CHAPTER 4 FREE VIBRATION OF BREAKWATER IN AIR

4.1 INITIAL CONFIGURATION

Before considering the motion of the standard case discussed in Chapter 3, a general case will be adopted to provide more flexibility in design alterations and to serve as an introduction to the dynamic characteristics. The general system consists of a cylindrical breakwater of length L, radius R, thickness t_l along the longitudinal axis, and end thickness t_e, along with one mooring line modeled by n-1 masses and n springs. Figure 4.1 shows the initial or "unstretched springs" configuration of the system. The XYZ system is fixed in space and is used to trace the motion of the cylinder's center of gravity and the point masses making up the mooring line. The xyz system is fixed to the cylinder with its origin at the cylinder's center of gravity and thus participates in the cylinder's motion. The vector $\mathbf{r}_{c,0}$ denotes the initial position of the cylinder's center of gravity, while the vector $\mathbf{r}_{i,j,0}$ describes the location of mass j for mooring line i. Mass $m_{i,j}$ is described by the initial location ($X_{i,j,0}$, $Y_{i,j,0}$, $Z_{i,j,0}$) where i and j indicate the line and mass number, respectively. As shown in Figure 4.1, point A, which corresponds to the location where the cylinder and mooring line connect, can also be described as the sum of the two vectors $\mathbf{r}_{c,0}$ and $\mathbf{A}_0$ or (X_{co}+A1, Y_{c0}+A2, Z_{c0}+A3) where vector **A** consists of the **I**, **J**, and **K** components A1, A2, and A3, respectively. This will be discussed further in Section 4.3 where a specific correlation between these vectors will be derived.

4.2 GENERAL CONFIGURATION

Figure 4.2 portrays the general configuration of the system using the same axes described in Section 4.1 where the subscript 0 has been removed from the coordinates to represent a new configuration. Thus, a description of the elements of Figure 4.2 will not be provided here.

4.3 CYLINDER ROTATION (EULER ANGLES)

It has been previously established that the XYZ and xyz axes can be used simultaneously to locate any point on the moving structure. Nevertheless, since the xyz system is allowed to rotate and translate with the cylinder, the orientation of this system must first be established. Hence, to trace the rotational motion of the body and likewise any point on the body other than the center of gravity, the Euler angles ψ, θ, and ϕ will be used. It is important to note here that the rotations will be performed independently and always in the order stated above. Should the rotations be performed inconsistently, the results would be misleading if not entirely invalid. Therefore, the six variables X_C, Y_C, Z_C, ψ, θ, and ϕ will be used to trace the body's motion.

Let the axes $x_0y_0z_0$ represent the initial or "unstretched springs" orientation of the xyz system. Now, consider the three successive rotations shown in Figure 4.3 as (A), (B), and (C) which result in the final xyz orientation of $x_3y_3z_3$. From (A) the transformation from $x_0y_0z_0$ to $x_1y_1z_1$ (a rotation of angle ψ about the z_0 axis) yields

$$\begin{aligned} x_1 &= x_0 \cos\psi + y_0 \sin\psi \\ y_1 &= y_0 \cos\psi - x_0 \sin\psi \\ z_1 &= z_0 \end{aligned} \tag{4.1}$$

or,

$$\begin{Bmatrix} x_1 \\ y_1 \\ z_1 \end{Bmatrix} = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} x_0 \\ y_0 \\ z_0 \end{Bmatrix} = [R_1(\psi)] \begin{Bmatrix} x_0 \\ y_0 \\ z_0 \end{Bmatrix} \tag{4.2}$$

where $R_1(\psi)$ denotes the first rotational matrix. Similarly, the second rotation, θ, about the y_1 axis, provides

$$\begin{aligned} x_2 &= x_1 \cos\theta - z_1 \sin\theta \\ y_2 &= y_1 \\ z_2 &= z_1 \cos\theta + x_1 \sin\theta \end{aligned} \tag{4.3}$$

or,

$$\begin{Bmatrix} x_2 \\ y_2 \\ z_2 \end{Bmatrix} = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \begin{Bmatrix} x_1 \\ y_1 \\ z_1 \end{Bmatrix} = [R_2(\theta)] \begin{Bmatrix} x_1 \\ y_1 \\ z_1 \end{Bmatrix} \tag{4.4}$$

and the final rotation ϕ about the x_2 axis gives

$$\begin{aligned} x_3 &= x_2 \\ y_3 &= y_2 \cos\phi + z_2 \sin\phi \\ z_3 &= z_2 \cos\phi - y_2 \sin\phi \end{aligned} \tag{4.5}$$

or,

$$\begin{Bmatrix} x_3 \\ y_3 \\ z_3 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \begin{Bmatrix} x_2 \\ y_2 \\ z_2 \end{Bmatrix} = [R_3(\phi)] \begin{Bmatrix} x_2 \\ y_2 \\ z_2 \end{Bmatrix} \tag{4.6}$$

The complete transformation from $x_0y_0z_0$ to $x_3y_3z_3$ is simply

$$\begin{Bmatrix} x_3 \\ y_3 \\ z_3 \end{Bmatrix} = [R_3][R_2][R_1] \begin{Bmatrix} x_0 \\ y_0 \\ z_0 \end{Bmatrix} = [R] \begin{Bmatrix} x_0 \\ y_0 \\ z_0 \end{Bmatrix} \tag{4.7}$$

or in expanded form,

$$\begin{Bmatrix} x_3 \\ y_3 \\ z_3 \end{Bmatrix} =$$

$$\begin{bmatrix} \cos\theta\cos\psi & \sin\psi\cos\theta & -\sin\theta \\ \sin\phi\sin\theta\cos\psi - \sin\psi\cos\phi & \sin\psi\sin\phi\sin\theta + \cos\phi\cos\psi & \sin\phi\cos\theta \\ \sin\theta\cos\phi\cos\psi + \sin\phi\sin\psi & \sin\theta\cos\phi\sin\psi - \sin\phi\cos\psi & \cos\theta\cos\phi \end{bmatrix} \begin{Bmatrix} x_0 \\ y_0 \\ z_0 \end{Bmatrix} \tag{4.8}$$

As a result, any point on the cylinder can be written as a function of the three aforementioned Euler angles alone using the original xyz system position. Hence, point A with coordinates a_1, a_2, a_3 in the xyz system can now be expressed mathematically as a function of the variables X_C, Y_C, Z_C, ψ, θ, and ϕ by simply adding the motion of the center of gravity of the cylinder to the transformed position of point A. However, this assumes that the initial configuration of the system is known and can be expressed using vectors. This is fine for our purpose, where the equilibrium or unstretched position of the springs causes the xyz axes to be parallel to the XYZ system. The vector relationship should now be written as

$$\bar{r}_A = \bar{r}_c + [R]^T \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} \tag{4.9}$$

Since the rotation matrices have been previously defined, $\mathbf{r}_A$ can be written in terms of the six unknowns:

$$\begin{aligned}\vec{r}_A &= [X_c + a_1(\cos\theta\cos\psi) + a_2(\sin\phi\sin\theta\cos\psi - \sin\psi\cos\phi) \\ &+ a_3(\sin\theta\cos\phi\cos\psi + \sin\phi\sin\psi)]\hat{I} + [Y_c + a_1(\sin\psi\cos\theta) \\ &+ a_2(\sin\psi\sin\phi\sin\theta + \cos\phi\cos\psi) + a_3(\sin\theta\cos\phi\sin\psi - \sin\phi\cos\psi)]\hat{J} \\ &+ [Z_c + a_1(-\sin\theta) + a_2(\sin\phi\cos\theta) + a_3(\cos\theta\cos\phi)]\hat{K}\end{aligned} \tag{4.10}$$

4.4 ANGULAR VELOCITIES

To begin, it should first be assumed that the angular velocity of the cylinder can be written as

$$\Omega = P\hat{i}_3 + Q\hat{j}_3 + R\hat{k}_3 \tag{4.11}$$

where P, Q, and R are simply functions of the Euler angles and $\mathbf{i_3}$, $\mathbf{j_3}$, and $\mathbf{k_3}$ are unit vectors along the x_3, y_3, and z_3 axes, respectively. (Similar notations will be used for the $x_2y_2z_2$ and $x_1y_1z_1$ systems). Remembering the order of the rotations, the angular velocities can be written relative to the previous transformation. In other words, the angular velocity of $x_3y_3z_3$ relative to $x_2y_2z_2$ is

$$\Omega_{x_3y_3z_3/x_2y_2z_2} = \dot{\phi}\hat{i}_3 \tag{4.12}$$

Similarly, the other angular velocities can be written as:

$$\Omega_{x_2y_2z_2/x_1y_1z_1} = \dot{\theta}\hat{j}_2 \tag{4.13}$$

$$\Omega_{x_1y_1z_1/x_0y_0z_0} = \dot{\psi}\hat{k}_1 \tag{4.14}$$

Now, the angular velocities should be expressed in terms of the final position of the xyz system. This can be done by transforming the angular velocities utilizing the previously determined rotation matrices. Letting $\mathbf{j}_2=a\mathbf{i}_3+b\mathbf{j}_3+c\mathbf{k}_3$**,** one can see that the unknowns a, b, and c can be obtained from

$$[R_3]\begin{Bmatrix}0\\1\\0\end{Bmatrix} = \begin{Bmatrix}a\\b\\c\end{Bmatrix} \tag{4.15}$$

which results in

$$\begin{Bmatrix}a\\b\\c\end{Bmatrix} = \begin{Bmatrix}0\\\cos\phi\\-\sin\phi\end{Bmatrix} \tag{4.16}$$

Likewise, letting $\mathbf{k}_1=d\mathbf{i}_3+e\mathbf{j}_3+f\mathbf{k}_3$ yields

$$[R_3][R_2]\begin{Bmatrix}0\\0\\1\end{Bmatrix}=\begin{Bmatrix}d\\e\\f\end{Bmatrix} \tag{4.17}$$

which results in

$$\begin{Bmatrix}d\\e\\f\end{Bmatrix}=\begin{Bmatrix}-\sin\theta\\ \sin\phi\cos\theta\\ \cos\theta\cos\phi\end{Bmatrix} \tag{4.18}$$

So the resulting angular velocity is

$$\Omega=(\dot{\phi}-\dot{\psi}\sin\theta)\hat{i}_3+(\dot{\theta}\cos\phi+\dot{\psi}\sin\phi\cos\theta)\hat{j}_3+(\dot{\psi}\cos\theta\cos\phi-\dot{\theta}\sin\phi)\hat{k}_3 \tag{4.19}$$

or in matrix form

$$\begin{Bmatrix}\Omega_{x_3}\\ \Omega_{y_3}\\ \Omega_{z_3}\end{Bmatrix}=\begin{bmatrix}1 & 0 & -\sin\theta\\ 0 & \cos\phi & \sin\phi\cos\theta\\ 0 & -\sin\phi & \cos\theta\cos\phi\end{bmatrix}\begin{Bmatrix}\dot{\phi}\\ \dot{\theta}\\ \dot{\psi}\end{Bmatrix} \tag{4.20}$$

Derivations similar to those of Sections 4.3 and 4.4 are considered by Clayton, et al. (1982) and Chakrabarti (1990).

4.5 MOORING LINE ENERGY

For the mooring line shown in Figures 4.1 and 4.2 made up of n-1 masses, the kinetic energy of the line, T_i, is described by

$$T_i=\frac{1}{2}\sum_{j=1}^{n-1}(m_{i,j})(\dot{X}_{i,j}^2+\dot{Y}_{i,j}^2+\dot{Z}_{i,j}^2) \tag{4.21}$$

The potential energy of the line, V_i, is found as

$$\begin{aligned}V_i &=\sum_{j=1}^{n}\frac{1}{2}K_j(|\vec{r}_{i,j}-\vec{r}_{i,j-1}|-l_j)^2-\sum_{j=1}^{n-1}w_{i,j}Y_{i,j}\\ &=\sum_{j=1}^{n}\frac{1}{2}K_j\{(X_{i,j}-X_{i,j-1})^2+(Y_{i,j}-Y_{i,j-1})^2+(Z_{i,j}-Z_{i,j-1})^2+l_j^2\\ &\quad-2l_j[(X_{i,j}-X_{i,j-1})^2+(Y_{i,j}-Y_{i,j-1})^2+(Z_{i,j}-Z_{i,j-1})^2]^{\frac{1}{2}}\}-\sum_{j=1}^{n-1}w_{i,j}Y_{i,j}\end{aligned} \tag{4.22}$$

where $w_{i,j}$ represents the nominal net buoyant force on mass $m_{i,j}$ and l_j is the length of spring j. It should be recalled that vector $\mathbf{r}_{i,n}$ can be written as the sum of the vectors $\mathbf{r}_c$ and $\mathbf{A}$. Thus V_i can also be written as

$$
\begin{aligned}
V_i = &\sum_{j=1}^{n-1} \frac{1}{2} K_j \{ (X_{i,j} - X_{i,j-1})^2 + (Y_{i,j} - Y_{i,j-1})^2 + (Z_{i,j} - Z_{i,j-1})^2 + l_j^2 \\
&- 2l_j [(X_{i,j} - X_{i,j-1})^2 + (Y_{i,j} - Y_{i,j-1})^2 + (Z_{i,j} - Z_{i,j-1})^2]^{\frac{1}{2}} \} \\
&+ \frac{1}{2} K_n \{ [X_c + a_1(\cos\theta\cos\psi) + a_2(\sin\phi\sin\theta\cos\psi - \sin\psi\cos\phi) \\
&+ a_3(\sin\theta\cos\phi\cos\psi + \sin\phi\sin\psi) - X_{i,n-1}]^2 + [Y_c + a_1(\sin\psi\cos\theta) \\
&+ a_2(\sin\psi\sin\phi\sin\theta + \cos\phi\cos\psi) + a_3(\sin\theta\cos\phi\sin\psi - \sin\phi\cos\psi) \\
&- Y_{i,n-1}]^2 + [Z_c + a_1(-\sin\theta) + a_2(\sin\phi\cos\theta) + a_3(\cos\theta\cos\phi) - Z_{i,n-1}]^2 + l_n^2 \\
&- 2l_n \{ [X_c + a_1(\cos\theta\cos\psi) + a_2(\sin\phi\sin\theta\cos\psi - \sin\psi\cos\phi) \\
&+ a_3(\sin\theta\cos\phi\cos\psi + \sin\phi\sin\psi) - X_{i,n-1}]^2 + [Y_c + a_1(\sin\psi\cos\theta) \\
&+ a_2(\sin\psi\sin\phi\sin\theta + \cos\phi\cos\psi) + a_3(\sin\theta\cos\phi\sin\psi - \sin\phi\cos\psi) \\
&- Y_{i,n-1}]^2 + [Z_c + a_1(-\sin\theta) + a_2(\sin\phi\cos\theta) + a_3(\cos\theta\cos\phi) - Z_{i,n-1}]^2 \}^{\frac{1}{2}} \} \\
&- \sum_{j=1}^{n-1} w_{i,j} Y_{i,j}
\end{aligned} \tag{4.23}
$$

4.6 ENERGY OF THE CYLINDER

The potential energy, V_c, of the cylinder can be found as

$$V_c = -w_c Y_c \tag{4.24}$$

where w_c denotes the net buoyant force on the cylinder.

The kinetic energy of the cylinder is more complicated as it involves not only translation but rotation as well. The xyz axes are the principal axes of inertia of the cylinder, with the corresponding principal moments of inertia I_{xx}, I_{yy}, and I_{zz}, respectively. Thus the kinetic energy, T_c, of the cylinder is found as

$$
\begin{aligned}
T_c = &\frac{1}{2}[m_c(\dot{X}_c^2 + \dot{Y}_c^2 + \dot{Z}_c^2) + I_{xx}(\dot{\phi} - \dot{\psi}\sin\theta)^2 + I_{yy}(\dot{\theta}\cos\phi + \dot{\psi}\sin\phi\cos\theta)^2 \\
&+ I_{zz}(\dot{\psi}\cos\theta\cos\phi - \dot{\theta}\sin\phi)^2]
\end{aligned} \tag{4.25}
$$

where I_{xx}, I_{yy}, and I_{zz} are all found with reference to Figure 4.4 as

$$I_{xx} = \frac{1}{2} M_0 R^2 - \frac{1}{2} M_1 (R - t_L)^2 \tag{4.26}$$

$$I_{yy} = I_{zz} = \frac{1}{4} M_0 R^2 + \frac{1}{12} M_0 L^2 - [\frac{1}{4} M_1 (R - t_L)^2 + \frac{1}{12} M_1 (L - 2t_e)^2] \tag{4.27}$$

where M_0 is the mass of a solid cylinder of radius R and length L, and M_1 is the mass of a solid cylinder of radius R-t_l and length L-2t_e, i.e.,

$$M_0 = \pi R^2 L \rho_c \tag{4.28}$$

and

$$M_1 = \pi (R - t_L)^2 (L - 2t_e)\rho_c \tag{4.29}$$

where the density, ρ_c, of the cylinder is known.

4.7 LAGRANGE'S EQUATIONS

Given the complexity of equations (4.21) through (4.25) along with the variations in point masses along the mooring line, the Lagrangian, L_i, for the mooring line will be found independently from the Lagrangian, L_c, of the cylinder. However, a summation of these and the development of Lagrange's equations will conclude this section.

Referencing Meirovitch (1986), for any system, the Lagrangian can be found by

$$L = T - V \tag{4.30}$$

where T and V are the kinetic and potential energies of the system, respectively. Now, for a conservative system defined by n degrees of freedom, Lagrange's Equations are found directly from

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{Q}_j}\right) - \frac{\partial L}{\partial Q_j} = 0 \qquad j = 1,2,3,\ldots,N \tag{4.31}$$

where each Q_j represents an independent variable. However, for the case of the inflatable breakwater,

$$L = L_i + L_c \tag{4.32}$$

and

$$\frac{d}{dt}\left(\frac{\partial L_i}{\partial \dot{Q}_j}\right) - \frac{\partial L_i}{\partial Q_j} + \frac{d}{dt}\left(\frac{\partial L_c}{\partial \dot{Q}_j}\right) - \frac{\partial L_c}{\partial Q_j} = 0 \qquad j = 1,2,3,\ldots,N \tag{4.33}$$

where N degrees of freedom define the system, which includes both the structure and the mooring lines (lumped masses).

For mooring line i made of n-1 masses, the Lagrangian, L_i, can be written as

$$L_i = T_i - V_i = \frac{1}{2}\sum_{j=1}^{n-1}(m_{i,j})(\dot{X}_{i,j}^2 + \dot{Y}_{i,j}^2 + \dot{Z}_{i,j}^2) - \sum_{j=1}^{n-1}\frac{1}{2}K_j\{(X_{i,j} - X_{i,j-1})^2$$
$$+ (Y_{i,j} - Y_{i,j-1})^2 + (Z_{i,j} - Z_{i,j-1})^2 + l_j^2$$
$$- 2l_j[(X_{i,j} - X_{i,j-1})^2 + (Y_{i,j} - Y_{i,j-1})^2 + (Z_{i,j} - Z_{i,j-1})^2]^{\frac{1}{2}}\}$$
$$-\frac{1}{2}K_n\{[X_c + a_1(\cos\theta\cos\psi) + a_2(\sin\phi\sin\theta\cos\psi - \sin\psi\cos\phi)$$
$$+ a_3(\sin\theta\cos\phi\cos\psi + \sin\phi\sin\psi) - X_{i,n-1}]^2 + [Y_c + a_1(\sin\psi\cos\theta)$$
$$+ a_2(\sin\psi\sin\phi\sin\theta + \cos\phi\cos\psi) + a_3(\sin\theta\cos\phi\sin\psi - \sin\phi\cos\psi)$$
$$- Y_{i,n-1}]^2 + [Z_c + a_1(-\sin\theta) + a_2(\sin\phi\cos\theta) + a_3(\cos\theta\cos\phi) - Z_{i,n-1}]^2$$
$$+ l_n^2 - 2l_n\{[X_c + a_1(\cos\theta\cos\psi) + a_2(\sin\phi\sin\theta\cos\psi - \sin\psi\cos\phi)$$
$$+ a_3(\sin\theta\cos\phi\cos\psi + \sin\phi\sin\psi) - X_{i,n-1}]^2 + [Y_c + a_1(\sin\psi\cos\theta)$$
$$+ a_2(\sin\psi\sin\phi\sin\theta + \cos\phi\cos\psi) + a_3(\sin\theta\cos\phi\sin\psi - \sin\phi\cos\psi)$$
$$- Y_{i,n-1}]^2 + [Z_c + a_1(-\sin\theta) + a_2(\sin\phi\cos\theta) + a_3(\cos\theta\cos\phi) - Z_{i,n-1}]^2\}^{\frac{1}{2}}\}$$
$$+ \sum_{j=1}^{n-1} w_{i,j}Y_{i,j} \tag{4.34}$$

For each mooring line mass, it can be shown that the accompanying Lagrange's equations only involve L_i (i.e., $X_{i,j}$, $Y_{i,j}$, and $Z_{i,j}$ will not be associated with the energy equations developed for the cylinder).

Hence, Lagrange's equations are obtained directly from L_i for each of the point masses $m_{i,j}$ as

$$\frac{d}{dt}\left(\frac{\partial L_i}{\partial \dot{X}_{i,j}}\right) - \frac{\partial L_i}{\partial X_{i,j}} = 0 \tag{4.35}$$

or,

$$m_{i,j}\ddot{X}_{i,j} + K_j(X_{i,j} - X_{i,j-1}) - K_j l_j[(X_{i,j} - X_{i,j-1})^2 + (Y_{i,j} - Y_{i,j-1})^2$$
$$+ (Z_{i,j} - Z_{i,j-1})^2]^{-\frac{1}{2}}(X_{i,j} - X_{i,j-1}) - K_{j+1}(X_{i,j+1} - X_{i,j}) + \tag{4.36}$$
$$K_{j+1}l_{j+1}[(X_{i,j+1} - X_{i,j})^2 + (Y_{i,j+1} - Y_{i,j})^2 + (Z_{i,j+1} - Z_{i,j})^2]^{-\frac{1}{2}}(X_{i,j+1} - X_{i,j}) = 0;$$

$$\frac{d}{dt}\left(\frac{\partial L_i}{\partial \dot{Y}_{i,j}}\right) - \frac{\partial L_i}{\partial Y_{i,j}} = 0 \tag{4.37}$$

or,

$$m_{i,j}\ddot{Y}_{i,j} + K_j(Y_{i,j} - Y_{i,j-1}) - K_j l_j[(X_{i,j} - X_{i,j-1})^2 + (Y_{i,j} - Y_{i,j-1})^2 + (Z_{i,j} - Z_{i,j-1})^2]^{-\frac{1}{2}}(Y_{i,j} - Y_{i,j-1}) - K_{j+1}(Y_{i,j+1} - Y_{i,j}) + K_{j+1}l_{j+1}[(X_{i,j+1} - X_{i,j})^2 + (Y_{i,j+1} - Y_{i,j})^2 + (Z_{i,j+1} - Z_{i,j})^2]^{-\frac{1}{2}}(Y_{i,j+1} - Y_{i,j}) - w_{i,j} = 0 \tag{4.38}$$

and

$$\frac{d}{dt}\left(\frac{\partial L_i}{\partial \dot{Z}_{i,j}}\right) - \frac{\partial L_i}{\partial Z_{i,j}} = 0 \tag{4.39}$$

or,

$$m_{i,j}\ddot{Z}_{i,j} + K_j(Z_{i,j} - Z_{i,j-1}) - K_j l_j[(X_{i,j} - X_{i,j-1})^2 + (Y_{i,j} - Y_{i,j-1})^2 + (Z_{i,j} - Z_{i,j-1})^2]^{-\frac{1}{2}}(Z_{i,j} - Z_{i,j-1}) - K_{j+1}(Z_{i,j+1} - Z_{i,j}) + K_{j+1}l_{j+1}[(X_{i,j+1} - X_{i,j})^2 + (Y_{i,j+1} - Y_{i,j})^2 + (Z_{i,j+1} - Z_{i,j})^2]^{-\frac{1}{2}}(Z_{i,j+1} - Z_{i,j}) = 0 \tag{4.40}$$

from j=1 to n-2 making up mooring line i.

However, for mass n-1, equation (4.23) leads to the equations

$$\begin{aligned}
&m_{i,n-1}\ddot{X}_{i,n-1} + K_{n-1}(X_{i,n-1} - X_{i,n-2}) - K_{n-1}l_{n-1}[(X_{i,n-1} - X_{i,n-2})^2 \\
&+ (Y_{i,n-1} - Y_{i,n-2})^2 + (Z_{,n-1} - Z_{i,n-2})^2]^{-\frac{1}{2}}(X_{,n-1} - X_{i,n-2}) - K_n[X_c + a_1(\cos\theta\cos\psi) \\
&+ a_2(\sin\phi\sin\theta\cos\psi - \sin\psi\cos\phi) + a_3(\sin\theta\cos\phi\cos\psi + \sin\phi\sin\psi) \\
&- X_{i,n-1}] + K_n l_n\{[X_c + a_1(\cos\theta\cos\psi) + a_2(\sin\phi\sin\theta\cos\psi - \sin\psi\cos\phi) \\
&+ a_3(\sin\theta\cos\phi\cos\psi + \sin\phi\sin\psi) - X_{i,n-1}]^2 + [Y_c + a_1(\sin\psi\cos\theta) \\
&+ a_2(\sin\psi\sin\phi\sin\theta + \cos\phi\cos\psi) + a_3(\sin\theta\cos\phi\sin\psi - \sin\phi\cos\psi) - Y_{i,n-1}]^2 \\
&+ [Z_c + a_1(-\sin\theta) + a_2(\sin\phi\cos\theta) + a_3(\cos\theta\cos\phi) - Z_{i,n-1}]^2\}^{-\frac{1}{2}}[X_c \\
&+ a_1(\cos\theta\cos\psi) + a_2(\sin\phi\sin\theta\cos\psi - \sin\psi\cos\phi) \\
&+ a_3(\sin\theta\cos\phi\cos\psi + \sin\phi\sin\psi) - X_{i,n-1}] = 0;
\end{aligned} \tag{4.41}$$

$$
\begin{aligned}
&m_{i,n-1}\ddot{Y}_{i,n-1} + K_{n-1}(Y_{i,n-1} - Y_{i,n-2}) - K_{n-1}l_{n-1}[(X_{i,n-1} - X_{i,n-2})^2 + (Y_{i,n-1} - Y_{i,n-2})^2 \\
&+ (Z_{,n-1} - Z_{i,n-2})^2]^{-\frac{1}{2}}(Y_{,n-1} - Y_{i,n-2}) - K_n[Y_c + a_1(\sin\psi\cos\theta) \\
&+ a_2(\sin\psi\sin\phi\sin\theta + \cos\phi\cos\psi) + a_3(\sin\theta\cos\phi\sin\psi - \sin\phi\cos\psi) \\
&- Y_{i,n-1}] + K_n l_n\{[X_c + a_1(\cos\theta\cos\psi) + a_2(\sin\phi\sin\theta\cos\psi - \sin\psi\cos\phi) \\
&+ a_3(\sin\theta\cos\phi\cos\psi + \sin\phi\sin\psi) - X_{i,n-1}]^2 + [Y_c + a_1(\sin\psi\cos\theta) \\
&+ a_2(\sin\psi\sin\phi\sin\theta + \cos\phi\cos\psi) + a_3(\sin\theta\cos\phi\sin\psi - \sin\phi\cos\psi) - Y_{i,n-1}]^2 \\
&+ [Z_c + a_1(-\sin\theta) + a_2(\sin\phi\cos\theta) + a_3(\cos\theta\cos\phi) - Z_{i,n-1}]^2\}^{-\frac{1}{2}}[Y_c \\
&+ a_1(\sin\psi\cos\theta) + a_2(\sin\psi\sin\phi\sin\theta + \cos\phi\cos\psi) + a_3(\sin\theta\cos\phi\sin\psi \\
&- \sin\phi\cos\psi) - Y_{i,n-1}] - w_{i,n-1} = 0
\end{aligned}
\tag{4.42}
$$

and

$$
\begin{aligned}
&m_{i,n-1}\ddot{Z}_{i,n-1} + K_{n-1}(Z_{i,n-1} - Z_{i,n-2}) - K_{n-1}l_{n-1}[(X_{i,n-1} - X_{i,n-2})^2 + (Y_{i,n-1} - Y_{i,n-2})^2 \\
&+ (Z_{,n-1} - Z_{i,n-2})^2]^{-\frac{1}{2}}(Z_{,n-1} - Z_{i,n-2}) - K_n[Z_c + a_1(-\sin\theta) + a_2(\sin\phi\cos\theta) \\
&+ a_3(\cos\theta\cos\phi) - Z_{i,n-1}] + K_n l_n\{[X_c + a_1(\cos\theta\cos\psi) + a_2(\sin\phi\sin\theta\cos\psi \\
&- \sin\psi\cos\phi) + a_3(\sin\theta\cos\phi\cos\psi + \sin\phi\sin\psi) - X_{i,n-1}]^2 + [Y_c \\
&+ a_1(\sin\psi\cos\theta) + a_2(\sin\psi\sin\phi\sin\theta + \cos\phi\cos\psi) + a_3(\sin\theta\cos\phi\sin\psi \\
&- \sin\phi\cos\psi) - Y_{i,n-1}]^2 + [Z_c + a_1(-\sin\theta) + a_2(\sin\phi\cos\theta) + a_3(\cos\theta\cos\phi) \\
&- Z_{i,n-1}]^2\}^{-\frac{1}{2}}[Z_c + a_1(-\sin\theta) + a_2(\sin\phi\cos\theta) + a_3(\cos\theta\cos\phi) - Z_{i,n-1}] = 0
\end{aligned}
\tag{4.43}
$$

For the rest of the variables, the Lagrangians for both the cylinder, L_c, and the mooring line, L_i, contribute to the general equation

$$
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{Q}}\right) - \frac{\partial L}{\partial Q} = 0 \tag{4.44}
$$

where Q denotes any of the unknowns X_c, Y_c, Z_c, ψ,θ, or ϕ.

The parts of Lagrange's equations (33) involving L_i are found as

$$
\frac{d}{dt}\left(\frac{\partial L_i}{\partial \dot{X}_c}\right) - \frac{\partial L_i}{\partial X_c} \tag{4.45}
$$

or,

$$
\begin{aligned}
&K_n[X_c + a_1(\cos\theta\cos\psi) + a_2(\sin\phi\sin\theta\cos\psi - \sin\psi\cos\phi) \\
&+ a_3(\sin\theta\cos\phi\cos\psi + \sin\phi\sin\psi) - X_{i,n-1}] - K_n l_n\{[X_c + a_1(\cos\theta\cos\psi) \\
&+ a_2(\sin\phi\sin\theta\cos\psi - \sin\psi\cos\phi) \\
&+ a_3(\sin\theta\cos\phi\cos\psi + \sin\phi\sin\psi) - X_{i,n-1}]^2 + [Y_c + a_1(\sin\psi\cos\theta) \\
&+ a_2(\sin\psi\sin\phi\sin\theta + \cos\phi\cos\psi) + a_3(\sin\theta\cos\phi\sin\psi - \sin\phi\cos\psi) - Y_{i,n-1}]^2 \\
&+ [Z_c + a_1(-\sin\theta) + a_2(\sin\phi\cos\theta) + a_3(\cos\theta\cos\phi) - Z_{i,n-1}]^2\}^{-\frac{1}{2}}[X_c \\
&+ a_1(\cos\theta\cos\psi) + a_2(\sin\phi\sin\theta\cos\psi - \sin\psi\cos\phi) \\
&+ a_3(\sin\theta\cos\phi\cos\psi + \sin\phi\sin\psi) - X_{i,n-1}];
\end{aligned}
\tag{4.46}
$$

$$
\frac{d}{dt}\left(\frac{\partial L_i}{\partial \dot{Y}_c}\right) - \frac{\partial L_i}{\partial Y_c} \tag{4.47}
$$

or,

$$
\begin{aligned}
&K_n[Y_c + a_1(\sin\psi\cos\theta) + a_2(\sin\psi\sin\phi\sin\theta + \cos\phi\cos\psi) \\
&+ a_3(\sin\theta\cos\phi\sin\psi - \sin\phi\cos\psi) - Y_{i,n-1}] - K_n l_n\{[X_c + a_1(\cos\theta\cos\psi) \\
&+ a_2(\sin\phi\sin\theta\cos\psi - \sin\psi\cos\phi) \\
&+ a_3(\sin\theta\cos\phi\cos\psi + \sin\phi\sin\psi) - X_{i,n-1}]^2 + [Y_c + a_1(\sin\psi\cos\theta) \\
&+ a_2(\sin\psi\sin\phi\sin\theta + \cos\phi\cos\psi) + a_3(\sin\theta\cos\phi\sin\psi - \sin\phi\cos\psi) \\
&- Y_{i,n-1}]^2 + [Z_c + a_1(-\sin\theta) + a_2(\sin\phi\cos\theta) + a_3(\cos\theta\cos\phi) - Z_{i,n-1}]^2\}^{-\frac{1}{2}}[Y_c \\
&+ a_1(\sin\psi\cos\theta) + a_2(\sin\psi\sin\phi\sin\theta + \cos\phi\cos\psi) + a_3(\sin\theta\cos\phi\sin\psi \\
&- \sin\phi\cos\psi) - Y_{i,n-1}];
\end{aligned}
\tag{4.48}
$$

$$
\frac{d}{dt}\left(\frac{\partial L_i}{\partial \dot{Z}_c}\right) - \frac{\partial L_i}{\partial Z_c} \tag{4.49}
$$

or,

$$
\begin{aligned}
&K_n[Z_c + a_1(-\sin\theta) + a_2(\sin\phi\cos\theta) + a_3(\cos\theta\cos\phi) - Z_{i,n-1}] - K_n l_n\{[X_c \\
&+ a_1(\cos\theta\cos\psi) + a_2(\sin\phi\sin\theta\cos\psi - \sin\psi\cos\phi) \\
&+ a_3(\sin\theta\cos\phi\cos\psi + \sin\phi\sin\psi) - X_{i,n-1}]^2 + [Y_c + a_1(\sin\psi\cos\theta) \\
&+ a_2(\sin\psi\sin\phi\sin\theta + \cos\phi\cos\psi) + a_3(\sin\theta\cos\phi\sin\psi - \sin\phi\cos\psi) \\
&- Y_{i,n-1}]^2 + [Z_c + a_1(-\sin\theta) + a_2(\sin\phi\cos\theta) + a_3(\cos\theta\cos\phi) \\
&- Z_{i,n-1}]^2\}^{-\frac{1}{2}}[Z_c + a_1(-\sin\theta) + a_2(\sin\phi\cos\theta) + a_3(\cos\theta\cos\phi) - Z_{i,n-1}];
\end{aligned}
\tag{4.50}
$$

$$
\frac{d}{dt}\left(\frac{\partial L_i}{\partial \dot{\psi}}\right) - \frac{\partial L_i}{\partial \psi} \tag{4.51}
$$

or,

$$
\begin{aligned}
&K_n[X_c + a_1(\cos\theta\cos\psi) + a_2(\sin\phi\sin\theta\cos\psi - \sin\psi\cos\phi) \\
&+ a_3(\sin\theta\cos\phi\cos\psi + \sin\phi\sin\psi) - X_{i,n-1}][-a_1\cos\theta\sin\psi \\
&- a_2(\sin\phi\sin\theta\sin\psi + \cos\psi\cos\phi) - a_3(\sin\theta\cos\phi\sin\psi - \sin\phi\cos\psi)] + K_n[Y_c \\
&+ a_1(\sin\psi\cos\theta) + a_2(\sin\psi\sin\phi\sin\theta + \cos\phi\cos\psi) + a_3(\sin\theta\cos\phi\sin\psi \\
&- \sin\phi\cos\psi) - Y_{i,n-1}][a_1\cos\theta\cos\psi + a_2(\cos\psi\sin\phi\sin\theta - \cos\phi\sin\psi) \\
&+ a_3(\sin\theta\cos\phi\cos\psi + \sin\phi\sin\psi)] - K_n l_n\{[X_c + a_1(\cos\theta\cos\psi) + \\
&a_2(\sin\phi\sin\theta\cos\psi - \sin\psi\cos\phi) \\
&+ a_3(\sin\theta\cos\phi\cos\psi + \sin\phi\sin\psi) - X_{i,n-1}]^2 + [Y_c + a_1(\sin\psi\cos\theta) \\
&+ a_2(\sin\psi\sin\phi\sin\theta + \cos\phi\cos\psi) + a_3(\sin\theta\cos\phi\sin\psi - \sin\phi\cos\psi) \\
&- Y_{i,n-1}]^2 + [Z_c + a_1(-\sin\theta) + a_2(\sin\phi\cos\theta) + a_3(\cos\theta\cos\phi) - Z_{i,n-1}]^2\}^{-\frac{1}{2}}\{[X_c \\
&+ a_1(\cos\theta\cos\psi) + a_2(\sin\phi\sin\theta\cos\psi - \sin\psi\cos\phi) \\
&+ a_3(\sin\theta\cos\phi\cos\psi + \sin\phi\sin\psi) - X_{i,n-1}][-a_1\cos\theta\sin\psi \\
&- a_2(\sin\phi\sin\theta\sin\psi + \cos\psi\cos\phi) - a_3(\sin\theta\cos\phi\sin\psi - \sin\phi\cos\psi)] \\
&+ [Y_c + a_1(\sin\psi\cos\theta) + a_2(\sin\psi\sin\phi\sin\theta + \cos\phi\cos\psi) \\
&+ a_3(\sin\theta\cos\phi\sin\psi - \sin\phi\cos\psi) - Y_{i,n-1}][a_1\cos\theta\cos\psi \qquad (4.52) \\
&+ a_2(\cos\psi\sin\phi\sin\theta - \cos\phi\sin\psi) + a_3(\sin\theta\cos\phi\cos\psi + \sin\phi\sin\psi)]\};
\end{aligned}
$$

$$
\frac{d}{dt}\left(\frac{\partial L_i}{\partial\dot{\theta}}\right) - \frac{\partial L_i}{\partial\theta} \qquad (4.53)
$$

or,

$$
\begin{aligned}
&K_n[X_c + a_1(\cos\theta\cos\psi) + a_2(\sin\phi\sin\theta\cos\psi - \sin\psi\cos\phi) \\
&+ a_3(\sin\theta\cos\phi\cos\psi + \sin\phi\sin\psi) - X_{i,n-1}][-a_1\cos\psi\sin\theta \\
&+ a_2(\sin\phi\cos\theta\cos\psi) + a_3(\cos\theta\cos\phi\cos\psi)] + K_n[Y_c + a_1(\sin\psi\cos\theta) \\
&+ a_2(\sin\psi\sin\phi\sin\theta + \cos\phi\cos\psi) + a_3(\sin\theta\cos\phi\sin\psi - \sin\phi\cos\psi) \\
&- Y_{i,n-1}][-a_1\sin\psi\sin\theta + a_2\sin\psi\sin\phi\cos\theta + a_3(\cos\theta\cos\phi\sin\psi)] \\
&+ K_n[Z_c + a_1(-\sin\theta) + a_2(\sin\phi\cos\theta) + a_3(\cos\theta\cos\phi) - Z_{i,n-1}](-a_1\cos\theta \\
&- a_2\sin\phi\sin\theta - a_3\sin\theta\cos\phi) - K_n l_n\{[X_c + a_1(\cos\theta\cos\psi) + \\
&a_2(\sin\phi\sin\theta\cos\psi - \sin\psi\cos\phi) \\
&+ a_3(\sin\theta\cos\phi\cos\psi + \sin\phi\sin\psi) - X_{i,n-1}]^2 + [Y_c + a_1(\sin\psi\cos\theta) \\
&+ a_2(\sin\psi\sin\phi\sin\theta + \cos\phi\cos\psi) + a_3(\sin\theta\cos\phi\sin\psi - \sin\phi\cos\psi) \\
&- Y_{i,n-1}]^2 + [Z_c + a_1(-\sin\theta) + a_2(\sin\phi\cos\theta) + a_3(\cos\theta\cos\phi) \\
&- Z_{i,n-1}]^2\}^{-\frac{1}{2}}\{[X_c + a_1(\cos\theta\cos\psi) + a_2(\sin\phi\sin\theta\cos\psi - \sin\psi\cos\phi) \\
&+ a_3(\sin\theta\cos\phi\cos\psi + \sin\phi\sin\psi) - X_{i,n-1}][-a_1\cos\psi\sin\theta \\
&+ a_2(\sin\phi\cos\theta\cos\psi) + a_3(\cos\theta\cos\phi\cos\psi)] + [Y_c + a_1(\sin\psi\cos\theta) \\
&+ a_2(\sin\psi\sin\phi\sin\theta + \cos\phi\cos\psi) + a_3(\sin\theta\cos\phi\sin\psi - \sin\phi\cos\psi) \\
&- Y_{i,n-1}][-a_1\sin\psi\sin\theta + a_2\sin\psi\sin\phi\cos\theta + a_3(\cos\theta\cos\phi\sin\psi)] \qquad (4.54) \\
&+ [Z_c + a_1(-\sin\theta) + a_2(\sin\phi\cos\theta) + a_3(\cos\theta\cos\phi) - Z_{i,n-1}](-a_1\cos\theta \\
&- a_2\sin\phi\sin\theta - a_3\sin\theta\cos\phi)\};
\end{aligned}
$$

and

$$\frac{d}{dt}\left(\frac{\partial L_i}{\partial\dot{\phi}}\right) - \frac{\partial L_i}{\partial\phi} \qquad (4.55)$$

or,

$$
\begin{aligned}
&K_n[X_c + a_1(\cos\theta\cos\psi) + a_2(\sin\phi\sin\theta\cos\psi - \sin\psi\cos\phi) \\
&+ a_3(\sin\theta\cos\phi\cos\psi + \sin\phi\sin\psi) - X_{i,n-1}][a_2(\cos\phi\sin\theta\cos\psi \\
&+ \sin\psi\sin\phi) + a_3(-\sin\theta\sin\phi\cos\psi)] + K_n[Y_c + a_1(\sin\psi\cos\theta) \\
&+ a_2(\sin\psi\sin\phi\sin\theta + \cos\phi\cos\psi) + a_3(\sin\theta\cos\phi\sin\psi \\
&- \sin\phi\cos\psi) - Y_{i,n-1}][a_2(\sin\psi\cos\phi\sin\theta - \sin\phi\cos\psi) \\
&- a_3(\sin\theta\sin\phi\sin\psi + \cos\phi\cos\psi)] + K_n[Z_c + a_1(-\sin\theta) \\
&+ a_2(\sin\phi\cos\theta) + a_3(\cos\theta\cos\phi) - Z_{i,n-1}][a_2\cos\phi\cos\theta \\
&- a_3\cos\theta\sin\phi] - K_n l_n\{[X_c + a_1(\cos\theta\cos\psi) + a_2(\sin\phi\sin\theta\cos\psi \\
&- \sin\psi\cos\phi) + a_3(\sin\theta\cos\phi\cos\psi + \sin\phi\sin\psi) - X_{i,n-1}]^2 + [Y_c \\
&+ a_1(\sin\psi\cos\theta) + a_2(\sin\psi\sin\phi\sin\theta + \cos\phi\cos\psi) \\
&+ a_3(\sin\theta\cos\phi\sin\psi - \sin\phi\cos\psi) - Y_{i,n-1}]^2 + [Z_c + a_1(-\sin\theta) \\
&+ a_2(\sin\phi\cos\theta) + a_3(\cos\theta\cos\phi) - Z_{i,n-1}]^2\}^{-\frac{1}{2}}\{[X_c \\
&+ a_1(\cos\theta\cos\psi) + a_2(\sin\phi\sin\theta\cos\psi - \sin\psi\cos\phi) \\
&+ a_3(\sin\theta\cos\phi\cos\psi + \sin\phi\sin\psi) - X_{i,n-1}][a_2(\cos\phi\sin\theta\cos\psi + \sin\psi\sin\phi) \\
&+ a_3(-\sin\theta\sin\phi\cos\psi)] + [Y_c + a_1(\sin\psi\cos\theta) \\
&+ a_2(\sin\psi\sin\phi\sin\theta + \cos\phi\cos\psi) + a_3(\sin\theta\cos\phi\sin\psi \\
&- \sin\phi\cos\psi) - Y_{i,n-1}][a_2(\sin\psi\cos\phi\sin\theta - \sin\phi\cos\psi) - a_3(\sin\theta\sin\phi\sin\psi \\
&+ \cos\phi\cos\psi)] + [Z_c + a_1(-\sin\theta) + a_2(\sin\phi\cos\theta) \\
&+ a_3(\cos\theta\cos\phi) - Z_{i,n-1}][a_2\cos\phi\cos\theta - a_3\cos\theta\sin\phi]\}
\end{aligned}
\tag{4.56}
$$

For the cylinder, the Lagrangian, L_c, can be written as

$$
\begin{aligned}
L_c = T_c - V_c = \frac{1}{2}[&m_c(\dot{X}_c^2 + \dot{Y}_c^2 + \dot{Z}_c^2) + I_{xx}(\dot{\phi} - \dot{\psi}\sin\theta)^2 + I_{yy}(\dot{\theta}\cos\phi \\
&+ \dot{\psi}\sin\phi\cos\theta)^2 + I_{zz}(\dot{\psi}\cos\theta\cos\phi - \dot{\theta}\sin\phi)^2] + w_c Y_c
\end{aligned}
\tag{4.57}
$$

from which the following parts of the Lagrange's equations are obtained:

$$\frac{d}{dt}\left(\frac{\partial L_c}{\partial \dot{X}_c}\right) - \frac{\partial L_c}{\partial X_c} \tag{4.58}$$

or,

$$m_c \ddot{X}_c; \tag{4.59}$$

$$\frac{d}{dt}\left(\frac{\partial L_c}{\partial \dot{Y}_c}\right) - \frac{\partial L_c}{\partial Y_c} \tag{4.60}$$

or,

$$m_c \ddot{Y}_c - w_c; \tag{4.61}$$

$$\frac{d}{dt}\left(\frac{\partial L_c}{\partial \dot{Z}_c}\right) - \frac{\partial L_c}{\partial Z_c} \tag{4.62}$$

or,

$$m_c \ddot{Z}_c; \tag{4.63}$$

$$\frac{d}{dt}\left(\frac{\partial L_c}{\partial \dot{\psi}}\right) - \frac{\partial L_c}{\partial \psi} \tag{4.64}$$

or,

$$\begin{aligned}
&- I_{xx}[\dot{\phi}\dot{\theta}\cos\theta + \ddot{\phi}\sin\theta - 2\dot{\psi}\dot{\theta}\sin\theta\cos\theta - \ddot{\psi}\sin^2\theta] \\
&+ I_{yy}[\dot{\theta}\cos\phi(-\dot{\theta}\sin\phi\sin\theta + \dot{\phi}\cos\theta\cos\phi) + \sin\phi\cos\theta(-\dot{\phi}\dot{\theta}\sin\phi \\
&+ \ddot{\theta}\cos\phi) + \ddot{\psi}\sin^2\phi\cos^2\theta + \dot{\psi}(-2\dot{\theta}\sin^2\phi\cos\theta\sin\theta \\
&+ 2\dot{\phi}\sin\phi\cos\phi\cos^2\theta)] + I_{zz}[\dot{\psi}(-2\dot{\phi}\cos\phi\sin\phi\cos^2\theta \\
&- 2\dot{\theta}\cos\theta\sin\theta\cos^2\phi) + \ddot{\psi}\cos^2\theta\cos^2\phi + (\dot{\theta}\sin\phi)(\dot{\phi}\cos\theta\sin\phi \\
&+ \dot{\theta}\cos\phi\sin\theta) - \cos\theta\cos\phi(\dot{\phi}\dot{\theta}\cos\phi + \ddot{\theta}\sin\phi)];
\end{aligned} \tag{4.65}$$

$$\frac{d}{dt}\left(\frac{\partial L_c}{\partial \dot{\theta}}\right) - \frac{\partial L_c}{\partial \theta} \tag{4.66}$$

or,

$$\begin{aligned}
&I_{yy}[(-2\dot{\phi}\dot{\theta}\cos\phi\sin\phi + \ddot{\theta}\cos^2\theta - \dot{\psi}\sin\phi(\dot{\phi}\cos\theta\sin\phi + \dot{\theta}\cos\phi\sin\theta) \\
&+ \cos\theta\cos\phi(\dot{\psi}\dot{\phi}\cos\phi + \ddot{\psi}\sin\phi)] - I_{zz}[\dot{\psi}\cos\theta(\dot{\phi}\cos^2\phi - \dot{\phi}\sin^2\phi) \\
&+ \cos\phi\sin\phi(\ddot{\psi}\cos\theta - \dot{\theta}\dot{\psi}\sin\theta) - 2\dot{\phi}\dot{\theta}\cos\phi\sin\phi - \ddot{\theta}\sin^2\theta] \\
&- [I_{xx}(\dot{\phi} - \dot{\psi}\sin\theta)(-\dot{\psi}\cos\theta) + I_{yy}(\dot{\theta}\cos\phi + \dot{\psi}\sin\phi\cos\theta)(-\dot{\psi}\sin\phi\sin\theta) \\
&+ I_{zz}(\dot{\psi}\cos\theta\cos\phi - \dot{\theta}\sin\phi)(-\dot{\psi}\cos\phi\sin\theta)];
\end{aligned} \tag{4.67}$$

and

$$\frac{d}{dt}\left(\frac{\partial L_c}{\partial \dot{\phi}}\right) - \frac{\partial L_c}{\partial \phi} \tag{4.68}$$

or,

$$\begin{aligned}
&I_{xx}(\ddot{\phi} - \dot{\psi}\dot{\theta}\cos\theta - \ddot{\psi}\sin\theta) - [I_{yy}(\dot{\theta}\cos\phi + \dot{\psi}\sin\phi\cos\theta)(\dot{\psi}\cos\theta\cos\phi - \dot{\theta}\sin\phi) \\
&- I_{zz}(\dot{\psi}\cos\theta\cos\phi - \dot{\theta}\sin\phi)(\dot{\psi}\cos\theta\sin\phi + \dot{\theta}\cos\phi)]
\end{aligned} \tag{4.69}$$

Hence, the resulting Lagrange's equations include

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{X}_c}\right)-\frac{\partial L}{\partial X_c}=0 \tag{4.70}$$

or,

$$\begin{aligned}
&m_c\ddot{X}_c+K_n[X_c+a_1(\cos\theta\cos\psi)+a_2(\sin\phi\sin\theta\cos\psi-\sin\psi\cos\phi)\\
&+a_3(\sin\theta\cos\phi\cos\psi+\sin\phi\sin\psi)-X_{i,n-1}]-K_n l_n\{[X_c+a_1(\cos\theta\cos\psi)\\
&+a_2(\sin\phi\sin\theta\cos\psi-\sin\psi\cos\phi)\\
&+a_3(\sin\theta\cos\phi\cos\psi+\sin\phi\sin\psi)-X_{i,n-1}]^2+[Y_c+a_1(\sin\psi\cos\theta)\\
&+a_2(\sin\psi\sin\phi\sin\theta+\cos\phi\cos\psi)+a_3(\sin\theta\cos\phi\sin\psi-\sin\phi\cos\psi)\\
&-Y_{i,n-1}]^2+[Z_c+a_1(-\sin\theta)+a_2(\sin\phi\cos\theta)+a_3(\cos\theta\cos\phi)\\
&-Z_{i,n-1}]^2\}^{-\frac{1}{2}}[X_c+a_1(\cos\theta\cos\psi)+a_2(\sin\phi\sin\theta\cos\psi-\sin\psi\cos\phi)\\
&+a_3(\sin\theta\cos\phi\cos\psi+\sin\phi\sin\psi)-X_{i,n-1}]=0;
\end{aligned} \tag{4.71}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{Y}_c}\right)-\frac{\partial L}{\partial Y_c}=0 \tag{4.72}$$

or,

$$\begin{aligned}
&m_c\ddot{Y}_c-w_c+K_n[Y_c+a_1(\sin\psi\cos\theta)+a_2(\sin\psi\sin\phi\sin\theta+\cos\phi\cos\psi)\\
&+a_3(\sin\theta\cos\phi\sin\psi-\sin\phi\cos\psi)-Y_{i,n-1}]-K_n l_n\{[X_c+a_1(\cos\theta\cos\psi)\\
&+a_2(\sin\phi\sin\theta\cos\psi-\sin\psi\cos\phi)\\
&+a_3(\sin\theta\cos\phi\cos\psi+\sin\phi\sin\psi)-X_{i,n-1}]^2+[Y_c+a_1(\sin\psi\cos\theta)\\
&+a_2(\sin\psi\sin\phi\sin\theta+\cos\phi\cos\psi)+a_3(\sin\theta\cos\phi\sin\psi-\sin\phi\cos\psi)\\
&-Y_{i,n-1}]^2+[Z_c+a_1(-\sin\theta)+a_2(\sin\phi\cos\theta)+a_3(\cos\theta\cos\phi)-Z_{i,n-1}]^2\}^{-\frac{1}{2}}[Y_c\\
&+a_1(\sin\psi\cos\theta)+a_2(\sin\psi\sin\phi\sin\theta+\cos\phi\cos\psi)+a_3(\sin\theta\cos\phi\sin\psi\\
&-\sin\phi\cos\psi)-Y_{i,n-1}]=0;
\end{aligned} \tag{4.73}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{Z}_c}\right)-\frac{\partial L}{\partial Z_c}=0 \tag{4.74}$$

or,

$$
\begin{aligned}
&m_c \ddot{Z}_c + K_n [Z_c + a_1(-\sin\theta) + a_2(\sin\phi\cos\theta) + a_3(\cos\theta\cos\phi) \\
&- Z_{i,n-1}] - K_n l_n \{[X_c + a_1(\cos\theta\cos\psi) + a_2(\sin\phi\sin\theta\cos\psi - \sin\psi\cos\phi) \\
&+ a_3(\sin\theta\cos\phi\cos\psi + \sin\phi\sin\psi) - X_{i,n-1}]^2 + [Y_c + a_1(\sin\psi\cos\theta) \\
&+ a_2(\sin\psi\sin\phi\sin\theta + \cos\phi\cos\psi) + a_3(\sin\theta\cos\phi\sin\psi - \sin\phi\cos\psi) \\
&- Y_{i,n-1}]^2 + [Z_c + a_1(-\sin\theta) + a_2(\sin\phi\cos\theta) + a_3(\cos\theta\cos\phi) - Z_{i,n-1}]^2\}^{-\frac{1}{2}} [Z_c \\
&+ a_1(-\sin\theta) + a_2(\sin\phi\cos\theta) + a_3(\cos\theta\cos\phi) - Z_{i,n-1}] = 0;
\end{aligned}
\tag{4.75}
$$

$$
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\psi}}\right) - \frac{\partial L}{\partial \psi} = 0\,, \tag{4.76}
$$

or,

$$
\begin{aligned}
&-I_{xx}[\phi\dot{\theta}\cos\theta + \ddot{\phi}\sin\theta - 2\dot{\psi}\dot{\theta}\sin\theta\cos\theta - \ddot{\psi}\sin^2\theta] + I_{yy}[\dot{\theta}\cos\phi(-\dot{\theta}\sin\phi\sin\theta \\
&+ \dot{\phi}\cos\theta\cos\phi) + \sin\phi\cos\theta(-\dot{\phi}\dot{\theta}\sin\phi + \ddot{\theta}\cos\phi) + \ddot{\psi}\sin^2\phi\cos^2\theta \\
&+ \dot{\psi}(-2\dot{\theta}\sin^2\phi\cos\theta\sin\theta + 2\dot{\phi}\sin\phi\cos\phi\cos^2\theta)] + I_{zz}[\dot{\psi}(-2\dot{\phi}\cos\phi\sin\phi\cos^2\theta \\
&- 2\dot{\theta}\cos\theta\sin\theta\cos^2\phi) + \ddot{\psi}\cos^2\theta\cos^2\phi + (\dot{\theta}\sin\phi)(\dot{\phi}\cos\theta\sin\phi + \dot{\theta}\cos\phi\sin\theta) \\
&- \cos\theta\cos\phi(\dot{\phi}\dot{\theta}\cos\phi + \ddot{\theta}\sin\phi)] + K_n[X_c + a_1(\cos\theta\cos\psi) + a_2(\sin\phi\sin\theta\cos\psi \\
&- \sin\psi\cos\phi) + a_3(\sin\theta\cos\phi\cos\psi + \sin\phi\sin\psi) - X_{i,n-1}][-a_1\cos\theta\sin\psi \\
&- a_2(\sin\phi\sin\theta\sin\psi + \cos\psi\cos\phi) - a_3(\sin\theta\cos\phi\sin\psi - \sin\phi\cos\psi)] \\
&+ K_n[Y_c + a_1(\sin\psi\cos\theta) + a_2(\sin\psi\sin\phi\sin\theta + \cos\phi\cos\psi) + a_3(\sin\theta\cos\phi\sin\psi \\
&- \sin\phi\cos\psi) - Y_{i,n-1}][a_1\cos\theta\cos\psi + a_2(\cos\psi\sin\phi\sin\theta - \cos\phi\sin\psi) \\
&+ a_3(\sin\theta\cos\phi\cos\psi + \sin\phi\sin\psi)] - K_n l_n\{[X_c + a_1(\cos\theta\cos\psi) + \\
&a_2(\sin\phi\sin\theta\cos\psi - \sin\psi\cos\phi) \\
&+ a_3(\sin\theta\cos\phi\cos\psi + \sin\phi\sin\psi) - X_{i,n-1}]^2 + [Y_c + a_1(\sin\psi\cos\theta) \\
&+ a_2(\sin\psi\sin\phi\sin\theta + \cos\phi\cos\psi) + a_3(\sin\theta\cos\phi\sin\psi - \sin\phi\cos\psi) - Y_{i,n-1}]^2 \\
&+ [Z_c + a_1(-\sin\theta) + a_2(\sin\phi\cos\theta) + a_3(\cos\theta\cos\phi) - Z_{i,n-1}]^2\}^{-\frac{1}{2}}\{[X_c \\
&+ a_1(\cos\theta\cos\psi) + a_2(\sin\phi\sin\theta\cos\psi - \sin\psi\cos\phi) \\
&+ a_3(\sin\theta\cos\phi\cos\psi + \sin\phi\sin\psi) - X_{i,n-1}][-a_1\cos\theta\sin\psi - a_2(\sin\phi\sin\theta\sin\psi \\
&+ \cos\psi\cos\phi) - a_3(\sin\theta\cos\phi\sin\psi - \sin\phi\cos\psi)] + [Y_c + a_1(\sin\psi\cos\theta) \\
&+ a_2(\sin\psi\sin\phi\sin\theta + \cos\phi\cos\psi) + a_3(\sin\theta\cos\phi\sin\psi \\
&- \sin\phi\cos\psi) - Y_{i,n-1}][a_1\cos\theta\cos\psi + a_2(\cos\psi\sin\phi\sin\theta - \cos\phi\sin\psi) \\
&+ a_3(\sin\theta\cos\phi\cos\psi + \sin\phi\sin\psi)]\} = 0;
\end{aligned}
\tag{4.77}
$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \tag{4.78}$$

or,

$$\begin{aligned}
&I_{yy}[-2\dot{\phi}\dot{\theta}\cos\phi\sin\phi + \ddot{\theta}\cos^2\theta - \dot{\psi}\sin\phi(\dot{\phi}\cos\theta\sin\phi + \dot{\theta}\cos\phi\sin\theta) \\
&+ \cos\theta\cos\phi(\dot{\psi}\dot{\phi}\cos\phi + \ddot{\psi}\sin\phi)] - I_{zz}[\dot{\psi}\cos\theta(\dot{\phi}\cos^2\phi - \dot{\phi}\sin^2\phi) \\
&+ \cos\phi\sin\phi(\ddot{\psi}\cos\theta - \dot{\theta}\dot{\psi}\sin\theta) - 2\dot{\phi}\dot{\theta}\cos\phi\sin\phi - \ddot{\theta}\sin^2\theta] - [I_{xx}(\dot{\phi} \\
&- \dot{\psi}\sin\theta)(-\dot{\psi}\cos\theta) + I_{yy}(\dot{\theta}\cos\phi + \dot{\psi}\sin\phi\cos\theta)(-\dot{\psi}\sin\phi\sin\theta) + I_{zz}(\dot{\psi}\cos\theta\cos\phi \\
&- \dot{\theta}\sin\phi)(-\dot{\psi}\cos\phi\sin\theta)] + K_n[X_c + a_1(\cos\theta\cos\psi) + a_2(\sin\phi\sin\theta\cos\psi \\
&- \sin\psi\cos\phi) + a_3(\sin\theta\cos\phi\cos\psi + \sin\phi\sin\psi) - X_{i,n-1}][-a_1\cos\psi\sin\theta \\
&+ a_2(\sin\phi\cos\theta\cos\psi) + a_3(\cos\theta\cos\phi\cos\psi)] + K_n[Y_c + a_1(\sin\psi\cos\theta) \\
&+ a_2(\sin\psi\sin\phi\sin\theta + \cos\phi\cos\psi) + a_3(\sin\theta\cos\phi\sin\psi - \sin\phi\cos\psi) - Y_{i,n-1}] \\
&[-a_1\sin\psi\sin\theta + a_2\sin\psi\sin\phi\cos\theta + a_3(\cos\theta\cos\phi\sin\psi)] \\
&+ K_n[Z_c + a_1(-\sin\theta) + a_2(\sin\phi\cos\theta) + a_3(\cos\theta\cos\phi) - Z_{i,n-1}](-a_1\cos\theta \\
&- a_2\sin\phi\sin\theta - a_3\sin\theta\cos\phi) - K_n l_n\{[X_c + a_1(\cos\theta\cos\psi) + \\
&a_2(\sin\phi\sin\theta\cos\psi - \sin\psi\cos\phi) \\
&+ a_3(\sin\theta\cos\phi\cos\psi + \sin\phi\sin\psi) - X_{i,n-1}]^2 + [Y_c + a_1(\sin\psi\cos\theta) \\
&+ a_2(\sin\psi\sin\phi\sin\theta + \cos\phi\cos\psi) + a_3(\sin\theta\cos\phi\sin\psi - \sin\phi\cos\psi) - Y_{i,n-1}]^2 \\
&+ [Z_c + a_1(-\sin\theta) + a_2(\sin\phi\cos\theta) + a_3(\cos\theta\cos\phi) - Z_{i,n-1}]^2\}^{-\frac{1}{2}}\{[X_c + \\
&a_1(\cos\theta\cos\psi) + a_2(\sin\phi\sin\theta\cos\psi - \sin\psi\cos\phi) \\
&+ a_3(\sin\theta\cos\phi\cos\psi + \sin\phi\sin\psi) - X_{i,n-1}][-a_1\cos\psi\sin\theta + a_2(\sin\phi\cos\theta\cos\psi) \\
&+ a_3(\cos\theta\cos\phi\cos\psi)] + [Y_c + a_1(\sin\psi\cos\theta) \\
&+ a_2(\sin\psi\sin\phi\sin\theta + \cos\phi\cos\psi) + a_3(\sin\theta\cos\phi\sin\psi - \sin\phi\cos\psi) - Y_{i,n-1}] \\
&[-a_1\sin\psi\sin\theta + a_2\sin\psi\sin\phi\cos\theta + a_3(\cos\theta\cos\phi\sin\psi)] \\
&+ [Z_c + a_1(-\sin\theta) + a_2(\sin\phi\cos\theta) + a_3(\cos\theta\cos\phi) - Z_{i,n-1}](-a_1\cos\theta \\
&- a_2\sin\phi\sin\theta - a_3\sin\theta\cos\phi)\} = 0;
\end{aligned} \tag{4.79}$$

and

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) - \frac{\partial L}{\partial \phi} = 0 \tag{4.80}$$

or,

$$
\begin{aligned}
& I_{xx}(\ddot{\phi} - \dot{\psi}\dot{\theta}\cos\theta - \ddot{\psi}\sin\theta) - [I_{yy}(\dot{\theta}\cos\phi + \dot{\psi}\sin\phi\cos\theta)(\dot{\psi}\cos\theta\cos\phi - \dot{\theta}\sin\phi) \\
& - I_{zz}(\dot{\psi}\cos\theta\cos\phi - \dot{\theta}\sin\phi)(\dot{\psi}\cos\theta\sin\phi + \dot{\theta}\cos\phi)] \\
& + K_n[X_c + a_1(\cos\theta\cos\psi) + a_2(\sin\phi\sin\theta\cos\psi - \sin\psi\cos\phi) \\
& + a_3(\sin\theta\cos\phi\cos\psi + \sin\phi\sin\psi) - X_{i,n-1}][a_2(\cos\phi\sin\theta\cos\psi + \sin\psi\sin\phi) \\
& + a_3(-\sin\theta\sin\phi\cos\psi)] + K_n[Y_c + a_1(\sin\psi\cos\theta) \\
& + a_2(\sin\psi\sin\phi\sin\theta + \cos\phi\cos\psi) + a_3(\sin\theta\cos\phi\sin\psi \\
& - \sin\phi\cos\psi) - Y_{i,n-1}][a_2(\sin\psi\cos\phi\sin\theta - \sin\phi\cos\psi) - a_3(\sin\theta\sin\phi\sin\psi \\
& + \cos\phi\cos\psi)] + K_n[Z_c + a_1(-\sin\theta) + a_2(\sin\phi\cos\theta) \\
& + a_3(\cos\theta\cos\phi) - Z_{i,n-1}][a_2\cos\phi\cos\theta - a_3\cos\theta\sin\phi] - K_n l_n\{[X_c \\
& + a_1(\cos\theta\cos\psi) + a_2(\sin\phi\sin\theta\cos\psi - \sin\psi\cos\phi) \\
& + a_3(\sin\theta\cos\phi\cos\psi + \sin\phi\sin\psi) - X_{i,n-1}]^2 + [Y_c + a_1(\sin\psi\cos\theta) \\
& + a_2(\sin\psi\sin\phi\sin\theta + \cos\phi\cos\psi) + a_3(\sin\theta\cos\phi\sin\psi - \sin\phi\cos\psi) - Y_{i,n-1}]^2 \\
& + [Z_c + a_1(-\sin\theta) + a_2(\sin\phi\cos\theta) + a_3(\cos\theta\cos\phi) - Z_{i,n-1}]^2\}^{-\frac{1}{2}}\{[X_c \\
& + a_1(\cos\theta\cos\psi) + a_2(\sin\phi\sin\theta\cos\psi - \sin\psi\cos\phi) \\
& + a_3(\sin\theta\cos\phi\cos\psi + \sin\phi\sin\psi) - X_{i,n-1}][a_2(\cos\phi\sin\theta\cos\psi + \sin\psi\sin\phi) \\
& + a_3(-\sin\theta\sin\phi\cos\psi)] + [Y_c + a_1(\sin\psi\cos\theta) \\
& + a_2(\sin\psi\sin\phi\sin\theta + \cos\phi\cos\psi) + a_3(\sin\theta\cos\phi\sin\psi \\
& - \sin\phi\cos\psi) - Y_{i,n-1}][a_2(\sin\psi\cos\phi\sin\theta - \sin\phi\cos\psi) - a_3(\sin\theta\sin\phi\sin\psi \\
& + \cos\phi\cos\psi)] + [Z_c + a_1(-\sin\theta) + a_2(\sin\phi\cos\theta) \\
& + a_3(\cos\theta\cos\phi) - Z_{i,n-1}][a_2\cos\phi\cos\theta - a_3\cos\theta\sin\phi]\} = 0
\end{aligned}
\tag{4.81}
$$

4.8 SMALL MOTIONS ABOUT EQUILIBRIUM (LINEARIZED EQUATIONS)

In contrast to the previously defined complete equations of motion (EOM's) for the system, we now look at the linearized equations in order to more easily examine the behavior of small motions about the equilibrium point. It can be shown from equations (4.36), (4.38), and (4.40) that the EOM's for each of the lowest n-2 (to exclude n-1) point masses $m_{i,j}$ making up mooring line i, where $X_{i,j}(t) = X_{i,j,new}(t) + X_{i,j,eq}$, $Y_{i,j}(t) = Y_{i,j,new}(t) + Y_{i,j,eq}$, and $Z_{i,j}(t) = Z_{i,j,new}(t) + Z_{i,j,eq}$, are found as

$$
\begin{aligned}
&m_{i,j}\ddot{X}_{i,j,new} + K_{i,j}(X_{i,j,new} - X_{i,j-1,new})(1 - \frac{l_{i,j}}{\sqrt{G_{i,j}}}) + \\
&\frac{K_{i,j}l_{i,j}H_{i,j}}{2G_{i,j}^{3/2}}(X_{i,j,eq} - X_{i,j-1,eq}) - \\
&K_{i,j+1}(X_{i,j+1,new} - X_{i,j,new})(1 - \frac{l_{i,j+1}}{\sqrt{G_{i,j+1}}}) - \\
&\frac{K_{i,j+1}l_{i,j+1}H_{i,j+1}}{2G_{i,j+1}^{3/2}}(X_{i,j+1,eq} - X_{i,j,eq}) = 0;
\end{aligned}
\tag{4.82}
$$

$$
\begin{aligned}
&m_{i,j}\ddot{Y}_{i,j,new} + K_{i,j}(Y_{i,j,new} - Y_{i,j-1,new})(1 - \frac{l_{i,j}}{\sqrt{G_{i,j}}}) + \\
&\frac{K_{i,j}l_{i,j}H_{i,j}}{2G_{i,j}^{3/2}}(Y_{i,j,eq} - Y_{i,j-1,eq}) - \\
&K_{i,j+1}(Y_{i,j+1,new} - Y_{i,j,new})(1 - \frac{l_{i,j+1}}{\sqrt{G_{i,j+1}}}) - \\
&\frac{K_{i,j+1}l_{i,j+1}H_{i,j+1}}{2G_{i,j+1}^{3/2}}(Y_{i,j+1,eq} - Y_{i,j,eq}) = 0;
\end{aligned}
\tag{4.83}
$$

and

$$
\begin{aligned}
&m_{i,j}\ddot{Z}_{i,j,new} + K_{i,j}(Z_{i,j,new} - Z_{i,j-1,new})(1 - \frac{l_{i,j}}{\sqrt{G_{i,j}}}) + \\
&\frac{K_{i,j}l_{i,j}H_{i,j}}{2G_{i,j}^{3/2}}(Z_{i,j,eq} - Z_{i,j-1,eq}) - \\
&K_{i,j+1}(Z_{i,j+1,new} - Z_{i,j,new})(1 - \frac{l_{i,j+1}}{\sqrt{G_{i,j+1}}}) - \\
&\frac{K_{i,j+1}l_{i,j+1}H_{i,j+1}}{2G_{i,j+1}^{3/2}}(Z_{i,j+1,eq} - Z_{i,j,eq}) = 0
\end{aligned}
\tag{4.84}
$$

where

$$G_{i,j} = X_{i,j,eq}^2 + X_{i,j-1,eq}^2 - 2X_{i,j,eq}X_{i,j-1,eq} + Y_{i,j,eq}^2 + Y_{i,j-1,eq}^2 - 2Y_{i,j,eq}Y_{i,j-1,eq}$$
$$Z_{i,j,eq}^2 + Z_{i,j-1,eq}^2 - 2Z_{i,j,eq}Z_{i,j-1,eq}$$
$$H_{i,j} = A_{i,j}X_{i,j,new} + B_{i,j}X_{i,j-1,new} + C_{i,j}Y_{i,j,new} + D_{i,j}Y_{i,j-1,new} + E_{i,j}Z_{i,j,new} + F_{i,j}Z_{i,j-1,new}$$
$$A_{i,j} = 2(X_{i,j,eq} - X_{i,j-1,eq})$$
$$B_{i,j} = -A_{i,j}$$
$$C_{i,j} = 2(Y_{i,j,eq} - Y_{i,j-1,eq})$$
$$D_{i,j} = -C_{i,j}$$
$$E_{i,j} = 2(Z_{i,j,eq} - Z_{i,j-1,eq})$$
$$F_{i,j} = -E_{i,j}$$
(4.85)

for all j except j=n. Here the subscript “eq” refers to the equilibrium state under a net bouyant force, and the subscript “new” refers to a quantity relative to the equilibrium state. However, for mass n-1, linearizing equations (4.41) through (4.43) by ignoring all higher order terms in the “new” variables and letting $X_c(t) = X_{c,new}(t) + X_{c,eq}$, $Y_c(t) = Y_{c,new}(t) + Y_{c,eq}$, and $Z_c(t) = Z_{c,new}(t) + Z_{c,eq}$ leads to

$$m_{i,n-1}\ddot{X}_{i,n-1,new} + K_{i,n-1}(X_{i,n-1,new} - X_{i,n-2,new})(1 - \frac{l_{i,n-1}}{\sqrt{G_{i,n-1}}}) +$$
$$\frac{K_{i,n-1}l_{i,n-1}H_{i,n-1}}{2G_{i,n-1}^{3/2}}(X_{i,n-1,eq} - X_{i,n-2,eq}) + K_{i,n}(X_{c,new} - a_{i,2}\psi + a_{i,3}\theta \tag{4.86}$$
$$- X_{i,n-1,new})(1 - \frac{l_{i,n}}{\sqrt{G_{i,n}}}) + \frac{K_{i,n}l_{i,n}H_{i,n}}{2G_{i,n}^{3/2}}(a_{i,1} - X_{i,n-1,eq}) = 0;$$

$$m_{i,n-1}\ddot{Y}_{i,n-1,new} + K_{i,n-1}(Y_{i,n-1,new} - Y_{i,n-2,new})(1 - \frac{l_{i,n-1}}{\sqrt{G_{i,n-1}}})$$
$$+ \frac{K_{i,n-1}l_{i,n-1}H_{i,n-1}}{2G_{i,n-1}^{3/2}}(Y_{i,n-1,new} - Y_{i,n-2,new}) + K_{i,n}(Y_{c,new} + a_{i,1}\psi \tag{4.87}$$
$$- a_{i,3}\phi - Y_{i,n-1,new})(1 - \frac{l_{i,n}}{\sqrt{G_{i,n}}}) + \frac{K_{i,n}l_{i,n}H_{i,n}}{2G_{i,n}^{3/2}}(Y_{c,eq} + a_{i,2} - Y_{i,n-1,eq}) = 0;$$

and

$$m_{i,n-1}\ddot{Z}_{i,n-1,new} + K_{i,n-1}(Z_{i,n-1,new} - Z_{i,n-2,new})(1-\frac{l_{i,n-1}}{\sqrt{G_{i,n-1}}})$$
$$+\frac{K_{i,n-1}l_{i,n-1}H_{i,n-1}}{2G_{i,n-1}^{3/2}}(Z_{i,n-1,new} - Z_{i,n-2,new}) + K_{i,n}(Z_{c,new} - a_{i,1}\theta \tag{4.88}$$
$$+a_{i,2}\phi - Z_{i,n-1,new})(1-\frac{l_{i,n}}{\sqrt{G_{i,n}}}) + \frac{K_{i,n}l_{i,n}H_{i,n}}{2G_{i,n}^{3/2}}(a_{i,3} - Z_{i,n-1,eq}) = 0$$

where

$$G_{i,n} = a_{i,1}^2 - 2a_{i,1}X_{i,n-1,eq} + X_{i,n-1,eq}^2 + Y_{c,eq}^2 + 2a_{i,2}Y_{c,eq} - 2Y_{c,eq}Y_{i,n-1,eq} + a_{i,2}^2$$
$$- 2a_{i,2}Y_{i,n-1,eq} + Y_{i,n-1,eq}^2 + a_{i,3}^2 - 2a_{i,3}Z_{i,n-1,eq} + Z_{i,n-1,eq}^2$$
$$H_{i,n} = A_{i,n}X_{c,new} + B_{i,n}X_{c,new} + C_{i,n}Y_{c,new} + D_{i,n}\psi + E_{i,n}\theta + F_{i,n}\phi + R_{i,n}X_{i,n-1,new}$$
$$+ S_{i,n}Y_{i,n-1,new} + T_{i,n}Z_{i,n-1,new}$$
$$A_{i,n} = 2(a_{i,1} - X_{i,n-1,eq})$$
$$B_{i,n} = 2(Y_{c,eq} + a_{i,2} - Y_{i,n-1,eq})$$
$$C_{i,n} = 2(a_{i,3} - Z_{i,n-1,eq})$$
$$D_{i,n} = 2(a_{i,2}X_{i,n-1,eq} + a_{i,1}Y_{c,eq} - a_{i,1}Y_{i,n-1,eq})$$
$$E_{i,n} = 2(-a_{i,3}X_{i,n-1,eq} + a_{i,1}Z_{i,n-1,eq})$$
$$F_{i,n} = 2(-a_{i,3}Y_{c,eq} + a_{i,3}Y_{i,n-1,eq} - a_{i,2}Z_{i,n-1,eq})$$
$$R_{i,n} = -A_{i,n}$$
$$S_{i,n} = -B_{i,n}$$
$$T_{i,n} = -C_{i,n} \tag{4.89}$$

Here, it is assumed that the system has sufficient symmetry such that the initial and equilibrium values of θ, ϕ, and ψ are zero. Hence, no subscript “new” will be used for these quantities. Likewise, $X_{c,eq}$ and $Z_{c,eq}$ are zero and these terms are omitted in all equations.

Using the same procedure as above, linearizing equations (4.71), (4.73), (4.75), (4.77), (4.79), and (4.81) yields

$$m_c \ddot{X}_{c,new} + K_{i,n}(X_{c,new} - a_{i,2}\psi + a_{i,3}\theta - X_{i,n-1,new})(1 - \frac{l_{i,n}}{\sqrt{G_{i,n}}}) +$$
$$\frac{K_{i,n} l_{i,n} H_{i,n}}{2G_{i,n}^{3/2}}(a_{i,1} - X_{i,n-1,eq}) = 0; \tag{4.90}$$

$$m_c \ddot{Y}_{c,new} + K_{i,n}(Y_{c,new} + a_{i,1}\psi - a_{i,3}\phi - Y_{i,n-1,new})(1 - \frac{l_{i,n}}{\sqrt{G_{i,n}}}) +$$
$$\frac{K_{i,n} l_{i,n} H_{i,n}}{2G_{i,n}^{3/2}}(Y_{c,eq} + a_{i,2} - Y_{i,n-1,eq}) = 0; \tag{4.91}$$

$$m_c \ddot{Z}_{c,new} + K_{i,n}(Z_{c,new} - a_{i,1}\theta + a_{i,2}\phi - Z_{i,n-1,new})(1 - \frac{l_{i,n}}{\sqrt{G_{i,n}}}) +$$
$$\frac{K_{i,n} l_{i,n} H_{i,n}}{2G_{i,n}^{3/2}}(a_{i,3} - Z_{i,n-1,eq}) = 0; \tag{4.92}$$

$$I_{zz}\ddot{\psi} + K_{i,n}(-a_{i,2}X_{c,new} + a_{i,1}X_{i,n-1,eq}\psi - a_{i,3}X_{i,n-1,eq}\phi + a_{i,2}X_{i,n-1,new} + a_{i,1}Y_{c,new}$$
$$- a_{i,1}Y_{i,n-1,new} - a_{i,2}\psi Y_{c,eq} + a_{i,3}\theta Y_{c,eq} + a_{i,2}\psi Y_{i,n-1,eq} - a_{i,3}\theta Y_{i,n-1,eq})(1 - \frac{l_{i,n}}{\sqrt{G_{i,n}}}) \tag{4.93}$$
$$+ \frac{K_{i,n} l_{i,n} H_{i,n}}{2G_{i,n}^{3/2}}(a_{i,2}X_{i,n-1,eq} + a_{i,1}Y_{c,eq} - a_{i,1}Y_{i,n-1,eq}) = 0;$$

$$I_{yy}\ddot{\theta} + K_{i,n}(a_{i,3}X_{c,new} - a_{i,3}X_{i,n-1,new} + a_{i,1}X_{i,n-1,eq}\theta - a_{i,2}X_{i,n-1,eq}\phi + a_{i,3}Y_{c,eq}\psi$$
$$- a_{i,3}Y_{i,n-1,eq}\psi - a_{i,1}Z_{c,new} + a_{i,1}Z_{i,n-1,new} + a_{i,3}Z_{i,n-1,eq}\theta)(1 - \frac{l_{i,n}}{\sqrt{G_{i,n}}}) \tag{4.94}$$
$$+ \frac{K_{i,n} l_{i,n} H_{i,n}}{2G_{i,n}^{3/2}}(-a_{i,3}X_{i,n-1,eq} + a_{i,1}Z_{i,n-1,eq}) = 0;$$

and

$$I_{xx}\ddot{\phi} + K_{i,n}(-a_{i,2}X_{i,n-1,eq}\theta - a_{i,3}Y_{c,new} - a_{i,1}a_{i,3}\psi + a_{i,3}Y_{i,n-1,new} - a_{i,2}Y_{c,eq}\phi$$
$$+ a_{i,2}Y_{i,n-1,eq}\phi - a_{i,2}Z_{c,new} - a_{i,2}Z_{i,n-1,new} + a_{i,3}Z_{i,n-1,eq}\phi)(1 - \frac{l_{i,n}}{\sqrt{G_{i,n}}}) \tag{4.95}$$
$$+ \frac{K_{i,n} l_{i,n} H_{i,n}}{2G_{i,n}^{3/2}}(a_{i,3}Y_{i,n-1,eq} - a_{i,3}Y_{c,eq} - a_{i,2}Z_{i,n-1,eq}) = 0$$

Equations (4.82) through (4.84), (4.86) through (4.88), and (4.90) through (4.95) can be nondimensionalized by letting $a_{i,j}$ = $a_{i,j}$/R, $X_{c,new}$ = $X_{c,new}$/R, $Y_{c,new}$ = $Y_{c,new}$/R, $Z_{c,new}$ = $Z_{c,new}$/R, $X_{i,j,new}$ = $X_{i,j,new}$/R, $Y_{i,j,new}$ = $Y_{i,j,new}$/R, $Z_{i,j,new}$ = $Z_{i,j,new}$/R, $m_{i,j}$ = $m_{i,j}/m_c$, $m_c = m_c/m_c = 1$, $K_{i,j}$ = $K_{i,j}R/(m_c g)$, $I_{xx} = I_{xx}/(m_c R^2)$, $I_{yy} = I_{yy}/(m_c R^2)$, $I_{zz} = I_{zz}/(m_c R^2)$, and $\tau = t(g/R)^{1/2}$.

If overdots now represent derivatives with respect to τ, these equations become:

$$
\begin{aligned}
& m_{i,j}\ddot{X}_{i,j,new} + K_{i,j}(X_{i,j,new} - X_{i,j-1,new})(1 - \frac{l_{i,j}}{\sqrt{G_{i,j}}}) + \\
& \frac{K_{i,j} l_{i,j} H_{i,j}}{2G_{i,j}^{3/2}}(X_{i,j,eq} - X_{i,j-1,eq}) - \\
& K_{i,j+1}(X_{i,j+1,new} - X_{i,j,new})(1 - \frac{l_{i,j+1}}{\sqrt{G_{i,j+1}}}) - \\
& \frac{K_{i,j+1} l_{i,j+1} H_{i,j+1}}{2G_{i,j+1}^{3/2}}(X_{i,j+1,eq} - X_{i,j,eq}) = 0;
\end{aligned}
\tag{4.96}
$$

$$
\begin{aligned}
& m_{i,j}\ddot{Y}_{i,j,new} + K_{i,j}(Y_{i,j,new} - Y_{i,j-1,new})(1 - \frac{l_{i,j}}{\sqrt{G_{i,j}}}) + \\
& \frac{K_{i,j} l_{i,j} H_{i,j}}{2G_{i,j}^{3/2}}(Y_{i,j,eq} - Y_{i,j-1,eq}) - \\
& K_{i,j+1}(Y_{i,j+1,new} - Y_{i,j,new})(1 - \frac{l_{i,j+1}}{\sqrt{G_{i,j+1}}}) - \\
& \frac{K_{i,j+1} l_{i,j+1} H_{i,j+1}}{2G_{i,j+1}^{3/2}}(Y_{i,j+1,eq} - Y_{i,j,eq}) = 0;
\end{aligned}
\tag{4.97}
$$

$$
\begin{aligned}
& m_{i,j}\ddot{Z}_{i,j,new} + K_{i,j}(Z_{i,j,new} - Z_{i,j-1,new})(1 - \frac{l_{i,j}}{\sqrt{G_{i,j}}}) + \\
& \frac{K_{i,j} l_{i,j} H_{i,j}}{2G_{i,j}^{3/2}}(Z_{i,j,eq} - Z_{i,j-1,eq}) - \\
& K_{i,j+1}(Z_{i,j+1,new} - Z_{i,j,new})(1 - \frac{l_{i,j+1}}{\sqrt{G_{i,j+1}}}) - \\
& \frac{K_{i,j+1} l_{i,j+1} H_{i,j+1}}{2G_{i,j+1}^{3/2}}(Z_{i,j+1,eq} - Z_{i,j,eq}) = 0;
\end{aligned}
\tag{4.98}
$$

$$m_{i,n-1}\ddot{X}_{i,n-1,new} + K_{i,n-1}(X_{i,n-1,new} - X_{i,n-2,new})(1-\frac{l_{i,n-1}}{\sqrt{G_{i,n-1}}}) + \frac{K_{i,n-1}l_{i,n-1}H_{i,n-1}}{2G_{i,n-1}^{3/2}}(X_{i,n-1,eq} - X_{i,n-2,eq}) + K_{i,n}(X_{c,new} - a_{i,2}\psi + a_{i,3}\theta - X_{i,n-1,new})(1-\frac{l_{i,n}}{\sqrt{G_{i,n}}}) + \frac{K_{i,n}l_{i,n}H_{i,n}}{2G_{i,n}^{3/2}}(a_{i,1} - X_{i,n-1,eq}) = 0; \tag{4.99}$$

$$m_{i,n-1}\ddot{Y}_{i,n-1,new} + K_{i,n-1}(Y_{i,n-1,new} - Y_{i,n-2,new})(1-\frac{l_{i,n-1}}{\sqrt{G_{i,n-1}}}) + \frac{K_{i,n-1}l_{i,n-1}H_{i,n-1}}{2G_{i,n-1}^{3/2}}(Y_{i,n-1,new} - Y_{i,n-2,new}) + K_{i,n}(Y_{c,new} + a_{i,1}\psi - a_{i,3}\phi - Y_{i,n-1,new})(1-\frac{l_{i,n}}{\sqrt{G_{i,n}}}) + \frac{K_{i,n}l_{i,n}H_{i,n}}{2G_{i,n}^{3/2}}(Y_{c,eq} + a_{i,2} - Y_{i,n-1,eq}) = 0; \tag{4.100}$$

$$m_{i,n-1}\ddot{Z}_{i,n-1,new} + K_{i,n-1}(Z_{i,n-1,new} - Z_{i,n-2,new})(1-\frac{l_{i,n-1}}{\sqrt{G_{i,n-1}}}) + \frac{K_{i,n-1}l_{i,n-1}H_{i,n-1}}{2G_{i,n-1}^{3/2}}(Z_{i,n-1,new} - Z_{i,n-2,new}) + K_{i,n}(Z_{c,new} - a_{i,1}\theta + a_{i,2}\phi - Z_{i,n-1,new})(1-\frac{l_{i,n}}{\sqrt{G_{i,n}}}) + \frac{K_{i,n}l_{i,n}H_{i,n}}{2G_{i,n}^{3/2}}(a_{i,3} - Z_{i,n-1,eq}) = 0; \tag{4.101}$$

$$\ddot{X}_{c,new} + K_{i,n}(X_{c,new} - a_{i,2}\psi + a_{i,3}\theta - X_{i,n-1,new})(1-\frac{l_{i,n}}{\sqrt{G_{i,n}}}) + \frac{K_{i,n}l_{i,n}H_{i,n}}{2G_{i,n}^{3/2}}(a_{i,1} - X_{i,n-1,eq}) = 0; \tag{4.102}$$

$$\ddot{Y}_{c,new} + K_{i,n}(Y_{c,new} + a_{i,1}\psi - a_{i,3}\phi - Y_{i,n-1,new})(1-\frac{l_{i,n}}{\sqrt{G_{i,n}}}) + \frac{K_{i,n}l_{i,n}H_{i,n}}{2G_{i,n}^{3/2}}(Y_{c,eq} + a_{i,2} - Y_{i,n-1,eq}) = 0; \tag{4.103}$$

$$\ddot{Z}_{c,new} + K_{i,n}(Z_{c,new} - a_{i,1}\theta + a_{i,2}\phi - Z_{i,n-1,new})(1-\frac{l_{i,n}}{\sqrt{G_{i,n}}}) + \frac{K_{i,n}l_{i,n}H_{i,n}}{2G_{i,n}^{3/2}}(a_{i,3} - Z_{i,n-1,eq}) = 0; \tag{4.104}$$

$$
\begin{aligned}
& I_{zz}\ddot{\psi} + K_{i,n}(-a_{i,2}X_{c,new} + a_{i,1}X_{i,n-1,eq}\psi - a_{i,3}X_{i,n-1,eq}\phi + a_{i,2}X_{i,n-1,new} \\
& + a_{i,1}Y_{c,new} - a_{i,1}Y_{i,n-1,new} - a_{i,2}\psi Y_{c,eq} + a_{i,3}\theta Y_{c,eq} + a_{i,2}\psi Y_{i,n-1,eq} \\
& - a_{i,3}\theta Y_{i,n-1,eq})(1-\frac{l_{i,n}}{\sqrt{G_{i,n}}}) + \frac{K_{i,n}l_{i,n}H_{i,n}}{2G_{i,n}^{3/2}}(a_{i,2}X_{i,n-1,eq} + a_{i,1}Y_{c,eq} \\
& - a_{i,1}Y_{i,n-1,eq}) = 0;
\end{aligned} \tag{4.105}
$$

$$
\begin{aligned}
& I_{yy}\ddot{\theta} + K_{i,n}(a_{i,3}X_{c,new} - a_{i,3}X_{i,n-1,new} + a_{i,1}X_{i,n-1,eq}\theta - a_{i,2}X_{i,n-1,eq}\phi \\
& + a_{i,3}Y_{c,eq}\psi - a_{i,3}Y_{i,n-1,eq}\psi - a_{i,1}Z_{c,new} + a_{i,1}Z_{i,n-1,new} \\
& + a_{i,3}Z_{i,n-1,eq}\theta)(1-\frac{l_{i,n}}{\sqrt{G_{i,n}}}) + \frac{K_{i,n}l_{i,n}H_{i,n}}{2G_{i,n}^{3/2}}(-a_{i,3}X_{i,n-1,eq} + a_{i,1}Z_{i,n-1,eq}) = 0;
\end{aligned} \tag{4.106}
$$

and

$$
\begin{aligned}
& I_{xx}\ddot{\phi} + K_{i,n}(-a_{i,2}X_{i,n-1,eq}\theta - a_{i,3}Y_{c,new} - a_{i,1}a_{i,3}\psi + a_{i,3}Y_{i,n-1,new} - a_{i,2}Y_{c,eq}\phi \\
& + a_{i,2}Y_{i,n-1,eq}\phi - a_{i,2}Z_{c,new} - a_{i,2}Z_{i,n-1,new} + a_{i,3}Z_{i,n-1,eq}\phi)(1-\frac{l_{i,n}}{\sqrt{G_{i,n}}}) \\
& + \frac{K_{i,n}l_{i,n}H_{i,n}}{2G_{i,n}^{3/2}}(a_{i,3}Y_{i,n-1,eq} - a_{i,3}Y_{c,eq} - a_{i,2}Z_{i,n-1,eq}) = 0
\end{aligned} \tag{4.107}
$$

where

$$
\begin{aligned}
G_{i,j} &= X_{i,j,eq}^2 + X_{i,j-1,eq}^2 - 2X_{i,j,eq}X_{i,j-1,eq} + Y_{i,j,eq}^2 + Y_{i,j-1,eq}^2 - 2Y_{i,j,eq}Y_{i,j-1,eq} \\
& Z_{i,j,eq}^2 + Z_{i,j-1,eq}^2 - 2Z_{i,j,eq}Z_{i,j-1,eq} \\
H_{i,j} &= A_{i,j}X_{i,j,new} + B_{i,j}X_{i,j-1,new} + C_{i,j}Y_{i,j,new} + D_{i,j}Y_{i,j-1,new} + E_{i,j}Z_{i,j,new} + F_{i,j}Z_{i,j-1,new} \\
A_{i,j} &= 2(X_{i,j,eq} - X_{i,j-1,eq}) \\
B_{i,j} &= -A_{i,j} \\
C_{i,j} &= 2(Y_{i,j,eq} - Y_{i,j-1,eq}) \\
D_{i,j} &= -C_{i,j} \\
E_{i,j} &= 2(Z_{i,j,eq} - Z_{i,j-1,eq}) \\
F_{i,j} &= -E_{i,j}
\end{aligned} \tag{4.108}
$$

and

$$
\begin{aligned}
&G_{i,n} = a_{i,1}^2 - 2a_{i,1}X_{i,n-1,eq} + X_{i,n-1,eq}^2 + Y_{c,eq}^2 + 2a_{i,2}Y_{c,eq} - 2Y_{c,eq}Y_{i,n-1,eq} + a_{i,2}^2 \\
&- 2a_{i,2}Y_{i,n-1,eq} + Y_{i,n-1,eq}^2 + a_{i,3}^2 - 2a_{i,3}Z_{i,n-1,eq} + Z_{i,n-1,eq}^2 \\
&H_{i,n} = A_{i,n}X_{c,new} + B_{i,n}X_{c,new} + C_{i,n}Y_{c,new} + D_{i,n}\psi + E_{i,n}\theta + F_{i,n}\phi + R_{i,n}X_{i,n-1,new} \\
&+ S_{i,n}Y_{i,n-1,new} + T_{i,n}Z_{i,n-1,new} \\
&A_{i,n} = 2(a_{i,1} - X_{i,n-1,eq}) \\
&B_{i,n} = 2(Y_{c,eq} + a_{i,2} - Y_{i,n-1,eq}) \\
&C_{i,n} = 2(a_{i,3} - Z_{i,n-1,eq}) \\
&D_{i,n} = 2(a_{i,2}X_{i,n-1,eq} + a_{i,1}Y_{c,eq} - a_{i,1}Y_{i,n-1,eq}) \\
&E_{i,n} = 2(-a_{i,3}X_{i,n-1,eq} + a_{i,1}Z_{i,n-1,eq}) \\
&F_{i,n} = 2(-a_{i,3}Y_{c,eq} + a_{i,3}Y_{i,n-1,eq} - a_{i,2}Z_{i,n-1,eq}) \\
&R_{i,n} = -A_{i,n} \\
&S_{i,n} = -B_{i,n} \\
&T_{i,n} = -C_{i,n}
\end{aligned}
\tag{4.109}
$$

4.9 STANDARD CASE

The previously stated linearized equations of motion can now be written in matrix form as an eigenvalue problem from which the vibration frequencies, mode shapes, and small motions of the system about its equilibrium state can be determined. However, given the size of each equation and the number of variables (6 for the cylinder and 3 per mass per mooring line), the procedure will be performed similarly to the one used in Chapter 3 where three separate cases were considered. These include: Case A, where the lines are modeled as massless springs; Case B, where the mass of each mooring line is lumped into one mass; and Case C, where the mass of each mooring line is lumped into some chosen finite number of equal masses. As detailed in Chapter 3, the structure is configured in such a way that only two spring stiffnesses will be considered. Hence, each line will contain a series of springs of a common and known stiffness. K_A will refer to the stiffness of each spring making up lines 1 and 4 (previously tagged i) and K_B will refer to those making up lines 2, 3, 5, and 6 as shown in Figure 4.5. In addition, the points where the mooring lines connect to the cylinder are known to have one value of each coordinate equal to zero, which will help simplify the EOM's. These are $a_{1,3}$, $a_{2,2}$, $a_{3,2}$, $a_{4,3}$, $a_{5,2}$, and $a_{6,2}$.

4.10 CASE A (MASSLESS LINES)

Case A, the most simple case to be considered, should still show much about the small motion of the cylinder. The system of linearized equations can now be written in matrix form as

$$[m]\{\ddot{q}\}+[s]\{q\}=0 \tag{4.110}$$

where q represents a general vector of variables and [m] and [s] are the inertia and stiffness matrices, respectively.

First, the vector $\{q\}^T$ is defined as

$$\{q\}^T=\{X_{c,new} \quad Y_{c,new} \quad Z_{c,new} \quad \psi \quad \theta \quad \phi\} \tag{4.111}$$

Here, only six EOM's will be involved, making the eigenvalue problem associated with this condition quite attractive. The inertia matrix [m] is easy to develop from the EOMs (4.102)-(4.107) with n=1 and $X_{i,0,new} = Y_{i,0,new} = Z_{i,0,new} = 0$ as

$$[m]=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{zz} & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{yy} & 0 \\ 0 & 0 & 0 & 0 & 0 & I_{xx} \end{bmatrix} \tag{4.112}$$

The stiffness matrix [s] can be easily obtained as well, but due to the complexity shown in equations (4.102) – (4.107) for one mooring line, a listing for six mooring lines becomes too exhaustive and only the numerical results will be provided.

The non-dimensional eigenvalues obtained by solving the eigenvalue problem for the standard case (L=30 ft, R=5 ft, a_T=10 ft, b_T= 15 ft, K_A=197,970 plf, and K_B=131,980 plf) are found in matrix form to be

$$[\omega^2]=[46.2 \quad 205.3 \quad 675.7 \quad 802.4 \quad 808.5 \quad 1{,}188.2]^T \tag{4.113}$$

which can be written as dimensional frequencies (rad/s) as

$$[\omega]=[17.3 \quad 36.4 \quad 66.0 \quad 71.9 \quad 72.2 \quad 87.5]^T \tag{4.114}$$

with the following normalized eigenvectors:

$$[u]=\begin{bmatrix} 0 & 0.99 & 0 & 0 & 0 & -0.52 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0.74 & 0 & 0 & 0.65 & 0 & 0 \\ 0 & 0.14 & 0 & 0 & 0 & 0.86 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ -0.68 & 0 & 0 & 0.76 & 0 & 0 \end{bmatrix} \tag{4.115}$$

where each column represents the eigenvector corresponding to the matching natural frequency. The sum of the squares of the elements of each eigenvector is unity. It is easy to see that each natural frequency causes the structure to vibrate in a unique, but fairly simple, fashion. The third eigenvalue results in a motion of the cylinder's center of mass only in the Y (heave) direction, whereas the fifth eigenvector describes a rotation only in the θ (pitch) direction. Since these motions are intuitive with reference to Figure 4.1, figures will not be shown here. The coupled motions occur in the first, second, fourth, and sixth eigenvectors, which are shown as Figures 4.6, 4.7, 4.8 and 4.9, respectively. Eigenvectors one and four involve sway and roll, whereas two and six involve surge and pitch. The solid lines represent the equilibrium configuration. Figures 4.6 and 4.8 show end views, while Figures 4.7 and 4.9 show side views.

Non-dimensionalizing the EOM's has allowed for solutions for various cases of L/R in addition to the standard case. As shown graphically in Figure 4.10, the fourth, fifth, and sixth natural frequencies (nondimensional) tend to decrease nonlinearly with respect to increasing L/R. The second and third natural frequencies increase nonlinearly for very low L/R. However, they reach a maximum for L/R less than that of the standard case (L/R = 6) and then decrease. The first natural frequency is hardly affected by L/R. For very low L/R values a slight increase can be seen. However, for the most part, it is constant after that. It should be noted that a change in L/R affects the entire inertia matrix. This is probably the reason for such sharp and contrasting curves. A plot of ω vs. $K/(\rho_c g R^2)$ is also shown as Figure 4.11. Here, all six natural frequencies increase nonlinearly with respect to increasing $K/(\rho_c g R^2)$.

4.11 CASE B (ONE MASS PER LINE)

For case B, each mooring line will be modeled as two springs with a central mass representing the weight of the entire line. Each mass contributes three new unknowns to the system, which results in a 24x24 eigenvalue problem. For convenience, only the non-zero elements of the mass matrix will be identified here. Here, the vector $\{q\}^T$ is defined as

$$\{q\}^T = \{ X_{c,new} \quad Y_{c,new} \quad Z_{c,new} \quad \psi \quad \theta \quad \phi \quad X_{1,1,new} \quad Y_{1,1,new} \quad Z_{1,1,new} \quad X_{2,1,new} \quad Y_{2,1,new} \quad Z_{2,1,new} \quad X_{3,1,new} \quad Y_{3,1,new} \quad Z_{3,1,new} \quad X_{4,1,new} \quad Y_{4,1,new} \quad Z_{4,1,new} \quad X_{5,1,new} \quad Y_{5,1,new} \quad Z_{5,1,new} \quad X_{6,1,new} \quad Y_{6,1,new} \quad Z_{6,1,new} \} \tag{4.116}$$

Using (4.99) through (4.107), the inertia matrix in (4.110) contains the elements

$$\begin{array}{lllll}
m_{1,1} = m_c & m_{2,2} = m_c & m_{3,3} = m_c & m_{4,4} = I_{ZZ} & m_{5,5} = I_{YY} \\
m_{6,6} = I_{XX} & m_{7,7} = m_{1,1} & m_{8,8} = m_{1,1} & m_{9,9} = m_{1,1} & m_{10,10} = m_{2,1} \\
m_{11,11} = m_{2,1} & m_{12,12} = m_{2,1} & m_{13,13} = m_{3,1} & m_{14,14} = m_{3,1} & m_{15,15} = m_{3,1} \\
m_{16,1} = m_{4,1} & m_{17,17} = m_{4,1} & m_{18,18} = m_{4,1} & m_{19,19} = m_{5,1} & m_{20,20} = m_{5,1} \\
m_{21,21} = m_{5,1} & m_{22,22} = m_{6,1} & m_{23,23} = m_{6,1} & m_{24,24} = m_{6,1} &
\end{array} \tag{4.117}$$

The values of K_A and K_B are twice as high as the corresponding values in Case A, since the springs are half as long. Solving this eigenvalue problem results in the following natural frequencies in rad/s:

$$\begin{array}{lllll}
\omega_1=17.21 & \omega_2=36.32 & \omega_3=65.89 & \omega_4=71.83 & \omega_5=71.98 \\
\omega_6=87.28 & \omega_7=191.22 & \omega_8=191.22 & \omega_9=191.22 & \omega_{10}=191.29 \\
\omega_{11}=191.36 & \omega_{12}=191.36 & \omega_{13}=191.39 & \omega_{14}=191.41 & \omega_{15}=286.54 \\
\omega_{16}=286.69 & \omega_{17}=286.70 & \omega_{18}=286.71 & \omega_{19}=1180.4 & \omega_{20}=1181.0 \\
\omega_{21}=1181.0 & \omega_{22}=1,881.1 & \omega_{23}=2,302.6 & \omega_{24}=2,302.9 &
\end{array} \tag{4.118}$$

The non-zero mode shape terms for the first six natural frequencies are as follows:

$$\begin{array}{lllll}
u_{3,1}=0.42 & u_{6,1}=-0.39 & u_{9,1}=0.041 & u_{11,1}=-0.20 & u_{12,1}=0.21 \\
u_{14,1}=0.20 & u_{15,1}=0.21 & u_{18,1}=0.041 & u_{20,1}=-0.20 & u_{21,1}=0.21 \\
u_{23,1}=0.20 & u_{24,1}=0.21 & & & \\
u_{1,2}=0.57 & u_{4,2}=0.08 & u_{7,2}=0.33 & u_{8,2}=-0.12 & u_{10,2}=0.30 \\
u_{11,2}=-0.12 & u_{13,2}=0.3 & u_{14,2}=-0.12 & u_{16,2}=0.33 & u_{17,2}=0.12 \\
u_{19,2}=0.29 & u_{20,2}=0.12 & u_{22,2}=0.29 & u_{23,2}=0.12 &
\end{array}$$

$u_{2,3}$=0.62 $u_{8,3}$=0.31 $u_{11,3}$=0.32 $u_{14,3}$=0.32 $u_{17,3}$=0.31
$u_{20,3}$=0.32 $u_{23,3}$=0.32
$u_{3,4}$=0.45 $u_{6,4}$=0.54 $u_{9,4}$=-0.04 $u_{11,4}$=0.27 $u_{6,4}$=0.23
$u_{14,4}$=-0.27 $u_{15,4}$=0.23 $u_{18,4}$=-0.04 $u_{20,4}$=0.27 $u_{21,4}$=0.23
$u_{23,4}$=-0.27 $u_{24,4}$=0.23
$u_{5,5}$=-0.23 $u_{9,5}$=-0.38 $u_{10,5}$=0.14 $u_{12,5}$=-0.38 $u_{13,5}$=-0.14
$u_{15,5}$=-0.38 $u_{18,5}$=0.38 $u_{19,5}$=0.14 $u_{21,5}$=0.38 $u_{22,5}$=-0.14
$u_{24,5}$=0.38
$u_{1,6}$=-0.14 $u_{4,6}$=0.23 $u_{7,6}$=0.06 $u_{8,6}$=-0.37 $u_{10, 6}$=-0.09
$u_{11,6}$=-0.39 $u_{13,6}$=-0.09 $u_{14,6}$=-0.39 $u_{16,6}$=0.06 $u_{17,6}$=0.37
$u_{19,6}$=-0.09 $u_{20,6}$=0.39 $u_{22,6}$=-0.09 $u_{23,6}$=0.39

(4.119)

where for $u_{i,j}$, i and j represent an independent variable and corresponding natural frequency, respectively. Here, given that the relatively high seventh natural frequency and above are of less interest, the mode shapes corresponding to these frequencies are not listed. On the other hand, it should be noted that the first six natural frequencies have decreased slightly as a result of the added masses and splitting a single spring into two springs with half the length and twice the stiffness (so that the total stiffness of the line doesn't change). Excluding the third mode shape where the cylinder and mass vibrate only in the Y direction, plots of the first six mode shapes are shown as Figures 4.12 through 4.16. Again, the solid lines represent the equilibrium configuration of the system. Figures 4.12 and 4.14 show end views, Figures 4.13 and 4.16 show side views, and Figure 4.15 is a top view of the breakwater. The small circles shown in all the figures are the lumped masses that are located at the center of each mooring line. It is helpful to visualize the mooring lines extending from the cylinder's ends through the masses, and to the ocean floor. In all the figures, the masses tend to "go with" the motion of the breakwater as expected.

4.12 CASE C (TWO MASSES PER LINE)

For case C, each mooring line will be modeled as three springs with two equal masses whose weights sum to equal the weight of the entire line. Each mass contributes three new unknowns to the system, which now yields a 42x42 eigenvalue problem. Following the same procedure discussed in Sections 4.10 and 4.11, the first six natural frequencies are found in rad/s to be

ω_1=17.23 ω_2=36.34 ω_3=65.92 ω_4=71.84 ω_5=72.01

ω_6=87.33 (4.120)

which are practically the same as previously obtained in Case B. Due to the fact that the mode shapes for these frequencies correspond to motions of the cylinder and mooring lines that are essentially the same as for Case B, they will not be shown here.

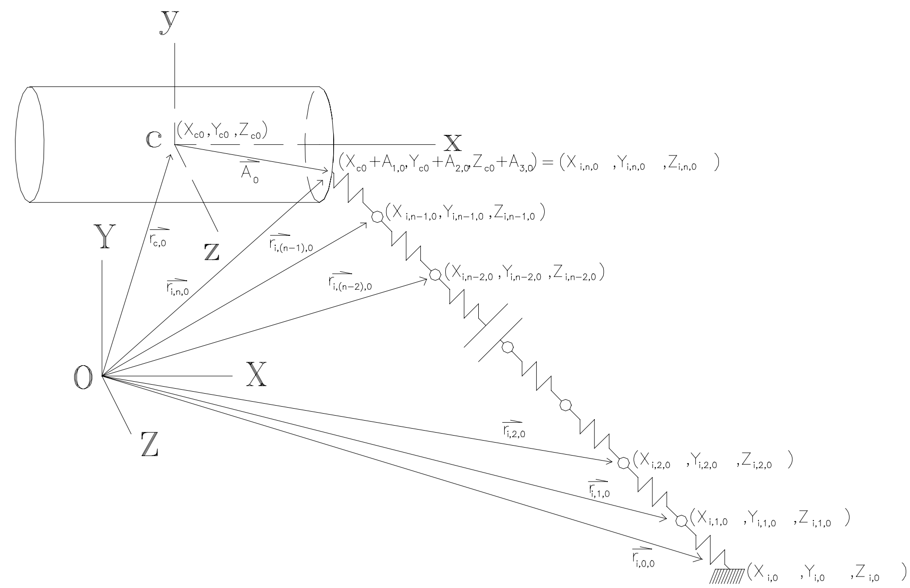

Figure 4.1 – “Unstretched Springs” Configuration – General System

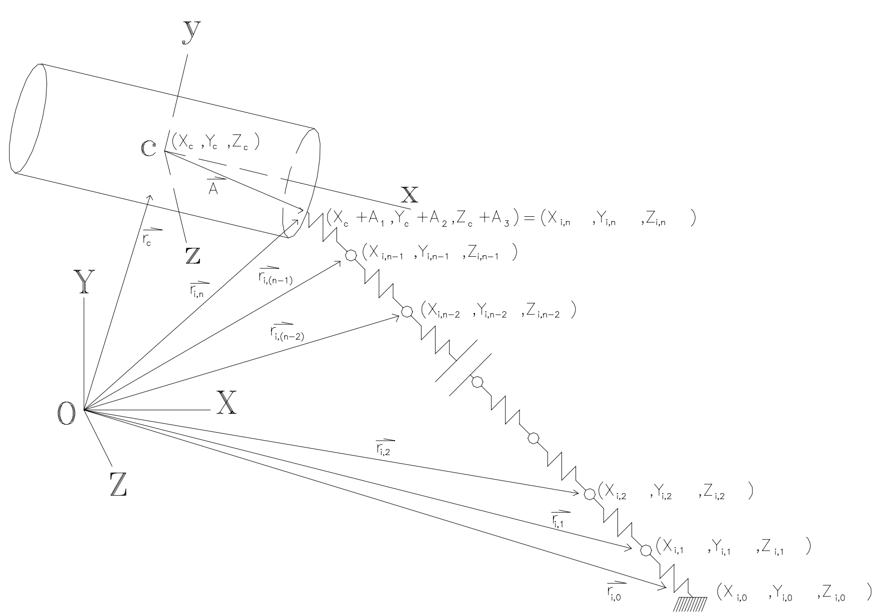

Figure 4.2 – General Configuration During Motion

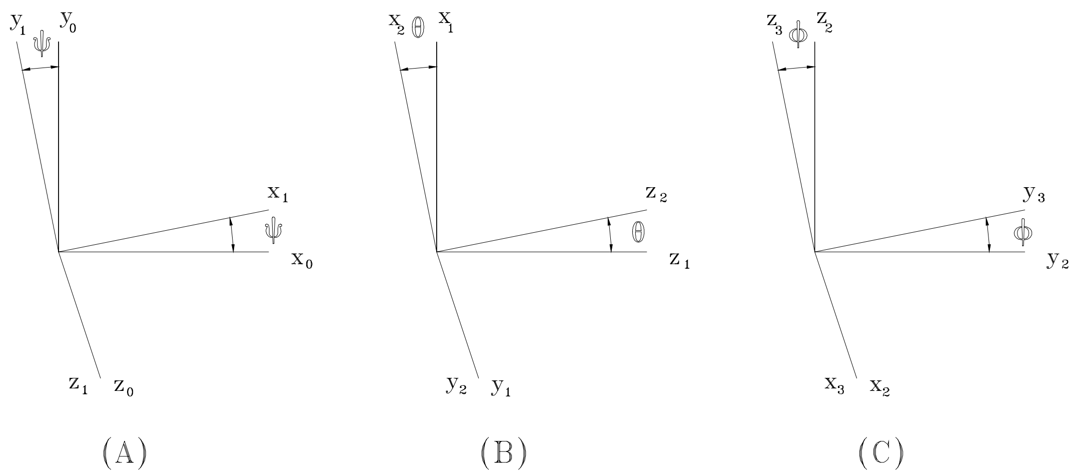

Figure 4.3 – Euler Angles

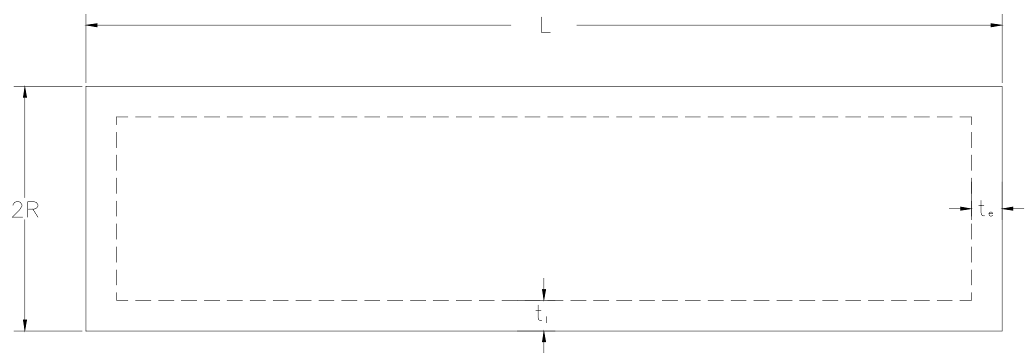

Figure 4.4 – Cylinder Dimensions

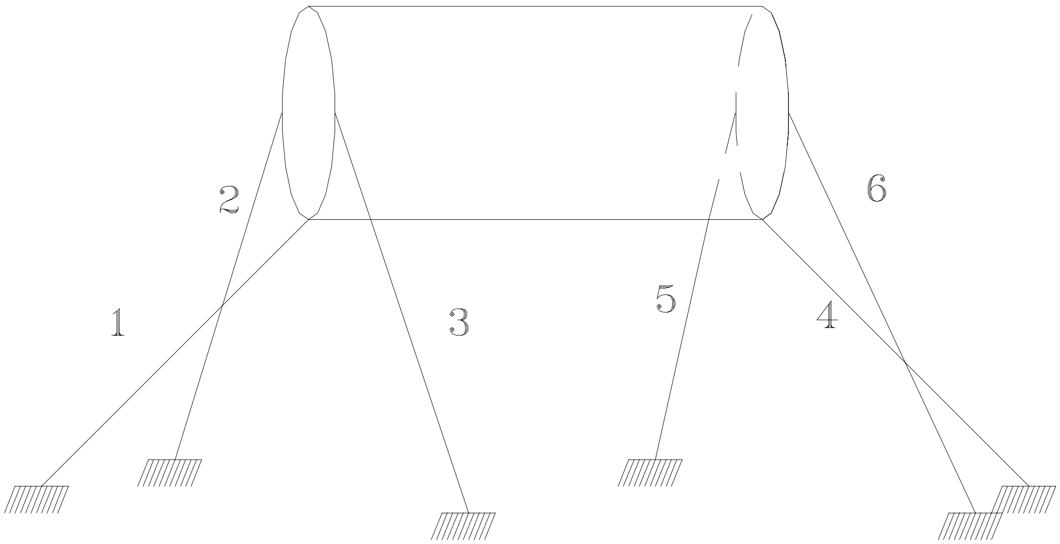

Figure 4.5 – Mooring Line Identification

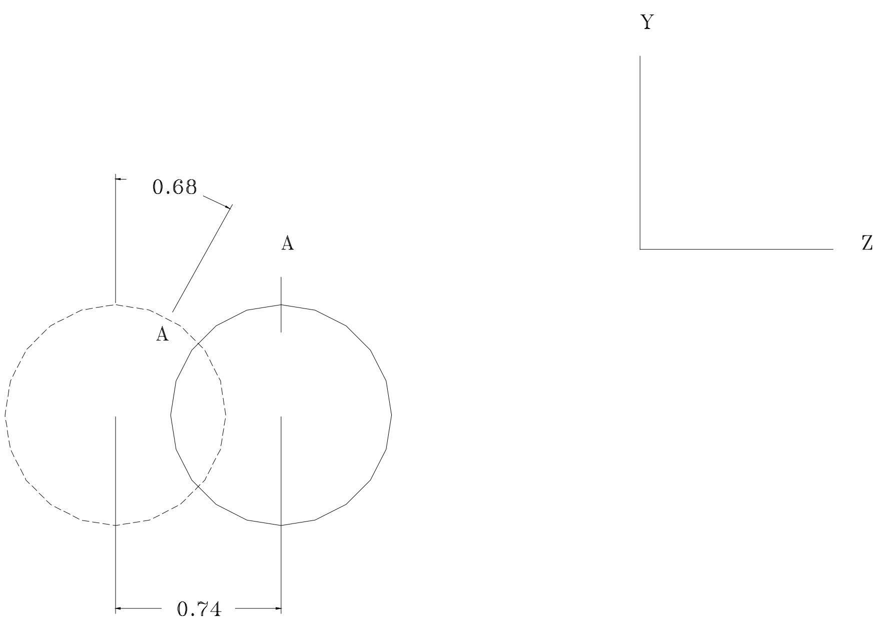

Figure 4.6 – Mode Shape (First Eigenvector – Case A)

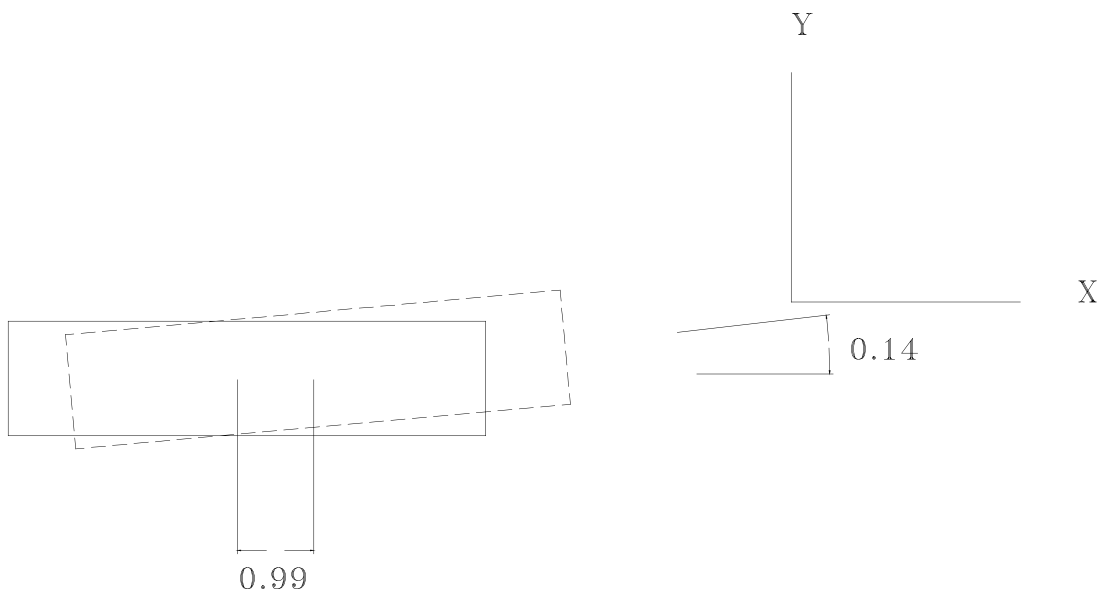

Figure 4.7 – Mode Shape (Second Eigenvector – Case A)

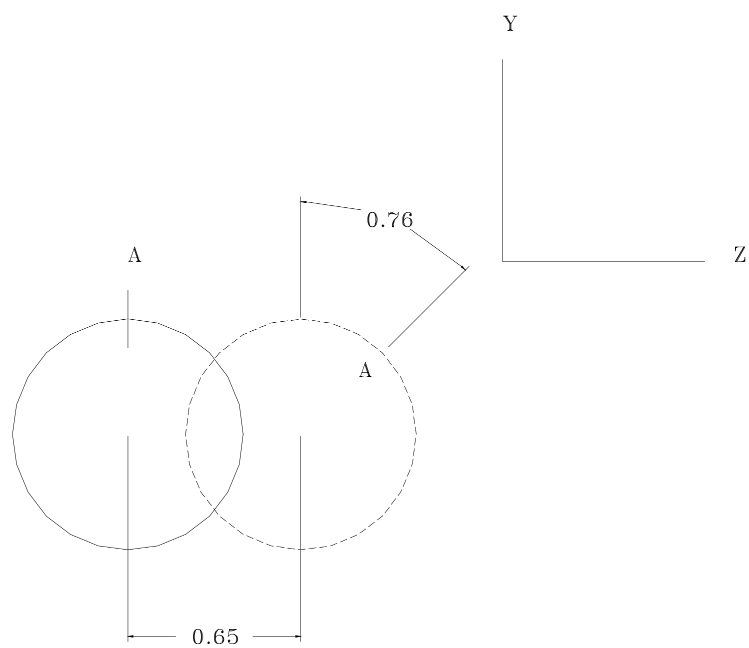

Figure 4.8 – Mode Shape (Fourth Eigenvector – Case A)

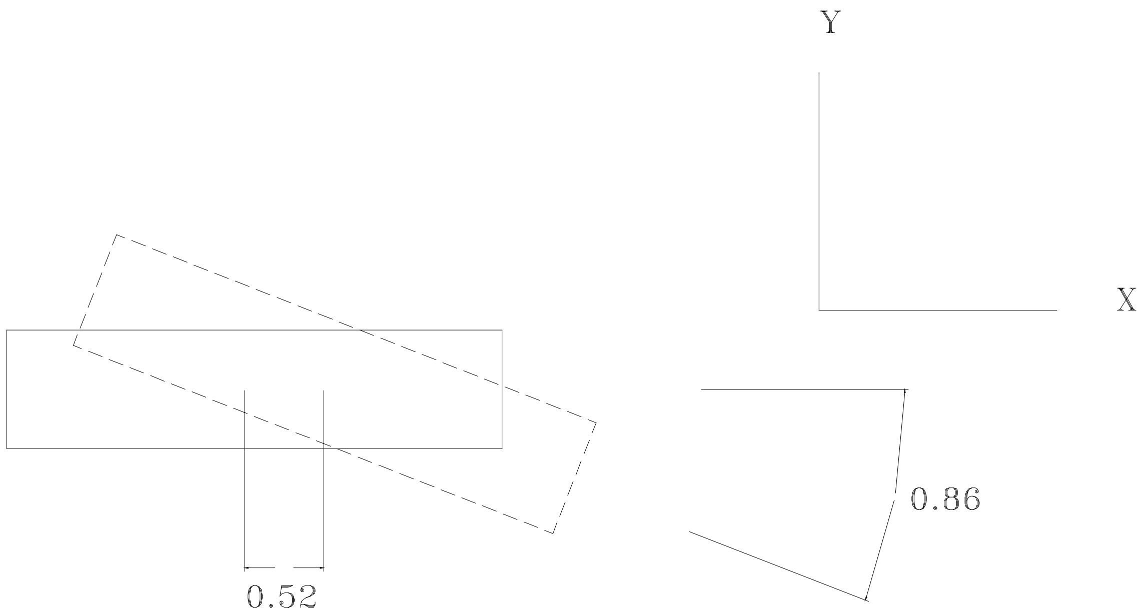

Figure 4.9 – Mode Shape (Sixth Eigenvector – Case A)

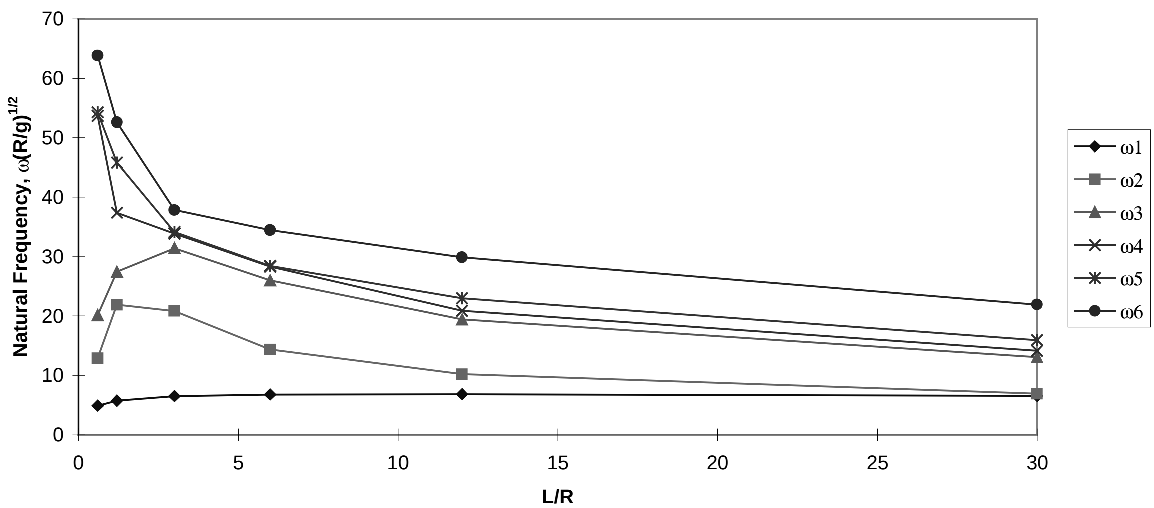

Figure 4.10 – Natural Frequency (Nondimensional) vs. L/R – Case A

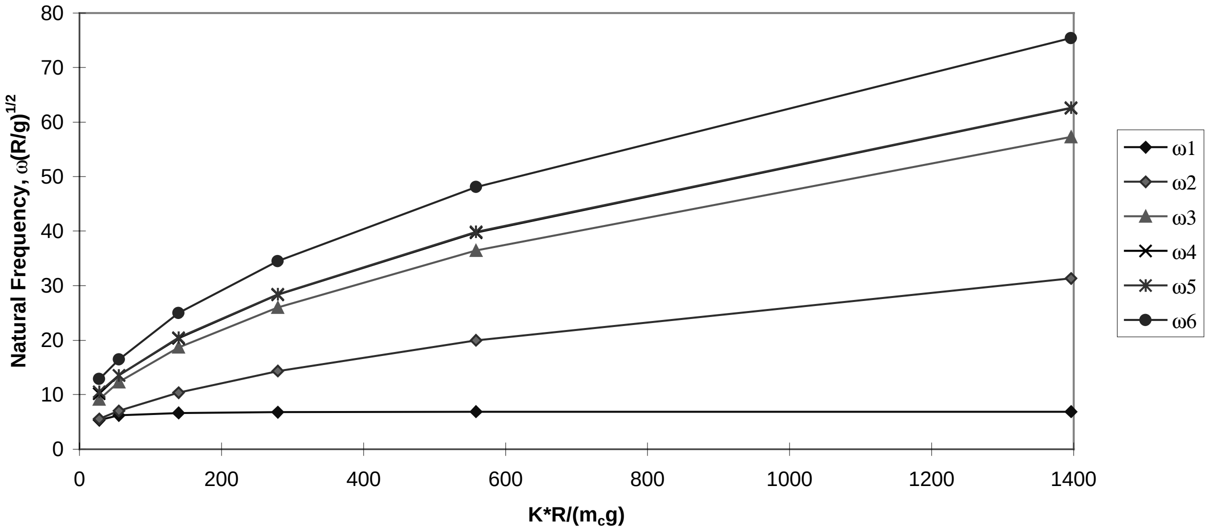

Figure 4.11 – Natural Frequency (Nondimensional) vs. K (Nondimensional)
Case A

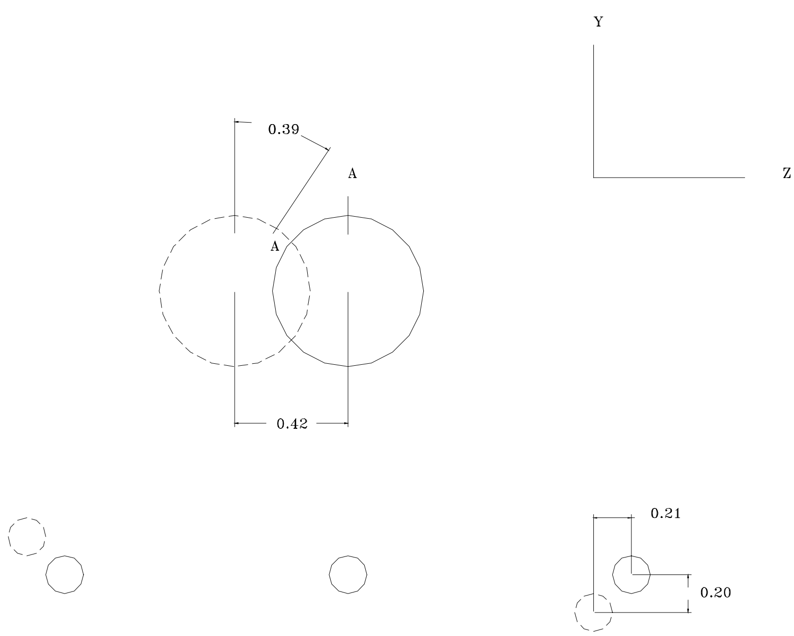

Figure 4.12 – Mode Shape (First Eigenvector – Case B)

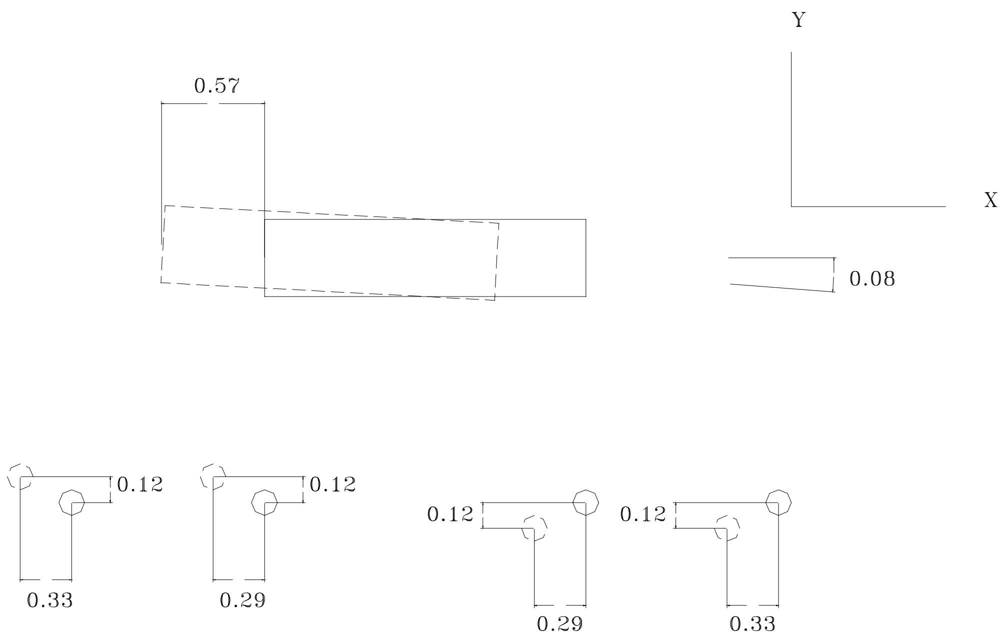

Figure 4.13 – Mode Shape (Second Eigenvector – Case B)

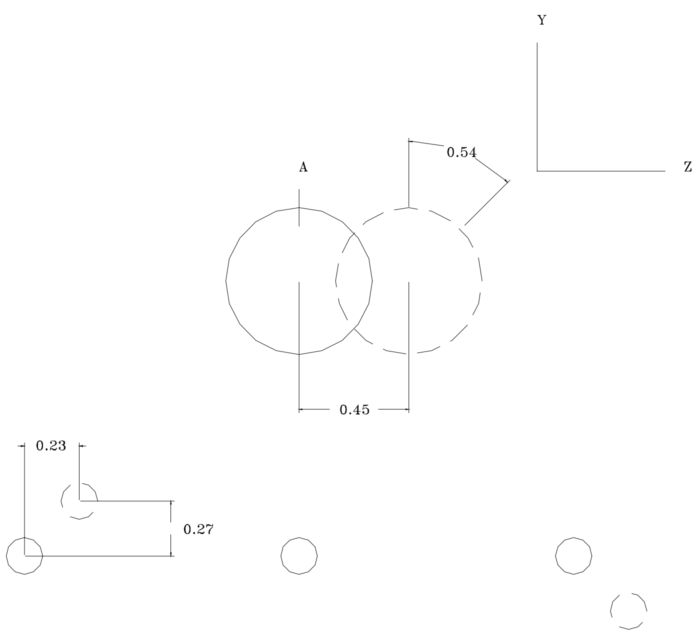

Figure 4.14 – Mode Shape (Fourth Eigenvector – Case B)

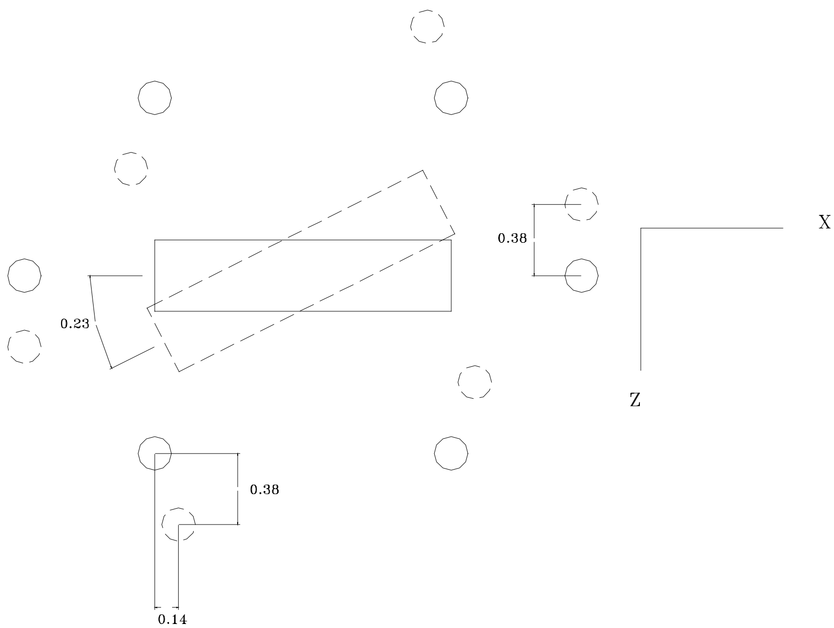

Figure 4.15 – Mode Shape (Fifth Eigenvector – Case B)

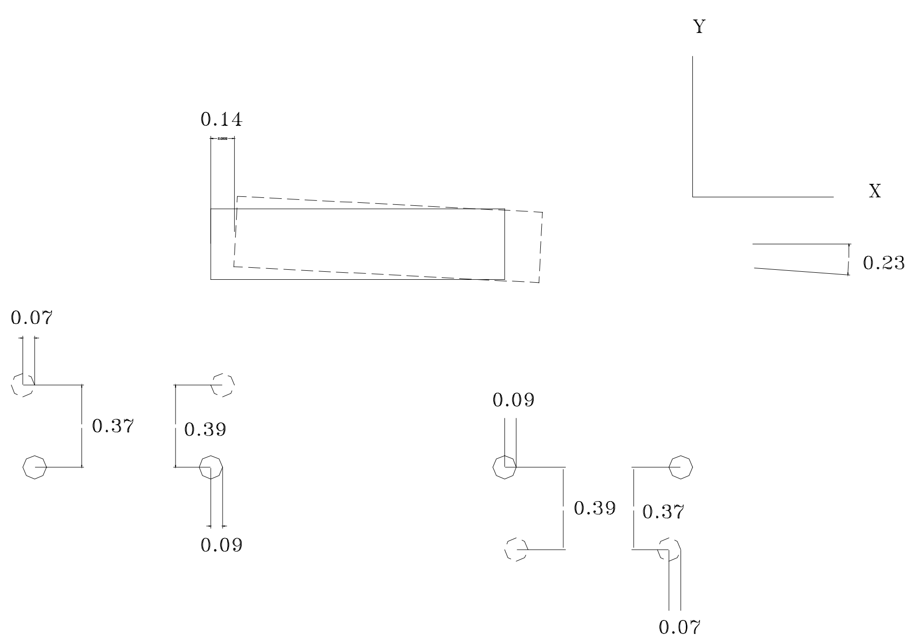

Figure 4.16 – Mode Shape (Sixth Eigenvector – Case B)