Algebraically growing waves in ducts with sheared mean flow

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Standing or traveling waves which vary algebraically with the axial distance in uniform ducts with
sheared mean velocity profiles are investigated. The results show that such waves are not possible for
ducts with uniform cross sections and fully developed mean flows.

INTRODUCTION

Recently, the problem of sound propagation through ducts with mean flow has received considerable attention as evidenced by the large number of contributions (see Refs. 1, 2, and 3). To predict the reduction of sound as it travels through a duct with a given length, most of the authors quoted in the above references assume that all acoustic flow quantities can be expressed in terms of the normal modes in the duct as

$$Q = \sum_{n=0}^{\infty} A_n\{q_n(y)\exp[i(k_n x - \omega t)]$$
$$+ \overline{q}_n(y)\exp[-i(k_n x - \omega t)]\}. \tag{1}$$

Here, Q stands for any of the acoustic flow quantities u (axial velocity), v (normal velocity), p (pressure), x is the axial coordinate, y is the normal coordinate, and t is time. The real and imaginary parts of k_n represent, respectively, the wavenumber and attenuation rate of the nth mode. In writing Eq. 1, the above authors assumed that the eigenfunctions between the brackets in Eq. 1 form a complete set.

In this paper we present an indirect argument that these eigenfunctions are complete by showing that (a) there cannot exist standing wave solutions of the acoustic equations of the form

$$Q = q(x,y)e^{i\omega t} + \overline{q}(x,y)e^{-i\omega t}, \tag{2}$$

where $q(x, y)$ is an algebraic function of x, and (b) whereas Nayfeh and Telionis[4] showed that algebraically varying traveling modes [i.e., $A_n = A_n(x)$ in Eq. 1] are possible in ducts with varying cross sections, such modes do not exist in uniform ducts.

I. STANDING WAVES

The differential equations that govern the propagation of sound in ducts with parallel generators were derived by Pridmore-Brown[5] from the Euler equations. For a two-dimensional flow with the x axis directed along the axis of the duct, these equations may be expressed in terms of the acoustic pressure p and the normal component v of the acoustic velocity as

$$(1 - M^2)\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} - \frac{2M}{c}\frac{\partial^2 p}{\partial x \partial t}$$
$$+ 2\rho_0 c\frac{dM}{dy}\frac{\partial v}{\partial x} - \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} = 0, \tag{3}$$

$$\frac{\partial v}{\partial t} + Mc\frac{\partial v}{\partial x} + \frac{1}{\rho_0}\frac{\partial p}{\partial y} = 0, \tag{4}$$

where ρ_0 is the density of the mean flow, c is the speed of sound, and M is the Mach number which is assumed to be a function of y only. These equations are valid only when the neglected viscous acoustic terms are small compared with retained invisid acoustic terms. Thus Eqs. 3 and 4 are not valid near the boundaries where acoustic boundary layers exist[6] and near critical layers[7] where the speed of propagation equals the fluid speed. None of the existing literature of the acoustic propagation in ducts with sheared mean flow takes critical layers into consideration. In this paper, we assume the mean flow speed to be small enough so that critical layers do not exist in the ducts under consideration. For lined walls the boundary conditions are

$$v/p = \pm \beta/\rho_0 c \quad \text{at} \quad y = \pm d, \tag{5}$$

where β is the acoustic admittance of the walls and $2d$ is the width of the duct.

In the farfield we seek asymptotic solutions to Eqs. 3 and 4 in the form

$$p = e^{i\omega t}x^r[F_0(y) + x^{-1}F_1(y) + x^{-2}F_2(y) + \cdots], \tag{6}$$

$$v = e^{i\omega t}x^r[G_0(y) + x^{-1}G_1(y) + x^{-2}G_2(y) + \cdots], \tag{7}$$

where r is any real number. We substitute Expansions 6 and 7 into Eqs. 3 and 4, equate coefficients of like powers of x to zero, and obtain

$$F_0'' + (\omega^2/c^2)F_0 = 0, \tag{8}$$

$$F_0' + i\rho_0\omega G_0 = 0; \tag{9}$$

$$F_1'' + (\omega^2/c^2)F_1 = 2i\omega r\frac{M}{c}F_0 - 2\rho_0 cr\frac{dM}{dy}G_0, \tag{10}$$

$$F_1' + i\rho_0\omega G_1 = -\rho_0 rM_c G_0; \tag{11}$$

$$F_m'' + (\omega^2/c^2)F_m = \xi_m, \quad \text{for } m \geq 2, \tag{12}$$

$$F_m' + i\rho_0\omega G_m = \zeta_m, \quad \text{for } m \geq 2, \tag{13}$$

where ξ_m and ζ_m are known functions of F_0, F_1, ..., F_{m-1} and G_0, G_1, ..., G_m.

Substituting Expansions 6 and 7 into Eq. 5 and again equating coefficients of like powers of x, we obtain

$$G_m(d)/F_m(d) = \beta/\rho_0 c, \quad m = 0, 1, 2, \ldots . \tag{14}$$

The symmetric solution of Eqs. 8 and 9 subject to the

Boundary Condition 14 with $m = 0$ is

$$F_0 = A_n \cos(\omega y / c),$$
$$G_0 = - (iA_n / \rho_0 c) \sin(\omega y / c), \tag{15}$$

provided that

$$\tan(\omega d / c) = i\beta. \tag{16}$$

The antisymmetric counterpart of Eqs. 15 and 16 is

$$F_0 = A_n \sin(\omega y / c),$$
$$G_0 = (iA_n / \rho_0 c) \cos(\omega y / c), \tag{17}$$
$$\cot(\omega d / c) = - i\beta. \tag{18}$$

Equations 16 and 18 can be satisfied only when β is purely imaginary; that is, only when the resistance of the lining material is zero. Lining materials with such a property, though, are considered unrealistic. Therefore, we conclude that standing waves which vary algebraically with the axial distance may be possible only for hard-walled ducts, provided that the frequency is an element of the infinite discrete set

$$\omega_n = n\pi c / d, \qquad \text{for symmetric modes,}$$
$$\omega_n = (n + \tfrac{1}{2})\pi c / d, \quad \text{for antisymmetric modes,} \tag{19}$$

With $\beta = 0$, an inspection of Eqs. 4–7 reveals that the Boundary Conditions 14 reduce to

$$F_m'(d) = 0, \quad m = 0, 1, 2, \ldots \quad . \tag{20}$$

Taking $\omega = \omega_n = n\pi c / d$, we can write Eq. 10 as

$$F_1'' + \frac{n^2 \pi^2}{d^2} F_1$$
$$= 2ir A_n \left[\frac{n\pi}{d} M \cos \frac{n\pi y}{d} + \frac{dM}{dy} \sin \frac{n\pi y}{d} \right] . \tag{21}$$

Equation 21 has a solution if, and only if, its non-homogeneous part is orthogonal to the solution of the adjoint homogeneous problem[8] $\cos(n\pi y / d)$; that is, if, and only if,

$$\frac{n\pi}{d} \int_0^d M(y) \cos^2 \frac{n\pi y}{d} \, dy + \frac{1}{2} \int_0^d \frac{dM}{dy} \sin \frac{2n\pi y}{d} \, dy = 0. \tag{22}$$

Integrating by parts, we can rewrite Eq. 22 as

$$S \equiv \int_0^d M(y) \, dy - \int_0^d M(y) \cos \frac{2n\pi y}{d} \, dy = 0. \tag{23}$$

The case $n = 0$ corresponds to a time independent solution which is not of interest. For $n \geq 1$, we assume that the mean Mach number profile is uniform and equal to M_0 in the core of the duct, and possesses a boundary layer profile $M(y)$ in a thin layer of thickness δ next to the walls. Then

$$S \equiv M_0(d - \delta) + \int_{d-\delta}^{\delta} M(y) \, dy - M_0 \frac{d}{2n\pi} \sin \frac{2n\pi(d - \delta)}{d}$$
$$- \int_{d-\delta}^{\delta} M \cos \frac{2n\pi y}{d} \, dy. \tag{24}$$

A lower bound on S is given by

$$\frac{S}{M_0 d} > 1 - \frac{\delta}{d} - \frac{1}{2n\pi} . \tag{25}$$

As long as δ / d is less than 0.85, which is a value very much larger than those encountered in practical applications, $S > 0$. Therefore, there cannot exist standing waves that vary algebraically with the axial distance; this conclusion appears to be different from that reached by Möhring.[9]

II. TRAVELING WAVES

Next let us determine if traveling waves that vary algebraically along the duct can exist. We seek expansions in the farfield of the form

$$p = e^{i\theta} x^r [F_0(y) + x^{-1} F_1(y) + \cdots], \tag{26}$$
$$v = e^{i\theta} x^r [G_0(y) + x^{-1} G_1(y) + \cdots], \tag{27}$$

where $\theta = kx - \omega t$. Substituting Expansions 26 and 27 into Eqs. 3 and 4 and equating coefficients of like powers of x, we obtain

$$F_0'' + \left[\frac{\omega^2}{c^2} - k^2(1 - M^2) + 2M \frac{\omega}{c} k \right] F_0 + 2i\rho_0 ck \frac{dM}{dy} G_0 = 0, \tag{28}$$

$$F_0' + i\rho_0(\omega - kMc)G_0 = 0, \tag{29}$$

$$F_1'' + \left[\frac{\omega^2}{c^2} - k^2(1 - M^2) + 2M \frac{\omega}{c} k \right] F_1 + 2\rho_0 c \frac{dM}{dy} ikG$$
$$= \frac{2Mi\omega}{c} r F_0 - 2\rho_0 cr \frac{dM}{dy} G_0 - 2(1 - M^2) ikr F_0, \tag{30}$$

$$F_1' + i\rho_0(\omega - kMc)G_1 = - Mcr G_0, \tag{31}$$

$$F_m'' + \left[\frac{\omega^2}{c^2} - k^2(1 - M^2) + 2M \frac{\omega}{c} k \right] F_m + 2\rho_0 c \frac{dM}{dy} ikG = \xi_m,$$
$$\text{for } m = 2, 3, \ldots, \tag{32}$$

$$F_m' + i\rho_0(\omega - kMc)G_m = \zeta_m . \tag{33}$$

For a mean velocity profile that vanishes at the walls, the Boundary Conditions 14 still hold.

Eliminating G_0 from Eqs. 28 and 29 we have

$$L(F_0) \equiv F_0'' - \frac{2kc}{\omega - Mkc} \frac{dM}{dy} F_0'$$
$$+ \left[\frac{\omega^2}{c^2} - k^2(1 - M^2) + 2M \frac{\omega}{c} k \right] F_0 = 0. \tag{34}$$

Note that Eq. 34 has a singularity at y_c where $M(y_c) = \omega / kc$; a critical layer exists near such a singularity and the acoustic viscous terms cannot be neglected in such a layer. In what follows, we assume the mean Mach number flow to be small enough so that $M < \omega / kc$ everywhere in the duct. For each frequency there exists an infinite number of eigenvalues k and eigenfunctions F_0 that satisfy Eq. 34, the Boundary Condition 14, and $F_0(0) = 0$ or $F_0'(0) = 0$. These eigenfunctions can be obtained either analytically or numerically. In what follows, we consider only the case of distinct eigenvalues.

Eliminating G_1 from Eqs. 30 and 31 and using Eq. 34, we obtain

$$L(F_1) = \psi_1[M(y), F_0(y)], \tag{35}$$

where ψ_1 is a known function of $M(y)$ and $F_0(y)$. For a given frequency, the nonhomogeneous first-order problem has a solution if, and only if, its nonhomogeneous

part is orthogonal to the solution of the adjoint homo-
geneous problem. This solvability condition imposes a
restriction on the mean Mach number profile. Carrying
the expansion to higher order, we find more restric-
tions on $M(y)$. If r is a positive integer, there will be
r restrictions on $M(y)$; otherwise, there will be an in-
finite number of restrictions on $M(y)$. Since $M(y)$ must
satisfy the Navier–Stokes equations, it will not satisfy,
in general, any other restriction.

III. RESULTS AND DISCUSSION

The results show that there cannot exist standing
waves which vary algebraically along a uniform duct if
the walls have a finite resistance. Since Nayfeh[6] showed
that the acoustic boundary-layer produces an effective
finite admittance at the wall even if it is rigid, we con-
clude that there are no standing waves which vary alge-
braically along a duct with uniform cross section.

Also the results show that traveling waves which vary
algebraically along the duct do not exist unless the mean
velocity profile satisfies one or more restrictions be-
sides satisfying the Navier–Stokes equations. Clearly
such velocity profiles are not realistic.

It should be mentioned that our analysis is valid only
when the flow speed is everywhere less than the speed
of propagation. If these speeds are equal anywhere in
the duct, the acoustic viscous terms must be included
within the so-called critical layer. Neglecting these
viscous terms, one may find a continuous spectrum
of eigenvalues due to the fact that the invisid equa-
tion has a singularity,[10] and hence, the initial value
problem might have algebraically growing traveling
waves.[11]

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