Pseudocompactifications

and

Pseudocompact Spaces

by

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CHAPTER 1

INTRODUCTION

Spaces on which every continuous real valued function is bounded were first introduced in 1906 under the name of extremal spaces by one of the founders of abstract topology, Maurice Fréchet. The systematic study of these spaces was begun in 1948 by Edwin Hewitt [34], who called them pseudocompact spaces. The term pseudocompact seems to have been introduced after Yu. M. Smirnov showed that extremality is equivalent to the following compactness property: Every locally finite covering has a finite refinement. It soon became apparent that this property was significantly weaker than compactness.

Pseudocompact spaces turn out to be an important class of spaces in analysis. Glicksburg [28] showed that they are precisely the spaces on which the theorems of Riesz, Dini and Ascoli hold. In this same paper he presented some analytical characterizations of pseudocompact spaces. In [29] he investigated products of pseudocompact spaces when solving the following problem: for which spaces does the equality $\beta(X \times Y) = \beta X \times \beta Y$ hold?

Bagley, Connell and McKnight [4] studied properties characterizing pseudocompactness without requiring the complete regularity of the spaces. They also investigated properties weaker than compactness but stronger than pseudocompactness. Stephenson expanded upon these results in [76]. Iseki and Kashara [43] characterized these spaces as those on which certain types of convergence of functions hold.

Vidossich [83] generalizes the concept of pseudocompactness with his pseudo-$\aleph$-compact spaces. Kennison [51] also generalizes with the idea of m-pseudocompactness. In his paper Vidossich also introduces a one-point pseudocompactification but does not investigate the properties of this pseudocompactification. This is the only mention of pseudocompactifications which appears in the literature.

Products of pseudocompact spaces have been extensively studied. Glicksburg [29], Frolik [24], Terasaka [82], Tamano [80], Noble [68] and Stephenson [76] all studied conditions under which pseudocompactness carries over to the product space. Out of the investigation of products grew the study of pseudocompactness under certain maps and inverse maps. Hanai and Okuyama [31], Isiwata [44] and [45], Woods [87] and Zenor [90] consider certain functions which characterize pseudocompactness.

Other pseudocompact properties appear in the investigation of z-filters. (e.g. Mandelker [53], Johnson and Mandelker [50] and Woods [89]), in the investigation of the remainder $\beta X - X$. (e.g. Fine and Gillman [19] and [20], Isiwata [48], Woods [86] and [88]) and in the investigation of real compact spaces (Dykes [17], Comfort [10] and [11], Negrapontis [66], and Woods [89]).

However in all the literature there is only one mention of pseudocompactifications and that is by Vidossich [83]. This is one area which we investigate in this paper. We first characterize pseudocompact spaces and derive a few new results. Then we discuss the pseudocompact subspaces of βX which contain X. Using a particular subspace of βX we can derive some further results concerning pseudocompact subspaces of the space X. The concept of relative pseudocompactness turns out to be very important when defining a pseudocompactification. It has appeared several times in the literature but has never been fully investigated. We provide results concerning relative pseudocompactness and its relation to pseudocompact subspaces of βX.

Other areas which seem natural to investigate but which for the most part do not appear in the literature are local pseudocompactness and σ-pseudocompactness. Results concerning each of these concepts are presented in an effort to "round out" the study of pseudocompactness.

We rely heavily on notation and background material from Gillman and Jerison [27]. Any set of the form $f^{-1}[0]$ for a continuous function f from X to the reals will be called a zero set. The family of zero sets of a space X will be denoted by Z(X). A cozero set is any set of the form $\{x: |f(x)| > 0\}$ where f is a continuous map from X to the reals. The closure of a set A is a space X will be denoted by $cl_X A$.

A filter base on a space X is a family $\mathcal{F} = \{F_\alpha : \alpha \in \mathcal{A}\}$ of subsets of X having the two properties:

1) for every $\alpha \in \mathcal{A}$, $F_\alpha \neq \emptyset$

2) for every α and $\beta \in \mathcal{A}$ there exists $\gamma \in \mathcal{A}$ such that $F_\gamma \subset F_\alpha \cap F_\beta$.

A collection of sets A_α is said to be locally finite if each point of the space has a neighborhood V such that $V \cap A_\alpha \neq \emptyset$ for at most finitely many indices α .

With this introduction and these statements on notation we begin our study of pseudocompact spaces.

CHAPTER 2

PSEUDOCOMPACTNESS AND OTHER PROPERTIES

We begin our study by relating pseudocompactness to other topological properties. In particular we investigate compactness and properties weaker than compactness which imply pseudocompactness and include conditions underwhich the implications may be reversed. Most of these results were known to Hewitt and many are due to Bagley, Connell and McKnight [4] and Stephenson [76].

Definition 2.1: A topological space X is pseudocompact if every continuous real valued function defined on X is bounded.

Definition 2.2: $C(X)$ ($C^*(X)$) denotes the ring of real valued (bounded) continuous functions on X.

Definition 2.3: a) A filter base $\mathcal{F}$ on a space X is *fixed* or *free* according to whether $\bigcap_{F \in \mathcal{F}} cl_X F$ is nonempty or empty.

b) Two filter bases $\mathcal{F}_1$ and $\mathcal{F}_2$ are said to be *equivalent* if for every $F_1 \in \mathcal{F}_1$ there corresponds an $F_2 \in \mathcal{F}_2$ such that $F_2 \subseteq F_1$ and vice versa.

c) A filter base $\mathcal{F}$ on X is said to be *open* *(closed)* if all the members of $\mathcal{F}$ are open (closed) sets.

d) An open filter base which is equivalent to a closed filter base is said to be *regular*.

e) An open filter base $\mathcal{F}$ on X is said to be *completely* regular provided that for each $F_1 \in \mathcal{F}$ there exists a set $F_2 \in \mathcal{F}$ and a function $f \in C(X)$ with $0 \leq f \leq 1$ such that f is equal to zero on F_2 and equal to 1 on $X - F_1$.

Definition 2.4: An open cover θ of X is said to be cocompletely regular if for each $O \in \theta$ there is an $O' \in \theta$ and $f \in C(X)$, $0 \leq f \leq 1$, such that f is equal to zero on O and f is equal to 1 on $X - O'$.

This first theorem can be found in Stephenson's paper [76].

Theorem 2.1: The following are equivalent for any topological space:

a) X is pseudocompact

b) For every space Y and every continuous function $f: X \rightarrow Y$, f(X) is pseudocompact.

c) For every $f \in C(X)$, f(X) is closed in R.

d) For every $f \in C(X)$, f(X) is compact.

e) For every $f \in C(X)$, there is a point $x \in X$ such that $\sup_{y \in X} f(y) = f(X)$.

f) Any countable collection of zero sets with the finite intersection property has non-empty intersection.

g) Every locally finite collection of cozero sets of X is finite.

h) Every completely regular countable filter base on X is fixed

i) Every countable cocompletely regular cover of X has a finite subcover.

We can add yet another condition which is equivalent to pseudocompactness in any topological space.

Theorem 2.2: X is pseudocompact if and only if every countable cover of X by cozero sets has a finite subcover.

Proof: Suppose the condition holds. Then for each $f \in C(X)$ $\{x: |f(x)| < n\}$ is a cozero set and $X \subset \bigcup_{n=1}^{\infty} \{X: |f(x)| < n\}$. Thus the collection forms a countable cozero set cover and must have a finite subcover. Therefore there must exist n such that $|f(x)| < n$ for all $x \in X$ for each function f. Thus each $f \in C(X)$ is bounded and X is pseudocompact.

Suppose X is pseudocompact. Let $\{C_i\}_{i=1}^{\infty}$ be a countable cover of X by cozero sets. Let $Z_1 = X - C_1$, $Z_2 = X - [C_1 \cup C_2], \ldots, Z_k = X - \bigcup_{i=1}^{k} C_i$, ... Each Z_i is a zero set of X. Now if there does not exist a finite subcover, then the collection $\{Z_i\}_{i=1}^{\infty}$ has the finite intersection property. Since $\{C_i\}_{i=1}^{\infty}$ covers X, $\bigcap_{i=1}^{\infty} Z_i = \emptyset$. But this contradicts (f) of Theorem 2.1. Therefore there exists a finite subcover.

We include a proof of the above theorem for the sake of completeness. This theorem appeared in 1957 in Annales Academiae Scientiarum Fennicae Series A. I. Mathematica in a paper by Jouko Väänänen. Further generalizations may be found in a paper by the same author in No. 559 (1973) of the same journal.

The following definition is due to Mardesić and Papić [54] and characterizes spaces which they called feebly compact. More recently the term lightly compact has been applied.

Definition 2.5: A topological space X is said to be lightly compact if every locally finite collection of open subsets of X is finite.

This property turns out to be equivalent to pseudocompactness in completely regular spaces but in general is stronger than pseudocompactness (example 2.1) and weaker than countable compactness (example 10.3). We now give some useful conditions equivalent to light compactness which can be found in [4].

Theorem 2.3: For any space X the following are equivalent:

a) X is lightly compact.

b) Every countable, locally finite collection of disjoint open subsets of X is finite.

c) Every countable family of disjoint sets has a cluster point.

d) For every countable open covering of X, and for every infinite subset A of X, the closure of some member of the cover contains infinitely many points of A.

e) Every countable open cover of X has a finite subcollection whose union is dense in X.

f) Every countable open filter base has an adherent point.

g) Every countable sequence of non-empty closed subsets of X with non-void interiors and with the finite intersection property has non-empty intersection.

The next theorem can be found in [76] and lists sufficient conditions for a space X to be pseudocompact. Additional hypotheses are stated so that each condition is also necessary. First we need the following definitions.

<u>Definition 2.6</u>: An open cover $\mathcal{O}$ of a space X is said to be <u>coregular</u> provided that for every set $O \in \mathcal{O}$ there is a set $P \in \mathcal{O}$ such that $cl_X O \subset P$.

<u>Definition 2.7</u>: A space X is <u>weakly normal</u> provided any two disjoint closed subsets, one of which is countable, are completely separated.

On any space X each of the following is a sufficient condition for X to be pseudocompact.

A(1) Every countable open cover has a finite subcover.

A(2) Every countable filter base on X has an adherent point.

B(1) Every locally finite system of open sets in X is finite.

B(2) Every countable, locally finite disjoint system of open subsets of X is finite.

B(3) If $\mathcal{U}$ is a countable open cover of X and A is an infinite subset of X, then the closure of some member of $\mathcal{U}$ contains infinitely many points of A.

B(4) If $\mathcal{U}$ is a countable open cover of X, then there is a finite subcollection of $\mathcal{U}$ whose closures cover X.

B(5) Every countable open filter base on X has an adherent point.

C(1) If $\mathcal{U} = \{U_n : n \in N\}$ is a collection of non-empty open subsets of X such that $cl_X U_i \cap cl_X U_j = \emptyset$ whenever $i \neq j$, then $\mathcal{U}$ is not locally finite.

D(1) Every countable coregular cover of X has a finite subcover.

D(2) Every countable regular filter base on X is fixed.

Theorem 2.4: On a space X the following hold.

a) A(1) and A(2) are equivalent.

b) B(1), B(2), B(3), B(4), and B(5) are equivalent.

c) D(1) and D(2) are equivalent.

d) A(2) implies B(5).

e) B(5) implies C(1).

f) C(1) implies D(2).

g) D(2) implies that X is pseudocompact.

h) If each point of X has a fundamental system of closed neighborhoods, then D(2) implies B(2).

i) If X in completely regular and pseudocompact then B(1) holds.

j) If X is weakly normal and pseudocompact then A(1) holds.

The next result includes some conditions weaker than compactness not listed in Stephenson's theorem. The step by step weakening of the conditions is more apparent and each condition is strictly weaker than the preceeding one in general spaces. This can be found in Stone's paper [79].

Theorem 2.5: In any topological space X each of the following properties implies the next.

a) X is compact.

b) Every open cover has a finite subcollection whose union is dense in X.

c) X is lightly compact.

d) Every locally finite open covering of X has a finite subcovering.

e) Every locally finite open covering of X has a finite subcollection whose union is dense in X.

f) Every countable regular filter base on X is fixed.

g) Every star finite open covering of X is finite.

h) X is pseudocompact.

None of the above implications can be reversed in general. However if X is regular g) implies c) and b) implies a). If X is completely regular h) implies c).

<u>Example 2.1</u>: A space which is pseudocompact but not lightly compact. Let $X = \{P : P \subseteq N,\ P \neq \emptyset,\ P \text{ is finite}\} \subseteq 2^N$. If $P \in X$, let a neighborhood of P be $2^P - \{\emptyset\}$. (This space is T_o but not T_1). If $\mathcal{U}$ is an open cover of X, then $\mathcal{U}$ must cover the point $P_n = \{1,2,\ldots,n\}$ for all $n \in N$. If $\mathcal{U}$ is locally finite, the points P_n must be contained in at most finitely many members of $\mathcal{U}$, since otherwise local finiteness is violated for the point $\{1\} \in X$. But then it is clear that the members of $\mathcal{U}$ which cover all the P_n will cover X, since any point of X is a finite subset of P_n for n sufficiently large. Thus X satisfies property d) of Theorem 2.5 and hence is pseudocompact. However the collection $\{\{n\}\}$, $n \in N$ is clearly locally finite (since no member of X can have infinitely many of the $\{n\}$ as subsets) but not finite. Therefore X is not lightly compact. This example also shows that property d) of Theorem 2.5 does

not imply c) of Theorem 2.5.

The next results can be found in [4] and follow easily from Lemma 2.1. They give conditions under which pseudocompact spaces are compact. The theorems are stated in terms of light compactness since we wish to assume as few separation axioms as possible. However in view of Theorem 2.4 i) is completely regular T_1 spaces we may replace light compactness with pseudocompactness.

Lemma 2.1: A lightly compact space is Lindelöf if and only if every open cover has a refinement which is σ-locally finite.

Proof: If X is Lindelöf every open cover will have a countable subcover which is clearly σ-locally finite. Conversely if every open cover has a σ-locally finite refinement, then each locally finite family must be finite by light compactness. Hence the σ-locally finite family is countable.

Theorem 2.6: X is compact if and only if X is countably compact and has the property that every open cover has a σ-locally finite refinement.

Theorem 2.7: X is compact if and only if X is paracompact and lightly compact.

Theorem 2.8: A metric space X is compact if and only if it is pseudocompact.

Let us pause for a moment and examine why the locally finite condition is a logical one to appear in the study of pseudocompact spaces. First we need a result from [27].

Definition 2.8: A subspace S of X is said to be C-embedded (C^*-embedded) in X provided every function in C(S) ($C^*(S)$) can be continuously extended to a function C(X) ($C^*(X)$).

Theorem 2.9: A space X is pseudocompact if and only if X contains no C-embedded copy of N.

If $\{V_\alpha : \alpha \in A\}$ is a family of disjoint sets in a completely regular space X with non-empty interiors and if for each index α, the set $\cup\{V_\sigma : \sigma \in A, \sigma \neq \alpha\}$ is closed. Then any set D formed by selecting one element from the interior of each V_α is C-embedded in X. [3L of [27]]

A locally finite collection of disjoint open subsets is closure preserving and hence the closures will satisfy the hypothesis. If A is an infinite index set, then we can form an infinite discrete C-embedded subset D of X. Any unbounded function on D has a continuous extension to all of X and hence X could not be pseudocompact.

Further light is shed on these conditions when we characterize pseudocompact spaces by the maximal ideals of C(X) and hence by the z-ultrafilters which they generate. We know that every residue class field of C or C^* (modulo a maximal ideal M) contains a copy of the real field R: the set of images of the constant maps, under the canonical homomorphism. When the canonical copy of R is the entire field C/M, then M is said to be a <u>real</u> ideal. Where the residue class field modulo M is not real. M is said to be a hyperreal ideal.

This next theorem can be found in [27] and leads to the definition of realcompact spaces.

<u>Theorem 2.10</u>: Every maximal ideal in C(X) is real if and only if X is pseudocompact.

<u>Definition 2.9</u>: A space X is realcompact if and only if every free maximal ideal in C(X) is hyperreal.

We might point out here that in completely regular spaces compactness, the second axiom of countability, and the Lindelöf property all imply realcompactness. The next theorem substantiates property f) of theorem 2.1 and also can be found in [27].

<u>Theorem 2.11</u>: The following are equivalent for any maximal ideal M of C(X).

a) M is real

b) Z[M] (The Z-filter generated by M) is closed under countable intersection

c) Z[M] has the countable intersection property

The ideas of regular and completely regular filter bases are generalizations of the z-filter concept.

Let us now proceed with some properties of pseudocompact spaces and their subspaces. The four theorems 2.12, 2.13, 2.16 and 2.17 can be found in Glicksburg [29].

Theorem 2.12: A completely regular space X is pseudocompact if and only if every sequence of non-void open sets has a cluster point.

Theorem 2.13: In a completely regular pseudocompact space X, the closure of every open subset is pseudocompact.

The hypothesis of complete regularity of X is necessary here as the following example shows.

Example 2.2: (Stephenson [76]) A pseudocompact space with a regular closed subspace which is not pseudocompact. Let $X = [0,1]$, $\mathcal{T}$ = usual topology. Let $\{X_n\}_{n=1}^{\infty}$ be a collection of dense, disjoint subsets of X whose union is X. Let $\mathfrak{w}$ be the topology on X generated by $\mathcal{T} \cup \{X_{2n-1} \cup X_{2n} \cup X_{2n+1}\}_{n=1}^{\infty} \cup \{X_{2n-1}\}_{n=1}^{\infty}$. To show $(X,\mathfrak{w})$ is pseudocompact we shall first show the following:

If $a,b \in R$; O,P are open subsets of R such that $cl_R P \subset O$ and if $f \in C(X,\mathfrak{w})$ with $f((a,b) \cap X_j) \subseteq P$ (where (a,b) denotes the open interval from a to b in R), then $f((a,b) \cap X) \subseteq O$. Note that for any $n \in N$, $W \in \mathfrak{w}$ the following hold.

$cl_{\mathfrak{w}}((a,b) \cap X_1) = [a,b] \cap (X_1 \cup X_2)$

$cl_{\mathfrak{w}}((a,b) \cap X_{2n+1}) = [a,b] \cap (X_{2n} \cup X_{2n+1} \cup X_{2n+2})$

$cl_{\mathfrak{w}}((a,b) \cap X_{2n}) = [a,b] \cap X_{2n}$

also if $(a,b)\cap X_1 \subset W$, then $[a,b]\cap (X_1 \cup X_2)\subseteq cl_{\mathfrak{W}} W$

if $(a,b)\cap X_{2n+1} \subset W$, then $[a,b]\cap [X_{2n+1}\cup X_{2n+2}]\subset cl_{\mathfrak{W}} W$

if $(a,b)\cap X_2 \subseteq W$, then $[a,b]\cap [X_1 \cup X_2 \cup X_3 \cup X_4] \subseteq cl_{\mathfrak{W}} W$

and if $(a,b)\cap X_{2n+2} \subseteq W$, then $[a,b]\cap (X_{2n} \cup X_{2n+1} \cup X_{2n+3} \cup X_{2n+4}) \subseteq cl_{\mathfrak{W}} W$.

Now since R is normal and since $cl_R P \subset 0$, we have for each $n \in N$ open sets V_n such that $cl_R P \subseteq V_n \subseteq cl_R V_n \subset V_{n+1} \subseteq 0$. Let $Y_n = f^{-1}(V_n)$ for each $n \in N$. Then $(a,b)\cap X_j \subseteq Y_1$, $Y_n \in \mathfrak{W}$ for each $n \in N$ and $cl_{\mathfrak{W}} Y_n \subseteq Y_{n+1}$ by continuity of f. If j is odd, then for every $n \geq \frac{j-2}{2}$, $(a,b)\cap (\bigcup_{k=1}^{j+2n+1} X_k) \subseteq Y_{2+n}$, and if j is even, then for every $n \geq \frac{j+2}{2}$, $(a,b)\cap (\bigcup_{k=1}^{j+2n-2} X_k) \subseteq Y_n$. These statements may be proved by induction on n, using the fact that $f(cl_{\mathfrak{W}} A) \subseteq cl_R[f(A)]$, and by using the above deduced formulae. It follows that $(a,b)\cap X = \bigcup_{n=\frac{j+2}{2}}^{\infty} \{(a,b)\cap (\bigcup_{k=1}^{j+2n-2} X_k)\} \subseteq \bigcup_{n=\frac{j+2}{n}}^{\infty} Y_{2+n} \subseteq \bigcup_{n=1}^{\infty} Y_n$.

Therefore $f((a,b) \cap X)\subseteq f(\bigcup_{n=1}^{\infty} Y_n) = \bigcup_{n=1}^{\infty} f(Y_n) \subseteq \bigcup_{n=1}^{\infty} V_n \subset 0$.

We now show X is pseudocompact in the $\mathfrak{W}$ topology. Let f be an unbounded positive continuous function on X. Since f is unbounded f is not continuous in the usual topology. Then for some open set $0 \subseteq R$, $f^{-1}(0)$ does not contain an interval. But let $P \subseteq R$ such that $cl_R P \subseteq 0$. Since f is continuous in the $\mathfrak{W}$ topology, $f^{-1}(P)$ must contain a member of $\mathfrak{W}$. ie. a set of the form $(a,b)\cap X_j$ for some $j \in N$. It follows that $(a,b) \subseteq f(0)$. This is a contradiction and hence X must be pseudocompact.

Let $a \in X_1 \cap [0,1)$. Consider the open subset $A = X_3 \cap (a,1]$.

The function $F(x) = \frac{1}{x-a}$ is continuous on $cl_{\omega}A$, since $a \notin cl_{\omega}A$. But f is unbounded so that $cl_{\omega}A$ is not pseudocompact.

This example also shows that in theorem 2.4, (h) does not imply (g) in general. $\{X_{2n-1} \cup X_{2n} \cup X_{2n+1}\}_{n=1}^{\infty}$ is a star finite open cover of X which has no finite subcover.

In any pseudocompact space however the following holds. The proof is essentially due to Mandelker [53].

Theorem 2.14: In any pseudocompact space the closure of a cozero set is pseudocompact.

Proof: Let Z be a zero set of space X and let $C = X - Z$. Let $S = cl_X C$. Suppose S is not pseudocompact. We know that S is relatively pseudocompact since X is pseudocompact. Now there exists $h \in C(S)$ such that $h \geq 1$ and is unbounded on S. Hence h is unbounded on C. Thus C contains a C-embedded subset D of S on which h is unbounded and D is completely separated from the zero set $W = Z \cap S$ of S. Therefore we may choose $g \in C(S)$ with $g = 0$ on D and $g = 1$ on W. Put $f = (\frac{1}{h}) \vee g$ on S and $f = 1$ on $X - S$. Clearly $f \in C(X)$, $f > 0$ and $1/f$ is unbounded on D. Thus X is not pseudocompact. We have a contradiction and it follows that S must be pseudocompact.

Note here that $1/f$ is unbounded on C also. Hence if C is relatively pseudocompact then S is pseudocompact.

The next theorem gives a characterization of pseudocompact zero sets in any space X. It is analogous to lemma 4.10 of [27] for compact zero sets.

Theorem 2.15: If a zero set Z of X is pseudocompact then Z belongs to no free hyperreal z-ultrafilter on X.

Proof: Let Z be pseudocompact. Suppose there exists a hyperreal z-ultrafilter $\mathcal{U}$ containing Z. Then $\mathcal{U}$ does not have the countable intersection property. ie. There exists a family $\{Z_i : i = 1,2,\ldots\}$ in $\mathcal{U}$ such that $\cap \{Z_i : i = 1,2,\ldots\} = \emptyset$. Now $Z \cap Z_i \neq \emptyset$ for all $i = 1, 2, \ldots$, and $Z \subseteq \sim \bigcap_{i=1}^{\infty} Z_i = \bigcup_{i=1}^{\infty} (\sim Z_i)$. Thus there exists a countable cover of Z by cozero sets. By theorem 2.2 there exists a finite subcover. That is $Z \subset \bigcup_{j=1}^{k} (\sim Z_{i_j})$ for some finite subcollection $Z_{i_1}, Z_{i_2}, \ldots Z_{i_k}$. This implies $Z \subset \sim (\bigcap_{i=1}^{k} Z_{i_j})$ and hence $Z \cap [\bigcap_{i=1}^{k} Z_{i_j}] = \emptyset$. This contradicts the properties of a filter. Thus Z is not contained in any hyperreal z-ultrafilter on X.

Theorem 2.16: If X and Y are pseudocompact spaces then $X \cup Y$ is pseudocompact.

Theorem 2.17: If X is dense in Y and X is pseucocompact then so is Y.

As a corollary to 2.17 we have the following.

Corollary 2.17: If A is pseudocompact subspace of X, then $cl_X A$ is pseudocompact.

We can slightly generalize theorem 2.17.

Definition 2.10: Let $A \subseteq X$. If every function in C(X) is bounded on A, then A is said to be <u>relatively pseudocompact</u>.

We might point out here that any pseudocompact subset is relatively pseudocompact but not conversely as every subset of a pseudocompact space is relatively pseudocompact.

<u>Theorem 2.18</u>: If X is dense in Y, then Y is pseudocompact if and only if X is relatively pseudocompact.

<u>Proof</u>: Suppose there exists a function $f \in C(Y)$ such that f is unbounded on Y. Since X is relatively pseudocompact, f is bounded on X. Therefore there exists a positive integer m such that $|f| < m$ for all $x \in X$. Hence $|f|^{-1}[(m,\infty)]$ is a non-empty open subset of Y which is disjoint from X. This is a contradiction. Therefore Y is pseudocompact. Conversely if Y is pseudocompact then every subset is relatively pesudocompact. Hence X is relatively pseudocompact.

We will return to this concept of relative pseudocompactness as it turns out to be very important in the area of pseudocompacifications.

Let ∂A denote boundary of A.

<u>Theorem 2.19</u>: Let X be pseudocompact and let A be a closed subset of X. If ∂A is pseudocompact then A is pseudocompact.

<u>Proof</u>: Suppose A is not pseudocompact. Then there exists $f \geq 1$ in C(A) such that f is unbounded on A. Now ∂A is pseudocompact, hence f is bounded on ∂A. Thus $g = \frac{1}{f}$ is bounded away from zero on ∂A. There exists a positive real number r such that $g(x) \geq r$ for all x

in ∂A. Let $W = \{x \in A : g(x) \geq r\}$. Then W is a zero set in A. Now A contains a C-embedded subset D of A on which f is unbounded and D is completely separated from the zero set W. Therefore we may choose $h \in C(A)$ with $h = 0$ on D and $h = 1$ on W. Put $k = (g \vee h)$ on A and $k = 1$ on $X - A$. Then $k \in C(X)$, $k > 0$ and $\frac{1}{k}$ is unbounded on D. Hence X is not pseudocompact. This is a contradiction and hence A must be pseudocompact.

We note here that if A is an open subspace of a pseudocompact space X and $cl_X A$ is pseudocompact, the boundary of A need not be pseudocompact. Consider the Tychonov plank of example 10.3. Let $A = T - N$ where N denotes the right edge $\{\omega_1\} \times N$. Then A is open in T, $cl_T A$ is T and $\partial A = N$ which is an infinite discrete subspace and hence not pseudocompact.

An analogous theorem to the one above might be if A is closed and ∂A is relatively pseudocompact, then A is relatively pseudocompact. This is false, however. Consider $A = \{x : x \in R \text{ and } x \geq 0\}$. Then $\partial A = 0$ which is relatively pseudocompact (in fact compact) yet A is not relatively pseudocompact. Also A is a zero set and hence even for a zero set Z, ∂Z relatively pseudocompact does not imply Z is relatively pseudocompact. This example also shows that theorem 2.19 is false if the space itself is not pseudocompact.

Since we know compactness, countable compactness and real compactness are preserved under finite intersections we would naturally investigate the intersection of two pseudocompact subsets of a pseudocompact space. In general the intersection of two pseudocompact spaces need not be pseudocompact, even if the space is compact and one of the subsets is compact. Consider the space ψ of example 10.4. D is a zero set of ψ but is not pseudocompact (this will be shown in a later section). Since ψ is pseudocompact $cl_{\beta\psi}D$ is a zero set in $\beta\psi$ and is compact and $D = cl_{\beta\psi}D \cap \psi$. Now ψ is pseudocompact and $cl_{\beta\psi}D$ is compact, yet D is not pseudocompact. Thus we see that pseudocompact subsets are not well behaved under intersection. However we do have the following results.

Lemma 2.2 If A and B are closed, then $\partial(A-B) \cap \partial(B-A) = \partial A \cap \partial B$.

Proof. Let $x \in \partial A \cap \partial B$. Since A and B are closed, $\partial A \subset A$ and $\partial B \subset B$. Therefore $x \in A \cap B$. Suppose $x \notin \partial(A-B)$. Now $\partial(A-B) = (\partial A - B) \cup (\partial B \cap A)$. Thus $x \notin (\partial A-B) \cup (\partial B \cap A)$. But A is closed so $\partial A-B \subset A-B$ and $x \notin A-B$. Thus $x \notin \partial A-B$, also $x \notin \partial B \cap A$. But we know $x \in A$ and by hypothesis $x \in \partial B$. This is a contradiction; therefore $x \in \partial(A-B)$. By symmetry $x \in \partial(B-A)$ also, and thus $\partial A \cap \partial B \subset \partial(A-B) \cap \partial(B-A)$.

Conversely, let $x \in \partial(A-B) \cap \partial(B-A)$. Since $x \in \partial(A-B)$ either $x \in \partial A$ or $x \in \partial B \cap A$ but not an element of ∂A. If $x \in \partial A \cap B$ but $x \notin \partial A$, then there exists N_x, a neighborhood of x, such that $N_x \subset A$. But $x \in \partial(B-A)$ and if $N_x \subset A$ then $N_x \cap (B-A) = \phi$. And hence $x \notin \partial(B-A)$. Therefore it must be that $x \in \partial A$. Similarily $x \in \partial B$ and the equality is established.

Theorem 2.20: Let A and B be closed and pseudocompact in a completely regular T_1 space. If $\partial A \cap \partial B$ is also pseudocompact, then $A \cap B$ is pseudocompact.

<u>Proof</u>: Let $\{G_2: i = 1, 2, \ldots\}$ be a countable cover of $A \cap B$. Then there exists $\{H_i: i = 1, 2, \ldots\}$ open in $A \cup B$ such that $H_i \cap (A \cap B) = G_i$ for every i. Now $\{H_i: i = 1,2,\ldots\} \cup (A-B) \cup (B-A)$ is a countable open cover of $A \cup B$. Since A is pseudocompact there exists a finite subcollection $H_{i_1}^{(A)}, H_{i_2}^{(A)}, \ldots, H_{i_{n_1}}^{(A)}$, $A - B$ such that $A \subseteq [\bigcup_{k=1}^{n_1} cl_{A \cap B} H_{i_k}^{(A)}] \cup cl_A (A-B)$. And since B is pseudocompact there exists a finite subcollection $H_{i_1}^{(B)}, H_{i_2}^{(B)}, \ldots H_{i_{n_2}}^{(B)}$, B-A such that $B \subset \bigcup_{k=1}^{n_2} (cl_{A \cap B} H_{i_k}^{(B)}) \cup (cl_B(B - A))$. Suppose that there exists $x \in A \cap B$ such that $x \notin (\bigcup_{k=1}^{n_1} cl_{A \cap B} H_{i_k}^{(A)}) \cup (\bigcup_{k=1}^{n_2} cl_{A \cap B} H_{i_k}^{(B)})$. Then

$x \in cl_B(B-A)$ or $x \in cl_A(A-B)$. If $x \in cl_B(B-A)$, then since $x \in A$ and $x \in \bigcup_{k=1}^{n_1} cl_{A \cap B} H_{i_k}^{(A)}$, $x \in cl_A(A-B)$. But $x \notin A - B$ and $x \notin B-A$. Therefore $x \in \partial(A-B) \cap \partial(B-A)$. By Lemma 2.2 $x \in \partial A \cap \partial B$. But $\cap A \cap \partial B \subset A \cap B$ and is therefore covered by the family $\{Gi: i = 1,2,\ldots\}$. Thus there exists a finite subcollection $G_{i_1}, G_{i_2}, \ldots, G_{i_{n_3}}$ such that $\partial A \cap \partial B \subset \bigcup_{k=1}^{n_3} cl_{A \cap B} G_{i_k}$. Let $H_{i_k}^{(A)} \cap (A \cap B) = G_{i_k}^{(A)}$ for $k = 1,2,\ldots, n_1$ and $H_{i_k}^{(B)} \cap (A \cap B) = G_{i_k}^{(B)}$, $k = 1,\ldots,n_2$. Then $A \cap B \subset (\bigcup_{k=1}^{n_3} cl_{A \cap B} G_{i_k}) \cup (\bigcup_{k=1}^{n_1} G_{i_k}^{(A)}) \cup (\bigcup_{k=1}^{n_2} G_{i_k}^{(B)})$ and hence $A \cap B$ is pseudocompact.

Coro 2.20: Let A and B be closed and pseudocompact sets with compact boundaries. Then $A \cap B$ is pseudocompact.

We may note here in passing that if A is compact and B is closed and pseudocompact. Then $A \cap B$ is in fact compact since it is a closed subspace of a compact space.

The following theorem is due to Stephenson [76]

Theorem 2.21: The following are equivalent for a completely regular space X.

i) X is pseudocompact

ii) whenever X is embedded in a T_2 (regular, completely regular) space Y with Y-X first countable, then X is closed in Y.

Pseudocompactness like compactness is an absolute property. ie. a pseudocompact space is pseudocompact in any space in which it is embedded. Hence this theorem implies that every pseudocompact subspace of a first countable space is closed.

An immediate corollary to this is this following.

Coro. 2.21: If X is pseudocompact and first countable, then no proper dense subspace of X is pseudocompact.

Using this corollary we can answer a question posed by Mardešic and Papić which was whether there exists a pseudocompact space having no everywhere dense countably compact subspace. Marjanovic [55] answers the question affirmatively and gives as an example the space ψ which we previously mentioned. By Corollary 2.21 we can make the more general statement that if X is pseudocompact and first countable then no proper dense subspace is even pseudocompact. Using as an example the space W of all ordinals less than the first uncountable ordinal with the usual order topology (Example 10.1) we have a countably compact space which has no everywhere dense pseudocompact subspace. This is a stronger result than that first asked for by Mardešic and Papić.

We have that every pseudocompact subspace of a first countable space is closed. The converse of this is not true as can be seen by the following example.

Example 2.3: Let X be an uncountable discrete space. Let p be any element not in X. Let $\hat{X} = X \cup \{p\}$ be the one point compactification of X. $\hat{X}$ is not first countable since the point $\{p\}$ has no countable base of neighborhoods. Let P be any pseudocompact subset of $\hat{X}$. Either $p \in P$ or $p \notin P$. If $p \notin P$, then P is discrete and hence must be finite in order to be pseudocompact. If $p \in P$ then X-P is first countable and therefore P is closed by Theorem 2.21. Thus every pseudocompact subset of X is closed, but X is not first countable.

It thus appears that first countable pseudocompact spaces have some nice properties. It turns out as we will now show, that a first countable pseudocompact space is maximally pseudocompact and minimally first countable in the class of completely regular spaces. The Lemma is due to Stephenson [76].

Lemma 2.3: Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be completely regular topologies on a set X and suppose that $(X,\mathcal{T}_2)$ is pseudocompact and $(X,\mathcal{T}_1)$ is first countable. Then $\mathcal{T}_1 \subset \mathcal{T}_2$ if and only if $\mathcal{T}_1 = \mathcal{T}_2$.

Theorem 2.22: A first countable pseudocompact space is maximally pseudocompact and minimally first countable in the class of completely regular spaces.

Proof: Let $(X,\mathcal{T})$ be first countable and pseudocompact. Suppose there exists a topology $\mathcal{T}_1$ on X which is strictly stronger than $\mathcal{T}$ and also pseudocompact then $\mathcal{T} \subset \mathcal{T}_1$ with $\mathcal{T}$ first countable and $\mathcal{T}_1$ pseudocompact. Hence by Lemma 2.3 $\mathcal{T}_1 = \mathcal{T}$. This is a contradiction. Hence $(X,\mathcal{T})$ is maximally pseudocompact.

Suppose there exists a topology $\mathcal{T}_2$ such that $\mathcal{T}_2 \subset \mathcal{T}_1$ with $\mathcal{T}_2$ first countable. Again by Lemma 2.3 $\mathcal{T}_2 = \mathcal{T}$. Thus $(X,\mathcal{T})$ is minimally first countable.

Theorem 2.23: Pseudocompactness is preserved under weakening the topology.

Proof: This follows from 2.1(b) which states that pseudocompactness is preserved under continuous functions.

In a number of instances results which hold true for closed subsets of a countably compact space hold true for zero sets in pseudocompact spaces. We will list some of these results to show the analogy and for later use. Most of these can be found in Gillman and Jerison [27] or Isiwata [44].

Theorem 2.24: Let X be a Hausdorff space. If of any two disjoint closed sets, at least one is countably compact, then X is countably compact.

Theorem 2.25: If of any two disjoint zero sets in X at least one is pseudocompact, then X is pseudocompact.

Theorem 2.26: X is countably compact provided every family of closed sets with the finite intersection property has the countable intersection property.

Theorem 2.27: X is pseudocompact if and only if every z-filter has the countable intersection property.

Theorem 2.28: X is countably compact if and only if every countable closed set is compact.

Theorem 2.29: If X is pseudocompact, then every countable zero set is compact.

More generally we have

Theorem 2.30: X is pseudocompact if and only if every countable C-embedded subset is compact.

Theorem 2.31: X is countably compact if and only if every closed subset of X has the inf property (Definition 4.1).

<u>Theorem 2.32</u>: X is pseudocompact if and only if every zero set of X has the inf property.

One might conclude from the above that since every closed subset of a countably compact space is countably compact, every zero set of a pseudocompact space is pseudocompact. This however is not the case as seen in Example 10.4.

We now give one final characterization of completely regular pseudocompact spaces which we will use repeatedly in the next chapter. In view of the fact that every zero set is a G_δ and that X is dense in βX we may apply Theorem 2.12 to obtain the following.

<u>Theorem 2.33</u>: X is pseudocompact if and only if every non-void zero set of βX has non-empty intersection with X.

We may characterize realcompact spaces analogously as Mrówka does in [65].

<u>Theorem 2.34</u>: A space X is realcompact provided for any $p \in \beta X - X$ there exists a zero set Z_p in βX containing p such that $Z_p \cap X = \emptyset$.

These two theorems yield immediately the following result.

<u>Theorem 2.35</u>: X is compact if and only if X is pseudocompact and real compact.

CHAPTER 3

PSEUDOCOMPACTIFICATIONS

In this chapter we shall begin our investigation of pseudocompactifications. We study basically two types of pseudocompactifications. One will consist of subspaces of βX containing X which are pseudocompact and the other will be one point pseudocompactifications. Using our construction we then derive some further properties of pseudocompact spaces. βX as usual will denote the Stone-Cech compactification and υX will denote the Hewitt realcompactification.

<u>Definition 3.1</u>: A pseudocompactification PX of a space X is a pseudocompact space in which X is embedded as a dense subspace.

<u>Definition 3.2</u>: For any completely regular space let $\alpha X = (\beta X - \upsilon X) \cup X$ $= \beta X - (\upsilon X - X)$.

The next theorem follows from results in [27].

<u>Theorem 3.1</u>: $X \subset \alpha X \subset \beta X$ implies the following:

i) $\beta(\alpha X) = \beta X$

ii) X is C*-embedded in αX.

<u>Theorem 3.2</u>: αX is pseudocompact.

<u>Proof</u>: We show that every non-empty zero set of βX meets X. Suppose there exists a zero set $Z \in \beta(\alpha X)$ such that $Z \cap \alpha X = \phi$. Then $Z \subset \upsilon X - X$. Let $f \in C(\beta X)$ such that $Z(f) = Z$. Then $f' = f|\upsilon X$ is an element of $C(\upsilon X)$ and $Z(f') = Z \cap \upsilon X$. Therefore $Z = Z \cap \upsilon X$ is a zero set in υX

and $Z \cap X = \emptyset$. This is a contradiction since every zero set in υX meets X. Thus every zero set in $\beta(\alpha X)$ meets αX and hence αX is pseudocompact by Theorem 2.31.

Theorem 3.3: X is pseudocompact if and only if $X = \alpha X$.

Proof: X is pseudocompact if and only if $\upsilon X = \beta X$. This is true if and only if $\beta X - \upsilon X = \emptyset$. And this follows if and only if $\alpha X = X$.

Theorem 3.4: X is realcompact if and only if $\alpha X = \beta X$.

Proof: X is realcompact if and only if $\upsilon X = X$, hence if and only if $\upsilon X - X = \emptyset$. This is true if and only if $\alpha X = \beta X$.

If X is completely regular and neither realcompact nor pseudocompact, then $X \subsetneq \alpha X \subsetneq \beta X$.

The next theorem follows from Theorem 6.4 of [27].

Theorem 3.5: The following properties hold for αX.

i) Every continuous map τ from X into a compact space Y has a continuous extension from αX into Y.

ii) X is C*-embedded in αX.

iii) Any two disjoint zero sets in X have disjoint closures in αX.

iv) For any two zero sets $Z_1, Z_2 \in X$, $cl_{\alpha X}(Z_1 \cap Z_2) = cl_{\alpha X} Z_1 \cap cl_{\alpha X} Z_2$.

v) Every point of αX is the limit of a unique z-ultrafilter on X.

In order that the theory develop analogously to that of compactifications and realcompactifications we would like to change i) to read as follows:

i)* Every continuous map τ from X into a pseudocompact space Y has a continuous extension from αX into Y.

This turns out to be impossible however, because if we did replace i) by i)* we would have the following theorem.

<u>Theorem F</u>: If $X = \prod_{\omega \in W} X_\omega$ where each X_ω is pseudocompact, then X is pseudocompact.

<u>Proof</u>: Let $X = \prod_{\omega \in W} X_\omega$ where X_ω is pseudocompact. Since the product of completely regular spaces is completely regular, X is completely regular. Each projection $\pi_\omega : X \to X_\omega$ has a continuous extension $\pi_\omega^\alpha : \alpha X \to X_\omega$ (by 1)*). Therefore X is closed in αX (by 6.13 (b) [27]) and so $X = \alpha X$. Hence X is pseudocompact.

We know however that Theorem F is false from an example by Novák [69] hence i)* cannot be true. In fact the example is of two countably compact spaces whose product is not pseudocompact. This prohibits us from strengthening i) by using a countably compact space instead of a pseudocompact space. That is, we cannot change i) to read that every continuous $\tau : X \to Y$ where Y is countably compact has a continuous extension $\tau^\alpha : \alpha X \to Y$. For then we could prove just as in Theorem F that the product of two countably compact spaces is pseudocompact. Thus we must be content with i) as it stands. We can however offer some results along this line.

Theorem 3.6: If τ is a continuous map from X onto Y, then τ has a continuous extension from αX onto a pseudocompact space T such that $Y \subset T \subset \beta Y$.

Proof: Let τ map X onto Y. Then τ has a continous extension $\tau^{\beta}:\beta X \to \beta Y$. Let $\tau^{\alpha} = \tau^{\beta}|\alpha X$. Thus τ^{α} maps αX into βY. Since the continuous image of a pseudocompact space is pseudocompact (Theorem 2.1 (b)), $\tau^{\alpha}(\alpha X) - T$ is pseudocompact. And τ was onto Y, hence $Y \subset \tau^{\alpha}(\alpha X) \subset \beta Y$.

Theorem 3.7: If Y is realcompact and $\tau:X \to Y$ is a continuous map, then there exists a continuous extension $\tau^{\alpha}: \alpha X \to \alpha Y$.

Proof: Let $\tau:X \to Y$. Then τ has a continuous extension extension $\tau^{\beta}:\beta X \to \beta Y$. Let $\tau^{\alpha} = \tau^{\beta}|\alpha X$. Now $\tau^{\alpha}:\alpha X \to \beta Y$ but since Y is realcompact $\alpha Y = \beta Y$ and so $\tau^{\alpha}: \alpha X \to \alpha Y$.

We now concern ourselves with pseudocompact subspaces of βX and αX.

Theorem 3.8: If $\alpha X \subset T \subset \beta X$, then T is pseudocompact.

Proof: αX is dense in βX and hence dense in T. It follows from Theorem 2.17 that T is pseudocompact.

Thus we see that every subspace between αX and βX is pseudocompact. In particular for every point $p \in \alpha X - X$, $\beta X - \{p\}$ is pseudocompact.

Theorem 3.9: Let $p \in \beta X - X$, then $\beta X - \{p\}$ is pseudocompact.

Proof: No point of $\beta X - X$ is a G_{δ} point. Hence no point of $\beta X - X$ is a zero set. Thus every zero set in $\beta(\beta X - \{p\}) = \beta X$ meets $\beta X - \{p\}$.

Therefore $\beta X - \{p\}$ is pseudocompact.

<u>Corollary 3.9</u>: X is the intersection of all pseudocompact subspaces of βX which contain X.

Thus it turns out that αX is not the intersection of all pseudocompact subspaces of βX which contain X as we had hoped. The analogous fact is true for realcompactifications and thus yields υX as the unique smallest realcompactification of X as a subspace of βX.

<u>Theorem 3.10</u>: Let A be any set of $\beta X - X$ such that $|A| \leq c$. Then $\beta X - A$ is pseudocompact.

<u>Proof</u>: If $\beta X - A$ is not pseudocompact there exists a zero set W in βX such that $W \subset A$. ie. $W \cap (\beta X - A) = \emptyset$. Therefore W is disjoint from X. But by Theorem 9.5 [27], $|W| \geq 2^c$. This is a contradiction since $|W| \leq |A| \leq c$.

<u>Theorem 3.11</u>: If X is not pseudocompact, then $|\alpha X - X| \geq 2^c$.

<u>Proof</u>: By 9 D [27] if X is realcompact then $|\beta X - X| \geq 2^c$. Now $\alpha X = (\beta X - \upsilon X)\upsilon X$. Therefore $|\alpha X - X| \geq 2^c$ whenever $\alpha X - X \neq \emptyset$.

Since for every set of points A in βX such that $|A| \leq c$, there exist a pseudocompactification of X; there exists at least 2^c pseudocompactifications of X none of which are compact, hence none of which are realcompact.

We should point out here that $|\beta X - X| \geq 2^c$ does not imply that X is realcompact. In fact there exist pseudocompact spaces X for which

$|\beta X - X| \geq 2^c$. For example the space $\Lambda = \beta R - (\beta N - N)$ is pseudocompact and not compact (hence not realcompact) and $|\beta\Lambda - \Lambda| \geq 2^c$.

<u>Theorem 3.12</u>: Let $p \in \alpha X - X$, then $\alpha X - \{p\}$ is pseudocompact.

<u>Proof</u>: Suppose there exists a zero set W_p in βX such that $p \in Wp$ and $Wp \cap (\alpha X - \{p\}) = \emptyset$. Since $p \in \alpha X - X$, $p \in \beta X - \upsilon X$ and hence there exists a zero set Zp in βX such that $p \in Zp$ and $Zp \subset \beta X - \upsilon X$. Then $Zp \cap Wp = \{p\}$ which implies $\{p\}$ is a zero set in $\beta X - X$. This is a contradiction. Therefore each zero set containing $\{p\}$ meets $\alpha X - \{p\}$ and hence $\alpha X - \{p\}$ is pseudocompact.

Note here that it is possible to have $Y \subset T \subset \beta Y$ such that $\alpha Y \not\subset \alpha T$. Consider any space Y which is neither pseudocompact nor realcompact (e.g. $Y = R \cup W$ where R is the real line and W is the space of ordinals less than the first uncountable ordinal). Then let $T = \alpha Y - \{p\}$ where $p \in \alpha Y - Y$. Then T is pseudocompact and $Y \subset T \subset \beta Y$. Now since T is pseudocompact $\alpha T = T$ and $\alpha Y \not\subset T$. Therefore we have $Y \subset T \subset \beta Y$ yet $\alpha Y \not\subset \alpha T$. Also there exist spaces Y which are realcompact and spaces T which are not realcompact such that $Y \subset T \subset \beta Y$. On the other hand if X is pseudocompact and T is any space such that $X \subset T \subset \beta X$, then T is always pseudocompact.

We might wonder if X is any pseudocompact space and $p \in X$, then is $X - \{p\}$ pseudocompact? Hewitt answers this question with Theorem 17 of [34] which states that if p is first countable and not isolated then $X - \{p\}$ is not pseudocompact. The converse of this statement is not true. In example 2.3 $X = \ddot{X} - \{p\}$ is not pseudocompact while the point p is nonisolated and non-first countable.

The preceding theorems point out the difficulty of finding a unique pseudocompactification of X as a subspace of βX. With βX when we remove a point $p \in \beta X - X$ the resulting space is no longer compact and for $p \in \upsilon X - X$, $\upsilon X - \{p\}$ is no longer realcompact. However, $\alpha X - \{p\}$ is pseudocompact for all $p \in \alpha X - X$. In the case of βX when we remove a point from $\beta X - X$ we leave a free z-ultrafilter on X which can be expanded to a free z-ultrafilter on $\beta X - \{p\}$. With $\upsilon X - \{p\}$ we leave a free real z-ultrafilter on X and this can be expanded to a free real z-ultrafilter on $\upsilon X - \{p\}$. However in $\alpha X - \{p\}$ there is a free hyperreal z-ultrafilter on X but the expanded z-ultrafilter on $\alpha X - \{p\}$ is no longer hyperreal. The zero sets of X which are in the free hyperreal z-ultrafilter do not have the countable intersection property, but when they are enlarged to become zero sets of $\alpha X - \{p\}$ they then have the countable intersection property in $\alpha X - \{p\}$. The problem arises from the fact that if a space has a free hyperreal z-ultrafilter defined on it, then it must have at least 2^c such z-ultrafilters. Thus we have added so many points that the free z-ultrafilters on the expanded space do not maintain the same character that they had in the original space.

We do have the following characterizations of our space αX.

Theorem 3.13: αX is the smallest pseudocompactification T of X contained in βX such that every free hyperreal z-ultrafilter on X is fixed on T.

Proof: Let T be a pseudocompactification of X which is properly contained in αX. Then there exists some $p \in \alpha X - X$ such that $p \notin T$. Since $p \in \alpha X - X$, there exists some free hyperreal z-ultrafilter on X which converges to p. This hyperreal z-ultrafilter on X is not fixed in T. Thus αX is the smallest pseudocompactification in βX with this property.

Theorem 3.14: αX is the largest pseudocompactification T of X contained in βX such that every point in $T - X$ is contained in a zero set which does not intersect X.

Proof: Suppose $X \subset \alpha X \subseteq T$. Then there exists a point $p \in T - \alpha X$. Thus $p \in \iota X - X$. Let Z be any zero set of T such that $p \in Z$. Suppose $Z \cap X = \emptyset$. Now there exists $f \in C^*(T)$ such that $Z(f) = Z$. Let $g = f|X$. Consider f^{ι}, the extension of f to $\iota(X)$. Then $f^{\iota} = f$ on $\iota X \cap T$. Hence $f^{\iota}(p) = 0$ and $Z(f^{\iota}) \cap X = \emptyset$. This is a contradiction since every non-void zero set of $\iota(X)$ meets X. Therefore $Z \cap X \neq \emptyset$. Now if $p \in \alpha X - X$ there exists $g \in C(X)$ such that $g > 0$ for all $x \in X$ and $g^{\beta}(p) = 0$. Hence every point in $\alpha X - X$ is contained in a zero set which doesn't meet X.

The following results are modifications of 3.1 in [19] and 3.1 in [20] and in the case where X is locally compact tell us a property of the zero sets in $\alpha X - X$.

Theorem 3.15: If X is locally compact, then each zero set of βX in

$\alpha X - X$ has a non-void interior in $\alpha X - X$.

Proof: Let Z be any zero set in βX such that $Z \subset \alpha X - X$. Let $f \in C^*(\beta X)$ such that $Z(f) = Z$. Let x_n be a sequence of distinct points in X on which f approaches zero. Choose disjoint compact neighborhoods V_n of x_n such that $|f(x) - f(x_n)| < \frac{1}{n}$ for $x \in V_n$. Now there exists $g_1 \in C^*(\beta X)$ such that $g_1(x_1) = 1$ and $g_1(x) = 0$ for $x \in X - V_1$. There exists $g_2 \in C^*(\beta X)$ such that $g_2(x_2) = 1$ and $g_2(x) = 0$ for all $x \in X - \bigcup_{i=1}^{2} V_i$. In general there exists functions $g_n \in C^*(\beta X)$ such that $g_n(x_n) = 1$ and $g_n(x) = 0$ for $x \in X - \bigcup_{i=1}^{n} V_i$. Let $g = \sum_{i=1}^{\infty} g_i$. Then g is a continuous function such that g is equal to 1 on each x_n and $g = 0$ on $X - \bigcup_{i=1}^{\infty} V_i$. If r is any point of $\alpha X - X$ at which $g(r) \neq 0$, then every neighborhood of r meets infinitely many of the compact sets V_n. Hence $f(r) = 0$. Thus Z(f) contains the non-void open subsets $(\alpha X - X) - Z(g)$ of $\alpha X - X$.

Corollary 3.15: If X is locally compact then each zero set of βX in $\alpha X - X$ is the closure of its interior in $\alpha X - X$.

Proof: Suppose there exists $p \in Z - \text{cl}(\text{int } Z)$. Then there exists some zero set Z_1 in βX which is disjoint from int Z, contains p and int $(Z_1 \cap Z) = \emptyset$. This contradicts the theorem.

Theorem 3.16: Let X be locally compact. Then $X \cup A$ is pseudocompact if and only if A is dense in $\alpha X - X$.

Proof: Suppose there exists a zero set $Z \in \beta X$ such that $Z \cap (X \cup A) = \emptyset$. Now $Z \subset \alpha X - X$ since any zero set in $\beta X - \alpha X$ meets X. Therefore by Theorem 3.13 Z contains a non-void open subset of $\alpha X - X$. Hence A is not dense in $\alpha X - X$. This is a contradiction and so every zero set in $\beta X - (X \cup A)$ meets $X \cup A$. Thus $X \cup A$ is pseudocompact. Conversely if A is not dense in $\alpha X - X$ then there exists a non-void open set U in $\alpha X - X$ disjoint from A. Write $U = V - X$ where V is open in βX. Let $p \in U$. There exists a zero set $Z \in \beta X$ such that $p \in Z \subset V$. Since $V \cap A = \emptyset$ $Z \cap A = \emptyset$. By the properties of αX there exists a zero set Z' in βX such that $p \in Z' \subset \alpha X - X$. Therefore $p \in Z \cap Z' \subset (\beta X - (X - A)) \cup A$ $\beta X - (X \cup A) = \beta(X \cup A) - (X \cup A)$. Thus there exists a zero set in $\beta(X \cup A)$ disjoint from $(X \cup A)$. Hence $X \cup A$ is not pseudocompact.

Note: Local compactness is essential here. See [19].

We may drop the condition of local compactness on X and obtain a similar result for the pseudocompact spaces T such that $X \subset T \subset X$.

Theorem 3.17: Let $X \subset T \subset \beta X$. Then T is pseudocompact if and only if every zero set of αX intersects T.

Proof: Since $X \subset T$ we need only consider the zero sets in $\alpha X - X$. Suppose there exists a zero set Z of αX contained in $\alpha X - X$ such that $Z \cap T = \emptyset$. Let $f \in C(\alpha X)$ such that $Z(f) = Z$. Then $g = f|X$ has a continuous extension g^T to T and $Z'(f) = Z(g^T) \cap \alpha X$. If $p \in Z(g^T)$ and $p \notin \alpha X$, then $Z(g^T) \cap X = \emptyset$ and p is contained in a zero set which doesn't meet X. This is a contridiction. Therefore $Z(g^T) = Z(f)$. Hence $\frac{1}{g^T}$ is continuous and unbounded on T, and T is not pseudocompact.

If T is not pseudocompact then there exists $f \in C(T)$ such that $f \geq 1$ and f is unbounded on T. Consider $g = \frac{1}{f}$. Now $g > 0$ for all $x \in T$ and g has a continuous extension g^β to βX. If $Z(g^\beta) = \phi$ then $g^\beta > 0$ on βX and $\frac{1}{g^\beta}$ is continuous and unbounded on βX. This is a contradiction. Hence $Z(g^\beta) \neq \phi$ and since $Z(g^\beta) \cap X = \phi$, $Z(g^\beta) \subseteq \alpha X - X$. Therefore there exists a zero set of αX which doesn't intersect T.

The importance of relatively pseudocompact subsets of X in our space X can be seen in the next few results. This next theorem was proved by the author and then later found in Isiwata's paper [48].

Theorem 3.18: For $A \subseteq X$, A is relatively pseudocompact if and only if $cl_{\beta X} A \subseteq \upsilon X$.

Corollary 3.18.1: Let X be realcompact, then A is relatively pseudocompact if and only if $cl_X A$ is compact.

Corollary 3.18.2: Let $A \subseteq X$. Then A is relatively pseudocompact if and only if $cl_{\alpha X} A = cl_X A$.

Proof: Let $A \subseteq X$ be relatively pseudocompact, then by Theorem 3.18 $cl_{\beta X} A \subseteq \upsilon X$. Thus $cl_{\alpha X} A = cl_{\beta X} A \cap \alpha X = cl_{\beta X} A \cap X = cl_X A$. Conversely, let $cl_X A = cl_{\alpha X} A$. Then $cl_{\beta X} A = cl_{\upsilon X} A \subseteq \upsilon X$. Hence A is relatively pseudocompact.

Corollary 3.18.3: If $A \subset X$ is closed and pseudocompact, then A is closed in αX.

Corollary 3.18.4: If $A \subset X$ is pseudocompact, then $cl_X A = cl_{\alpha X} A$.

Proof: Since A is pseudocompact, $cl_X A$ is pseudocompact by Corollary 2.17. Therefore $cl_X A$ is closed in αX and hence $cl_X A = cl_{\alpha X} A$.

Corollary 3.18.5: If X is first countable and $A \subset X$ is pseudocompact, then A is closed in αX.

Proof: By Theorem 2.21 A is closed in X. Hence by the theorem, A is closed in αX.

In [10] Comfort has the following theorem which he attributes to Hager and Johnson. We can now give a simple proof of this theorem and its converse.

Theorem 3.19: Let U be an open subset of X and suppose $cl_{\iota X} U$ is compact, then $cl_X U$ is pseudocompact.

Proof: Suppose $cl_{\iota X} U$ is compact. Then $cl_{\beta X} U \subset \iota X$. Hence $cl_X U = cl_{\alpha X} U$. Since X is dense in αX, U is also open in αX. Thus $cl_X U$ is the closure of an open subset of a completely regular pseudocompact space and is therefore pseudocompact.

Theorem 3.20: Let $U \subset X$. If $cl_X U$ is relatively pseudocompact then $cl_{\iota X} U$ is compact.

Proof: Since $cl_X U$ is relatively pseudocompact then $cl_X U$ is closed in αX. Hence $cl_{\beta X} U - cl_X U \subset \beta X - \alpha X = \iota X - X$. Therefore $cl_{\beta X} U \subset \iota X$ and $cl_{\beta X} U = cl_{\iota X} U$. Thus $cl_{\iota X} U$ is compact.

Corollary 3.20.1: If $cl_X U$ is pseudocompact, then $cl_{\iota X} U$ is compact.

Corollary 3.20.2: If U is any open relatively pseudocompact subset of X, then $cl_X U$ is pseudocompact.

Proof: $cl_X U = cl_{\alpha X} U$ by Corollary 3.18.1. Hence U is open in αX and so $cl_{\alpha X} U = cl_X U$ is pseudocompact.

Let us summarize what we have concerning closures of sets being pseudocompact.

1) In a completely regular space the closure of any relatively pseudocompact open set is pseudocompact.

2) (Corollary to 1)) In a completely regular pseudocompact space the closure of any open set is pseudocompact.

3) In any space the closure of a relatively pseudocompact cozero set is pseudocompact.

4) (Corollary to 3)) In any pseudocompact space the closure of any cozero set is pseudocompact.

Thus we see that what holds for open sets in completely regular spaces holds for cozero sets in arbitrary spaces.

Theorem 3.21: Let A be a pseudocompact subset of X which is C*-embedded in X, then A is closed in αX.

Proof: If A is pseudocompact and C*-embedded, then A is C-embedded in X. Therefore $cl_{\upsilon X}A = \upsilon A$. Also since A is C*-embedded in X, $cl_{\beta X}A = \beta A$ [27]. Now $cl_{\beta X}A = \beta A - A = \alpha A - A$. But $cl_{\alpha X} = (cl_{\beta X}A - cl_{\upsilon X}A) \cup A$. Thus $cl_{\alpha X}A = \alpha A$. Now A is pseudocompact and hence $\alpha A = A$. Therefore $cl_{\alpha X}A = A$ and hence A is closed in αX.

Theorem 3.22: Let X be normal and $A \subseteq X$ closed, then $cl_{\alpha X}A = \alpha A$.

Proof: Since X is normal and A is closed, A is both C- and C*-embedded in X. Therefore $cl_{\beta X}A = \beta A$ and $cl_{\upsilon X}A = \upsilon A$. Thus $cl_{\beta X}A - cl_{\upsilon X}A = \beta A - \upsilon A$ while $cl_{\beta X}A - cl_{\upsilon X}A = cl_{\alpha X}A - A$ and $\beta A - \upsilon A = \alpha A - A$. Therefore $cl_{\alpha X}A = \alpha A$.

Theorem 3.23: If the support of $f \in C(X)$ is pseudocompact then $f^{\alpha}(X) = 0$ for all $X \in \alpha X - X$.

Proof: If S(f), the support of f, is pseudocompact then $S(f) = cl_X(C(f))$ where C(f) is the cozero set of f. Since the support is pseudocompact we have $cl_X(C(f)) = cl_{\alpha X}(C(f)) \supset cl_{\alpha X}(C(f^{\alpha})) = S(f^{\alpha})$. Therefore $S(f^{\alpha}) \subset cl_X (C(f))$ and hence $f^{\alpha}(x) = 0$ for all $x \in \alpha X - X$.

Theorem 3.24: Let X be a completely regular T_1 space and $A \subset X$. If A is relatively pseudocompact and not pseudocompact, then A is not C-embedded in X.

Proof: Since A is relatively pseudocompact $cl_{\upsilon X}A = cl_{\beta X}A$. Now $cl_{\beta X}A$ is compact, but since Ais not pseudocompact, υA is not compact.

Therefore $cl_{\iota X}A \neq \iota A$. Hence A is not C-embedded by 8.10 (a) [27].

CHAPTER 4

RELATIVE PSEUDOCOMPACTNESS

As we have already seen, relatively pseudocompact subsets play an important role in the space of αX. We wish to investigate the property of relative pseudocompactness further.

Theorem 4.1: The continuous image of a relatively pseudocompact subset is relatively pseudocompact.

Proof: Let $f:X \to Y$ be continuous and let $F \subset X$ be relatively pseudocompact. Suppose $f[F]$ is not relatively pseudocompact. Then there exists $g \in C(Y)$ such that g is unbounded on $f[F]$. But $g \circ f \in C(X)$ and $g \circ f$ is unbounded on F. This is a contradiction so $f[F]$ must be relatively pseudocompact.

Theorem 4.2: A set $A \subset X$ is relatively pseudocompact if and only if $f[A]$ is compact for every $f \in C(X)$.

Proof: By Theorem 4.1 f(A) is relatively pseudocompact. Thus since every relatively pseudocompact subset of R is compact f(A) is compact. Conversely if f(A) is compact for every $f \in C(X)$, then A is pseudocompact. Therefore A is relatively pseudocompact.

Theorem 4.3: Let X be normal and let A be relatively pseudocompact subset of X, then $cl_X A$ is pseudocompact and hence countably compact.

Proof: Let $A \subset X$ such that A is relatively pseudocompact. Suppose that $cl_X A$ is not pseudocompact. Then there exists a function $g \in C(cl_X A)$ such that g is unbounded on $cl_X A$. $cl_X A$ is a closed subset of a normal space and as such is C-embedded in X. Hence g can be extended to $g^* \in C(X)$. Since g^* is unbounded on $cl_X A$ it is unbounded on A, but this is a contradiction. Therefore $cl_X A$ is pseudocompact and as a closed subspace of a normal space is also normal. Together normality and pseudocompactness imply countable compactness.

Theorem 4.4: X is pseudocompact if and only if every proper zero set is relatively pseudocompact.

Proof: If X is pseudocompact, then every subset of X is relatively pseudocompact. Suppose X is not pseudocompact, then there exists a function $f \in C(X)$ such that f is unbounded on X. Let $Z = \{x: |f(x)| \geq m\}$ where $m \in N$. Then Z is a zero set which is not relatively pseudocompact.

In [44] Isiwata defines property (*) which we shall denote as the inf property.

Definition 4.1: A subset F of X has the inf property provided $\inf\{f(x): x \in F\} > 0$ for every $f \in C(X)$ which is positive on F.

Theorem 4.5: Each of the following statements implies the next for any subset F of a space X.

i) F is pseudocompact.

ii) F has the inf property.

iii) F is relatively pseudocompact.

Proof: Immediate from definitions.

Theorem 4.6: If F has the inf property, then F is completely separated from every disjoint zero set.

Proof: Suppose F has the inf property. Let Z(f) be a zero set of X such that $Z(f) \cap F = \emptyset$. Then $\inf\{f(x): x \in F\} = r > 0$. Thus $\{x: f(x) = \frac{2}{3} r\}$ is a zero set containing F and disjoint from Z(f). Thus F and Z(f) are completely separated.

Corollary 4.6.1: If $F \subset X$ is pseudocompact, then F is completely separated from every disjoint zero set.

Corollary 4.6.2: Let X be pseudocompact. Then every subset A of X which is C*-embedded and has the inf property in pseudocompact.

Proof: By Theorem 4.6 A is completely separated from every disjoint zero set. Hence A is C-embedded and must be pseudocompact.

Theorem 4.7: X is countably compact if and only if every closed subset of X has the inf property.

Proof: If X is not countably compact, let $\{x_n\}_{n=1}^{\infty}$ be an infinite discrete subset of X. We can find a positive $f \in C(X)$ such that for some sequence $\{r_n\}_{n=1}^{\infty}$ in R which converges to zero, $f(x_n) = r_n$. Hence $\{x_n\}$ is a closed subset of X which does not have the inf property. Conversely if X is countably compact every closed set is pseudocompact and hence by Theorem 4.5 has the inf property.

Corollary 4.7: X is countably compact if and only if every closed subset of X is pseudocompact. If every closed subset is pseudocompact, then every closed subset has the inf property by Theorem 4.5. Thus by the theorem X is countably compact.

Definition 4.2: (Zenor [90]) The space X has property Z if every closed set H is completely separated from every zero set disjoint from H.

The following Theorem is due to Zenor and can be found in [90].

Theorem 4.8: X is countably compact if and only if X has property Z and is pseudocompact.

Perhaps it would seem logical to conjecture that any closed subset of a pseudocompact space which is completely separated from every disjoint zero set is pseudocompact. However, every zero set of a pseudocompact space has this property and hence every zero set would be pseudocompact. This is not the case as seen in example 10.4.

The next Theorem can be found in Isiwata's paper [44]

Theorem 4.9: Every zero set of a pseudocompact space has the inf property.

Combining Theorems 4.4 and 4.6 we have the following result.

Theorem 4.10: X is pseudocompact if and only if every zero set of X has the inf property.

This is analogous to Corollary 4.7 for countably compact spaces.

Theorem 4.11: Let F be a C-embedded subset of X. Then F is relatively pseudocompact if and only if F is pseudocompact.

Proof: If F is pseudocompact then, F is relatively pseudocompact. Assume F is not pseudocompact. Then there exists $f \in C(F)$ which is unbounded. Now f has a continuous extension f^* on X. Thus f^* is a continuous function on X which is unbounded on F.

Corollary 4.11: Every relatively pseudocompact C*-embedded zero set is pseudocompact.

Proof: Every C*-embedded zero set is C-embedded.

Theorem 4.12: Let PX be any pseudocompactification of a space X. If $A \subset X$ is relatively pseudocompact, then A is completely separated from every zero set in $PX - X$.

Proof: Suppose A is not completely separated from every zero set in $PX - X$. Then there exists a function $f \in C(PX)$ such that $Z(f) \subset PX - X$ and $Z(f) \cap A = \emptyset$ but f is not bounded away from zero on A. Let $f|X = g$. Then $h = \frac{1}{g}$ is continuous on X and unbounded on A. Thus A is not relatively pseudocompact.

In terms of coverings by cozero sets we obtain a theorem analogous to Theorem 2.2 for relatively pseudocompact subsets of a space X.

Theorem 4.13: Let X be completely regular T_1 space and let T be any pseudocompactification of X such that $X \subset T \subset \beta X$. Then if $A \subset X$, A is relatively pseudocompact if and only if A is completely separated from every zero set of T which is contained in $T - X$.

Proof: Necessity follows from the above Theorem.

Let $p \in T - X$ such that $p \in \alpha X - X$. Then there exists a zero set Zp of αX and hence of T such that $Zp \subset \alpha X - X \subset T - X$. By hypothesis there exists zero set Z_A of T such that $A \subset Z_A$ and $Z_p \cap Z_A = \emptyset$. Therefore $p \notin cl_{\alpha X}A$ for any $p \in T$ such that $p \in \alpha X - X$. Then there exists a zero set Z_x of αX such that $Z_x \cap X = \emptyset$. Since T is pseudocompact $Z_x \cap T = Z_x' \neq \emptyset$ and Z_x' is a zero set of T. Hence there exists a zero set Z_A such that $Z_A \cap Z_x' = \emptyset$. If $Z_A \cap Z_x \neq \emptyset$. Then $Z_A \cap Z_x = Z$ is a zero set of αX such that $Z \cap T = \emptyset$. This is a contradiction of Theorem 3.17. Thus $Z_A \cap Z_x = \emptyset$ and so $x \notin cl_{\alpha X}A$. Hence $cl_{\alpha X}A = cl_X A$ and by Corollary 3.18.2 A is relatively pseudocompact.

Theorem 4.14: If every countable cover of A by cozero sets of X has a finite subcover, then A is relatively pseudocompact.

Proof: For every $f \in C(X)$, $\{x : |f(x)| < n\}_{n=1}^{\infty}$ is a countable cozero set cover of A. This has a finite subcover and hence there exists some n, such that $A \subset \{x : |f(x)| < n\}$. That is, f is bounded on A for all $f \in C(x)$. Thus A is relatively pseudocompact.

In terms of the free z-filters on a space X we may characterize the relatively pseudocompact zero sets as follows:

<u>Theorem 4.15</u>: A zero set Z that belongs to no free z-filter is relatively pseudocompact.

<u>Proof</u>: Suppose Z is not relatively pseudocompact. Then there exists a function $g \in C(X)$ which is unbounded on Z. Let $Z_n = \{x : |g(x)| \geq n\}$. Each Z_n is a zero set and $Z_n \cap Z = \emptyset$ for all n. This collection along with Z has the finite intersection property and thus there exists a z-ultrafilter W on X containing Z and all the Zn. Now $\bigcap_{n=1}^{\infty} Zn = \emptyset$. Hence W is free.

The converse of this result is false however. Let X be any pseudocompact, noncompact, completely regular T_1 space (for example, W of example 10.1). Then X is itself a relatively pseudocompact zero set which does belong to a free z-filter. We do have the following converse however.

<u>Theorem 4.16</u>: A zero set Z is relatively pseudocompact if and only if Z is contained in no hyperreal z-ultrafilter.

<u>Proof</u>: Theorem 4.14 gives the only if part since any hyperrreal z-ultrafilter is free.

Conversely let Z be a relatively pseudocompact zero set of X. Then $cl_{\alpha X} Z = Z$. Hence $cl_{\beta} Z \subset \upsilon X$. Now Z belongs to a free z-filter for each $p \in cl_{\beta} Z - Z$. There exists a one-to-one correspondence between the points in $\beta X - X$ and the free z-ultrafilters on X. If $p \in \upsilon X - X$, then p is the limit of a real z-ultrafilter. If $p \in \beta X - \upsilon X = \alpha X - X$, then p is the limit of a hyperreal z-ultrafilter on X. Now for each p the z-ultrafilter is of the form $Zp = \{Z \in Z(X): p \in cl_\beta Z\}$. If Z is relatively pseudocompact then $cl_\beta Z \cap (\alpha X - X) = \emptyset$. Therefore Z is not in any z-ultrafilter whose limit is in $\alpha X - X$. That is Z is not in any z-ultrafilter which is hyperreal.

Notice that if Z is relatively pseudocompact but not pseudocompact then Z is not contained in any hyperreal z-ultrafilter. Therefore we cannot get a converse of Theorem 2.15.

Using this result we can get a converse to Theorem 4.13 for zero sets.

<u>Theorem 4.17</u>: If Z is a relatively pseudocompact zero set of X, then every countable cover of Z by cozero sets of X has a finite subcover.

Proof: Let Z be a zero set of X and let $\{C_i\}_{i=1}^{\infty}$ be a collection of cozero sets which cover Z but have no finite subcover. Let $Zn = X - (\bigcup_{i=1}^{n} C_i)$ for each n. Then $\{Zn\}_{n=1}^{\infty} \cup Z$ is a collection of zero sets of X with the finite intersection property. Since the cozero sets cover Z, the intersection of this family is empty. Therefore there exists a free z-ultrafilter on X containing Z which is hyperreal. Thus Z is not relatively pseudocompact.

One question which concerns us is when is a zero set of X a zero set of αX. The following theorem answers this question.

<u>Theorem 4.18</u>: $Z(f)$ is a zero set of αX if and only if $Z(f) = Z(f^{\alpha})$.

Proof: If $Z(f) = Z(f^{\alpha})$ then Z is a zero set of αX. If Z is a zero set of αX, then there exists some $f \in C(\alpha X)$ such that $Z(f) = Z$ then $Z = Z(f|X)$ and $(f|X)^{\alpha} = f$ so that $Z(f|X) = Z((f|X)^{\alpha}) = Z(f) = Z$.

Theorem 4.19: If Z(f) is a zero set of αX, then Z is relatively pseudocompact.

Proof: $Z(f) \subset cl_{\alpha X}Z(f) \subset Z(f^{\alpha}) = Z(f)$. Therefore $cl_{\alpha X}Z(f) \subset X$ and Z(f) is relatively pseudocompact.

The next lemma can be found in [48].

Lemma 4.1: If $Z(f^{\beta}) \subset \upsilon X$ then $Z(f^{\beta}) = cl_{\beta X}Z$.

Theorem 4.20: Let $f \in C^{*}(X)$ such that $Z(f) \neq \emptyset$. Then $Z(f^{\alpha}) = cl_{\alpha X}Z(f)$ if and only if $Z(f^{\beta}) = cl_{\beta X}Z$.

Proof: $f^{\alpha} = f^{\beta}|\alpha X$ and $cl_{\alpha X}\, Z = cl_{\beta X}\, Z \cap \alpha X$. Therefore $cl_{\beta X}Z = Z(f^{\beta})$ implies $cl_{\alpha X}\, Z = Z(f^{\alpha})$.

Now suppose $cl_{\alpha X}\, Z = Z(f^{\alpha})$. Then $(f^{\alpha})^{\beta} = f^{\beta}$ and $cl_{\beta X}(cl_{\alpha X}Z) = cl_{\beta X}Z$. Therefore $cl_{\beta X}Z(f^{\alpha}) = cl_{\beta X}Z$. Now $Z\,((f^{\alpha})^{\beta}) \subset \upsilon(\alpha X) = \beta X$. Thus $Z((f^{\alpha})^{\beta}) = cl_{\beta}\,[Z(f^{\alpha})]$ by Lemma 4.1. Thus $Z(f^{\beta}) = cl_{\beta X}Z$.

Thus if $Z(f^{\beta}) \neq cl_{\beta}Z$, then the difference must occur in the remainder $\alpha X - X$. And if the two sets agree on $\alpha X - X$, then they agree on all of βX.

Theorem 4.21: A is a relatively pseudocompact subset of X if and only if $cl_X A$ is relatively pseudocompact.

Proof: If $cl_X A$ is relatively pseudocompact, then every subset of $cl_X A$ is relatively pseudocompact. Suppose $cl_X A$ is not relatively pseudocompact. Any function unbounded on $cl_X A$ must be unbounded on A. Thus A is not relatively pseudocompact. We now state formally a theorem mentioned after the proof of Theorem 2.14. The result is due to Mandelker [53].

Theorem 4.22: Let A be a relatively pseudocompact cozero set of X, then $cl_X A$ is pseudocompact.

Proof: Since A is open this follows from Corollary 3.20.2.

Corollary 4.22: Let A be a cozero set of X, then $cl_X A$ is pseudocompact if and only if A is relatively pseudocompact.

As pointed out in Chapter 2, the intersection of two pseudocompact subsets need not be pseudocompact. However, their intersection must be relatively pseudocompact as the next theorem shows.

Theorem 4.23: Let A and B be relatively pseudocompact subsets of X, then $A \cap B$ is relatively pseudocompact.

Proof: $A \cap B \subset A$ hence $A \cap B$ is relatively pseudocompact.

Corollary 4.23: Let A and B be pseudocompact subsets of X. Then $A \cap B$ is relatively pseudocompact.

CHAPTER 5

LOCALLY PSEUDOCOMPACT SPACES

Definition 5.1: A space X is locally pseudocompact provided every point of X has a pseudocompact neighborhood.

This definition was introduced in the literature by Comfort in [11] in 1966. He required that every point have a local base of neighborhoods which were pseudocompact. When we are dealing with completely regular topological spaces it is obvious that the two definitions are equivalent. Vidossich [84] in 1971 also gives the above definition. We now investigate the properties of locally pseudocompact spaces.

Trivially if a completely regular T_1 space is pseudocompact it is locally pseudocompact. The closure of every open neighborhood of a point would be a pseudocompact neighborhood. The converse is not true however.

Theorem 5.1: Let X be a dense subspace of first countable completely regular T_1 space T, then every pseudocompact neighborhood in X of a point p in X is a neighborhood in T of p.

Proof: Let U be the interior of a pseudocompact neighborhood of p in X. Then $cl_X U$ is pseudocompact in X and hence pseudocompact in T. T is first countable and therefore $cl_X U$ is closed in T. Thus $cl_T U = cl_X U$. Let V be open in T such that $V \cap X = U$. Since X is dense in T we have $cl_T(V) = cl_T(V \cap X) = cl_T(U) = cl_X(U) \subset X$. Therefore $V \subset X$ and $V = U$ so U is open in T.

Corollary 5.1: If X is locally pseudocompact and dense in the first countable completely regular T_1 space T, then X is open in T.

Proof: X contains a neighborhood about each of its points.

Theorem 5.2: Let X be dense in first countable pseudocompact space T. Then X is locally pseudocompact if and only if X is open in T.

Proof: Necessity is implied by corollary 5.1.

Let $p \in X$, then there exists neighborhood N_p such that $N_p \subset cl_T N_p \subset X$. Since T is pseudocompact, $cl_T N_p$ is pseudocompact. Therefore X is locally pseudocompact.

It should be noted here that X is locally pseudocompact but not pseudocompact since by orollary 2.21 no proper dense subspace of T is pseudocompact.

Theorem 5.3: If X is realcompact then X is locally pseudocompact if and only if X is locally compact.

Proof: Local compactness implies local pseudocompactness.

Let $p \in X$ and let N_p be a pseudocompact neighborhood of p. There exists $N_p^* \in \mathfrak{n}\, p$ such that $p \in N_p^* \subset cl_X N_p^* \subset N_p$. Therefore $cl_X N_p^*$ is pseudocompact. and as a closed subspace of a realcompact is realcompact. Hence by Theorem 2.35 $cl_X N_p^*$ is compact.

The next two theorems show that some properties of local pseudocompactness are analogous to those for local compactness.

Theorem 5.4: Every open subspace of a locally pseudocompact completely regular T_1 space is locally pseudocompact.

Proof: Let A be open in X and let $p \in A$. There exists a neighborhood N_p of p in X such that N_p is pseudocompact. Now $(\text{int } N_p) \cap A = V$ is open in X. There exists N_p^* such that $p \in N_p^* \subset cl_X N_p^* \subset V$. Therefore $cl_X N_p^* \subset N_p$ and is pseudocompact by theorem 2.13. Since $p \in cl_X N_p^* \subset A$, A is locally pseudocompact.

Theorem 5.5: Let X be locally pseudocompact and f an open map. Then f[X] is locally pseudocompact.

Proof: Let $p \in f[X]$. Then there exists some $x \in X$ such that $p = f(x)$. Since X is locally pseudocompact there exists a pseudocompact neighborhood V_x of x. $f[V_x]$ is pseudocompact and is a neighborhood of p. Therefore f[X] is locally pseudocompact.

Theorem 5.6: Let PX be any pseudocompactification of a completely regular T_1 space X such that closed pseudocompact subsets of X are closed in PX. Then X is open in PX if and only if X is locally pseudocompact.

Proof: Let $x \in X$. There exists a neighborhood N_x of X such that N_x is pseudocompact. By complete regularity of X there exists N_x^* such that $x \in N_x^* \subset cl_X N_x^* \subset N_x$. Now $cl_X N_x^*$ is a closed pseudocompact subspace of X. Hence it is closed inPX. Let V be open in PX such that

$V \cap X = N_x^*$. Since X is dense in PX, $cl_{PX}(V) = cl_{PX}(V \cap X) = cl_{PX}(N_x^*)$. Hence $cl_{PX}V \subset X$ and so $V \subset X$. We must have $V = N_x^*$. Therefore X contains a PX open subset about each of its points and hence is open in PX.

Notice that this proof does not require the complete regularity of PX only of X.

To prove the converse we need only observe that since PX is pseudocompact it is locally pseudocompact and by theorem 5.4, X is locally pseudocompact.

Corollary 5.6.1: X is locally pseudocompact if and only if X is open in αX.

Proof: By corollary 3.18.3 X satisfies the hypothesis of the theorem.

Corollary 5.6.2: Let X be locally pseudocompact and let U be open in X, then $cl_{\alpha X}U$ is pseudocompact.

Proof: If U is open in X, then U is open in αX. Hence its closure is pseudocompact.

Corollary 5.6.3: If $\alpha X - X$ is pseudocompact and X is first countable, then X is locally pseudocompact.

Proof: By theorem 2.21 $\alpha X - X$ is closed in αX. Hence X is open and locally pseudocompact.

Example 5.1: A space which is locally pseudocompact but not locally compact. In example 9.1 the space $N \cup \Sigma_p$ is pseudocompact but its product with pseudocompact space $N \cup \Sigma_q$ is not pseudocompact. Hence in view of Theorem 9.5 $N \cup \Sigma_p$ is not a k-space and hence not locally compact. Yet since $N \cup \Sigma_p$ is pseudocompact it is locally pseudocompact.

Example 5.2: A space which is locally pseudocompact but not locally compact or pseudocompact. Let Δ be the disjoint union of a countably infinite number of copies of $N \cup \Sigma_p$. Then Δ is not pseudocompact or locally compact but is locally pseudocompact.

The next results are given in an effort to show that the conclusion of Corollary 5.6.3 cannot be strengthened to locally compact.

Theorem 5.7: Let X and Y be any completely regular topological spaces. Let T be the disjoint union of X and Y, then $\beta T = \beta X \cup \beta Y$.

Proof: X is a C-embedded zero set of T. Hence $cl_{\beta T} X = \beta X$ and similarily $cl_{\beta T} Y = \beta Y$. Closures of disjoint zero sets in T are disjoint in βT. Therefore $\beta T = cl_{\beta T}(X \cup Y) = cl_{\beta T} X \cup cl_{\beta T} Y = \beta X \cup \beta Y$.

By a similar argument $\upsilon T = \upsilon X \cup \upsilon Y$.

Theorem 5.8: Under the hypothesis of 5.7 $\alpha T = \alpha X \cup \alpha Y$.

Proof: $\alpha(X \cup Y) = [\beta(X \cup Y) - \upsilon(X \cup Y) \cup (X \cup Y)$

$$= [\beta X \cup \beta Y) - (\upsilon X \cup \upsilon Y)] \cup (X \cup Y)$$

$$= [(\beta X - \upsilon Y) \cup (\beta Y - \upsilon Y)] \cup (X \cup Y)$$

since all subsets are disjoint. Continuing the equality we have:

$\alpha(X \cup Y) = [(\beta X - \iota X) \cup X] \cup [(\beta Y - \iota Y) \cup Y] = \alpha X \cup \alpha Y$.

Now let X be first countable, pseudocompact and not locally compact (See example 10.8). Let R be the reals. Then $X \cup R$ is first countable and non-locally compact and $\alpha(X \cup R) = \alpha X \cup \alpha R = X \cup \beta R$. Now $\alpha(X \cup R) - (X \cup R) = \beta R - R$ which is compact and closed in $\beta(X \cup R)$. But $X \cup R$ is not countably compact and not locally compact. Hence we cannot strengthen the conclusion of corollary 5.6.3 to say that X must be locally compact. Here is a space T such that $\alpha T - T$ is closed in βT, hence T is open in αT but not open in βT.

Definition 5.2: A space X is locally relatively pseudocompact if each point $p \in X$ has a relatively pseudocompact neighborhood.

Theorem 5.9: In a completely regular space locally relatively pseudocompact is equivalent to locally pseudocompact.

Proof: Any pseudocompact neighborhood is relatively pseudocompact. Let X be locally relatively pseudocompact. Then for each $p \in X$. There exists open set U such that $p \in U$ and U is relatively pseudocompact. Then $cl_X U$ is pseudocompact by Corollary 3.20.2. Therefore every point has a pseudocompact neighborhood.

Definition 5.3: X is a P-space provided every prime ideal in C(X) is maximal.

Theorem 5.10: Every P-space is locally relatively pseudocompact.

Proof: Every function in C(X) is constant on a neighborhood which is relatively pseudocompact.

Corollary 5.10.1: Every completely regular P-space is locally pseudocompact.

Corollary 5.10.2: Every normal P-space is locally countably compact.

In [26] corollary 5.4 states that every locally countably compact P-space is discrete. Hence we have the following.

Corollary 5.10.3: A normal P-space is discrete.

W. W. Comfort in [10] gives the following characterization of a locally pseudocompact space.

Theorem 5.11: X is locally pseudocompact if and only if there exists a locally compact space Y such that $X \subset Y \subset \upsilon X$.

CHAPTER 6

PSEUDOCOMPACTNESS AND C*-EMBEDDINGS

Here we will discuss when a C*-embedded subspace is pseudocompact and when pseudocompact subspaces are C*-embedded. We also include in the chapter some weak normality properties and relate them to pseudocompact and countably compact spaces. We briefly examine pseudocompact P-spaces and F-spaces and αX for such spaces. Recall definition 2.8.

Theorem 6.1: Every C*-embedded Zero Set of a pseudocompact space is pseudocompact.

Proof: Let Z be a C*-embedded zero set of X. Then by 1.18 of [27] Z is C-embedded. Hence Z must be pseudocompact.

We will give an example later to show that a C*-embedded subset of a pseudocompact space need not be pseudocompact (example 9.5).

Theorem 6.2: If A is C*-embedded in a first countable space X, then A is closed in X.

Proof: Since A is C*-embedded $cl_{\beta X}A = \beta A$(6.9a) [27]). βA-A is never first countable. Therefore either $cl_{\beta X}A - A = \emptyset$ or $cl_{\beta X}A - A \subset \beta X - X$. In either case A is closed in X.

Corollary 6.2.1: Let X be countably compact and first countable, then if A is C*-embedded in X, A is countably compact.

Proof: By the theorem A is closed and hence countably compact.

Definition 6.1: [27] A space T is extremally disconnected if every open set has an open closure.

Corollary 6.2.2: If X is first countable and extremely disconnected, then X contains no proper dense subspace.

Proof: Any proper dense subspace of an extremely disconnected space is C*-embedded 6M2 [27] and hence closed.

Corollary 6.2.3: A first countable, extremally disconnected pseudocompact space is discrete.

Proof: For each $p \in X$, $X - \{p\}$ is not dense. Hence $\{p\}$ is open.

Corollary 6.2.4: A first countable, extremally disconnected pseudocompact space is finite.

Proof: By above corollary X is discrete. Hence if X is pseudocompact, X is finite.

Theorem 6.3: Let X be a first countable normal space. If A is pseudocompact, then A is C*-embedded in X.

Proof: If A is pseudocompact, then A is closed and every closed subspace of a normal space is C*-embedded.

Theorem 6.4: Let A be a pseudocompact subspace of X. If for any two completely separated sets S and T in A there exists zero sets Z_1 and Z_2

in X with $S \subset Z_1$ and $T \subset Z_2$ and $(Z_1 \cap Z_2) \cap A = \emptyset$, then A is C*-embedded in X.

Proof: Let A be a pseudocompact subset of X satisfying the hypothesis. Let S and T be completely separated in A. Then there exist zero sets Z_1 and Z_2 in X with $S \subset Z_1$ and $T \subset Z_2$ and $(Z_1 \cap Z_2) \cap A = \emptyset$. Now $Z_1 \cap Z_2$ is a zero set in X and hence must be completely separated from A by Corollary 4.6. Therefore there exists a zero set Z_3 in X such that $A \subset Z_3$ and $Z_3 \cap [Z_1 \cap Z_2] = \emptyset$. Now $S \subset Z_1 \cap Z_3$ and $T \subset Z_2 \cap Z_3$ and $[Z_1 \cap Z_3] \cap [Z_2 \cap Z_3] = (Z_1 \cap Z_2) \cap Z_3 = \emptyset$. Therefore S and T are completely separated in X and hence A is C*-embedded in X.

Theorem 6.5: If X is pseudocompact and first countable and $|\beta X - X| < 2^c$, then every C*-embedded subset of X is pseudocompact.

Proof: Let A be C*-embedded in X. Then $\beta A = cl_{\beta X} A$. Since X is first countable A is closed by X. Hence $\beta A = A \subset \beta X - X$. If A is not pseudocompact then $\upsilon A \neq \beta A$ and so $\beta A - \upsilon A \neq \emptyset$. By 9D [27], $|\beta A - \upsilon A| \geq 2^c$. Hence $|\beta A - A| \geq |\beta A - \upsilon A| \geq 2^c$. But $\beta A - A \subset \beta X - X$ implies $|\beta X - X| \geq 2^c$. This is a contradiction and therefore A must be pseudocompact.

We can change the hypothesis of theorem 6.5 slightly and obtain the same conclusion by a similar argument.

Theorem 6.6: If X is pseudocompact and $|X| \leq c$ and $|\beta X - X| < 2^c$, then every C*-embedded subset is pseudocompact.

Proof: Let A be C*-embedded in X. Then $\beta A = cl_{\beta X} A$. If A is not pseudocompact $|\beta A - \upsilon A| \geq 2^c$. Hence $\beta A - A \subset \beta X$ and $|\beta A - A| \geq 2^c$

while $|\beta X| < 2^c$. This is a contradiction hence A must be pseudocompact.

<u>Theorem 6.7</u>: If every subspace of a pseudocompact Hausdorff space X is pseudocompact, then X is finite.

<u>Proof</u>: Let Z be a zero set of X such that $X - Z \neq \emptyset$. Then Z and X - Z are pseudocompact. Therefore X - Z is completely separated from Z. Hence X - Z must be a zero set and so X - Z are open and closed. Therefore X - Z is C-embedded in X. Since every cozero set is C-embedded in X, X is a P-space (4J [27]). But a pseudocompact P space must be finite.

A generalization of the above theorem may be found in [79].

<u>Theorem 6.8</u>: Let PX be any pseudocompactification of X; then X is C*-embedded in PX if and only if $X \subset PX \subset \beta X$.

<u>Proof</u>: If X is C*-embedded in PX, then $X \subset PX \subset \beta X$ by 6.7 of [27]. Conversely if $X \subset PX \subset \beta X$, then X is C*-embedded in PX.

<u>Corollary 6.8</u>: X is C*-embedded in its pseudocompactification PX if and only if $\beta X = \beta(PX)$.

Recall definition 2.7. We now give a proof of Theorem 2.4 j) The Theorem is due to P. Zenor.

<u>Theorem 6.9</u>: X is pseudocompact and weakly normal if and only if X is countably compact.

<u>Proof</u>: If X is countably compact, then X is pseudocompact. Let A and B be disjoint closed sets with B countable. Since the space is countably

compact, B is compact. For every $x_i \in B$ there exists zero set neighborhoods Z_{x_i} and $Z_A(x_i)$ of x_i and A respectively which are disjoint. The family $\{Z_{x_i} : x_i \in B\}$ is cover of B and hence there exists a finite subcover say $Z_{x_{i_1}}, Z_{x_{i_2}}, \ldots, Z_{x_{i_k}}$ such that $\bigcup_{j=1}^{k} Z_{x_{i_j}} = Z_B$ is a zero set containing A and $Z_A \cap Z_B = \emptyset$. Therefore A and B are completely separated.

Now suppose X is pseudocompact and weakly normal but X is not countably compact. Then there is a set $H = \{x_1, x_2, x_3, \ldots\}$ of distinct points such that H has no limit point. Let $\{U_1, U_2 \ldots\}$ be a collection of mutually exclusive open sets such that $x_i \in U_i$ for each i. For each n let f_n be a mapping from X into $[0,1]$ such that $f(x_n) = \frac{1}{n}$ and $X - U_n \subset f_n^{-1}(0)$. Let $f = \sup\{f_1, f_2, \ldots\}$, f is continuous and $f^{-1}(0) \cap H = \emptyset$. By hypothesis H and $f^{-1}(0)$ are completely separated therefore there exists a mapping $h: X \to [0,1]$ such that $H \subset h^{-1}(0)$ and $f^{-1}(0) \subset h^{-1}(1)$. Let $g = f + h$. Then $\frac{1}{g}$ is a mapping on X which is unbounded. This is a contradiction since X is pseudocompact. Hence X must be countably compact.

In [52] Mack gives the following definitions which generalize weak normality.

Definition 6.2: A set A in a topological space will be called a G_δ (regular G_δ) set provided it is the intersection of at most $\aleph_0$ open sets (closed sets whose interiors contain A).

Definition 6.3: A topological space is δ-normal if each pair of disjoint closed sets, one of which is a regular G_δ-set have disjoint neighborhoods.

Definition 6.4: A space will be called δ-normally separated (property Z of Zenor in [90]) if each closed set and each zero set disjoint from it are completely separated.

Definition 6.5: A space X will be termed weakly δ-normally separated if each regular closed set and zero set disjoint from it are completely separated.

It is obvious from Corollary 4.6 and Theorem 2.13 that every completely regular pseudocompact space is weakly δ-normally separated and that any countably compact space is δ-normally separated. The author in [49] points out that in arbitrary spaces the properties δ-normal and δ-normally separated are not comparable. In completely regular pseudocompact spaces however they are equivalent and imply countable compactness. Combining the results of Mack and Zenor [52] and [90] respectively we have the following Theorem.

Theorem 6.10: In a completely regular space the following are equivalent.

i) X is countably compact

ii) X is pseudocompact and δ-normal

iii) X is pseudocompact and δ-normally separated.

In [87] Woods defines regular δ-normally separated as follows:

Definition 6.6: A space is regularly δ-normally separated if each regular closed subset of X is completely separated from each disjoint closed subset of X.

Obviously regular δ-normally separated implies weakly δ-normally separated, but not conversely even in countably compact spaces. Hence neither δ-normal nor δ-normally separated implies regularly δ-normally separated. However normality does imply regular δ-normal separation, but not conversely. Each extremally disconnected space is regularly δ-normally separated and there certainly exist non-normal extremally disconnected space. Woods example of a countably compact non-regularly δ-normally separated space is as follows: Let W be the space of ordinals less than the first uncountable ordinal and let W^* be its one-point compactification. Then $W \times W^*$ is countably compact hence both δ-normal and δ-normally separated but is not regularly δ-normally separated. For let $A = \{(\alpha,\alpha): \alpha \in W\}$ and $B = \{(\alpha,\omega_1): \alpha \in W\}$. Then A is regular closed and B is closed and disjoint from A, but A and B are not completely separated in $W \times W^*$.

A natural question to arise here is the following: If a space X is pseudocompact and regularly δ-normally separated is X countably compact? All we need to show that this is not true is the existence of a pseudocompact extremally disconnected space which is not countably compact. For any extremally disconnected space X, αX is extremally disconnected and pseudocompact. There exist subspaces of βN which are pseudocompact and not countably compact. These spaces will be extremally disconnected as dense subspaces of βN. Hence they are regularly δ-normally separated and pseudocompact but not countably compact.

The question which now arises is: when is αX countably compact? Obviously if αX is normal, weakly normal or δ-normal then αX is countably compact. We look therefore for conditions on X and $\alpha X - X$ which assure countable compactness.

Theorem 6.11: Let X be locally compact and normal. If $\alpha X - X$ is a closed subset of $\beta X - X$, then αX is countably compact.

Proof: Since X is locally compact, $\beta X - X$ is compact. Hence $\alpha X - X$ is compact. $\alpha X = \{\alpha X - X\} \cup X$ where $\alpha X - X$ is compact and X is open and normal. Hence αX is normal (3D5 [27]), and pseudocompact. Therefore αX is countably compact.

Corollary 6.11: Let X be locally compact and normal. Then $X \cup cl_{\beta X}(\alpha X - X)$ is countably compact.

CHAPTER 7

ONE-POINT PSEUDOCOMPACTIFICATIONS

Here we discuss two 1-point pseudocompactifications of a space X. One turns out not necessarily to be completely regular but does have the property that a closed pseudocompact subset of the space is closed in the one-point pseudocompactification. The other is completely regular but may lack this property.

Definition 7.1: Let X be a locally pseudocompact completely regular space. Let $\mathcal{P}$ be the collection of all closed pseudocompact subspaces of X. Let ∞ be a point not in X, and let $\hat{X} = X \cup \{\infty\}$. Let $\mathcal{T}_0$ be the topology on $\hat{X}$ having as a basis $\{U \subset X: U \text{ is open in } X\} \cup \{\ \ - P: P \in \mathcal{P}\}$.

Theorem 7.1: $(\hat{X},\mathcal{T}_0)$ is pseudocompact.

Proof: Let $\{G_i: i = 1, 2,...\}$ be any open cover of $\hat{X}$. Then there exists some i say i_1 such that $\{\infty\} \in G_{i_1}$.

By definition there exists an open set T such that $\{\infty\} \in T \subset G_{i_1}$ and $\hat{X}$ - T is pseudocompact. Now $\{G_i: i = 1, 2,...\}$ covers $\hat{X}$ - T and so there exists a finite subcollection G_{i_2}, g_{i_3},..., G_{i_k} such that $X - T \subset \bigcup_{k=2}^{k} cl_X G_{i_j}$. Thus $X \subset \bigcup_{k=1}^{k} cl_{\hat{X}} G_{i_j}$ and so $\hat{X}$ is pseudocompact.

Theorem 7.2: $(\hat{X},\mathcal{T}_0)$ is Hausdorff.

Proof: Let $x_1, x_2 \in \ddot{X}$ such that $x_1 \neq x_2$. If neither point is $\{\infty\}$, then since X is completely regular and T_1 we have it. Therefore let $x_2 = \{\infty\}$. There exist a neighborhood of x_1, N_{x_1}, which is pseudocompact. Therefore $X - N_{x_1}$ must be an open set containing $\{\infty\}$. By the regularity of X, there exists neighborhood $N^*_{x_1}$ of x_1 such that $x_1 \in N^*_{x_1} \subset cl_X N^*_{x_1} \subset N_{x_1}$. Therefore $N^*_{x_1}$ and $X - N_{x_1}$ are the disjoint neighborhoods of x_1 and x_2.

By construction any subset of X which is closed and pseudocompact in X will be closed in $(\hat{X},\mathcal{T}_0)$. By theorem 5.6, X is open in $\hat{X}$. If X is realcompact then one point pseudocompactification is precisely the one point compactification, for any closed pseudocompact subset of a realcompact space is compact, and by theorem 5.3 local pseudocompactness is equivalent to local compactness. Hence the constructions are identical.

<u>Definition 7.2</u>: A locally pseudocompact space X is said to be σ-pseudocompact provided X is the union of at most countably many pseudocompact sets.

<u>Theorem 7.3</u>: If X is σ-pseudocompact, then in $\hat{X}$, $\{\infty\}$ is a G_δ-point.

<u>Proof</u>: Let $X = \bigcup_{i=1}^{\infty} Y_i$ where each Y_i is pseudocompact. Then $cl_X Y_i$ is pseudocompact for each i, by corollary 2.17. Let $W_1 = \hat{X} - cl_X Y_1$, $W_2 = \hat{X} - (cl_X(Y_1 \cup Y_2)), \ldots, W_n = \hat{X} - [cl_X(\bigcup_{i=1}^{n} Y_i)], \ldots$. For each n, $\bigcup_{i=1}^{n} cl_X Y_i$ is pseudocompact by theorem 2.16. Thus each W_i, i = 1, 2,...

is an open set containing ∞ and $\bigcap_{i=1}^{\infty} W_i = \{\infty\}$. Hence ∞ is a G_δ- point.

Definition 7.3: Let $\mathcal{T}$ be the weak topology on $\hat{X}$ induced by $C((\hat{X},\mathcal{T}_0))$.

Theorem 7.4: $(\hat{X},\mathcal{T})$ is a completely regular pseudocompact space.

Proof: Since $\mathcal{T} \subset \mathcal{T}_0$, $(\hat{X},\mathcal{T})$ is pseudocompact. Since X is embedded homeomorphically in $\hat{X}$, the functions on $\hat{X}$ must separate the points of X. Thus suppose that for some $x_0 \in X$, $f(x_0) = f(\infty)$ for every $f \in C(\hat{X})$. Define $\tau : X \to X$ by $\tau(x) = x$ for every $x \in X$ and $\tau(\infty) = x_0$. For every $f \in C(\hat{X})$ associate a function $g \in R^X$ as follows: $g(x) = f(x)$ for all $x \in X$. Therefore $f = g \circ \tau$. Let C' denote the family of all such functions g. ie. $g \in C'$ if and only if $g \circ \tau \in C(\hat{X})$. Now endow X with the weak topology induced by C'. By definition every function in C' is continuous on x. ie. $C' \subset C(X)$. Therefore by theorem 3.8 [27], τ is continuous. It is evident that if x and x' are distinct points of X then there exists $g \in C'$ such that $g(x) \neq g(x')$. Therefore X is a completely regular T_2 space by 3.7 [27]. Now consider any function $h \in C(X)$. Since τ is continuous $h \circ \tau$ is continuous on X. Therefore $h \in C'$. Thus $C' = C(X)$ and the mapping $g \to g \circ \tau$ is an isomorphism from $C(\hat{X})$ to $C(X)$. Since $C(\hat{X}) = C^*(\hat{X})$ we must have $C(X) = C^*(X)$. Thus X is already pseudocompact. This is a contradiction. Hence for every $x \in X$, there exists $f \in C(X)$ such that $f(x) \neq f(\infty)$. Thus the weak topology on $\hat{X}$ is T_2 and hence completely regular.

Theorem 7.5: X is open in $(\hat{X},\mathcal{T})$.

Proof: Suppose every open set in $\hat{X}$ which contains x, also contains $\{\infty\}$ for some $x \in X$. Then $f(x) = f(\infty)$ for every $f \in C(\hat{X})$. But this can't happen. Therefore there exists $f \in C(X)$ such that $f(x) \neq f(\infty)$. Hence there exist open sets in R, V_1 and V_2 such that $f(x) \in V_1$ and $f(\infty) \in V_2$ with $V_1 \cap V_2 = \emptyset$. Therefore $f^{-1}(V_1)$ is an open set in X containing x and X is open in X.

Theorem 7.6: $\mathcal{T} = \mathcal{T}_0$ if and only if every closed pseudocompact subset of X is closed in $(\hat{X},\mathcal{T})$.

Proof: We know that $\mathcal{T} \subset \mathcal{T}_0$. Let $V \in \mathcal{T}_0$. Then either V is open in X or X - V is closed and pseudocompact. If V is open in X then V is open in $(\hat{X},\mathcal{T})$ since $(x,\mathcal{T}_X)$ and $(X,\mathcal{T}_{0_X})$ are homeomorphic and X is open in either space.

If $\hat{X}$ - V is closed and pseudocompact in X then by hypothesis $\hat{X}$ - V is closed in $(X,\mathcal{T})$ and again V is open in $(X,\mathcal{T})$. Therefore $\mathcal{T} = \mathcal{T}_0$.

If $\mathcal{T} = \mathcal{T}_0$ then since every closed pseudocompact subset of X is closed in $(\hat{X},\mathcal{T}_0)$ it is also closed in $(\hat{X},\mathcal{T})$.

Theorem 7.7: If $\{\infty\}$ is a G_δ-point in $(X,\mathcal{T})$, then $\mathcal{T} = \mathcal{T}_0$.

Proof: Let A be closed and pseudocompact in X, then A is completely separated from every zero set disjoint from A. $\{\infty\}$ is a zero set and hence is completely separated from A. Therefore A is closed in $(\hat{X},\mathcal{T})$. Hence by the above theorem $\mathcal{T} = \mathcal{T}_0$.

Michael Henry has the following result in his paper [33]: A locally compact T_2 space is first countable if and only if each point is a G_δ. Using this we see from Theorem 7.3 if X is σ-pseudocompact, locally compact and each point of X is a G_δ-point then $\hat{X}$ is first countable.

<u>Theorem 7.8</u>: Let X be a locally pseudocompact non-pseudocompact completely regular T_1 space. Then X is not C*-embedded in its one-point pseudocompactification $(\hat{X},\mathcal{T})$.

<u>Proof</u>: Suppose X is C*-embedded in $\hat{X}$, then $X \subset \ddot{X} \subset \beta X$ and $\beta\hat{X} = \beta X$. Since X is not pseudocompact $\alpha X - X \neq \emptyset$. Let $y \in \alpha X - X$, then there exists a zero set, Zy, in βX such that $Zy \cap X = \emptyset$. Since the zero sets of βX form a base for the closed sets there exists Zy' such that $\infty \notin Zy'$. Therefore $Z = Zy \cap Zy'$ is a zero set in βX such that $Z \cap \hat{X} = \emptyset$. This is a contradiction since X is pseudocompact. Thus in order for X to be C*-embedded in $\hat{X}$ we must have $\hat{X} = \beta X$. But $|\beta X - X| \leq 1$ implies X is pseudocompact. Again we have a contradiction. Thus X is not C*-embedded in $\hat{X}$.

CHAPTER 8

PSEUDOCOMPACT SPACES AND CONTINUOUS FUNCTIONS

Much study has been directed toward the preservation of certain properties under inverse mappings of continuous functions. The basic idea is what conditions must be imposed on a map $f: X \to Y$ so that if Y has property P then X must also have property P. A good deal of this material grew out of the study of product spaces. In this chapter we will look at certain types of maps and present some survey material. We also include results concerning convergence of sequences of functions in pseudocompact spaces. The following is a list of pertinent references: [5], [9], [13], [17], [24], [28], [29], [31], [40], [44], [45], [46], [59], [60], [61], [68], [69], [74], [80], [87], [90].

We begin with a list of definitions.

Definition 8.1: A map f from X onto Y is called a pk-map (rpk-map) provided $f^{-1}(y)$ is pseudocompact (relatively pseudocompact) in X for every $y \in Y$.

Definition 8.2: A map f from X onto Y is called a peripheral pk-map (peripheral rpk-map) provided $\partial f^{-1}(y)$ is pseudocompact (relatively pseudocompact) for every $y \in Y$.

Definition 8.3: A map f from X onto Y is called a quasi-k map provided $f^{-1}(y)$ is countably compact for every $y \in Y$.

Theorem 8.1: For any map f we have the following implications:

i) f is quasi-k $\Rightarrow$ f is pk $\Rightarrow$ f is rpk

ii) f is peripherally pk $\Rightarrow$ f is peripherally rpk.

Proof: Obviously from definitions.

Theorem 8.2: If X is pseudocompact then every peripherally pk-map is a pk-map.

Proof: $f^{-1}(y)$ is closed and $\partial f^{-1}(y)$ is pseudocompact. Hence by Theorem 2.19 $f^{-1}(y)$ is pseudocompact.

Theorem 8.3: If X is normal and f: X $\to$ Y is a rpk-map. Then f is a quasi-k map.

Proof: $f^{-1}(y)$ is relatively pseudocompact and closed in normal space X. Hence by Theorem 4.3 $f^{-1}(y)$ is countably compact.

In [59] Michael has the following definition and Theorem.

Definition 8.4: A point $y \in Y$ is called a q-point if it has a sequence of neighborhoods N_i such that if $y_i \in N_i$ and the y_i are all distinct, then $\{y_i\}_{i=1}^{\infty}$ has an accumulation point in Y. Y is a q-space if every $y \in Y$ is a q-point.

Theorem 8.4: Let f: X $\to$ Y be continuous closed and onto where Y is T_1. If Y is a q-space then f is a peripheral rpk-map.

Corollary 8.4: If X is normal in theorem 8.4, then $\partial f^{-1}(y)$ is countably compact.

Proof: $\partial f^{-1}(y)$ is closed and relatively pseudocompact in a normal space. Hence by Theorem 4.3 $\partial f^{-1}(y)$ is countably compact.

Theorem 8.5: Let $f: X \to Y$ be continuous, closed and onto, where X is realcompact. If y is a q-point of Y, then $\partial f^{-1}(y)$ is compact.

Proof: By Theorem 8.4 $\partial f^{-1}(y)$ is relatively pseudocompact. Hence by corollary 3.18.1 $\partial f^{-1}(y)$ is compact.

Definition 8.5: [74] A map $f: X \to Y$ is countably bi-quotient, if whenever $y \in Y$ and $\{Un\}_{n=1}^{\infty}$ is an increasing cover of $f^{-1}(y)$ by open subsets of X, then $y \in \text{int } f(Un)$ for some n.

Theorem 8.6: If $f: X \to Y$ is countably bi-quotient, then f is quasi-k and hence a pk-map.

Proof: Given $f^{-1}(y)$. Let $\{G_i\}_{i=1}^{\infty}$ be any countable cover of $f^{-1}(y)$. Let $H_1 = G_1$, $H_2 = G_1 \cup G_2$, $H_3 = G_1 \cup G_2 \cup G_3, \ldots, H_n = \bigcup_{i=1}^{n} G_i$, then $\{H_i\}_{i=1}^{\infty}$ is an increasing cover of $f^{-1}(y)$. Therefore there exists k such that $y \in \text{int } f(H_k)$. Then H_k covers $f^{-1}(y)$ and so the family $G_1, G_2, \ldots, G_k$ covers $f^{-1}(y)$. Hence $f^{-1}(y)$ is countably compact.

Theorem 8.7: If $f: X \to Y$ is closed and quasi-k, then f is countably bi-quotient.

Proof: Let $\{U_n\}_{n=1}^{\infty}$ be an increasing cover of $f^{-1}(y)$. Since $f^{-1}(y)$ is countably compact there exists k such that $f^{-1}(y) \subset U_k$. Now

$f(X - U_k)$ is closed and $Y - f[U_k] \subset f(X - U_k)$. Since $f^{-1}(y) \subset U_k$ then $y \notin f(X - U_k$. Therefore $y \in \sim f[X - U_k] \subset f[U_k]$. Since y is an open set contained in $f[U_k]$, $y \in \text{int } f[U_k]$.

Theorem 8.8: If X is countably compact and Y is first countable, then every continuous map from X onto Y is a closed quasi-k map.

Proof: Let $F \subset X$ be closed, then F is pseudocompact, hence $f[F]$ is pseudocompact. Since Y is first countable $f[F]$ must be closed. Hence f is a closed map. Let $y \in Y$, then $f^{-1}(y)$ is closed in X and must be countably compact.

Henriksen and Isbell in [32] give an example to show that pseudocompactness is not a perfect property. A property Q is a perfect property if and only if the following holds: If $p:X \to Y$ is a perfect map, then X has property Q if and only if Y has property Q. Compactness, countable compactness, the Lindelöf property and local compactness are a few examples of perfect properties.

Definition 8.6: A map f is pseudo-perfect (quasi-perfect) provided it is a closed pk-map (quasi-k map).

Lemma 8.1: Let $p: X \to Y$ be an open onto continuous map. Let W be open in X and V open in Y such that $p^{-1}(V) \subset W$ then $p^{-1}(\text{cl}_Y V) \subset \text{cl}_X W$.

Proof: Suppose $p^{-1}(\text{cl}_Y V) \not\subset \text{cl}_X W$, then there exists a neighborhood N_x of x such that $N_x \cap W = \emptyset$. $p(N_x)$ is open in Y. Let $y \in Y$ such that $p(x) = y$. Then $p(x) \in p(p^{-1}(\text{cl}_Y V)) = \text{cl}_Y V$. Therefore $y \in \text{cl}_Y V$

and hence $p(N_x) \cap V \neq \emptyset$. Let $z \in p(N_x) \cap V$, then there exists $w \in N_x$ such that $p(w) = z$, but $z \in V$ implies $w \in p^{-1}(V)$ which implies $N_x \cap p^{-1}(V) \neq \emptyset$. This is a contradiction since $p^{-1}(V) \subset W$ and $N_x \cap W = \emptyset$.

<u>Theorem 8.9</u>: Let $p: X \to Y$ be an open pseudoperfect map onto Y with Y completely regular T_1, then X is pseudocompact if and only if Y is pseudocompact.

<u>Proof</u>: If X is pseudocompact, then Y is pseudocompact as the continuous image of a pseudocompact space. Assume Y is pseudocompact and let $\{W_i\}_{i=1}^{\infty}$ be a countable coregular cover (Definition 2.6) of X. For every $n \in N$, let $V_n = Y - p(X - \bigcup_{i=1}^{n} W_i)$ then the $\{V_i\}_{i=1}^{\infty}$ form an open cover of Y. For since $p^{-1}(y)$ is pseudocompact $p^{-1}(y) \subset \bigcup_{i=1}^{m} W_i$ for some m. Hence $p^{-1}(y) \not\subset X - \bigcup_{i=1}^{m} W_i$ and so $y \notin p(X - \bigcup_{i=1}^{m} W_i)$. It follows that $y \in Y - p(X - \bigcup_{i=1}^{m} W_i)$. Since Y is pseudocompact there exists a finite subfamily of the $\{V_i\}_{i=1}^{\infty}$ whose closures cover Y, say $V_1, V_2, \ldots, V_k$. Since p is open and $p^{-1}(V_i) \subset W_i$ for $i = 1, 2, \ldots, k$, by lemma 6.1 $p^{-1}(cl_Y V_i) \subset cl_X W_i$. Since $\bigcup_{i=1}^{k} cl_Y V_i = Y$ and p is onto $\bigcup_{i=1}^{k} cl_X W_i = X$. By coregularity of the cover there exist members of the cover W_i' such that $cl_X W_i \subset W_i'$ for $i = 1, 2, \ldots, k$. Therefore $\{W_i'\}_{i=1}^{k}$ is a finite subfamily of the cover which covers X. Hence X is pseudocompact.

<u>Corollary 8.9</u>: Let $p: X \to Y$ be an open perfect map onto completely regular T_1 space Y, then X is pseudocompact if and only if Y is pseudocompact.

Proof: Any perfect map is a pseudocompact map.

Theorem 8.10: Let $f: X \to Y$ be a pseudoperfect map onto Y. If Y is countably compact, then X is pseudocompact.

Proof: Let $\{W_n\}_{n=1}^{\infty}$ be any countable coregular cover of X, then the open sets $V_n = Y - f(X - \bigcup_{i=1}^{n} W_i)$, $n \in N$, cover Y. For given $y \in Y$, the pseudocompactness of $f^{-1}(y)$ implies $f^{-1}(y) \subset \bigcup_{i=1}^{m} W_i$ for some $m \in N$. Therefore $y \notin f^{-1}[X - \bigcup_{i=1}^{m} W_i]$. Since there is a finite subcover $V_1, V_2, \ldots, V_k$ for Y and $f^{-1}(V_i) \subset W_i$, the sets $W_1, W_2, \ldots, W_k$ are a finite subcover of X. It follows that X is pseudocompact.

We would like to carry this another step and say if $f: X \to Y$ is pseudoperfect and onto and Y is compact then X is countably compact. This is not true however. Map any pseudocompact non-countably compact space to a single point. The map is pseudoperfect, Y is compact but X is not countably compact.

Theorem 8.11: If X is countably compact and $f \in C(X)$, then f is quasi-perfect.

Proof: Let $f \in C(X)$, then f(F) is realcompact and pseudocompact for every closed set F of X, since every subset of R is realcompact. Hence f[F] is closed and compact. Therefore f is a closed map. For every $y \in R$, $f^{-1}(y)$ is closed in X and hence countably compact.

Theorem 8.12: If X is realcompact then any pseudoperfect map from X onto any space Y is perfect.

<u>Proof</u>: Let $\tau : X \to Y$ be pseudoperfect. Then $\tau^{-1}(y)$ is pseudocompact and closed for every $y \in Y$. Hence $\tau^{-1}(y)$ is realcompact and pseudocompact and thus is compact.

One would like to investigate the class of pseudoperfect and quasiperfect properties. Let $\mathcal{P}^*$ denote the class of pseudoperfect properties, Q denote the class of quasiperfect properties and $\mathcal{P}$ denote the class of perfect properties. From the following implications: f is a perfect map $\Rightarrow$ f is a quasi-perfect map $\Rightarrow$ f is a pseudoperfect map, we have the following inclusions: $\mathcal{P}^* \subset Q \subset \mathcal{P}$. At this point we have no properties which are in $\mathcal{P}^*$ but we can produce a number of properties which are in $\mathcal{P}$ but not in $\mathcal{P}^*$. Also we can show $Q \neq \emptyset$.

i) Compactness is not a quasi-perfect property and hence is not a pseudoperfect property.

Let K be a compact first countable space and let W be the space of ordinals (Example 10.1) less than the first uncountable ordinal. Then $W \times K$ is countably compact and the projection map $\pi : W \times K \to K$ is a quasi-k map. π is also a closed map since the image of every closed set is pseudocompact and hence closed in K. It follows that π is a quasiperfect map from $W \times K$ onto K when K is compact but $W \times K$ is not. Therefore compactness is not a quasi-perfect property.

ii) Realcompactness is not a quasi-perfect property.

Consider any $f \in C(W)$, then $f(W) \subset R$ and is realcompact. By Theorem 8.11 f is a quasi-perfect map and f(W) is realcompact while W is not.

iii) The Lindelöf property is not quasi-perfect.

In ii) f(W) is Lindelöf while W is not.

Theorem 8.13: Countable compactness $\in Q$.

Proof: The proof here is essentially the same as theorem 8.10. Let $\{W_n\}_{n=1}^{\infty}$ be any countable cover of X. Let f: X → Y be quasi-perfect map onto Y. Then the open sets $V_n = Y - f(X - \bigcup_{i=1}^{n} W_i)$ $n \in N$, cover Y by the countable compactness of each $f^{-1}(y)$. Since there exists a finite subcover $V_1, V_2, \ldots, V_k$ for Y and $f^{-1}(V_1) \subset W_i$, the sets $W_1, W_2, \ldots, W_k$ from a finite subcover of X. Hence X is countably compact.

iv) Local compactness and first countability are not pseudoperfect properties.

Let $X = N \cup \Sigma p$ of example 9.1 and K be a first countable, compact space, then $X \times K$ is pseudocompact and $\pi: X \times K \to K$ as in i) is a pseudoperfect map. X is neither locally compact nor first countable hence $X \times K$ is neither locally compact nor first countable.

Let us now proceed to several types of maps introduced by Isiwata [44], Frolik [23] and Woods [87].

Definition 8.7: (Isiwata [44]) The mapping $\varphi: X \to Y$ is a WZ-mapping if $cl_{\beta X}\varphi^{-1}(y) = \Phi^{-1}(y)$ for each $y \in \varphi(X)$, where Φ is the stone extension of φ taking βX into βY.

Definition 8.8: (Frolik [23]) The mapping $\varphi: X \to Y$ is a z-mapping if the images of zero sets are closed.

The next three theorems are due to Isiwata and can be found in [44].

Theorem 8.14: X is pseudocompact if and only if any continuous map from X onto a first countable space Y is a z-mapping.

Theorem 8.15: X is pseudocompact if and only if $\pi: X \times Y \to Y$ is a Z-map for any first countable space Y.

Isiwata introduced WZ-mappings as an extension of the notion of Z-maps and showed that every Z-mapping is a WZ-mapping.

Theorem 8.16: Let $\varphi: X \to Y$ be a WZ-mapping. If either X is normal or $\partial\varphi^{-1}(y)$ is compact for every $y \in Y$, then φ is a closed map.

The next two theorems can be found in [90].

Theorem 8.17: X is normal if and only if every Z-mapping is closed.

Theorem 8.18: A pseudocompact space X is countably compact if and only if every WZ-mapping defined on X is a Z-mapping.

In [87] Woods defines a WN-map as follows.

Definition 8.9: A map $\varphi: X \to Y$ which is onto is called a WN-map if $cl_{\beta X}\varphi^{-1}(Z) = \varphi^{-1}(cl_{\beta Y}Z)$ for each zero set Z of Y.

Woods simply replaces a point of Y by a zero set of Y and obtains a class of maps which turn out to behave quite differently than WZ-maps. In general the classes of WN-maps and WZ-maps are not countable.

Theorem 8.19: A space X is pseudocompact if and only if any map from X onto any space Y is a WN-map.

Theorem 8.20: If each point of Y is a G_δ-set, then each WN-map from X onto Y is a WZ-map.

Proof: Each G_δ-point is a zero set.

CHAPTER 9

PRODUCT SPACES

In this chapter we present a survey of results obtained on the product of pseudocompact spaces. The first example to appear in the literature to show that the product of two pseudocompact spaces need not be pseudocompact was given in 1952 by Terasaka [82]. In 1953 Novak [69] gave an example of two countably compact spaces whose product was not pseudocompact. Novak's example was simplified by Frolik in 1959 [21]. The simplified example appears in Engelking's book [18]. Both examples described are subspaces of βN. In 1967 W. W. Comfort [9] gave an example of a non-pseudocompact product space whose finite subproducts were pseudocompact. This came about because Glicksburg [29] observed that a product space $\pi_{\alpha \in A} X_\alpha$ is pseudocompact provided each subproduct $\pi_{\alpha \in \beta} X_\alpha$, with β a countable subset of A, is pseudocompact. The example showed that the word "countable" cannot legitimately be replaced by the word finite in Glicksburg's theorem. Again the example is a subspace of βN.

The study first arose from the following consideration: If $\{X_\alpha\}$ is a family of spaces, then $\pi\beta X_\alpha$ is a compactification of πX_α. However it is not necessarily the Stone-Čech compactification, as in the case of $R \times R$ where $\beta(R \times R)$ is not equal to $\beta R \times \beta R$. The question then arose as to when is it ture that $\beta(\pi X_\alpha)$ is $\pi(\beta X_\alpha)$ or equivalently when will πX_α be C*-embedded in $\pi(\beta X_\alpha)$. In 1959 Glicksburg [29] showed that for infinite spaces, this will be the case exactly when

πX_α is pseudocompact. Thus began the study of when πX_α will be pseudocompact.

In order to proceed with the first theorem we need the following definitions and clarifications: When a topology on $C^*(X)$ is referred to, it is the metric topology induced by the norm $\|f\|_x = \sup\{|f(x)|: x \in X\}$, the topology of uniform convergence on X. In theorem 9.1(8), $C(X, C^*(Y))$ is the space of continuous functions from X to $C^*(Y)$ with the topology of uniform convergence on X. The subspace of bounded functions is normed by $\|\psi\| = \sup\{\|\psi(x)\|_Y : x \in X\}$, and becomes a metric space. For $f \in C^*(X)$ and $S \subset X$, the set $osc_s(f) = \sup\{|f(a) - f(b)|; a, b \in S\}$. When X, Y and Z are sets and f maps $X \times Y$ into Z, then f_y (for $y \in Y$) is defined on X by the rule $f_y(x) \equiv f(x,y)$. Similarily $_xf$ (for $x \in X$) is defined on Y by $_xf(y) = f(x,y)$. If $f \in C^*(X \times Y)$, the family $\{f_y : y \in Y\}$ is said to be equicontinuous at a point $x_o \in X$ provided that for each $\epsilon > 0$, there is a neighborhood U of x_o for which $|f_y(x_o) - f_y(x)| < \epsilon$ whenever $(x,y) \in U \times Y$. The theorem can be found in Comfort and Hager's paper [13].

<u>Theorem 9.1</u>: The following conditions on the product space $X \times Y$ are equivalent:

1) The projection π_X from $X \times Y$ onto X is a Z-mapping.

2) If Z is a zero set in $X \times Y$, then $cl\, Z = \cup\{cl(Z \cap (\{x\} \times Y)) : x \in X\}$, each closure being taken in $X \times \beta Y$.

3) Each function in $C^*(X \times Y)$ can be extended continuously over $X \times \beta Y$.

4) If $f \in C^*(X \times Y)$, then $F(x) \equiv \sup\{f(x,y): y \in Y\}$ defines a continuous function F on X (and similarily for $\inf\{f(x,y): y \in Y\}$).

5) If $f \in C^*(X \times Y)$, then $y(x_1,x_2) = \{|f(x_1,y)-f(x_2,y)| : y \in Y\}$ defines a continuous pseudometric y for X.

6) If $f \in C^*(X \times Y)$, then $\{f_y : y \in Y\}$ is an equicontinuous family on X.

7) If $f \in C^*(X \times Y)$, then $\Phi(f)(x) \equiv {}_x f$ defines a continuous mapping $\Phi(f)$ from X into $C^*(Y)$.

8) Φ (as defined in (7)) is a homeomorphism (indeed, an isometry) of $C^*(X \times Y)$ onto the space of bounded functions in $C(X,C^*(Y))$.

9) If $f \in C^*(X \times Y)$, $x_o \in X$ and $\epsilon > 0$, then there is a neighborhood U of x_o and $g \in C^*(Y)$ such that $|f(x,y)-g(y)| < \epsilon$ whenever $x \in U$ and $y \in Y$.

10) If $f \in C^*(X \times Y)$ and $\epsilon > 0$, then there is an open cover $\mathcal{U}$ of X, and for $U \in \mathcal{U}$ a finite open cover $\mathcal{V}(U)$ of Y such that $osc_{U \times V}(f) < \epsilon$ whenever $U \in \mathcal{U}$ and $V \in \mathcal{V}(U)$.

Tamano has the next theorem in [80].

<u>Theorem 9.2</u>: The following are equivalent.

1) $X \times Y$ is pseudocompact

2) X and Y are pseudocompact and π_X is a Z-mapping.

The following theorem is the very important characterization given by Glicksburg in [29].

<u>Theorem 9.3</u>: For infinite X and Y, the following are equivalent.

1) $X \times Y$ is pseudocompact

2) $\beta(X \times Y) = \beta X \times \beta Y$.

<u>Definition 9.1</u>: Two points x_1 and x_2 of a space X are of the same type provided there exists a homeomorphism $h:X \to X$ such that $h(x_1) = h(x_2)$.

<u>Definition 9.2</u>: A space X is homogeneous if any two points of the space are of the same type.

Rudin's paper [71] shows that $\beta N - N$ is not homogeneous. Thus using the fact that the orbit of any point $\beta N - N$ is dense in βN we can now construct pseudocompact spaces X and Y such that $X \times Y$ is not pseudocompact.

<u>Example 9.1</u>: Choose two points p and q of βN-N which are of different types. Let Σp and Σq denote their orbits under automorphisms of βN. Then Σp and Σq are disjoint dense subspaces of $\beta N - N$. Hence by Theorem 3.16 $N \cup \Sigma p$ and $N \cup \Sigma q$ are pseudocompact. However $(N \cup \Sigma p) \times (N \cup \Sigma q)$ contains the diagonal $\Delta = \{(n,n): n \in N\}$ as a clopen and therefore C-embedded copy of N. Hence the product is not pseudocompact.

<u>Theorem 9.4</u>: The pseudocompactness of the product is equivalent to the pseudocompactness of every countable partial product.

Comfort's example in [9] shows that the word "countable" in Glicksburg's theorem refers only to "countable infinite" partial products.

<u>Theorem 9.5</u>: $X \times Y$ is pseudocompact if X and Y are pseudocompact and either X or Y is a k-space.

This result of Tamano [80] generalizes the result of Glicksburg which states that $X \times Y$ is pseudocompact provided X and Y are pseudocompact and one of the spaces is locally compact. Also generalized is the result of Henriksen and Isbell [32] which requires that one of the spaces be first countable. As a corollary to this theorem we may obtain the well-known result that if X is pseudocompact and Y is compact, then $X \times Y$ is pseudocompact.

<u>Definition 9.3</u>: (Franklin) A space X is sequential if every sequentially closed subset of X is closed.

<u>Theorem 9.6</u>: If X and Y are pseudocompact, one of which is sequential, then $X \times Y$ is pseudocompact.

The last theorem is in [76] and is due to Stephenson.

In [24] Frolik defines the class $\mathcal{P}$ and gives the following characteristics.

<u>Definition 9.4:</u> (Frolik) Let $\mathcal{P}$ be the class of all completely regular spaces X such that for every completely regular pseudocompact space Y, the product $X \times Y$ is a pseudocompact space.

<u>Theorem 9.7</u>: Let X be a pseudocompact space such that each point of X has a neighborhood belonging to $\mathcal{P}$; then X belongs to $\mathcal{P}$.

<u>Theorem 9.8</u>: A space X belongs to $\mathcal{P}$ if it satisfies the following condition: If $\mathcal{U}$ is an infinite disjoint family of non-void open subsets of X, then for some compact subset of X, the intersection $K \cap A$ is non-void for an infinite number of sets A belonging to $\mathcal{U}$.

<u>Theorem 9.9</u>: $X \in \mathcal{P}$ if and only if X satisfies the following condition:

If $\mathcal{U}$ is an infinite disjoint family of non-void open subsets of X, then there exists a disjoint sequence $\{U_n\}$ in $\mathcal{U}$ such that for every filter $\mathcal{F}$ of infinite subsets of N, we have $\bigcap_{F_1 \in \mathcal{F}} cl_X[\bigcup_{n \in F_1} U_n] \neq \emptyset$.

Frolik in [25] gives an example which shows that for each integer n there exists a topological space X such that X^n is pseudocompact but X^{n+1} is not pseudocompact. For n = 2 this answers negatively the following question: If X,Y and Z are pseudocompact spaces such that $X \times Y$, $Y \times Z$ and $X \times Z$ are all pseudocompact, then is it true that $X \times Y \times Z$ is pseudocompact?

We wish to apply some of this to determine when $\alpha X \times \alpha Y$ is pseudocompact.

<u>Theorem 9.10</u>: If αX is open in βX, then $\alpha X \times \alpha Y$ is pseudocompact for any completely regular space Y.

<u>Proof</u>: If αX is open in βX, then αX is locally compact and hence by Theorem 9.5, $\alpha X \times \alpha Y$ is pseudocompact.

<u>Theorem 9.11</u>: If $\alpha X \times \alpha Y = \alpha(X \times Y)$ then $\alpha X \times \alpha Y$ is pseudocompact.

<u>Proof</u>: $\alpha(X \times Y)$ is pseudocompact.

<u>Theorem 9.12</u>: If X is locally compact and $\alpha X - X$ is closed in $\beta X - X$, then $\alpha X \times \alpha Y$ is pseudocompact for any Y.

<u>Proof</u>: Since X is locally compact $\beta X - X$ is compact and hence locally compact. Now $\alpha X - X$ is locally compact as a closed subspace of a locally

compact space. Thus αX is locally compact and invoking Theorem 9.5 again we have the result.

CHAPTER 10

EXAMPLES

In this chapter we will look at a list of examples and indicate their relevance to certain theorems in this text. We will show where strengthening is not possible or where the converse statements are invalid.

Example 10.1: We have already used this example but now we will discuss it in detail. Let W be the set of all ordinals less than the first uncountable ordinal ω_1. The set of ordinals less than a given ordinal α is denoted by $W(\alpha)$: $W(\alpha) = \{\sigma : \sigma < \alpha\}$. Provide W with the interval topology by taking as a subbase for the open sets the family of all rays $\{x: x > \alpha\}$ and $\{x: x < \sigma\}$. A point of $W(\alpha)$ is an isolated point if and only if it is not a limit ordinal (ie. it is 0 or has an immediate predecessor). The space $W(\omega)$ of all finite ordinals is homeomorphic with N.

We will now list the properties of W.

i) W is normal

ii) W is pseudocompact and hence countably compact.

iii) W is not compact and hence W is not realcompact.

iv) W is C-embedded in its one-point compactification W^*. Hence $W^* = \beta W$.

v) W is first countable but not second countable or metrizable.

vi) W contains an everywhere dense realcompact subspace. Let $X = W - L$ where L is the set of all limit ordinals of W. Then X is discrete and hence realcompact and X is dense in W.

vii) W does not contain an everywhere dense pseudocompact subspace.

(Corollary 2.21).

Example 10.2: Let $\Omega^* = W^* \times W^*$, denote the corner point (ω_1, ω_1) by ω and define $\Omega = \Omega^* - \{\omega\}$.

i) Ω is C-embedded in Ω^* so that Ω is pseudocompact and $\Omega^* = \beta\Omega$.

ii) Ω is not normal. The diagonal and an edge are not contained in disjoint open sets.

iii) Ω is countably compact.

iv) Ω is not first countable. Any point on the edge does not have a countable base.

v) Every closed G_δ in Ω is a C-embedded zero set.

vi) Every zero set is pseudocompact.

Since Ω is not normal there exists a closed subset of which is not C*-embedded in Ω.Ω is countably compact and hence this closed subset is pseudocompact. Thus a non first countable pseudocompact space may contain a closed pseudocompact subset which is not C*-embedded. Each edge of Ω is C-embedded but their union is not C*-embedded.

vii) Ω is locally compact.

Example 10.3: The tychonoff plank. Let $T^* = W^* \times N^*$ where N^* denotes the one point compactification $N \cup \{\omega\}$ of N. Let $t = (\omega_1, \omega)$. Then

i) T is pseudocompact and T is C-embedded in T* and $\beta T = T^*$.

ii) T is not realcompact.

iii) T is not countably compact since $N = \{\omega_1\} \times N$ has no limit point in T.

iv) T is not normal or even weakly normal since T is not countably compact.

v) T is not first countable.

vi) Every closed G_δ-set is a C-embedded zero set. Hence every zero set in T is pseudocompact.

vii) T is locally compact.

Example 10.4: (J. Isbell) There exists an infinite maximal family $\mathcal{E}$ of infinite subsets of N such that the intersection of any two is finite. Let $D = \{\omega_E : E \in \mathcal{E}\}$ be a new set of distinct points, and define $\psi = N \cup D$, with the following topology: The points of N are isolated, while a neighborhood of a point ω_E is any set contiaining ω_E and all but a finite number of points of E. Thus $E \cup \{\omega_E\}$ is the one point compactification of E and N is dense in ψ.

i) ψ is completely regular.

ii) ψ is pseudocompact.

iii) ψ is not normal and not countably compact.

iv) ψ is not realcompact.

v) D is an infinite discrete zero set in ψ.

vi) D is not C*-embedded in ψ. D is realcompact since D is discrete. If D is a C*-embedded zero set, then D must be C-embedded and hence pseudocompact. But D is not pseudocompact for if it were then D would be compact and C-embedded. Thus a pseudocompact space may contain a non-pseudocompact zero set.

vii) Ψ is first countable.

viii) Ψ is locally compact.

ix) Every subset of Ψ is a G_δ.

<u>Example 10.5</u>: (Katetov) Let $\Lambda = \beta R - (\beta N - N)$.

i) $\beta\Lambda = \beta R$

ii) Λ is pseudocompact.

iii) Λ is not normal.

iv) Λ is not countably compact.

v) Λ is not real compact.

vi) N is closed and C*-embedded in Λ but is not C-embedded.

Hence a pseudocompact space may contain a countable closed C*-embedded subset which is not pseudocompact. Also N is not a zero set.

vii) Λ is not countably compact.

<u>Example 10.6</u>: Let $X = W \times N$

i) X is not pseudocompact since N is not pseudocompact.

ii) X is not real compact since W is not realcompact.

iii) X is not σ-compact since X is not pseudocompact.

iv) X is σ-pseudocompact since $X = \bigcup \{W \times \{n\} : n \in N\}$.

v) X is locally compact since W and N are locally compact.

vi) X is first countable since W and N are first countable.

Here we have a σ-pseudocompact space which is not σ-compact or realcompact or pseudocompact.

Example 10.7: Let $\Sigma = N \cup \{\sigma\}$ where $\sigma \in \beta N - N$. Define a topology Σ as follows: all points of N are isolated and the neighborhoods of σ are the sets $U \cup \{\sigma\}$ for $U \in \mathcal{U}$, where $\mathcal{U}$ is any free ultrafilter on N. N is dense in Σ. Every set containing $\{\sigma\}$ is closed hence every subspace of Σ is open or closed.

Σ is normal.

Σ is extremally disconnected.

Every subspace of Σ is C*-embedded.

Σ is realcompact.

Σ is not pseudocompact.

Σ is not first countable.

Σ is not a P-space.

Example 10.8: This is an example of a completely regular Hausdorff space which is pseudocompact, first countable but not locally compact. Let W be as in example 10.1. Let N be the natural numbers with the discrete topology. Let p be a point not in $W \times N$. Let $X = W \times N \cup \{p\}$. Call a set $V \subset X$ open if and only if

1) $V \cap (W \times N)$ is open in the product space $W \times N$ and
2) if $p \in V$ then for some $k \in N$, $W \times \{n : n \geq k\} \subset V$. This space is completely regular, Hausdorff, pseudocompact and first countable but p has no compact neighborhood.

i) X is not locally compact.

ii) X is not realcompact

iii) X is countably compact. Any countable open cover must contain p and hence one member of the cover must contain all but a finite number of copies of W. W is countably compact and hence there will be a finite subcover for each copy and a finite subcover for the entire space.

We wish to relate some well-known properties to pseudocompactness.

i) There exist locally compact spaces which are not pseudocompact. This is easy to see as the reals satisfy the requirements.

ii) There exist pseudocompact spaces which are not locally compact. This is somewhat more difficult as the spaces in 10.1, 10.2, 10.3, 10.4, and 10.6 are all locally compact. However example 9.1 in view of Theorem 9.5 must not be locally compact.

iii) There exist pseudocompact spaces which aren't first countable.

Example 10.5

iv) There exist pseudocompact spaces which are first countable.

Example 10.1

v) A closed locally compact subspace of a pseudocompact space need not be pseudocompact. The subset N of Λ in example 10.5.

vi) A closed C*-embedded subset of a pseudocompact space need not be pseudocompact. Example 10.5:

vii) A pseudocompact space may contain a zero set which is not pseudocompact and hence not C*-embedded.

Example 10.4.

viii) A pseudocompact space may contain a pseudocompact subset which is not C*-embedded. Let X be weakly normal and pseudocompact but not normal. (Ω of example 10.2). Then since X is not normal there exists a closed subset F in X such that F is not C*-embedded in X. This subset will be pseudocompact as a closed subset of a countably compact space.

ix) First countable pseudocompact completely regular Hausdorff spaces need not be locally compact. (Example 10.8).

CHAPTER 11

UNSOLVED PROBLEMS

First countable pseudocompact completely regular T_1 spaces enjoy some nice properties. Several questions arose concerning these spaces which I was unable to answer. Recall that if $|\beta X - X| < 2^c$. Then X must be pseudocompact. The converse is not true however. In example 10.5 the space Λ is pseudocompact and yet $|\beta\Lambda - \Lambda| \geq 2^c$. The problem is as follows:

Problem 1: If X is first countable and pseudocompact, then is $|\beta X - X| < 2^c$?

A second question which is related to the first is:

Problem 2: If a subset of a first countable pseudocompact space is C*-embedded in the space, then is the subset pseudocompact?

If we could show the subset had the inf property (Definition 4.1), pseudocompactness would follow. We also note the logical relationship here that if (1) is true then (2) is true.

In chapter 8 we defined the class of properties $\mathcal{P}^*$. We did not show however that $\mathcal{P}^* \neq \emptyset$. Therefore we have the following problem.

Problem 3: Find a pseudo perfect property.

Another question which was raised is the following.

Problem 4: Is every pseudocompact space with a point countable base H-closed?

Now a completely regular H-closed space is compact. Hence the question in the setting of completely regular spaces becomes the following:

Is a pseudocompact space with a point-countable base compact. Or equivalently does a pseudocompact space with a point countable base necessarily have a countable base. Corson and Michael [15] show using a proof suggested by M. E. Rudin that for countably compact spaces the answer is in the affirmative.

In [83] Vidossich gives the following result: A pseudocompact uniformizable space whose points are G_δ-sets is second countable. He then states that the proof of this is more difficult than for the compact case. Indeed the proof would be very difficult, as this is not true. Any completely regular spaces which is second countable is Lindelöf and hence realcompact. Thus any space which is pseudocompact and each of whose points is a G_δ would be compact. Consider however the space Ψ of Example 10.4. Every subset of Ψ is a G_δ, Ψ is completely regular and hence uniformizable let Ψ is pseudocompact but not compact. Hence Ψ is neither realcompact nor second countable.

Problem 5 The question arose in this study as to just when is $|\beta X - \alpha X| = 1$. Or equivalently when does X have a one-point realcompactification. In [27] Gillman and Jerison give a list of conditions which are equivalent to $|\beta X - X| \leq 1$. It is noted that X must be pseudocompact. The question then becomes when does the pseudocompactification αX of a space X have a one-point compactification.

In [11] Comfort gives an example of a locally compact σ-pseudocompact space with a one-point realcompactification.

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PSEUDOCOMPACTIFICATIONS AND PSEUDOCOMPACT SPACES

by

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(ABSTRACT)

We begin this paper with a survey of characterizations of pseudocompact spaces and relate pseudocompactness to other forms of compactness such as light compactness, countable compactness, weak compactness, etc. Some theorems on properties of subspaces of pseudocompact spaces are presented. In particular, conditions are given for the intersection of two pseudocompact spaces to be pseudocompact. First countable pseudocompact spaces are investigated and turn out to be maximally pseudocompact and minimally first countable in the class of completely regular spaces.

We define a pseudocompactification of a space X to be a pseudocompact space in which X is embedded as a dense subspace. In particular, for a completely regular space X, we consider the pseudocompactification $\alpha X = (\beta X - \upsilon X) \cup X$. We investigate this space and in general all pseudocompact subspaces of βX which contain X. There are many pseudocompact spaces between X and βX, but we may characterize αX as follows:

1) αX is the smallest subspace of βX containing X such that every free hyperreal z-ultrafilter on X is fixed in αX.

2) αX is the largest subspace of βX containing X such that

every point in $\alpha X - X$ is contained in a zero set which doesn't intersect X.

The space αX also has the nice property that any subset of X which is closed and relatively pseudocompact in X is closed in αX.

The relatively pseudocompact subspaces of a space are important and are investigated in Chapter 4. We further relate relative pseudocompactness to the hyperreal z-ultrafilter on X and obtain the following characterizations of a relatively pseudocompact zero set:

1) A zero set Z is relatively pseudocompact if and only if Z is contained in no hyperreal z-ultrafilter.

2) A zero set Z is relatively pseudocompact if and only if every countable cover of Z by cozero sets of X has a finite subcover.

In the next chapter we consider locally pseudocompact spaces and obtain results analogous to those for locally compact spaces. Then we relate pseudocompactness and the property of being C^*- or C-embedded in a space X. Included in this is a study of certain weak normality properties and their relationship to pseudocompact spaces.

We develop two types of one-point pseudocompactifications and investigate the properties of each. It turns out that a space X is never C^*-embedded in its one-point pseudocompactification. Also one space has the property that closed pseudocompact subsets are closed in the one-point pseudocompactification while the other may not have this property but will be completely regular.

We present survey material on products of pseudocompact spaces and unify these results. As an outgrowth of this study we investigate certain functions which are related to pseudocompactness.