

AN INTRODUCTION TO GEOMETRICALLY
NONLINEAR FINITE ELEMENT STIFFNESS ANALYSIS
USING THE INCREMENTAL STEP AND EQUILIBRIUM METHODS

by

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CHAPTER I

INTRODUCTION

Nonlinear structural analysis is divided into two categories:

1. physically nonlinear structural analysis, and
2. geometrically nonlinear structural analysis.

The former is characterized by nonlinear or piecewise linear constitutive relationships; the latter is characterized by nonlinear strain-displacement relations and equilibrium formulations based on the deformed state. The two nonlinear effects are independent of each other. Examples of each are:

1. creep and elastic-plastic transition, and
2. large-deflections of beams and beam-columns.

This thesis will be concerned with geometrically nonlinear analysis of plane frame structures.

The field of geometrically nonlinear finite element stiffness analysis has been investigated over the past twelve years and many authors have contributed to the wealth of knowledge that has accumulated. In many cases the important concepts have been developed simultaneously by two or more authors, causing some duplicity and occasional discrepancies among the publications.

The intent of this thesis is to abstract the work of other authors in a consistent and uniform manner. The stiffness matrices for a beam-column element are derived, and two basic methods (the incremental step method and the equilibrium method) are presented.

The beam-column element is chosen for two reasons:

1. The derivation of the matrices for this element is intricate enough to illustrate the underlying principles involved, but not so complicated as to be impractical for this study.
2. The stiffness method, using the flexural beam element and loaded only at the nodes, is considered to be exact, i.e. discretization errors associated with higher dimensional elements (e.g. approximate displacement functions and discontinuities at element boundaries) are not present in the beam element. Thus the only errors developed in this non-linear analysis are caused by the numerical solution process. The roundoff errors are virtually eliminated by the use of appropriate programming techniques.

Many small but significant details that are not dealt with in the original publications are brought into focus. Consistency in the assumptions and restrictions imposed on the derivation of element matrices is maintained. Two algorithms are presented to highlight the basic steps of these two methods. Finally, three example problems follow which illustrate and further clarify the procedures. The problems are solved using a computer program based on the algorithms presented. It is assumed that the reader has a basic knowledge of the stiffness method and is familiar with the notation involved. It is hoped that the reader, after studying the material in this thesis, will have a better insight into the problems and solutions of geometrically nonlinear finite element analysis and a greater appreciation for the underlying original works.

CHAPTER II

HISTORY OF THE GEOMETRICALLY NONLINEAR FINITE ELEMENT METHOD

The history of the development of the incremental step method and of the equilibrium method will now be given.

The Incremental Step Method

The pioneering publication in the field of geometrically nonlinear finite element analysis [1] is the work of M. J. Turner, et al. In this paper, the concept of the incremental step and the geometric stiffness matrix are introduced. The elements considered are the simple truss member and the plane stress triangle. The authors' development of the force-displacement relation consists of direct substitution of the governing compatibility and constitutive equations into the Taylor series expansion of the axial force function and formulation of the conditions of equilibrium.

Additional early investigations of the incremental step method are presented in unpublished reports to The Boeing Company [2,3,4] between 1960 and 1962. H. C. Martin [5] relates these early developments in an excellent review of the literature through mid-1965.

Also, Ref. 5 develops the stiffness matrices for the truss member, the beam-column element, and the plane stress triangle. The concept of the incremental step is illustrated and the stiffness matrices are derived using energy considerations.

In a report to The Jet Propulsion Laboratories [6], the nonlinear strain-displacement relation is studied in detail. Explicit numerical examples are also presented; the stiffness formulation is similar to that described by Martin in Ref. 5.

J. S. Przemieniecki [7] presents essentially the same material for the two one-dimensional elements discussed in Ref. 5. However, his formulation is a more simplified alternative to the formulations presented in Refs. 5 and 6, and the format is that of a student text. Also included is a detailed discussion of the incremental step procedure. In another publication, Przemieniecki presents a general method for developing arbitrary nonlinear elements [8].

Finally, J. H. Argyris [9] discusses the general incremental approach and extends applications to plate and shell elements [10,11].

The Equilibrium Method

The second widely used method for geometrically nonlinear analysis is the equilibrium approach. The most common variation, which is essentially the Newton-Raphson method for solving the nonlinear equilibrium equations, uses the unbalanced nodal force vector corresponding to a prescribed set of nodal forces in solving for a new set of nodal displacements. This particular variation will be called the Newton-Raphson equilibrium method. Zienkiewicz [12] gives a basic description of the Newton-Raphson equilibrium method; and other authors [13,14] develop versions of the equilibrium method for specific elements. Murray and Wilson [14] use a version similar to

the Newton-Raphson equilibrium method for determining solutions of plate problems and present a detailed algorithm for the entire method. Finally, Livesly [15] presents a detailed discussion of the equilibrium method and the more popular variations.

Other Methods

Connor, et al. [16], describe a method using both the incremental approach and an iterative equilibrium correction procedure at the end of each incremental step. This method is implemented in the Strudl System [17].

Stricklin, et al. [18], give a brief history and comparison of elements and procedures used up to 1970. This paper also mentions some of the lesser known procedures available.

In this thesis, the derivation of the beam-column element matrices and the application of the incremental step method as found in Refs. 5, 6, and 7 are used. Also, the equilibrium method as found in Refs. 14 and 15 is modified for use with the beam-column element.

CHAPTER III

DEVELOPMENT OF THE GEOMETRICALLY NONLINEAR STIFFNESS

ELEMENT, THE INCREMENTAL STEP METHOD, AND

THE EQUILIBRIUM METHOD

This chapter will present the development of the geometrically nonlinear beam-column element matrices based upon the nonlinear strain-displacement equation. An energy approach will be used. The incremental step method and the equilibrium method will also be described. The same beam-column element will be used in both methods.

Geometrical Nonlinearities and the Geometrically

Nonlinear Strain-Displacement Equation

Geometrical nonlinearities may be introduced into the formulation of a mathematical model of a structure in two ways [6]:

1. by including the effects of rotation of the differential element in the strain-displacement equations, and
2. by establishing equilibrium for the rotated differential element.

The nonlinear effects in (1) above may be illustrated in the following manner. Suppose a differential rectangular element is elongated along the x-axis in pure strain, i.e. the element does not go through any rotations. The classical theory of elasticity assumes the strain is small as compared to unity (hence, $(\frac{\partial u}{\partial x})^2 \approx 0$), and the resulting expression for the strain is

$$\epsilon_{xx} = \frac{ds_f - ds_i}{ds_i} = \frac{\partial u}{\partial x}$$

where ds_f is the length of the element after deformation, and ds_i is the length of the element before deformation. Now, the element is elongated and rotated (strain and rotation are still considered small); the final configuration of the element will be a skewed parallelogram. The length, ds_f , is a function of the mean orientation of the element, and the dependence of the strain on the rotations of the element is evident from

$$\epsilon_{xx} = \frac{ds_f - ds_i}{ds_i}$$

The nonlinear effects in (2) may best be illustrated by examining a simple axially loaded truss member. If the element rotates and the nodal axial forces translate with the nodes, but do not rotate, then an unbalanced moment develops; this moment is equal to the product of the axial force and the relative transverse nodal displacements. In classical theory, this moment is considered negligible and does not enter into the formulation of the equilibrium equations. However, in nonlinear theory this moment is considered appreciable and is, therefore, included in the development of the equilibrium equations. To restore equilibrium to the deformed (rotated) member, equal and opposite transverse nodal forces must produce a couple to balance the moment. The model must include provisions for developing these transverse forces.

Strain Considerations

Only the axial extensional strains and the axial bending strains will be considered significant in the development of the beam-column element. All other strains will be neglected. The total strain may be expressed by

$$\epsilon = \epsilon_a + \epsilon_b \quad (1)$$

where ϵ_a is the axial extensional strain and ϵ_b is the bending strain parallel to the longitudinal axis.

Only the final expression for ϵ_a will be given in this thesis. Novozhilov has published an introductory text on nonlinear elasticity [19], and Martin has abstracted this work with respect to the finite elements he develops in Ref. 6.

The expression given by these authors is

$$\epsilon_a = \frac{\partial u}{\partial x} + \frac{1}{2} \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial x} \right)^2 + \left(\frac{\partial w}{\partial x} \right)^2 \right] \quad (2)$$

where u, v, w , are the displacements in the x, y, z directions, respectively.

Eq. 2 is simplified by noting

$$\frac{\partial u}{\partial x} \ll 1 \Rightarrow \left(\frac{\partial u}{\partial x} \right)^2 \approx 0$$

and specialized for the plane by setting $\partial w / \partial x = 0$. Eq. 2 becomes

$$\epsilon_a = \frac{\partial u}{\partial x} + \frac{1}{2} \left(\frac{\partial v}{\partial x} \right)^2 \quad (3)$$

In developing the nonlinear expression for ϵ_a , certain restrictions on the deformations and rotations of the differential element have been imposed. These restrictions must also be imposed upon the derivation of the beam-column element and the development of the incremental step and equilibrium methods. These restrictions are:

1. Strains are assumed to be small when compared with unity.
2. The angles of rotation are assumed to be small when compared with unity.

Classical bending theory is used to determine ϵ_b ,

$$\epsilon_b = -\frac{y}{\rho} = \frac{-y \frac{\partial^2 v}{\partial x^2}}{[1 + (\frac{\partial v}{\partial x})^2]^{3/2}} \approx -y \frac{\partial^2 v}{\partial x^2} \quad (4)$$

$$\frac{\partial v}{\partial x} \ll 1 \implies (\frac{\partial v}{\partial x})^2 \approx 0$$

where $\frac{1}{\rho}$ is the curvature and y is distance from the neutral axis to the strained fiber of the element. Positive y is associated with negative strains for positive curvature.

In simplifying Eq. 4, the restriction that the bending rotations are small is imposed by setting $(\partial v / \partial x)^2 \approx 0$. This restriction seems to be inconsistent with expression for ϵ_a , where $(\partial v / \partial x)^2$ is considered appreciable. Further examination shows that consistency between Eq. 3 and Eq. 4 is preserved. Eq. 3 is the sum of two small quantities, neither of which is negligible with respect to the other; both, however, are small in comparison to unity. The denominator in the

middle term of Eq. 4 is the sum of a very small number and unity. In this case $(\partial v / \partial x)^2$ is negligible with respect to the other term in the denominator. The decision whether or not to include a small quantity in a summation of terms is based upon the relative magnitude of the term in question as compared with the remaining terms in the expression.

The expression for the total strain, ϵ , is obtained by substituting Eq. 3 and Eq. 4 into Eq. 1,

$$\epsilon = \frac{\partial u}{\partial x} + \frac{1}{2} \left(\frac{\partial v}{\partial x} \right)^2 - y \frac{\partial^2 v}{\partial x^2} \quad (5)$$

Further development of the beam-column element will use Eq. 5 as the expression for strain in the element.

Large Rotations of Assembled Elements

In large deflections of beams, large rotations usually occur. If the restriction on rotations, outlined in the previous section, is imposed upon the beam-column element, then another aspect of the stiffness method must provide for the large rotations. This is accomplished by finite discontinuities of the slope between elements. Thus, the orientation of the stiffness element represents the mean rotation of that particular segment of the continuum the element represents. The bending of the stiffness element provides for local variations from the mean rotation along the length of the element. Significant error may be introduced at the nodes by this method. As Fig. 1b shows, the axial forces have an outward component at the intersection of two

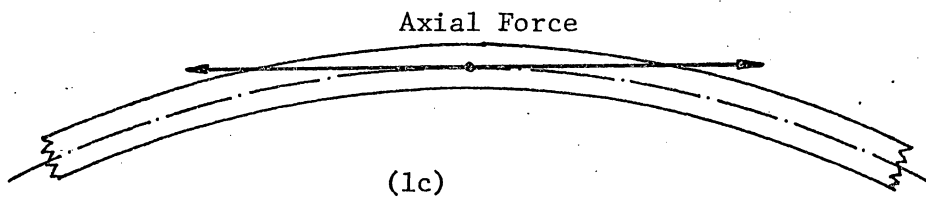
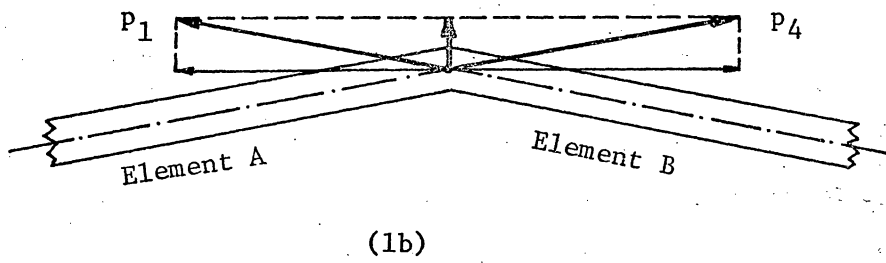
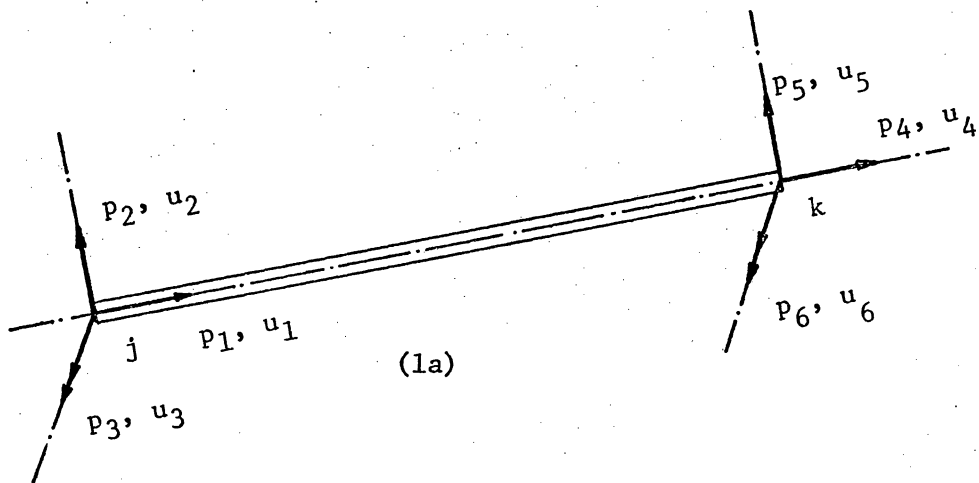


Figure 1 The Beam-Column Element

elements. As the number of elements in the idealization increase, this outward component decreases. At the limit, the continuum representation of the structure (Fig. 1c) is obtained, and no outward component is produced.

Development of the Beam-Column Element

Both Martin [5,6] and Przemieniecki [7] have derived the beam-column element using similar procedures. In this thesis, a combination of the two approaches will be used.

The Displacement Function

Both authors have selected the displacement functions as

$$\begin{aligned} v\{x\} &= \alpha_1 + \alpha_2 x + \alpha_3 x^2 + \alpha_4 x^3 \\ u\{x\} &= \alpha_5 + \alpha_6 x \\ \theta\{x\} &= \frac{\partial v}{\partial x} = \alpha_2 + 2\alpha_3 x + 3\alpha_4 x^2 \end{aligned} \quad (6)$$

where $v\{x\}$ is the transverse displacement, $u\{x\}$ is the axial displacement, $\theta\{x\}$ is the rotation along the element and α_1, α_2 , etc. are as yet undetermined coefficients.

Both authors cite the basic requirement for a stiffness element displacement function; i.e. the function must admit rigid body motions (α_1, α_5) , constant shear forces and constant axial strains $(\frac{\partial v}{\partial x}, \frac{\partial u}{\partial x} : \alpha_2, \alpha_6)$, and linearly varying moments $(\frac{\partial^2 v}{\partial x^2} : \alpha_4 x)$. However, in selecting the displacement functions, neither author considers

the possible coupling effects between axial and transverse displacements.¹ Thus, the resulting displacement functions, $v\{x\}$ and $u\{x\}$, are independent of each other. The error produced by neglecting the coupling effects is negligible [16].

The number of coefficients in the displacement functions is determined by the number of nodal displacements for each element. For the beam-column element, the total number of nodal displacements is six; these displacements are shown in the positive sense in Fig. 1a.

Other mathematical models may have displacement functions, which may be represented by polynomials of higher order than the cubic function. An example of such a model is Euler's differential equation solution for buckling loads of simple columns. In this case, the deformation of the column is described by the sine function. Expanding the sine function in the Maclaurin series,²

$$v_E\{x\} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

is obtained.

¹Connor, et al [16] develop a more exact expression for the force-displacement relation of this element which includes bending and axial-displacement coupling. The resulting equation is impossible to solve; it is, therefore, simplified by neglecting the coupling effects; the stiffness matrices are essentially the same as Martin's and Przemieniecky's matrices.

²Any continuous displacement function that has continuous first n derivatives may be written in polynomial form by expanding the function in a Taylor series.

The cubic function $v\{x\}$ can only approximate the first two terms of the Maclaurin form of the sine displacement function. Evaluating Eqs. 6 at the nodes by setting $x = 0$ and $x = l$, and solving the resulting two simultaneous equations associated with $u\{x\}$ for α_5 , α_6 , and solving the resulting four simultaneous equations associated with $v\{x\}$ and $\theta\{x\}$ for $\alpha_1, \alpha_2, \alpha_3, \alpha_4$, and finally substituting the expressions for the α 's back into Eqs. 6 and collecting like terms, the following expressions are obtained

$$u\{x\} = (1 - \frac{x}{l})u_1 + \frac{x}{l}u_4 \quad (7)$$

$$v\{x\} = (1 - \frac{3x^2}{l^2} + \frac{2x^3}{l^3})u_2 + (x - \frac{2x^2}{l} + \frac{x^3}{l^2})u_3 + (\frac{3x^2}{l^2} - \frac{2x^3}{l^3})u_5 + (-\frac{x^2}{l} + \frac{x^3}{l^2})u_6$$

where u_1, u_2 , etc. are the six nodal displacements.

Force-Displacement Relation by Castigliano's First Theorem

Castigliano's first theorem may be used to relate the nodal forces, p , to the nodal displacements, u . This theorem states that, in general,

$$p_i = \frac{\partial U_\epsilon}{\partial u_i} \quad (8)$$

where p_i is force in the same direction as the displacement u_i , and U_ϵ is the total internal strain energy for the deformed body. If the total internal strain energy is expressed in terms of the nodal displacements, each p_i may be found by differentiating U_ϵ with respect to the corresponding nodal displacement u_i , $i=1, 2, \dots 6$.

Total Internal Strain Energy for the Beam-Column Element

The total internal strain energy may be expressed by

$$U_{\epsilon} = \frac{1}{2} \int_{Vol} \sigma \epsilon \, dVol = \frac{E}{2} \int \epsilon^2 \, dVol \quad (9)$$

where $\sigma = \epsilon E$, E is Young's Modulus, and integration is over the volume of the element.

Substituting Eq. 5 for ϵ in Eq. 9 and squaring the expression as indicated,

$$U_{\epsilon} = \frac{E}{2} \int_{Vol} \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial^2 v}{\partial x^2} \right)^2 y^2 + \frac{1}{4} \left(\frac{\partial v}{\partial x} \right)^4 - 2 \frac{\partial u}{\partial x} y \frac{\partial^2 v}{\partial x^2} - y \frac{\partial^2 v}{\partial x^2} \left(\frac{\partial v}{\partial x} \right)^2 + \frac{\partial u}{\partial x} \left(\frac{\partial v}{\partial x} \right)^2 \right] dA \, dx$$

is obtained.

Neglecting the fourth order term, realizing that terms of the form $\int y \, dA$ are first moment of area expressions and are equal to zero when y is measured from the neutral axis, and integrating over the section area,

$$U_{\epsilon} = \frac{EA}{2} \int_0^L \left(\frac{\partial u}{\partial x} \right)^2 dx + \frac{EI}{2} \int_0^L \left(\frac{\partial^2 v}{\partial x^2} \right)^2 dx + \frac{EA}{2} \int_0^L \frac{\partial u}{\partial x} \left(\frac{\partial v}{\partial x} \right)^2 dx \quad (10)$$

is obtained.

Taking the partial derivatives of Eqs. 7 with respect to x , the required partial derivatives in Eq. 10 may be expressed in terms of

the nodal displacements:

$$\begin{aligned}\frac{\partial u\{x\}}{\partial x} &= \frac{1}{\ell} (u_4 - u_1) \\ \frac{\partial v\{x\}}{\partial x} &= \left(-\frac{6x}{\ell^2} - \frac{6x^2}{\ell^3}\right)u_2 + \left(1 - \frac{4x}{\ell} + \frac{3x^2}{\ell^2}\right)u_3 + \left(\frac{6x}{\ell^2} - \frac{6x^2}{\ell^3}\right)u_5 \\ &\quad + \left(-\frac{2x}{\ell} + \frac{3x^2}{\ell^2}\right)u_6\end{aligned}\quad (11)$$

$$\begin{aligned}\frac{\partial^2 v\{x\}}{\partial x^2} &= \left(-\frac{6}{\ell^2} + \frac{12x}{\ell^3}\right)u_2 + \left(-\frac{4}{\ell} + \frac{6x}{\ell^2}\right)u_3 + \left(\frac{6}{\ell^2} - \frac{12x}{\ell^3}\right)u_5 \\ &\quad + \left(-\frac{2}{\ell} + \frac{6x}{\ell^2}\right)u_6\end{aligned}$$

Substituting Eqs. 11 into Eq. 10 and integrating over the length of the element, the strain energy is obtained in the form

$$\begin{aligned}U_\epsilon &= \frac{EA}{2\ell} (u_1^2 - 2u_1u_4 + u_4^2) \\ &\quad + \frac{2EI}{\ell^3} (3u_2^2 + \ell^2u_3^2 + 3u_5^2 + \ell^2u_6^2 + 3\ell u_2u_3 - 6u_2u_5 + 3\ell u_2u_6 \\ &\quad - 3\ell u_2u_5 + \ell^2u_3u_6 - 3\ell u_5u_6) \\ &\quad + \frac{EA}{\ell^2} (u_4 - u_1) \left(\frac{3}{5}u_2^2 + \frac{1}{15}\ell^2u_3^2 + \frac{3}{5}u_5^2 + \frac{1}{15}\ell^2u_6^2 + \frac{1}{10}\ell u_2u_3 \right. \\ &\quad \left. - \frac{6}{5}u_2u_5 + \frac{1}{10}\ell u_2u_6 - \frac{1}{10}\ell u_3u_5 - \frac{1}{30}\ell^2u_3u_6 - \frac{1}{10}\ell u_5u_6\right)\end{aligned}\quad (12)$$

Derivation of the Conventional and Geometrical Stiffness

Matrices from the Internal Strain Energy

In the last term of Eq. 12, the expression $\frac{EA}{l} (u_4 - u_1)$ is equal to the axial force, p_x , in the member for small displacements. This force is considered to be constant over an increment of load (for the incremental step) or an iteration of displacement (for the equilibrium method).

Applying Eq. 8 to Eq. 12, $i=1,2, \dots 6$, the following force-displacement relation is obtained,

$$p = [k_o + k_i]u = ku$$

$$k_i = k_i\{p_x\}$$

where p is a 6 by 1 column vector of nodal forces, u is a 6 by 1 column vector of nodal displacements, k_o is the conventional stiffness matrix, and k_i is the geometrical stiffness matrix.

Explicitly, k_o and k_i are given as

$$k_o = \begin{bmatrix} \frac{EA}{l} & 0 & 0 & -\frac{EA}{l} & 0 & 0 \\ 0 & \frac{12EI}{l^3} & \frac{6EI}{l^2} & 0 & -\frac{12EI}{l^3} & \frac{6EI}{l^2} \\ 0 & \frac{6EI}{l^2} & \frac{4EI}{l} & 0 & -\frac{6EI}{l^2} & \frac{2EI}{l} \\ -\frac{EA}{l} & 0 & 0 & \frac{EA}{l} & 0 & 0 \\ 0 & -\frac{12EI}{l^3} & -\frac{6EI}{l^2} & 0 & \frac{12EI}{l^3} & -\frac{6EI}{l^2} \\ 0 & \frac{6EI}{l^2} & \frac{2EI}{l} & 0 & -\frac{6EI}{l^2} & \frac{4EI}{l} \end{bmatrix}$$

and

$$k_i = p_x \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{6}{5\ell} & \frac{1}{10} & 0 & \frac{-6}{5\ell} & \frac{1}{10} \\ 0 & \frac{1}{10} & \frac{2\ell}{15} & 0 & \frac{-1}{10} & \frac{-\ell}{30} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{-6}{5\ell} & \frac{-1}{10} & 0 & \frac{6}{5\ell} & \frac{-1}{10} \\ 0 & \frac{1}{10} & \frac{-\ell}{30} & 0 & \frac{-1}{10} & \frac{2\ell}{15} \end{bmatrix}$$

where $p_x = \frac{EA}{\ell} (u_4 - u_1)$.

The stiffness matrices k_o and k_i may now be transformed from local to global coordinates using the transformation matrix, L , developed from

$$\left. \begin{array}{l} p = LP \\ p = ku \\ u = L^T U \end{array} \right\} \Rightarrow P = L^T k L U = k^* U$$

The stiffness matrices, k^* , may now be assembled in accordance with standard stiffness method procedures to yield the stiffness matrix of the structure, K .

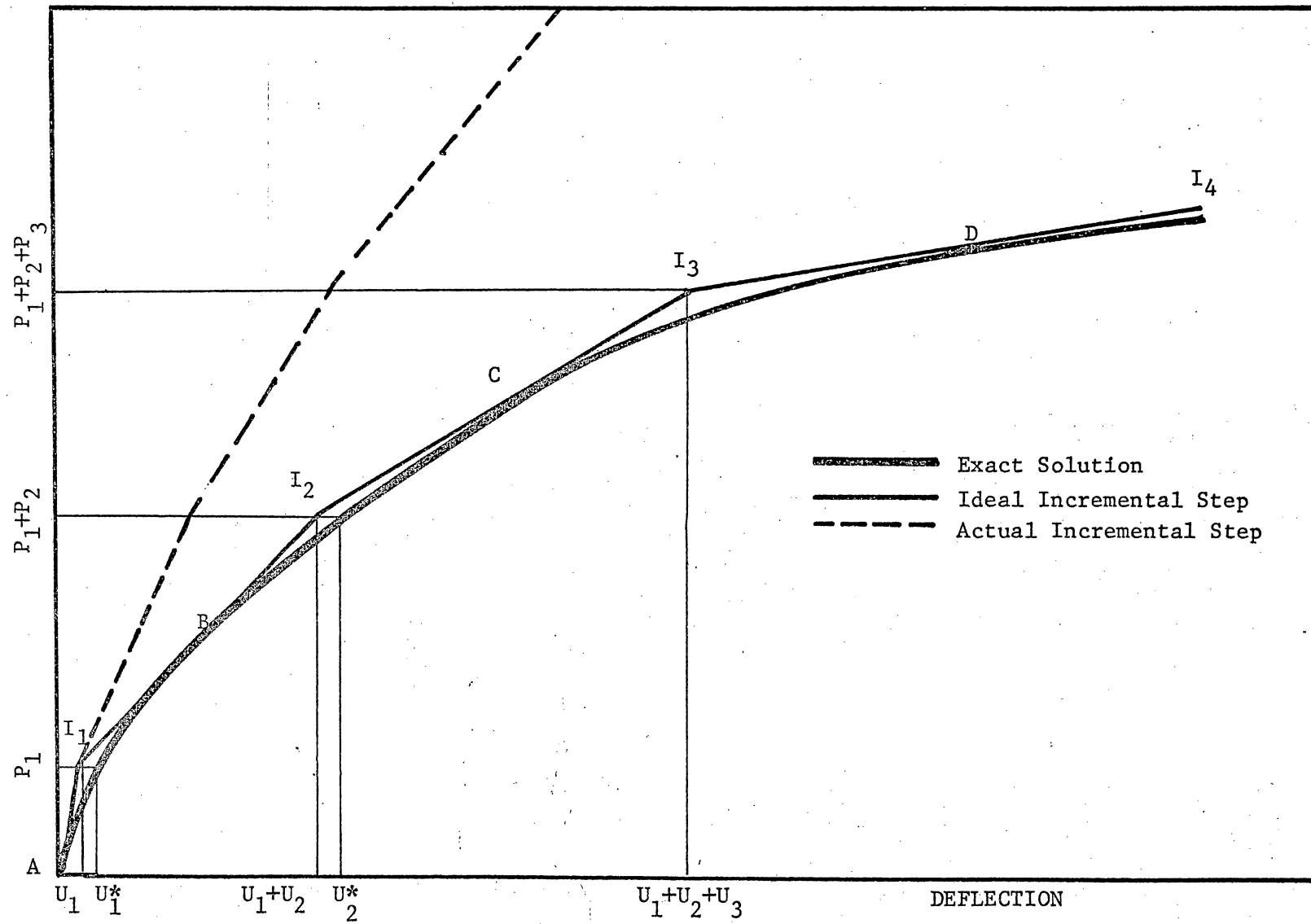


Figure 2 The Incremental Step Method

The Incremental Step Method

In this section an energy approach will be used to develop the incremental step method. An algorithm describing an application of the method will also be given.

An Intuitive Concept of the Incremental Step

An imaginary mathematical model of a one-degree-of-freedom structural system has been evaluated, and the resulting monotonic nonlinear load-deflection curve is plotted in Fig. 2. This curve will be considered the exact solution for the structure. The slope of a tangent to the exact curve is the instantaneous stiffness of the structure at the point of tangency. The incremental step method starts with the division of the total load, P , into a number of incremental loads; each increment of load is P_i . In an ideal incremental step, the instantaneous stiffness of the structure at the point A (Fig. 2) is used to calculate an increment of displacement produced by an increment of load. In the next ideal incremental step, the instantaneous stiffness at point B (the location of B is a function of I_1) is used in calculating the next increment of displacement produced by a second increment of load. The point I_2 is obtained by summing the first and second incremental displacements and by summing the first and second applied incremental loads. If this procedure is continued until the last P_i is applied, the ideal tracking of the nonlinear curve by the tangents in Fig. 2 is obtained. If the number of increments in P is increased, each tangent line is smaller and

more closely represents the nonlinear curve within the increment. As the size of each incremental step approaches zero, the plotted load-displacement relation for the entire assembly of increments resembles the exact load-displacement curve.

The ideal tracking in Fig. 2 is usually never achieved. The broken-line curve is more representative of the curve that would be plotted from points produced by the incremental step method.

The reason is that the instantaneous stiffness at point B is not known at the end of the first increment, point I_1 . The stiffness calculated at the beginning of the second increment, which is not related to the stiffness associated with any point on the exact curve, is a function of the displacements associated with I_1 . The same reasoning will explain the deviation of the remaining increments. A more detailed investigation of the errors developed by this method will be presented in Chapter IV.

Large Rotations

As briefly mentioned earlier, the incremental step method must idealize the large mean rotations of the segment of the continuum the element is representing. This is accomplished by calculating the new orientation of the element after each increment and formulating equilibrium for this configuration. The corresponding global stiffness matrix of the element is then $k^* = L^T k L$, where L is the coordinate transformation matrix defined previously.

An Algorithm For the Incremental Step Method

An algorithm is now presented that illustrates the incremental step in detail. Comments are listed in the right-hand column to explain the function of various steps.

1. Evaluate and assemble into K each k^* ; k^* is based on the original orientation of the element,

$$(k^*)^1 = (L^T k L)^1 = (L^T (k_o + k_i) L)^1$$
 where the superscript is the number of the incremental step.
 If there is no initial prestress, then $k_i = 0$.
 This step calculates the stiffness matrices for the undeformed structure and accounts for any modification of the stiffness due to prestress.
2. Solve for the nodal displacements,

$$U^1 = (K^{-1} P)^1$$
 where P is the vector of applied incremental loads and U is the global joint displacements.
3. Convert the six member end displacements from global to local coordinates

$$u^1 = (L U^*)^1$$
 where U^* is a 6 by 1 vector of global member-end displacements.

4. Find the internal member forces for each element,

$$p^1 = (ku)^1$$

5. Initialize a summation of the member forces for each element and a summation of the joint displacements, U ,

$$p_t^1 = p^1$$

$$U_t^1 = U^1$$

6. For the next increment and every increment thereafter, re-evaluate and reassemble K from each k^* . Note that

$$(k_i)^i = k_i \{p_{t,4}^{i-1}\}$$

where the subscript 4 indicates that the total nodal force p_4 is used for p_x ; and

$$(k^*)^i = (L^T[k_o + k_i]L)^i$$

where L is based on the new geometry.

7. Solve for nodal displacements

$$U^i = (K^{-1} P)^i$$

The internal member forces and the joint displacements at the end of any incremental step are equal to the sum of the internal member forces and the sum of the joint displacements for all completed increments, respectively.

k_i is a function of the total axial force, $p_{t,4}$ in the member in the previous incremental step.

8. Convert the global displacements
of the nodes for each element
into local displacements,

$$u^i = (L U^*)^i$$

9. Find internal member forces for
each element,

$$p^i = (ku)^i$$

10. Add the internal member forces and
the global displacements for the
 i^{th} increment to p_t and U_t ,
respectively,

$$p_t^i = p_t^{i-1} + p^i$$

$$U_t^i = U_t^{i-1} + U^i$$

11. Continue with steps 6 through 10
until the final incremental load,
 P is applied.

The Equilibrium Method

The pertinent features of the equilibrium method will be presented in this section from equilibrium considerations. Two variations of the equilibrium method will be explained: these are (1) the Newton-Raphson equilibrium method, and (2) a first-order convergence method similar to (1). An algorithm will be given in this section that describes an application of the two variations of the equilibrium method.

Equilibrium of a System of Elements

The nodal equilibrium equation for a system of finite elements corresponding to the exact nodal displacements is given by

$$P - p^* = 0 \quad (15)$$

where P is the vector of applied nodal forces and p^* is the corresponding member force vector, expressed in the global coordinate system. If the system is not in equilibrium, a residual vector is defined by

$$R = P - p^* \quad (16)$$

The internal member forces are found from the member distortions. With the method presented in this thesis (cf. Appendix I), three distortions are calculated from which three internal forces are determined. The remaining three nodal forces can be determined from the equilibrium requirements for the element. The member distortions are found from the vector of global nodal displacements.

An approximate set of nodal displacements is evaluated on the basis of equilibrium. If Eq. 15 is not satisfied, a correction to the nodal displacements must be applied. This correction is then added to the nodal displacements and a new approximate set of displacements is generated. The procedure is then repeated, and each repetition will be called an iteration. Practically, the procedure terminates when R is negligible.

Large Rotations

As in the incremental step method, the equilibrium method provides for large rotations of the system of elements by discontinuities of the slope between elements. The equilibrium method is highly sensitive to the mesh size since the outward component of the axial force (cf. Fig. 1b) in the elements directly enters Eq. 16 and affects the residual vector calculated.

Calculation of Approximate Nodal Displacements

A method for calculating a correction to an approximate set of nodal displacements is required. The governing criterion for the method is that the output from Eq. 16 must correctly influence the calculations of the new nodal displacements so that, eventually, the residual vector tends toward zero. Two correctional procedures will now be given without proof. The proof for these methods may be found in Ref. 16.

Procedure 1 (Newton-Raphson equilibrium method):

After each iteration, the stiffness matrix, K , which is a function of the current displacements, is updated and used to calculate a new correction to the displacements by

$$\Delta U^i = K\{U^i\}^{-1} R^i$$

where the superscript, i , is the number of the iteration.

The $i + 1$ approximation of the global displacements is calculated by

$$U^{i+1} = \Delta U^i + U^i$$

Procedure 2:

A similar procedure is used, however K is only evaluated for the original configuration of the structure, i.e.

$$\Delta U^i = K\{o\}^{-1} R^i$$

and

$$U^{i+1} = \Delta U^i + U^i$$

Algorithm for the Two Variations of the Equilibrium Method

An algorithm for the equilibrium method using either of the two correctional procedures outlined in the previous section will now be given. Comments are again given in the right-hand column.

- | | |
|--|--|
| <p>1. Assume a set of global displacements U has been found on which L is based.</p> | <p>In general, the first approximation may be calculated by any means; however, the approximation should be sufficiently close to the exact displacements so that problems in convergence will not occur. A convenient approximation is the displacements given by the linear analysis of the structure for the load P.</p> |
| <p>2. Determine the three member distortions u from the global displacements U.</p> | <p>See Appendix I for the equations to calculate the member distortions.</p> |

3. Find the axial force in the member from $\Delta u_x AE/l$, where Δu_x is the axial distortion. Using this determine k_i and evaluate the two remaining nodal forces using $k = k_o + k_i$, i.e. $p = ku$; note that k_o and k_i are 3 by 3 matrices relating the 3 by 1 vector of nodal forces to the 3 by 1 vector of member distortions. Find the other three nodal forces by imposing equilibrium on the element.
4. Express member force vector in terms of global coordinates, and form the residual vector as indicated by Eq. 16, $R = P - p^*$.
- 5.A (Procedure 1) Compute $k^* = L^T (k_o + k_i) L$, where L is the coordinate transformation matrix from the current local to the global system. Assemble each k^* into K and solve for the correction to the nodal displacements,

$$\Delta U = K^{-1} R$$

Use either 5A or 5B

B. (Procedure 2) Compute $k^* =$

$L^T (k_0 + k_1) L$ for the

original structure only;

assemble k^* into K . This

evaluation of K may be used

for all iterations. Solve

for ΔU ,

$$\Delta U = K^{-1} R$$

6. Add ΔU to the current nodal displacements

$$U^{i+1} = \Delta U^i + U^i$$

where the superscript, i , is

the iteration number. Repeat

7. Repeat steps 2 through 6 until a desired degree of accuracy has been attained; i.e. R or ΔU has become sufficiently small.

CHAPTER IV

APPLICATIONS & CONCLUSIONS

Three example problems will be presented in this chapter. The first problem will be solved using both the incremental step and the equilibrium methods. The solution of the second and third problems will be based on the incremental step method and the equilibrium method, respectively.

A discussion of the three problems will follow the statement of the problems, citing possible reasons for the results obtained, and comparisons between the two methods will be made. Each example will demonstrate salient features of the respective methods. Finally, a concluding paragraph will relate the goals set forth in the Introduction with the results and comments in this and the previous chapter. Recommendations for future study will be made.

The Example Problems

The following problems are investigated:

1. A simple beam-column loaded with a concentrated transverse load at midspan and an axial load.
2. A single bay frame with joint loads.
3. A cantilevered beam subjected to large displacements by a uniformly distributed load.

In cases (1) and (2), the magnitudes of all axial loads remain below the corresponding critical buckling loads. The beam in case (3) is considered to be restrained against out-of-plane buckling.

The Beam-Column

A mathematical model of the beam-column in Fig. 3a, limited to small axial loads relative to the Euler buckling load, S_c , and small displacements, is presented in reference 20. The corresponding mid-span deflection is

$$y = \frac{PL^3}{48EI} \cdot \frac{3 \tan(h) - h}{h^3} \quad (17)$$

$$h = \frac{|S| L^2}{4EI}$$

where P is the transverse load, L is the length of the beam-column, EI is the flexural rigidity, and S is the axial load. The deflections obtained from a finite element analysis will be compared with the respective values given by Eq. 17. The numerical values of the properties of the beam-column used in this example are specified in Fig. 3a.

A. Incremental Step Method With Three Loading Patterns

1. The transverse load $\lambda P_{\max}$ and the axial load λS_c are applied simultaneously in incremental steps; $P_{\max}$ is the maximum transverse load ($P_{\max} = 100K$, $S_c = 9850K$), and λ is the load parameter that varies incrementally from 0 to 1. A two-element idealization is used. The load-deflection curve defined by Eq. 17 is shown in Fig. together with four numerical solutions based on different load increments.

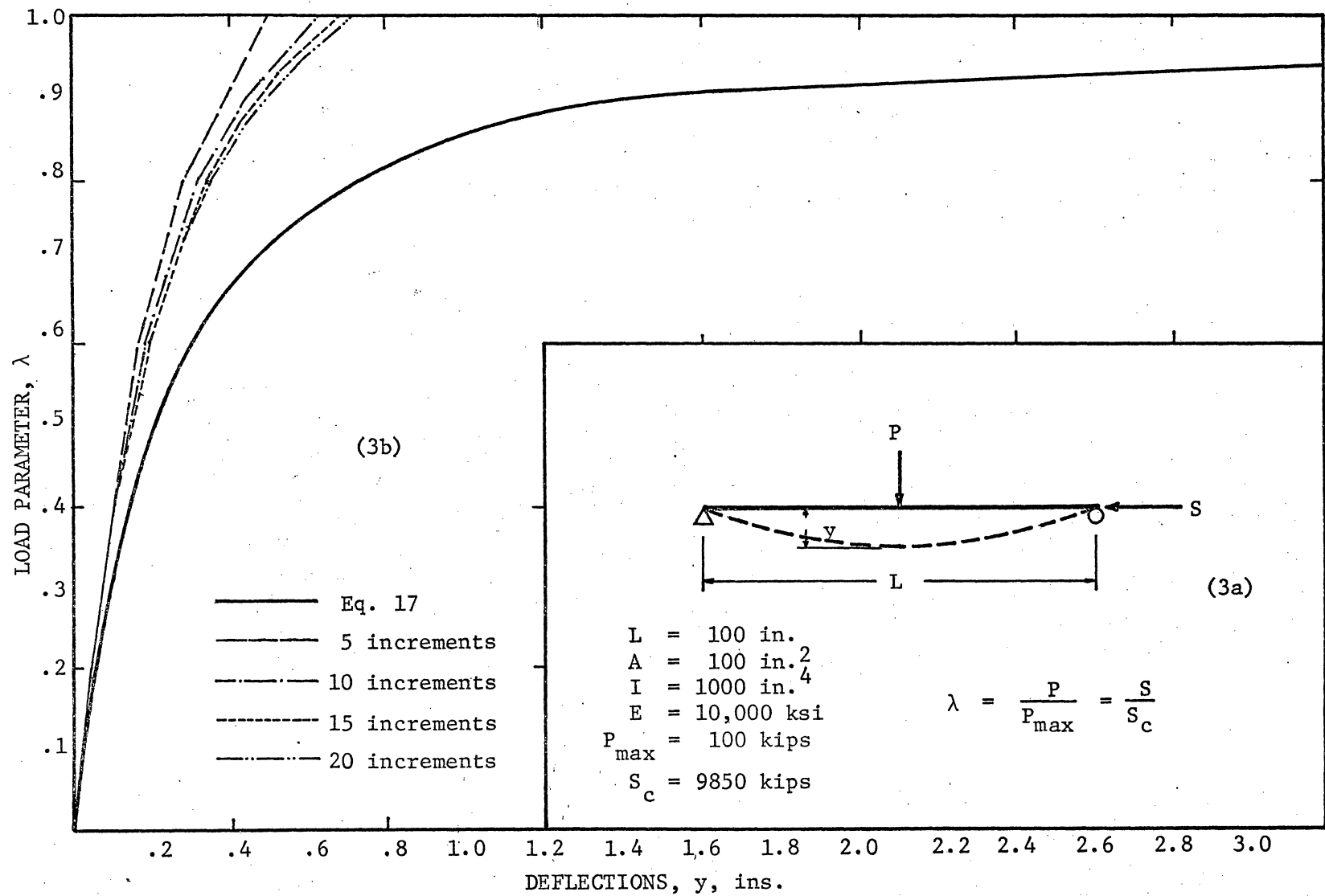


Figure 3 Beam Column

- 2a. The axial load S is used to initialize k_1 in each element before any displacements are calculated. P is then applied in a single step. Again a two-element idealization is used. Tabulated values of y found by this approach (for various values of S and P) are compared in Table Ia with the respective values determined by Eq. 17.
 - 2b. The axial load S is again applied as a "prestress" force; however, P is applied in a number of increments. Again a two-element idealization is used. Table Ib shows that the results are identical with those in Table Ia.
- B. Equilibrium Method With Two Correctional Procedures
1. The axial load λS_c and the transverse load $\lambda P_{\max}$ are applied simultaneously; $\lambda = .2, .3, .4, .6, \text{ and } .8$. Correctional procedure 2 is used and displacement corrections are carried out for 20 iterations (30 where noted). Four idealizations are studied using 2, 4, 6, and 8 elements, respectively. All elements in an idealization are identical. Table II presents values of y for the 1st, 5th, 10th, 15th, 20th (25th and 30th where noted) increments and respective values determined by Eq. 17.
 2. The axial load λS_c and the transverse load λP are applied simultaneously; $\lambda = .2, .3, .4, .6, \text{ and } .8$. Correctional procedure 1 is used. The same number of iterations as in B1 are carried out. Table III shows difference in rate of convergence between correctional procedures 1 and 2 for the

TABLE Ia. Comparison of Deflections Given by Eq. 17 and a Piece-wise Linear Incremental Step Method for the Beam-Column in Fig. 3a

S Kips	Eq. 17		Incremental Step Method			
	P = 30K	P = 60K	P = 30K		P = 60K	
	y	y	y	% error	y	% error
1970	.0773	.1547	.0776	.38	.1552	.32
2955	.0879	.1757	.0883	.46	.1765	.50
3940	.1017	.2034	.1024	.68	.2048	.69
5910	.1492	.2984	.1509	1.14	.3017	1.11
7880	.2820	.5639	.2875	1.95	.5750	1.97

TABLE Ib. Comparison of Deflections Given by Two Piece-wise Linear Incremental Step Methods for the Beam-Column in Fig. 3a

S Kips	1 step to		3 steps to		6 steps to	
	P = 30K	P = 60K	P = 30K		P = 60K	
	y	y	y		y	
1970	.0776	.1552	.0776		.1552	
2955	.0883	.1765	.0883		.1765	
3940	.1024	.2048	.1024		.2048	
5910	.1509	.3017	.1509		.3017	
7880	.2875	.5750	.2875		.5751	

TABLE II. Comparison of the Displacements Given by Eq. 17 and the Equilibrium Method for the Beam-Column in Fig. 3a.

the Beam Column in Fig. 5a.

$\lambda P_{\max}$ Kips	λS_c Kips	Iteration Number													
		1		5		10		15		20		25		30	
		y	% error	y	% error	y	% error	y	% error	y	% error	y	% error	y	% error
2-element															
20	1970	.04167	19.2	.05153	.06	.05155	.03	.05155	.03	.05155	.03				
30	2955	.06250	28.9	.08760	.28	.08777	.09	.08777	.09	.08777	.09				
40	3940	.08333	38.6	.13423	1.00	.13538	.18	.13539	.17	.13539	.17				
60	5910	.12500	58.1	.22708	7.10	.29529	1.00	.29649	.62	.29657	.60				
80	7880	.16667	77.8	.53069	29.40	.67873	9.70	.72012	4.20	.73168	2.70				
4-element															
20	1970	.04167	19.2	.05155	.02	.05156	--	.05156	--	.05156	--				
30	2955	.06250	28.9	.08766	.22	.08784	.01	.08784	.01	.08784	.01				
40	3940	.08333	38.6	.13442	.89	.13561	.01	.13562	.01	.13562	.01				
60	5910	.12500	58.1	.27799	6.80	.29685	.50	.29814	.07	.29823	.04				
80	7880	.16667	77.8	.53373	29.00	.68771	8.50	.73234	2.60	.74528	.88				
6-element															
20	1970	.04167	19.2	.05155	.02	.05156	--	.05156	--	.05156	--				
30	2955	.06250	28.9	.08766	.22	.08785	--	.08785	--	.08785	--				
40	3940	.8333	38.6	.13442	.89	.13562	.01	.13563	--	.13563	--				
60	5910	.12500	58.1	.27798	6.80	.29694	.47	.29825	.03	.29834	--				
80	7880	.16667	77.8	.53352	29.00	.68801	8.40	.73301	2.50	.74612	.76	.74994	.25	.75105	.10
8-element															
20	1970	.04167	19.2	.05155	.02	.05156	--	.05156	--	.05156	--				
30	2955	.06250	28.9	.08766	.21	.08785	--	.08785	--	.08785	--				
40	3940	.08333	38.6	.13441	.90	.13562	--	.13563	--	.13563	--				
60	5910	.12500	58.1	.27787	6.90	.29694	.47	.29826	.03	.29835	--	.29836	--	.29836	--
80	7880	.16667	77.8	.53291	29.10	.68769	8.50	.73296	2.50	.74621	.75	.75008	.24	.75122	.09

-- indicates less than .01% error.

TABLE III. Comparison of the Rate of Convergence
of the Two Correctional Procedures for
the Equilibrium Method

$\lambda P_{\max}$	λS_c	Procedure	ERROR AFTER ITERATION NUMBER					
Kips	Kips	Number	5	10	15	20	25	30
2-elements								
20	1970	2	.06	.03	.03	.03		
20	1970	1	.03	.03				
30	2955	2	.28	.09	.09	.09		
30	2955	1	.09	.09	.09	.09		
40	3940	2	1.00	.18	.17	.17		
40	3940	1	.17	.17	.17	.17		
60	5910	2	7.10	1.00	.62	.60		
60	5910	1	.60	.60	.60	.60		
80	7880	2	29.40	9.70	4.20	2.70		
80	7880	1	2.10	2.10	2.10	2.10		
6-elements								
20	1970	2	.02	--	--	--		
20	1970	1	--	--	--	--		
30	2955	2	.22	--	--	--		
30	2955	1	--	--	--	--		
40	3940	2	.89	.01	--	--		
40	3940	1	--	--	--	--		
60	5910	2	6.80	.47	.03	--		
60	5910	1	--	--	--	--		
80	7880	2	29.00	4.20	1.30	.76	.25	.10
80	7880	1	.09	.05	.05	.05	.05	.05

-- indicates less than .01% error.

two- and six-element idealizations.

The Single Bay Frame, The Incremental Step Method

The frame in Fig. 4a has a load-displacement curve shown in Fig. 4b. This curve is given in Ref. 21 and has been determined to within a 5% accuracy by another variation of the equilibrium method. The problem is solved by initializing k_1 for the column elements with S and solving for the transverse deflection, Δ , by applying P in a single step. The resulting configurations corresponding to various load levels are shown in Fig. 4b.

The Cantilevered Beam, The Equilibrium Method

The cantilever beam in Fig. 5a has a load-displacement curve depicted in Fig. 5b. This curve is plotted from the solution of the differential equation model for the large deflections of a cantilevered beam found in Ref. 22. The uniformly distributed load w is replaced by statically equivalent concentrated loads, and the resulting displacements are identified with data points along the curve.

Discussion

The Beam-Column, Incremental Step Method

- A1. The deviation of the solution obtained with the incremental step method from the solution given by Eq. 17, does occur as predicted and explained in Chapter III. As the number of

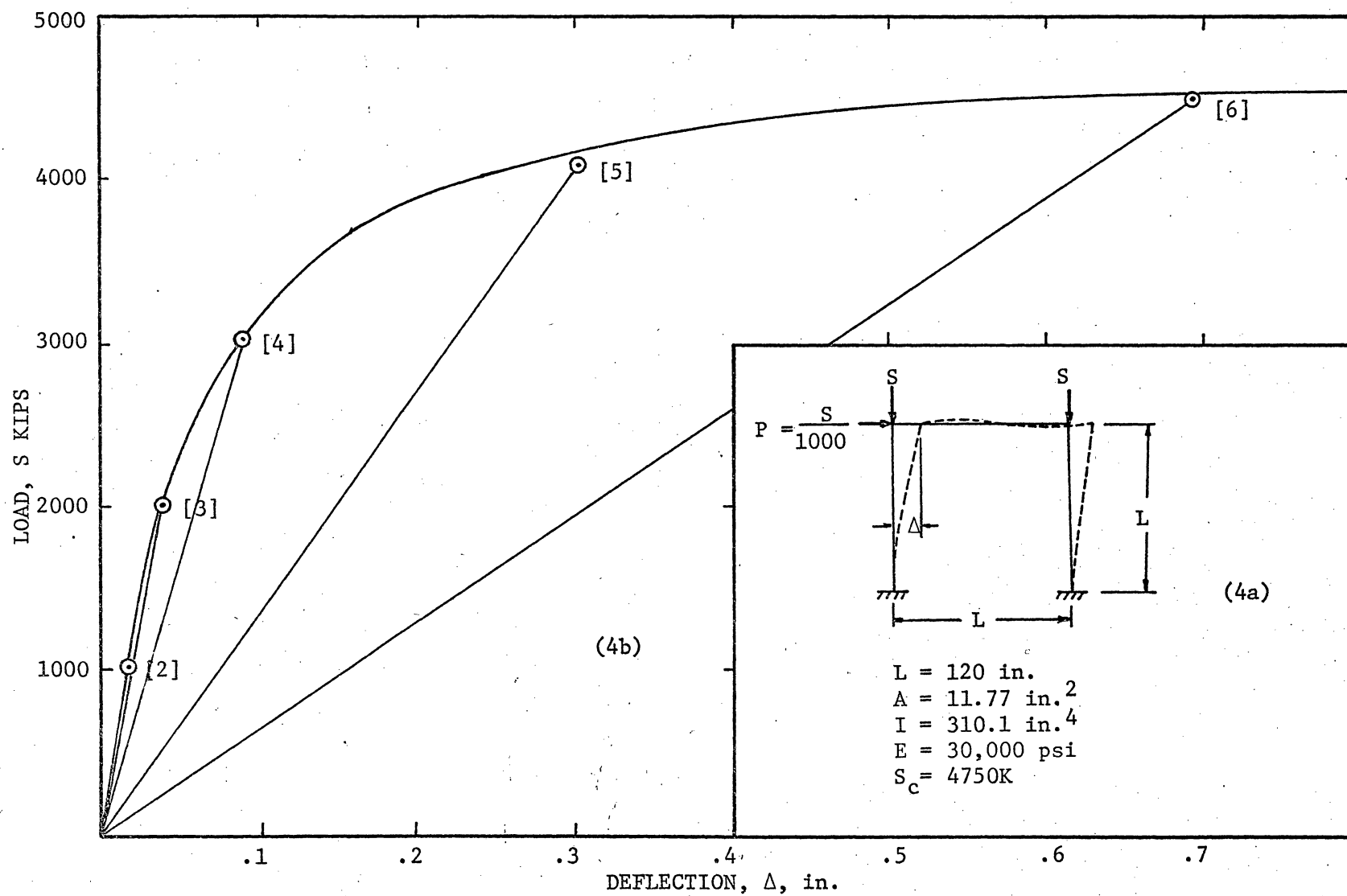


Figure 4 Plane Frame

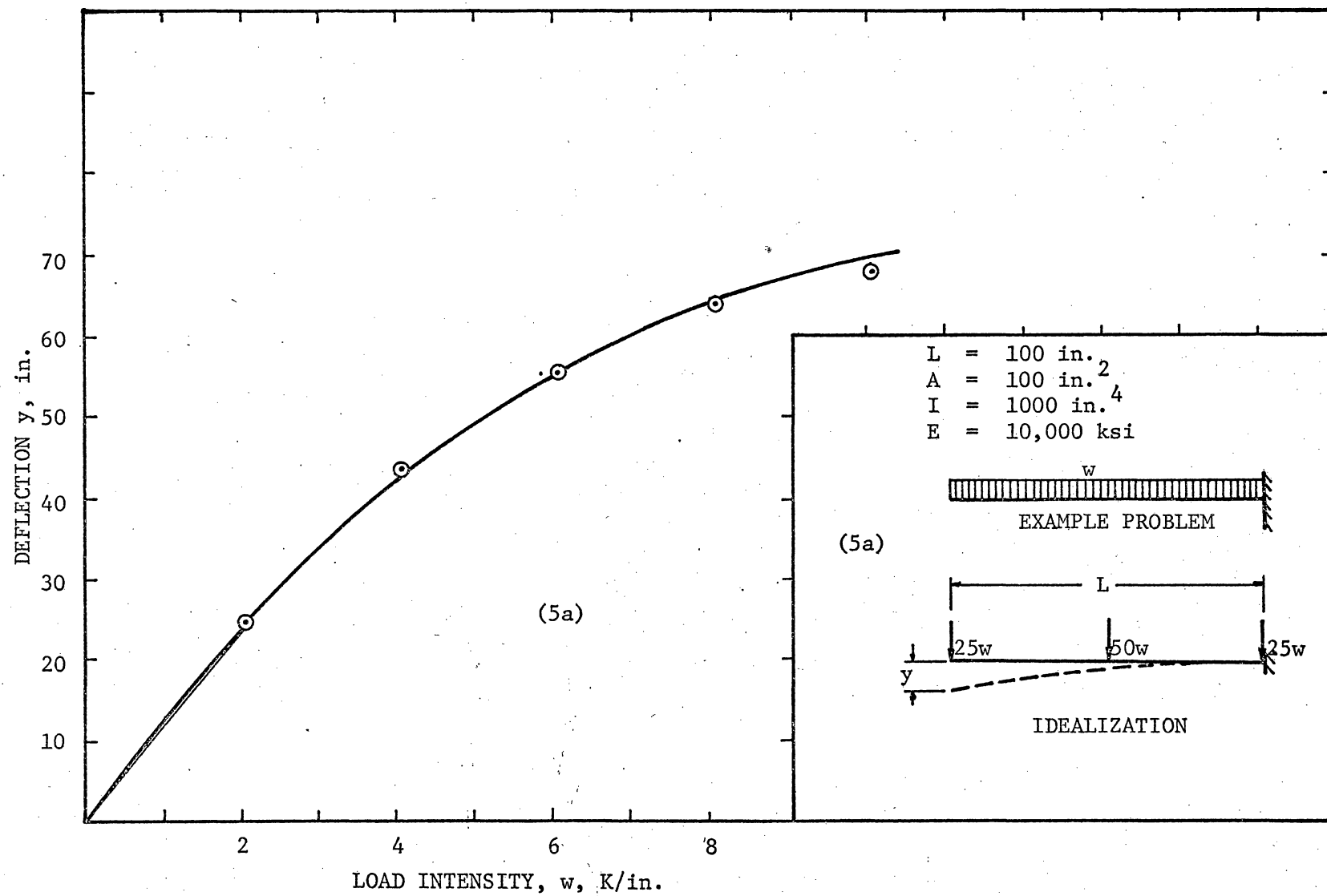


Figure 5 Cantilevered Beam

increments increase, however, the accuracy does improve; but for all cases considered the accuracy is quite poor.

A2. Contrary to A1, quite good results are obtained by using the axial load as a prestress and only one increment. The reason for this accuracy can be explained by investigating the behavior of the structure described by Eq. 17. This equation is the product of two terms. The first is a linear expression for a simple beam loaded at midspan and is proportional to the transverse load P . The second term is the nonlinear magnification factor dependent on the axial load S . If a series of load deflection curves is plotted where P varies from 0 to $P_{\max}$ and S is held constant for each evaluation of P over the specified range, Fig. 6 is obtained. These straight lines lie on the surface (first quadrant only) defined by Eq. 17. The load-deflection path produced by applying S and P simultaneously is shown by the nonlinear curve traced on the surface generated in Fig. 6. This curve can only be approximated by tangents to the curve. Path OSA on the surface in Fig. 6, on the other hand, is piecewise linear. The end point is still the same deflection, y_A . The incremental step method is more suited for tracking this path than for tracking a nonlinear path. Path OS is traced when the prestressing force S is applied, path SA is produced when the transverse load is applied. Since there is no change in the stiffness of the structure along the path SA, it does not matter whether one step or many

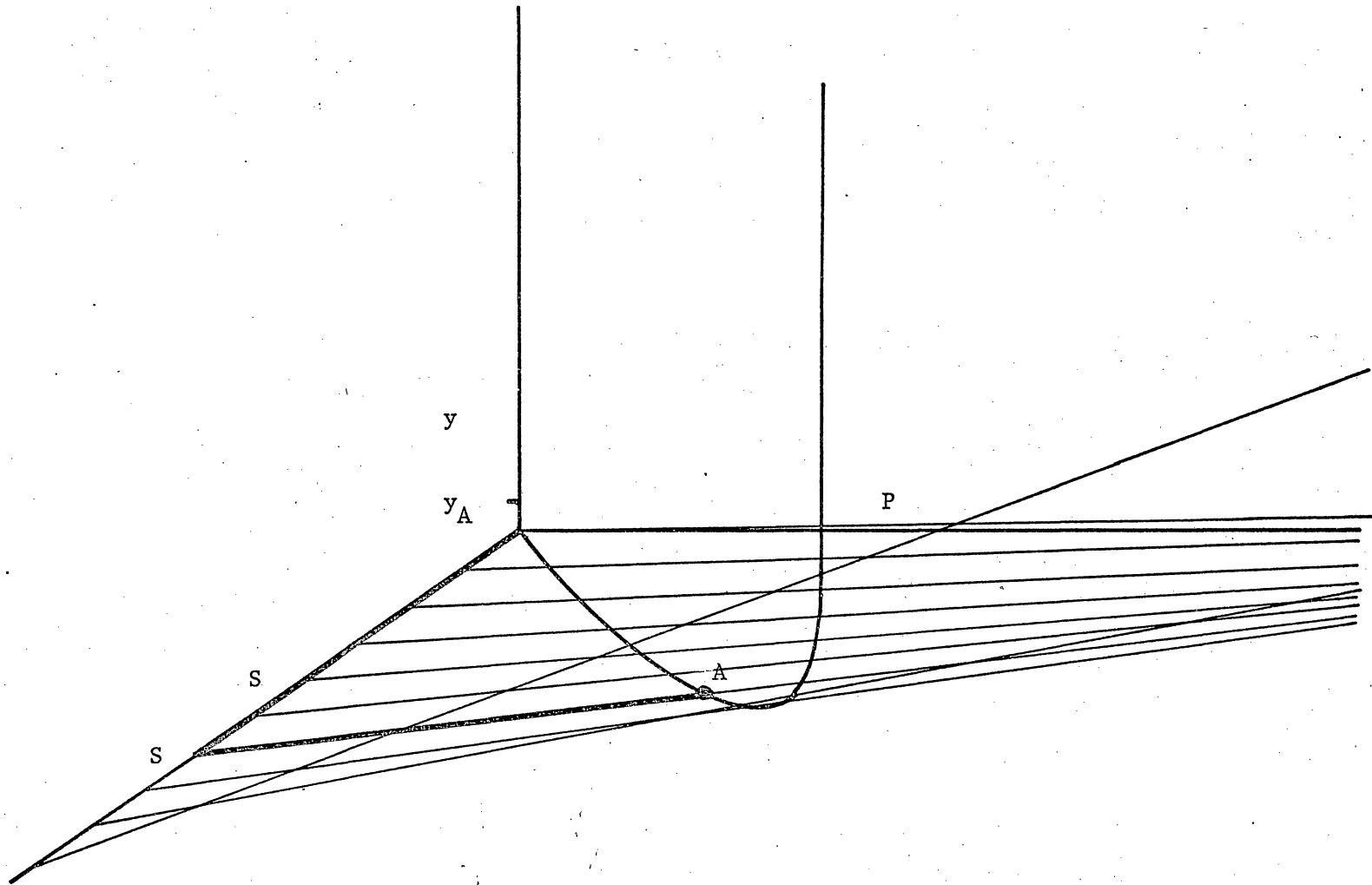


Figure 6 Load-Deflection Surface for the Beam-Column

steps are used to attain point A - the results are identical. From another viewpoint, after S is used to initialize k_i , the transverse load P does not affect (for small displacements) the axial distortions of the elements, thus k_i remains constant over any number of increments. This example would indicate that any generally nonlinear structure that has a piecewise linear load-displacement function can be accurately analyzed by the incremental step method. The single bay frame is an example of such a structure. The biggest disadvantage with this particular method is that the behavior of the structure must be known before the method can be applied.

The Beam Column, Equilibrium Method

- B1. Final displacements, as determined by the equilibrium method and shown in Table II, are quite accurate. The equilibrium method establishes equilibrium (to within a certain degree of accuracy) for the system of elements and is, hence, self-corrective. As mentioned earlier, the residual vector, R, is affected by the outward axial force component at the nodes. Table II indicates that a greater degree of accuracy is possible with more elements in the idealization. The results also show that the error increases with increasing deflection. This fact can also be related to the mesh size. As the deflection increases, the discontinuity between elements is more pronounced and the outward force component is larger.

Again, a decrease in the mesh size improves the accuracy of the method.

B2. Table III compares the convergence rate of the two correctional procedures; the maximum accuracy obtainable is a function of the idealization as illustrated in the paragraph before and is independent of the correctional procedure used. Thus, the maximum degree of accuracy is only obtained earlier in the iteration process. Procedure 1 converges considerably faster than procedure 2, since procedure 1 uses a more accurate approximation of the stiffness of the structure to calculate each new correction to the nodal displacements.

The Single Bay Frame

This example is included to show how the incremental step method may be used to advantage over an equilibrium method. The number in brackets along the curve in Fig. 4b is the number of iterations used in Ref. 21 to converge to the point on the curve. The incremental step method uses only one increment, however. The savings in computational effort is evident.

The Cantilevered Beam

This example is included to illustrate two features of the equilibrium method:

1. loadings between nodes, i.e. on the element, may be idealized

by concentrated forces at the nodes,³ and

2. large rotations of the elements are allowed.

As Fig. 5b indicates, the method will produce reasonable results; however, the accuracy is not quite as good as in the beam-column example. The reasons for this fact are twofold: (1) There is a certain amount of error introduced by the idealization of the uniformly distributed load, especially at the ends of the beam where symmetry on each side of the nodes ends. As the number of nodes increases, the set of concentrated nodal forces more closely approximates the uniform load. (2) The other factor accounting for error in this problem is, again, the error produced by the discontinuities, which are now more pronounced than in the small-displacement beam-column problem.

Conclusions

As stated in the introduction, the stiffness matrices for the beam-column element, and the incremental step and equilibrium methods have been developed in sufficient detail so that a computer program may be written directly on the basis of Chapter III. The material in Chapter III was originally presented by the authors cited therein; however, in presenting a detailed compilation of their work, many questions arise that require investigation. Of importance is the question concerning consistency of the assumptions and restrictions for the matrices derived and the methods that use these matrices.

³ A complete investigation into the loadings between nodes is beyond the scope of this thesis. This example is included to show only qualitatively how the problem might be approached.

Pertinent formulas and equations used in the derivations in Chapter III have been explained with regard to the restrictions imposed upon them.

Another important question that has been answered concerns the occurrence of large rotations even though the beam-column element is restricted to small rotations. The effect of these large rotations on the accuracy of the methods have been discussed.

Finally, three example problems are given to illustrate the methods and to compare the relative accuracy of the methods. The incremental step method is shown to be accurate in solving problems with piecewise linear load-displacement relations and inaccurate in solving general nonlinear problems. The incremental step method is more efficient in terms of computational effort than the equilibrium method for the piecewise linear problem. In general, however, the equilibrium method is shown to be superior in accuracy to the incremental step method.

Further Study

1. Correctional procedure 1 for the equilibrium method converges faster than correctional procedure 2. However, the computational effort for procedure 1 can be considerably greater than that for procedure 2 for systems with many, degrees-of-freedom. Further evaluation of the relationship between degrees-of-freedom and computational effort is required. Livesly [15] suggests that perhaps the most efficient correctional procedure is one that combines procedures 1 and 2, so that the stiffness of the structure

is re-evaluated after a number of iterations instead of after every iteration as in procedure 1. How should procedures 1 and 2 be combined for optimal results?

2. The incremental step method and the equilibrium method may be combined to form an efficient and accurate method for solving geometrically nonlinear problems. In this case, the equilibrium method provides an independent check on (and correction to) the displacements produced at the end of a number of increments. How can this method be best developed and utilized?

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APPENDIX I

TRANSFORMATION OF k_i AND k_o FROM LOCAL TO GLOBAL COORDINATES

It is convenient when transcribing the incremental step method and the equilibrium method algorithms into computer code to have the algebraic multiplications indicated by

$$k^* = L^T k L \quad (AI.1)$$

completely carried out before numeric quantities are introduced. Multiplication of numeric matrices for each element is too time consuming for efficient programming.

L is given by Ref. 23 for the six-degree-of-freedom plane flexural element as

$$L = \begin{bmatrix} \cos \alpha & \sin \alpha & 0 & 0 & 0 & 0 \\ -\sin \alpha & \cos \alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos \alpha & \sin \alpha & 0 \\ 0 & 0 & 0 & -\sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

where α is shown in Fig. 1a.

Using the rules of matrix algebra, Eq. AI.1 can be expressed as

$$k^* = L^T k L = L^T (k_o + k_i) L = L^T k_o L + L^T k_i L \quad (AI.2)$$

Substituting the expansions of the appropriate matrices into Eq. AI.2 and performing the matrix multiplication, the explicit form of k^* is obtained:

$$k^* = \begin{bmatrix} \frac{EA}{\ell} C_X^2 + \frac{12EI}{\ell^3} C_Y^2 & (\frac{EA}{\ell} - \frac{12EI}{\ell^3}) C_X C_Y & -\frac{6EI}{\ell^2} C_Y & -(\frac{EA}{\ell} C_X^2 + \frac{12EI}{\ell^3} C_Y^2) & -(\frac{EA}{\ell} - \frac{12EI}{\ell^3}) C_X C_Y & \frac{6EI}{\ell^2} C_Y \\ (\frac{EA}{\ell} - \frac{12EI}{\ell^3}) C_X C_Y & \frac{EA}{\ell} C_Y^2 + \frac{12EI}{\ell^3} C_X^2 & \frac{6EI}{\ell^2} C_X & -(\frac{EA}{\ell} - \frac{12EI}{\ell^3}) C_X C_Y & -(\frac{EA}{\ell} C_Y^2 + \frac{12EI}{\ell^3} C_X^2) & \frac{6EI}{\ell^2} C_X \\ -\frac{6EI}{\ell^2} C_Y & \frac{6EI}{\ell^2} C_X & \frac{4EI}{\ell} & \frac{6EI}{\ell^2} C_Y & -\frac{6EI}{\ell^2} C_X & \frac{2EI}{\ell} \\ -(\frac{EA}{\ell} C_X^2 + \frac{12EI}{\ell^3} C_Y^2) & -(\frac{EA}{\ell} - \frac{12EI}{\ell^3}) C_X C_Y & \frac{6EI}{\ell^2} C_Y & \frac{EA}{\ell} C_X^2 + \frac{12EI}{\ell^3} C_Y^2 & (\frac{EA}{\ell} - \frac{12EI}{\ell^3}) C_X C_Y & \frac{6EI}{\ell^2} C_Y \\ -(\frac{EA}{\ell} - \frac{12EI}{\ell^3}) C_X C_Y & -(\frac{EA}{\ell} C_Y^2 + \frac{12EI}{\ell^3}) C_X C_Y & -\frac{6EI}{\ell^2} C_X & (\frac{EA}{\ell} - \frac{12EI}{\ell^3}) C_X C_Y & \frac{EA}{\ell} C_Y^2 + \frac{12EI}{\ell^3} C_X^2 & -\frac{6EI}{\ell^2} C_X \\ -\frac{6EI}{\ell^2} C_Y & \frac{6EI}{\ell^2} C_X & \frac{2EI}{\ell} & \frac{6EI}{\ell^2} C_Y & -\frac{6EI}{\ell^2} C_X & \frac{4EI}{\ell} \end{bmatrix}$$

+

$$\begin{bmatrix}
 \frac{C_Y C_Y 6F}{5l} & - \frac{C_Y C_X 6F}{5l} & - \frac{C_Y F}{10} & - \frac{C_Y C_Y 6F}{5l} & \frac{C_Y C_X 6F}{5l} & - \frac{C_Y F}{10} \\
 - \frac{C_X C_Y 6F}{5l} & \frac{C_X C_Y 6F}{5l} & \frac{C_X F}{10} & \frac{C_X C_Y 6F}{5l} & - \frac{C_X C_X 6F}{5l} & \frac{C_X F}{10} \\
 - \frac{C_Y F}{10} & \frac{C_X F}{10} & \frac{2Fl}{15} & \frac{C_Y F}{10} & - \frac{C_X F}{10} & - \frac{Fl}{30} \\
 - \frac{C_Y C_Y 6F}{5l} & \frac{C_Y C_X 6F}{5l} & \frac{C_Y F}{10} & \frac{C_Y C_Y 6F}{5l} & - \frac{C_Y C_X 6F}{5l} & \frac{C_Y F}{10} \\
 \frac{C_Y C_X 6F}{5l} & - \frac{C_X C_X 6F}{5l} & - \frac{C_X F}{10} & - \frac{C_X C_Y 6F}{5l} & \frac{C_X C_X 6F}{5l} & - \frac{C_X F}{10} \\
 - \frac{C_Y F}{10} & \frac{C_X F}{10} & - \frac{Fl}{30} & \frac{C_Y F}{10} & - \frac{C_X F}{10} & \frac{2Fl}{15}
 \end{bmatrix}$$

MEMBER DISTORTIONS FROM THE GLOBAL DISPLACEMENTS

The equations relating the member distortions to the global displacements will be derived from the following argument. Consider the undeformed element jk in Fig. AI.1. The orientation α and the original length ℓ_0 are known. During deformation, the nodes j and k move through the displacement vectors $\vec{U}_j$ and $\vec{U}_k$, respectively; in addition the nodes also rotate through the angle θ_j and θ_k , respectively. The new orientation of the element, α' , is based in the cord drawn between the nodes. The undeformed element is now translated and rotated in rigid-body fashion so that the element axis coincides with the cord. This virtual element is shown by the broken line in Fig. AI.1.

The required distortions are: (1) the axial elongation, u_x , (2) the rotation of node j , θ_j , and (3) the rotation of the node k , θ_k ; they are shown in the figure. For small bending rotations, u_x is the difference between the length of the cord, $j'k'$, and the original length, ℓ_0 . The rotational distortions θ_j and θ_k are the angles that the nodes rotate through with respect to the cord, $j'k'$. If the change in orientation of the element is

$$\Delta\alpha = \alpha' - \alpha$$

then the nodal rotation distortions are simply

$$\theta_j = \theta_j - \Delta\alpha$$

$$\theta_k = \theta_k - \Delta\alpha$$

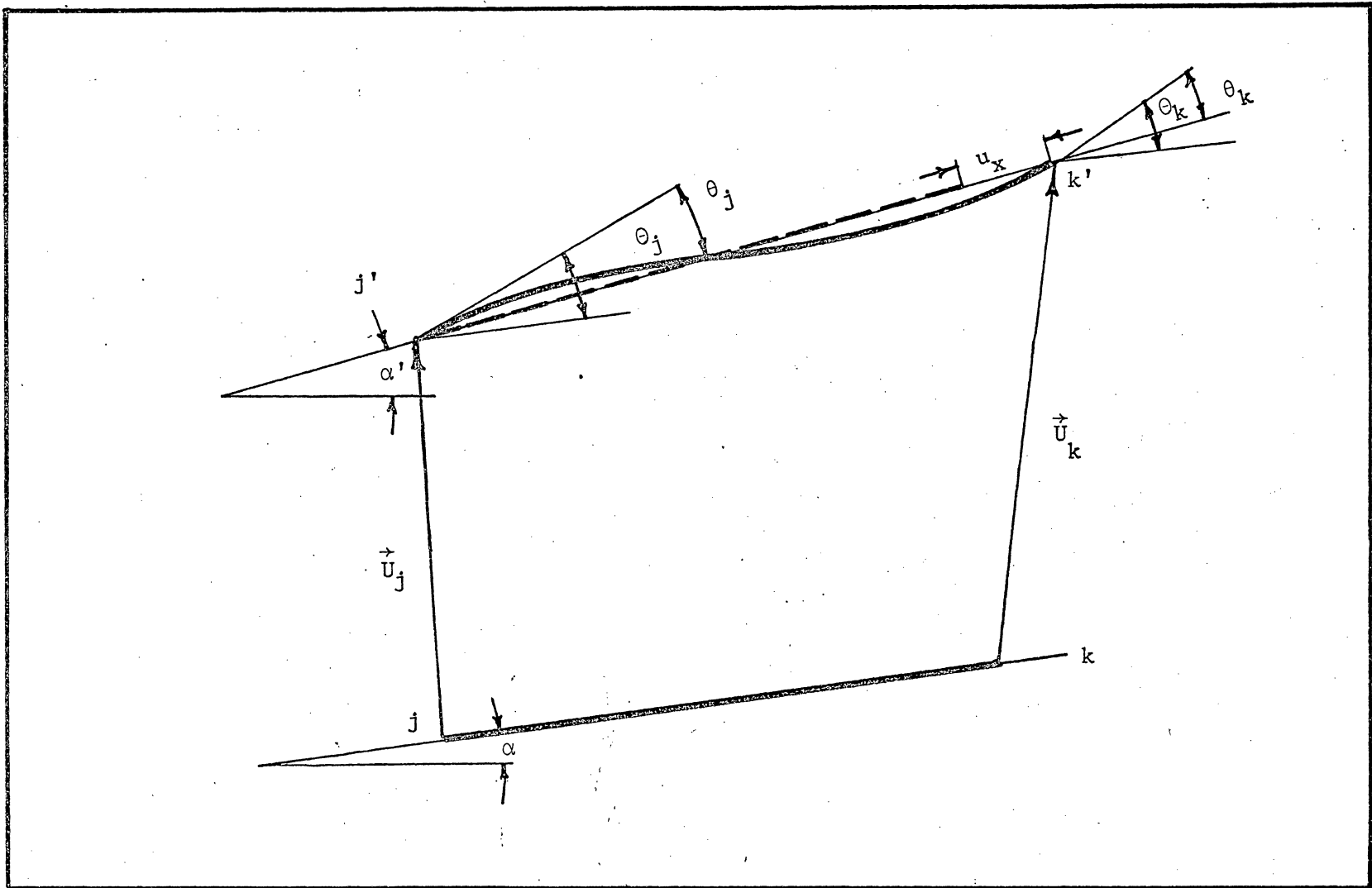


Figure 7. Member Distortions from Global Displacements

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AN INTRODUCTION TO GEOMETRICALLY NONLINEAR FINITE
ELEMENT STIFFNESS ANALYSIS USING THE
INCREMENTAL STEP AND EQUILIBRIUM METHODS

by

Warren Averill Stewart

(ABSTRACT)

The derivation of the beam-column element matrices and development of the incremental step method and of the equilibrium method are abstracted from other authors. Emphasis is placed on the consistency of the assumptions and restrictions imposed upon the element and the two methods. Two algorithms are developed describing the two methods and are compatible with the element. Three example problems are solved illustrating the features of the two methods. A comparison of the solutions obtained for each method is made.