d\vec{r}\,''_o \int\limits_{S_o} d\vec{r}\,'''_o$$

$$* \; S_i(\vec{r}\,'_o;\hat{k}_i)\; S_p(\vec{r}\,'_o;\hat{k}_i;\hat{k}_1)\; S_s(\vec{r}_o;\hat{k}_1;\hat{k})\; S_i(\vec{r}\,'''_o;\hat{k}_i)\; S_p(\vec{r}\,'''_o;\hat{k}_i;\hat{k}_1)\; S_s(\vec{r}\,''_o;\hat{k}_1;\hat{k})$$

$$* \left\{\hat{k}\times\hat{k}\times\left\{ \vec{N}(\vec{r}_o) \times\left\{[\vec{N}(\vec{r}\,'_o)\times\hat{h}_o]\times\hat{k}_1 \right\}\right\}\right\}$$

$$* \left\{\hat{k}\times\hat{k}\times\left\{ \vec{N}(\vec{r}\,''_o) \times\left\{[\vec{N}(\vec{r}\,'''_o)\times\hat{h}_o]\times\hat{k}'_1 \right\}\right\}\right\}$$

$$* \frac{1}{|\vec{r}_s-\vec{r}\,'_s|}\,\frac{1}{|\vec{r}\,''_s-\vec{r}\,'''_s|}\; exp\left\{jk_o(\hat{k}_1-\hat{k}_i)\cdot\vec{r}\,'_s\right\}\, exp\left\{jk_o(\hat{k}-\hat{k}_1)\cdot\vec{r}_s\right\}$$

$$* \; exp\left\{-jk_o(\hat{k}'_1-\hat{k}_i)\cdot\vec{r}\,'''_s\right\}\, exp\left\{-jk_o(\hat{k}-\hat{k}'_1)\cdot\vec{r}\,''_s\right\}, \qquad (7.11)$$

where $\hat{k}'_1 = \hat{u}(\vec{r}\,''_s-\vec{r}\,'''_s)$. The solution goes to zero except when $\vec{r}_s \approx \vec{r}\,''_s$ and $\vec{r}\,'_s \approx \vec{r}\,'''_s$ due to the fact that the electromagnetic fields decorrelate very rapidly as the points move away from each other as in the single bounce case (we assert this without proof). Therefore, $\hat{k}_1 \approx \hat{k}'_1$ and

$$\vec{E}_s^{(2)}(\vec{r}) \cdot \vec{E}_s^{*(2)}(\vec{r}) = k_o^4 \left(\frac{\eta_o H_o}{4\pi^2 r}\right)^2 \int_{S_o} d\vec{r}_o \int_{S_o} d\vec{r}'_o \int_{S_o} d\vec{r}''_o \int_{S_o} d\vec{r}'''_o$$

$$* \; S_i^2(\vec{r}'_o;\widehat{k}_i)\; S_p^2(\vec{r}'_o;\widehat{k}_i;\widehat{k}_1)\; S_s^2(\vec{r}_o;\widehat{k}_1;\widehat{k})$$

$$* \left\{\widehat{k} \times \widehat{k} \times \left\{ \vec{N}(\vec{r}_o) \times \left\{[\vec{N}(\vec{r}'_o) \times \widehat{h}_o] \times \widehat{k}_1 \right\}\right\}\right\}^2$$

$$* \frac{1}{|\vec{r}_s - \vec{r}'_s|^2} \; exp\left\{jk_o(\widehat{k}_1 - \widehat{k}_i) \cdot (\vec{r}'_s - \vec{r}'''_s)\right\} exp\left\{jk_o(\widehat{k} - \widehat{k}_1) \cdot (\vec{r}_s - \vec{r}''_s)\right\}. \quad (7.12)$$

If we let $\vec{q} = \widehat{k}_1 - \widehat{k}_i = q_1\widehat{x} + q_2\widehat{y} + q_3\widehat{z}$, $\vec{q}' = \widehat{k} - \widehat{k}_1 = q'_1\widehat{x} + q'_2\widehat{y} + q'_3\widehat{z}$, and expand $\zeta(\vec{r}_o)$ about $\vec{r}''_o$ and $\zeta(\vec{r}'_o)$ about $\vec{r}'''_o$, we get

$$\zeta_o(x_o,y_o) = \zeta_o(x''_o,y''_o) + \zeta_{o_x}(x''_o,y''_o)\,(x_o - x''_o) + \zeta_{o_y}(x''_o,y''_o)\,(y_o - y''_o) +$$
$$\frac{1}{2}\,\zeta_{o_{xx}}(x''_o,y''_o)\,(x_o - x''_o)^2 + \frac{1}{2}\,\zeta_{o_{yy}}(x''_o,y''_o)\,(y_o - y''_o)^2 +$$
$$\zeta_{o_{xy}}(x''_o,y''_o)\,(x_o - x''_o)\,(y_o - y''_o) + \text{H.O.T.}^{(1)} \quad (7.13)$$

and

$$\zeta_o(x'_o,y'_o) = \zeta_o(x'''_o,y'''_o) + \zeta_{o_x}(x'''_o,y'''_o)\,(x'_o - x'''_o) + \zeta_{o_y}(x'''_o,y'''_o)\,(y'_o - y'''_o) +$$
$$\frac{1}{2}\,\zeta_{o_{xx}}(x'''_o,y'''_o)\,(x'_o - x'''_o)^2 + \frac{1}{2}\,\zeta_{o_{yy}}(x'''_o,y'''_o)\,(y'_o - y'''_o)^2 +$$
$$\zeta_{o_{xy}}(x'''_o,y'''_o)\,(x'_o - x'''_o)\,(y'_o - y'''_o) + \text{H.O.T.}^{(2)}. \quad (7.14)$$

Letting

$$\alpha'_x = \zeta_{o_x}(x''_o, y''_o),$$
$$\alpha'_y = \zeta_{o_y}(x''_o, y''_o),$$
$$\alpha_x = \zeta_{o_x}(x'''_o, y'''_o)$$
$$\alpha_y = \zeta_{o_y}(x'''_o, y'''_o),$$

for convenience, we arrive at

$$\vec{E}_s^{(2)}(\vec{r}) \cdot \vec{E}_s^{*(2)}(\vec{r}) = k_o^4 \left(\frac{\eta_o H_o}{4\pi^2 r}\right)^2 \int_{S_o} d\vec{r}_o \int_{S_o} d\vec{r}'_o \int_{S_o} d\vec{r}''_o \int_{S_o} d\vec{r}'''_o$$

$$* \; S_i^2(\vec{r}'_o; \widehat{k}_i) \; S_p^2(\vec{r}'_o; \widehat{k}_i; \widehat{k}_1) \; S_s^2(\vec{r}_o; \widehat{k}_1; \widehat{k})$$

$$* \left\{ \widehat{k} \times \widehat{k} \times \left\{ \vec{N}(\vec{r}_o) \times \left\{ [\vec{N}(\vec{r}'_o) \times \widehat{h}_o] \times \widehat{k}_1 \right\}\right\}\right\}^2 \frac{1}{|\vec{r}_s - \vec{r}'_s|^2}$$

$$* \; exp\left\{ jk_o \left\{ q'_1(x_o - x''_o) + q'_2(y_o - y''_o) + q'_3[\alpha'_x(x_o - x''_o) + \alpha'_y(y_o - y''_o) + \text{H.O.T.}^{(1)}]\right\}\right\} *$$

$$exp\left\{ jk_o \left\{ q_1(x'_o - x'''_o) + q_2(y'_o - y'''_o) + q_3[\alpha_x(x'_o - x'''_o) + \alpha_y(y'_o - y'''_o) + \text{H.O.T.}^{(2)}]\right\}\right\}. \qquad (7.15)$$

If we make the change of variables, and let

$$u' = k_o(x_o - x''_o) \qquad du' = -k_o dx''_o$$

$$v' = k_o(y_o - y''_o) \qquad dv' = -k_o dy''_o$$

$$u = k_o(x'_o - x'''_o) \qquad du = -k_o dx'''_o$$

$$v = k_o(y'_o - y'''_o) \qquad dv = -k_o dy'''_o,$$

we are left with

$$\vec{E}_s^{(2)}(\vec{r}) \cdot \vec{E}_s^{*(2)}(\vec{r}) = \left(\frac{\eta_o H_o}{4\pi^2 r}\right)^2 \int\limits_{S_o} d\vec{r}_o \int\limits_{S_o} d\vec{r}'_o$$

$$* \int\limits_{k_o(x_o - S_{o_{xmin}})}^{k_o(x_o - S_{o_{xmax}})} du' \int\limits_{k_o(x_o - S_{o_{ymin}})}^{k_o(x_o - S_{o_{ymax}})} dv' \int\limits_{k_o(x'_o - S_{o_{xmin}})}^{k_o(x'_o - S_{o_{xmax}})} du \int\limits_{k_o(y'_o - S_{o_{ymin}})}^{k_o(y'_o - S_{o_{ymax}})} dv$$

$$* \; S_i^2(\vec{r}'_o; \widehat{k}_i) \; S_p^2(\vec{r}'_o; \widehat{k}_i; \widehat{k}_1) \; S_s^2(\vec{r}_o; \widehat{k}_1; \widehat{k})$$

$$* \left\{ \widehat{k} \times \widehat{k} \times \left\{ \vec{N}(\vec{r}_o) \times \left\{ [\vec{N}(\vec{r}'_o) \times \widehat{h}_o] \times \widehat{k}_1 \right\}\right\}\right\}^2 \frac{1}{|\vec{r}_s - \vec{r}'_s|^2}$$

$$* \; exp\left\{ j\left\{ (q'_1 + \alpha'_x q'_3)u' + (q'_2 + \alpha'_y q'_3)v' + O\left(\frac{1}{k_o}\right)\right\}\right\}$$

$$* \; exp\left\{ j\left\{ (q_1 + \alpha_x q_3)u + (q_2 + \alpha_y q_3)v + O\left(\frac{1}{k_o}\right)\right\}\right\}. \qquad (7.16)$$

where we have again assumed a rectangular integration area such that

$$S_{o_{xmin}} < x_o < S_{o_{xmax}}$$

and

$$S_{o_{ymin}} < y_o < S_{o_{ymax}}.$$

As $k_o \to \infty$, $O\left(\frac{1}{k_o}\right) \to 0$, and

$$\begin{aligned}
k_o(x_o - S_{o_{xmax}}) &\to -\infty \\
k_o(x_o - S_{o_{xmin}}) &\to \infty \\
k_o(y_o - S_{o_{ymax}}) &\to -\infty \\
k_o(x_o - S_{o_{xmin}}) &\to \infty \\
k_o(x'_o - S_{o_{xmax}}) &\to -\infty \\
k_o(x'_o - S_{o_{xmin}}) &\to \infty \\
k_o(y'_o - S_{o_{ymax}}) &\to -\infty \\
k_o(x'_o - S_{o_{xmin}}) &\to \infty.
\end{aligned}$$

We have from (2.33) that

$$\int_{-\infty}^{\infty} du\ exp\{j(q_1 + \alpha_x q_3)u\} = 2\pi\delta(q_1 + \alpha_x q_3),$$

$$\int_{-\infty}^{\infty} dv\ exp\left\{j(q_2 + \alpha_y q_3)v\right\} = 2\pi\delta(q_2 + \alpha_y q_3),$$

$$\int_{-\infty}^{\infty} du'\ exp\left\{j(q'_1 + \alpha'_x q'_3)u'\right\} = 2\pi\delta(q'_1 + \alpha'_x q'_3),$$

and

$$\int_{-\infty}^{\infty} dv'\ exp\left\{j(q'_2 + \alpha'_y q'_3)v'\right\} = 2\pi\delta(q'_2 + \alpha'_y q'_3).$$

Therefore, we have

$$\vec{E}_s^{(2)}(\vec{r})\cdot\vec{E}_s^{*(2)}(\vec{r}) = (2\pi)^4\left(\frac{\eta_o H_o}{4\pi^2 r}\right)^2 \int\limits_{S_o} d\vec{r}_o \int\limits_{S_o} d\vec{r}\,'_o$$

$$* \; S_i^2(\vec{r}\,'_o;\hat{k}_i)\; S_p^2(\vec{r}\,'_o;\hat{k}_i;\hat{k}_1)\; S_s^2(\vec{r}_o;\hat{k}_1;\hat{k})$$

$$* \left\{\hat{k}\times\hat{k}\times\left\{\;\vec{N}(\vec{r}_o)\;\times\left\{[\vec{N}(\vec{r}\,'_o)\times\hat{h}_o]\times\hat{k}_1\right\}\right\}\right\}^2 \frac{1}{|\,\vec{r}_s-\vec{r}\,'_s\,|^2}$$

$$* \; \delta(q'_1+\alpha'_x q'_3)\;\delta(q'_2+\alpha'_y q'_3)\;\delta(q_1+\alpha_x q_3)\;\delta(q_2+\alpha_y q_3) \qquad (7.17)$$

Now taking the ensemble average inside the coordinate integrations, we have

$$\langle\vec{E}_s^{(2)}(\vec{r})\cdot\vec{E}_s^{*(2)}(\vec{r})\rangle = (2\pi)^4\left(\frac{\eta_o H_o}{4\pi^2 r}\right)^2 \int\limits_{S_o} d\vec{r}_o \int\limits_{S_o} d\vec{r}\,'_o \int\limits_{-\infty}^{\infty} d\zeta'_o \int\limits_{-\infty}^{\infty} dS_i \int\limits_{-\infty}^{\infty} dS_p \int\limits_{-\infty}^{\infty} dS_s$$

$$* \int\limits_{-\infty}^{\infty} d\alpha_x \int\limits_{-\infty}^{\infty} d\alpha_y \int\limits_{-\infty}^{\infty} d\alpha_{x'} \int\limits_{-\infty}^{\infty} d\alpha_{y'}\; S_i^2(\vec{r}\,'_o;\hat{k}_i)\; S_p^2(\vec{r}\,'_o;\hat{k}_i;\hat{k}_1)\; S_s^2(\vec{r}_o;\hat{k}_1;\hat{k})$$

$$* \left\{\hat{k}\times\hat{k}\times\left\{\;\vec{N}(\vec{r}_o)\;\times\left\{[\vec{N}(r'_o)\times\hat{h}_o]\times\hat{k}_1\right\}\right\}\right\}^2 \frac{1}{|\,\vec{r}_s-\vec{r}\,'_s\,|^2}$$

$$* \; \delta(q'_1+\alpha'_x q'_3)\;\delta(q'_2+\alpha'_y q'_3)\;\delta(q_1+\alpha_x q_3)\;\delta(q_2+\alpha_y q_3)$$

$$* \; \mathrm{p}(S_i, S_p, S_s, \alpha'_x, \alpha'_y, \alpha_x, \alpha_y, \zeta_o, \zeta'_o). \qquad (7.18)$$

Here we note that the height at $\vec{r}_s$, $\zeta(x_o, y_o)$, may depend on the height and slope at $\vec{r}\,'_s$. Letting

$$x'_1 = q'_1 + \alpha'_x q'_3 \quad dx'_1 = q'_3 \, d\alpha'_x$$

$$x'_2 = q'_2 + \alpha'_y q'_3 \quad dx'_2 = q'_3 \, d\alpha'_y$$

$$x_1 = q_1 + \alpha_x q_3 \quad dx_1 = q_3 \, d\alpha_x$$

$$x_2 = q_2 + \alpha_y q_3 \quad dx_2 = q_3 \, d\alpha_y$$

we arrive at

$$\langle \vec{E}_s^{(2)}(\vec{r}) \cdot \vec{E}_s^{*(2)}(\vec{r}) \rangle = (2\pi)^4 \left(\frac{\eta_o H_o}{4\pi^2 r} \right)^2 \int_{S_o} d\vec{r}_o \int_{S_o} d\vec{r}'_o \int_{-\infty}^{\infty} d\zeta'_o \int_{-\infty}^{\infty} dS_i \int_{-\infty}^{\infty} dS_p \int_{-\infty}^{\infty} dS_s$$

$$* \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \int_{-\infty}^{\infty} dx'_1 \int_{-\infty}^{\infty} dx'_2 \; \frac{1}{q_3^2} \; \frac{1}{q_3'^2}$$

$$S_i^2(\vec{r}'_o; \widehat{k}_i) \; S_p^2(\vec{r}'_o; \widehat{k}_i; \widehat{k}_1) \; S_s^2(\vec{r}_o; \widehat{k}_1; \widehat{k})$$

$$* \left\{ \widehat{k} \times \widehat{k} \times \left\{ \vec{N}(\frac{x_1 - q_1}{q_3}, \frac{x_2 - q_2}{q_3}) \times [\vec{N}(\frac{x'_1 - q'_1}{q'_3}, \frac{x'_2 - q'_2}{q'_3}) \times \widehat{h}_o] \times \widehat{k}_1 \right\}^2 \right\}$$

$$* \frac{1}{|\vec{r}_s - \vec{r}'_s|^2} \; \delta(x'_1) \; \delta(x'_2) \; \delta(x_1) \; \delta(x_2)$$

$$* \; \mathrm{p}(S_i, S_p, S_s, \frac{x'_1 - q'_1}{q'_3}, \frac{x'_2 - q'_2}{q'_3}, \frac{x_1 - q_1}{q_3}, \frac{x_2 - q_2}{q_3}, \zeta_o, \zeta'_o) \tag{7.19}$$

This results in

$$\langle \vec{E}_s^{(2)}(\vec{r}) \cdot \vec{E}_s^{*(2)}(\vec{r}) \rangle = (2\pi)^4 \left(\frac{\eta_o H_o}{4\pi^2 r} \right)^2 \int_{S_o} d\vec{r}_o \int_{S_o} d\vec{r}'_o \int_{-\infty}^{\infty} d\zeta'_o \int_{-\infty}^{\infty} dS_i \int_{-\infty}^{\infty} dS_p \int_{-\infty}^{\infty} dS_s$$

$$* \frac{1}{q_3^2} \frac{1}{q_3'^2} S_i^2(\vec{r}'_o; \widehat{k}_i)\ S_p^2(\vec{r}'_o; \widehat{k}_i; \widehat{k}_1)\ S_s^2(\vec{r}_o; \widehat{k}_1; \widehat{k})$$

$$* \left\{ \widehat{k} \times \widehat{k} \times \left\{ \vec{N}(-\frac{q_1}{q_3}, -\frac{q_2}{q_3}) \times [\vec{N}(-\frac{q'_1}{q'_3}, -\frac{q'_2}{q'_3}) \times \widehat{h}_o] \times \widehat{k}_1 \right\}^2 \right\}$$

$$* \frac{1}{|\vec{r}_s - \vec{r}'_s|^2} \cdot \delta(x'_1)\ \delta(x'_2)\ \delta(x_1)\ \delta(x_2)$$

$$* \mathrm{p}(S_i, S_p, S_s, -\frac{q'_1}{q'_3}, -\frac{q'_2}{q'_3}, -\frac{q_1}{q_3}, -\frac{q_2}{q_3}, \zeta_o, \zeta'_o) \tag{7.20}$$

The integrand will be non-zero only if $S_i = S_p = S_s = 1$, so we write as in [Sancer, 1969]

$$\mathrm{p}(S_i, S_p, S_s, -\frac{q'_1}{q'_3}, -\frac{q'_2}{q'_3}, -\frac{q_1}{q_3}, -\frac{q_2}{q_3}, \zeta_o, \zeta'_o) = \delta(S_i - 1)\delta(S_p - 1)\delta(S_s - 1) \cdot$$

$$\mathrm{P}(\widehat{k}_i, \widehat{k}_1, \widehat{k}, -\frac{q'_1}{q'_3}, -\frac{q'_2}{q'_3}, -\frac{q_1}{q_3}, -\frac{q_2}{q_3}, \zeta_o, \zeta'_o). \tag{7.21}$$

Inserting this into the above integrals we get

$$\langle \vec{E}_s^{(2)}(\vec{r}) \cdot \vec{E}_s^{*(2)}(\vec{r}) \rangle = (2\pi)^4 \left(\frac{\eta_o H_o}{4\pi^2 r} \right)^2 \int_{S_o} d\vec{r}_o \int_{S_o} d\vec{r}'_o \int_{-\infty}^{\infty} d\zeta'_o \frac{1}{q_3^2} \frac{1}{q_3'^2}$$

$$* \left\{ \widehat{k} \times \widehat{k} \times \left\{ \vec{N}(-\frac{q_1}{q_3}, -\frac{q_2}{q_3}) \times [\vec{N}(-\frac{q'_1}{q'_3}, -\frac{q'_2}{q'_3}) \times \widehat{h}_o] \times \widehat{k}_1 \right\}^2 \right\}$$

$$* \frac{1}{|\vec{r}_s - \vec{r}'_s|^2} \mathrm{P}(\widehat{k}_i, \widehat{k}_1, \widehat{k}, -\frac{q'_1}{q'_3}, -\frac{q'_2}{q'_3}, -\frac{q_1}{q_3}, -\frac{q_2}{q_3}, \zeta_o, \zeta'_o). \tag{7.22}$$

$P(\widehat{k}_i, \widehat{k}_1, \widehat{k}, -\frac{q_1'}{q_3'}, -\frac{q_2'}{q_3'}, -\frac{q_1}{q_3}, -\frac{q_2}{q_3}, \zeta_o, \zeta_o')$ can be written in a more convenient form (following [Lynch and Wagner, 1970a)

$$P(\widehat{k}_i, \widehat{k}_1, \widehat{k}, -\frac{q_1'}{q_3'}, -\frac{q_2'}{q_3'}, -\frac{q_1}{q_3}, -\frac{q_2}{q_3}, \zeta_o, \zeta_o') = P(\widehat{k} \mid \widehat{k}_1, \widehat{k}_i, -\frac{q_1'}{q_3'}, -\frac{q_2'}{q_3'}, -\frac{q_1}{q_3}, -\frac{q_2}{q_3}, \zeta_o, \zeta_o')$$

$$* P(\frac{q_1}{q_3}, -\frac{q_2}{q_3}, \zeta_o' \mid \widehat{k}_i, \widehat{k}_1, -\frac{q_1'}{q_3'}, -\frac{q_2'}{q_3'}, \zeta_o)$$

$$* P(\widehat{k}_1 \mid \widehat{k}_i, -\frac{q_1'}{q_3'}, -\frac{q_2'}{q_3'}, \zeta_o)$$

$$* P(\widehat{k}_i \mid -\frac{q_1'}{q_3'}, -\frac{q_2'}{q_3'}, \zeta_o)\, P(\zeta_o, -\frac{q_1'}{q_3'}, -\frac{q_2'}{q_3'}) , \tag{7.23}$$

where $P(\zeta_o', -\frac{q_1'}{q_3'}, -\frac{q_2'}{q_3'})$ is the joint PDF of the height and slope at $\vec{r}_s'$ and $P(\widehat{k}_i \mid -\frac{q_1'}{q_3'}, -\frac{q_2'}{q_3'}, \zeta_o')$ is the probability that given the height and slope at the point $\vec{r}_s'$ that it will be illuminated by $\widehat{k}_i$, both of which are known from our previous work. $P(\widehat{k}_1 \mid \widehat{k}_i, -\frac{q_1'}{q_3'}, -\frac{q_2'}{q_3'}, \zeta_o')$ is the probability that $\widehat{k}_1$ will strike the surface at $\vec{r}_s$ given that $\widehat{k}_i$ illuminates the point at $\vec{r}_s'$ and the height and slope at $\vec{r}_s'$. $P(\widehat{k} \mid \widehat{k}_1, \widehat{k}_i, -\frac{q_1'}{q_3'}, -\frac{q_2'}{q_3'}, -\frac{q_1}{q_3}, -\frac{q_2}{q_3}, \zeta_o, \zeta_o')$ is the probability that $\widehat{k}$ will exit the surface given that (1) $\widehat{k}_i$ has illuminated the point at $\vec{r}_s'$ and has reflected off that point to $\vec{r}_s$ without being blocked by another point on the surface, and (2) the heights and slopes at $\vec{r}_s$ and $\vec{r}_s'$.

This concludes the development of the scattered intensity for doubly scattered rays. We in no way represent this method as being the best approach to solving this problem, merely a logical extension of Sancer's work provided in Chapter 2. Certainly we have left open many questions concerning this result, particularly the importance of the interference of the single bounce and double bouncing rays, the neglect of the cross terms, and the validity of setting $\widehat{k}_1 \approx \widehat{k}_1'$. These topics would certainly need to be examined further, along with the very difficult problem of computing the shadowing function involved in this case. It was only our intention here to show how this method could be extended further to include multiple scattering.

8.0 CONCLUSIONS

8.1 Overview

This thesis investigated the effects of shadowing on randomly rough surfaces. The shadow corrected Kirchhoff approximation was used to estimate the surface current which was then used to calculate the scattered electric field and the scattered intensity. The high frequency limit was used in all calculations. These assumptions simplified matters considerably and provide us with a check against which future, more exact results, may be compared.

The shadowing function was computed numerically using a infinite series of integrals based on the work of Ricciardi and Sato [Ricciardi and Sato, 1986]. Several analytic expressions which approximate the shadowing function were compared against our numerical results, specifically the work of Wagner [Wagner, 1967] and Smith [Smith, 1967b]. Monte Carlo simulations were also performed and compared, where possible, against the numerical and analytic results.

8.2 List of Original Contributions

The major contributions of this thesis include:

(1) Proper accounting for scatter shadowing.

(2) Rigorous extension of Sancer's analysis [Sancer, 1969] to bistatic shadowing.

(3) Extension of the range, in incident and observation angle, in which Wagner's [Wagner, 1967] and Smith's shadowing functions [Smith, 1967b] can be verified.

(4) Defined Wagner's approximations [Wagner, 1967] in developing the "no correlation" shadowing function in terms of the physics of the problem. Solved the first passage in time problem for a Gaussian process, with a Gaussian correlation function, crossing a ray (ramp), when the shadowed and shadowing points (intersection points of the ray with the surface) are uncorrelated.

(5) Identified an error in Wagner's "correlated" shadowing function [Wagner, 1967].

(6) Computed the first three terms in the series developed by Ricciardi and Sato [Ricciardi and Sato, 1986], which approximates the exact solution to the first passage in time problem of a Gaussian process, with a Gaussian correlation function, crossing a ray (ramp).

8.3 Summary of Results

(1) This work has shown that scatter shadowing can be properly accounted for by nulling those current elements that give rise to multiply scattered rays. Previously, scatter shadowing was accounted for by nulling only those current elements that gave rise to scattering in the direction of the observer. This produced the unphysical situation of having surface currents which depend on the direction of the observer.

(2) Using the above scatter shadowing function, Sancer's work [Sancer, 1969] was formally extended to the case of bistatic shadowing, i.e. when the source and receiver are not located at the same point.

(3) The range of validity, in incident and observation angle, in which the analytical derivations of Wagner [Wagner, 1967] and Smith [Smith, 1967b] can be checked was extended. Previously, these works were compared to Monte Carlo computer simulations, which suffer from the limitation that it is difficult to compute shadowing close to grazing incidence, since the number of reflection points, in which the shadowing calculation is based, are so few.

(4) We have defined the approximations made by Wagner [Wagner, 1967] in deriving the "no correlation" shadowing function based on the physics of the problem. Wagner is able to develop a closed form solution to the shadowing problem by making approximations to a function related to the first passage in time function. However, the assumptions were made for mathematical

convenience, and the meaning of these assumptions were not established based on the physics of the problem. Using the infinite series of integrals developed by Ricciardi and Sato [Ricciardi and Sato, 1986], we have shown that these approximations are equivalent to neglecting the correlation between the shadowed point and the shadowing points. Thus, the first passage in time problem is solved for a Gaussian process, with a Gaussian correlation function, crossing a ray (ramp), in which the shadowed point and the shadowing points (intersection points) are uncorrelated (Figure 11).

(5) In the course of comparing our results to that of Wagner [Wagner, 1967] we have discovered an error in his result concerning the "no correlation" shadowing function. This shadowing function has been excepted in the literature as the authoritative work on shadowing theory, and was purported to be more rigorous than previous works which neglected correlation between the shadowed point and the shadowing points. However, we have found that, in fact, due to the error, the result is less accurate that previous derivations.

(6) We have computed the first three term in the series developed by Ricciardi and Sato [Ricciardi and Sato, 1986] using numerical integration with no other assumptions. Thus, we have solved the first passage in time problem for a Gaussian process, with a Gaussian correlation function, crossing a ray (ramp), in the cases in which the exact solution can be approximated by the first three terms in the series.

8.4 Conclusions of Numerical Results

Our work has shown that for gently sloping surfaces the no correlation assumption between the shadowing points and the shadowed point is adequate to describe the effects of shadowing on the surface at grazing angles (below 85 degrees incidence). At larger angles we see some disagreement between our results and the others, with Wagner's no correlation shadowing function being the closest to our work. The reason for this disagreement has been traced to the fact that the no correlation assumption, invalid near the origin ($t \rightarrow 0^+$), has been incorporated into both Smith's work, which explicitly neglects it, and Wagner's more rigorous work for which this correlation was partially accounted for, but improperly incorporated into the derivation. We also see that the Monte Carlo simulations agreed well with our results where the number of reflection points was large enough to make the outcome accurate. For very rough surfaces ($\sigma_2^2 = 2.0$) Smith's work appears more accurate than Wagner's, however, we were not able to present results below 76° incidence since our series had not properly converged at that point. We further note that our work has significantly extended the range (except when $\sigma_2^2 = 2.0$), in which a check of Wagner's and Smith's work was possible with Monte Carlo simulations.

Furthermore, we note that although we have freely altered the correlation length of the surface to check the regions of validity of the shadowing functions, the Kirchhoff approximation becomes invalid as multiple scattering effects become important. Thus, as the correlation length becomes smaller we must explicitly account for these effects, as outlined in Chapter 7. This thesis is a first step in in attempting to study the effects of correlation of the shadowing points and the shadowed point as might occur on the surface when multiple scattering is considered.

Lastly, we point out that the results presented in this thesis are valid only for surfaces with a Gaussian height spectrum. The sensitivity of the results to correlation function may become extremely important, particularly at low grazing angles. Thus, the effects of correlation between the shadowing points and the shadowed point may change in the event that the surface spectrum differs from that of Gaussian.

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APPENDIX A

In this section we derive the conditional means and conditional variances for the probability density functions in equations (6.6) and (6.7).

If we define a vector $\underline{x} = \begin{bmatrix} \underline{x}_1 \\ \underline{x}_2 \end{bmatrix}$, where $\underline{x}_1$ and $\underline{x}_2$ are vectors of Gaussian random variables, with mean

$$\underline{\mu} = \begin{bmatrix} \underline{\mu}_1 \\ \underline{\mu}_2 \end{bmatrix}$$

and covariance matrix

$$\underline{c} = \begin{bmatrix} \underline{c}_{11} & \underline{c}_{12} \\ \underline{c}_{21} & \underline{c}_{22} \end{bmatrix}$$

($\underline{c}_{21}$ = $\underline{c}_{12}^T$ since Gaussian covariance matrices are symmetric), a conditional probability density function can be written in which the resulting distribution will also be Gaussian. The probability density function (PDF) of $\underline{x}_1$ conditioned on $\underline{x}_2$ is

$$p(\underline{x}_1 \,|\, \underline{x}_2) = N(\underline{\mu}_1 + \underline{c}_{12}\underline{c}_{22}^{-1}(\underline{x}_2 - \underline{\mu}_2)\ ,\ \underline{c}_{11} - \underline{c}_{12}\underline{c}_{22}^{-1}\underline{c}_{12}^T), \tag{A.1}$$

where the resulting conditional mean is

$$\underline{m}_1 = \underline{\mu}_1 + c_{12}\underline{c}_{22}^{-1}(\underline{x}_2 - \underline{\mu}_2) \tag{A.2}$$

and the conditional covariance is

$$\underline{k}_1 = \underline{c}_{11} - \underline{c}_{12}\underline{c}_{22}^{-1}\underline{c}_{12}^T. \tag{A.3}$$

If we let

$$\underline{x} = \begin{bmatrix} \underline{x}_1 \\ \underline{x}_2 \end{bmatrix} = \begin{bmatrix} \zeta'(t) \\ S(t) \\ \zeta_{osp} \\ \zeta'_{osp} \end{bmatrix} \tag{A.4}$$

for $p(\zeta'(t)\ |S(t),\ \zeta_{osp},\ \zeta'_{osp})$ in (6.6), where

$$\underline{x}_1 = \zeta'(t) \tag{A.5}$$

and

$$\underline{x}_2 = \begin{bmatrix} S(t) \\ \zeta_{osp} \\ \zeta'_{osp} \end{bmatrix}, \tag{A.6}$$

we have

$$\underline{c} = \begin{bmatrix} E[\zeta'^2(t)] & E[\zeta'(t)S(t)] & E[\zeta'(t)\zeta_{osp}] & E[\zeta'(t)\zeta'_{osp}] \\ E[\zeta'(t)S(t)] & E[S^2(t)] & E[S(t)\zeta_{osp}] & E[S(t)\zeta'_{osp}] \\ E[\zeta'(t)\zeta_{osp}] & E[S(t)\zeta_{osp}] & E[\zeta^2_{osp}] & E[\zeta_{osp}\zeta'_{osp}] \\ E[\zeta'(t)\zeta'_{osp}] & E[S(t)\zeta'_{osp}] & E[\zeta_{osp}\zeta'_{osp}] & E[\zeta'^2_{osp}] \end{bmatrix}, \tag{A.7}$$

where

$$c_{11} = E[\zeta'^2(t)] \tag{A.8a}$$

$$c_{12} = \begin{bmatrix} E[\zeta'(t)S(t)] & E[\zeta'(t)\zeta_{osp}] & E[\zeta'(t)\zeta'_{osp}] \end{bmatrix} \tag{A.8b}$$

$$\underline{c}_{21} = \underline{c}^T_{12} \tag{A.8c}$$

and

$$
\underline{c}_{22} = \begin{bmatrix} E[S^2(t)] & E[S(t)\zeta_{osp}] & E[S(t)\zeta'_{osp}] \\ E[S(t)\zeta_{osp}] & E[\zeta^2_{osp}] & E[\zeta_{osp}\zeta'_{osp}] \\ E[S(t)\zeta'_{osp}] & E[\zeta_{osp}\zeta'_{osp}] & E[\zeta'^2_{osp}] \end{bmatrix}. \tag{A.8d}
$$

$E[S^2(t)] = E[\zeta^2_{osp}] = \sigma_1^2$ and $E[\zeta^2_{osp}] = \sigma_2^2$ are the variance of the heights and slopes, respectively, and $E[\zeta_{osp}\zeta'_{osp}] = E[S(t)\zeta'(t)] = 0$. Also, $\underline{\mu}_1 = \underline{\mu}_2 = 0$, since the heights and slopes are a zero mean process. If we define

$$
\rho_{23} \doteq E[S(t)\zeta_{osp}] = E[\zeta_{osp}S(t)] = R(t) = \sigma_1^2 \; exp\left\{-\frac{t^2}{l^2}\right\} \tag{A.9a}
$$

$$
\rho_{24} \doteq E[S(t)\zeta'_{osp}] = E[\zeta'_{osp}S(t)] = -E[\zeta'(t)\zeta_{osp}] = -E[\zeta_{osp}\zeta'(t)] \tag{A.9b}
$$

$$
= -\frac{dR(t)}{dt} = t\sigma_2^2 \; exp\left\{-\frac{t^2}{l^2}\right\}
$$

and

$$
\rho_{14} \doteq E[\zeta'(t)\zeta'_{osp}] = -\frac{d^2R(t)}{dt^2} = (\sigma_2^2 - \frac{\sigma_2^4}{\sigma_1^2}\,t^2) \; exp\left\{-\frac{t^2}{l^2}\right\}, \tag{A.9c}
$$

where $R(t)$ is the autocorrelation function, l is the correlation length, and t is the horizontal distance between two points on the surface, we arrive at (after much algebra),

$$m_1 = -\frac{1}{\sigma_1^2\rho_{24}^2 - \sigma_1^4\sigma_2^2 + \sigma_2^2\rho_{23}^2}\Big\{S(t)(\sigma_1^2\rho_{14}\rho_{24} - \sigma_2^2\rho_{23}\rho_{24}) + \zeta_{osp}[\rho_{24}(\sigma_1^2\sigma_2^2 - \rho_{24}^2) -$$

$$\rho_{14}\rho_{23}\rho_{24}] + \zeta'_{osp}[\rho_{23}\rho_{24}^2 - \rho_{14}(\sigma_1^4 - \rho_{23}^2)]\Big\} \tag{A.10}$$

and

$$k_1 = \sigma_2^2 + \frac{\rho_{24}^2(\sigma_1^2\sigma_2^2) - \rho_{14}\rho_{23}\rho_{24}^2 - \rho_{14}\rho_{23}\rho_{24}^2 + \rho_{14}^2(\sigma_1^4 - \rho_{23}^2)}{\sigma_1^2\rho_{24}^2 + (\rho_{23}^2 - \sigma_1^4)\sigma_2^2}, \tag{A.11}$$

which correspond to equations (6.8) and (6.10).

Performing a similar analysis for $p(S(t)|\ \zeta_{osp}, \zeta'_{osp})$ in (6.7) we have, using Baye's theorem,

$$p(S(t)\ |\zeta_{osp}, \zeta'_{osp}) = \frac{p(S(t), \zeta_{osp}, \zeta'_{osp})}{p(\zeta_{osp}, \zeta'_{osp})}. \tag{A.12}$$

If we let

$$\hat{\underline{x}} = \begin{bmatrix} \hat{\underline{x}}_1 \\ \hat{\underline{x}}_2 \end{bmatrix} = \begin{bmatrix} S(t) \\ \zeta_{osp} \\ \zeta'_{osp} \end{bmatrix}, \tag{A.13}$$

where

$$\hat{\underline{x}}_1 = S(t)$$

and

$$\underline{\widehat{x}}_2 = \begin{bmatrix} \zeta_{osp} \\ \zeta'_{osp} \end{bmatrix},$$

and with mean

$$\underline{\widehat{\mu}} = \begin{bmatrix} \underline{\widehat{\mu}}_1 \\ \underline{\widehat{\mu}}_2 \end{bmatrix} = \begin{bmatrix} E[\underline{\widehat{x}}_1] \\ E[\underline{\widehat{x}}_2] \end{bmatrix} = 0$$

and covariance matrix,

$$\underline{\widehat{c}} = \begin{bmatrix} \widehat{c}_{11} & \widehat{c}_{12} \\ \widehat{c}_{21} & \widehat{c}_{22} \end{bmatrix}$$

of $p(S(t), \zeta_{osp}, \zeta'_{osp})$ equal to

$$\underline{\widehat{c}} = \begin{bmatrix} E[S^2(t)] & E[S(t)\zeta_{osp}] & E[S(t)\zeta'_{osp}] \\ E[\zeta_{osp}S(t)] & E[\zeta^2_{osp}] & E[\zeta_{osp}\zeta'_{osp}] \\ E[\zeta'_{osp}S(t)] & E[\zeta'_{osp}\zeta_{osp}] & E[\zeta'^2_{osp}] \end{bmatrix} = \begin{bmatrix} \sigma_1^2 & \rho_{23} & \rho_{24} \\ \rho_{23} & \sigma_1^2 & 0 \\ \rho_{24} & 0 & \sigma_2^2 \end{bmatrix}, \quad \text{(A.14)}$$

where we have

$$\widehat{c}_{11} = \sigma_1^2 \quad \text{(A.15a)}$$

$$\widehat{c}_{12} = \left[\rho_{23}\ \rho_{24}\right] \quad \text{(A.15b)}$$

$$\hat{c}_{21} = \hat{c}_{12}^{T} \tag{A.15c}$$

$$\hat{c}_{22} = \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix}. \tag{A.15d}$$

Thus, $p(S(t)\ |\zeta_{osp}, \zeta'_{osp}) = N(\hat{\underline{\mu}}_1 + \hat{c}_{12}\hat{c}_{22}^{-1}(\hat{\underline{x}}_2 - \hat{\underline{\mu}}_2),\ \hat{c}_{11} - \hat{c}_{12}\hat{c}_{22}^{-1}\hat{c}_{12}^{T})$, where

$$m_2 = \hat{\underline{\mu}}_1 + \hat{c}_{12}\hat{c}_{22}^{-1}(\hat{\underline{x}}_2 - \hat{\underline{\mu}}_2) \tag{A.16}$$

is the conditional mean and

$$k_2 = \hat{c}_{11} - \hat{c}_{12}\hat{c}_{22}^{-1}\hat{c}_{12}^{T}, \tag{A.17}$$

is the conditional covariance matrix. We find from this that

$$\hat{m}_1 = \frac{\zeta_{osp}\rho_{23}}{\sigma_1^2} + \frac{\zeta'_{osp}\rho_{24}}{\sigma_2^2} \tag{A.18}$$

and

$$\hat{k}_1 = \sigma_1^2 - \frac{\rho_{23}^2}{\sigma_1^2} - \frac{\rho_{24}^2}{\sigma_2^2}. \tag{A.19}$$

These correspond to equations (6.9) and (6.11).

APPENDIX B

In this appendix we derive equation 6.23 in chapter 6.

As mentioned in Appendix A, a conditional PDF made up of Gaussian random variables is also Gaussian, so we know

$$p(S(t_1), S(t), \zeta_{osp} \,|\zeta'(t_1), \zeta'(t), \zeta'_{osp}) = N(\underline{\hat{\hat{m}}}, \underline{\hat{\hat{c}}}), \tag{B.1}$$

where $\underline{\hat{\hat{m}}}$ is the conditional mean and $\underline{\hat{\hat{c}}}$ is the conditional covariance matrix for this PDF. If we define

$$\underline{\hat{\hat{x}}} = \begin{bmatrix} \underline{\hat{\hat{x}}}_1 \\ \underline{\hat{\hat{x}}}_2 \end{bmatrix}, \tag{B.2}$$

where

$$\hat{\hat{\underline{x}}}_1 = \begin{bmatrix} S(t_1) \\ S(t) \\ \zeta_{osp} \end{bmatrix} \quad \text{(B.3)}$$

and

$$\hat{\hat{\underline{x}}}_2 = \begin{bmatrix} \zeta'(t_1) \\ \zeta'(t) \\ \zeta'_{osp} \end{bmatrix} \quad \text{(B.4)}$$

with

$$\hat{\hat{\underline{\mu}}} = \begin{bmatrix} \hat{\hat{\underline{\mu}}}_1 \\ \hat{\hat{\underline{\mu}}}_2 \end{bmatrix} = E[\hat{\hat{\underline{x}}}] = \underline{0}$$

and

$$\hat{\hat{\underline{c}}} = \begin{bmatrix} \hat{\hat{\underline{c}}}_{11} & \hat{\hat{\underline{c}}}_{12} \\ \hat{\hat{\underline{c}}}_{21} & \hat{\hat{\underline{c}}}_{22} \end{bmatrix},$$

we get

$$p(\hat{\hat{\underline{x}}}_1 \mid \hat{\hat{\underline{x}}}_2) = N(\hat{\hat{\underline{\mu}}}_1 + \hat{\hat{\underline{c}}}_{12}\hat{\hat{\underline{c}}}_{22}^{-1}(\hat{\hat{\underline{x}}}_2 - \hat{\hat{\underline{\mu}}}_2) \ , \ \hat{\hat{\underline{c}}}_{11} - \hat{\hat{\underline{c}}}_{12}\hat{\hat{\underline{c}}}_{22}^{-1}\hat{\hat{\underline{c}}}_{12}^{T}). \quad \text{(B.5)}$$

Since the heights are a zero mean process $\hat{\hat{\underline{\mu}}}_1 = \hat{\hat{\underline{\mu}}}_2 = 0$, the conditional mean,

$$\hat{\underline{m}} = \hat{\underline{c}}_{12}\hat{\underline{c}}_{22}^{-1}\hat{\underline{x}}_2$$

$$= \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} \zeta'(t_1) \\ \zeta'(t) \\ \zeta'_{osp} \end{bmatrix}$$

$$= \begin{bmatrix} \hat{m}_1 \\ \hat{m}_2 \\ \hat{m}_3 \end{bmatrix}. \tag{B.6}$$

Here, we have used A_{ij}, which will be used in computing subsequent integrations over $\zeta'(t)$. The conditional covariance matrix for $\hat{\underline{x}}_1$ is $\hat{\underline{c}} = \hat{\underline{c}}_{11} - \hat{\underline{c}}_{12}\hat{\underline{c}}_{22}^{-1}\hat{\underline{c}}_{12}^{T}$, where

$$\hat{\underline{c}}_{11} = \begin{bmatrix} E[S^2(t_1)] & E[S(t_1)S(t)] & E[S(t_1)\zeta_{osp}] \\ E[S(t_1)S(t)] & E[S^2(t)] & E[\zeta_{osp}S(t)] \\ E[S(t_1)\zeta_{osp}] & E[\zeta_{osp}S(t)] & E[\zeta^2_{osp}] \end{bmatrix}, \tag{B.7a}$$

$$\hat{\underline{c}}_{12} = \begin{bmatrix} E[S(t_1)\zeta'(t_1)] & E[S(t_1)\zeta'(t)] & E[S(t_1)\zeta'_{osp}] \\ E[\zeta(t)\zeta'(t_1)] & E[\zeta(t)\zeta'(t)] & E[\zeta(t)\zeta'_{osp}] \\ E[\zeta_{osp}\zeta'(t_1)] & E[\zeta_{osp}\zeta'(t)] & E[\zeta_{osp}\zeta'_{osp}] \end{bmatrix}, \tag{B.7b}$$

$$\hat{\hat{\underline{c}}}_{21} = \hat{\hat{\underline{c}}}_{12}^{T}, \tag{B.7c}$$

and

$$\hat{\hat{\underline{c}}}_{22} = \begin{bmatrix} E[\zeta'^2(t_1)] & E[\zeta'(t_1)\zeta'(t)] & E[\zeta'(t_1)\zeta'_{osp}] \\ E[\zeta'(t)\zeta'(t_1)] & E[\zeta'^2(t)] & E[\zeta'(t)\zeta'_{osp}] \\ E[\zeta'_{osp}\zeta'(t_1)] & E[\zeta'_{osp}\zeta'(t)] & E[\zeta'^2_{osp}] \end{bmatrix}, \tag{B.7d}$$

where we again have $E[\zeta'^2(t_1)] = E[\zeta'^2(t)] = E[\zeta'^2_{osp}] = \sigma_2^2$ and $E[\zeta_{osp}\zeta'_{osp}] = E[\zeta'_{osp}\zeta_{osp}] = E[S(t_1)\zeta'(t_1)] = E[\zeta'(t_1)S(t_1)] = 0$. If we now define

$$\hat{\hat{\underline{c}}}^{-1} = \begin{bmatrix} \tilde{c}_{11} & \tilde{c}_{12} & \tilde{c}_{13} \\ \tilde{c}_{21} & \tilde{c}_{22} & \tilde{c}_{23} \\ \tilde{c}_{31} & \tilde{c}_{32} & \tilde{c}_{33} \end{bmatrix} \tag{B.8}$$

we end up with

$$p(S(t_1), S(t_2), \zeta_{osp} \,|\zeta'(t_1), \zeta'(t_2), \zeta'_{osp}) = \frac{1}{(2\pi)^{\frac{3}{2}} \, [det(\hat{\hat{\underline{c}}})]^{\frac{1}{2}}} *$$

$$exp\left\{ -\frac{1}{2}\left[(\zeta_{osp} + \eta_0 t_1 - \hat{\hat{m}}_1)\,(\zeta_{osp} + \eta_0 t - \hat{\hat{m}}_2) \quad (\zeta_{osp} - \hat{\hat{m}}_3) \right] \begin{bmatrix} \tilde{c}_{11} & \tilde{c}_{12} & \tilde{c}_{13} \\ \tilde{c}_{21} & \tilde{c}_{22} & \tilde{c}_{13} \\ \tilde{c}_{31} & \tilde{c}_{32} & \tilde{c}_{33} \end{bmatrix} \cdot \begin{bmatrix} (\zeta_{osp} + \eta_0 t_1 - \hat{\hat{m}}_1) \\ (\zeta_{osp} + \eta_0 t - \hat{\hat{m}}_2) \\ (\zeta_{osp} - \hat{\hat{m}}_3) \end{bmatrix} \right\}. \tag{B.9}$$

This corresponds to equation (6.23).

APPENDIX C

In this appendix we derive equation 6.54 in chapter 6.

If we define

$$\underset{\sim}{\tilde{x}} = \begin{bmatrix} \underset{\sim}{\tilde{x}}_1 \\ \underset{\sim}{\tilde{x}}_2 \end{bmatrix} = \begin{bmatrix} S(t_1) \\ S(t_2) \\ S(t) \\ \zeta_{osp} \\ \zeta'(t_1) \\ \zeta'(t_2) \\ \zeta'(t) \\ \zeta'_{osp} \end{bmatrix}, \tag{C.1}$$

with

$$\underset{\sim}{\tilde{\mu}} = \begin{bmatrix} \underset{\sim}{\tilde{\mu}}_1 \\ \underset{\sim}{\tilde{\mu}}_2 \end{bmatrix} = E[\underset{\sim}{\tilde{x}}] \tag{C.2}$$

and

$$\underset{\sim}{\hat{c}} = \begin{bmatrix} \underset{\sim}{\hat{c}}_{11} & \underset{\sim}{\hat{c}}_{12} \\ \underset{\sim}{\hat{c}}_{21} & \underset{\sim}{\hat{c}}_{22} \end{bmatrix}, \tag{C.3}$$

where

$$\underset{\sim}{\tilde{x}}_1 = \begin{bmatrix} \mathrm{S(t_1)} \\ \mathrm{S(t_2)} \\ \mathrm{S(t)} \\ \zeta_{\mathrm{osp}} \end{bmatrix} \quad \text{and} \quad \underset{\sim}{\tilde{x}}_2 = \begin{bmatrix} \zeta'(\mathrm{t_1}) \\ \zeta'(\mathrm{t_2}) \\ \zeta'(\mathrm{t}) \\ \zeta'_{\mathrm{osp}} \end{bmatrix}, \tag{C.4}$$

then

$$\underset{\sim}{\tilde{m}} = \begin{bmatrix} \tilde{m}_1 \\ \tilde{m}_2 \\ \tilde{m}_3 \\ \tilde{m}_4 \end{bmatrix} = \underset{\sim}{\tilde{\mu}}_1 + \underset{\sim}{\hat{c}}_{12} \underset{\sim}{\hat{c}}_{22}^{-1} (\underset{\sim}{\tilde{x}}_2 - \underset{\sim}{\mu}_2), \tag{C.5}$$

where

$$\underset{\sim}{\hat{c}}_{11} = \begin{bmatrix} \sigma_1^2 & R_2 & R_4 & R_1 \\ R_2 & \sigma_1^2 & R_5 & R_3 \\ R_4 & R_5 & \sigma_1^2 & R_6 \\ R_1 & R_3 & R_6 & \sigma_1^2 \end{bmatrix}, \tag{C.6a}$$

$$\underset{\sim}{\hat{c}}_{12} = \begin{bmatrix} 0 & R_2' & R_4' & -R_1' \\ -R_2^2 & 0 & R_5' & -R_3' \\ -R_4' & -R_5' & 0 & -R_6' \\ R_1' & R_3' & R_6' & 0 \end{bmatrix}, \tag{C.6b}$$

$$\widehat{\underset{\sim}{c}}_{21} = \widehat{\underset{\sim}{c}}_{12}^{T} \tag{C.6c}$$

and

$$\widehat{\underset{\sim}{c}}_{22} = \begin{bmatrix} \sigma_2^2 & -R_2'' & -R_4'' & -R_1'' \\ -R_2'' & \sigma_2^2 & -R_5'' & -R_3'' \\ -R_4'' & -R_5'' & \sigma_2^2 & -R_6'' \\ -R_1'' & -R_3'' & -R_6'' & \sigma_2^2 \end{bmatrix}, \tag{C.6d}$$

with

$$R_1 = \mathrm{E}[\zeta_{\mathrm{osp}}\ \mathrm{S}(\mathrm{t}_1)] = \sigma_1^2\ exp\left\{\frac{\mathrm{t}_1^2}{l^2}\right\}, \tag{C.7a}$$

$$R_2 = \mathrm{E}[\mathrm{S}(\mathrm{t}_1)\ \mathrm{S}(\mathrm{t}_2)] = \sigma_1^2\ exp\left\{\frac{(\mathrm{t}_2-\mathrm{t}_1)^2}{l^2}\right\}, \tag{C.7b}$$

$$R_3 = \mathrm{E}[\zeta_{\mathrm{osp}}\ \mathrm{S}(\mathrm{t}_2)] = \sigma_1^2\ exp\left\{\frac{\mathrm{t}_2^2}{l^2}\right\}, \tag{C.7c}$$

$$R_4 = \mathrm{E}[\mathrm{S}(\mathrm{t}_1)\ \mathrm{S}(\mathrm{t})] = \sigma_1^2\ exp\left\{\frac{(\mathrm{t}-\mathrm{t}_1)^2}{l^2}\right\}, \tag{C.7d}$$

$$R_5 = \mathrm{E}[\mathrm{S}(\mathrm{t}_2)\ \mathrm{S}(\mathrm{t})] = \sigma_1^2\ exp\left\{\frac{(\mathrm{t}-\mathrm{t}_2)^2}{l^2}\right\}, \tag{C.7e}$$

$$R_6 = \mathrm{E}[\zeta_{\mathrm{osp}}\ \mathrm{S}(\mathrm{t})] = \sigma_1^2\ exp\left\{\frac{\mathrm{t}^2}{l^2}\right\}, \tag{C.7f}$$

$$R_1' = \frac{dR_1(\mathrm{t}_1)}{d\mathrm{t}_1} = -\mathrm{t}_1\sigma_2^2\ exp\left\{-\frac{\mathrm{t}_1^2}{l^2}\right\}, \tag{C.7g}$$

$$R_2' = \frac{dR_2(\mathrm{t}_2-\mathrm{t}_1)}{d(t_2-t_1)} = -(\mathrm{t}_2-\mathrm{t}_1)\sigma_2^2\ exp\left\{-\frac{(\mathrm{t}_2-\mathrm{t}_1)^2}{l^2}\right\}, \tag{C.7h}$$

$$R_3' = \frac{dR_3(\mathrm{t}_2)}{d\mathrm{t}_2} = -\,\mathrm{t}_2\sigma_2^2\ exp\left\{-\frac{\mathrm{t}_2^2}{l^2}\right\}, \tag{C.7i}$$

$$R_4' = \frac{dR_4(\mathrm{t}-\mathrm{t}_1)}{d(\mathrm{t}-t_1)} = -(\mathrm{t}-\mathrm{t}_1)\sigma_2^2\ exp\left\{-\frac{(\mathrm{t}-\mathrm{t}_1)^2}{l^2}\right\}, \tag{C.7j}$$

$$R_5' = \frac{dR_5(\mathrm{t}-\mathrm{t}_2)}{d(\mathrm{t}-\mathrm{t}_2)} = -(\mathrm{t}-\mathrm{t}_2)\sigma_2^2\ exp\left\{-\frac{(\mathrm{t}-\mathrm{t}_2)^2}{l^2}\right\}, \tag{C.7k}$$

$$R_6' = \frac{dR_3(\mathrm{t})}{d\mathrm{t}} = -\,\mathrm{t}\sigma_2^2\ exp\left\{-\frac{\mathrm{t}^2}{l^2}\right\}, \tag{C.7l}$$

$$R_1'' = \frac{d^2R(\mathrm{t}_1)}{d\mathrm{t}_1^2} = -\ (\sigma_2^2 - \frac{\sigma_2^4}{\sigma_1^2}\,\mathrm{t}_1^2)\ exp\left\{-\frac{\mathrm{t}_1^2}{l^2}\right\}, \tag{C.7m}$$

$$R_2'' = \frac{d^2R(\mathrm{t}_2-\mathrm{t}_1)}{d(\mathrm{t}_2-\mathrm{t}_1)^2} = -\ (\sigma_2^2 - \frac{\sigma_2^4}{\sigma_1^2}\,(\mathrm{t}_2-\mathrm{t}_1)^2)\ exp\left\{-\frac{(\mathrm{t}_2-\mathrm{t}_1)^2}{l^2}\right\}, \tag{C.7n}$$

$$R_3'' = \frac{d^2R(\mathrm{t}_2)}{d\mathrm{t}_2^2} = -\ (\sigma_2^2 - \frac{\sigma_2^4}{\sigma_1^2}\,\mathrm{t}_2^2)\ exp\left\{-\frac{\mathrm{t}_2^2}{l^2}\right\}, \tag{C.7o}$$

$$R_4'' = \frac{d^2R(\mathrm{t}-\mathrm{t}_1)}{d(\mathrm{t}-\mathrm{t}_1)^2} = -\ (\sigma_2^2 - \frac{\sigma_2^4}{\sigma_1^2}\,(\mathrm{t}-\mathrm{t}_1)^2)\ exp\left\{-\frac{(\mathrm{t}-\mathrm{t}_1)^2}{l^2}\right\}, \tag{C.7p}$$

$$R_5'' = \frac{d^2R(\mathrm{t}-\mathrm{t}_2)}{d(\mathrm{t}-\mathrm{t}_2)^2} = -\ (\sigma_2^2 - \frac{\sigma_2^4}{\sigma_1^2}\,(\mathrm{t}-\mathrm{t}_2)^2)\ exp\left\{-\frac{(\mathrm{t}-\mathrm{t}_2)^2}{l^2}\right\}, \tag{C.7q}$$

$$R_6'' = \frac{d^2R(\mathrm{t})}{d\mathrm{t}^2} = -\ (\sigma_2^2 - \frac{\sigma_2^4}{\sigma_1^2}\,\mathrm{t}^2)\ exp\left\{-\frac{\mathrm{t}^2}{l^2}\right\}. \tag{C.7r}$$

Since $\tilde{\underline{\mu}}_1 = \tilde{\underline{\mu}}_2 = 0$, and

$$\underline{\tilde{m}} = \hat{\underline{\tilde{c}}}_{12}\hat{\underline{\tilde{c}}}_{22}^{-1}\hat{\underline{\tilde{x}}}_2 = \hat{\underline{\tilde{c}}}_{12}\hat{\underline{\tilde{c}}}_{22}^{-1}\begin{bmatrix}\zeta'(t_1)\\ \zeta'(t_2)\\ \zeta'(t)\\ \zeta'_{osp}\end{bmatrix} = \begin{bmatrix}A_{11} & A_{12} & A_{13} & A_{14}\\ A_{21} & A_{22} & A_{23} & A_{24}\\ A_{31} & A_{32} & A_{33} & A_{34}\\ A_{41} & A_{42} & A_{43} & A_{44}\end{bmatrix}\begin{bmatrix}\zeta'(t_1)\\ \zeta'(t_2)\\ \zeta'(t)\\ \zeta'_{osp}\end{bmatrix} \tag{C.8}$$

This gives us

$$p(S(t_1),S(t_2),S(t),\zeta_{osp}\,|\zeta'(t_1),\zeta'(t_2),\zeta'(t),\zeta'_{osp}) =$$

$$\frac{1}{(2\pi)^2(det\hat{\underline{\tilde{c}}}\,)^{\frac{1}{2}}}$$

$$* \; exp\Bigg\{-\frac{1}{2}\Big[(\zeta_{osp}+\eta_0 t_1-\tilde{m}_1)\;\;(\zeta_{osp}+\eta_0 t_2-\tilde{m}_2)\;\;(\zeta_{osp}+\eta_0 t-\tilde{m}_3)\;\;(\zeta_{osp}-\tilde{m}_4)\Big]\cdot \begin{bmatrix}\tilde{\tilde{c}}_{11} & \tilde{\tilde{c}}_{12} & \tilde{\tilde{c}}_{13} & \tilde{\tilde{c}}_{14}\\ \tilde{\tilde{c}}_{21} & \tilde{\tilde{c}}_{22} & \tilde{\tilde{c}}_{23} & \tilde{\tilde{c}}_{24}\\ \tilde{\tilde{c}}_{31} & \tilde{\tilde{c}}_{32} & \tilde{\tilde{c}}_{33} & \tilde{\tilde{c}}_{34}\\ \tilde{\tilde{c}}_{41} & \tilde{\tilde{c}}_{42} & \tilde{\tilde{c}}_{43} & \tilde{\tilde{c}}_{44}\end{bmatrix}\begin{bmatrix}\zeta_{osp}+\eta_0 t_1-\tilde{m}_1\\ \zeta_{osp}+\eta_0 t_2-\tilde{m}_2\\ \zeta_{osp}+\eta_0 t-\tilde{m}_3\\ \zeta_{osp}-\tilde{m}_4\end{bmatrix}\Bigg\}, \tag{C.9}$$

where

$$\hat{\underline{\tilde{c}}}^{\,-1} \doteq \begin{bmatrix}\tilde{\tilde{c}}_{11} & \tilde{\tilde{c}}_{12} & \tilde{\tilde{c}}_{13} & \tilde{\tilde{c}}_{14}\\ \tilde{\tilde{c}}_{21} & \tilde{\tilde{c}}_{22} & \tilde{\tilde{c}}_{23} & \tilde{\tilde{c}}_{24}\\ \tilde{\tilde{c}}_{31} & \tilde{\tilde{c}}_{32} & \tilde{\tilde{c}}_{33} & \tilde{\tilde{c}}_{34}\\ \tilde{\tilde{c}}_{41} & \tilde{\tilde{c}}_{42} & \tilde{\tilde{c}}_{43} & \tilde{\tilde{c}}_{44}\end{bmatrix}. \tag{C.10}$$

If we let

$$\begin{bmatrix} \zeta_{osp} + \eta_0 t_1 - \widetilde{m}_1 \\ \zeta_{osp} + \eta_0 t_2 - \widetilde{m}_2 \\ \zeta_{osp} + \eta_0 t - \widetilde{m}_3 \\ \zeta_{osp} - \widetilde{m}_4 \end{bmatrix} = \zeta_{osp} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} \tag{C.11}$$

to isolate ζ_{osp}. Equation (C.9) becomes

$$p(S(t_1), S(t_2), S(t), \zeta_{osp} \,|\zeta'(t_1), \zeta'(t_2), \zeta'(t), \zeta'_{osp}) =$$

$$\frac{1}{(2\pi)^2 (det\widehat{\underline{c}}\,)^{\frac{1}{2}}} \, exp\left\{ -\frac{1}{2}\, \zeta_{osp}^2 c \; -\frac{1}{2}\zeta_{osp}(W + Y) \; -\frac{1}{2}Z \right\}, \tag{C.12}$$

where

$$c = \sum_{i,j} \widehat{\underline{c}}_{\,ij}\,, \tag{C.13}$$

$$W = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \widehat{\underline{c}}^{\,-1} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}, \tag{C.14}$$

$$Y = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \end{bmatrix} \widehat{\underline{c}}^{\,-1} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \tag{C.15}$$

and $$Z = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \end{bmatrix} \underline{\widehat{\widetilde{c}}}^{\,-1} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}. \tag{C.16}$$

Since we desire to isolate $\zeta'(\mathrm{t})$ also, we let

$$\underline{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \underline{b} + \underline{d}\zeta'(\mathrm{t}), \tag{C.17}$$

so now $W + Y = E + F\zeta'(\mathrm{t})$, where

$$E = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \underline{\widehat{\widetilde{c}}}^{\,-1} \underline{b} + \underline{b}^{\mathrm{T}} \underline{\widehat{\widetilde{c}}}^{\,-1} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \tag{C.18}$$

$$F = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \underline{\widehat{\widetilde{c}}}^{\,-1} \underline{d} + \underline{d}^{\mathrm{T}} \underline{\widehat{\widetilde{c}}}^{\,-1} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \tag{C.19}$$

and

$$Z = \underline{b}^{\mathrm{T}}\, \widehat{\underline{c}}^{\,-1}\underline{b} + \zeta'(\mathrm{t})\left[\underline{d}^{\mathrm{T}}\, \widehat{\underline{c}}^{\,-1}\underline{b} + \underline{b}^{\mathrm{T}}\, \widehat{\underline{c}}^{\,-1}\underline{d}\right] + \mathrm{S}^2(\mathrm{t})\, \underline{d}^{\mathrm{T}}\, \widehat{\underline{c}}^{\,-1}\, \underline{d}. \tag{C.20}$$

So $Z = \mathrm{Q} + \zeta'(\mathrm{t})P + \zeta'^2(\mathrm{t})R$, where $Q = \underline{b}^{\mathrm{T}}\, \widehat{\underline{c}}^{\,-1}\underline{b}$, $P = \left[\underline{d}^{\mathrm{T}}\, \widehat{\underline{c}}^{\,-1}\underline{b} + \underline{b}^{\mathrm{T}}\, \widehat{\underline{c}}^{\,-1}\underline{d}\right]$, and $R = \underline{d}^{\mathrm{T}}\, \widehat{\underline{c}}^{\,-1}\, \underline{d}$. (6.51) can now be written as

$$C^{(2)} \doteq \int_0^\infty \mathrm{dt} \int_0^{\mathrm{t}} \mathrm{dt}_1 \int_{\mathrm{t}_1}^{\mathrm{t}} \mathrm{dt}_2 \int_{\eta_0}^\infty d\zeta_1' \int_{\eta_0}^\infty d\zeta_2' \int_{\eta_0}^\infty d\zeta'\ (\zeta'(\mathrm{t}) - \eta_0)(\zeta'(\mathrm{t}_2) - \eta_0)$$

$$* \ (\zeta'(\mathrm{t}_1) - \eta_0)\mathrm{p}(\zeta'(\mathrm{t}_1), \zeta'(\mathrm{t}_2), \zeta'(\mathrm{t}) | \zeta'_{\mathrm{osp}})$$

$$* \ \frac{1}{(2\pi)^2 (det\widehat{\underline{c}}\,)^{\frac{1}{2}}} exp\left\{-\frac{1}{2}Z\right\} \int_{-\infty}^{\infty} d\zeta_{\mathrm{osp}}\ exp\left\{-\frac{1}{2}\left[\zeta_{\mathrm{osp}}^2 c + (W+Y)\zeta_{\mathrm{osp}}\right]\right\} \tag{C.21}$$

Performing the inner integral, we have

$$C^{(2)} \doteq \int_0^\infty \mathrm{dt} \int_0^{\mathrm{t}} \mathrm{dt}_1 \int_{\mathrm{t}_1}^{\mathrm{t}} \mathrm{dt}_2 \int_{\eta_0}^\infty d\zeta_1' \int_{\eta_0}^\infty d\zeta_2' \int_{\eta_0}^\infty d\zeta'\ (\zeta'(\mathrm{t}) - \eta_0)(\zeta'(\mathrm{t}_2) - \eta_0)$$

$$* \ (\zeta'(\mathrm{t}_1) - \eta_0)\mathrm{p}(\zeta'(\mathrm{t}_1), \zeta'(\mathrm{t}_2), \zeta'(\mathrm{t}) | \zeta'_{\mathrm{osp}})$$

$$* \ \frac{1}{(2\pi)^2 (det\widehat{\underline{c}}\,)^{\frac{1}{2}}} exp\left\{-\frac{1}{2}Z\right\} \sqrt{\frac{2\pi}{c}}\ exp\left\{\frac{(W+Y)^2}{8c}\right\}. \tag{C.22}$$

Using Baye's theorem we write

$$\mathrm{p}(\zeta'(\mathrm{t}_1), \zeta'(\mathrm{t}_2), \zeta'(\mathrm{t}) | \zeta'_{\mathrm{osp}}) = \mathrm{p}(\zeta'(\mathrm{t})\ | \zeta'(\mathrm{t}_1), \zeta'(\mathrm{t}_2), \zeta'_{\mathrm{osp}})\mathrm{p}(\zeta'(\mathrm{t}_1), \zeta'(\mathrm{t}_2) | \zeta'_{\mathrm{osp}}). \tag{C.23}$$

Stated explicitly:

$$\mathrm{p}(\zeta'(\mathrm{t})\ |\zeta'(\mathrm{t}_1),\zeta'(\mathrm{t}_2),\zeta'_{\mathrm{osp}}) = \frac{1}{\sqrt{2\pi\tilde{\tilde{c}}_3}}\, exp\left\{ -\frac{\left(\zeta'(\mathrm{t}) - \tilde{\tilde{m}}_3\right)^2}{2\tilde{\tilde{c}}_3} \right\}, \qquad \text{(C.24)}$$

where

$$\tilde{\tilde{m}}_3 = \begin{bmatrix} -R''_4 & -R''_5 & -R''_6 \end{bmatrix} \begin{bmatrix} \sigma_2^2 & -R''_2 & -R''_1 \\ -R''_2 & \sigma_2^2 & -R''_3 \\ -R''_1 & -R''_3 & \sigma_2^2 \end{bmatrix}^{-1} \qquad \text{(C.25)}$$

and

$$\tilde{\tilde{c}}_3 = \sigma_2^2 + \begin{bmatrix} -R''_4 & -R''_5 & -R''_6 \end{bmatrix} \begin{bmatrix} \sigma_2^2 & -R''_2 & -R''_1 \\ -R''_2 & \sigma_2^2 & -R''_3 \\ -R''_1 & -R''_3 & \sigma_2^2 \end{bmatrix}^{-1} \begin{bmatrix} -R''_4 \\ -R''_5 \\ R''_6 \end{bmatrix}. \qquad \text{(C.26)}$$

Also,

$$\mathrm{p}(\zeta'(\mathrm{t}_1),\zeta'(\mathrm{t}_2)|\zeta'_{\mathrm{osp}}) = \frac{1}{2\pi\, det(\underline{c})^{\frac{1}{2}}}$$

$$* \ exp\left\{ -\frac{1}{2}\begin{bmatrix} \zeta'(\mathrm{t}_1) - \tilde{\tilde{\mathrm{m}}}_1 & \zeta'(\mathrm{t}_2) - \tilde{\tilde{\mathrm{m}}}_2 \end{bmatrix} \underline{c}^{-1} \begin{bmatrix} \zeta'(\mathrm{t}_1) - \tilde{\tilde{\mathrm{m}}}_1 \\ \zeta'(\mathrm{t}_2) - \tilde{\tilde{\mathrm{m}}}_2 \end{bmatrix} \right\},$$

(C.27)

where

$$\tilde{\tilde{\mathrm{m}}} = \begin{bmatrix} \tilde{\tilde{\mathrm{m}}}_1 \\ \tilde{\tilde{\mathrm{m}}}_2 \end{bmatrix} = \begin{bmatrix} -R_1'' \\ -R_3'' \end{bmatrix} \frac{1}{\sigma_2^2} \zeta'_{\mathrm{osp}} \tag{C.28}$$

and
$$\underline{c} = \begin{bmatrix} \sigma_2^2 - \frac{R_1''}{\sigma_2^2} & -R_2'' - \frac{R_1'' R_3''}{\sigma_2^2} \\ -R_2'' - \frac{R_1'' R_3''}{\sigma_2^2} & \sigma_2^2 - \frac{R_1''}{\sigma_2^2} \end{bmatrix}. \tag{C.29}$$

Now, using the expressions $Z = \mathrm{Q} + \zeta'(\mathrm{t})P + \zeta'^2(\mathrm{t})R$ and $W + Y = E + F\zeta'(\mathrm{t})$ in (C.22), we can write

$$C^{(2)} \doteq \int_0^\infty \mathrm{dt} \int_0^{\mathrm{t}} \mathrm{dt}_1 \int_{\mathrm{t}_1}^{\mathrm{t}} \mathrm{dt}_2 \int_{\eta_0}^{\infty} d\zeta_1' \int_{\eta_0}^{\infty} d\zeta_2' \int_{\eta_0}^{\infty} d\zeta' \, (\zeta'(\mathrm{t}) - \eta_0)(\zeta'(\mathrm{t}_2) - \eta_0)$$
$$* \, (\zeta'(\mathrm{t}_1) - \eta_0) \frac{1}{\sqrt{2\pi \tilde{\tilde{c}}_3}} \, exp\left\{ - \frac{\left(\zeta'(\mathrm{t}) - \tilde{\tilde{\mathrm{m}}}_3\right)^2}{2\tilde{\tilde{c}}_3} \right\} \mathrm{p}(\zeta'(\mathrm{t}_1), \zeta'(\mathrm{t}_2) | \zeta'_{\mathrm{osp}})$$
$$* \, \frac{1}{(2\pi)^2 (det\hat{\underline{c}})^{\frac{1}{2}}} exp\left[-\frac{1}{2}\left\{ \mathrm{Q} + \zeta'(\mathrm{t})P + \zeta'^2(\mathrm{t})R \right\} \right] \sqrt{\frac{2\pi}{c}} \, exp\left\{ \frac{(E + F\zeta'(\mathrm{t}))^2}{8c} \right\}. \tag{C.30}$$

this leads to

$$C^{(2)} \doteq \int_0^\infty dt \int_0^t dt_1 \int_{t_1}^t dt_2 \int_{\eta_0}^\infty d\zeta_1' \int_{\eta_0}^\infty d\zeta_2' \,(\zeta'(t_2) - \eta_0)$$

$$* \,(\zeta'(t_1) - \eta_0)\frac{1}{\sqrt{2\pi\tilde{\tilde{c}}_3}} p(\zeta'(t_1),\zeta'(t_2)|\zeta'_{osp}) \frac{1}{(2\pi)^2 (det\hat{\underline{\underline{c}}})^{\frac{1}{2}}} \sqrt{\frac{2\pi}{c}}\, exp\left\{ -\frac{\tilde{\tilde{m}}_3^{\,2}}{2\tilde{\tilde{c}}_3} - \frac{Q}{2} + \frac{E^2}{8c} \right\}$$

$$* \int_{\eta_0}^\infty d\zeta_3' \,(\zeta'(t) - \eta_0)\, exp\left\{ \left[-\frac{1}{2\tilde{\tilde{c}}_3} - \frac{R}{2} + \frac{F^2}{8c} \right] \zeta'^2(t) + \left[\frac{\tilde{\tilde{m}}_3}{\tilde{\tilde{c}}_3} - \frac{P}{2} + \frac{2EF}{8c} \right] \zeta'(t) \right\} \tag{C.31}$$

Performing the inner integral we get, finally,

$$C^{(2)} \doteq \int_0^\infty dt \int_0^t dt_1 \int_{t_1}^t dt_2 \int_{\eta_0}^\infty d\zeta_1' \int_{\eta_0}^\infty d\zeta_2' \,(\zeta'(t_2) - \eta_0)$$

$$* \,(\zeta'(t_1) - \eta_0)\frac{1}{\sqrt{2\pi\tilde{\tilde{c}}_3}} p(\zeta'(t_1),\zeta'(t_2)|\zeta'_{osp}) \frac{1}{(2\pi)^2 (det\hat{\underline{\underline{c}}})^{\frac{1}{2}}} \sqrt{\frac{2\pi}{c}}\, exp\left\{ -\frac{\tilde{\tilde{m}}_3^{\,2}}{2\tilde{\tilde{c}}_3} - \frac{Q}{2} + \frac{E^2}{8c} \right\}$$

$$* \, exp\left\{\frac{\beta^2}{4\gamma}\right\}\left\{ \frac{1}{2\gamma} exp\left\{ -\gamma \left[\eta_0 - \frac{\beta}{2\gamma} \right]^2 \right\} + \left(\frac{\beta}{2\gamma} - \eta_0 \right) \sqrt{\frac{\pi}{\gamma}}\, \frac{1}{2}\, erfc\left[\sqrt{\gamma}\left(\eta_0 - \frac{\beta}{2\gamma} \right) \right] \right\} \tag{C.32}$$

where

$$\gamma = -\left[-\frac{1}{2\tilde{\tilde{c}}_3} - \frac{R}{2} + \frac{F^2}{8c} \right] \text{ and } \beta = \frac{\tilde{\tilde{m}}_3}{\tilde{\tilde{c}}_3} - \frac{P}{2} + \frac{2EF}{8c}. \tag{C.33}$$

This corresponds to equation (6.54) in chapter 6.

VITA

David Anthony Kapp was born in Fulton, New York on July 13, 1963. In 1977, he was co-owner of the "Stamp and Coin Packet," in the Syracuse Mall in Syracuse, New York. He lived in Liverpool, New York, with his family until graduation from Liverpool High School in 1981. He attended classes at the Rochester Institute of Technology (R.I.T.) in Rochester, New York, and obtained a Bachelor of Science degree in Mechanical Engineering in 1986. While attending R.I.T., he worked at the General Electric Company in Syracuse, New York, as a Student Engineer and Co-op student in Drafting Documentation, Replacement Parts, and Purchased Materials Quality Control. In 1986 he enrolled at Virginia Polytechnic Institute and State University in Electrical Engineering. His first year of graduate study was spent at the Fiber and Electro-Optics Research Center under director Dr. Richard O. Claus. In the Summer of 1989, he was employed by Lockheed Aerospace Company in Saugus, California, where he worked on Fiber Optic sensors. For the past three years, he has worked under the direction of Dr. Gary S. Brown in the area of electromagnetic scattering from random rough surfaces. David was a Bradley Fellow with the Bradley Department of Electrical Engineering from 1989-1992 and is currently working there as a research assistant, pursuing a Doctorate degree in Electrical Engineering.

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