# ESTIMATION OF THE STRENGTH OF A RADIOACTIVE SOURCE

by

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# I. INTRODUCTION AND SUMMARY

This thesis is concerned with the estimation of the strength of a radioactive source when decay is rapid so that the usual assumption of a Poisson distribution of counts is not applicable. We distinguish two main cases according as the background radiation is small or large.

In the former case the common procedure of simply subtracting off a constant background radiation is adequate. If the background radiation is neglected altogether the method of maximum likelihood may be used to estimate both source strength and decay-constant for counts taken at any arbitrary times. The large-sample variances and covariances of these estimates are obtained and the procedure is illustrated on an actual set of experimental results. Some comparisons are made with simpler estimates.

When the background radiation is large it is clearly not altogether appropriate to apply a simple constant subtractive correction to the observed counts. The random fluctuations in the background radiation should be taken into account. It is not difficult to determine the exact distribution of counts as a sum of two independent variates, one a binomial and one a Poisson variate. However, this distribution is difficult to handle and a normal approximation with the same mean and variance will often suffice. Problems of estimation are nevertheless difficult in this case. We draw attention to a procedure due to Tandberg [6] who considered the problem of obtaining a single optimum count; that is to say, he determined the time interval which would give best results if only a single count is taken. Such an optimal time exists because if counting is delayed too long the background radiation will swamp the results. The method of maximum likelihood applied in conjunction with the assumption of normality of total counts is compared with the Tandberg procedure in a numerical example, where the difference turns out to be unimportant. The possibility of taking two counts is also considered and the source strength is estimated by a least squares approach.

Attention has been concentrated in this thesis on problems caused by the rapid decay of the radioactive source. Other commonly encountered complications, such as the geometry of the counter and corrections for the dead period of the counter, have been ignored in the present study.

## II. ESTIMATION WHEN BACKGROUND RADIATION IS NEGLIGIBLE

### 2.1 The Law of Radioactive Decay

The number of radioactive atoms of a given isolated species which is not simultaneously regenerated by any process decreases exponentially with time.

The disintegration law common to all radioactive processes is normally written

$$\frac{dN}{dt} = -\lambda N \quad , \tag{2.1.1}$$

where N is the number of radioactive atoms present at time t , and $\lambda$ is the disintegration constant characteristic of the radio-isotope considered: $\lambda$ is expressed in reciprocal time units, e.g., if t is in seconds, $\lambda$ is in units of $\text{sec.}^{-1}$ , and if t is in minutes $\lambda$ is in $\text{min.}^{-1}$ .

Equation (2.1.1) may be integrated to give

$$N_t = N_0 e^{-\lambda t} \quad ,$$

where $N_0$ is the number of atoms present at any arbitrary zero time and $N_t$ is the number remaining after the lapse of a further time interval t.

### 2.2 Estimation for a Rapidly Decaying Substance

From the law of radioactive decay it follows that in a time-interval t, a radioactive atom will decay with probability

$$p = 1 - e^{-\lambda t} \quad .$$

Thus the distribution of the number of emissions is binomial

$(p, N_0)$. If the decay is slow p will be small and a Poisson approximation to the binomial can be used. Here we will retain the exact binomial distribution.

Suppose now that $z_i$ $(i=1,2,\ldots,k)$ particles are counted at times $t_i$. Let $p_i$ denote the probability that a particle radioactive at time $t_{i-1}$ will decay in the time-interval $(t_{i-1}, t_i)$. Then

$$p_i = 1 - e^{-\lambda(t_i - t_{i-1})} \quad .$$

We have, writing $q_i$ for $1-p_i$ ,

$$\left.\begin{aligned}
f(z_1) &= \binom{N_0}{z_1} p_1^{z_1} q_1^{N_0-z_1} \quad , \\
f(z_2|z_1) &= \binom{N_0-z_1}{z_2} p_2^{z_2} q_2^{N_0-z_1-z_2} \quad , \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots , \\
f(z_k|z_1,z_2,z_3,\ldots,z_{k-1}) &= \\
&\binom{N_0-z_1-z_2-\cdots-z_{k-1}}{z_k} p_k^{z_k} q_k^{N_0-\sum_{i=1}^{k} z_i}
\end{aligned}\right\} \quad (2.2.1) \quad .$$

Putting $Z = \sum_{i=1}^{k} z_i$ and $Z_r = \sum_{i=1}^{r} z_i$ $(r=1,2,\ldots,k)$, we have on multiplying together equations (2.2.1), that the joint density function of $z_1, z_2, \ldots, z_k$, is

$$L = \frac{N_0!}{(N_0-Z)!} \prod_{i=1}^{k} \frac{p_i^{z_i} q_i^{N_0-Z_i}}{z_i!} \quad , \qquad (2.2.2)$$

where $p_i = 1 - e^{-\lambda(t_i - t_{i-1})}$ and $q_i = 1 - p_i$ .

Hence

$$\log L = c + \log N_0! - \log(N_0 - Z)! + \sum_{i=1}^{k} z_i \log[1 - e^{-\lambda(t_i - t_{i-1})}]$$

$$- \sum_{i=1}^{k} (N_0 - Z_i)\lambda(t_i - t_{i-1}) \quad , \qquad (2.2.3)$$

where $c = - \sum_{i=1}^{k} \log z_i!$ .

2.2.1 Estimation of $N_0$ for known $\lambda$.

Differentiating (2.2.3) with respect to $N_0$ we obtain

$$\frac{d \log L}{d N_0} = \frac{d \log N_0!}{d N_0} - \frac{d \log (N_0 - Z)!}{d (N_0 - Z)} \frac{d (N_0 - Z)}{d N_0} - \lambda t_k \quad .$$

Thus the maximum likelihood estimate $\hat{N}_0$ is the solution of the equation

$$\psi(\hat{N}_0) - \psi(\hat{N}_0 - Z) = \lambda t_k \quad , \qquad (2.2.4)$$

where $\psi(x) \equiv d \log x!/dx$ is tabulated by Davis [2] for $x < 450$. For large x one can use the asymptotic formula

$$\psi(x) = \log x - \frac{1}{2x} - \frac{B_2}{2x^2} - \frac{B_4}{4x^4} - \dots - \frac{B_{2r}}{(2x^2)r} - \dots , \qquad (2.2.5)$$

where the B's are the Bernoulli numbers $B_2 = 1/6$, $B_4 = -1/30$, $B_6 = 1/42$, etc.

A simpler estimate $N_0^*$ is obtained from the fact that z is binomial with probability parameter $1 - e^{-\lambda t_k}$ , viz.,

$$N_0^* = \frac{Z}{1 - e^{-\lambda t_k}} \quad . \tag{2.2.6}$$

This estimate is clearly unbiased and follows also directly from the law of radioactivity

$$N_{t_k} = N_0 \, e^{-\lambda t_k} \quad ,$$

since $N_{t_k} = N_0 - Z$. $N_0^*$ may be used as a first estimate of $N_0$ to facilitate the solution of (2.2.4).

Actually Z is sufficient for $N_0$, since the likelihood function L of (2.2.2) factors into

$$L = \frac{N_0!}{(N_0 - Z)!} \prod_{i=1}^{k} q_i^{N_0} \cdot \prod_{i=1}^{k} \frac{p_i^{z_i} \, q_i^{-z_i}}{z_i!} \quad ,$$

the first factor being a function of $N_0$ and Z only, while the second factor is a function of the observations only. Consequently there is no point in finding the m.l. estimate in this case since $N_0^*$ is optimal, being that function of Z which is unbiased for $N_0$. There are, in general, no sufficient statistics in the following two cases.

### 2.2.2 Estimation of λ for known $N_0$.

On differentiating (2.2.3) with respect to λ we have

$$\frac{d \log L}{d \lambda} = \sum_{i=1}^{k} \frac{z_i}{p_i} \frac{d\, p_i}{d\, \lambda} - \sum_{i=1}^{k} (N_0 - Z_i)(t_i - t_{i-1})$$

$$= \sum_{i=1}^{k} \frac{z_i(t_i-t_{i-1})e^{-\lambda(t_i-t_{i-1})}}{1 - e^{-\lambda(t_i-t_{i-1})}} - N_0 t_k$$

$$+ \sum_{i=1}^{k} Z_i(t_i-t_{i-1})$$

$$= \sum_{i=1}^{k} \frac{z_i(t_i-t_{i-1})}{e^{\lambda(t_i-t_{i-1})} - 1} - N_0 t_k$$

$$+ \sum_{i=1}^{k} Z_i(t_i-t_{i-1}) \quad . \qquad (2.2.7)$$

This equation may be solved numerically for $\lambda$. A first approximation to $\lambda$ is minus the slope of the regression line of log $(N_0 - Z_i)$ on $t_i$.

If counts are made at equal intervals we may take $t_i$ = it when (2.2.7) simplifies to

$$\frac{d \log L}{d \lambda} = t\left(\frac{Z}{e^{\lambda t} - 1} - A\right) \quad ,$$

where $A = kN_0 - \sum_{i=1}^{k} Z_i$. Setting d log L/d$\lambda$ equal to zero we obtain

$$e^{\hat{\lambda} t} = \frac{Z + A}{A} \quad ,$$

or $\hat{\lambda} = \frac{1}{t}$ [log (Z+A) - log A], is the maximum likelihood estimate of $\lambda$.

### 2.2.3 Simultaneous estimation of $\lambda$ and $N_0$.

We now have to solve simultaneously the equations

$$\frac{\partial \log L}{\partial N_0} = 0, \quad \frac{\partial \log L}{\partial \lambda} = 0,$$

that is, equations (2.2.4) and (2.2.7).

Those equations can be solved by Newton's iterated method for two variables or by the Gauss-Seidel method. A first approximation to $\lambda$ is minus the slope of the regression line of $\log \bar{z}_i$ on $t_i - \frac{1}{2}(t_i - t_{i-1})$, where $\bar{z}_i$ is the average count during the time interval $(t_i - t_{i-1})$. This approximation comes directly from the law of radioactive decay. If we draw a histogram against time, we will have an exponential curve. The midpoint of each time interval will be $t_i - \frac{1}{2}(t_i - t_{i-1})$ and points on the curve represent the average count.

If counts are made at equal time-intervals the equations reduce to

$$\psi(\hat{N}_0) - \psi(\hat{N}_0 - Z) = k\hat{\lambda}t$$

and $\hat{\lambda} = \frac{1}{t} [\log (Z + k\hat{N}_0 - \sum_{i=1}^{k} Z_i) - \log (k\hat{N}_0 - \sum_{i=1}^{k} Z_i)]$ .

The numerical solution is illustrated by an example in Section 2.3.

### 2.2.4 Large-sample variances and covariances of the maximum likelihood estimators.

The second partial derivatives of log L with respect to the parameters $N_0$ and $\lambda$ are

$$\frac{\partial^2 \log L}{\partial N_0^2} = \psi' (N_0) - \psi' (N_0 - Z),$$

$$\frac{\partial^2 \log L}{\partial N_0 \, \partial \lambda} = - t_k \quad ,$$

$$\frac{\partial^2 \log L}{\partial \lambda^2} = \sum_{i=1}^{k} \frac{- z_i (t_i - t_{i-1})^2 e^{\lambda(t_i - t_{i-1})}}{(e^{\lambda(t_i - t_{i-1})} - 1)^2} \quad . (2.2.8)$$

Hence the elements of the information matrix can be found on taking expectations. We have

$$i_{11} = E(- \frac{\partial^2 \log L}{\partial N_0^2}) = E[\psi'(N_0 - Z) - \psi'(N_0)]$$

$$= E[ \sum_{j=0}^{Z-1} \frac{1}{(N_0 - j)^2} ]$$

$$= \sum_{Z=1}^{N_0} \binom{N_0}{Z} (1 - e^{-\lambda t_k})^Z \, (e^{-\lambda t_k})^{N_0 - Z} \sum_{j=0}^{Z-1} \frac{1}{(N_0 - j)^2} \quad .$$

Setting $1+r = e^{\lambda t_k}$ we obtain (cf. Fisher, 1940, [3])

$$i_{11} = (1+r)^{-N_0} \{ N_0 r \frac{1}{N_0^2} + \frac{N_0(N_0-1)}{2} r^2 [\frac{1}{N_0^2} + \frac{1}{(N_0-1)^2}]$$

$$+ \frac{N_0(N_0-1)(N_0-2)}{6} r^3 [\frac{1}{N_0^2} + \frac{1}{(N_0-1)^2} + \frac{1}{(N_0-2)^2}]$$

$$+ \ldots\ldots \}$$

$$= \frac{r}{N_0} + \frac{r^2}{2N_0(N_0-1)} + \frac{4r^3}{6N_0(N_0-1)(N_0-2)} + \ldots$$

$$= \sum_{j=1}^{N_0} \frac{r^j}{j} \quad \frac{(j-1)! \, (N_0 - j)!}{N_0!} \quad .$$

Also

$$i_{12} = i_{21} = E\left(- \frac{\partial^2 \log L}{\partial N_0 \, \partial \lambda}\right) = t_k \quad .$$

Since

$$E(z_i) = E(Z_i) - E(Z_{i-1})$$

$$= N_0(e^{-\lambda t_{i-1}} - e^{-\lambda t_i}) \quad ,$$

we have

$$i_{22} = E\left( - \frac{\partial^2 \log L}{\partial \lambda^2}\right)$$

$$= N_0 \sum_{i=1}^{k} \frac{(t_i - t_{i-1})^2 \, e^{-\lambda t_{i-1}}}{[e^{\lambda(t_i - t_{i-1})} - 1]}$$

The large-sample variances and covariances follow an inversion of the information matrix

$$I = \begin{bmatrix} i_{11} & i_{12} \\ i_{21} & i_{22} \end{bmatrix} \quad ,$$

being the elements of $I^{-1}$. To estimate them the m.l. estimates of $N_0$ and $\lambda$ are substituted for the unknown parameters. This procedure is also illustrated in the example of the following section.

## 2.3 Numerical Example

### 2.3.1 Estimation of $N_0$ and $\lambda$ for Indium.

The data of counts were taken in the VPI Physics Department by John Rogers in 1957. Assume that subtracting off a constant background radiation is adequate.

The first estimate for $\lambda$ is minus the slope of the regression line Y on X, by the Table I.

$$S_{xx} = 3,892,174.07 \quad ,$$

$$SP_{xy} = -720.8264 \quad ,$$

$$\frac{SP_{xy}}{S_{xx}} = -.000185 \quad ,$$

giving $\lambda = .000185$.

Now we have two equations (2.2.4) and (2.2.7) and two unknowns to solve for. In this example we used the Gauss-Seidel method.

A first estimate of $\lambda$ is $\lambda_0 = .000185$, and from Table II, we obtain $t_k = 1670$, $Z = 648,573$ and $\sum_{i=1}^{k} Z_i(t_i - t_{i-1}) = 616,722,570$.

Equation (2.2.6) now gives a first estimate $N_{00}$ of $N_0$ as

$$N_{00} = N_0{*} = \frac{648,573}{1 - e^{-1670\lambda}} \quad . \qquad (2.3.1)$$

From equation (2.2.7) we have

$$\frac{\partial \log L}{\partial \lambda} = \sum_{i=1}^{27} \frac{z_i(t_i - t_{i-1})}{e^{\lambda(t_i - t_{i-1})} - 1} - N_0 1670 + 616,722,570 \quad ,$$

and similarly $\partial^2 \log L / \partial \lambda^2$ from (2.2.8).

Using Newton's iterative method for one variable we can solve for $\lambda$ from

$$\lambda_{r+1} = \lambda_r - \frac{f(\lambda_r)}{f'(\lambda_r)}$$

Table I

| | | B | | | X |
|---|---|---|---|---|---|
| Time in sec. | Count per 10 sec. | Count - Bg. per 10 sec. | Y log B. | $t_i - t_{i-1}$ | $t_i - \frac{1}{2}(t_i - t_{i-1})$ |
| 10 | 5102 | 5052 | 8.5275 | 10 | 5 |
| 30 | 4736 | 4686 | 8.4523 | 20 | 20 |
| 50 | 4472 | 4422 | 8.3943 | 20 | 40 |
| 70 | 4612 | 4562 | 8.4255 | 20 | 60 |
| 90 | 4369 | 4319 | 8.3708 | 20 | 80 |
| 110 | 4368 | 4318 | 8.3705 | 20 | 100 |
| 130 | 4363 | 4313 | 8.3694 | 20 | 120 |
| 150 | 4323 | 4273 | 8.3601 | 20 | 140 |
| 170 | 4334 | 4284 | 8.3626 | 20 | 160 |
| 190 | 4450 | 4400 | 8.3894 | 20 | 180 |
| 210 | 4223 | 4173 | 8.3364 | 20 | 200 |
| 230 | 4270 | 4220 | 8.3476 | 20 | 220 |
| 260 | 4135 | 4085 | 8.3151 | 30 | 245 |
| 290 | 4217 | 4164 | 8.3342 | 30 | 275 |
| 320 | 4360 | 4310 | 8.3687 | 30 | 305 |
| 350 | 4347 | 4297 | 8.3657 | 30 | 335 |
| 380 | 4219 | 4169 | 8.3354 | 30 | 365 |
| 410 | 4258 | 4208 | 8.3447 | 30 | 395 |
| 470 | 4311 | 4261 | 8.3573 | 60 | 440 |
| 530 | 4211 | 4161 | 8.3335 | 60 | 500 |
| 590 | 4216 | 4166 | 8.3347 | 60 | 560 |
| 710 | 4129 | 4079 | 8.3136 | 120 | 650 |
| 830 | 4026 | 3976 | 8.2880 | 120 | 770 |
| 950 | 3826 | 3776 | 8.2364 | 120 | 890 |
| 1070 | 3851 | 3801 | 8.2430 | 120 | 1010 |
| 1370 | 3638 | 3588 | 8.1854 | 300 | 1220 |
| 1670 | 3421 | 3371 | 8.1230 | 300 | 1520 |
| Σ | | | 225.1851 | | 10805 |

## Table II

|  | $t_i-t_{i-1}$ | $z_i$ | $Z_i$ |
|---|---|---|---|
|  | 10 | 5052 | 5052 |
|  | 20 | 9372 | 14424 |
|  | 20 | 8844 | 23268 |
|  | 20 | 9124 | 32392 |
|  | 20 | 8638 | 41030 |
|  | 20 | 8636 | 49666 |
|  | 20 | 8626 | 58292 |
|  | 20 | 8546 | 66838 |
|  | 20 | 8568 | 75406 |
|  | 20 | 8800 | 84206 |
|  | 20 | 8346 | 92552 |
|  | 20 | 8440 | 100992 |
|  | 30 | 12255 | 113247 |
|  | 30 | 12492 | 125739 |
|  | 30 | 12930 | 138669 |
|  | 30 | 12891 | 151560 |
|  | 30 | 12507 | 164067 |
|  | 30 | 12624 | 176691 |
|  | 60 | 25566 | 202257 |
|  | 60 | 24966 | 227223 |
|  | 60 | 24996 | 252219 |
|  | 120 | 48948 | 301167 |
|  | 120 | 47712 | 348879 |
|  | 120 | 45312 | 394191 |
|  | 120 | 45612 | 439803 |
|  | 300 | 107640 | 547443 |
|  | 300 | 101130 | 648573 |
| Σ | 1670 | 648573 |  |

where $f(\lambda_r) = \left(\frac{\partial \log L}{\partial \lambda}\right)_{\lambda=\lambda_r}$ ,

and $f'(\lambda_r) = \left(\frac{\partial^2 \log L}{\partial \lambda^2}\right)_{\lambda=\lambda_r}$ .

To start with, $N_{00}$ was substituted for $N_0$ in these formulae. The iterations for $\lambda$ were done on an IBM 650 computer until the difference between successive $\lambda$'s was less than $5 \times 10^{-7}$ . With this value of $\lambda$ a new estimate $N_{01}$ was obtained from equation (2.3.1). The procedure was repeated until sufficient accuracy for both $\lambda$ and $N_0$ was attained. The final values, which are the required maximum likelihood estimates, came out to be

$$N_0 = 2,388,887 \qquad \lambda = .000189678 .$$

These may be compared with the first estimates

$$N_{00} = 2,439,178 \qquad \lambda = .000185 .$$

2.3.2 Variance-covariance matrix.

We have estimates

$$\hat{\lambda} = .000189678$$

$$\hat{N}_0 = 2,388,887 .$$

The information matrix is

$$I = \begin{bmatrix} i_{11} & i_{12} \\ i_{21} & i_{22} \end{bmatrix} .$$

We obtain

$$i_{11} = \frac{r}{N_0} + \frac{r^2}{2N_0(N_0-1)} + \frac{4r^3}{6N_0(N_0-1)(N_0-2)} + \dots + \dots$$

$$= \frac{.372324}{2,388,887} + \frac{(.372324)^2}{2(2,388,887)(2,388,886)} + \dots + \dots$$

$$\sim .15585657 \times 10^{-6}$$

where $r = e^{\lambda t_k} - 1 = e^{.31676226} - 1 = .372324$

$$i_{12} = i_{21} = t_k = 1670$$

$$i_{22} = N_0 \left( \sum_{i=1}^{k} \frac{(t_i-t_{i-1})^2 e^{-\lambda t_{i-1}}}{(e^{\lambda(t_i-t_{i-1})} - 1)} \right)$$

$$= (2.388887)(7.545457) \times 10^{12} \quad .$$

The variance-covariance matrix is the inverse of the information matrix. We obtain,

$$I^{-1} = \begin{bmatrix} 8.812261 \times 10^{8} & -.816437 \times 10^{-2} \\ -.816437 \times 10^{-2} & 7.619585 \times 10^{-12} \end{bmatrix} .$$

# III. ESTIMATION WHEN BACKGROUND RADIATION IS NOT NEGLIGIBLE

## 3.1 Exact Distribution of Counts

So far we have ignored the presence of background radiation. If this is small there is little harm in merely adjusting each count for the background radiation, assumed constant throughout the experiment and determined separately. Such a procedure was used in the example of Section 2.3. In this chapter we will attempt to take into account the random fluctuations of the background radiation.

If x is the number of particles emitted by the source, then the distribution f(x) of x is binomial, and if y is the number of particles in the background radiation, then y has a Poisson distribution g(y), say. Thus the distribution which we are dealing with will be h(z), where z=x+y.

The exact distribution of z is

$$h(z) = \sum_{x=0}^{z} f(x)g(z-x) = \sum_{y=0}^{z} f(z-y)g(y) \quad ,$$

since f(x) and g(y) are independent of each other.

The distribution of x is binomial with

$$E(x) = N_0(1 - e^{-\lambda t}) \quad ,$$

$$\mathrm{Var}\ (x) = N_0(1 - e^{-\lambda t})e^{-\lambda t} \quad ,$$

and $$f(x) = \binom{N_0}{x}(1 - e^{-\lambda t})^x(e^{-\lambda t})^{N_0-x} \qquad x=0,1,2,\ldots,N_0 \quad .$$

The distribution of y is Poisson

$$g(y) = e^{-bt} \frac{(bt)^y}{y!} \quad , \qquad y=0,1,2,\ldots \quad .$$

Thus h(z) can be written as

$$h(z) = \sum_{x=0}^{\min(z,N_0)} \binom{N_0}{x} p^x (1-p)^{N_0-x} \; e^{-bt} \frac{(bt)^{z-x}}{(z-x)!}$$

$$z=0,1,2,\ldots \quad ,$$

where $p=(1-e^{-\lambda t})$.

Also $E(z) = N_0(1-e^{-\lambda t})+bt$ and $Var(z) = N_0(1-e^{-\lambda t})e^{-\lambda t} + bt$ .

## 3.2 Estimation of Source Strength

3.2.1 The problem of measuring the intensity of a radioactive source in the presence of background radiation was discussed by J. Tandberg [6].

With his method of estimating $N_0$, supposing we have an unknown number $N_0$ of impulses, the decay-constant $\lambda$ and the background of b impulses per minute being known, we shall get during t minutes x impulses from the source with

$$\mu_x = N_0(1 - e^{-\lambda t}) \quad ,$$

$$\sigma_x = [N_0(1 - e^{-\lambda t})e^{-\lambda t}]^{\frac{1}{2}} \quad ,$$

and y impulses from the background with

$$\mu_y = bt \quad ,$$

$$\sigma_y = [bt]^{\frac{1}{2}} \quad .$$

Thus we have during t minutes a total number of impulses, with

$$\mu = N_0(1-e^{-\lambda t})+bt \quad ,$$

$$\sigma = [N_0(1-e^{-\lambda t})e^{-\lambda t}+bt]^{\frac{1}{2}} \quad .$$

An unbiased estimate of $N_0$ is therefore given by

$$N_0{*} = \frac{Z - bt}{1 - e^{-\lambda t}} \quad . \qquad (3.2.1)$$

### 3.2.2 Estimate from normalization of exact distribution.

Since f(x) and g(y) are binomial and Poisson distributions, we will now assume that h(z) is approximately normally distributed with $E(z) = N_0(1-e^{-\lambda t})+bt$ and $Var(z) = N_0(1-e^{-\lambda t})e^{-\lambda t} + bt$ .

Then the distribution of z will be

$$h(z) = \frac{1}{(2\pi)^{\frac{1}{2}}\ \sigma} \exp\ [- \frac{1}{2\sigma^2}\ (z-\mu)^2] \quad .$$

Using the method of maximum likelihood to estimate $N_0$, we have

$$\text{Log } h(z) = c - \frac{1}{2}\ \ln \sigma^2 - \frac{1}{2\sigma^2}\ (z-\mu)^2 \quad ,$$

$$\frac{\partial L}{\partial N_0} = - \frac{1}{2\sigma^2}\ (1-e^{-\lambda t})e^{-\lambda t} + \frac{(z-\mu)^2}{2\sigma^4}\ (1-e^{-\lambda t})e^{-\lambda t} + \frac{(z-\mu)(1-e^{-\lambda t})}{\sigma^2} \quad .$$

Set this equal to zero and obtain

$$e^{-\lambda t} = \frac{(z-\mu)^2}{\sigma^2}\ e^{-\lambda t} + 2(z-\mu) \quad ,$$

$$e^{-\lambda t} = \frac{[z-N_0(1-e^{-\lambda t})-bt]e^{-\lambda t}}{N_0(1-e^{-\lambda t})e^{-\lambda t} + bt} + 2[z-N_0(1-e^{-\lambda t}) - bt] \quad .$$

For simplification let us call, $e^{-\lambda t} = w$, $(1-e^{-\lambda t}) = a$ , and $bt = d$. Let us solve for $N_0$, and obtain the following equation.

$$N_0 = \frac{-(w^2+w-2wz+2wd+2d)+[(w^2+w-2wz+2wd+2d)^2-4(2w)(2wd-wz-2dz+2d^2)]^{\frac{1}{2}}}{4wa} \tag{3.2.2}$$

To show how close h(z) is to a normal distribution, let us find the coefficients $\gamma_1$ and $\gamma_2$,

where $$\gamma_1 = \frac{k_3}{(k_2)^{3/2}} \quad , \quad \gamma_2 = \frac{k_4}{(k_2)^2} \quad .$$

Clearly $k_{x2} = N_0 pq$, and $k_{y2} = bt$. From Kendall and Stuart [4] we obtain also

$$k_{x3} = N_0 pq(q-p) \quad , \quad k_{x4} = N_0 pq(1-6pq)$$
$$k_{y3} = bt \quad , \quad k_{y4} = bt \quad .$$

Since f(x) and g(y) are independent of each other we can get

$$k_{z2} = k_{(x+y)2} = k_{x2} + k_{y2} = N_0 pq + bt \quad ,$$
$$k_{z3} = N_0 pq(q-p) + bt \quad ,$$
$$k_{z4} = N_0 pq(1-6pq) + bt \quad .$$

Thus,

$$\gamma_1 = \frac{N_0 pq(q-p) + bt}{(N_0 pq + bt)^{3/2}} \quad ,$$

$$\gamma_2 = \frac{N_0 pq(1-6pq) + bt}{(N_0 pq + bt)^2} \quad .$$

Clearly $\gamma_1$ and $\gamma_2$ tend to zero if at least one of $N_0$ and t tends to infinity.

## 3.3 The Tandberg Procedure for Obtaining a Single Optimal Count

A problem in measuring the intensity of a radioactive source by means of a Geiger-Müller tube arises in determining the optimum period of counting time. This determination will depend of course, on the average life of the radioactive source and the intensity of the background interference. If the counting were carried on for a long period of time, the background fluctuations would certainly cause confusion. For a very short series, on the other hand, the fluctuations of the source itself may easily predominate and the result will accordingly be uncertain.

In order to obtain the least erroneous value of $N_0$ we will set out to find the time $t^*$ for which the mean relative fluctuation

$$\frac{[N_0(1-e^{-\lambda t})e^{-\lambda t} + bt]^{\frac{1}{2}}}{N_0(1 - e^{-\lambda t})}$$

is a minimum. Let us denote the square of this expression by $G(t)$, and differentiate $G(t)$ with respect to $t$.

$$G(t) = \frac{N_0(1-e^{-\lambda t})e^{-\lambda t} + bt}{N_0^2(1 - e^{-\lambda t})^2}$$

$$= \frac{1}{N_0} \left[\frac{e^{-\lambda t}}{1-e^{-\lambda t}} + \frac{bt}{N_0(1-e^{-\lambda t})^2}\right] .$$

$$G'(t) = \frac{1}{N_0} \left[ \frac{(1-e^{-\lambda t})e^{-\lambda t}(-\lambda) - e^{-\lambda t}(e^{-\lambda t})\lambda}{(1-e^{-\lambda t})^2} + \frac{(1-e^{-\lambda t})^2 b - 2bt(1-e^{-\lambda t})\lambda(e^{-\lambda t})}{N_0(1-e^{-\lambda t})^4} \right] .$$

Set this equal to zero, and transforming the equation we get,

$$N_0(1-e^{-\lambda t^*})^2 \, [(1-e^{-\lambda t^*})e^{-\lambda t^*}(-\lambda)-(e^{-\lambda t^*})(e^{-\lambda t^*})\lambda]$$
$$+(1-e^{-\lambda t^*})^2 b-2b\lambda t^*(1-e^{-\lambda t^*})e^{-\lambda t^*} = 0 \quad ,$$

$$N_0\lambda(1-e^{-\lambda t^*})^2 \, [-(1-e^{-\lambda t^*})e^{-\lambda t^*} - (e^{-\lambda t^*})(e^{-\lambda t^*})]$$
$$= b[2\lambda t^*(1-e^{-\lambda t^*})e^{-\lambda t^*} - (1-e^{-\lambda t^*})^2]$$

$$\frac{N_0\lambda}{b} = \frac{(1-e^{-\lambda t^*}) \, [2\lambda t^* e^{-\lambda t^*} - (1-e^{-\lambda t^*})]}{(1-e^{-\lambda t^*})^2(-e^{-\lambda t^*})}$$

$$= \frac{2\lambda t^* e^{-\lambda t^*} - (1-e^{-\lambda t^*})}{(1-e^{-\lambda t^*})(-e^{-\lambda t^*})}$$

$$= e^{\lambda t^*} - \frac{2\lambda t^*}{1 - e^{-\lambda t^*}} \quad .$$

By inspection of the second derivative of G(t), t* corresponds to a minimum of G(t).

### 3.4 Comparison of Two Methods of Estimation for Optimal Counting Time

If following Tandberg we denote $\lambda t^*$ by $\theta$ we may regard $\theta$ as the optimum time of counting, expressed in terms of the average life ($1/\lambda$) of the radioactive substance.

Now $N_0$ represents the initial intensity of the source, corrected for the effect of the background. Thus the quotient $N_0\lambda/b$ may be considered as the relative initial intensity of the radioactive source in terms of that of the background. The graph in Fig. 1 gives the relation between $N_0\lambda/b$ and $\theta$ for small values of $\theta$. If we consider a numerical example with radioactive indium, for which $\lambda = 0.01274$ min.$^{-1}$, having an initial activity equal to the background, it will be seen that it is of no use to continue the measurement longer than $\theta = 1.6$ or about 2 hours. An extended counting of impulses would give results of decreasing accuracy.

As a numerical example, let us take $\lambda = 0.01274$, and $\lambda t^* = \theta = 1.6$ from Fig. 1, and take $N_0^*\lambda = b$.

Then using (3.2.1)

$$N_0^* = \frac{z - bt^*}{1-e^{-\lambda t^*}} = \frac{z - N_0^*\lambda t^*}{1 - e^{-\lambda t^*}} \quad .$$

Let us take $z = 100$, and solve for $N_0^*$ .

$$\lambda t^* = \theta = 1.6 \text{ gives } t^* = 125.5887 \quad ,$$

$$N_0^*(1 - e^{-\lambda t^*}) = z - N_0^*\lambda t^* \quad ,$$

$$N_0^*(1 - e^{-\lambda t^*}) + N_0^*\lambda t^* = z \quad ,$$

$$N_0^*(1 - e^{-\lambda t^*} + \lambda t^*) = z \quad ,$$

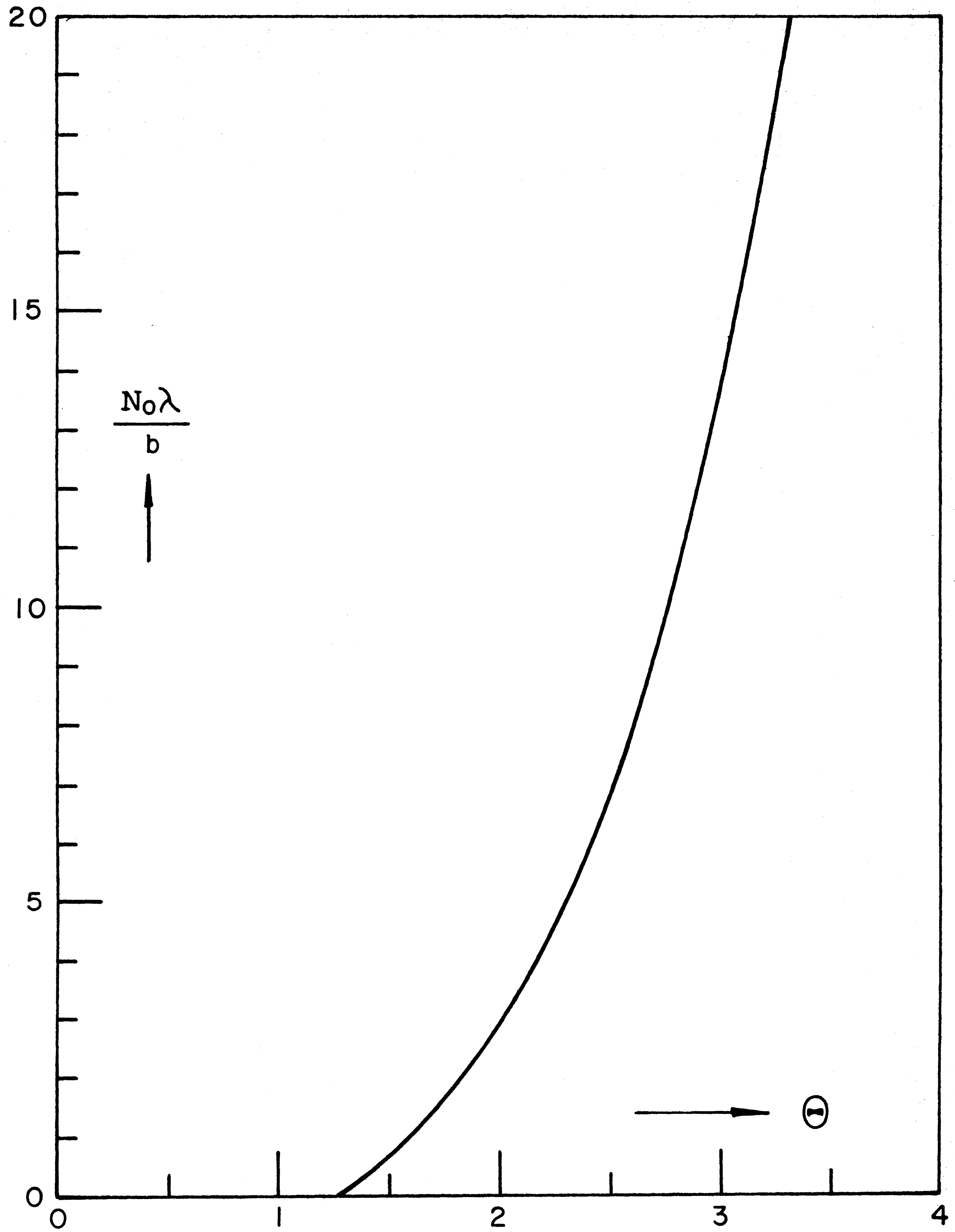

Fig. 1

$$N_0^* = \frac{z}{1 - e^{-\lambda t^*} + \lambda t^*}$$

$$= \frac{100}{1 - e^{-1.6} \pm 1.6}$$

$$= 41.7014 \quad .$$

$$N_0^*\lambda = b^* = (41.7014)(0.01274) = 0.531276 \quad .$$

Let us use this b*, put it into equation (3.2.2), and solve for $N_0$ . We find

$$\hat{N}_0 = 41.5910 \quad .$$

Further investigation with different values of $\theta$ show that estimating $N_0$ by Tandberg's method and maximum likelihood leads to almost the same results.

Table III shows the estimation of $N_0$ for different values of $\theta$ .

Table III

| $N_0^*\lambda$ | $\lambda \underline{t} = \theta$ | t | $N_0^*$ | $\hat{N}_0$ |
|---|---|---|---|---|
| 2b | 1.88 | 147.5667 | 55.9503 | 55.8643 |
| 3b | 2.1 | 164.8352 | 63.3874 | 63.3111 |
| 5b | 2.3 | 180.5338 | 73.5456 | 73.4607 |
| 15b | 3.1 | 243.328 | 86.0807 | 86.0605 |

Assumed z = 100

### 3.5 Estimation When Two Counts Can be Made

Now let us consider the case when two counts are made at equal time-intervals, $z_1$ for the interval $(0,t)$, $z_2$ for $(t,2t)$.

To estimate $N_0$, we minimize the following expression with respect to $N_0$

$$S = (\underline{z} - \underline{\mu})' \, V^{-1} \, (\underline{z} - \underline{\mu}) \quad ,$$

where

$$V = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix} \quad ,$$

and

$$V^{-1} = \frac{1}{\sigma_{11}\,\sigma_{22} - \sigma_{12}^2} \begin{bmatrix} \sigma_{22} & -\sigma_{12} \\ -\sigma_{12} & \sigma_{11} \end{bmatrix} \quad .$$

Thus

$$S = \frac{1}{\sigma_{11}\,\sigma_{22} - \sigma_{12}^2} [(z_1-\mu_1)^2\sigma_{22} - 2(z_1-\mu_1)(z_2-\mu_2)\sigma_{12} + (z_2-\mu_2)\sigma_{11}^2] \quad ,$$

where

$$\mu_1 = E(z_1) = E(x_1+y_1) = N_0(1-e^{-\lambda t})+bt = N_0 p + bt \quad ,$$

$$\begin{aligned} \mu_2 = E(z_2) = E(x_2+y_2) &= N_0[(1-e^{-2\lambda t})-(1-e^{-\lambda t})]+b(2t-t) \\ &= N_0 e^{-\lambda t}(1-e^{-\lambda t}) + bt \\ &= N_0 pq + bt \quad , \end{aligned}$$

$$\sigma_{11} = N_0(1-e^{-\lambda t})e^{-\lambda t} + bt = N_0pq + bt \quad .$$

To evaluate $\sigma_{22}$, consider $E(x_1{}^2)$. Let us call probability of decay in interval $t_1$, p and $q = 1-p$. Call probability of decay in interval $(t,2t)$ $p_2$ and $q_2 = 1-p_2$ respectively.

$$E(x_1{}^2) = N_0pq + (N_0p)^2 \quad .$$

Then

$$E(x_2^2 \mid x_1) = E[(N_0-x_1)p_2q_2 + (N_0-x_1)^2p_2{}^2] \quad .$$

Then

$$E(x_2{}^2) = (N_0-N_0p)p_2q_2 + (N_0{}^2 - 2N_0{}^2p + N_0pq + N_0{}^2p^2)p_2{}^2$$

$$= N_0qp_2q_2 + N_0{}^2q^2p_2{}^2 + N_0pqp_2{}^2 \quad .$$

Then

$$\sigma_{22} = E(x_2{}^2) - N_0{}^2p^2q^2 + bt$$

$$= N_0qp_2q_2 + N_0{}^2q^2p_2{}^2 + N_0pqp_2{}^2 - N_0{}^2p^2q^2 + bt$$

$$= N_0(qp_2q_2 + pqp_2{}^2) + N_0{}^2(q^2p_2{}^2 + p^2q^2) + bt \quad .$$

But $p_2 = 1 - e^{-\lambda(2t-t)} = 1-e^{-\lambda t} = p,\ q_2 = q \quad .$

$$\sigma_{22} = N_0(q^2p + p^3q) + N_0{}^2(q^2p^2 - p^2q^2) + bt$$

$$= N_0pq(q + p^2) + bt$$

$$= N_0A + bt \quad ,$$

where $A = pq(1 - pq) \quad .$

Now we must find the expression for $\sigma_{12}$ in order to differentiate S.

$$\mathrm{Var}(z_1 \pm z_2) = \sigma_{11} \pm \sigma_{22} \pm 2\sigma_{12} \quad .$$

But

$$\mathrm{Var}(z_1 + z_2) = N_0(1 - e^{-2\lambda t})\, e^{-2\lambda t} + 2bt \quad .$$

Therefore

$$\begin{aligned}\sigma_{12} &= \tfrac{1}{2}[N_0(1 - e^{-2\lambda t})\, e^{-2\lambda t} + 2bt - \sigma_{11} - \sigma_{22}] \\ &= \tfrac{1}{2}[N_0(1 - e^{-2\lambda t})\, e^{-2\lambda t} - N_0pq - N_0pq(1 - pq)] \\ &= \tfrac{1}{2}N_0[(1 - q^2)q^2 - pq - pq + p^2q^2] \\ &= -\, N_0p^2q \quad .\end{aligned}$$

Differentiating S with respect to $N_0$ we get

$$\begin{aligned}\frac{\partial S}{\partial N_0} = \frac{1}{\sigma_{11}\sigma_{22}-\sigma_{12}^2}\, \frac{\partial}{\partial N_0}\, &[(z_1-\mu_1)^2\sigma_{22}-2(z_1-\mu_1)(z_2-\mu_2)\sigma_{12} \\ &+(z_2-\mu_2)^2\sigma_{11}] + [(z_1-\mu_1)^2\sigma_{22}-2(z_1-\mu_1)(z_2-\mu_2)\sigma_{12} \\ &+(z_2-\mu_2)^2\sigma_{11}]\, \frac{\partial\,(\sigma_{11}\sigma_{22}-\sigma_{12}^2)}{\partial N_0}\end{aligned}$$

$$\begin{aligned}\frac{\partial\,(z_1-\mu_1)^2\sigma_{22}}{\partial N_0} &= -2(z_1-\mu_1)\sigma_{22}p+(z_1-\mu_1)^2A = -2z_1p\sigma_{22} \\ &\quad +2\mu_1p\sigma_{22}+z_1^2A-2z_1\mu_1A+\mu_1^2A\end{aligned}$$

$$\begin{aligned}-2\,\frac{\partial\,(z_1-\mu_1)(z_2-\mu_2)\sigma_{12}}{\partial N_0} &= -2[-p(z_2-\mu_2)\sigma_{12}-pq(z_1-\mu_1)\sigma_{12} \\ &\quad -(z_1-\mu_1)(z_2-\mu_2)p^2q]\end{aligned}$$

$$= 2z_2p\sigma_{12}-2p\sigma_{11}\sigma_{12}+2z_1pq\sigma_{12}$$
$$-2\mu_1pq\sigma_{12}+2z_1z_2p^2q-2z_1\sigma_{11}p^2q$$
$$-2z_2\mu_1p^2q+2\mu_1\sigma_{11}p^2q$$

$$\frac{\partial\ (z_2-\mu_2)^2\sigma_{11}}{\partial\ N_0} = -2(z_2-\sigma_{11})\sigma_{11}pq+(z_2-\sigma_{11})^2pq = -2z_2pq\sigma_{11}$$
$$+2pq\sigma_{11}{}^2+z_2{}^2pq-2z_2\sigma_{11}pq\ +\sigma_{11}^2pq$$

$$\frac{\partial\ (\sigma_{11}\sigma_{22}-\sigma_{12}{}^2)^{-1}}{\partial\ N_0} = -\frac{1}{(\sigma_{11}\sigma_{22}-\sigma_{12}{}^2)^2}\ [\sigma_{22}pq+\sigma_{11}A+2\sigma_{12}p^2q]$$

$$[\sigma_{11}\sigma_{22}-\sigma_{12}{}^2][-2z_1p\sigma_{22}+2\mu_1p\sigma_{22}+z_1{}^2A-2z_1\mu_1A+\mu_1{}^2A+2z_2p\sigma_{12}$$
$$-2p\sigma_{11}\sigma_{12}+2z_1pq\sigma_{12}-2\mu_1pq\sigma_{12}+2z_1z_2p^2q$$
$$-2z_1\sigma_{11}p^2q-2z_2\mu_1p^2q+2\mu_1\sigma_{11}p^2q-2z_2pq\sigma_{11}$$
$$+3pq\sigma_{11}{}^2+z_2{}^2pq-2z_2\sigma_{11}pq]$$

$$-\ [\sigma_{22}pq+\sigma_{11}A+2\sigma_{12}p^2q][z_1{}^2\sigma_{22}-2z_1\mu_1\sigma_{22}+\mu_1^2\sigma_{22}-2z_1z_2\sigma_{12}$$
$$+2z_1\sigma_{11}\sigma_{12}+2z_2\mu_1\sigma_{12}-2\mu_1\sigma_{11}\sigma_{12}$$
$$+z_2{}^2\sigma_{11}-2z_2\sigma_{11}{}^2+\sigma_{11}{}^3] = 0$$

The equation $\partial\ S/\partial\ N_0 = 0$ reduces to the following quartic for $N_0$:

$$N_0{}^4p^5q^4(1+q)+N_0{}^3 2(bt)p^4q^2(2+q+q^3)+N_0{}^4[-2z_1bt\ p^4q^2(2-q^2p)$$
$$-z_1{}^2p^3q^4(1-pq)-2z_1z_2p^4q^4-2z_2bt\ p^4q^2(2+pq)-z_2{}^2p^3q^4$$

$$+(bt)^2p^3q(5+10q-3q^2-3q^3+4q^4-q_5)]$$

$$+N_0[-4z_1(bt)^2p^2q(1-q^2)-2bt\ p^2q^3(z_1{}^2+z_2{}^2)-4z_2(bt)^2p^3q(1+q)$$

$$+2(bt)^3p^2(1+4q+q^2-2q^3)]$$

$$+[-2z_1(bt)^3p(1-q^2)-2z_2(bt)^3pq(1-q^2)+2z_1z_2(bt)^2p^2q$$

$$-z_1{}^2(bt)^2pq-z_2{}^2(bt)^2pq(1-pq)+(bt)^4p(2+2q-q^2q^3)] = 0$$

$$(3.5.1)$$

For our numerical example we have $z_1 = 60$, $z_2 = 45$

$q = e^{-\lambda t}$ where $t = 60$, $\lambda = .01274$ .

Equation (3.5.1) becomes

$$.0030N_0{}^4 + 2.7226N_0{}^3 + 285.2945N_0{}^2 - 4{,}721.8225N_0$$

$$- 1{,}198{,}955.8408 = 0 \quad .$$

Inspection of this equation tells us that it has one positive root and three negative roots. Solving for $N_0$ we get

$N_0 = 57$ .

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# ABSTRACT

This thesis is concerned with the estimation of the strength of a radioactive source when decay is rapid so that the usual assumption of a Poisson distribution of counts is not applicable. We distinguish two main cases according as the background radiation is small or large.

In the former case the common procedure of simply subtracting off a constant background radiation is adequate. If the background radiation is neglected altogether the method of maximum likelihood may be used to estimate both source strength and decay-constant for counts taken at any arbitrary times. The large-sample variances and covariances of these estimates are obtained and the procedure is illustrated on an actual set of experimental results.

When the background radiation is large their randomness should be taken into account. The exact distribution of counts is the sum of two independent variates, one a binomial and one a Poisson. However, a normal approximation with the same mean and variance will often suffice. We draw attention to a procedure due to Tandberg who considered the problem of obtaining a single optimum count. The method of maximum likelihood applied in conjunction with the assumption of normality of total counts is compared with his method in a numerical example. The possibility of taking two counts is also considered and the source strength is estimated by a least squares approach.