RATIONAL AND HARMONIC APPROXIMATION ON F.P.A. SETS

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(ABSTRACT)

Let K be a compact subset of complex N-dimensional space, where $N \geq 1$. Let $H(K)$ denote the functions analytic in a neighborhood of K. Let $R(K)$ denote the closure of $H(K)$ in $C(K)$. We study the problem: What is $R(K)$?

The study of $R(K)$ is contained in the field of rational approximation. In a set of lecture notes, T. Gamelin [6] has shown a certain operator to be essential to the study of rational approximation. We study properties of this operator.

Now let K be a compact subset of real N-dimensional space, where $N \geq 2$. Let harm$K$ denote those functions harmonic in a neighborhood of K. Let $h(K)$ denote the closure of harm$K$ in $C(K)$. We also study the problem: What is $h(K)$?

The study of $h(K)$ is contained in the field of harmonic approximation. As well as obtaining harmonic analogues to our results in rational approximation, we also produce a harmonic analogue to the operator studied in Gamelin's notes.

TABLE OF CONTENTS

Preface

In this section, we outline the format of the text.

We begin with an introduction: We will discuss the types of problems which we will study (in the main body of the text). The Introduction is not an outline of the results which we will obtain. Instead, we present a few known facts concerning our subject. We attempt (in the introduction) only to interest the reader.

The Introduction will be followed by a "Preliminaries" section. Although the reader should be familiar with distributions, we list a few (indispensable) facts concerning them. In this section, we also set down (some of) the basic notation we will use. More advanced notation will be given when necessary. Certain abbreviations will occur frequently throughout the text; the convention we follow (concerning these abbreviations) will be discussed in this section.

The main body (8 chapters) of the text follows. We will not discuss the content of these chapters here: An outline can be found in the Introduction.

Each chapter is preceded by a short abstract. This abstract is not meant as a chapter summary. Instead, we discuss the goal of the chapter. These abstracts are included so that the reader can – in one short paragraph – determine whether the current chapter is of interest. It should be mentioned that chapter abstracts are terse. They are devoid of definitions, for example. The main body of each chapter will have the necessary details and explanations.

We will use the symbol $\blacksquare$ to denote the end of a proof. We will also use $\blacksquare$ to denote the end of any complete idea, be it a remark, definition, example, or section.

We will use the symbol $\square$ to denote the end of a portion of a larger whole. For example, if the hypotheses of a theorem get "out of hand," then we will separate the hypotheses and the conclusion with a $\square$. If we break the proof of a theorem into steps, then the steps will be separated by a $\square$.

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Introduction

In this introductory chapter, we shall discuss the problems to be studied in the main body of the text. We will try to keep the discussion as simple as possible, leaving the necessary difficulties and technicalities for later.

Let K be a compact Hausdorff space and let $C(K)$ denote the continuous complex-values functions on K. For $f \in C(K)$, set $\|f\| = \max\{|f(x)| : x \in K\}$. Under this norm, $C(K)$ is a C^*- algebra.

Let M be a subspace of $C(K)$. Let $\overline{M}$ denote the closure of M with respect to the above norm.

What is $\overline{M}$?

(1) Given $f \in C(K)$, find conditions which insure that $f \in \overline{M}$. Optimally, these conditions should be easy to describe and easy to check. □

(2) Conversely, given $f \in \overline{M}$, what properties does the function f posess? Again it is hoped that these properties are basic and easily understood. □

Finally, can we get the conditions (in **(1)**) and the properties (in **(2)**) to coincide? If so, then we will have achieved our goal, which is to answer the original question "What is $\overline{M}$?"

Such questions will occupy us for the first four chapters. In addition, a certain operator is shown (in [6]) to be essential to answering such questions. It is the usefulness of this operator which has inspired the remaining four chapters.

That is, directly or indirectly, the question "What is $\overline{M}$?" will occupy us the entire text. ■

We begin our discussion with Chapter I.

Let K denote an arbitrary compact subset of the complex plane $\mathbb{C}$. Let Rat$K$ denote the rational functions all of whose poles lie in $\mathbb{C} \setminus$ K. Let $R(K)$ denote the closure of Rat$K$ in $C(K)$.

(3) Let $\Omega \subseteq \mathbb{C}$ be open, with $K \subseteq \Omega$. By Runge's Theorem (Theorem 13.6 [12]), if f is analytic in Ω, then $f \in R(K)$. □

[We will let $H(\Omega)$ denote those functions analytic in Ω.]

How satisfying is the result **(3)**? As we refer back to **(1)**, we note some good and some bad. On the positive side, the condition of analyticity is easy to understand and to check. Or is it? Note we are given $f \in C(K)$ and we are asked if f has an analytic extension to an open set containing K. What if K has no interior? What if the boundary of K has positive area?

Unfortunately, one aspect of the result **(3)** will be typical in our results: We need to know about continuous extensions of f to a neighborhood of K.

(4) Let $\Omega \subseteq \mathbb{C}$ be open. If $f \in H(\Omega)$, then $\overline{\partial} f \equiv 0$ in Ω, where $\overline{\partial} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$. □

Note that if f is as in **(3)**, then **(4)** shows that $\overline{\partial} f \equiv 0$ in K.

Let $f \in R(K)$. At the expense of revealing the plot (too soon), we state our goal: Show that $\overline{\partial} f = 0$ in K.

We must be careful: From a regularity standpoint, all we know is that $f \in C(K)$. That is, we must make sense of the expression $\overline{\partial} f$.

Since $f \in R(K)$, there exists a sequence (r_n) in Rat$K$ with $r_n \rightrightarrows f$ in K. (We write $\rightrightarrows$ to denote uniform convergence.) It follows (Theorem 10.28 [12]) that

$f \in H(\text{int}K)$ and hence, $\overline{\partial} f \equiv 0$ in $\text{int}K$. (We let $\text{int}K$ denote the interior of the set K.) We record this result:

(5) If $f \in R(K)$, then $\overline{\partial} f \equiv 0$ in $\text{int}K$. □

What can we say about the result (5)? Is *it* satisfying? If K is a "nice" set, then the above result may suffice. (For the moment, let us pretend that "nice" means that $\text{int}K$ is nonempty and that K has a boundary consisting of finitely many smooth Jordan curves.) The problem with such nice sets is that the analysis is too easy and hence, uninteresting.

What if K has empty interior? What if the boundary of K has positive area? The analysis becomes much more difficult on such sets K. (More interesting, too!)

We return to the problem of making sense of the expression $\overline{\partial} f = 0$ in K. (Recall: we are assuming $f \in R(K)$.) By Tietze's Extension Theorem, we may extend f to be continuous in the entire plane. This extended function f is a distribution in the plane. (We will not discuss the Theory of Distributions at this time. For a few facts concerning distributions, see the "Preliminaries" section. For a proper introduction to distributions, see Chapter 6 [11].) Distributions are infinitely differentiable; hence, we may consider $\overline{\partial} f$. Unfortunately, $\overline{\partial} f$ is not necessarily a function (defined) in the plane. In general, all we can say is that $\overline{\partial} f$ is itself a distribution (in the plane). In this general setting it is difficult to say much. The phrase "consider $\overline{\partial} f$ on K" has no meaning. (We will, however, not allow such difficulty to deter us: See Theorem 150.)

Assume that $\overline{\partial} f$ may be identified with a function (defined) in the plane. Call this function g. If g is nice enough, then there is a lot we can say. (Here "nice" means: For $2 < p < \infty$ and any bounded Borel set $B \subseteq \mathbb{C}$, we have

$\iint_B |g|^p \, dA < \infty$.) Our goal will be to conclude that $g = 0$ almost everywhere in K (with respect to Lebesgue area measure $dm_2 = dA$).

We now state one of the goals of Chapter I.

Let $\mathcal{K}$ be a collection of compact subsets in the plane. Let $\mathcal{F}$ be a subspace of $C(\mathbb{C})$. (We let $C(\mathbb{C})$ denote the complex-valued functions which are continuous on all of $\mathbb{C}$.)

(6) Let $K \in \mathcal{K}$ and $f \in \mathcal{F}$. Then $f \in R(K)$ iff $\overline{\partial} f = 0$ a.e. $[m_2]$ in K. □

The result **(6)** is satisfying in that we get a property (of the function f) equivalent to the condition that $f \in R(K)$. This property is both easy to check and easy to state. (Of course, we hope to "maximize" $\mathcal{K}$ and $\mathcal{F}$ in **(6)**.)

In Chapter I, we will also present a generalization of **(6)**. Along the way, we will also generalize and extend results which fall under the category of rational approximation.

Many of the results we obtain (in Chapter I) are extensions of those of Khavinson ([8]). As we progress through Chapter I, we will mention exactly what Khavinson *did* prove. Some of our results are proved in the same manner (as Khavinson's). ■

In Chapter IV, we will attempt to extend the results from Chapter I to the setting of N-dimensional complex space $\mathbb{C}^N$, where $N > 1$. As we shall discuss in the abstract to Chapter IV, the results in Chapter IV will appear weaker than their Chapter I analogues.

In Chapter IV, we will meet a certain p.d.e., called the the $\overline{\partial}$-equation. The $\overline{\partial}$-equation concerns N equations in one unknown. If $N = 1$, as is the case in Chapter I, the equation is easy to solve and our results appear strong. In Chapter IV, when $N > 1$, the equation is not always solvable.

Throughout the text, we will extend definitions of transforms to distributions. We will generate many formulas which these distributions satisfy. These will be in the form of solutions of p.d.e.s. All will involve transforms.

In Chapter IV, we will introduce new transforms in an attempt to cope with the problems inherent to several complex variables. Our goal is to overcome these problems, as we attempt to obtain results analogous to our Chapter I results. ■

In Chapter II, we obtain results in harmonic approximation. These results are (somewhat) analogous to our (Chapter I) results in rational approximation.

(7) Let $\Omega \subseteq \mathbb{C}$ be open and let $f : \Omega \longrightarrow \mathbb{C}$. We say f is *harmonic* in Ω if $\triangle f = 0$ in Ω, where $\triangle = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$.

We immediately notice a problem: What (exactly) is meant by $\triangle f$? If f is continuous, then f is a distribution in Ω, as is $\triangle f$. The statement that "$\triangle f$ *is* the zero distribution in Ω" makes perfect sense. We will return (in Chapter II) to the idea of considering $\triangle f$ as a distribution in Ω. For now, in order to avoid confusion, we will clarify the above definition by changing (7) to read: $\triangle f \equiv 0$ in Ω. By this, we mean: For each $z_0 \in \Omega$, the quantity $(\triangle f)(z_0)$ exists (in the usual sense) and equals zero.

Let $K \subseteq \mathbb{C}$ be compact. We denote by harm$K$ those functions which are harmonic in a neighborhood of K. That is, $g \in$ harm$K$ if there exists an open set $\Omega \subseteq \mathbb{C}$ containing K with $\triangle g \equiv 0$ in Ω.

Note the relationship between the space harm$K$ and the differential operator $\triangle$ is analogous to that between Rat$K$ and $\overline{\partial}$: If $r \in$ Rat$K$, then $\overline{\partial} r \equiv 0$ in a neighborhood of K.

We let $h(K)$ denote the closure of harm$K$ in $C(K)$. Note $h(K)$ is the harmonic analogue of $R(K)$.

If we look back at **(6)** and match analogies, we arrive at the following conjecture:

(8) Let $K \in \mathcal{K}$ and $f \in \mathcal{F}$. Then $f \in h(K)$ iff $\triangle f = 0$ a.e. $[m_2]$ in K. □

Unfortunately, conjecture **(8)** is not valid: The property "$\triangle f = 0$ a.e. $[m_2]$ in K" is much stronger than the property "$f \in h(K)$."

In Chapter II, we find the proper equivalence to "$\triangle f = 0$ a.e. $[m_2]$ in K" (see Theorem 220). We also further generalize this result (Theorem 218 and Theorem 224). The key to our results is an integral formula (Theorem 208). This integral formula will allow us to guess the correct equivalence to "$\triangle f = 0$ a.e. $[m_2]$ in K."

In Chapter II, we are not extending previous results. Instead, we will be obtaining analogues of our Chapter I results. The proofs are more difficult in Chapter II, but are all easily manageable.

We will experience difficulties as we extend our results from "$\mathbb{C}$" to "$\mathbb{C}^N$" (as we make the transition from Chapter I to Chapter IV). We will not have similar problems when we extend our Chapter II results from $\mathbb{C}$, which may be identified with $\mathbb{R}^2$, to $\mathbb{R}^N$, where $N \geq 3$.

We attempt now to outline the main factor behind this ease of extension. This factor will be familiar to those who have studied harmonic analysis or p.d.e.s.

The function $\log|z|$ plays a large role in harmonic approximation (in the plane). Which properties of $\log|z|$ are most essential (to the results of Chapter II)?

(9) The function $\log|z|$ is harmonic in $\mathbb{C} \setminus \{0\}$, is locally integrable, and appears prominently (as a convolution kernel) in the essential integral formula (Theorem 208). □

By locally integrable, we mean: $\iint\limits_B |\log|z||\, dA < \infty$ for all bounded Borel sets $B \subseteq \mathbb{C}$.

Let $N \geq 3$ and set $\Delta = \sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2}$. The following is the analogue of **(9)** in higher dimensions.

(10) The function $\frac{1}{|x|^{N-2}}$ is harmonic in $\mathbb{R}^N \setminus \{0\}$, is locally integrable, and appears prominently (as a convolution kernel) in the $\left(\mathbb{R}^N-\right)$ integral formula Theorem 305, the analogue of Theorem 208. □

Here locally integrable means $\int\limits_B \frac{dm_N}{|x|^{N-2}} < \infty$ for all bounded Borel sets $B \subseteq \mathbb{R}^N$, where m_N denotes N-dimensional Lebesgue measure on $\mathbb{R}^N$.

As we shall see, our Chapter II results become Chapter III results as we replace $\log|z|$ by $\frac{1}{|x|^{N-2}}$. ■

The Cauchy transform plays an important role in Chapters I and II. (We will also define generalizations of the Cauchy transform in Chapters III and IV.)

For $f : \mathbb{C} \longrightarrow \mathbb{C}$, define its Cauchy transform $\hat{f}(z_0) = \iint\limits_{\mathbb{C}} \frac{f(z)}{z-z_0}\, dA(z)$.

Even before the dust settles, many questions arise. Is the integral absolutely convergent? What sort of regularity (of f) is required to insure that the integral converge for a fixed $z_0 \in \mathbb{C}$? For a.e. $[m_2]$ $z_0 \in \mathbb{C}$?

What is needed to insure that $\hat{f}$ has certain regularity properties (integrability, continuity, differentiability, etc.)? What properties does the map $f \mapsto \hat{f}$ possess? Is it bounded? Is it compact?

A major question with transforms involves the "removal" of the transform. Is the Cauchy transform left-invertible? Invertible? In what sense?

After reading through Chapters I and II, where Cauchy transforms play a major role, we find that such questions should be answered. We answer these questions in Chapter V.

Without giving the definition here, we should mention that it will be necessary to extend the definition of the Cauchy transform (given above) to distributions (in the plane). We will also prove results concerning these transformed distributions (in Chapter V). ■

In Chapter VI, we shall discuss properties of the T_ϕ operator. This operator plays an essential role in rational approximation (see [6]).

Let $\phi \in C_c^1(\mathbb{C})$. By this, we mean: $\phi, \frac{\partial\phi}{\partial x}$, and $\frac{\partial\phi}{\partial y}$ are continuous in $\mathbb{C}$ and $\{z \in \mathbb{C} : \phi(z) \neq 0\}$ is a bounded subset of $\mathbb{C}$. We write suppϕ to denote the closure of the above set, the support of ϕ. (The subscript c above denotes compact support.)

Let $f : \mathbb{C} \longrightarrow \mathbb{C}$. Define $T_\phi f = \phi f + \frac{1}{\pi}\left(f\overline{\partial}\phi\right)^{\wedge}$.

This formula generates another long list of questions (as was the case with the Cauchy transform definition). We will not repeat that list here. It will suffice to say the questions are similar.

If we look back at the definition (of T_ϕ), we notice T_ϕ is the sum of two operators. The first is a multiplication (by ϕ) operator. The second is the composition of a multiplication (by $\overline{\partial}f$) operator followed by the Cauchy transform operator.

Such multiplication operators are well understood. We spend Chapter V studying the Cauchy transform operator. Is this the proper way to view the operator T_ϕ?

Yes and no.

We must, of course, try to extract as much knowledge as possible by viewing T_ϕ as the sum of two operators, as above. But viewing T_ϕ in this manner alone would ignore essential properties of this operator.

The following result is from [6].

(11) Let $\Omega \subseteq \mathbb{C}$ be open and bounded. Let $f : \Omega \longrightarrow \mathbb{C}$ be bounded and analytic. Then $T_\phi f : \Omega \longrightarrow \mathbb{C}$ is bounded and analytic. □

The separate terms (decomposed as above) do not necessarily give analytic functions (where f is as in **(11)**). Using the product rule and **(2)**, we know ϕf is rarely analytic. Corollary 520 shows $\left(f\overline{\partial}\phi\right)^\wedge$ is rarely analytic, as well. But the sum is always analytic!

See [6] for many applications involving the property **(11)**.

Consider the following question:

(12) How large are certain quantities involving $T_\phi f$ (in terms of properties of f)?

□

In **(12)**, "large" means whatever is relevant, be it an L^1-norm of $T_\phi f$ or an L^∞-bound on $\overline{\partial}(T_\phi f)$. The "best" answers to **(12)** cannot be proved by viewing T_ϕ as the sum of two operators and to then find bounds on each term separately. To view T_ϕ in this manner ignores cancellations which lead to the best answers to **(12)**. (Combining the product rule with **(2)** and Corollary 121 shows how cancellation between terms gives **(11)**.) For the "answers" to **(12)**, see [6].

Once **(12)** is "answered," the following questions (when answered) lead to many applications in rational approximation (see [6]).

Let $K \subseteq \mathbb{C}$ be compact and let $f : K \longrightarrow \mathbb{C}$. Let $\mathcal{U} = \{U_j : 1 \leq j \leq n\}$ and $\mathcal{W} = \{W_j : 1 \leq j \leq n\}$ be open covers (in $\mathbb{C}$) of K. Assume each $\overline{U}_j \subseteq W_j$. For each j, let $\phi_j \in C_c^1(\mathbb{C})$ satisfy :$\phi_j \equiv 1$ in $\overline{U}_j$ and supp ϕ_j is a compact subset of W_j.

Consider $g = \sum\limits_{j=1}^{n} T_{\phi_j} f$.

(13) How does g behave (compared to f)? □

For example, how does $\sup_{z\in\mathbb{C}} |g(z)|$ compare with $\sup_{z\in K} |f(z)|$? If $\delta > 0$ is fixed, how does

$$\sup \{|g(z_1) - g(z_2)| : |z_1 - z_2| \leq \delta \text{ and } z_1, z_2 \in \mathbb{C} \}$$

compare with

$$\sup \{|f(z_1) - f(z_2)| : |z_1 - z_2| \leq \delta \text{ and } z_1, z_2 \in \mathbb{C} \}?$$

How does $\sup_{z\in K} |f(z) - g(z)|$ compare with the sizes of the elements of $\mathcal{U}$ and $\mathcal{W}$?

The above questions are considered in [6]. □

Suppose for each $\delta > 0$, we let $\mathcal{U}_\delta$ and $\mathcal{W}_\delta$ be open and covers of K (as above) depending on δ. (For example, suppose each element of $\mathcal{U}_\delta$ and $\mathcal{W}_\delta$ has diameter less than δ and 2δ, respectively.) Form $g = g(\delta)$ as above.

(14) How does $g(\delta)$ behave as $\delta \searrow 0$? How does $g(\delta)$ approximate f as $\delta \searrow 0$?

□

For example, does there exist $C > 0$ such that $\sup_{z\in\mathbb{C}} \left|\left[\overline{\partial} g(\delta)\right](z)\right| \leq \frac{C}{\delta}$ for all small enough δ? (Here we are assuming $\overline{\partial} g$ is defined pointwise throughout $\mathbb{C}$.) How does $\sup_{z\in\mathbb{C}} |f(z) - g(\delta)(z)|$ compare with δ?

The answers to **(14)** appear in [6]. We do not wish to repeat the answers and the applications here (from [6]). Instead, we mention that the applications in [6] all involve the sup-norm. In Chapter VI, we concentrate on properties of the T_ϕ operator with respect to L^p-norms, where $1 \leq p < \infty$. ■

In Chapter VII, we generalize the T_ϕ operator. We now discuss the motivation behind Chapter VII.

Look at the definition of T_ϕ. Is it necessary that $\phi \in C^1_c(\mathbb{C})$? Multiplication by bounded functions is well-understood. Would it be enough to assume that ϕ

and $\overline{\partial}\phi$ are bounded functions? (Here we think of $\overline{\partial}\phi$ as a distribution and we assume $\overline{\partial}\phi$ can be identified with a function in the plane.) If we write M_g to denote multiplication by the function g and if we write $\wedge$ to denote the Cauchy transform operator (the map $f \mapsto \hat{f}$), then we may write $T_\phi = M_\phi + \frac{1}{\pi}\left[\wedge \circ M_{\overline{\partial}\phi}\right]$.

As long as we are multiplying by bounded function, why don't we just let g and h be arbitrary bounded functions and consider the operator $M_g + \frac{1}{\pi}\left[\wedge \circ M_h\right]$?

We will consider such questions in Chapter VII. ■

In Chapter VIII, we introduce a new operator, h_ϕ. We hope the operator h_ϕ will be the harmonic analogue of T_ϕ. We have the following result.

(15) Let $\Omega \subseteq \mathbb{C}$ be open and bounded. Let $f : \Omega \longrightarrow \mathbb{C}$ be a bounded harmonic function. Then $h_\phi f : \Omega \longrightarrow \mathbb{C}$ is harmonic. □

This is the harmonic analogue of **(11)**. Note that unlike **(11)**, however, we cannot conclude that h_ϕ is bounded. If the first partials of f are bounded in Ω, then h_ϕ is bounded. (We can do better than this, actually: See Theorem 805.) We should mention that $\phi \in C_c^2(\mathbb{C})$ in **(15)**.

■

Notation, Conventions, and Preliminaries

We assume the reader is familiar with the basics of real, complex, and functional analysis (as found in [12] and [11]).

WLOG means that a reduction has taken place; such reductions will be easily justified. *The following are equivalent* will be abbreviated TFAE.

If $1 \leq p \leq \infty$ and if q "appears" without notice, then $\frac{1}{p} + \frac{1}{q} = 1$.

The following theorems will be used often, but we will rarely mention their usage: Holder's Inequality, Fubini's Theorem, the Riesz Representation Theorems, and the Hahn-Banach Theorem. Their usage will always be clear.

Throughout the text, we will define many sets. If the description of a particular set is extremely technical and if this set is only used for a short period, then we will denote this set by the letter A. Although there will be different sets called A throughout the text, this will cause no confusion as these sets will be "spaced far apart." □

We will write $\mathbb{N} = \{1, 2, 3, \cdots\}$ and $\mathbb{N}^* = \mathbb{N} \bigcup \{0\}$. We will also write $\mathbb{Z} = \{0, \pm 1, \pm 2, \cdots\}$.

Let $a = (a_n)_{n=0}^{\infty}$ denote an arbitrary sequence of complex numbers. For $0 < p < \infty$, we set $\ell^p(\mathbb{N}^*) = \left\{a : \sum_{n=0}^{\infty} |a_n|^p < \infty\right\}$. We also set $\ell^\infty(\mathbb{N}^*) = \{a : \sup |a_n| < \infty\}$.

Let X be any set. Let (f_α) be a net of complex-valued functions defined on X. Let $f : X \longrightarrow \mathbb{C}$. We will write $f_\alpha \rightrightarrows f$ in X to denote uniform convergence. □

Let X and Y be topological vector spaces. We let $L(X, Y)$ denote the continuous linear maps from X to Y. We write $L(X)$ to denote $L(X, X)$.

For $T \in L(X, Y)$, we set $\ker T = \{x \in X : Tx = 0\}$.

If M is a closed subspace of X, then we will abbreviate this (fact) by writing $M \leq X$.

We denote the dual of X by X^*. By definition, $X^* = L(X, \mathbb{C})$. For $\phi \in X^*$ and $x \in X$, we let $\phi(x)$ denote the complex number obtained when ϕ acts on x.

See Theorem 6.16 and Theorem 6.19 of [12] for two standard dual space (Riesz) representations, which we abbreviate here by $(L^p)^* = L^q$ and $(C_0)^* = M$.

We let $w - *$ denote the weak-star topology on a dual space X^* (see Section 3.14 [11]). □

Let X be a topological space. We let $C(X)$ denote the complex-valued functions which are continuous in X. If $S \subseteq X$, then int$S$ and $\overline{S}$ denote its interior an closure, respectively. We let ∂S denote the boundary of S. We let $X \setminus S$ denote the complement of S (in X). □

Let X be a locally compact Hausdorff space. We let $M(X)$ denote the Banach space of complex Borel measures on X. For $\mu \in M(X)$, we set $\|\mu\| = |\mu|(X)$. □

Except in Chapters III and IV, we will be working in the complex plane, denoted $\mathbb{C}$. We will let $dm_2 = dA$ denote Lebesgue area measure on $\mathbb{C}$. Let S be any m_2-measurable subset of $\mathbb{C}$. For $1 \leq p < \infty$, we let $L^p(S)$ denote those (complex-valued, measurable) functions f satisfying: $\iint_S |f|^p \, dA < \infty$. We let $L^\infty(S)$ denote the essentially bounded functions (on S). If the set S is (clearly) understood, then we write $\| \ \|_p$ to denote the norm in $L^p(S)$. For $1 \leq p \leq \infty$, we let $L^p_{\text{loc}}(S)$ denote those f for which $f \in L^p(K)$ for all compact sets $K \subseteq S$.

For $z_0 \in \mathbb{C}$ and $u, v : \mathbb{C} \longrightarrow \mathbb{C}$, we write

$$(u * v)(z_0) = \iint_{\mathbb{C}} u(z_0 - z)\, v(z)\, dA(z).$$

For $z_0 \in \mathbb{C}$ and $r > 0$, we set $D(z_0; r) = \{z \in \mathbb{C} : |z - z_0| < r\}$.

We write $\mathbb{D} = \mathrm{D}(0;1)$ and $\mathbf{T} = \{z \in \mathbb{C} : |z| = 1\}$. For $r > 0$, we write $r\mathbb{D} = \mathrm{D}(0;r)$ and $r\mathbf{T} = \{z \in \mathbb{C} : |z| = r\}$.

For the next few paragraphs, fix an open set $\Omega \subseteq \mathbb{C}$.

We let $H(\Omega)$ denote those functions which are analytic in Ω.

We let $\mathcal{D}(\Omega)$ denote those $f \in C^{\infty}(\Omega)$ having compact support in Ω. We let $\mathcal{D}'(\Omega)$ denote the distributions in Ω. For $\Lambda \in \mathcal{D}'(\Omega)$ and $\phi \in \mathcal{D}(\Omega)$, we let $\Lambda\phi$ (or $\Lambda(\phi)$) denote the complex number obtained when Λ acts on ϕ. We will state a few facts concerning distributions; for more detail, see Chapter 6 [11].

The space of distributions is the set of all continuous linear operators from $\mathcal{D}(\Omega)$ into $\mathbb{C}$. For a definition of the topology of $\mathcal{D}(\Omega)$, see Chapter 6 [11].

The following two examples (of distributions) will be used throughout this book.

Let $f \in L^1_{\text{loc}}(\Omega)$. We may think of $f \in \mathcal{D}'(\Omega)$: For $\phi \in \mathcal{D}(\Omega)$, set $f(\phi) = \iint\limits_{\Omega} f\phi\, dA$. At times, to avoid confusion, we will write Λ_f to denote the above distribution.

If $\mu \in M(\Omega)$ or if μ is a positive measure (on Ω) under which each compact subset of Ω has finite measure, then we may think of μ as an element of $\mathcal{D}'(\Omega)$: For $\phi \in \mathcal{D}(\Omega)$, set $\mu(\phi) = \int \phi\, d\mu$. Occasionally we will write Λ_μ to denote this above distribution.

Let $\Lambda \in \mathcal{D}'(\Omega)$ and set $D = \frac{\partial^{j+\ell}}{\partial x^j \partial y^\ell}$. We define the distribution $D\Lambda$ as follows: For $\phi \in \mathcal{D}(\Omega)$, we set $(D\Lambda)\,\phi = (-1)^{j+\ell}\Lambda\,(D\phi)$.

Note that $D\Lambda$ is defined by the action of Λ on $D\phi$, where $\phi \in \mathcal{D}(\Omega)$. Since $D\phi$ is independent of the order of differentiation, we see $D\Lambda$ is also.

Let $f \in L^1_{\text{loc}}(\Omega)$ and think of $f \in \mathcal{D}'(\Omega)$. Write Λ_f to denote this distribution. Set $D = \frac{\partial^{j+\ell}}{\partial x^j \partial y^\ell}$. Then $D\Lambda_f$ exists (as a distribution). If $g \in L^1_{\text{loc}}(\Omega)$ and if $\Lambda_g = D\Lambda_f$ in $\mathcal{D}'(\Omega)$, then we call g the D-derivative of f. [Note $D\Lambda_f$ always exists as a distribution; it is a special case when $D\Lambda_f$ can be represented as a function.] We should also note: If Df exists in the "usual" sense and if $f \in C^{j+\ell}(\Omega)$, then $\Lambda_{Df} = D\Lambda_f$ in $\mathcal{D}'(\Omega)$.

For $f \in L^1_{\text{loc}}(\Omega)$ and D as above, we will write Df to denote $D\Lambda_f$. As noted in the last sentence of the above paragraph, this will cause no confusion.

Let $\Lambda \in \mathcal{D}'(\Omega)$. Define the support of Λ by

$\text{supp}\Lambda = \Omega \setminus [\bigcup\{\text{U} \subseteq \Omega : \text{U is open and } \Lambda = 0 \text{ in } \mathcal{D}'(\text{U})\}]$.

For $f \in L^1_{\text{loc}}(\Omega)$ and $\mu \in M(\Omega)$, we may think of f and μ as elements of $\mathcal{D}'(\Omega)$ and define each of their supports as in the above paragraph.

Although distributions cannot (in general) be multiplied, $\mathcal{D}'(\Omega)$ does form a module over $C^\infty(\Omega)$: Let $\psi \in C^\infty(\Omega)$ and $\Lambda \in \mathcal{D}'(\Omega)$. Define $\psi\Lambda : \mathcal{D}(\Omega) \longrightarrow \mathbb{C}$ by $(\psi\Lambda)\,\phi = \Lambda\,(\phi\psi)$. The resulting product $\psi\Lambda$ *is* a distribution. With respect to this multiplication, the product rule (for differentiation) holds: For example,
if $D = \frac{\partial}{\partial x}$, then $D\,(\psi\Lambda) = \psi D\Lambda + (D\psi)\,\Lambda$.

Set $\mathcal{E}'(\Omega) = \{\Lambda \in \mathcal{D}'(\Omega) : \text{supp}\Lambda \subseteq \Omega \text{ is compact}\}$. Every element of $\mathcal{E}'(\Omega)$ has a continuous extension to $C^\infty(\Omega)$. [See Section 1.46 [11] for a definition of the topology of $C^\infty(\Omega)$.] In fact, $\mathcal{E}'(\Omega)$ *is* the space of continuous linear functionals on $C^\infty(\Omega)$.

The subscript c will denote compact support. So $L^p_c(\Omega)$ and $M_c(\Omega)$ will denote those elements of $L^p(\Omega)$ and $\mu \in M(\Omega)$, respectively, having compact support in Ω. We let $C^n_c(\Omega)$ denote those elements of

$$C^n(\Omega) \equiv \left\{ f \in C(\Omega) : \frac{\partial^{j+\ell} f}{\partial x^j \partial y^\ell} \in C(\Omega) \ \text{ whenever } \ j + \ell \leq n \right\}$$

having compact support in Ω.

Let $\Lambda \in \mathcal{D}'(\Omega)$ and $n \in \mathbb{N}^*$. We say Λ has *order n* if given any compact $K \subseteq \Omega$, there exists $C > 0$ satisfying: If $\phi \in \mathcal{D}(\Omega)$ and if supp $\phi \subseteq$ K, then $|\Lambda\phi| \le C \max\left\{\left|\frac{\partial^{j+\ell}\phi}{\partial x^j \partial y^\ell}(z)\right| : j+\ell \le n \text{ and } z \in K\right\}$. We will let $\mathcal{D}'_n(\Omega)$ denote those distributions having order n.

Place the following topology on $C^n(\Omega)$: $\phi_\alpha \longrightarrow 0$ in $C^n(\Omega)$ iff there exists a compact set $K \subseteq \Omega$ containing each supp ϕ_α and $D\phi_\alpha \rightrightarrows 0$ in Ω for each $D = \frac{\partial^{j+\ell}}{\partial x^j \partial y^\ell}$, where $j + \ell \le n$.

Let $\Lambda \in \mathcal{D}'_n(\Omega)$. Then Λ has a continuous extension to $C^n_c(\Omega)$. In fact, $\mathcal{D}'_n(\Omega) = [C^n_c(\Omega)]^*$.

We note that $\mathcal{D}'_n(\Omega)$ is a module over $C^n(\Omega)$: Let $\psi \in C^n(\Omega)$ and $\Lambda \in \mathcal{D}'_n(\Omega)$. Define $\psi\Lambda : C^n_c(\Omega) \longrightarrow \mathbb{C}$ by $(\psi\Lambda)\,\phi = \Lambda\,(\psi\phi)$. Note $\psi\Lambda \in [C^n_c(\Omega)]^*$.

Note $L^1_{\text{loc}}(\Omega) \bigcup M(\Omega) \subseteq \mathcal{D}'_0(\Omega)$. For easy examples of elements of $\mathcal{D}'_1(\Omega)$, choose $\Lambda \in \mathcal{D}'_0(\Omega)$ and consider both $\frac{\partial\Lambda}{\partial x}$ and $\frac{\partial\Lambda}{\partial y}$. ■

Except in Chapters III and IV, the underlying measure space will be Lebesgue area measure $dm_2 = dA$ on $\mathbb{C}$. If we write "a.e.," it is understood we mean "a.e. $[m_2]$ in $\mathbb{C}$." We will abbreviate $L^p(\mathbb{C}, \text{dA}), C(\mathbb{C}), \mathcal{D}(\mathbb{C}), \mathcal{D}'(\mathbb{C})$, and $\mathcal{E}'(\mathbb{C})$ by $L^p, C, \mathcal{D}, \mathcal{D}'$, and $\mathcal{E}'$, respectively.

Unless mentioned otherwise, S will denote an m_2-measurable subset of $\mathbb{C}$. We let $L^p(S)$ denote $L^p(S, dA)$. We abbreviate "a.e. $[m_2]$ in S" by writing "a.e. in S." □

In Chapter III, we fix $N > 2$. The underlying measure space is N-dimensional Lebesgue measure m_N on $\mathbb{R}^N$. Otherwise, we use the above convention with $(\mathbb{C}, \text{m}_2)$ replaced by $\left(\mathbb{R}^N, \text{m}_N\right)$. For example, $\mathcal{D}'$ denotes the distributions in $\mathbb{R}^N$ and we write $\mathcal{D} = C^\infty_c\left(\mathbb{R}^N\right)$. The (basic) Theory of Distributions is independent of N: We will not repeat all the above definitions and facts. In this (Chapter III) setting, $\mathcal{D}'$ *is* the space of continuous linear functionals on $\mathcal{D}$. □

In Chapter IV, we fix $N > 1$. The underlying measure space is $(2N)$-dimensional

Lebesgue measure m_{2N} on $\mathbb{C}^N$. We use the convention (above) with $(\mathbb{C}, \mathrm{m}_2)$ replaced by $\left(\mathbb{C}^{\mathrm{N}}, \mathrm{m}_{2\mathrm{N}}\right)$. ■

Chapter I
Rational Approximation on F.P.A. Sets

Abstract

We wish to extend results in rational approximation found in [8].

Let $2 < p \leq \infty$ and $f \in C_c(\mathbb{C})$, with $\overline{\partial} f \in L^p(\mathbb{C})$. Let $S \subseteq \mathbb{C}$ be bounded and m_2–measurable. Consider the following conditions:

(i) $f \in R(S)$.

(ii) $\overline{\partial} f = 0$ a.e. $[m_2]$ in S. □

Our goal is to find the largest class of sets for which (i) and (ii) are equivalent. Our best result is the following: If S is an inner f.p.a. set, then (i) and (ii) are equivalent. ■

Important: Except for Chapters III and IV, the (unmentioned) measure space is $(\mathbb{C}, \text{dA})$, where $dm_2 = dA$ denotes Lebesgue area measure. $S \subseteq \mathbb{C}$ will always denote an m_2– measurable set. We write $L^p = L^p(\mathbb{C}, \text{dA})$ and $L^p(S) = L^p(S, dA)$; "a.e." means "a.e. $[m_2]$ in $\mathbb{C}$" and "a.e. in S" means "a.e. $[m_2]$ in S."

Definition: For $\mu \in M = M(\mathbb{C})$, define the Cauchy transform $\hat{\mu}$ of μ as follows: For those $z_0 \in \mathbb{C}$ for which $\int \frac{d|\mu|(z)}{|z-z_0|} < \infty$, set $\hat{\mu}(z_0) = \int \frac{d\mu(z)}{z-z_0}$.

Remark: For $f \in L^1 = L^1(\mathbb{C})$, we let $\hat{f}$ denote the Cauchy transform of the measure $f\,dA$.

Proposition 100 *Let* $\mu \in M$ *and* $1 \leq p < 2$. *Define* $f : \mathbb{C} \longrightarrow [0, \infty]$ *by* $f(\zeta) = \int \frac{d|\mu|(z)}{|z-\zeta|^p}$. *Then* $f \in L^1_{\text{loc}} = L^1_{\text{loc}}(\mathbb{C})$. *It follows that* $\hat{\mu}$ *is defined a.e. and that* $\hat{\mu} \in L^p_{\text{loc}}$.

Proof. Let $R > 0$.

$$\iint\limits_{R\mathbb{D}} f\, dA = \int \left(\iint\limits_{R\mathbb{D}} \frac{dA(\zeta)}{|z-\zeta|^p} \right) d|\mu|(z)$$

$$\leq \int \left(\int\!\!\int\limits_{D(z;2R)} \frac{dA(\zeta)}{|z-\zeta|^p} \right) d|\mu|(z) = \frac{2\pi}{2-p}(2R)^{2-p}\|\mu\|.$$

■

Definition: Let $\gamma : [0,1] \longrightarrow \mathbb{C}$ be continuous. We call γ a *rectifiable Jordan path* if

(i) $\gamma\,|_{[0,1)}$ is $1-1$ and

(ii) there exist $0 = t_0 < t_1 < \ldots < t_n = 1$ satisfying: $\gamma \in C^1(t_{j-1}, t_j)$ and
$\ell(\gamma) \equiv \sum\limits_{j=1}^{n} \int\limits_{t_{j-1}}^{t_j} |\gamma'(s)|\, ds < \infty.$ □

γ is a *closed, rectifiable Jordan path* if also $\gamma(0) = \gamma(1)$.

Section 101 For a closed, rectifiable Jordan path γ, we give γ the positive orientation: Let U and V denote the bounded and unbounded components of $\mathbb{C} \setminus \gamma$, respectively. [Note we write γ to (also) denote the image of the function γ.] By positive orientation, we mean

$$\frac{1}{2\pi i} \int\limits_{\gamma} \frac{dz}{z - z_0} = \begin{cases} 1 & \text{if } z_0 \in U, \\ 0 & \text{if } z_0 \in V. \end{cases}$$

We call U and V the interior and exterior of γ, respectively. We write $U = \operatorname{int} \gamma$ and $V = \operatorname{ext} \gamma$.

We write $-\gamma$ to denote the negative orientation of γ:

$$\frac{1}{2\pi i} \int\limits_{-\gamma} \frac{dz}{z - z_0} = \begin{cases} -1 & \text{if } z_0 \in U, \\ 0 & \text{if } z_0 \in V. \end{cases}$$

■

Definition: [8] Let $S \subseteq \mathbb{C}$ be (m_2-) measurable. We call S a set of finite perimeter (f.p.) if there exists $\mu \in M$ satisfying: $\hat{\mu} = \chi_S$ a.e..

Theorem 102 *Let $(\gamma_n)_{n=0}^{\infty}$ be a sequence of closed, rectifiable Jordan paths. Set $U_n = int\,\gamma_n$. Assume the following:*

(i) $\overline{U}_n \subseteq U_0 \qquad (n \in \mathbb{N})$.

(ii) $\overline{U}_j \bigcap \overline{U}_n = \phi$ *whenever* $j \neq n$ *and* $j, n \in \mathbb{N}$.

(iii) $\sum\limits_{n=0}^{\infty} \ell(\gamma_n) < \infty$. □

Set $K = \overline{U}_0 \setminus (\bigcup\limits_{n=1}^{\infty} U_n)$. Then K is a compact set having finite perimeter.

Proof. Set $\mu = \frac{1}{2\pi i}\left(dz\,|_{\gamma_0} + \sum\limits_{n=1}^{\infty} dz\,|_{-\gamma_n}\right)$. Fix $1 < p < 2$. Let q satisfy: $\frac{1}{p} + \frac{1}{q} = 1$. Set $E_1 = \left\{z_0 \in \mathbb{C} : \int \frac{d|\mu|(z)}{|z-z_0|^p} = \infty\right\}$, $\quad E_2 = \bigcup\limits_{n=0}^{\infty} \gamma_n$, and $E = E_1 \bigcup E_2$. By Proposition 100, we note $m_2 E = 0$.

If $z_0 \in \mathbb{C} \setminus$ K, then $\hat{\mu}(z_0) = 0$.

So fix $z_0 \in K \setminus E$. For each N, Cauchy's Theorem shows

$$1 = \frac{1}{2\pi i}\left(\int\limits_{\gamma_0} \frac{dz}{z - z_0} - \sum_{n=1}^{N} \int\limits_{\gamma_n} \frac{dz}{z - z_0}\right)$$

and hence,

$$|\hat{\mu}(z_0) - 1| = \left| \int\limits_{\cup\{\gamma_n : n \geq N+1\}} \frac{dz}{z - z_0} \right| \leq \left(\sum_{n=N+1}^{\infty} \ell(\gamma_n)\right)^{\frac{1}{q}} \left(\int \frac{d|\mu|}{|z - z_0|^p}\right)^{\frac{1}{p}} \longrightarrow 0.$$

We just showed: $\hat{\mu} = \chi_K$ in $\mathbb{C} \setminus$ E. ■

Definition: [8] Let $S \subseteq \mathbb{C}$ be measurable and let $z_0 \in \mathbb{C}$. We say S has an *exterior normal* at z_0 if there exists a unit vector $n = n(z_0)$ satisfying :

$$\lim_{\delta \searrow 0} \frac{m_2(S \bigcap D_+(z_0;\delta))}{m_2(D_+(z_0;\delta))} = 0 \text{ and}$$

$$\lim_{\delta \searrow 0} \frac{m_2(S \bigcap D_-(z_0;\delta))}{m_2(D_-(z_0;\delta))} = 1, \text{ where}$$

$$D_+(z_0;\delta) = \{z \in D(z_0;\delta) : (z - z_0) \cdot n \geq 0\},$$
$$D_-(z_0;\delta) = \{z \in D(z_0;\delta) : (z - z_0) \cdot n \leq 0\},$$

and where we think of $z - z_0$ and n as being vectors in $\mathbb{R}^2$.

Set $B_S = \{z \in \mathbb{C} : \text{S possesses an exterior normal at } z\}$, the *reduced boundary* of S.

Remark: B_S is the "nice" portion of the (topological) boundary of S. ■

Example 103 **(i)** Set $S = [0,1] \times [0,1]$.

Note $B_S = (\partial S) \setminus \{(0,0),(1,0),(0,1),(1,1)\}$, the boundary of S "minus" the corners.

Note S has finite perimeter: Set $d\mu = \frac{1}{2\pi i}\, dz\,|_{\partial S}$, where we think of ∂S as a closed, rectifiable Jordan path (positively oriented). Note $\hat{\mu} = \chi_{\text{int} S}$ in $\mathbb{C} \setminus \partial S$ and hence, $\hat{\mu} = \chi_S$ a.e..

(ii) Set $S = \mathbb{D} \setminus (0,1)$, the open unit "slit" disc. Set $d\mu = \frac{1}{2\pi i}\, dz\,|_{\mathbf{T}}$. Note $B_S = \mathbf{T}$ and $\hat{\mu} = \chi_{\mathbb{D}}$ in $\mathbb{C} \setminus \mathbf{T}$. So $\hat{\mu} = \chi_S$ a.e. and hence, S has f.p..

(iii) We simplify the underlying set of Theorem 102: Assume each γ_n is a circle. By Theorem 102, we note K has f.p.. By Theorem 104 (below), $\mathcal{H}^1 (B_K \Delta [\bigcup \gamma_n]) = 0$. [Here $\mathcal{H}^1$ denotes one-dimensional Hausdorff measure and $S_1 \Delta S_2 = (S_1 \setminus S_2) \bigcup (S_2 \setminus S_1)$. For a precise definintion of Hausdorff measure, see [9, p. 403].] ■

Terminology: Let $\Omega \subseteq \mathbb{C}$ be open, bounded, and connected. Assume $\partial\Omega$ consists of finitely many pairwise disjoint C^1, closed, rectifiable Jordan paths. We shall call such a set Ω a *finitely connected C^1 Jordan domain.*

Remark: Let Ω be as above. Let $(\gamma_n)_{n=0}^N$ be the components of $\partial\Omega$. Assume $\gamma_n \subseteq \operatorname{int}\gamma_0 \quad (1 \le n \le N)$. We define the measure dz on $\partial\Omega$: $dz\,|_{\partial\Omega} = dz\,|_{\gamma_0} + \sum_{n=1}^{N} dz\,|_{-\gamma_n}$. That is, we orient the paths γ_n so that

$$\frac{1}{2\pi i}\int_{\partial\Omega} \frac{dz}{z - z_0} = \begin{cases} 1 & \text{if } z_0 \in \Omega, \\ 0 & \text{if } z_0 \in \mathbb{C} \setminus \overline{\Omega}. \end{cases}$$

■

Part **(a)** of the following theorem relates the measure associated with a set of finite perimeter with its reduced boundary; part **(b)** shows how such a set can be approximated by "nice" open sets.

Theorem 104 *[8] Let S have finite perimeter. By definition, there exists $\mu \in M$ with $\hat{\mu} = \chi_S$ a.e..*

(a) **(i)** $\mathcal{H}^1(B_S) = 2\pi\|\mu\|$.

(ii) If $B \subseteq \mathbb{C}$ is a Borel set, then $\mu(B) = \mu(B \cap B_S) = \frac{1}{2\pi}\int_{B\cap B_S} n(z)\, d\mathcal{H}^1(z)$.

(iii) There exists a set $E_1 \subseteq B_S$ satisfying:

$$\mathcal{H}^1(E_1) = 0 \text{ and } \lim_{\delta \searrow 0} \frac{\mathcal{H}^1(B_S \cap D(z_0;\delta))}{2\delta} = 1 \qquad (z_0 \in B_S \setminus E_1).$$

(iv) There exists a set $E_2 \subseteq B_S$ and a sequence of rectifiable Jordan paths (γ_n) satisfying :

$$\mathcal{H}^1(E_2) = 0, \quad B_S \setminus E_2 \subseteq \bigcup \gamma_n, \text{ and } d\mu = \frac{1}{2\pi i}\, dz\,|_{B_S}.$$

(b) There exists a sequence (Ω_n) satisfying:

(i) Each Ω_n is a finite union of pairwise disjoint finitely connected C^1 Jordan domains.

(ii) $m_2(S\Delta\Omega_n)\longrightarrow 0$.

(iii) $\chi_{\Omega_n}\longrightarrow\chi_S$ pointwise a.e..

(iv) $\|d\mu - \frac{1}{2\pi i}\, dz \,|_{\partial\Omega_n}\|\longrightarrow 0$. ■

We assume the following standard fact [7, page 26]):

Lemma 105 (Cauchy-Green Formula) *Let $f \in C^1 = C^1(\mathbb{C})$ and $\Omega \subseteq \mathbb{C}$ be a finitely connected C^1 Jordan domain. Then*

$$\frac{1}{2\pi i}\int_{\partial\Omega}\frac{f}{z-z_0}\,dz - \frac{1}{\pi}\iint_{\Omega}\frac{\overline{\partial} f}{z-z_0}\,dA = \begin{cases} f(z_0) & \text{if } z_0 \in \Omega \\ 0 & \text{if } z_0 \in \mathbb{C}\setminus\overline{\Omega}, \end{cases}$$

where $\overline{\partial}$ denotes $\frac{\partial}{\partial \bar{z}} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$. ■

Khavinson extends the Cauchy-Green formula to a wider class of sets:

Lemma 106 *[8, Theorem 2.1] Let $f \in C^1_c$ and $K \subseteq \mathbb{C}$ be a compact set having finite perimeter. Then there exists $E \subseteq K$ satisfying: $m_2E = 0$ and*

$$\frac{1}{2\pi i}\int_{B_K}\frac{f}{z-z_0}\,dz - \frac{1}{\pi}\iint_{K}\frac{\overline{\partial} f}{z-z_0}\,dA = \begin{cases} f(z_0) & \text{if } z_0 \in K\setminus E, \\ 0 & \text{if } z_0 \in \mathbb{C}\setminus \mathrm{K}. \end{cases}$$

Proof. Let (Ω_n) be as in Theorem 104 (with S replaced by K).
Step 1: Set $F_1 = \left\{z_0 \in \mathbb{C}\setminus \mathrm{K} : \chi_{\overline{\Omega}_n}(z_0)\longrightarrow 0\right\}$ and $E_1 = (\mathbb{C}\setminus\mathrm{K})\setminus \mathrm{F}_1$. By Theorem 104, we see $m_2E_1 = 0$.

Let $z_0 \in F_1$. Set $\delta = \frac{1}{2}\mathrm{dist}(z_0, K)$ and $K^\delta = \bigcup_{z\in K} D(z;\delta)$. By Theorem 104 (b)(iv), $\overline{\Omega}_n \subseteq K^\delta$ whenever $n \geq n_1$, say. By Lemma 105, for $n \geq n_1$,

$$\frac{1}{2\pi i}\int_{\partial\Omega_n}\frac{f}{z-z_0}\,dz - \frac{1}{\pi}\iint_{\Omega_n}\frac{\overline{\partial} f}{z-z_0}\,dA = 0.$$

By Theorem 104, again for $n \geq n_1$,

$$\left| \int\limits_{B_K} \frac{f}{z - z_0} dz - \int\limits_{\partial\Omega_n} \frac{f}{z - z_0} dz \right| \leq \frac{\|f\|_\infty}{\delta} \| \, dz \, |_{B_K} - dz|_{\partial\Omega_n} \|.$$

and
$$\left| \iint\limits_K \frac{\overline{\partial} f}{z - z_0} dA - \iint\limits_{\Omega_n} \frac{\overline{\partial} f}{z - z_0} dA \right| \leq \frac{\|\overline{\partial} f\|_\infty}{\delta} m_2(K \Delta \Omega_n).$$

Hence,

$$\frac{1}{2\pi i} \int\limits_{B_K} \frac{f}{z - z_0} dz - \frac{1}{\pi} \iint\limits_K \frac{\overline{\partial} f}{z - z_0} dA = 0.$$

Set $g(z_0) = \frac{1}{2\pi i} \int\limits_{B_K} \frac{f}{z - z_0} dz - \frac{1}{\pi} \iint\limits_K \frac{\overline{\partial} f}{z - z_0} dA$. Note $g \in H(\mathbb{C} \setminus \mathrm{K})$ and $g \equiv 0$ in F_1. Since $m_2 E_1 = 0$, we note F_1 has a limit point in each component of $\mathbb{C} \setminus \mathrm{K}$. So $g \equiv 0$ in $\mathbb{C} \setminus \mathrm{K}$.

Step 2: Set $F = \{z_0 \in K : \chi_{\Omega_n}(z_0) \longrightarrow 1, \int\limits_{B_K} \frac{d|z|}{|z - z_0|} < \infty$, and $\frac{1}{2\pi i} \int\limits_{B_K} \frac{dz}{z - z_0} = 1 \}$ and $E = K \setminus F$. By Proposition 100, Theorem 104, and Lemma 105, we see $m_2 E = 0$.

Let $z_0 \in F$. Set $C = \sup \left\{ \left| \frac{f(z) - f(z_0)}{z - z_0} \right| : z \neq z_0 \right\}$. For n large enough,

$$f(z_0) = \frac{1}{2\pi i} \int\limits_{\partial\Omega_n} \frac{f}{z - z_0} dz - \frac{1}{\pi} \iint\limits_{\Omega_n} \frac{\overline{\partial} f}{z - z_0} dA.$$

By Theorem 104,

$$\left| \iint\limits_K \frac{\overline{\partial} f}{z - z_0} dA - \iint\limits_{\Omega_n} \frac{\overline{\partial} f}{z - z_0} dA \right| \leq \|\overline{\partial} f\|_\infty \int\int\limits_{K \Delta \Omega_n} \frac{dA}{|z - z_0|},$$

and

$$\left| \frac{1}{2\pi i} \int\limits_{B_K} \frac{f}{z - z_0} dz - \frac{1}{2\pi i} \int\limits_{\partial\Omega_n} \frac{f}{z - z_0} dz \right|$$
$$= \left| \frac{1}{2\pi i} \int\limits_{B_K} \frac{f - f(z_0)}{z - z_0} dz - \frac{1}{2\pi i} \int\limits_{\partial\Omega_n} \frac{f - f(z_0)}{z - z_0} dz \right|$$
$$\leq \frac{C}{2\pi} \| dz \, |_{B_K} - dz|_{\partial\Omega_n} \|,$$

we see

$$f(z_0) = \frac{1}{2\pi i} \int\limits_{B_K} \frac{f}{z - z_0}\, dz - \frac{1}{\pi} \iint\limits_{K} \frac{\overline{\partial} f}{z - z_0}\, dA.$$

■

Definition: Let $F \subseteq \mathbb{C}$ be closed. Define $\mathrm{Lip}_1(F) = \{ f : F \longrightarrow \mathbb{C} \mid$ there exists $C > 0$ satisfying: $|f(z)| \leq C$ and $|f(z_1) - f(z_2)| \leq C|z_1 - z_2| \quad (z, z_1, z_2, \in F) \}$.

Remark: [14] $\mathrm{Lip}_1(F)$ is a Banach space with norm

$$\|f\|_{\mathrm{Lip}_1(F)} = \max \left\{ \sup_{(z \in F)} |f(z)|, \sup_{\substack{(z_1, z_2 \in F) \\ (z_1 \neq z_2)}} \left| \frac{f(z_1) - f(z_2)}{z_1 - z_2} \right| \right\}.$$

■

Khavinson extends the Cauchy-Green formula to a wider class of functions.

Theorem 107 *[8, Theorem 2.1] Let $f \in Lip_1(\mathbb{C})$ and $K \subseteq \mathbb{C}$ be a compact f.p. set. Then there exists $E \subseteq K$ satisfying: $m_2 E = 0$ and*

$$\frac{1}{2\pi i} \int\limits_{B_K} \frac{f}{z - z_0}\, dz - \frac{1}{\pi} \iint\limits_{K} \frac{\overline{\partial} f}{z - z_0}\, dA = \begin{cases} f(z_0) & \text{if } z_0 \in K \setminus E, \\ 0 & \text{if } z_0 \in \mathbb{C} \setminus \mathrm{K}. \end{cases}$$

Remark: Here $\overline{\partial} f$ is the distributional partial derivative of f with respect to $\overline{z}$. Khavinson shows $\overline{\partial} f \in L^\infty$. See [8] for the proof. ■

Definition: Let $\Omega \subseteq \mathbb{C}$ be open and $1 \leq p \leq \infty$. Set $W^{1,p}(\Omega) = \{ f \in L^p(\Omega) : f_x, f_y \in L^p(\Omega) \}$, where f_x and f_y denote distributional partial derivatives. Such spaces are Sobolev spaces. See [1] for more on Sobolev spaces.

Remark: [1, Theorem 3.1] If we set

$$\|f\|_{W^{1,p}(\Omega)} = \begin{cases} \left(\|f\|^p_{L^p(\Omega)} + \|f_x\|^p_{L^p(\Omega)} + \|f_y\|^p_{L^p(\Omega)} \right)^{\frac{1}{p}} & \text{if } 1 \leq p < \infty, \\ \max \left\{ \|f\|_{L^\infty(\Omega)}, \quad \|f_x\|_{L^\infty(\Omega)}, \quad \|f_y\|_{L^\infty(\Omega)} \right\} & \text{if } p = \infty, \end{cases}$$

then $W^{1,p}(\Omega)$ is a Banach space. ■

In order to extend (Khavinson's) Theorem 107 to a wider class of functions, we will need the following two results. Lemma 109 is an example of a Sobolev embedding theorem.

Notation: For an open set $\Omega \subseteq \mathbb{C}$, we let $\mathcal{D}(\Omega)$ denote the (complex-valued) infinitely differentiable functions having compact support in Ω.

Lemma 108 *[1, Theorem 3.18] Let* $1 \leq p < \infty$. *Then* $\mathcal{D} = \mathcal{D}(\mathbb{C})$ *is dense in* $W^{1,p} = W^{1,p}(\mathbb{C})$. ■

Lemma 109 *[1, Theorem 5.4] Let* $2 < p < \infty$. *Then* $W^{1,p} \hookrightarrow C \bigcap L^{\infty}$.

Explanation: The conclusion of Lemma 109 means the following: There exists $A > 0$ with the property that if $f \in W^{1,p}$, then $f \in C = C(\mathbb{C})$ and

$\|f\|_{\infty} \equiv \sup_{z\in\mathbb{C}} |f(z)| \leq A\|f\|_{W^{1,p}}$. ■

We now extend Khavinson's result (Theorem 107) to $W^{1,p}$.

Theorem 110 *Let* $2 < p < \infty$. *Let* $f \in W^{1,p}$ *and let* K *have f.p.. Then there exists* $E \subseteq K$ *satisfying:* $m_2 E = 0$ *and*

$$\frac{1}{2\pi i}\int\limits_{B_K} \frac{f}{z - z_0}\,dz - \frac{1}{\pi}\iint\limits_{K} \frac{\overline{\partial} f}{z - z_0}\,dA = \begin{cases} f(z_0) & \text{if } z_0 \in K \setminus E, \\ 0 & \text{if } z_0 \in \mathbb{C} \setminus \mathrm{K}. \end{cases}$$

Proof. By Lemma 108, there exists a sequence (ϕ_n) in $\mathcal{D}$ with $\phi_n \longrightarrow f$ in $W^{1,p}$. By Lemma 109, we note $\|f - \phi_n\|_{\infty} \longrightarrow 0$.

Set

$$F = \left\{ z_0 \in K : \int\limits_{B_K} \frac{d|z|}{|z - z_0|} < \infty \quad \text{and for each } n, \right.$$

$$\left. \phi_n(z_0) = \frac{1}{2\pi i}\int\limits_{B_K} \frac{\phi_n}{z - z_0}\,dz - \frac{1}{\pi}\iint\limits_{K} \frac{\overline{\partial}\phi_n}{z - z_0}\,dA \right\}$$

and $E = K \setminus F$. By Proposition 100 and Lemma 106, we get $m_2 E = 0$.

Let $z_0 \in \mathbb{C} \setminus \mathrm{E}$. Then

$$\phi_n(z_0) \longrightarrow f(z_0),$$

$$\left| \int_{B_K} \frac{f - \phi_n}{z - z_0}\, dz \right| \leq \|f - \phi_n\|_{B_K} \int_{B_K} \frac{d|z|}{|z - z_0|},$$

$$\text{and} \quad \left| \iint_K \frac{\overline{\partial}(f - \phi_n)}{z - z_0}\, dA \right| \leq \|\overline{\partial}(f - \phi_n)\|_{L^p(K)} \left(\iint_K \frac{dA}{|z - z_0|^q} \right)^{\frac{1}{q}}.$$

By Lemma 106, we are done. ■

Remark: Theorem 110 holds also for $f \in W^{1,\infty}$: Choose $\psi \in \mathcal{D}$, with $\psi \equiv 1$ in a nhd of K. Then $\psi f \in W^{1,3}$ and $f = \psi f$ in Ω. ■

We now proceed toward showing : Theorem 110 *is* an extension of Theorem 107.

Definition [11, p. 157] Let $h \in \mathcal{D}$ satisfy: $h \geq 0$ and $\iint_{\mathbb{C}} h\, dA = 1$. For $n \in \mathbb{N}$, set $h_n(z) = n^2 h(nz)$. We call (h_n) an approximate identity on $\mathbb{C}$. ■

The following results are standard and will be cited for reference purposes only.

Proposition 111 *[1, Lemma 2.18 and Lemma 3.15] Let* $1 \leq p < \infty$ *and* $\Omega \subseteq \mathbb{C}$ *be open. Let* $K \subseteq \mathbb{C}$ *be compact, with* $K \subseteq \Omega$. *Let* $f : \Omega \longrightarrow \mathbb{C}$. *Extend the domain of* f *to all of* $\mathbb{C}$ *by setting* $f \equiv 0$ *in* $\mathbb{C} \setminus \mathrm{K}$.

(a) If $f \in L^1_{\mathrm{loc}}$, then each $h_n * f \in C^\infty$. If also supp (f) is bounded, then each $h_n * f \in \mathcal{D}$.

(b) If $f \in L^p(\Omega)$, then $h_n * f \longrightarrow f$ in $L^p(\Omega)$.

(c) If $f \in W^{1,p}(\Omega)$, then $h_n * f \longrightarrow f$ in $W^{1,p}(\text{int}K)$.

(d) If $f \in C(\Omega)$, then $h_n * f \rightrightarrows f$ in K. ■

Lemma 112 *Let $U \subseteq \mathbb{C}$ be open and $F \subseteq \mathbb{C}$ be compact, with $\overline{U} \subseteq intF$. Then $Lip_1(F) \hookrightarrow W^{1,\infty}(U)$.*

Proof. Let (h_n) be an approximate identity on $\mathbb{C}$. Let $f \in \text{Lip}_1(F)$. Set $f \equiv 0$ in $\mathbb{C} \setminus$ F and set $f_n = h_n * f$. By Proposition 111, we note $f_n \in \mathcal{D}$ and $f_n \rightrightarrows f$ in $\overline{U}$.

Let $\xi \in \mathbf{T}$. There exists $n_0 \in \mathbb{N}$ satisfying: If $a \in \left\{z \in \mathbb{C} : 0 < |z| < \frac{1}{n_0}\right\}$ and $n \geq n_0$, then $z_0 + a\xi - \frac{1}{n}z$ and $z_0 - \frac{1}{n}z$ belong to F for all $z_0 \in U$. Hence, for such z_0, a, and n,

$$\left|\frac{f_n(z_0 + a\xi) - f_n(z_0)}{a}\right| =$$
$$\frac{n^2}{|a|}\left|\iint h\left(n\left[z_0 + a\xi - z\right]\right)\, f(z)\, dA(z) - \iint h\left(n[z_0 - z]\right) f(z)\, dA(z)\right|$$
$$\leq \frac{1}{|a|}\iint h(w)\left|f(z_0 + a\xi - \frac{w}{n}) - f(z_0 - \frac{w}{n})\right|\, dA(w) \leq \|f\|_{\text{Lip}_1(F)}.$$

Therefore there exists a subsequence (f_{n_j}) and $g \in L^\infty(U)$ satisfying: $D_\xi f_{n_j} \longrightarrow g \;\; w-*$, where D_ξ denotes the appropriate directional differential operator. □

We show: $g = D_\xi f$ in $\mathcal{D}'(U)$.

Let $\phi \in \mathcal{D}(U)$.

$$\begin{aligned}(D_\xi f)\phi = -f(D_\xi \phi) &= -\iint f D_\xi \phi\, dA = -\lim \iint f_{n_j} D_\xi \phi\, dA \\ &= \lim \iint (D_\xi f_{n_j})\phi\, dA = \iint g\phi\, dA = g(\phi).\end{aligned}$$

□

Now $\|D_\xi f\|_{L^\infty(U)} = \sup\limits_{\substack{(\tau \in L^1(U))\\(\|\tau\|_1 = 1)}} \left|\iint_U \tau g\, dA\right| = \sup\limits_{\tau} \lim\limits_{j} \left|\iint \tau D_\xi f_{n_j}\, dA\right| \leq \|f\|_{\text{Lip}_1(F)}$.

In the line above, take $\xi = 1$ (first) and then take $\xi = i$: We see $f \in W^{1,\infty}(U)$ and

$$\|f\|_{W^{1,\infty}(U)} = \max\left\{\|f\|_{L^\infty(U)}, \|f_x\|_{L^\infty(U)}, \|f_y\|_{L^\infty(U)}\right\} \leq \|f\|_{\mathrm{Lip}_1(F)}.$$

■

Lemma 113 *[14, Chapter VI, Theorem 3] Let $F \subseteq \mathbb{C}$ be closed. Then there exists a continuous linear extension operator E :$Lip_1(F) \longrightarrow Lip_1(\mathbb{C})$.* ■

Remarks:

(i) Extension means: If $f \in \mathrm{Lip}_1(F)$, then $Ef \equiv f$ in F.

(ii) Let K be as in Theorem 107. Assume $K \subseteq R\mathbb{D}$. Set $F = \overline{R\mathbb{D}}$.

Let $f \in \mathrm{Lip}_1(\mathbb{C})$. For the conclusion of Theorem 107 (only the values of f in a neighborhood of K matter), we may restrict f and think of $f \in \mathrm{Lip}_1(F)$.

Using Lemma 113, we may, in the hypotheses of Theorem 107, replace $\mathrm{Lip}_1(\mathbb{C})$ by $\mathrm{Lip}_1(F)$. ■

Lemma 114 *Let $z_0 \in \mathbb{C}$ and $r > 0$. Set $\Omega = D(z_0; r)$. If $1 \leq p < \infty$, then $W^{1,\infty}(\Omega) \subsetneq W^{1,p}(\Omega)$.*

Proof. $W^{1,\infty}(\Omega) \subseteq W^{1,p}(\Omega)$ is clear.

Set $f = [z - (z_0 + r)]^\alpha$, where $1 - \frac{1}{p} < \alpha < 1$. In Ω, note $\partial f = \alpha[z - (z_0 + r)]^{\alpha - 1}$ and $\overline{\partial} f = 0$. $[\partial = \frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}).]$ So $f \in W^{1,p}(\Omega) \setminus W^{1,\infty}(\Omega)$. ■

Lemma 115 *[1, Theorem 4.26] With the notation of Lemma 114, there exists a continuous linear extension operator E : $W^{1,p}(\Omega) \longrightarrow W^{1,p}(\mathbb{C})$.*

Remark: Extension means: If $f \in W^{1,p}(\Omega)$, then $Ef = f$ a.e. in Ω. ∎

Notation: Let X be a locally compact Hausdorff space. Let $C_0(X)$ denote those continuous $f : X \longrightarrow \mathbb{C}$ which "vanish at infinity" (see [12, p. 72]) and set

$$C_c(X) = \{f \in C_0(X) : \text{supp f} \subseteq \text{X is compact}\}.$$

Recall: With the maximum norm on both the above spaces:

(i) [12, Theorem 6.19] $(C_0(X))^* = M(X)$.

(ii) [12, Theorem 3.17] $C_c(X)$ is dense in $C_0(X)$. ∎

Lemma 116 *Let $\mu \in M$ and $\Omega \subseteq \mathbb{C}$ be open. If $\hat{\mu} = 0$ a.e. in Ω, then $|\mu|(\Omega) = 0$.*

Proof. Recall: **(iii)** $\mathcal{D}(\Omega)$ is dense in $C_c(\Omega)$. [To see this, convolve with an approximate identity (as in Proposition 111).]

Let $f \in \mathcal{D}(\Omega)$. Assume $(\text{supp } \mu) \bigcup (\text{supp f}) \subseteq R\mathbb{D}$. For all $|z| < R$, by Lemma 105,

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int\limits_{R\mathbb{T}} \frac{f(\zeta)}{\zeta - z}\, d\zeta - \frac{1}{\pi} \iint\limits_{R\mathbb{D}} \frac{(\overline{\partial} f)(\zeta)}{\zeta - z}\, dA(\zeta) \\ &= 0 - \frac{1}{\pi} \iint\limits_{R\mathbb{D}} \frac{(\overline{\partial} f)(\zeta)}{\zeta - z}\, dA(\zeta). \end{aligned}$$

Hence,

$$\begin{aligned} \int f(z)\, d\mu(z) &= -\frac{1}{\pi} \iint\limits_{R\mathbb{D}} (\overline{\partial} f)(\zeta) \left[\int \frac{d\mu(z)}{\zeta - z} \right] dA(\zeta) \\ &= \iint\limits_{R\mathbb{D}} (\overline{\partial} f)(\zeta) \hat{\mu}(\zeta)\, dA(\zeta). \end{aligned}$$

Now $\overline{\partial} f \equiv 0$ in $\mathbb{C} \setminus \Omega$ and $\hat{\mu} = 0$ a.e. in Ω. That is, $\int f(z)\, d\mu(z) = 0$.

By Hahn-Banach, (i), (ii), and (iii), $|\mu|(\Omega) = 0$. ■

Notation: Let $K \subseteq \mathbb{C}$ be compact. Denote by Rat$K$ those rational functions all of whose poles lie off K. Let $R(K)$ denote the closure of Rat$K$ in $C(K)$.

Lemma 117 *[7, Hartogs-Rosenthal, Chapter II, Corollary 8.4] Let $K \subseteq \mathbb{C}$ be compact with $m_2 K = 0$. Then $R(K) = C(K)$.*

Proof. Let $\mu \in [R(K)]^{\perp} \subseteq M(K)$. Then $\hat{\mu} \equiv 0$ in $\mathbb{C} \setminus \mathrm{K}$. That is, $\hat{\mu} = 0$ a.e. in $\mathbb{C}$. By Lemma 116, we see $\mu = 0$. Now use Hahn-Banach and the Riesz representation theorem. ■

Section 118 Let $2 < p < \infty$ and $K \subseteq \mathbb{C}$ be a compact set having finite perimeter. Assume $K \subseteq R\mathbb{D} \equiv \mathrm{U}$ and set $F = \overline{(2R)\mathbb{D}}$.

By Lemma 115 and Remark (ii) (following Lemma 113), we may replace $W^{1,p}(\mathbb{C})$ by $W^{1,p}(U)$ in the hypotheses of Theorem 110.

Let $f \in \mathrm{Lip}_1(F)$. By Lemma 112, we note $f \in W^{1,\infty}(U)$ and hence, $f \in W^{1,p}(U)$. Hence, we see Theorem 110 is an extension of Theorem 107.

(a) Suppose int$K \neq \phi$, say $D(z_0; r) \subseteq K$. Let f be as in the proof of Lemma 114. By Lemma 115, we may extend f to be in $W^{1,p}(\mathbb{C})$. But by Lemma 112 and Lemma 114, f cannot even be extended to $\mathrm{Lip}_1(K)$.

That is, if int$K \neq \phi$, then Theorem 110 applies to a *strictly* larger class of functions than does Theorem 107.

(b) The underlying set K in Theorem 102 is a standard example of an infinitely connected compact set having finite perimeter. [Infinitely connected means: K is connected but $\mathbb{C} \setminus \mathrm{K}$ has infinitely many components.]

If $\mathrm{int}K \neq \phi$, then **(a)** applies. So assume such a set K has empty interior. Let μ be as in Theorem 102. Note $\hat{\mu} \equiv 0$ in $\mathbb{C} \setminus \mathrm{K}$. That is, $\mu \perp R(K)$. Since μ is not the zero measure, $R(K) \neq C(K)$. By Lemma 117, we see $m_2 K > 0$. Hence there exist $z_0 \in \mathbb{C}$, $\xi \in \mathbf{T}$, and $r > 0$ satisfying : if $1 - \frac{1}{p} < \alpha < 1$, $f = [z - (z_0 + r\xi)]^\alpha$, and $D = D(z_0; r)$, then $f \in W^{1,p}(D) \setminus W^{1,\infty}(D)$. [If not, we would have $\lim_{\delta \searrow 0} \frac{m_2[K \bigcap D(z_0;\delta)]}{\pi \delta^2} = 0$ for all $z_0 \in \mathbb{C}$ and hence $m_2 K = 0$.]

We conclude: for such sets K, Theorem 110 applies to a strictly larger class of functions than does Theorem 107. ■

We recall the moral of Theorem 104: finite perimeter sets are nice. We show now, however, that the boundary of a finite perimeter set can have positive area.

Example 119 Suppose we choose (γ_n) in Theorem 102 so that $\mathrm{int}K = \phi$. With μ as in the proof of Theorem 102, we see $\mu \perp R(K)$. Hence $R(K) \neq C(K)$. By Hartogs-Rosenthal, $m_2 K > 0$. Since $K = \partial K$, we see $m_2(\partial K) > 0$. ■

We now extend Theorem 110.

Theorem 120 *Let $f \in C_c = C_c(\mathbb{C})$, with $\overline{\partial} f \in L^1$. Let $K \subseteq \mathbb{C}$ be a compact set having finite perimeter. Then there exists $E \subseteq K$ satisfying : $m_2 E = 0$ and*

$$\frac{1}{2\pi i} \int_{B_K} \frac{f}{z - z_0}\, dz - \frac{1}{\pi} \iint_K \frac{\overline{\partial} f}{z - z_0}\, dA = \begin{cases} f(z_0) & \text{if } z_o \in K \setminus E, \\ 0 & \text{if } z_0 \in \mathbb{C} \setminus \mathrm{K}. \end{cases}$$

Proof. Let (h_n) be an approximate identity on $\mathbb{C}$. Assume $K \bigcup \mathrm{supp}\ \mathrm{f} \subseteq R\mathbb{D}$. Set $\phi_n = h_n * f$, $\quad g_n = \chi_K \overline{\partial} \phi_n$, $\quad g = \chi_K \overline{\partial} f$, $\quad \beta(z) = -\frac{1}{z} \chi_{(3R)\mathbb{D}}(z)$, and $\gamma(z_0) = \frac{1}{2\pi i} \int_{B_K} \frac{f}{z - z_0}\, dz - \frac{1}{\pi} \iint_K \frac{\overline{\partial} f}{z - z_0}\, dA$.

By Proposition 111, we see $\phi_n \rightrightarrows f$ in $\mathbb{C}$. Since $\overline{\partial}\phi_n = h_n * (\overline{\partial} f)$, we have $\overline{\partial}\phi_n \longrightarrow \overline{\partial} f$ in $L^1 = L^1(\mathbb{C})$, also by Proposition 111. By Theorem 4.30 [1], $\|\beta * (g - g_n)\|_1 \leq \|\beta\|_1 \|g - g_n\|_1 \longrightarrow 0$, where $\| \ \|_1$ denotes the norm in L^1. WLOG, assume $\beta * g_n \longrightarrow \beta * g$ a.e. in $\mathbb{C}$. Set $E_1 = \{z_0 \in \mathbb{C} : (\beta * g_n)(z_0) \not\longrightarrow (\beta * g)(z_0)\}$. For $z_0 \in (R\mathbb{D}) \setminus E_1$,

$$\iint_K \frac{\overline{\partial}\phi_n}{z - z_0}\, dA = (\beta * g_n)(z_0) \longrightarrow (\beta * g)(z_0) = \iint_K \frac{\overline{\partial} f}{z - z_0}\, dA.$$

$$\text{Set} \quad E_2 = \left\{ z_0 \in K : \iint_{B_K} \frac{d|z|}{|z - z_0|} = \infty \right\} \bigcup \left[\bigcup_n \left\{ z_0 \in K : \phi_n(z_0) \neq \frac{1}{2\pi i} \int_{B_K} \frac{\phi_n}{z - z_0}\, dz - \frac{1}{\pi} \iint_K \frac{\overline{\partial}\phi_n}{z - z_0}\, dA \right\} \right]$$

and $E = (E_1 \cap K) \cup E_2$. By Proposition 100 and Lemma 105, we note $m_2 E = 0$. For $z_0 \in (R\mathbb{D}) \setminus E$, we have

$$\left| \int_{B_K} \frac{f - \phi_n}{z - z_0}\, dz \right| \leq \|f - \phi_n\|_{B_K} \int_{B_K} \frac{d|z|}{|z - z_0|} \longrightarrow 0.$$

So far, we have

$$\gamma(z_0) = \begin{cases} f(z_0) & \text{if } z_0 \in K \setminus E \\ 0 & \text{if } z_0 \in (R\mathbb{D}) \setminus (K \cup E_1). \end{cases}$$

But $\gamma \in H(\mathbb{C} \setminus K)$. Since $(R\mathbb{D}) \setminus (K \cup E_1)$ has a limit point in each component of $\mathbb{C} \setminus K$, we see $\gamma \equiv 0$ in $\mathbb{C} \setminus K$. ■

Remark: By Lemma 109, we see Theorem 120 is an extension of Theorem 110.

Notation: We shall abbreviate the conclusion of Theorem 120 by writing (f, K, E). By this, we mean: f has such an integral representation with respect to the f.p. set K, where E is the exceptional set (having measure zero). ■

Corollary 121 *Let $f \in C_c$, with $\overline{\partial} f \in L^1$. Then for a.e. $z_0 \in \mathbb{C}$,*

$$f(z_0) = -\frac{1}{\pi} \iint_{\mathbb{C}} \frac{\overline{\partial} f}{z - z_0} \, dA.$$

If $2 < p \leq \infty$ and $\overline{\partial} f \in L^p$, then the above holds for all $z_0 \in \mathbb{C}$.

Proof. Assume supp f $\subseteq R\mathbb{D}$ and take $K = \overline{R\mathbb{D}}$ in Theorem 120. If $\overline{\partial} f \in L^p$, then $\left(\overline{\partial} f\right)^{\wedge} \in C$. [This will be proved later (Lemma 132).] ■

Section 122 Let $S \subseteq \mathbb{C}$ be measurable. Assume there exists a sequence (K_n) of compact sets having f.p. satisfying: $\chi_{K_n} \longrightarrow \chi_S$ a.e. in $\mathbb{C}$. We shall call such a set S an *f.p.a. set (finite perimeter approximable)* and write "$K_n \longrightarrow S$ f.p.a.." If, in addition, each $K_n \subseteq S$, then we call S an *inner f.p.a. set.* □

Let $K_n \longrightarrow S$ f.p.a.. Let $f \in C = C(\mathbb{C})$, with $\overline{\partial} f \in L^1$. By Theorem 120, for each n, we may write (f, K_n, E_n). Set

$E = \{z_0 \in \mathbb{C} : \chi_{K_n}(z_0) \not\longrightarrow \chi_S(z_0)\} \cup [\cup E_n] \cup \left\{ z_0 \in \mathbb{C} : \iint_{\mathbb{C}} \left| \frac{\overline{\partial} f}{z - z_0} \right| dA = \infty \right\}.$

By Theorem 120, we note $m_2 E = 0$.

Let $z_0 \in S \setminus E$. Taking limits gives

$$f(z_0) = \frac{1}{2\pi i} \lim \int_{B_{K_n}} \frac{f}{z - z_0} \, dz - \frac{1}{\pi} \iint_S \frac{\overline{\partial} f}{z - z_0} \, dA.$$

We shall write $\int_{\partial S} \frac{f}{z - z_0} \, dz$ to denote $\lim \int_{B_{K_n}} \frac{f}{z - z_0} \, dz$. This is merely notation; we do not suggest that $dz \,|_S$ is a measure.

For $z_0 \in (\mathbb{C} \setminus S) \setminus E$, we see (similarly)

$$0 = \frac{1}{2\pi i} \int_{\partial S} \frac{f}{z - z_0} \, dz - \frac{1}{\pi} \iint_S \frac{\overline{\partial} f}{z - z_0} \, dA.$$

Note that the choice of the sequence (K_n) determines the exceptional set E but does not effect the value of the expression $\lim \int_{B_{K_n}} \frac{f}{z - z_0} \, dz$ for $z_0 \in \mathbb{C} \setminus E$. It is in this sense that we consider the definition of $\int\limits_{\partial S} \frac{f}{z-z_0}\, dz$ to be independent of the choice of the sequence (K_n).

We summarize:

Theorem 123 *Let $f \in C$, with $\overline{\partial} f \in L^1$. Let $S \subseteq \mathbb{C}$ be an f.p.a. set. Then there exists $E \subseteq \mathbb{C}$ satisfying: $m_2 E = 0$ and*

$$\frac{1}{2\pi i}\int\limits_{\partial S} \frac{f}{z-z_0}\, dz - \frac{1}{\pi}\iint\limits_{S} \frac{\overline{\partial} f}{z-z_0}\, dA = \begin{cases} f(z_0) & \text{if } z_0 \in S \setminus E \\ 0 & \text{if } z_0 \in (\mathbb{C} \setminus \mathrm{S}) \setminus \mathrm{E}. \end{cases}$$

■

Corollary 124 *If, in additon, S is inner, then $E \subseteq S$.* ■

Example 125 For each n, set $D_n = D(z_n; r_n)$. Assume $\sum r_n < \infty$ and that $(\overline{D}_n)$ is a pairwise disjoint sequence in $\mathbb{D}$ accumulating on a compact set $J \subseteq \mathbb{D}$, but accumulating nowhere else. Set $\Omega = \mathbb{D} \setminus \mathrm{J}$. Note $\Omega \subseteq \mathbb{C}$ is open and $\partial\Omega = \mathbf{T} \bigcup J$.

For each n, set $K_n = \left[\left(1+\frac{1}{n}\right)\overline{\mathbb{D}}\right] \setminus \left(\bigcup\limits_{j=n}^{\infty} D_j\right)$. Note $K_n \longrightarrow \Omega$ f.p.a..

Let $f \in C_c$, with $\overline{\partial} f \in L^1$. Fix $z_0 \in \Omega$. For some $\delta > 0$ and n_0, $|z - z_0| > \delta$ whenever $z \in \overline{D}_n$ and $n \geq n_0$. Hence for $n \geq n_0$,

$$\begin{aligned} \left|\frac{1}{2\pi i}\left(\int\limits_{B_{K_n}} \frac{f}{z-z_0}\, dz - \int\limits_{(1+\frac{1}{n})\mathbf{T}} \frac{f}{z-z_0}\, dz\right)\right| &= \left|\frac{1}{2\pi i}\sum_{j=n}^{\infty}\int\limits_{\partial D_j} \frac{f}{z-z_0}\, dz\right| \\ &\leq \frac{\|f\|_{\overline{\mathbb{D}}}}{\delta}\sum_{j=n}^{\infty} r_j \longrightarrow 0. \end{aligned}$$

By Theorem 123,

$$f(z_0) = \frac{1}{2\pi i}\int\limits_{\mathbf{T}} \frac{f}{z-z_0}\, dz - \frac{1}{\pi}\iint\limits_{\Omega} \frac{\overline{\partial} f}{z-z_0}\, dA.$$

■

We address the following problem: Which sets are f.p.a.?

Notation: [12, Chapter 10] Let $\gamma_1, \cdots, \gamma_n$ be closed, rectifiable Jordan paths. Let Γ denote their formal sum: If $f \in C$, then $\int_{\Gamma} f\, dz = \sum_{j=1}^{N} \int_{\gamma_j} f\, dz$. We call Γ a cycle.

Assume $\frac{1}{2\pi i} \int_{\Gamma} \frac{dz}{z - z_0} \in \{0, 1\}$ for all $z_0 \in \mathbb{C} \setminus \Gamma$. We shall set $\text{int}\Gamma = \{ z_0 \in \mathbb{C} \setminus \Gamma : \frac{1}{2\pi i} \int_{\Gamma} \frac{dz}{z - z_0} = 1 \}$ and $\text{ext}\Gamma = \left\{ z_0 \in \mathbb{C} \setminus \Gamma : \frac{1}{2\pi i} \int_{\Gamma} \frac{dz}{z - z_0} = 0 \right\}$. ■

Proposition 126 *Every open set $\Omega \subseteq \mathbb{C}$ is inner f.p.a..*

Proof. We use Theorems 13.3 and 13.5 of [12]: Choose compact sets (L_n) with $L_n \subseteq \text{int} L_{n+1}$ and $\Omega = \bigcup L_n$. For each n, choose a cycle Γ_n with $L_n \subseteq \text{int}\Gamma_n$ and $(\mathbb{C} \setminus \text{int} L_{n+1}) \subseteq \text{ext}\Gamma_n$. Set $K_n = \overline{\text{int}\Gamma_n}$. Then K_n has f.p. $(B_{K_n} = \Gamma_n)$ and $K_n \longrightarrow \Omega$ inner f.p.a.. ■

Remarks: Note, in fact, $\chi_{\text{int} K_n} \longrightarrow \chi_\Omega$ everywhere in $\mathbb{C}$.

By Proposition 126, if $S \subseteq \mathbb{C}$ is measurable and if $m_2(\partial S) = 0$, then S is inner f.p.a.. ■

Proposition 127 *Every bounded G_δ- set G is f.p.a..*

Proof. WLOG, assume $G = \bigcap \Omega_n$, where each $\Omega_n \subseteq \mathbb{C}$ is open and $m_2(\Omega_n \setminus G) < \frac{1}{n}$. We will use Exercise 18, Chapter 3 [12]: By Proposition 126, for each n, there exists a sequence (L_{n_j}) of compact f.p. sets with $\chi_{L_{n_j}} \longrightarrow \chi_{\Omega_n}$ everywhere, and hence converging in measure. So there exists an f.p. set K_n satisfying: $K_n \subseteq \Omega_n$ and $m_2(\Omega_n \setminus K_n) < \frac{1}{n}$. Since (χ_{K_n}) converges to χ_G in measure, some subsequence converges a.e.. ■

Proposition 128 *Every bounded measurable set S is f.p.a..*

Proof. Choose a G_δ- set G with $S \subseteq G$ and $m_2(G \setminus S) = 0$ [12, Theorem 2.20]. Now use Proposition 127. ■

Proposition 129 *Let $S \subseteq \mathbb{C}$ be bounded and measurable and let $f \in C_c$, with $\overline{\partial} f \in L^1$. Then $\iint_S \overline{\partial} f \, dA = \frac{1}{2i} \int_{\partial S} f \, dz$.*

Remark: Again we warn: the boundary integral has meaning only as a limit.

Proof. Assume $S \subseteq R\mathbb{D}$. With G as in Proposition 127 and Proposition 128, we may choose (in Proposition 127) each $\Omega_n \subseteq (R+1)\mathbb{D}$. Choose z_0, with $|z_0| > R+1$. Set $g = (z - z_0)f$. Then $\overline{\partial} g = (z - z_0)\overline{\partial} f$ in $\mathcal{D}' = \mathcal{D}'(\mathbb{C})$. By Section 122 ($z_0 \notin E$) and Proposition 128,

$$\iint_S \overline{\partial} f \, dA = \iint_S \frac{\overline{\partial} g}{z - z_0} \, dA = \frac{1}{2i} \int_{\partial S} \frac{g}{z - z_0} \, dz = \frac{1}{2i} \int_{\partial S} f \, dz.$$

■

We turn now to rational approximation.

Lemma 130 *[6] Let $2 < p \leq \infty$ and $\mu \in M$. Let $f \in C_c$, with $\overline{\partial} f \in L^p$. Then $\int f \, d\mu = \frac{1}{\pi} \iint_{\mathbb{C}} \hat{\mu} \overline{\partial} f \, dA$.*

Proof. By Proposition 100, we may use Fubini; by Corollary 121,

$$\begin{aligned}
\int f(z) \, d\mu(z) &= -\frac{1}{\pi} \int \left[\iint_{\mathbb{C}} \frac{(\overline{\partial} f)(\zeta)}{\zeta - z} \, dA(\zeta) \right] d\mu(z) \\
&= -\frac{1}{\pi} \iint_{\mathbb{C}} \left[\int \frac{d\mu(z)}{\zeta - z} \right] (\overline{\partial} f)(\zeta) \, dA(\zeta) = \frac{1}{\pi} \iint_{\mathbb{C}} \hat{\mu}(\zeta)(\overline{\partial} f)(\zeta) \, dA(\zeta).
\end{aligned}$$

■

Theorem 131 *Let $2 < p \leq \infty$ and K be compact. Let $f \in C_c$.*

(a) *If* $\overline{\partial} f \in L^p$ *and if* $\overline{\partial} f = 0$ *a.e. in* K*, then* $f \in R(K)$.

(b) *If* $\overline{\partial} f \in L^1$*, if* $f \in R(K)$*, and if* K *is inner f.p.a., then* $\overline{\partial} f = 0$ *a.e. in* K.

Remark: Khavinson [8, Theorem 3.1] proves the above for $f \in \mathrm{Lip}_1(\mathbb{C})$ and (in (b)) K f.p..

Proof.

(a) Let $\mu \perp R(K)$. Since $\hat{\mu} \equiv 0$ in $\mathbb{C} \setminus$ K, Lemma 130 gives:

$$\int f\, d\mu = \frac{1}{\pi} \iint_{\mathbb{C}} \hat{\mu} \overline{\partial} f\, dA = 0.$$

(b) Let (r_j) be a sequence in Rat$K$, with $r_j \rightrightarrows f$ in K. Write (f, K_n, E_n) and (r_j, K_n, E_{n_j}), where $K_n \longrightarrow K$ inner f.p.a.. Set $E = \{z_0 \in K : \chi_{K_n}(z_0) \not\longrightarrow \chi_K(z_0)\} \cup [\cup E_n] \cup \left[\cup E_{n_j}\right] \cup \left\{ z_0 \in K : \iint_K \left|\frac{\overline{\partial} f}{z - z_0}\right| dA = \infty \right\}$.
By Theorem 120, we see $m_2 E = 0$.

Let $z_0 \in K \setminus E$. With n fixed and $z_0 \in K_n$,

$$0 = r_j(z_0) - \frac{1}{2\pi i} \int_{B_{K_n}} \frac{r_j}{z - z_0}\, dz \longrightarrow f(z_0) - \frac{1}{2\pi i} \int_{B_{K_n}} \frac{f}{z - z_0}\, dz.$$

By Theorem 120, we see $\iint_{K_n} \frac{\overline{\partial} f}{z - z_0}\, dA = 0$. Taking limits gives: $\iint_K \frac{\overline{\partial} f}{z - z_0}\, dA = 0$.

Let $z_0 \in \mathbb{C} \setminus$ K. With n again fixed,

$$0 = \frac{1}{2\pi i} \int_{B_{K_n}} \frac{r_j}{z - z_0}\, dz \longrightarrow \frac{1}{2\pi i} \int_{B_{K_n}} \frac{f}{z - z_0}\, dz = \frac{1}{\pi} \iint_{K_n} \frac{\overline{\partial} f}{z - z_0}\, dA$$

and hence $\iint_K \frac{\overline{\partial} f}{z - z_0}\, dA = 0$.

We just showed: $(\chi_K \overline{\partial} f)^\wedge = 0$ a.e. in $\mathbb{C}$. By Lemma 116, we get that $\overline{\partial} f = 0$ a.e. in K. ∎

Remark: The above proof follows the reasoning of [8, Theorem 3.1]. **(b)** can be proved more easily: By Theorem 3.1, we see that $\overline{\partial} f = 0$ a.e. in each K_n. The above proof is self-contained, however, and will be referenced later. ∎

Lemma 132 *If $2 < p < \infty$ and $f \in L^p_c$, then $\hat{f} \in C$.*

Proof. Let $z_n \longrightarrow z_0$ in $\mathbb{C}$. Assume $\{z_n : n \in \mathbb{N}\} \bigcup \text{supp f} \subseteq R\mathbb{D}$. For $w \in \mathbb{C}$, set $(\tau_w f)(z) = f(z-w)$. Using [12, Theorem 9.5] (which applies to $\mathbb{R}$, but whose proof works in $\mathbb{R}^N$, as well),

$$\begin{aligned} \left|\hat{f}(z_0) - \hat{f}(z_n)\right| &= \left| \int\int_{(2R)\mathbb{D}} \frac{f(z+z_0) - f(z+z_n)}{z} \, dA(z) \right| \\ &\leq \|\tau_{-z_0} f - \tau_{-z_n} f\|_p \left[\frac{2\pi (2R)^{2-q}}{2-q}\right]^{\frac{1}{q}} \longrightarrow 0. \end{aligned}$$

∎

Remark: If $f \in L^\infty_c$, then $\hat{f} \in C$. ∎

We will not need the following result at this time. This seems, however, the appropriate spot to set it down.

Theorem 133 *Let $\Omega \subseteq \mathbb{C}$ be open and bounded, $2 < p \leq \infty$, and $f \in (L^1 \bigcap L^p_{\text{loc}})(\Omega)$. Then $\hat{f} \in C(S^2 \setminus \partial\Omega)$.*

Proof. Fix $z_0 \in \Omega$. Choose $r > 0$, with $\overline{D(z_0; r)} \subseteq \Omega$. Set $\mu = \chi_{D(z_0;r)} f m_2$ and $\nu = \chi_{\Omega \setminus D(z_0;r)} f m_2$. Note $\hat{f} = \hat{\mu} + \hat{\nu}$ and $\hat{\nu} \in H(D(z_0; r))$. By Lemma 132, we note $\hat{\mu} \in C$. Hence $\hat{f} \in C(D(z_0; r))$. ∎

Lemma 134 *[6, Theorem 2.7] Let $\mu \in M$. Then $\overline{\partial}\hat{\mu} = -\pi\mu$ in $\mathcal{D}' = \mathcal{D}'(\mathbb{C})$.*

Proof. Let $\phi \in \mathcal{D} = \mathcal{D}(\mathbb{C})$. By Lemma 130,

$$(\overline{\partial}\hat{\mu})\phi = -\hat{\mu}(\overline{\partial}\phi) = -\iint_{\mathbb{C}} \hat{\mu}\overline{\partial}\phi \, dA = -\pi \int \phi \, d\mu = -\pi\mu(\phi).$$

■

Section 135 Let $2 < p \leq \infty$ and K be inner f.p.a.. Let $g \in L^p(K)$. [We set $g \equiv 0$ in $\mathbb{C} \setminus$ K.] Set $f = \hat{g}$. By Lemma 132, we note $f \in C$. By Lemma 134, we see $\overline{\partial} f = -\pi g \in L^p$. By Theorem 131,

$$f \in R(K) \textit{ iff } g = 0 \textit{ a.e. in } K.$$

□

Assume now $g \equiv 0$ in int$K$. Then $f \in A(K) \equiv C(K) \bigcap H(\text{int}K)$. So if $m_2(\partial K) > 0$, then $R(K) \subsetneq A(K)$. ■

Remark: Such functions f play an important role in rational approximation [6].

■

Proposition 136 *Let $K \subseteq \mathbb{C}$ be compact, with $m_2(\partial K) > 0$. Write $\mathbb{C} \setminus \mathrm{K} = \bigcup \mathcal{O}_n$, where each $\mathcal{O}_n$ is a component of $\mathbb{C} \setminus$ K. Assume $\partial K \subseteq \bigcup \partial \mathcal{O}_n$.*

Then K is not inner f.p.a..

Proof. By [7, Chapter VIII, Corollary 8.4], $R(K) = A(K)$. Section 135 now shows that K is not inner f.p.a.. ■

Definition For $2 < p \leq \infty$ and a bounded (measurable) set of $S \subseteq \mathbb{C}$, let $R(S;p)$ denote the closure in $C(\overline{S})$ of $\{\hat{g}\,|_{\overline{S}} : g \in L^p_c \text{ and } g = 0 \text{ a.e. in } S\}$.

Remark: By Lemma 132, such $\hat{g} \in C$. ■

Lemma 137 *Let $2 < p \leq \infty$. Let S be bounded and $\mu \in M(\overline{S})$. Then*

$$\mu \perp R(S;p) \quad \textit{iff} \quad \hat{\mu} = 0 \text{ a.e. in } \mathbb{C} \setminus S.$$

Proof.

" $\Longrightarrow$ " Fix a bounded measurable set $T \subseteq \mathbb{C}$, with $S \bigcap T = \phi$. Note

$$0 = \int_{\overline{S}} \hat{\chi_T}\, d\mu = \int_{\overline{S}} \left(\iint_T \frac{dA(\zeta)}{\zeta - z} \right) d\mu(z) = - \iint_T \hat{\mu}(\zeta)\, dA(\zeta).$$

By Proposition 100 and [12, Theorem 1.39], $\hat{\mu} = 0$ a.e. in $\mathbb{C} \setminus S$.

" $\Longleftarrow$" Let $g \in L^p_c$, with $g = 0$ a.e. in S. Set $\hat{g}_n = h_n * \hat{g}$, where (h_n) is an approximate identity on $\mathbb{C}$. By Lemma 134, we see $\overline{\partial}\hat{g}_n = h_n * (\overline{\partial}\hat{g}) = -\pi(h_n * g)$. By Proposition 111, we note $\hat{g}_n \rightrightarrows \hat{g}$ in $\overline{S}$ and $h_n * g \longrightarrow g = 0$ in $L^p(S)$. By Proposition 100, we have $\hat{\mu} \in L^q(S)$. Hence, by Proposition 129 and Lemma 130,

$$\int \hat{g}\, d\mu = \lim \int \hat{g}_n\, d\mu = \frac{1}{\pi} \lim \iint_{\mathbb{C}} \hat{\mu} \overline{\partial} \hat{g}_n\, dA = - \lim \iint_S \hat{\mu}(h_n * g)\, dA = 0.$$

■

Theorem 138 *Let $2 < p < \infty$ and S be bounded. Then $R(S;p) = R(S;\infty)$.*

Proof. Clearly $R(S;p) \supseteq R(S;\infty)$. Let $\mu \in M(\overline{S})$, with $\mu \perp R(S;\infty)$.

Let $g \in L^p_c$, with $g = 0$ a.e. in S. By Lemma 137, we note $g\hat{\mu} = 0$ a.e. in $\mathbb{C}$. By Proposition 100, we may use Fubini:

$$\int \hat{g}\, d\mu = \int \left(\iint_{\mathbb{C}} \frac{g(\zeta)}{\zeta - z}\, dA(\zeta) \right) d\mu(z) = - \iint_{\mathbb{C}} g(\zeta)\hat{\mu}(\zeta)\, dA(\zeta) = 0.$$

■

Proposition 139 *[6, Theorem 3.3] Let $K \subseteq \mathbb{C}$ be compact.*

Set $A = \{ \hat{g}\,|_K : g \in L^\infty_c$ and $g = 0$ a.e. in $K \}$ and let $\overline{A}$ denote its closure in $C(K)$. Then $\overline{A} = R(K)$.

Proof.

"$\subseteq$" Let $g \in L^\infty_c$, with $g = 0$ a.e. in K. Let $\mu \in M(K)$, with $\mu \perp R(K)$. Since $\hat{\mu} \equiv 0$ in $\mathbb{C} \setminus$ K, we see $g\hat{\mu} = 0$ a.e. in $\mathbb{C}$. As in the end of the proof of Theorem 138, we get $\int \hat{g}\, d\mu = -\iint_{\mathbb{C}} g\hat{\mu}\, dA = 0$. By Hahn-Banach, $\hat{g} \in R(K)$.

"$\supseteq$" Let $r \in \text{Rat}K$. WLOG, assume $r \in \mathcal{D}$. [That is, extend r to be in $\mathcal{D}$ while retaining the originial values of r in a neighborhood of K.] For $z_0 \in \mathbb{C}$, Corollary 121 gives

$$r(z_0) = -\frac{1}{\pi} \iint_{\mathbb{C}} \frac{\overline{\partial} r}{z - z_0}\, dA = -\frac{1}{\pi} (\overline{\partial} r)^\wedge(z_0).$$

Now $\overline{\partial} r \in \mathcal{D}$, $\overline{\partial} r \equiv 0$ in K, and $r \equiv -\frac{1}{\pi}(\overline{\partial} r)^\wedge$ in $\mathbb{C}$. Hence $r \in A$. ∎

Notation: For $2 < p \leq \infty$ and bounded S, we write $R(S)$ to denote $R(S; p)$.

Summary:

(a) By Theorem 138, we note $R(S)$ is well–defined.

(b) If S is compact, then the above definition agrees with the usual one (Proposition 139).

(c) $R(\overline{S}) \subseteq R(S)$ is always true; if $m_2(\overline{S} \setminus S) = 0$, then $R(\overline{S}) = R(S)$. ∎

We prove the analogue of Theorem 131:

Theorem 140 *Let $2 < p \leq \infty$ and S be bounded. Let $f \in C_c$.*

(a) *If* $\overline{\partial} f \in L^p$ *and* $\overline{\partial} f = 0$ *a.e. in* S*, then* $f \in R(S)$.

(b) *If* $\overline{\partial} f \in L^1$*, if* $f \in R(S)$*, and if* S *is inner f.p.a., then* $\overline{\partial} f = 0$ *a.e. in* S.

Proof.

(a) Let $\mu \in M(\overline{S})$, with $\mu \perp R(S)$. By Corollary 121, we see $f \equiv -\frac{1}{\pi}(\overline{\partial} f)^\wedge$. Since $(\overline{\partial} f)^\wedge \in R(S)$, we see $\int f \, d\mu = -\frac{1}{\pi}\int (\overline{\partial} f)^\wedge \, d\mu = 0$.

(b) Let (g_n) be a sequence in L_c^∞, with each $g_n = 0$ a.e. in S and $\hat{g}_n \rightrightarrows f$ in $\overline{S}$.

Fix n. By Lemma 134, we note $\overline{\partial} \hat{g}_n = -\pi g_n \in L_c^\infty$. Note $\overline{\partial} \hat{g}_n = 0$ a.e. in S. Using Theorem 120, we see that if K has f.p., then for a.e. $z_0 \in \mathbb{C}$,

$$(\chi_K \hat{g}_n)(z_0) = \frac{1}{2\pi i} \int_{B_K} \frac{\hat{g}_n}{z - z_0} \, dz$$

Now mimic the proof of Theorem 131. ■

Remark: Note Theorem 131 is contained in Theorem 140: Take S to be compact. That would save space, but much of the proof of Theorem 131 would need to be included above. ■

We now generalize Cauchy's Theorem to our new settings. Note that f need only be (defined and) continuous on $\overline{S}$.

Theorem 141 *Let* S *be bounded and innner f.p.a.. Let* $f \in R(S)$.
Then $\int_{\partial S} f \, dz = 0$.

Proof. Let $K_n \longrightarrow S$ inner f.p.a.. Let (g_j) be a sequence in L_c^∞, with each $g_j = 0$ a.e. in S and $\hat{g}_j \rightrightarrows f$ in $\overline{S}$.

Fixing n and noting that $\overline{\partial} \hat{g}_j = -\pi g_j = 0$ a.e. in S, we see each $\frac{1}{2i} \int_{B_{K_n}} \hat{g}_j \, dz = \iint_{K_n} \overline{\partial} \hat{g}_j \, dA = 0$, hence, $\int_{B_{K_n}} f \, dz = 0$, and so $\int_{\partial S} f \, dz = 0$. ■

Theorem 142 *Let $2 < p \leq \infty$ and $f \in C_c$, with $\overline{\partial} f \in L^p$. Let S be bounded, with int $S = \phi$. Assume $\int\limits_{\partial S} \frac{f}{z-z_0}\, dz = 0$ for a.e. $z_0 \in \mathbb{C} \setminus S$. Then $\overline{\partial} f = 0$ a.e. in S (and hence, $f \in R(S)$).*

Remark: Khavinson [8, Theorem 3.2] proves the above for $f \in \mathrm{Lip}_1(\mathbb{C})$ and K f.p..

Proof. For a.e. $z_0 \in \mathbb{C} \setminus S$, Theorem 123 and Proposition 128 show: $\frac{1}{2\pi i} \int\limits_{\partial S} \frac{f}{z-z_0}\, dz - \frac{1}{\pi} \iint\limits_{S} \frac{\overline{\partial} f}{z-z_0}\, dA = 0$. Using the hypothesis, we then have $(\chi_S \overline{\partial} f)^\wedge = 0$ a.e. in $\mathbb{C} \setminus S$. Since $(\chi_S \overline{\partial} f)^\wedge \in C$ and int$S = \phi$, we see $(\chi_S \overline{\partial} f)^\wedge \equiv 0$ in $\mathbb{C}$ and hence, $\overline{\partial} f = 0$ a.e. in S. By Theorem 140, we get $f \in R(S)$. ■

Remarks:

(a) The above proof follows the reasoning of [8, Theorem 3.2].

(b) If S is constructed as was the set K in Example 119, then $m_2 S > 0$. Essential to the proof, therefore, is that $(\chi_S \overline{\partial} f)^\wedge \in C$. ■

In order to motivate Theorem 150, we will need some preliminaries concerning distributions.

Notation: Let $\Omega \subseteq \mathbb{C}$ be open. For $\phi \in \mathcal{D}(\Omega)$ and $\Lambda \in \mathcal{D}'(\Omega)$, we will let $\Lambda\phi$ (or $\Lambda(\phi)$) denote the complex number obtained when Λ acts on ϕ.

Set $\mathcal{E}'(\Omega) = \{\Lambda \in \mathcal{D}'(\Omega) : \Lambda$ has compact support in $\Omega\}$.

Proposition 143 *Let $\phi \in \mathcal{D}$. Then $\hat{\phi} \in C^\infty$.*

Proof. For $z_0 \in \mathbb{C}$,

$$\hat{\phi}(z_0) = \iint \frac{\phi}{z - z_0}\, dA = -\left(\frac{1}{z} * \phi\right)(z_0).$$

Let $D = \frac{\partial^{j+n}}{\partial x^j \partial j^n}$. Since supp $\phi \subseteq \mathbb{C}$ is compact and $\frac{1}{z} \in L^1_{\text{loc}}$, we have $D\hat{\phi} = -\left(\frac{1}{z} * D\phi\right) \in C$. ■

Definition: For $\Lambda \in \mathcal{D}'$ and $\phi \in \mathcal{D}$, define $\Lambda * \phi : \mathbb{C} \longrightarrow \mathbb{C}$ by $(\Lambda * \phi)(z) = \Lambda[\phi(z - \cdot)]$, where $[\phi(z - \cdot)](\zeta) = \phi(z - \zeta)$.

Proposition 144 *(Theorem 6.33 [11]) Let $\Lambda \in \mathcal{D}'$ and define Γ on $\mathcal{D}$ by $\Gamma\phi = \Lambda * \phi$. Then $\Gamma : \mathcal{D} \longrightarrow C^\infty$ is continuous (and linear).* ■

Proposition 145 *Let (ϕ_n) be a sequence in $\mathcal{D}$. If $\phi_n \longrightarrow 0$ in $\mathcal{D}$, then $\hat{\phi}_n \longrightarrow 0$ in C^∞.*

Proof. Note $\hat{\phi}_n = -\frac{1}{z} * \phi_n$. Now use Proposition 144. ■

Motivation 146 *Let $\mu \in M_c$. Then $\hat{\mu} \in L^1_{\text{loc}}$ and we may think of μ and $\hat{\mu}$ as distributions. For $\phi \in \mathcal{D}$,*

$$\begin{aligned} \hat{\mu}(\phi) &= \iint \hat{\mu}(z)\phi(z)\,dA(z) = \iint \left(\int \frac{d\mu(\zeta)}{\zeta - z}\right) \phi(z)\,dA(z) \\ &= -\int \left(\iint \frac{\phi(z)}{\zeta - z}\,dA(z)\right) d\mu(\zeta) = -\mu(\hat{\phi}). \end{aligned}$$

This motivates the following:

Definition: For $\Lambda \in \mathcal{E}'$, define $\hat{\Lambda} : \mathcal{D} \longrightarrow \mathbb{C}$ by $\hat{\Lambda}\phi = -\Lambda\hat{\phi}$. We call $\hat{\Lambda}$ the *Cauchy transform* of Λ.

Remark: By Proposition 143 and Proposition 145, we note $\hat{\Lambda} \in \mathcal{D}'$. By Motivation 146, both concepts of Cauchy transform agree in M_c. ■

Proposition 147 *Let $\Lambda \in \mathcal{E}'$. Then $(\overline{\partial}\Lambda)^\wedge = -\pi\Lambda$ in $\mathcal{D}'$.*

Proof. Let $\phi \in \mathcal{D}$. By Lemma 134, we have $\overline{\partial}\hat{\phi} = -\pi\phi$. Hence,

$$(\overline{\partial}\Lambda)^\wedge(\phi) = -(\overline{\partial}\Lambda)\hat{\phi} = \Lambda(\overline{\partial}\hat{\phi}) = -\pi\Lambda\phi.$$

■

Proposition 148 *Let $\Lambda \in \mathcal{E}'$. Then $\hat{\Lambda} = 0$ in $\mathcal{D}'$ iff $\Lambda = 0$ in $\mathcal{D}'$.*

Proof. Let $\phi \in \mathcal{D}$.

" $\Longrightarrow$ " By Corollary 121, we have $\Lambda\phi = -\frac{1}{\pi}\Lambda\left((\overline{\partial}\phi)^\wedge\right) = \frac{1}{\pi}\hat{\Lambda}\left(\overline{\partial}\phi\right) = 0$.

" $\Longleftarrow$ " Note $\hat{\Lambda}\phi = -\Lambda\hat{\phi} = 0$. ■

Section 149 Let K be compact and let $f \in R(K)$. Then $\overline{\partial}f \in \mathcal{D}'$. We would like to believe that $\overline{\partial}f$ is, in some sense, zero in K. □

Let K have f.p.. Let $f \in C_c$, with $\overline{\partial}f \in L^1$. By Proposition 147 and Theorem 120, note

$$-\frac{1}{\pi}(\overline{\partial}(\chi_K f))^\wedge = \chi_K f = \left(\frac{1}{2\pi i} f\, dz\,|_{B_K} - \frac{1}{\pi}\chi_K\overline{\partial}f\right)^\wedge \quad \text{in} \quad \mathcal{D}'.$$

Using Proposition 148 and the above equality, we get

$$\chi_K\overline{\partial}f = \overline{\partial}(\chi_K f) + \frac{1}{2i} f\, dz\,|_{B_K},$$

a distribution equality. Even if f were merely in $C(K)$, the right-hand side above is a distribution.

This motivates the following:

Theorem 150 *Let K have f.p. and $f \in R(K)$.*
Then $\overline{\partial}f + \frac{1}{2i} f\, dz\,|_{B_K} = 0$ in $\mathcal{D}'$.

Remark: As mentioned at the end of Section 149, we are trying to say: $\chi_K \overline{\partial} f = 0$ in $\mathcal{D}'$. Of course, $\chi_K \overline{\partial} f$ need not be an element of $\mathcal{D}'$.

Proof. Let $\phi \in \mathcal{D}$. Let μ denote the measure $-\chi_K \overline{\partial}\phi\, dA + \frac{1}{2i}\phi\, dz\mid_{B_K}$. Note

$$\left(\overline{\partial} f + \frac{1}{2i} f\, dz \mid_{B_K}\right)\phi = -\iint\limits_K f\overline{\partial}\phi\, dA + \frac{1}{2i}\int\limits_{B_K} f\phi\, dz = \int f\, d\mu.$$

We show: $\hat{\mu} \equiv 0$ in $\mathbb{C} \setminus$ K. This proves $\mu \perp R(K)$ and hence, $\int\limits_K f\, d\mu = 0$.

For $z_0 \in \mathbb{C} \setminus$ K,

$$\frac{1}{\pi}\hat{\mu}(z_0) = -\frac{1}{\pi}\iint\limits_K \frac{\overline{\partial}\phi}{z - z_0}\, dA + \frac{1}{2\pi i}\int\limits_{B_K} \frac{\phi}{z - z_0}\, dA = 0,$$

by Lemma 105. ■

We give now an application to partial differential equations:

Section 151 Consider the following boundary value problem: let $g \in C(B_K)$ and $h \in L^1(K)$. Solve

$$\begin{cases} f = g & \text{in } B_K, \\ \overline{\partial} f = h & \text{a.e. in } K \end{cases} \tag{1}$$

subject to the regularity restraints: $f \in C_c$ and $\overline{\partial} f \in L^1$. □

Suppose f solves Equation 1. By Theorem 120,

$\left(\frac{1}{2\pi i} g\, dz \mid_{B_K} - \frac{1}{\pi} h\right)^\wedge \equiv 0$ in $\mathbb{C} \setminus$ K. That is, $\left(\frac{1}{2i} g\, dz \mid_{B_K} - h\right) \perp R(K)$.

By Section 149, we have $\overline{\partial}(\chi_K f) = h - \frac{1}{2i} g\, dz \mid_{B_K}$ in $\mathcal{D}'$.

Set $M = \{\, \overline{\partial}(\chi_K f) : f \in C_c \text{ and } \overline{\partial} f \in L^1 \,\}$, a subspace of $\mathcal{E}'$.

We state these results:

Theorem 152 **(a)** *If* $\left(h - \frac{1}{2i} g\, dz \mid_{B_K}\right) \notin [R(K)]^\perp$, *then Equation 1 has no solution.*

(b) *Equation 1 is solvable iff* $\left(h - \frac{1}{2i} g\, dz\,|_{B_K}\right) \in M$. ■

Definition: Let $S \subseteq \mathbb{C}$ be any set. We define $H(S) = \{f : S \longrightarrow \mathbb{C} \mid$ there exist an open set Ω containing S and $F \in H(\Omega)$, with $F\,|_S = f\,\}$.

Definition: Let S be bounded. Set $(H\mathcal{D})(S) = \mathcal{D}(\mathbb{C}) \bigcap \mathrm{H(S)}$ and $[(H\mathcal{D})(S)]^{\perp} = \{\Lambda \in \mathcal{D}'(\mathbb{C}) : \Lambda\phi = 0$ whenever $\phi \in (H\mathcal{D})(S)\}$.

Proposition 153 *Let* $\Lambda \in \mathcal{E}'$ *and* K *be compact. Then* $\Lambda \perp (H\mathcal{D})(K)$ *iff* $\hat{\Lambda} = 0$ *in* $\mathcal{D}'(\mathbb{C} \setminus \mathrm{K})$.

Proof.

"$\Longrightarrow$" Let $\phi \in \mathcal{D}(\mathbb{C} \setminus \mathrm{K})$. Note $\overline{\partial}\hat{\phi} = -\pi\phi = 0$ in a neighborhood of K, hence, $\hat{\phi} \in H(K)$, and so $\hat{\Lambda}\phi = -\Lambda\hat{\phi} = 0$.

"$\Longleftarrow$" Let $\phi \in \mathcal{D}(\mathbb{C}) \bigcap \mathrm{H(K)}$. By Corollary 121, we have $(\overline{\partial}\phi)^\wedge = -\pi\phi$. Since $\overline{\partial}\phi \in \mathcal{D}(\mathbb{C} \setminus \mathrm{K})$, we note $\Lambda\phi = -\frac{1}{\pi}\Lambda\left((\overline{\partial}\phi)^\wedge\right) = \frac{1}{\pi}\hat{\Lambda}(\overline{\partial}\phi) = 0$. ■

Chapter II
Harmonic Approximation on F.P.A. Sets

Abstract

Let $K \subseteq \mathbb{C}$ be a compact set having finite perimeter and let $f \in C(K)$. We would like to find conditions equivalent to the following:
$\Delta f = 0$ in K (in some sense).

With minimal regularity assumptions, we characterize those f satisfying $\Delta f = 0$ a.e. in K. We then solve the same problem with K replaced by a bounded open set. Finally, we return to the compact case and "almost" solve the problem when f has a "bit" of regularity missing. ■

Important: We use the same (abbreviated) notation as in Chapter I.

Definition: For $\mu \in M$, define $(\log * \mu)(z_0) = \int \log |z_0 - z| \, d\mu(z)$, the logarithmic transform of μ. For $f \in L^1$, we will denote the log transform of the measure $f\,dA$ by $\log * f$. ■

The following is the analogue of Proposition 100.

Proposition 200 *Let $\mu \in M$ and $1 \le p < \infty$. Set*
$f(z) = \int |\log|\zeta - z||^p \, d|\mu|(\zeta)$. Then $f \in L^1_{\mathrm{loc}}$ and hence, $\log * \mu \in L^p_{\mathrm{loc}}$.

Proof. Fix $R > 0$. Assume supp $\mu \subseteq R_1\mathbb{D}$. Note $\int_0^T r|\log r|^p \, dr < \infty$ for all $T > 0$.

$$\begin{aligned}
\iint_{R\mathbb{D}} f(z)\,dA(z) &= \int \left(\iint_{R\mathbb{D}} |\log|z-\zeta||^p \, dA(z) \right) d\,|\mu|\,(\zeta) \\
&\le \int \left(\int \int_{D(\zeta;R+R_1)} |\log|z-\zeta||^p \, dA(z) \right) d|\mu|(\zeta) \\
&\le 2\pi \|\mu\| \int_0^{R+R_1} r|\log r|^p \, dr < \infty.
\end{aligned}$$

■

Notation:

$\partial = \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$; $d\bar{z} = dx - i\,dy$.

$\triangle = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$.

Remarks:

(i) $\overline{\partial f} = \overline{\partial}\bar{f}$.

(ii) $\partial f = f'$ if f is analytic.

(iii) $\triangle = 4\left(\partial \circ \overline{\partial}\right)$. ■

Lemma 201 *Let S be bounded and $f \in C_c$, with $\overline{\partial} f \in L^1$. Then*

$$\iint_S \partial f \, dA = \frac{i}{2} \int_{\partial S} f \, d\bar{z}.$$

Proof. By Proposition 129,

$$\iint_S \partial f \, dA = \overline{\iint_S \overline{\partial}\bar{f} \, dA} = \overline{\frac{1}{2i}\int_{\partial S} \bar{f} \, dz} = \frac{i}{2}\int_{\partial S} f \, d\bar{z}$$

■

Lemma 202 *For $z \neq 0$, we have $\partial \log |z| = \frac{1}{2z}$ and $\overline{\partial} \log |z| = \frac{1}{2\bar{z}}$.*

Proof. Writing $z = x + iy$, we note $\partial \log |z| = \frac{1}{2}(\partial_x - i\partial_y)\frac{1}{2}\log(x^2 + y^2)$

$$= \frac{1}{4}\left[\frac{2x - 2iy}{(x+iy)(x-iy)}\right] = \frac{1}{2}\,\frac{1}{x+iy} \quad \text{and} \quad \overline{\partial}\log|z| = \overline{\partial \log |z|} = \frac{1}{2\bar{z}}$$

■

The following is an extension of [3, Lemma 3.4.3] .

Theorem 203 *Let* $1 < p \leq \infty$ *and* $\mu \in M$. *Let* $f \in C_c$, *with* $\overline{\partial} f \in C$ *and* $\Delta f \in L^p$. *Then*

$$\int f\,d\mu = \frac{1}{2\pi}\iint_{\mathbb{C}} (\Delta f)(\log * \mu)\,dA.$$

Proof. By Corollary 121,

$$\int f(z)\,d\mu(z) = -\frac{1}{\pi}\int \left[\iint_{\mathbb{C}} \frac{(\overline{\partial} f)(\zeta)}{\zeta - z}\,dA(\zeta)\right] d\mu(z).$$

Fix $z \in \operatorname{supp}\mu$ and choose $R > 0$ satisfying $\mathrm{supp\ f} \subseteq \mathrm{D(z;R)}$. For $\epsilon > 0$, set $A_\epsilon = \{\zeta \in \mathbb{C} : \epsilon < |\zeta - \mathrm{z}| < \mathrm{R}\}$.

$$\begin{aligned}\iint_{A_\epsilon} \frac{(\overline{\partial} f)(\zeta)}{\zeta - z}\,dA(\zeta) &= 2\iint_{A_\epsilon} (\overline{\partial} f)(\zeta)\frac{\partial}{\partial \zeta}\log|\zeta - z|\,dA(\zeta)\\ &= -2\iint_{A_\epsilon} (\partial\overline{\partial} f)(\zeta)\log|\zeta - z|\,dA(\zeta) + i\int_{\partial D(z;\epsilon)} (\overline{\partial} f)(\zeta)\log|\zeta - z|\,d\overline{\zeta},\end{aligned}$$

where we have integrated by parts and used Lemma 201. Note $\partial\overline{\partial} f = \frac{1}{4}\Delta f$,

$$\begin{aligned}\left|\int_{\partial D(z;\epsilon)} (\overline{\partial} f)(\zeta)\log|\zeta - z|\,d\overline{\zeta}\right| &\leq 2\pi\|\overline{\partial} f\|_\infty \epsilon\log\epsilon \longrightarrow 0, \quad \text{and}\\ \iint_{\mathbb{C}} \frac{(\overline{\partial} f)(\zeta)}{\zeta - z}\,dA(\zeta) &= \lim_{\epsilon \searrow 0}\iint_{A_\epsilon} \frac{(\overline{\partial} f)(\zeta)}{\zeta - z}\,dA(\zeta).\end{aligned}$$

Along with Proposition 200, we therefore have

$$\begin{aligned}\int f(z)\,d\mu(z) &= -\frac{1}{\pi}\int \left[(-2)\iint_{\mathbb{C}} \frac{1}{4}(\Delta f)(\zeta)\log|\zeta - z|\,dA(\zeta)\right] d\mu(z)\\ &= \frac{1}{2\pi}\iint_{\mathbb{C}} \left[\int \log|\zeta - z|\,d\mu(z)\right](\Delta f)(\zeta)\,dA(\zeta).\end{aligned}$$

■

Corollary 204 *Let $\mu \in M$. Then $\Delta(\log * \mu) = 2\pi\mu$ in $\mathcal{D}'$.*

Proof. For $\phi \in \mathcal{D}$, Theorem 203 gives:

$$\begin{aligned}\left[\frac{1}{2\pi}\Delta(\log * \mu)\right]\phi &= \frac{(-1)^2}{2\pi}(\log * \mu)(\Delta\phi) \\ &= \frac{1}{2\pi}\iint_{\mathbb{C}}(\Delta\phi)(\log * \mu)\,dA = \int \phi\,d\mu = \mu(\phi).\end{aligned}$$

■

Theorem 205 *Let $\mu \in M$ and $\Omega \subseteq \mathbb{C}$ be open. If $\log * \mu = 0$ a.e. in Ω, then $|\mu|(\Omega) = 0$.*

Proof. By Theorem 203, we note $\int f\,d\mu = 0$ for all $f \in C_c^2(\Omega)$. Since $C_c^2(\Omega)$ is dense in $C_0(\Omega)$, we are done. ■

Definition: For compact $K \subseteq \mathbb{C}$, set harm$K = \{h : h$ is harmonic in a neighborhood of $K\}$ and let $h(K)$ denote the closure of harm$K$ in $C(K)$.

Theorem 206 *[3, Lemma 3.4.5] Let K be compact and $\mu \in M(K)$. Then $\mu \perp h(K)$ iff $\log * \mu \equiv 0$ in $\mathbb{C} \setminus$ K.*

Proof.

" $\Longrightarrow$ " Note $\log|z_0 - z| \in$ harm$K$ for all $z_0 \in \mathbb{C} \setminus$ K. Hence,

$$(\log * \mu)(z_0) = \int \log|z_0 - z|\,d\mu(z) = 0.$$

" $\Longleftarrow$ " Let $h \in$ harm$K$. WLOG, assume $h \in \mathcal{D}$. By Theorem 203,

$$\int h\,d\mu = \frac{1}{2\pi}\iint_{\mathbb{C}}(\Delta h)(\log * \mu)\,dA = 0.$$

■

Lemma 207 *Let $f \in C^2 = C^2(\mathbb{C})$ and $\Omega \subsetneq \mathbb{C}$ be a finitely connected C^1 Jordan domain. Then*

$$\frac{1}{2\pi i}\left[\int_{\partial\Omega} \frac{f}{z - z_0}\, dz \quad + \quad 2\int_{\partial\Omega} (\overline{\partial} f) \log|z_0 - z|\, d\bar{z}\right]$$
$$+ \quad \frac{1}{2\pi}\iint_{\Omega} (\Delta f) \log|z_0 - z|\, dA = \begin{cases} f(z_0) & \text{if } z_0 \in \Omega, \\ 0 & \text{if } z_0 \in \mathbb{C} \setminus \overline{\Omega}. \end{cases}$$

Proof. Fix $z_0 \in \mathbb{C} \setminus (\partial\Omega)$. For $\epsilon > 0$ small enough, set $\Omega_\epsilon = \Omega \setminus D(z_0; \epsilon)$.

$$\begin{aligned} \iint_{\Omega} \frac{\overline{\partial} f}{z - z_0}\, dA &= 2 \lim_{\epsilon \searrow 0} \iint_{\Omega_\epsilon} (\overline{\partial} f) \partial \log|z_0 - z|\, dA \\ &= 2\left[-\iint_{\Omega} (\partial\overline{\partial} f) \log|z_0 - z|\, dA + \frac{i}{2} \lim_{\epsilon \searrow 0} \int_{\partial\Omega_\epsilon} (\overline{\partial} f) \log|z_0 - z|\, d\bar{z}\right]. \end{aligned}$$

Note $\Delta = 4\partial\overline{\partial}$. If $z_0 \in \mathbb{C} \setminus \overline{\Omega}$, then $\partial\Omega_\epsilon = \partial\Omega$. If $z_0 \in \Omega$, then

$$\lim_{\epsilon \searrow 0} \int_{\partial D(z_0;\epsilon)} \left(\overline{\partial} f\right) \log|z_0 - z|\; d\bar{z} = 0.$$

Now use Lemma 105. ■

We prove the harmonic analogue of Theorem 120.

Theorem 208 *Let $f \in C_c$, with $\overline{\partial} f \in C$ and $\Delta f \in L^1$. Let K have f.p.. Then there exists $E \subsetneq K$ satisfying $m_2 E = 0$ and*

$$\frac{1}{2\pi i}\left[\int_{B_K} \frac{f}{z - z_0}\, dz \quad + \quad 2\int_{B_K} \left(\overline{\partial} f\right) \log|z_0 - z|\, d\bar{z}\right]$$
$$+ \quad \frac{1}{2\pi}\iint_{K} (\Delta f) \log|z_0 - z|\, dA = \begin{cases} f(z_0) & \text{if } z_0 \in K \setminus E, \\ 0 & \text{if } z_0 \in \mathbb{C} \setminus \mathrm{K}. \end{cases}$$

Proof. Let (h_n) be an approximate identity on $\mathbb{C}$. Using Lemma 207 and proceeding as in the proof of Lemma 106, we see the above formula holds for each $h * f$.

Now proceed as in the proof of Theorem 120. Note

$$h_n * f \rightrightarrows f \text{ in } B_K,$$

$$\overline{\partial}(h_n * f) \rightrightarrows \overline{\partial} f \text{ in } B_K,$$

$$\text{and} \qquad \Delta(h_n * f) \longrightarrow \Delta f \text{ in } L^1.$$

It is important to regard the area integral as the convolution of two L^1- functions (as in the proof of Theorem 120). (The function $\log |z|$ must be "cut off ", as was the function $\frac{1}{z}$ in the proof of Theorem 120.) To finish up, recall the identity principle holds (locally) for harmonic functions. ■

We prepare to motivate Theorem 218, the analogue of Theorem 150. We will need some preliminaries.

Proposition 209 *Let* $\phi \in \mathcal{D}$. *Then* $\log * \phi \in C^\infty$.

Proof. Let $D = \frac{\partial^{j+n}}{\partial x^j \partial y^n}$. Note

$$(\log * \phi)(z_0) = \iint_{\mathbf{C}} \phi(z) \log |z_0 - z| \, dA(z) = \iint \phi(z_0 + w) \log |w| \, dA(w).$$

Since supp $\phi \subseteq \mathbb{C}$ is compact and $\log |w| \in L^1_{\text{loc}}$, we have
$D(\log * \phi) = \log * (D\phi) \in C$. ■

Proposition 210 *Let* (ϕ_n) *be a sequence in* $\mathcal{D}$*, with* $\phi_n \longrightarrow 0$ *in* $\mathcal{D}$*.*

Then $\log * \phi_n \longrightarrow 0$ *in* C^∞*.*

Proof. Use Propositon 144. ■

Motivation 211 Let $\mu \in M_c$. Then $\log * \mu \in L^1_{\text{loc}}$ and we may think of both μ and $\log * \mu$ as distributions. For $\phi \in \mathcal{D}$,

$$\begin{aligned}
(\log * \mu)\,\phi &= \iint_{\mathbb{C}} (\log * \mu)\,(z)\phi(z)\,dA(z) \\
&= \iint_{\mathbb{C}} \left(\int \log|z-\zeta|\,d\mu(\zeta)\right)\phi(z)\,dA(z) \\
&= \int (\log * \phi)\,(\zeta)\,d\mu(\zeta) = \mu(\log * \phi).
\end{aligned}$$

■

This motivates the following:

Definition: For $\Lambda \in \mathcal{E}'$, define $\log * \Lambda : \mathcal{D} \longrightarrow \mathbb{C}$ by $(\log * \Lambda)\phi = \Lambda(\log * \phi)$. We call $\log * \Lambda$ the *log transform* for Λ.

Remark: By Proposition 209 and Proposition 210, we see $\log * \Lambda$ *is* a distribution. By Motivation 211, both concepts of log transform agree on M_c. ■

Lemma 212 *Let* $\Lambda \in \mathcal{E}'$*. Then* $\log * \Lambda = 0$ *in* $\mathcal{D}'$ *iff* $\Lambda = 0$ *in* $\mathcal{D}'$*.*

Proof.

" $\Longrightarrow$ " Let $\phi \in \mathcal{D}$. Note

$$0 = (\log * \Lambda)(\Delta\phi) = \Lambda\left[\log * (\Delta\phi)\right] = \Lambda\left[\Delta(\log * \phi)\right] = 2\pi\Lambda\phi,$$

where we have used Corollary 204. ■

Definition: Let $K \subseteq \mathbb{C}$ be compact. Set

$\mathcal{D}(\text{harm}K) = \{\phi \in \mathcal{D} : \Delta\phi \equiv 0$ in a neighborhood of $K\}$. For $\Lambda \in \mathcal{D}'$, we write $\Lambda \perp \mathcal{D}(\text{harm}K)$ if $\Lambda\phi = 0$ whenever $\phi \in \mathcal{D}(\text{harm}K)$.

Lemma 213 *Let K be compact and $\Lambda \in \mathcal{E}'$.*

Then $\Lambda \perp \mathcal{D}(harmK)$ iff $\log * \Lambda = 0$ in $\mathcal{D}'(\mathbb{C} \setminus \mathrm{K})$.

Proof.

" $\Longrightarrow$ " Let $\phi \in \mathcal{D}(\mathbb{C} \setminus \mathrm{K})$. Then $\log * \phi \in \mathcal{D}(\text{harm}K)$ and hence,

$$(\log * \Lambda)\phi = \Lambda(\log * \phi) = 0.$$

" $\Longleftarrow$ " Let $\phi \in \mathcal{D}(\text{harm}K)$. Now $\Delta\phi \in \mathcal{D}(\mathbb{C} \setminus \mathrm{K})$, hence, by Corollary 204,

$$\Lambda\phi = \tfrac{1}{2\pi}\Lambda\left[\Delta(\log * \phi)\right] = \tfrac{1}{2\pi}\Lambda\left[\log * (\Delta\phi)\right] = \tfrac{1}{2\pi}(\log * \Lambda)(\Delta\phi) = 0.$$ ■

Lemma 214 *Let $\Lambda \in \mathcal{E}'$. Then $\Delta(\log * \Lambda) = \log * (\Delta\Lambda) = 2\pi\Lambda$ in $\mathcal{D}'$.*

Proof. For $\phi \in \mathcal{D}$, note

$$\left[\Delta(\log * \Lambda)\right]\phi = (\log * \Lambda)(\Delta\phi) = \Lambda\left[\log * (\Delta\phi)\right] = \Lambda\left[\Delta(\log * \phi)\right].$$

The latter quantity equals $(\Delta\Lambda)(\log * \phi) = \left[\log * (\Delta\Lambda)\right]\phi$.

By Corollary 204, it also equals $(2\pi\Lambda)\phi$. ■

Section 215 Let $f \in C_c$, with $\overline{\partial} f \in C$ and $\Delta f \in L^1$. Let K have f.p..

Note

$$-\int_{B_K} \frac{f}{z - z_0}\, dz = \left[\frac{1}{z} * \left(f\, dz\,|_{B_K}\right)\right](z_0).$$

Now

$$\frac{1}{z} * \left(f\, dz\,|_{B_K}\right) = (2\partial \log|z|) * \left(f\, dz\,|_{B_K}\right) = 2\log * \left[\partial\left(f\, dz\,|_{B_K}\right)\right],$$

where we have convolved an element of $\mathcal{D}'$ with one of $\mathcal{E}'$ (see [11, Chapter 6] .) By Theorem 208,

$$\log * \left\{\frac{1}{\pi i}\left[(\overline{\partial} f)\, d\bar{z} \mid_{B_K} - \partial(f\, dz \mid_{B_K})\right] + \frac{1}{2\pi}(\chi_K \Delta f)\right\} = 0 \qquad (1)$$

in $\mathcal{D}'(\mathbb{C} \setminus \mathrm{K})$. □

By Lemma 214, we have $\chi_K f = \frac{1}{2\pi} \log * [\Delta(\chi_K f)]$. Hence, by Theorem 208,

$$\log * \left\{\frac{1}{\pi i}\left[(\overline{\partial} f)\, d\bar{z} \mid_{B_K} - \partial\left(f\, dz \mid_{B_K}\right)\right] + \frac{1}{2\pi}\left[\chi_K \Delta f - \Delta(\chi_K f)\right]\right\} = 0$$

in $\mathcal{D}'$. By Lemma 212, we have

$$\Delta(\chi_K f) = \chi_K \Delta f + 2i\left[\partial\left(f\, dz \mid_{B_K}\right) - (\overline{\partial} f)\, d\bar{z} \mid_{B_K}\right]$$

in $\mathcal{D}'$. □

Set $N = \{\Delta(\chi_K f) : f \in C_c,\ \overline{\partial} f \in C, \text{ and } \Delta f \in L^1\}$, a subspace of $\mathcal{E}'$.

Consider the following boundary value problem: let $g_1, g_2 \in C(B_K)$ and $g_3 \in L^1(K)$. Solve

$$\begin{cases} f = g_1 & \text{in } B_K \\ \overline{\partial} f = g_2 & \text{in } B_K \\ \Delta f = g_3 & \text{a.e. in } K, \end{cases} \qquad (2)$$

subject to the following regularity restrictions: $f \in C_c$, $\overline{\partial} f \in C$, and $\Delta f \in L^1$.

This brings us to the analogue of Theorem 152:

Theorem 216 *Let K have f.p..*

(a) *If* $\left\{2\left[g_2\, d\bar{z} \mid_{B_K} - \partial(g_1\, dz \mid_{B_K})\right] + i g_3\right\} \notin [\,\mathcal{D}(harm K)\,]^{\perp}$, *then Equation 2 has no solution.*

(b) *Equation 2 has a solution iff* $\left\{g_3 + 2i\left[\partial(g_1\, dz \mid_{B_K}) - g_2\, d\bar{z} \mid_{B_K}\right]\right\} \in N$.

Proof. Use Equation 1 and Lemma 213 for **(a)**. ■

Definition: Let K have f.p.. Set $A = \{ f \in C_c : \overline{\partial} f \in C$ and there exists a sequence (g_j) in harm$K$ satisfying: $g_j \rightrightarrows f$ in K and $\left[\overline{\partial}(f - g_j)\right] d\bar{z} \mid_{B_K} \longrightarrow 0$ in $\mathcal{D}' \}$.

Section 217 Let K have f.p. and let $f \in h(K)$. We wish to believe that Δf is, in some sense, zero in K. But how should we phrase this when Δf need only be in $\mathcal{D}'$? □

Let $f \in C_c$, with $\overline{\partial} f \in C$ and $\Delta f \in L^1$. By Section 215,

$$\chi_K \Delta f = \Delta(\chi_K f) + 2i\left[(\overline{\partial} f)\, d\bar{z} \mid_{B_K} - \partial(f\, dz \mid_{B_K})\right] \text{ in } \mathcal{D}'. \tag{3}$$

Even if only $\Delta f \in \mathcal{D}'$, the right-hand side of Equation 3 is a distribution.

This motivates the following:

Theorem 218 *Let $1 < p \leq \infty$ and $f \in C_c$, with $\overline{\partial} f \in C$. Let K have f.p. and set* $\Lambda = \Delta(\chi_K f) + 2i\left[(\overline{\partial} f)\, d\bar{z} \mid_{B_K} - \partial(f\, dz \mid_{B_K})\right]$.

(a) *If $f \in A$, then $\Lambda = 0$ in $\mathcal{D}'$.*

(b) *If $\Delta f \in L^p$ and if $\Lambda = 0$ in $\mathcal{D}'$, then $f \in A$.*

Remark: In **(a)**, we would like to write: $\Lambda = \chi_K \Delta f$ (see Equation 3). But $\chi_K \Delta f$ need not be a distribution. **(b)** shows the hypothesis $f \in A$ is not "too" strong.

In a (very) loose sense, Theorem 218 states: the integral formula of Theorem 208 holds for $f \in A$.

Proof.

(a) Fix $\phi \in \mathcal{D}$. Let (g_j) be as in the definition of A. By Proposition 129, Lemma 201, and the definition of A,

$$\begin{aligned}
\Lambda\phi &= \lim_j \left\{ 4 \iint\limits_K g_j \overline{\partial}\partial\phi\, dA + 2i \left[\int\limits_{B_K} \phi\overline{\partial} g_j\, d\bar{z} + \int\limits_{B_K} g_j \partial\phi\, dz \right] \right\} \\
&= \lim_j \left\{ -4 \iint\limits_K (\overline{\partial} g_j)\partial\phi\, dA + 2i \int\limits_{B_K} \phi\overline{\partial} g_j\, d\bar{z} \right\} \\
&= \lim_j \iint\limits_K \phi\Delta g_j\, dA = 0.
\end{aligned}$$

(b) By Section 217, we note $\Lambda = \chi_K \Delta f$. By hypothesis and Theorem 205, $\Delta f = 0$ a.e. in K. Let $\mu \perp h(K)$. By Theorem 203 and Theorem 206,

$$\int f\, d\mu = \frac{1}{2\pi} \iint\limits_{\mathbb{C}} (\Delta f)(\log * \mu)\, dA = 0.$$

So $f \in h(K)$. Hence there exists a sequence (g_j) in harm$K$ satisfying: $g_j \rightrightarrows f$ in K. WLOG, assume each $g_j \in \mathcal{D}$. Equation 3 shows $\left[\overline{\partial}(f - g_j)\right] d\bar{z}\,|_{B_K} \longrightarrow 0$ in $\mathcal{D}'$. [$\chi_K \Delta g_j = \chi_K \Delta f = 0$ in $\mathcal{D}'$ takes care of the left-hand side of Equation 3; the other two terms on the right-hand side "cancel" since $g_j \rightrightarrows f$ in K.].
So $f \in A$. ■

The following is the analogue of Theorem 123.

Theorem 219 *Let $S \subseteq \mathbb{C}$ be bounded (and m_2-measurable). Let $f \in C_c$, with $\overline{\partial} f \in C$ and $\Delta f \in L^1$. Then there exists $E \subseteq \mathbb{C}$ satisfying $m_2 E = 0$ and*

$$\frac{1}{2\pi i}\left[\iint\limits_{\partial S} \frac{f}{z - z_0}\, dz + 2\int\limits_{\partial S} (\overline{\partial} f) \log|z_0 - z|\, d\bar{z} \right] + \frac{1}{2\pi} \iint\limits_S (\Delta f) \log|z_0 - z|\, dA = \begin{cases} f(z_0) & \text{if } z_0 \in S \setminus E, \\ 0 & \text{if } z_0 \in (\mathbb{C} \setminus S) \setminus E. \end{cases}$$

Proof. Using Proposition 128, let $K_n \longrightarrow S$ f.p.a.. Using Theorem 123, we know the limits, $\lim \iint\limits_{K_n} (\Delta f) \log|z_0 - z|\, dA$ and $\lim \int\limits_{B_{K_n}} \frac{f}{z - z_0}\, dz$, exist for a.e. $z_0 \in \mathbb{C}$. By Theorem 208, so does $\lim \int\limits_{B_{K_n}} (\overline{\partial} f) \log|z_0 - z|\, d\bar{z}$, which we denote by $\int\limits_{\partial S} (\overline{\partial} f) \log|z_0 - z|\, d\bar{z}$. ■

Definition: Let $1 \le s < \infty$ and $\Omega \subseteq \mathbb{C}$ be open. Let (f_n) be a sequence in $L^s_{\text{loc}}(\Omega)$. We write $f_n \longrightarrow 0$ in $L^s_{\text{loc}}(\Omega)$ if given any compact $K \subseteq \Omega$, it follows that $\lim \iint\limits_K |f_n|^s\, dA = 0$.

Definition: Let K have f.p.. Let $f_1 \in C(K)$ and $f_2 \in C(B_K)$. We write $(f_1, f_2) \in h(K; \log^s)$ if there exists a sequence (g_n) in harm$K$ satisfying: $g_n \rightrightarrows f_1$ in K and $\log * \left[(f_2 - \overline{\partial} g_n)\, d\bar{z}\, |_{B_K} \right] \longrightarrow 0$ in $L^s_{\text{loc}} = L^s_{\text{loc}}(\mathbb{C})$.

Let $f \in C$, with $\overline{\partial} f \in C$. We write $f \in h(K; \log^s)$ if $(f, \overline{\partial} f) \in h(K; \log^s)$.

Theorem 220 *Let $f \in C_c$, with $\overline{\partial} f \in C$ and $\Delta f \in L^1$. Let $1 < p \le \infty$, $1 \le s < 2$, and K have f.p..*

(a) *If $\Delta f \in L^p$ and if $\Delta f = 0$ a.e. in K, then $f \in h(K; \log^s)$.*

(b) *If $f \in h(K; \log^s)$, then $\Delta f = 0$ a.e. in K.*

Proof.

(a) Let $\mu \in M(K)$, with $\mu \perp h(K)$. By Theorem 203 and Theorem 206,

$$\int f\, d\mu = \frac{1}{2\pi} \iint\limits_{\mathbb{C}} (\Delta f)(\log * \mu)\, dA = 0.$$

Hence, $f \in h(K)$ and so $g_n \rightrightarrows f$ in K for some sequence (g_n) in harm$K$. WLOG, assume each $g_n \in \mathcal{D}'$. Taking limits in the integral representation of

Theorem 208 gives:

$$\lim \int_{B_K} \left[\overline{\partial}(f-g_n)\right] \log|z_0 - z|\, d\bar{z} = 0 \quad \text{for a.e.} \quad z_0 \in \mathbb{C}.$$

Let $R > 0$. We show $\log * \left\{\left[\overline{\partial}(f-g_n)\right] d\bar{z}\,|_{B_K}\right\} \longrightarrow 0$ in $L^s(R\mathbb{D})$.

WLOG, assume $K \subseteq R\mathbb{D}$. By the integral representation of Theorem 208, for a.e. $z \in R\mathbb{D}$,

$$2\left(\log * \left\{\left[\overline{\partial}(f-g_n)\right] d\bar{z}\,|_{B_K}\right\}\right)(z) = 2\pi i\left[\chi_K(f-g_n)\right](z) - \int_{B_K} \frac{(f-g_n)}{\zeta - z}\, d\zeta.$$

Writing $\frac{1}{S} + \frac{1}{t} = 1$, we see

$$\iint_{R\mathbb{D}} \left|\int_{B_K} \frac{(f-g_n)(\zeta)}{\zeta - z}\, d\zeta\right|^S dA(z)$$

$$\leq \|\, d\zeta\,|_{B_K}\|^{\frac{S}{t}} \iint_{R\mathbb{D}} \left(\int_{B_K} \left|\frac{(f-g_n)(\zeta)}{\zeta - z}\right|^S d|\zeta|\right) dA(z)$$

$$\leq \|\, d\zeta\,|_{B_K}\|^{\frac{S}{t}} \int_{B_K} |(f-g_n)(\zeta)|^S \left(\int\int_{D(\zeta;2R)} \frac{dA(z)}{|\zeta - z|^S}\right) d|\zeta|$$

$$\leq \frac{2\pi}{2-S}(2R)^{2-S}\,\|\, d\zeta\,|_{B_K}\|^{\frac{S}{t}}\,\|f-g_n\|_{B_K}^S \longrightarrow 0.$$

(b) WLOG, we may assume (g_n) is a sequence in harm$K$ satisfying: $g_n \rightrightarrows f$ in K and $\log * \left\{\left[\overline{\partial}(f-g_n)\right] d\bar{z}\,|_{B_K}\right\} \longrightarrow 0$ a.e. in $\mathbb{C}$. Taking limits (in Theorem 208, again) gives $\log * (\chi_K \Delta f) = 0$ a.e. in $\mathbb{C}$. By Theorem 205, $\Delta f = 0$ a.e. in K. ■

The following is the analogue of Proposition 139.

Theorem 221 *Let $K \subseteq \mathbb{C}$ be compact. Then $h(K)$ equals the closure of $\{\log * g : g \in L^\infty_c$ and $g = 0$ a.e. in $K\}$ in $C(K)$.*

Proof.

" $\subseteq$ " Let $h \in \text{harm}K$. WLOG, assume $h \in \mathcal{D}$. Then $\Delta h \in \mathcal{D}$,

$\Delta h \equiv 0$ in K, and $\frac{1}{2\pi}\log * (\Delta h) = \frac{1}{2\pi}\Delta(\log * h) = h$ in $\mathbb{C}$, by Corollary 204.

" $\supseteq$ " Let $\mu \perp h(K)$ and let $g \in L_c^\infty$, with $g = 0$ a.e. in K. By Theorem 206,

$$\begin{aligned}\int(\log * g)(z)\,d\mu(z) &= \int\left(\iint\limits_{\mathbb{C}} g(\zeta)\log|z-\zeta|\,dA(\zeta)\right)d\mu(z)\\ &= \iint\limits_{\mathbb{C}} g(\zeta)(\log * \mu)(\zeta)\,dA(\zeta) = 0.\end{aligned}$$

■

Definition: For bounded (m_2- measurable) $S \subseteq \mathbb{C}$, define $h(S)$ to be the closure of $\{(\log * g)\,|_{\overline{S}} : g \in L_c^\infty \text{ and } g = 0 \text{ a.e. in } S\}$ in $C(\overline{S})$.

Remarks:

(a) If $1 < p \leq \infty$ and $g \in L_c^p$, then $\log * g \in C$. [See Theorem 805 for a sketch of this proof.]

(b) By Theorem 221, if S is compact, then the two definitions of $h(S)$ agree.

(c) By Theorem 205 and Theorem 221, $h(S) = h(\overline{S})$ iff $m_2(\overline{S} \setminus S) = 0$. ■

Lemma 222 *Let S be bounded and $\mu \in M(\overline{S})$.*

Then $\mu \perp h(S)$ iff $\log * \mu = 0$ a.e. in $\mathbb{C} \setminus$ S.

Proof.

" $\Longrightarrow$ " Let $T \subseteq \mathbb{C} \setminus$ S be bounded and m_2-measurable. Since $\log * \chi_T \in h(S)$,

$\iint\limits_T(\log * \mu)(z)\,dA(z) = \iint\limits_T\left(\int \log|z-\zeta|\,d\mu(\zeta)\right)dA(z) = \int(\log * \chi_T)\,d\mu = 0.$

" $\Longleftarrow$ " Let $g \in L_c^\infty$, with $g = 0$ a.e. in S. Then

$$\begin{aligned}\int (\log * g)(z)\, d\mu(z) &= \int \left(\iint_{\mathbb{C}} g(\zeta) \log|z-\zeta|\, dA(\zeta) \right) d\mu(z) \\ &= \iint_{\mathbb{C}\setminus S} g(\zeta)(\log * \mu)(\zeta)\, dA(\zeta) = 0.\end{aligned}$$

■

Theorem 223 *Let* $1 < p \le \infty$ *and* S *be bounded. Let* $h(S;p)$ *denote the closure* $\{(\log * g)\,|_{\overline{S}} : g \in L_c^p$ *and* $g = 0$ *a.e. in* $S\}$ *in* $C(\overline{S})$.

Then $h(S;p) = h(S)$.

Remark: Note this is the analogue of Theorem 138.

Proof. Clearly $h(S;p) \subseteq h(S)$.

Let $\mu \in M(\overline{S})$, with $\mu \perp h(S)$. Let $g \in L_c^p$, with $g = 0$ a.e. in S. By Lemma 222 and the proof of Theorem 221, we see $\int (\log * g)\, d\mu = \iint_{\mathbb{C}} g(\log * \mu)\, dA = 0$. ■

Definition: Let $1 \le s < \infty$ and let $\Omega \subseteq \mathbb{C}$ be open and bounded. Let $f_1 \in C(\overline{\Omega})$ and $f_2 \in C(\Omega)$. We write $(f_1, f_2) \in h\left(\Omega; \log^s_{\text{f.p.}}\right)$ if there exists a sequence (g_n) in L_c^∞ satisfying: each $g_n = 0$ a.e. in Ω, $\log * g_n \rightrightarrows f_1$ in $\overline{\Omega}$, and, given any compact set $K \subseteq \Omega$ having finite perimeter,

$$\log * \left\{ \left[f_2 - \overline{\partial}(\log * g_n) \right] d\bar{z}\, |_{B_K} \right\} \longrightarrow 0 \ \text{ in } \ L^s_{\text{loc}}(\mathbb{C}).$$

Let $f \in C$, with $\overline{\partial} f \in C$. We write $f \in h\left(\Omega; \log^s_{\text{f.p.}}\right)$ if $(f, \overline{\partial} f) \in h\left(\Omega; \log^s_{\text{f.p.}}\right)$.

Theorem 224 *Let* $f \in C_c$, *with* $\overline{\partial} f \in C$ *and* $\Delta f \in L^1$. *Let* $1 < p \le \infty$, $1 \le s < 2$, *and* $\Omega \subseteq \mathbb{C}$ *be open and bounded.*

(a) *If* $\Delta f \in L^p$ *and* $\Delta f = 0$ *a.e. in* Ω, *then* $f \in h\left(\Omega; \log^*_{\text{f.p.}}\right)$.

(b) *If* $f \in h\left(\Omega; \log^*_{\text{f.p.}}\right)$. *then* $\Delta f = 0$ *a.e. in* Ω.

Proof.

(a) Let $\mu \in M(\overline{\Omega})$, with $\mu \perp h(\Omega)$. By Lemma 222,

$\log * \mu = 0$ a.e. in $\mathbb{C} \setminus \Omega$ and hence, by Theorem 203,

$\int f\, d\mu = \frac{1}{2\pi} \iint_{\mathbb{C}} (\Delta f)(\log * \mu)\, dA = 0.$

So $f \in h(\Omega)$. Hence there exists a sequence (g_j) in L^∞_c satisfying:

each $g_j = 0$ a.e. in Ω and $\log * g_j \rightrightarrows f$ in $\overline{\Omega}$.

Let K have f.p.. For all z_0 in a neighborhood of K,

$$\left[\overline{\partial}(\log * g_j)\right](z_0) = \frac{1}{2} \iint_{\mathbb{C}\setminus\Omega} \frac{g_j}{\bar{z} - \bar{z}_0}\, dA$$

and $[\Delta(\log * g_j)](z_0) = 0$. Hence, we may assume each $\log * g_j$ satisfies the hypotheses of Theorem 208. Now proceed as in the proof of Theorem 220 **(a)**.

(b) As in the proof of Theorem 220 **(b)**, one can show that if K has f.p., then $\Delta f = 0$ a.e. in K. ∎

Although we will not use the following result in this chapter, this seems the appropriate place to record it.

Proposition 225 *If* $\Lambda \in \mathcal{E}'$, *then* $\partial(\log * \Lambda) = -\frac{1}{2}\hat{\Lambda}$ *in* $\mathcal{D}'$.

Proof. Let $\phi \in \mathcal{D}$. For $z_0 \in \mathbb{C}$,

$$\begin{aligned}\hat{\phi}(z_0) = \iint_{\mathbb{C}} \frac{\phi}{z - z_0}\, dA &= 2 \iint \phi \partial \log |z_0 - z|\, dA \\ &= -2 \iint (\partial\phi) \log |z_0 - z|\, dA = -2\left[\log * (\partial\phi)\right](z_0)\end{aligned}$$

and hence,

$$\hat{\Lambda}\phi = -\Lambda\hat{\phi} = 2\Lambda\left[\log *\ (\partial\phi)\right] = 2\left(\log * \Lambda\right)\left(\partial\phi\right) = -2\left[\partial\left(\log * \Lambda\right)\right]\phi.$$

■

Chapter III
Harmonic Approximation in $\mathbb{R}^N$

Abstract

We extend the results from Chapter II to $\mathbb{R}^N$, where $N \geq 3$. ■

Important: We will omit any proof which does not differ from its corresponding Chapter II proof.

Fix $N \geq 3$. Throughout Chapter III, the unmentioned measure space is N-dimensional Lebesgue measure on $\mathbb{R}^N$, denoted m_N.

For $x = (x_1, \cdots, x_N) \in \mathbb{R}^N$, we write $|x| = \left(\sum_{j=1}^{N} x_j^2\right)^{\frac{1}{2}}$. For $x_0 \in \mathbb{R}^N$ and $r > 0$, set $B(x_0; r) = \left\{x \in \mathbb{R}^N : |x - x_0| < r\right\}$. Set $\mathbb{B}_N = B(0;1)$ and $C_{N-1} = (N-2) \int_{\partial \mathbb{B}_N} dm_{N-1}$. ■

Our first goal is to extend Theorem 120 to $\mathbb{R}^N$. The following will be our starting point.

Proposition 300 (Green's Theorem) *[5, Exercise 7.7(b), Chapter V]*
Let $X \subseteq \mathbb{R}^N$ be a compact N-manifold-with-boundary. Let $\Omega \subseteq \mathbb{R}^N$ be open, with $X \subseteq \Omega$, and let $f, g \in C^2(\Omega)$. Then

$$\int_X (f\Delta g - g\Delta f)\, dm_N = \int_{\partial X} \left(f\frac{\partial g}{\partial n} - g\frac{\partial f}{\partial n}\right) dm_{N-1},$$

where $\frac{\partial \phi}{\partial n} = (\nabla \phi) \cdot \vec{n}$ and $\vec{n}$ is the outward unit normal. ■

We will need the following easy calculation:

Proposition 301 *[15, Example (3), page 38]*
$|x|^{2-N}$ is harmonic in $\mathbb{R}^N \setminus \{0\}$. ■

Section 302 Let X be as in Proposition 300 and assume $0 \in \mathbb{R}^N \setminus \partial X$. For $\epsilon > 0$ small enough, set $X_\epsilon = X \setminus (\epsilon \mathbb{B}_N)$. If $g \in C^2 = C^2\left(\mathbb{R}^N\right)$ and if $f = |x|^{2-N}$, then Proposition 300 gives

$$\int_{X_\epsilon} \frac{\Delta g}{|x|^{N-2}}\, dm_N = \int_{\partial X_\epsilon} \left(\frac{\frac{\partial g}{\partial n}}{|x|^{N-2}} - g \frac{\partial}{\partial n} \frac{1}{|x|^{N-2}} \right) dm_{N-1}.$$

Note $\triangle |x|^{2-N} = \frac{2-N}{|x|^N} x$ for $x \neq 0$. If $0 \in \mathbb{R}^N \setminus X$, then $\partial X_\epsilon = \partial X$. So assume $0 \in \text{int}X$. On $\partial(\epsilon \mathbb{B}_N)$, we note $\triangle |x|^{2-N} = \frac{N-2}{\epsilon^{N-1}}$ since $n(\zeta) = -\frac{\zeta}{|\zeta|}$ is the outward unit normal to X_ϵ (at $\zeta \in \partial(\epsilon \mathbb{B}_N)$). Note

$$\left| \int_{\partial(\epsilon \mathbb{B}_N)} \frac{\frac{\partial g}{\partial n}}{|x|^{N-2}}\, dm_{N-1} \right| \leq \frac{\| \nabla g \|_\infty}{\epsilon^{N-2}} \int_{\partial(\epsilon \mathbb{B}_N)} dm_{N-1} \longrightarrow 0$$

$$\text{and} \quad \int_{\partial(\epsilon \mathbb{B}_N)} g \frac{\partial}{\partial n} \frac{1}{|x|^{N-2}}\, dm_{N-1} = \frac{N-2}{\epsilon^{N-1}} \int_{\partial(\epsilon \mathbb{B}_N)} g\, dm_{N-1} \longrightarrow g(0) C_{N-1}.$$

□

Replacing x by $x - x_0$ in the above gives:

$$\frac{1}{C_{N-1}} \left\{ \int_{\partial X} \left[\frac{1}{|\zeta - x_0|^{N-2}} (\nabla g)(\zeta) + \frac{(N-2) g(\zeta)}{|\zeta - x_0|^N} (\zeta - x_0) \right] \cdot n(\zeta)\, dm_{N-1}(\zeta) - \iint_X \frac{(\Delta g)(x)}{|x - x_0|^{N-2}}\, dm_N(x) \right\} = \begin{cases} g(x_0) & \text{if } x_0 \in \text{int}X \\ 0 & \text{if } x_0 \in \mathbb{R}^N \setminus X, \end{cases}$$

where we have used the fact that $\frac{1}{|x|} \in L^s_{\text{loc}}$ for $1 \leq s < N$. ■

Definition: For $\mu_1, \cdots, \mu_N \in M = M(\mathbb{R}^N)$, set $\mu = (\mu_1, \cdots, \mu_N)$ and write $\mu \in M^N$. Define the Cauchy transform of μ, by

$$\hat{\mu}(x_0) = \sum_{j=1}^N \int \frac{(x - x_0)_j}{|x - x_0|^N}\, d\mu_j(x),$$

where $(x - x_0)_j$ denotes the jth component of $x - x_0$. We will also write $\hat{\mu}(x_0) = \int \frac{1}{|x - x_0|^N} (x - x_0) \cdot d\mu(x)$.

Proposition 303 *Let $\mu \in M^N$ and $1 \le p < \frac{N}{N-1}$. Set*

$$g(x_0) = \sum_{j=1}^{N} \int \left(\frac{|(x - x_0)_j|}{|x - x_0|^N} \right)^p d|\mu_j|(x).$$

Then $g \in L^1_{\text{loc}} = L^1_{\text{loc}}(\mathbb{R}^N)$ and hence, $\hat{\mu} \in L^p_{\text{loc}}$.

Proof. Use $\frac{1}{|x|} \in L^s_{\text{loc}}$ for $1 \le s < N$. ∎

Definition: Let $S \subseteq \mathbb{R}^N$ be (m_N-)measurable.

(i) We let B_S, the reduced boundary of S, denotes those $x_0 \in \mathbb{R}^N$ at which S has an exterior normal.

(ii) We say S has f.p. if there exists $\mu \in M^N$ satisfying:
$\hat{\mu} = \chi_S$ a.e. ($[m_N]$ in $\mathbb{R}^N$).

Proposition 304 *Let X be a compact N-manifold-with boundary. Then X has f.p..*

Proof. Take $g \equiv 1$ and $d\mu(\zeta) = \frac{N-2}{C_{N-1}} n(\zeta)\, dm_{N-1}(\zeta)\, |_{\partial X}$ in the formula at the end of Section 302.

Definition: Let S have f.p.. By definition, there exists $\mu \in M^N$ with $\hat{\mu} = \chi_S$ a.e.. We shall call S a *nice f.p. set (n.f.p.)* if all of the following hold:

(a) (i) $\mathcal{H}_{N-1}(B_S) = \frac{C_{N-1}}{N-2} \|\mu\|$, where $\mathcal{H}_{N-1}$ denotes $(N-1)$-dimensional Hausdorff measure.

(ii) If $B \subseteq \mathbb{R}^N$ is a Borel set, then
$$\mu(B) = \mu(B \cap B_S) = \frac{N-2}{C_{N-1}} \sum_{j=1}^{N} \int_{B \cap B_S} n_j(\zeta)\, d\mathcal{H}_{N-1}(\zeta),$$
where we have written $n(\zeta) = (n_1(\zeta), \cdots, n_N(\zeta))$.

(iii) There exists a set $E_1 \subseteq B_S$ satisfying $\mathcal{H}_{N-1}(E_1) = 0$ and if $x_0 \in B_S \setminus E_1$, then

$$\lim_{\delta \searrow 0} \frac{\mathcal{H}_{N-1}(B_S \cap B(x_0;\delta))}{\delta^{N-1} C} = 1, \quad \text{where} \quad C = \int_{\mathbb{B}_N \cap \{x_N = 0\}} dm_{N-1}.$$

(iv) There exist a set $E_2 \subseteq B_S$ and a sequence of $(N-1)$-manifolds (Y_j) satisfying: $\mathcal{H}_{N-1}(E_2) = 0$,

$B_S \setminus E_2 \subseteq \bigcup Y_j$, and $d\mu(\zeta) = \frac{N-2}{C_{N-1}} n(\zeta)\, dm_{N-1}(\zeta) \mid_{B_S}$.

(b) There exists a sequence of sets (X_j) satisfying:

(i) Each X_j is a finite union of pairwise disjoint compact N-manifolds-with-boundary.

(ii) $m_N(S \Delta X_j) \longrightarrow 0$.

(iii) $\chi_{X_j} \longrightarrow \chi_S$ a.e..

(iv) $\| d\mu(\zeta) - \frac{N-2}{C_{N-1}} n(\zeta)\, dm_{N-1}(\zeta) \mid_{\partial X_j} \| \longrightarrow 0$. ■

Remark: We conjecture: f.p.$\Longrightarrow$ n.f.p.. (See Theorem 104.) ■

Theorem 305 *Let $f \in C_c$, with $\nabla f \in C$ and $\Delta f \in L^1$. Let $K \subseteq \mathbb{R}^N$ be a compact n.f.p. set. Then there exists $E \subseteq K$ satisfying $m_N E = 0$ and*

$$\frac{1}{C_{N-1}} \left\{ \int_{B_K} \left[\frac{1}{|\zeta - x_0|^{N-2}} (\nabla f)(\zeta) + \frac{(N-2) f(\zeta)}{|\zeta - x_0|^N} (\zeta - x_0) \right] \cdot n(\zeta)\, dm_{N-1}(\zeta) \right.$$
$$\left. - \int_K \frac{\Delta f}{|x - x_0|^{N-2}}\, dm_N \right\} = \begin{cases} f(x_0) & \text{if } x_0 \in K \setminus E, \\ 0 & \text{if } x_0 \in \mathbb{R}^N \setminus K. \end{cases}$$

Proof. Proceed as in the proofs of Lemma 106 and Theorem 120.

Note Proposition 303 and the n.f.p. hypothesis are used in place of Proposition 100 and Theorem 104, respectively. ■

Definition: For $\mu \in M$, define the logarithmic transform of μ by

$$(\log * \mu)(x_0) = \int \frac{d\mu(x)}{|x - x_0|^{N-2}}.$$

Remark: This appears to be a confusing choice of notation, since there is no logarithm in sight; but recall: The kernels $\log|z|$ and $\frac{1}{|x|^{N-2}}$ behave similarly on $\mathbb{C}$ and $\mathbb{R}^N$, respectively. Both are harmonic away from the origin; a peek at Theorem 305 shows we can (easily) obtain results analogous to Theorem 203, Theorem 205, and Theorem 206.

Proposition 306 *Let $\mu \in M$ and $1 \leq p < \frac{N}{N-2}$. Set*

$$g(x_0) = \int \frac{d|\mu|(x)}{(|x - x_0|^{N-2})^p}.$$

Then $g \in L^1_{\text{loc}}$ and hence, $\log * \mu \in L^p_{\text{loc}}$. ■

The following is the analogue of Theorem 203:

Theorem 307 *Let $\frac{N}{2} < p \leq \infty$ and $f \in C_c$, with $\nabla f \in C$ and $\Delta f \in L^p$. Let $\mu \in M$. Then*

$$\int f \, d\mu = -\frac{1}{C_{N-1}} \int_{\mathbb{R}^N} (\Delta f)(\log * \mu) \, dm_N.$$

Proof. Choose $R > 0$, with supp f $\subseteq R\mathbb{B}_N$. Take $K = \overline{R\mathbb{B}_N}$ in Theorem 305 and note K is n.f.p.. ■

Corollary 308 *If $\mu \in M$, then $\Delta(\log * \mu) = -\mu C_{N-1}$ in $\mathcal{D}' = \mathcal{D}'(\mathbb{R}^N)$.*

Proof. Use Theorem 307. ■

Definition: Let $K \subseteq \mathbb{R}^N$ be compact. Set harm$K = \{h : \Delta h \equiv 0$ in a neighborhood of $K\}$ and let $h(K)$ denote the closure of harm$K$ in $C(K)$. ■

Note the following is the (higher dimension) analogue of Theorem 206.

Theorem 309 *Let $K \subseteq \mathbb{R}^N$ be compact and $\mu \in M(K)$.*

Then $\mu \perp h(K)$ iff $\log * \mu \equiv 0$ in $\mathbb{R}^N \setminus K$.

Proof.

" $\Longrightarrow$ " If $x_0 \in \mathbb{R}^N \setminus K$, then $\frac{1}{|x-x_0|^{N-2}} \in \text{harm}K$.

" $\Longleftarrow$ " Let $h \in \text{harm}K$. WLOG, $h \in \mathcal{D}$. By Theorem 307,

$$\int h\,d\mu = -\frac{1}{C_{N-1}} \int_{\mathbb{R}^N} (\Delta h)(\log * \mu)\,dm_N = 0$$

■

The following is the analogue of Theorem 205.

Theorem 310 *Let $\mu \in M$ and $\Omega \subseteq \mathbb{R}^N$ be open.*

If $\log * \mu = 0$ a.e. in Ω, then $|\mu|(\Omega) = 0$.

Proof. Let $\phi \in \mathcal{D}(\Omega)$. Then $\int \phi\,d\mu = -\frac{1}{C_{N-1}} \int_{\mathbb{R}^N} (\Delta\phi)(\log * \mu)\,dm_N = 0$, where we have used Theorem 307. ■

Definition: Let $1 \le s < \frac{N}{N-1}$ and $f \in C_c$, with $\nabla f \in C$. Let K have f.p.. We write $f \in h(K;\log^s)$ if there exists a sequence (h_j) in harm$K$ satisfying: $h_j \rightrightarrows f$ in K and $\log * \left\{\left[\frac{\partial}{\partial n}(f - h_j)\right] dm_{N-1}|_{B_K}\right\} \longrightarrow 0$ in L^s_{loc}.

Theorem 311 *Let $f \in C_c$, with $\nabla f \in C$ and $\Delta f \in L^1$. Let $\frac{N}{2} < p \le \infty$, $1 \le s < \frac{N}{N-1}$, and let K be n.f.p..*

(a) *If $\Delta f \in L^p$ and $\Delta f = 0$ a.e. in K, then $f \in h(K;\log^s)$.*

(b) *If $f \in h(K;\log^s)$, then $\Delta f = 0$ a.e. in K.*

Proof.

(a) By Theorem 307 and Theorem 309, we see $f \in h(K)$. This gives a sequence (h_j) in harm$K$, with $h_j \rightrightarrows f$ in K. Proceed as in the proof of Theorem 220: Use the integral representation (Theorem 305) and use $\frac{1}{|x|^{N-1}} \in L^s_{\text{loc}}$.

(b) Proceed as in the proof of Theorem 220 to get: $\log * (\chi_K \Delta f) = 0$ a.e.. By Theorem 310, we get $\Delta f = 0$ a.e. in K. ∎

Lemma 312 *Let* $\frac{N}{N-2} < p \leq \infty$ *and* $g \in L^p_c$. *Then* $\log * g \in C$.

Remark: $\log * g$ denotes the log transform of $g\, dm_N$.

Proof. Proceed as in the proof of Lemma 132, noting $\frac{1}{|x|^{N-2}} \in L^q_{\text{loc}}$, where $\frac{1}{p} + \frac{1}{q} = 1$. ∎

Note the follwoing is the analogue of Theorem 221.

Theorem 313 *$h(K)$ is the closure of*
$\{ \log * g : g \in L^\infty_c$ and $g = 0$ a.e. in $K \}$ in $C(K)$.

Proof.

" $\subseteq$ " For $h \in$ harm$K$, we may assume $h \in \mathcal{D}$. By Corollary 308 and Lemma 312, we see $h \equiv -\frac{1}{C_{N-1}}\Delta(\log * h) = -\frac{1}{C_{N-1}} \log * (\Delta h)$.

" $\supseteq$ " Fix $g \in L^\infty_c$, with $g = 0$ a.e. in K. Let $\mu \in M(K)$ with $\mu \perp h(K)$.
By Theorem 309,

$$\int (\log * g)(x)\, d\mu(x) = \int \left(\int_{\mathbb{R}^N} \frac{g(y)}{|y-x|^{N-2}}\, dm_N(y) \right) d\mu(x)$$
$$= \int_{\mathbb{R}^N} (\log * \mu)(y) g(y)\, dm_N(y) = 0.$$

Since $\frac{1}{|x|^{N-2}} \in L^1_{\text{loc}}$, the use of Fubini is justified. ■

This motivates the following definition:

Definition: Let S be bounded (and m_N-measurable). Let $h(S)$ denote the closure of $\{ \log * g : g \in L^\infty_c \text{ and } g = 0 \text{ a.e. in } S \}$ in $C(\overline{S})$.

Lemma 314 *Let S be bounded and $\mu \in M(\overline{S})$.*

Then $\mu \perp h(S)$ iff $\log * \mu = 0$ a.e. in $\mathbb{R}^N \setminus S$.

Proof. Exactly as in Lemma 222. ■

Definition: Let S be bounded. Let $1 \leq s < \infty$ and $f \in C_c$, with $\overline{\partial} f \in C$. We write $f \in h(S; \log^s)$ if there exists a sequence (g_j) in L^∞_c satisfying: each $g_j = 0$ a.e. in S, $\log * g_j \rightrightarrows f$ in $\overline{S}$, and, given $B = B(x_0; r)$ with $\overline{B} \subseteq \text{int} S$,

$$\log * \left\{ \left[\frac{\partial}{\partial n}(f - \log * g_j) \right] dm_{N-1} \Big|_{\partial B} \right\} \longrightarrow 0 \text{ in } L^s_{\text{loc}}.$$

■

Theorem 315 *Let $f \in C_c$, with $\nabla f \in C$ and $\Delta f \in L^1$. Let $\frac{N}{2} < p \leq \infty$, $1 \leq s < \frac{N}{N-1}$, and S be bounded.*

(a) *If $\Delta f \in L^p$ and $\Delta f = 0$ a.e. in S, then $f \in h(S; \log^s)$.*

(b) *If $m_N(S \setminus \text{int} S) = 0$ and $f \in h(S; \log^s)$, then $\Delta f = 0$ a.e. in S.*

Proof.

(a) Using Lemma 314, we see $f \in h(S)$. Proceed now as in the proof of Theorem 224 **(a)**.

(b) As in the proof of Theorem 224 **(b)**, we see:

$\Delta f = 0$ a.e. in $B = B(x_0; r)$ whenever $\overline{B} \subseteq \text{int} S$. ∎

Remark: Note Theorem 224 could have been phrased in this more general manner.

∎

We will finish this chapter with analogues to Theorem 218 and to Theorem 219, respectively.

Theorem 316 *Let $f \in C_c$, with $\nabla f \in C$ and $\Delta f \in L^1$. Let S be bounded. Assume there exists a sequence (K_j) of n.f.p. sets satisfying: $\chi_{K_j} \longrightarrow \chi_S$ a.e.. Then there exists a set $E \subseteq \mathbb{R}^N$ satisfying $m_N E = 0$ and*

$$\frac{1}{C_{N-1}} \left\{ \int_{\partial S} \left[\frac{1}{|\zeta - x_0|^{N-2}} (\nabla f)(\zeta) + \frac{(N-2) f(\zeta)}{|\zeta - x_0|^N} (\zeta - x_0) \right] \cdot n(\zeta)\, dm_{N-1}(\zeta) \right.$$
$$\left. - \int_S \frac{\Delta f}{|x - x_0|^{N-2}}\, dm_N \right\} = \begin{cases} f(x_0) & \text{if } x_0 \in S \setminus E, \\ 0 & \text{if } x_0 \in (\mathbb{R}^N \setminus S) \setminus E. \end{cases}$$

Remark: By $\int_{\partial S} (g_1 + g_2)\, dm_{N-1}$, we mean the existence of $\lim \int_{B_{K_j}} (g_1 + g_2)\, dm_{N-1}$. Note that unlike Theorem 219, we cannot imply the existence of both $\lim \int_{B_{K_j}} g_1\, dm_{N-1}$ and $\lim \int_{B_{K_j}} g_2\, dm_{N-1}$. Recall that in Theorem 219 we used Theorem 123 (which stated one of the limits already existed). ∎

Definition: For $\Lambda \in \mathcal{E}' = \mathcal{E}'(\mathbb{R}^N)$, define $\log * \Lambda \in \mathcal{D}'$ by $(\log * \Lambda)\phi = \Lambda(\log * \phi)$.

Remark: We have omitted the usual preliminaries: If $\phi \in \mathcal{D}$, then $\log * \phi \in C^\infty$; if $\phi_j \longrightarrow 0$ in $\mathcal{D}$, then $\log * \phi_j \longrightarrow 0$ in C^∞. ∎

Section 317 Let $f \in C_c$, with $\nabla f \in C$ and $\Delta f \in L^1$. Let K be n.f.p.. By Section 302 and Theorem 305,

$$\chi_K f C_{N-1} = \frac{1}{|x|^{N-2}} * \left(\frac{\partial f}{\partial n}\, dm_{N-1} \,|_{B_K} - \chi_K \Delta f \right) + \left(\frac{\partial}{\partial n} \frac{1}{|x|^{N-2}} \right) * \left(f\, dm_{N-1} \,|_{B_K} \right)$$

in $\mathcal{D}'$. □

Let $\Lambda \in \mathcal{E}'$. For $\phi \in \mathcal{D}$,

$[\log * (\Delta\Lambda)]\,\phi = (\Delta\Lambda)(\log * \phi) = \Lambda\,[\Delta(\log * \phi)] = -\,C_{N-1}\Lambda\phi,$

by Corollary 308. That is, $\log * (\Delta\Lambda) = -\,C_{N-1}\Lambda$.

Hence, $\chi_K f C_{N-1} = -\log * \;[\Delta(\chi_K f)] = -\,\frac{1}{|x|^{N-2}} * [\Delta(\chi_K f)]$.

Similar to Lemma 212, we may use Corollary 308 to show: for $\Lambda \in \mathcal{E}'$, $\log * \Lambda = 0$ in $\mathcal{D}'$ iff $\Lambda = 0$ in $\mathcal{D}'$.

Combining all the above gives:

$$\chi_K \Delta f = \Delta\left(\chi_K f\right) + \frac{\partial f}{\partial n}\, dm_{N-1}\,|_{B_K} + \frac{\partial}{\partial n}\left(f\, dm_{N-1}\,|_{B_K}\right) \quad \text{in} \quad \mathcal{D}',$$

where $\left[\frac{\partial}{\partial n}(f\, dm_{N-1}\,|_{B_K})\right]\phi = -\int\limits_{B_K} f\frac{\partial\phi}{\partial n}\, dm_{N-1}$. ■

Lemma 318 *Let (f_j) be a sequence in L^1_c. If $\log * f_j \longrightarrow 0$ in L^1_{loc}, then $f_j \longrightarrow 0$ in $\mathcal{D}'$.*

Proof. Let $\phi \in \mathcal{D}$. By Corollary 308,

$$\begin{aligned} f_j(\phi) = -\,\frac{1}{C_{N-1}} f_j\,[\log * (\Delta\phi)] \quad &= \quad -\,\frac{1}{C_{N-1}} (\log * f_j)(\Delta\phi) \\ &= \quad -\,\frac{1}{C_{N-1}} \int\limits_{\mathbb{R}^N} (\log * f_j)\Delta\phi\, dm_N \longrightarrow 0. \end{aligned}$$

Definition: Let K have f.p.. Set $A = \{\, f \in C : \nabla f \in C$ and there exists a sequence (h_j) in harm$K$ satisfying: $h_j \rightrightarrows f$ in K and $\left[\frac{\partial}{\partial n}(f - h_j)\right] dm_{N-1}\,|_{B_K} \longrightarrow 0$ in $\mathcal{D}'$ $\}$.

Theorem 319 *Let $f \in C_c$, with $\nabla f \in C$. Let $\frac{N}{2} < p \le \infty$ and K be n.f.p.. Set $\Lambda = \Delta(\chi_K f) + \frac{\partial f}{\partial n}\, dm_{N-1}\,|_{B_K} + \frac{\partial}{\partial n}\left(f\, dm_{N-1}\,|_{B_K}\right)$.*

(a) *If $f \in A$, then $\Lambda = 0$ in $\mathcal{D}'$.*

(b) *If $\Delta f \in L^p$ and $\Lambda = 0$ in $\mathcal{D}'$, then $f \in A$.*

Proof.

(a) Let $\phi \in \mathcal{D}$. Applying the usual approximation scheme to extend Proposition 300 to n.f.p. sets, we get

$$\begin{aligned}
\Lambda\phi &= \lim_j \left[\int_{B_K} \left(\phi \frac{\partial h_j}{\partial n} - h_j \frac{\partial \phi}{\partial n} \right) dm_{N-1} + \int_K h_j \Delta\phi \, dm_N \right] \\
&= -\lim_j \int_K \phi \Delta h_j \, dm_N = 0.
\end{aligned}$$

(b) By Section 317, we see $\Lambda = \chi_K \Delta f$. Hence, $\Delta f = 0$ a.e. in K. By Theorem 311, we have $f \in h(K; \log^1)$. Now use Lemma 318. ■

Chapter IV
"Rational" Approximation on $\mathbb{C}^N$

Abstract

Fix $N > 1$. For a compact set $K \subseteq \mathbb{C}^N$, we will write $H(K)$ to denote those functions analytic in a neighborhood of K and we will let $R(K)$ denote the closure of $H(K)$ in $C(K)$.

We will do our best to extend our Chapter I results to this new setting. Unfortunately, the results will appear weaker. The stumbling block is the following: If $\phi \in \mathcal{D}(\mathbb{C})$, then $\phi \equiv -\frac{1}{\pi}\overline{\partial}\hat{\phi}$ in $\mathbb{C}$. That is, the $\overline{\partial}$-equation is always solvable in $\mathbb{C}$. Let $f = \sum_{j=1}^{N} f_j\, d\overline{z}_j$, where each $f_j \in \mathcal{D}(\mathbb{C}^N)$. The $\overline{\partial}$ - equation is to find u satisfying: $\overline{\partial}u = f$, where $\overline{\partial}u = \sum_{j=1}^{N} \frac{\partial u}{\partial \overline{z}_j}\, d\overline{z}_j$. Since $\left(\overline{\partial} \circ \overline{\partial}\right) u = 0$, the $\overline{\partial}$- equation (in $\mathbb{C}^N$) is not solvable whenever $\overline{\partial} f \neq 0$. ■

Important: We fix $N > 1$ and the unmentioned measure space will be normalized $(2N)$-dimensional Lebesgue measure on $\mathbb{R}^{2N} = \mathbb{C}^N$. (This normalization will be explained below.)

Notation 400 We write $z = (z_1, \cdots, z_N) = (x_1 + iy_1, \cdots, x_N + iy_N) \in \mathbb{C}^N$ and $\frac{\partial}{\partial \overline{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right)$. If $\Omega \subseteq \mathbb{C}^N$ is open and $f : \Omega \longrightarrow \mathbb{C}$, then f is *analytic* in Ω if $\frac{\partial f}{\partial \overline{z}_j} \equiv 0$ in Ω for each $1 \leq j \leq N$.

Let $\mu_1, \cdots, \mu_N \in M = M(\mathbb{C}^N)$. Set $\mu = (\mu_1, \cdots, \mu_N)$ and write $\mu \in M^N$. For $z_0 \in \mathbb{C}^N$, define the Cauchy transform of μ,

$$\hat{\mu}(z_0) = \sum_{j=1}^{N} \int \frac{(\overline{z} - \overline{z}_0)_j}{|z - z_0|^{2N}}\, d\mu_j(z).$$

We will also write

$$\hat{\mu}(z_0) = \int \frac{(d\mu, z - z_0)}{|z - z_0|^{2N}}.$$

[The standard inner product on $\mathbb{C}^N$, being conjugate linear in the second slot, suggests such notation.]

We will find it necessary to use differential forms (see [9, Appendices] or [13, Chapter 16]).

Set

$$\begin{aligned}
w(z) &= dz_1 \wedge \cdots \wedge dz_N, \\
w_j(z) &= dz_1 \wedge \cdots \wedge dz_{j-1} \wedge dz_{j+1} \wedge \cdots \wedge dz_N, \\
\eta(z) &= \sum_{j=1}^{N} (-1)^{j+1} z_j w_j(z), \\
\mathbb{B}_{2N} &= \left\{ z \in \mathbb{C}^N : |z| \equiv \sqrt{\sum_{j=1}^{N} z_j \bar{z}_j} < 1 \right\}, \\
\text{and} \quad d_N &= \int_{\mathbb{B}_{2N}} w(\bar{z}) \wedge w(z).
\end{aligned}$$

We will take $\frac{1}{N\, d_N} w(\bar{z}) \wedge w(z)$ as our normalized measure on $\mathbb{C}^N$, which we shall abbreviate $\frac{1}{N\, d_N} \bar{w} \wedge w$. All L^p-norms, distributions, etc., will be with respect to this measure. Note $w(\bar{z}) \wedge w(z) = (-1)^{\frac{N(N-1)}{2}} (2i)^N \, dm_{2N}$. ∎

Proposition 401 *Let $1 \leq p < \frac{2N}{2N-1}$ and $\mu \in M^N$. Set*

$$g(z_0) = \sum_{j=1}^{N} \int \left(\frac{|(z - z_0)_j|}{|z - z_0|^{2N}} \right)^p d|\mu_j|(z).$$

Then $g \in L^1_{\text{loc}}$ and hence $\hat{\mu} \in L^p_{\text{loc}}$. ∎

Theorem 402 *(Bochner-Martinelli) [9, p. 16] Let $f \in C^1 = C^1(\mathbb{C}^N)$ and let $\Omega \subseteq \mathbb{C}^N$ be open and bounded, with $\partial\Omega \in C^1$. Then*

$$\frac{1}{N\, d_N} \left\{ \int_{\partial\Omega} \frac{f(z)\, \eta(\bar{z} - \bar{z}_0) \wedge w(z)}{|z - z_0|^{2N}} - \int_{\Omega} \frac{(\bar{\partial} f)(z) \wedge \eta(\bar{z} - \bar{z}_0) \wedge w(z)}{|z - z_0|^{2N}} \right\}$$
$$= \begin{cases} f(z_0) & \text{if } z_0 \in \Omega, \\ 0 & \text{if } z_0 \in \mathbb{C}^N \setminus \bar{\Omega}. \end{cases}$$

∎

Definition: Let $S \subseteq \mathbb{C}^N$ be $\bar{w} \wedge w$-measurable. We say S *has f.p.* if there exists $\mu \in M^N$ satisfying $\hat{\mu} = \chi_S$ a.e. $\left([\bar{w} \wedge w] \text{ in } \mathbb{C}^N\right)$.

Proposition 403 *Let $\Omega \subseteq \mathbb{C}^N$ be open and bounded, with $\partial\Omega \in C^1$. Then Ω has f.p..*

Proof. Note $\eta(\bar{z} - \bar{z}_0) \wedge w(z) = \sum_{j=1}^{N} (-1)^{j+1} (\bar{z} - \bar{z}_0)_j \, w_j(\bar{z}) \wedge w(z)$. Set $d\mu_j(z) = \frac{(-1)^{j+1}}{N\, d_N} w_j(\bar{z}) \wedge w(z)\Big|_{\partial\Omega}$ and $\mu = (\mu_1, \cdots, \mu_N)$. Taking $f \equiv 1$ in Theorem 402 gives: $\hat{\mu} = \chi_\Omega$ in $\mathbb{C}^N \setminus (\partial\Omega)$. ∎

Notation: For $f : \mathbb{C}^N \longrightarrow \mathbb{C}$, write $\overline{\nabla} f = \left(\frac{\partial f}{\partial \bar{z}_1}, \cdots, \frac{\partial f}{\partial \bar{z}_N}\right)$. We shall write $\overline{\nabla} f \in L^p$ if each $\frac{\partial f}{\partial \bar{z}_j} \in L^p$. ∎

Regarding $\mathbb{C}^N$ as $\mathbb{R}^{2N}$, we may define the reduced boundary and n.f.p. sets as in Chapter III. Using the usual approximation scheme on Theorem 402 gives:

Theorem 404 *Let $f \in C_c$, with $\overline{\nabla} f \in L^1$. Let $K \subseteq \mathbb{C}^N$ be a compact n.f.p. set. Then there exists $E \subseteq K$ satisfying: $(\bar{w} \wedge w)(E) = 0$ and*

$$\frac{1}{N\, d_N} \left\{ \int_{B_K} \frac{f(z)\, \eta(\bar{z} - \bar{z}_0) \wedge w(z)}{|z - z_0|^{2N}} - \int_K \frac{(\bar{\partial} f)(z) \wedge \eta(\bar{z} - \bar{z}_0) \wedge w(z)}{|z - z_0|^{2N}} \right\} = \begin{cases} f(z_0) & \text{if } z_0 \in K \setminus E, \\ 0 & \text{if } z_0 \in \mathbb{C}^N \setminus \mathrm{K}. \end{cases}$$

∎

Notation: Let $\mu \in M$. For $1 \leq j \leq N$, set $\hat{\mu}_j(z_0) = \int \frac{(\bar{z} - \bar{z}_0)_j}{|z - z_0|^{2N}}\, d\mu(z)$. Set $\vec{\mu} = (\hat{\mu}_1, \cdots, \hat{\mu}_N)$.

Theorem 405 *Let $2N < p \leq \infty$, $\mu \in M$, and $f \in C_c$, with $\overline{\nabla} f \in L^p$. Then*

$$\int f\, d\mu = \frac{1}{N\, d_N} \int_{\mathbb{C}^N} \left[\left(\overline{\nabla} f\right) \cdot \vec{\mu} \right] \bar{w} \wedge w,$$

where we have written $\left(\overline{\nabla} f\right) \cdot \vec{\mu} = \sum\limits_{j=1}^{n} \frac{\partial f}{\partial \bar{z}_j} \hat{\mu}_j$.

Proof. Assuming supp f $\subseteq$ $R\mathbb{B}_{2N}$, we take $K = \overline{R\mathbb{B}_{2N}}$ in Theorem 404:

$$\int f(z)\, d\mu(z) = -\frac{1}{N\, d_N} \int \left(\int_{\mathbb{C}^N} \frac{(\overline{\partial} f)(\zeta) \wedge \eta(\bar{\zeta} - \bar{z}) \wedge w(\zeta)}{|\zeta - z|^{2N}} \right) d\mu(z).$$

Note $(\overline{\partial} f)(\zeta) \wedge \eta(\bar{\zeta} - \bar{z}) \wedge w(\zeta) = \left[\sum\limits_{j=1}^{N} \frac{\partial f}{\partial \bar{z}_j}(\zeta)(\bar{\zeta} - \bar{z})_j \right] w(\bar{\zeta}) \wedge w(\zeta)$.

By Proposition 401, we may use Fubini:

$$\begin{aligned}
\int f(z)\, d\mu(z) &= \frac{1}{N\, d_N} \sum_{j=1}^{N} \int_{\mathbb{C}^N} \frac{\partial f}{\partial \bar{z}_j}(\zeta) \left(\int \frac{(\bar{z} - \bar{\zeta})_j}{|z - \zeta|^{2N}}\, d\mu(z) \right) w(\bar{\zeta}) \wedge w(\zeta) \\
&= \frac{1}{N\, d_N} \int_{\mathbb{C}^N} \left[(\overline{\nabla} f) \cdot \vec{\mu} \right] \bar{w} \wedge w.
\end{aligned}$$

■

Notation: For $f_1, \cdots, f_N : \mathbb{C}^N \longrightarrow \mathbb{C}$, set $\overline{\text{div}}\,(f_1, \cdots, f_N) = \sum\limits_{j=1}^{N} \frac{\partial f_j}{\partial \bar{z}_j}$.

Corollary 406 *Let* $\mu \in M$. *Then* $\overline{\text{div}}\ \vec{\mu} = -\mu$ *in* $\mathcal{D}'$.

Proof. Let $\phi \in \mathcal{D}$. By Theorem 405,

$$\begin{aligned}
\mu(\phi) = \sum_{j=1}^{N} \hat{\mu}_j \left(\frac{\partial \phi}{\partial \bar{z}_j} \right) &= -\left(\sum_{j=1}^{N} \frac{\partial \hat{\mu}_j}{\partial \bar{z}_j} \right) \phi \\
&= -\left(\overline{\text{div}}\ \vec{\mu} \right) \phi.
\end{aligned}$$

■

Lemma 407 *[13, Theorem 16.3.4] Let* $f = \sum\limits_{j=1}^{N} f_j\, d\bar{z}_j$, *where each* $f_j \in C_c^1$. *Set* K = supp f *and assume* $\overline{\partial} f = 0$. *Let* Ω *denote the unbounded component of* $\mathbb{C}^N \setminus$ K.

Then there exists a unique $u \in C_c^1$ *satisfying* $\overline{\partial} u = f$ *and* $u \equiv 0$ *in* Ω. ■

Remarks:

(i) supp f $\equiv \bigcup$ supp f_j

(ii)

$$0 = \overline{\partial} f = \sum_{j=1}^{N}\sum_{\ell=1}^{N} \frac{\partial f_\ell}{\partial \overline{z}_j} d\overline{z}_\ell \wedge d\overline{z}_j \quad \text{iff} \quad \frac{\partial f_j}{\partial \overline{z}_\ell} = \frac{\partial f_\ell}{\partial \overline{z}_j} \quad (1 \le j, \ell \le N).$$

■

Section 408 Let $1 \le p < \frac{2N}{2N-1}$ and $\mu \in M$. Note $\vec{\mu} \in (L^p_{\text{loc}})^N$. Letting $\frac{1}{p} + \frac{1}{q} = 1$, we see that if $g = (g_1, \cdots, g_N) \in (L^q_c)^N$, then $\sum_{j=1}^{N} \int_{\mathbb{C}^N} g_j \hat{\mu}_j (\overline{w} \wedge w) \equiv \int_{\mathbb{C}^N} (g \cdot \vec{\mu}) \overline{w} \wedge w$ converges. We will therefore think of $\vec{\mu}$ as being a linear functional on the vector space $(L^q_c)^N$. Recalling our notational conventions for this chapter, we will write $\vec{\mu}(g) = \frac{1}{N d_N} \int_{\mathbb{C}^N} (g \cdot \vec{\mu}) \overline{w} \wedge w$ for $g \in (L^q_c)^N$.

Theorem 409 *Let $K \subseteq \mathbb{C}^N$ be compact and $\mu \in M(K)$. Set*
$A = \left\{ g = (g_1, \cdots, g_N) \,:\, \text{each } g_j \in C^1_c\left(\mathbb{C}^N \setminus K\right) \text{ and } \frac{\partial g_j}{\partial \overline{z}_\ell} = \frac{\partial g_\ell}{\partial \overline{z}_j} \quad (1 \le j, \ell \le N) \right\}$.
Then $\mu \perp R(K)$ iff $\vec{\mu} \equiv 0$ in A.

Remark: We are using the notation introduced in Section 408.

Proof.

"$\Longrightarrow$" Let $g \in A$. By Lemma 407 (and the remark following it), there exists $v \in C^1_c$ with $\overline{\nabla} v = g$. Note $v \in H(K)$. By Theorem 405, we have $\vec{\mu}(g) = \int v \, d\mu = 0$.

"$\Longleftarrow$" Let $f \in H(K)$. WLOG, assume $\overline{\nabla} f \in A$. By Theorem 405,

$$\int f \, d\mu = \vec{\mu}\left(\overline{\nabla} f\right) = 0.$$

■

We abbreviate the conclusion of Theorem 404.

Section 410 Let f and K be as in Theorem 404. Note

$$\left(\overline{\partial} f\right)(z) \wedge \eta\left(\overline{z}-\overline{z}_0\right) \wedge w(z) = \left[\sum_{j=1}^{N}\left(\overline{z}-\overline{z}_0\right)_j \frac{\partial f}{\partial \overline{z}_j}(z)\right] w(\overline{z}) \wedge w(z).$$

Let $\left[\chi_K \overline{\nabla} f\right]^{\wedge}$ denote the Cauchy transform of $(\mu_1, \cdots, \mu_N)$, where

$$\mu_j = \frac{1}{N\, d_N} \chi_K \frac{\partial f}{\partial \overline{z}_j}\, \overline{w} \wedge w.$$

Using the above calculation, we get

$$\frac{1}{N\, d_N} \int_K \frac{\left(\overline{\partial} f\right)(z) \wedge \eta\left(\overline{z} \wedge \overline{z}_0\right) \wedge w(z)}{|z-z_0|^{2N}} = \left[\chi_K \overline{\nabla} f\right]^{\wedge}(z_0).$$

Set $d\mu_{j,B_K} = \frac{(-1)^{j+1}}{N\, d_N} w_j\left(\overline{z}\right) \wedge w(z)\, |_{B_K}$ and $f\mu_{B_K} = \left(f\mu_{1,B_K}, \cdots, f\mu_{N,B_K}\right)$. Note

$$\frac{1}{N\, d_N} \int_{B_K} \frac{\left(f(z)\eta\left(\overline{z}-\overline{z}_0\right) \wedge w(z)\right)}{|z-z_0|^{2N}} = \left[f\mu_{B_K}\right]^{\wedge}(z_0).$$

With the above notation, we may write the conclusion of Theorem 404 as

$$\left[f\mu_{B_K} - \chi_K \overline{\nabla} f\right]^{\wedge}(z_0) = \begin{cases} f(z_0) & \text{if } z_0 \in K \setminus E, \\ 0 & \text{if } z_0 \in \mathbb{C}^N \setminus \mathrm{K}, \end{cases}$$

where E is as in Theorem 404.

Theorem 411 *Let $f \in C_c$, with $\overline{\nabla} f \in L^1$. Then $f = -\left[\overline{\nabla} f\right]^{\wedge}$ a.e.. If also $\overline{\nabla} f \in L^p$, where $2N < p \leq \infty$, then the above equality holds everywhere.*

Proof. Use Section 410.

As in the proof of Lemma 132, we see $\left[\chi_K \overline{\nabla} f\right]^{\wedge} \in C$. Here we are using $\frac{1}{|z|^{2N-1}} \in L^q$, where $\frac{1}{p} + \frac{1}{q} = 1$. ■

Notation: Let $\Lambda = (\Lambda_1, \cdots, \Lambda_N) \in (\mathcal{D}')^N$. For $\phi = (\phi_1, \cdots, \phi_N) \in \mathcal{D}^N$, set $\Lambda \cdot \phi = \sum_{j=1}^{N} \Lambda_j \phi_j$.

Remarks:

(i) Note $\Lambda \cdot (\ )$ is continuous from $\mathcal{D}^N$ into $\mathbb{C}$.

(ii) Note we can define $\Lambda \cdot (\ )$ if each $\Lambda_j \in \mathcal{E}'$ and each $\phi_j \in C^\infty$. In this case, $\Lambda \cdot (\ )$ will be continuous from $(C^\infty)^N$ into $\mathbb{C}$. ■

Motivation 412 Let $2N < p \leq \infty$ and $g = (g_1, \cdots, g_N) \in (L^p)^N$. Since $\hat{g} \in L^1_{\text{loc}}$, we may think of $\hat{g} \in \mathcal{D}'$.

Let $\phi \in \mathcal{D}$. Then

$$\begin{aligned}
\hat{g}(\phi) &= \frac{1}{N\, d_N} \int_{\mathbb{C}^N} \hat{g}(z)\phi(z) w(\overline{z}) \wedge w(z) \\
&= \frac{1}{N\, d_N} \sum_{j=1}^{N} \int_{\mathbb{C}^N} \left(\frac{1}{N\, d_N} \int_{\mathbb{C}^N} \frac{(\overline{\zeta} - \overline{z})_j g_j(\zeta)}{|\zeta - z|^{2N}} w(\overline{\zeta}) \wedge w(\zeta) \right) \phi(z) w(\overline{z}) \wedge w(z) \\
&= -\frac{1}{N\, d_N} \sum_{j=1}^{N} \int_{\mathbb{C}^N} \left(\frac{1}{N\, d_N} \int_{\mathbb{C}^N} \frac{(\overline{z} - \overline{\zeta})_j}{|z - \zeta|^{2N}} \phi(z) w(\overline{z}) \wedge w(z) \right) g_j(\zeta) w(\overline{\zeta}) \wedge w(\zeta) \\
&= -\frac{1}{N\, d_N} \sum_{j=1}^{N} \int_{\mathbb{C}^N} \hat{\phi}_j(\zeta) g_j(\zeta) w(\overline{\zeta}) \wedge w(\zeta) \\
&= -\frac{1}{N\, d_N} \sum_{j=1}^{N} \int_{\mathbb{C}^N} \left(\vec{\phi} \cdot g \right) \overline{w} \wedge w = -\, g\left(\vec{\phi} \right).
\end{aligned}$$

This motivates the following definition.

Definition: Let $\Lambda = (\Lambda_1, \cdots, \Lambda_N) \in (\mathcal{E}')^N$. Define $\hat{\Lambda} : \mathcal{D} \longrightarrow \mathbb{C}$ by

$$\hat{\Lambda}\phi = -\sum_{j=1}^{N} \Lambda_j \left(\hat{\phi}_j \right) \equiv -\Lambda \cdot \vec{\phi}.$$

Remark: If we use Proposition 144, we see $\hat{\phi}_{\alpha j} \longrightarrow 0$ in C^∞ whenever $\phi_\alpha \longrightarrow 0$ in $\mathcal{D}$. That is, $\hat{\Lambda}$ is a distribution. ■

Motivation 413 Let $2N < p \leq \infty$ and $g \in L^p_c$. Since $\vec{g} \in (L^1)^N$, we may think of $\vec{g} \in (\mathcal{D}')^N$.

Let $\phi = (\phi_1, \cdots, \phi_N) \in \mathcal{D}^N$. Then

$$\begin{aligned}
\vec{g} \cdot \phi &\equiv \frac{1}{N\,d_N} \sum_{j=1}^{N} \int_{\mathbb{C}^N} \hat{g}_j(z)\phi_j(z) w(\overline{z}) \wedge w(z) \\
&= \frac{1}{N\,d_N} \sum_{j=1}^{N} \int_{\mathbb{C}^N} \left(\frac{1}{N\,d_N} \int_{\mathbb{C}^N} \frac{(\overline{\zeta} - \overline{z})_j}{|\zeta - z|^{2N}} g(\zeta) w(\overline{\zeta}) \wedge w(\zeta) \right) \phi_j(z) w(\overline{z}) \wedge w(z) \\
&= -\frac{1}{N\,d_N} \int_{\mathbb{C}^N} \left(\frac{1}{N\,d_N} \sum_{j=1}^{N} \int_{\mathbb{C}^N} \frac{(\overline{z} - \overline{\zeta})_j \phi_j(z)}{|z - \zeta|^{2N}} w(\overline{z}) \wedge w(z) \right) g(\zeta) w(\overline{\zeta}) \wedge w(\zeta) \\
&= -\frac{1}{N\,d_N} \int_{\mathbb{C}^N} \hat{\phi}\, g\, \overline{w} \wedge w = -g\left(\hat{\phi}\right)
\end{aligned}$$

■

Definition: Let $\Lambda \in \mathcal{E}'$. Define $\vec{\Lambda} : \mathcal{D}^N \longrightarrow \mathbb{C}$ as follows:

For $\phi = (\phi_1, \cdots, \phi_N) \in \mathcal{D}^N$, set $\vec{\Lambda} \cdot \phi = -\Lambda\hat{\phi}$.

Remarks:

(i) Using Motivation 413, we see the above definition makes sense.

(ii) If we use Proposition 144 again, we see $\vec{\Lambda} \cdot (\ \ )$ is continuous. ■

Theorem 414 *Let $\Lambda \in \mathcal{E}'$. Then $\left(\overline{\nabla}\Lambda\right)^{\wedge} = -\Lambda$ in $\mathcal{D}'$.*

Proof. Let $\phi \in \mathcal{D}$. By Corollary 406,

$$\begin{aligned}
\left(\overline{\nabla}\Lambda\right)^{\wedge} \phi &= -\left(\overline{\nabla}\Lambda\right) \vec{\phi} = -\sum_{j=1}^{N} \frac{\partial \Lambda}{\partial \overline{z}_j} \hat{\phi}_j \\
&= \Lambda \left(\sum_{j=1}^{N} \frac{\partial \hat{\phi}_j}{\partial \overline{z}_j} \right) = \Lambda\left(\overline{\text{div}}\ \vec{\phi}\right) = -\Lambda\phi.
\end{aligned}$$

■

Theorem 415 *Let $f \in C_c$, with $\overline{\nabla} f \in L^1$. Let $K \subseteq \mathbb{C}^N$ be a compact n.f.p. set. Then*

$$\left[f\mu_{B_K} + \overline{\nabla}(\chi_K f) - \chi_K \overline{\nabla} f\right]^{\wedge} = 0 \quad \text{in} \quad \mathcal{D}'.$$

Remark: Note we are using the notation of Section 410.

Proof. By Theorem 414, we have $\chi_K f = -\left[\overline{\nabla}(\chi_K f)\right]^{\wedge}$ in $\mathcal{D}'$. Now use the last equality from Section 410. ■

Remark: We would like to remove the transform in Theorem 415 and conclude that $f\mu_{B_K} + \overline{\nabla}(\chi_K f) = \chi_K \overline{\nabla} f$ in $\mathcal{D}'$. Unfortunately, we cannot (see this chapter's abstract).

Corollary 416 *Let f and K be as in Theorem 415. Set $\vec{\mathcal{D}} = \left\{\vec{\phi} : \phi \in \mathcal{D}\right\}$, a subspace of $(C^\infty)^N$.*

Then $f\mu_B + \overline{\nabla}(\chi_K f) = \chi_K \overline{\nabla} f$ in $\vec{\mathcal{D}}$.

Proof. For $\Lambda \in (\mathcal{E}')^N$ and $\phi \in \mathcal{D}$, we have $\hat{\Lambda}\phi = -\Lambda\vec{\phi}$, by definition. Now use Theorem 415. ■

Section 417 A good problem is to characterize $\vec{\mathcal{D}}$. That is, fix $\psi = (\psi_1, \cdots, \psi_N) \in (C^\infty)^N$ and ask whether we can solve $\vec{\phi} = \psi$ for $\phi \in \mathcal{D}$.

This appears to be a very difficult problem. Note we are trying to invert a transform. Another trouble spot: We have N equations for the one unknown. But we shall overcome!

By Corollary 406, we see $\overline{\text{div}}\ \vec{\phi} = -\phi$. That is, $\phi = -\overline{\text{div}}\ \psi$ is a solution of $\vec{\phi} = \psi$. Or is it? Note that for $\psi \in (C^\infty)^N$, it does not follow that $\overline{\text{div}}\ \psi \in \mathcal{D}$.

Theorem 418 $\vec{\mathcal{D}} = \left\{\overline{\text{div}}\ \psi : \psi \in (C^\infty)^N \text{ and } \overline{\text{div}}\ \psi \in \mathcal{D}\right\}$. ■

We could, of course, restate Corollary 416 in these terms. Instead, we will continue to use the notation $\vec{\mathcal{D}}$ and ask the reader to keep Theorem 418 in mind while reading these results.

Theorem 419 *Let* $f \in C_c$*, with* $\overline{\nabla} f \in L^1$*. Let* $2N < p \leq \infty$ *and* $K \subseteq \mathbb{C}^N$ *be compact.*

(a) If $\overline{\nabla} f \in L^p$ and if $\overline{\nabla} f = 0$ a.e. in K, then $f\mu_{B_K} + \chi_K \overline{\nabla} f = 0$ in $\vec{\mathcal{D}}$.

(b) If $f \in R(K)$ and if K is n.f.p., then $\chi_K \overline{\nabla} f = 0$ in $\vec{\mathcal{D}}$.

Remark: This theorem appears very weak compared to its Chapter I analogue (Theorem 131). As stated in the abstract, this is the nature of studying such problems in $\mathbb{C}^N$.

Proof.

(a) Use Corollary 416.

(b) Since $f \in R(K)$, there exists a sequence (f_j) in $H(K)$ with $f_j \rightrightarrows f$ in K. By Section 410, we see $(f_j\mu_{B_K})^\wedge = \chi_K f_j$ a.e.. Letting $j \nearrow \infty$, we get $(f\mu_{B_K})^\wedge = \chi_K f$ a.e.. By Section 410, we see $\left(\chi_K \overline{\nabla} f\right)^\wedge = 0$ a.e.. Now use the proof of Corollary 416. ■

We will prove the analogue of Theorem 150, but first a Lemma.

Lemma 420 *Let* $\Lambda \in \mathcal{E}'$ *and* $\phi = (\phi_1, \cdots, \phi_N) \in \mathcal{D}^N$*. Then* $\left(\overline{\nabla}\Lambda\right)\cdot\phi = -\Lambda\left(\overline{\text{div}}\ \phi\right)$.

Proof.

$$\left(\overline{\nabla}\Lambda\right) \cdot \phi = \sum_{j=1}^{N} \frac{\partial \Lambda}{\partial \overline{z}_j}\phi_j = -\Lambda\left(\sum_{j=1}^{N} \frac{\partial \phi_j}{\partial \overline{z}_j}\right) = -\Lambda\left(\overline{\text{div}}\ \phi\right).$$

■

Theorem 421 *Let $K \subseteq \mathbb{C}^N$ be a compact n.f.p. set and let $f \in R(K)$. Then $f\mu_{B_K} + \overline{\nabla} f = 0$ in $\vec{\mathcal{D}}$.*

Remark: From a regularity standpoint, we are only assuming $f \in C(K)$. Although $\chi_K \overline{\nabla} f$ is not necessarily a distribution, we know that if $f \in C_c$, with $\overline{\nabla} f \in L^1$, then by Theorem 419, we get $\chi_K \overline{\nabla} f = 0$ in $\vec{\mathcal{D}}$. Along with Corollary 416, this motivates the conclusion of Theorem 421.

Proof. Let $\phi \in \mathcal{D}$. By Lemma 420,

$$\left(f\mu_{B_K} + \overline{\nabla} f\right) \cdot \vec{\phi} = \left(f\mu_{B_K}\right) \cdot \vec{\phi} \; - f\left(\overline{\text{div}}\; \vec{\phi}\right) \equiv \int f \, d\mu,$$

where

$$\mu = \frac{1}{N\, d_N} \left\{ \left[\sum_{j=1}^{N} (-1)^{j+1} \hat{\phi}_j w_j(\overline{z}) \wedge w(z) \mid_{B_K} \right] - \left(\overline{\text{div}}\; \vec{\phi}\right) \overline{w} \wedge w \mid_K \right\}.$$

Note $\mu \in M(K)$. By Theorem 409, we need only show $\vec{\mu} \equiv 0$ in A, where A is as in Theorem 409.

So let $g \in A$. By Lemma 407, there exists $h \in C_c^1\left(\mathbb{C}^N \setminus \mathrm{K}\right)$ with $\overline{\nabla} h = g$. By Theorem 405,

$$\begin{aligned} \vec{\mu}(g) &= \frac{1}{N\, d_N} \int\limits_{\mathbb{C}^N \setminus \mathrm{K}} (g \cdot \vec{\mu})\, \overline{w} \wedge w = \frac{1}{N\, d_N} \int\limits_{\mathbb{C}^N \setminus \mathrm{K}} \left[\left(\overline{\nabla} h\right) \cdot \vec{\mu}\right] \overline{w} \wedge w \\ &= \int h \, d\mu \end{aligned}$$

Since each $\hat{\phi}_j$ is a convolution,

$$\frac{\partial \hat{\phi}_j}{\partial \overline{z}_j}(z) = \frac{1}{N\, d_N} \int\limits_{\mathbb{C}^N} \frac{\left(\overline{\zeta} - \overline{z}\right)_j}{|\zeta - z|^{2N}} \frac{\partial \phi}{\partial \overline{z}_j}(\zeta) w(\overline{z}) \wedge w(\zeta).$$

Since $\overline{\nabla} h \equiv 0$ in K, Theorem 404 shows: For a.e. ζ,

$$\left(\chi_K h\right)(\zeta) = \frac{1}{N\, d_N} \sum_{j=1}^{N} \int\limits_{B_K} \frac{\left(\overline{z} - \overline{\zeta}\right)_j}{|z - \zeta|^{2N}} h(z) w_j(\overline{z}) \wedge w(z).$$

By Theorem 411 and Fubini,

$$
\begin{aligned}
\int h\,d\mu
&= \frac{1}{N\,d_N}\sum_{j=1}(-1)^{j+1}\int_{B_K}\left(\frac{1}{N\,d_N}\int_{\mathbb{C}^N}\frac{(\overline{z}-\overline{\zeta})_j}{|\zeta-z|^{2N}}\phi(\zeta)w(\overline{\zeta})\wedge w(\zeta)\right)\; h(z)w_j(\overline{z})\wedge w(z)\\
&\qquad -\frac{1}{N\,d_N}\int_{K}\left(\frac{1}{N\,d_N}\sum_{j=1}^{N}\int_{\mathbb{C}^N}\frac{(\overline{\zeta}-\overline{z})_j}{|\zeta-z|^{2N}}\,\frac{\partial\phi}{\partial\overline{z}_j}(\zeta)w(\overline{\zeta})\wedge w(\zeta)\right)\; h(z)w(\overline{z})\wedge w(z)\\
&= -\frac{1}{N\,d_N}\int_{\mathbb{C}^N}\left(\frac{1}{N\,d_N}\sum_{j=1}^{N}(-1)^{j+1}\int_{B_K}\frac{(\overline{z}-\overline{\zeta})_j}{|z-\zeta|^{2N}}\,h(z)w_j(\overline{z})\wedge w(z)\right)\phi(\zeta)w(\overline{\zeta})\wedge w(\zeta)\\
&\qquad +\frac{1}{N\,d_N}\int_{K}\phi(z)h(z)w(\overline{z})\wedge w(z)\\
&= 0.
\end{aligned}
$$

■

Chapter V
The Cauchy Transform

Abstract

We return to the complex plane to discuss properties of the Cauchy transform. Consider the following questions:

(i) Given f, is $\hat{f}$ differentiable? In what sense? What are its partial derivatives?

(ii) Is $f \mapsto \hat{f}$ continuous? Is it compact? ■

Important: Throughout this chapter, the underlying measure space will be Lebesgue area measure on a measurable set $S \subseteq \mathbb{C}$. ■

We begin by looking at the (pointwise) differentiability of $\hat{f}$.

Lemma 500 *[10, p. 155] Let* $r > 0$. *Then*

$$-\frac{1}{\pi}\iint_{r\mathbb{D}} \frac{dA}{z - z_0} = \begin{cases} \overline{z}_0 & \textit{if } |z_0| < r, \\ \frac{r^2}{z_0} & \textit{if } |z_0| > r. \end{cases}$$

Proof. Assume $|z_0| \neq 0, r$.

$$\begin{aligned} \frac{1}{2\pi i}\int_{r\mathbb{T}} \frac{\overline{z}}{z - z_0}\,dz &= \frac{r^2}{2\pi i}\int_{r\mathbb{T}} \frac{dz}{z(z - z_0)} \\ &= \frac{r^2}{2\pi i z_0}\int_{r\mathbb{T}} \left(\frac{1}{z - z_0} - \frac{1}{z}\right) dz = \begin{cases} 0 & \text{if } 0 < |z_0| < r, \\ -\frac{r^2}{z_0} & \text{if } |z_0| > r. \end{cases} \end{aligned}$$

The above also holds for $z_0 = 0$ (by continuity). Now use

$$\frac{1}{2\pi i}\int_{r\mathbb{T}} \frac{\overline{z}}{z - z_0}\,dz - \frac{1}{\pi}\iint_{r\mathbb{D}} \frac{dA}{z - z_0} = \begin{cases} \overline{z}_0 & \text{if } |z_0| < r, \\ 0 & \text{if } |z_0| > r. \end{cases}$$

■

Theorem 501 *Fix* $0 \leq |z_0| < R$ *and* $f \in L^1_c(R\mathbb{D})$.

Set $g(z) = \frac{f(z)-f(z_0)}{z-z_0}\chi_{R\mathbb{D}}(z)$ *and assume* $\hat{g}$ *is continuous at* $z = z_0$.

$$\begin{aligned} \text{Then} \quad (\partial \hat{f})(z_0) &= \iint\limits_{R\mathbb{D}} \frac{f - f(z_0)}{(z-z_0)^2}\, dA \\ \text{and} \quad (\overline{\partial} \hat{f})(z_0) &= -\pi f(z_0). \end{aligned}$$

Proof. Define $h(z) = \hat{f}(z) + \pi f(z_0)\overline{z}$. By Lemma 500, for $|z_0| < |z_0| + |a| < R$,

$$\begin{aligned} \frac{h(z_0+a) - h(z_0)}{a} &= \frac{1}{a}\Bigg\{ \iint\limits_{R\mathbb{D}} \frac{f}{z - z_0 - a}\, dA + \pi f(z_0)\left[-\frac{1}{\pi} \iint\limits_{R\mathbb{D}} \frac{dA}{z - z_0 - a}\right] \\ &- \iint\limits_{R\mathbb{D}} \frac{f}{z - z_0}\, dA - \pi f(z_0)\left[-\frac{1}{\pi}\iint\limits_{R\mathbb{D}} \frac{dA}{z-z_0}\right]\Bigg\} \\ &= \iint\limits_{R\mathbb{D}} \frac{f - f(z_0)}{(z-z_0)(z-z_0-a)}\, dA. \end{aligned}$$

But

$$\frac{h(z_0+a) - h(z_0)}{a} - \iint\limits_{R\mathbb{D}} \frac{f - f(z_0)}{(z-z_0)^2}\, dA$$

$$= \hat{g}(z_0 + a) - \hat{g}(z_0) \longrightarrow 0 \text{ as } a \longrightarrow 0.$$

Hence,

$$(\partial \hat{f})(z_0) = (\partial h)(z_0) = h'(z_0) = \iint\limits_{R\mathbb{D}} \frac{f - f(z_0)}{(z-z_0)^2}\, dA.$$

Since $(\overline{\partial} h)(z_0) = 0$, we see $(\overline{\partial}\hat{f})(z_0) = -\pi f(z_0)$. ∎

Corollary 502 *Let* $0 \leq |z_0| < R$ *and* $f \in L^1_c(R\mathbb{D})$. *Assume there exist* $r, \delta, C > 0$ *satisfying* $|f(z) - f(z_0)| \leq C\,|z - z_0|^\delta$ *whenever* $|z - z_0| < r$.

$$\begin{aligned} \textit{Then} \quad (\partial \hat{f})(z_0) &= \iint\limits_{R\mathbb{D}} \frac{f - f(z_0)}{(z-z_0)^2}\, dA \\ \textit{and} \quad \left(\overline{\partial} \hat{f}\right)(z_0) &= -\pi f(z_0). \end{aligned}$$

Proof. We will show: $\hat{g}$ is continuous at $z = z_0$, where $g(z) = \frac{f(z)-f(z_0)}{z-z_0}\chi_{R\mathbb{D}}(z)$.

Define $\alpha : (r\mathbb{D}) \setminus 0 \longrightarrow \mathbb{C}$ by $\alpha(a) = \hat{g}(z_0 + a) - \hat{g}(z_0)$. By the proof of Theorem 501, we get

$$\alpha(a) = a \iint_{R\mathbb{D}} \frac{dA}{(z - z_0)^2 (z - z_0 - a)}.$$

Writing $R\mathbb{D} = \mathrm{D}(z_0; r) \bigcup [(R\mathbb{D}) \setminus \mathrm{D}(z_0; r)]$ and using the Lip_δ-condition gives

$$|\alpha(a)| \le |a| \left\{ C \iint_{D(z_0;r)} \frac{dA}{|z - z_0|^{2-\delta} |z - z_0 - a|} + \frac{\|f\|_1 + \pi R^2 |f(z_0)|}{r^2 (r - |a|)} \right\}.$$

Set $D_0 = D(z_0; r)$, $D_1 = D\left(z_0; \frac{|a|}{2}\right)$, $D_2 = D\left(z_0 + a; \frac{|a|}{2}\right)$, and $A = \left\{ z \in \mathbb{C} : \frac{|a|}{2} < |z - z_0| < r \right\}$. For $z \in D_0 \setminus (D_1 \bigcup D_2)$, note that $|z - z_0 - a| \ge \frac{1}{3} |z - z_0|$. Writing $D_0 = D_1 \bigcup D_2 \bigcup [D_0 \setminus (D_1 \bigcup D_2)]$ and using $[D_0 \setminus (D_1 \bigcup D_2)] \subseteq A$ gives:

$$\begin{aligned}
&\iint_{D_0} \frac{dA}{|z - z_0|^{2-\delta} |z - z_0 - a|} \\
&\quad \le \frac{2}{|a|} \iint_{D_1} \frac{dA}{|z - z_0|^{2-\delta}} + \left(\frac{2}{|a|}\right)^{2-\delta} \iint_{D_2} \frac{dA}{|z - z_0 - a|} + \frac{1}{3} \iint_A \frac{dA}{|z - z_0|^{3-\delta}} \\
&\quad = 2\pi \left\{ \frac{2}{|a|} \left(\frac{|a|}{2}\right)^{\delta} \frac{1}{\delta} + \left(\frac{2}{|a|}\right)^{2-\delta} \frac{|a|}{2} + \frac{1}{3} \left[\left(\frac{|a|}{2}\right)^{\delta-1} - r^{\delta-1} \right] \right\},
\end{aligned}$$

where we have assumed (WLOG) $0 < \delta < 1$.

Hence $|\alpha(a)| \le C|a| \left(1 + |a|^{\delta-1}\right) \longrightarrow 0$ as $a \longrightarrow 0$. Now use Theorem 501. ■

We have two more results concerning the pointwise differentiability of $\hat{f}$. We will state both hypotheses in the manner of Theorem 501: It will be assumed that a corresponding $\hat{g}$ is continuous at z_0. Keep in mind, however, that if a Lip_δ-condition is satisfied (as in Corollary 502), then $\hat{g}$ is continuous at z_0 (as the proof of Corollary 502 showed).

Theorem 503 *Let $0 \le |z_0| < R$ and $f \in C_c(R\mathbb{D})$, with $\partial f \in L^1(\mathbb{C})$. Set $g(z) = \frac{f(z)-f(z_0)}{z-z_0}\chi_{R\mathbb{D}}(z)$ and assume $\hat{g}$ is continuous at $z = z_0$.*

Then $\left(\partial \hat{f} - \widehat{\partial f}\right)(z_0) = \pi\,(\partial f)\,(z_0)\overline{z}_0$.

Proof. For small enough $\epsilon > 0$, set $\Omega_\epsilon = (R\mathbb{D}) \setminus \mathrm{D}(z_0;\epsilon)$. By Lemma 500, Lemma 201, and Theorem 501,

$$\begin{aligned}
\widehat{\partial f}(z_0) + \pi(\partial f)(z_0)\overline{z}_0 &= \iint\limits_{R\mathbb{D}} \frac{\partial f}{z-z_0}\,dA + \pi(\partial f)(z_0)\left[-\frac{1}{\pi}\iint\limits_{R\mathbb{D}} \frac{dA}{z-z_0}\right] \\
&= \iint\limits_{R\mathbb{D}} \frac{\partial f - (\partial f)(z_0)}{z-z_0}\,dA = \lim_{\epsilon\searrow 0}\iint\limits_{\Omega_\epsilon}\left[\partial\left(\frac{f-f(z_0)}{z-z_0}\right) + \frac{f-f(z_0)}{(z-z_0)^2}\right] dA \\
&= \frac{i}{2}\lim_{\epsilon\searrow 0}\int\limits_{\partial\Omega_\epsilon} \frac{f-f(z_0)}{z-z_0}\,d\overline{z} + \left(\partial\hat{f}\right)(z_0).
\end{aligned}$$

If $z_0 \neq 0$, then

$$\begin{aligned}
\int\limits_{RT} \frac{f-f(z_0)}{z-z_0}\,d\overline{z} &= -f(z_0)\int\limits_{RT}\frac{d\overline{z}}{z-z_0} = R^2 f(z_0)\int\limits_{RT}\frac{dz}{(z-z_0)z^2} \\
&= \frac{R^2 f(z_0)}{z_0^2}\int\limits_{RT}\left(\frac{1}{z-z_0} - \frac{1}{z} - \frac{z_0}{z^2}\right)dz = 0,
\end{aligned}$$

which (clearly) holds for $z_0 = 0$ (also).

By continuity of f at z_0,

$$\lim_{\epsilon\searrow 0}\int\limits_{\partial D(z_0;\epsilon)} \frac{f-f(z_0)}{z-z_0}\,d\overline{z} = 0.$$

Hence,

$$\lim_{\epsilon\searrow 0}\int\limits_{\partial\Omega_\epsilon} \frac{f-f(z_0)}{z-z_0}\,d\overline{z} = 0$$

and we are done. ■

Theorem 504 *Let $f \in L^1_c(\mathbb{C})$, with $\partial f \in L^1(\mathbb{C})$. Let $z_0 \in \mathbb{C}$ and $\delta > 0$. For $\zeta \in D(z_0;\delta)$, set $g_\zeta(z) = \frac{f(z)-f(\zeta)}{z-\zeta}$ and assume $\hat{g}_\zeta(z)$ is continuous at $z = \zeta$. Set*

$h(z) = \frac{\widehat{\partial f}(z) - \widehat{\partial f}(z_0)}{z - z_0}$ *and assume* $\hat{h}$ *is continuous at* $z = z_0$. *Assume* $\left(\overline{\partial}\partial f\right)(z_0)$ *exists. Then* $\left(\overline{\partial}\partial \hat{f}\right)(z_0) = \pi \overline{z}_0 \left(\overline{\partial}\partial f\right)(z_0)$.

Proof. By Theorem 503, we know $\partial \hat{f} = \widehat{\partial f} + \pi \overline{z} \partial f$ in $D(z_0; \delta)$.

By Theorem 501,

$$\begin{aligned}\left(\overline{\partial}\partial \hat{f}\right)(z_0) &= -\pi (\partial f)(z_0) + \pi \left[\overline{z}_0 \left(\overline{\partial}\partial f\right)(z_0) + (\partial f)(z_0)\right] \\ &= \pi \overline{z}_0 \left(\overline{\partial}\partial f\right)(z_0).\end{aligned}$$

■

Theorem 505 *Let* $f \in C_c^2(\mathbb{C})$. *Then* $\iint_{\mathbb{C}} \left|\partial \hat{f}\right|^2 dA = -\pi^2 \left(\overline{\partial} f, zf\right)$, *where* $(\ ,\ )$ *denotes the inner product in* $L^2(\mathbb{C})$.

Proof. Step 1: We show: $\iint_{\mathbb{C}} \overline{\partial}\left(\overline{\hat{f}}\partial \hat{f}\right) dA = 0$.

Assume supp f $\subseteq R_0\mathbb{D}$. For $|z_0| > R_0$, we see $\left|\hat{f}(z_0)\right| \leq \frac{\|f\|_1}{|z_0| - R_0}$.

By Corollary 502,

$$\begin{aligned}\left|\left(\partial \hat{f}\right)(z_0)\right| = \left|\int\int_{(1+|z_0|)\mathbb{D}} \frac{f - f(z_0)}{(z - z_0)^2} dA\right| &\leq 2\|f\|_\infty \int\int_{R_0\mathbb{D}} \frac{dA}{|z - z_0|^2} \\ &\leq \|f\|_\infty \frac{\pi R_0^2}{(|z_0| - R_0)^2}.\end{aligned}$$

By Proposition 129,

$$\iint_{\mathbb{C}} \overline{\partial}\left(\overline{\hat{f}}\partial \hat{f}\right) dA = \lim_{R \nearrow \infty} \iint_{R\mathbb{D}} \overline{\partial}\left(\overline{\hat{f}}\partial \hat{f}\right) dA = \frac{1}{2i} \lim_{R \nearrow \infty} \int_{R\mathbb{T}} \overline{\hat{f}}\partial \hat{f}\, dz.$$

For $R > R_0$,

$$\int_{R\mathbb{T}} \left|\hat{f}\right| \left|\partial \hat{f}\right| d|z| \leq 2\pi R \frac{\|f\|_1}{R - R_0} \|f\|_\infty \frac{\pi R_0^2}{(R - R_0)^2} \longrightarrow 0 \quad \text{as} \quad R \nearrow \infty.$$

□

By Step 1,

$$\begin{aligned}\iint_{\mathbb{C}} \left|\partial \hat{f}\right|^2 dA &= \iint \left(\partial \hat{f}\right) \overline{\partial} \overline{\hat{f}}\, dA \\ &= \iint \left[\overline{\partial}\left(\overline{\hat{f}} \partial \hat{f}\right) - \overline{\hat{f}}\, \overline{\partial}\partial \hat{f}\right] dA = -\iint \overline{\hat{f}}\, \overline{\partial}\partial \hat{f}\, dA.\end{aligned}$$

By Theorem 504, we see $\overline{\partial}\partial \hat{f} = \pi \overline{z} \partial \overline{\partial} f = \pi \partial \left(\overline{z} \overline{\partial} f\right)$. Hence,

$$\begin{aligned}\iint_{\mathbb{C}} \left|\partial \hat{f}\right|^2 dA = -\iint \overline{\hat{f}}\, \overline{\partial}\partial \hat{f} &= -\pi \iint \overline{\hat{f}} \partial \left(\overline{z} \overline{\partial} f\right) dA \\ &= -\pi \iint \left[\partial \left(\overline{\hat{f}} \overline{z} \overline{\partial} f\right) - \left(\partial \overline{\hat{f}}\right) \overline{z} \overline{\partial} f\right] dA.\end{aligned}$$

By Lemma 201, we get $\iint \partial \left(\overline{\hat{f}} \overline{z} \overline{\partial} f\right) dA = 0$.

By Corollary 502, we have $\partial \overline{\hat{f}} = \overline{\overline{\partial} \hat{f}} = -\pi \overline{f}$. Hence,

$$\iint_{\mathbb{C}} \left|\partial \hat{f}\right|^2 dA = -\pi^2 \iint \overline{f} \overline{z} \overline{\partial} f\, dA = -\pi^2 \left(\overline{\partial} f, z f\right).$$

■

Remark: By the proof of Corollary 502, we note $C_c^2(\mathbb{C})$-functions satisfy the hypotheses of Theorem 504. Except for one instance, we could have weakened the hypotheses to $f \in \mathrm{Lip}_\delta(\mathbb{C})$ having compact support, with $\partial f \in \mathrm{Lip}_\delta(\mathbb{C})$. (Such f satisfy the hypotheses of Theorem 504.) The one instance: We used $\overline{\partial}\partial f = \partial \overline{\partial} f$. ■

Instead of looking at derivatives evaluated at a fixed point z_0, we now look at derivatives in the sense of distributions.

Proposition 506 *Let* $\Lambda \in \mathcal{E}'(\mathbb{C})$. *Then* $\widehat{\triangle \Lambda} = -\pi \left(\overline{z} \triangle \Lambda + 4 \partial \Lambda\right)$.

Proof. Let $\phi \in \mathcal{D}(\mathbb{C})$. Note

$$\triangle \left(\overline{z} \phi\right) = 4 \overline{\partial} \partial \left(\overline{z} \phi\right) = 4 \overline{\partial} \left(\overline{z} \partial \phi\right) = 4 \left(\overline{z} \overline{\partial} \partial \phi + \partial \phi\right) = \overline{z} \triangle \phi + 4 \partial \phi.$$

By Theorem 504,

$$\begin{aligned}\widehat{\triangle\Lambda}\phi &= -(\triangle\Lambda)\hat{\phi} = -\Lambda\left(\triangle\hat{\phi}\right) = -\pi\Lambda\left(\overline{z}\triangle\phi\right) \\ &= -\pi\Lambda\left[\triangle\left(\overline{z}\phi\right) - 4\partial\phi\right] = -\pi\left(\overline{z}\triangle\Lambda + 4\partial\Lambda\right)\phi.\end{aligned}$$

■

Notation: For $\phi \in \mathcal{D}(\mathbb{C})$, set $L\phi = \partial\hat{\phi}$.

Remarks:

(a) By Proposition 143, we note $L : \mathcal{D}(\mathbb{C}) \longrightarrow C^\infty(\mathbb{C})$.

(b) By Corollary 502, for $z_0 \in \mathbb{C}$ and for large enough $R > 0$,

$$(L\phi)(z_0) = \iint\limits_{R\mathbb{D}} \frac{\phi - \phi(z_0)}{(z - z_0)^2}\, dA.$$

(c) Using Theorem 505, we see $\|L\phi\|_2^2 = -\,\pi^2\left(\overline{\partial}\phi, z\phi\right)$. ■

Motivation 507 Let $f, \phi \in \mathcal{D}(\mathbb{C})$. Regard f as the distribution Λ_f. By Remark **(a)**, $\Lambda_{Lf} \in \mathcal{D}'(\mathbb{C})$. Assume supp f $\bigcup$ supp $\phi \subseteq$ R$\mathbb{D}$.

Writing $f(z) = [f(z) - f(\zeta)] + f(\zeta)$ gives:

$$\begin{aligned}\Lambda_f(L\phi) &= \iint\limits_{R\mathbb{D}}\left[\iint\limits_{R\mathbb{D}} \frac{\phi(\zeta) - \phi(z)}{(\zeta - z)^2}\, dA(\zeta)\right] dA(z) \\ &= \iint \phi(\zeta)\left[\iint \frac{f(z) - f(\zeta)}{(z - \zeta)^2}\, dA(z)\right] dA(\zeta) \\ &\qquad + \iint \phi(z)\left[\iint \frac{f(\zeta) - f(z)}{(\zeta - z)^2}\, dA(\zeta)\right] dA(z) \\ &\qquad - \iint f(\zeta)\left[\iint \frac{\phi(z) - \phi(\zeta)}{(z - \zeta)^2}\, dA(z)\right] dA(\zeta) \\ &= \Lambda_{Lf}\phi + \Lambda_{Lf}\phi - \Lambda_f(L\phi).\end{aligned}$$

That is, $\Lambda_{Lf}\phi = \Lambda_f(L\phi)$. ■

Definition: For $\Lambda \in \mathcal{E}'(\mathbb{C})$, define $L\Lambda \in \mathcal{D}'(\mathbb{C})$ by $(L\Lambda)\,\phi = \Lambda\,(L\phi)$.

Remarks:

(a) By Proposition 145 and Remark **(a)** (following Proposition 506), $L\Lambda$ *is* a distribution.

(b) By Motivation 507, for $\phi \in \mathcal{D}(\mathbb{C})$, we see $L(\Lambda_\phi) = \Lambda_{L\phi}$ in $\mathcal{D}'(\mathbb{C})$. ■

Proposition 508 *Let* $\Lambda \in \mathcal{E}'(\mathbb{C})$. *Then* $L\Lambda = \widehat{\partial\Lambda}$.

Proof. For $\phi \in \mathcal{D}(\mathbb{C})$,

$$(L\Lambda)\,\phi = \Lambda\,(L\phi) = \Lambda\left(\partial\hat{\phi}\right) = -\,(\partial\Lambda)\,\hat{\phi} = (\partial\Lambda)^\wedge\,\phi.$$

■

Proposition 509 *Let* $\Lambda \in \mathcal{E}'(\mathbb{C})$. *Then* $\partial\hat{\Lambda} = L\Lambda + \pi\overline{z}\partial\Lambda$.

Proof. Let $\phi \in \mathcal{D}(\mathbb{C})$. By Theorem 503,

$$\begin{aligned}\left(\partial\hat{\Lambda}\right)\phi &= -\hat{\Lambda}\,(\partial\phi) = \Lambda\left[(\partial\phi)^\wedge\right] \\ &= \Lambda\left(\partial\hat{\phi} - \pi\overline{z}\partial\phi\right) = \Lambda\,(L\phi) - \pi\Lambda\left[\partial\left(\overline{z}\phi\right)\right] = \left(L\Lambda + \pi\overline{z}\partial\Lambda\right)\phi.\end{aligned}$$

■

Proposition 510 *Let* $\Lambda \in \mathcal{E}'(\mathbb{C})$. *Then* $\triangle\hat{\Lambda} = -4\pi\partial\Lambda$.

Proof. Let $\phi \in \mathcal{D}(\mathbb{C})$. For $\psi \in \mathcal{D}(\mathbb{C})$, Corollary 121 shows $\psi = -\,\frac{1}{\pi}\widehat{\overline{\partial}\psi}$. Hence, $\left(\triangle\hat{\Lambda}\right)\phi = \hat{\Lambda}\,(\triangle\phi) = -\Lambda\left(\widehat{\triangle\phi}\right) = -4\Lambda\left(\overline{\partial}\partial\phi\right)^\wedge = 4\pi\Lambda\,(\partial\phi) = -4\pi\,(\partial\Lambda)\,\phi.$ ■

We now consider properties of $f \mapsto \hat{f}$.

Theorem 511 *Let $S \subseteq \mathbb{C}$ be m_2-measurable, with $m_2 S < \infty$. Let $1 \le p \le \infty$ and $f \in L^p(S)$. Then $\|\hat{f}\|_p \le 2\sqrt{\pi m_2 S}\|f\|_p$.*

Proof. Assume first that $1 \le p < \infty$. For $z \in S$, set $C_z = \iint\limits_S \frac{dA(\zeta)}{|\zeta - z|}$. Choosing $R > 0$ so that $\pi R^2 = m_2 S$, we get $C_z \le \iint\limits_{R\mathbb{D}} \frac{dA}{|\zeta|} = 2\sqrt{\pi m_2 S}$. By Jensen's Inequality,

$$\begin{aligned}
\|\hat{f}\|_p^p &\le \iint\limits_S C_z^p \left(\frac{1}{C_z} \iint\limits_S \left| \frac{f(\zeta)}{\zeta - z} \right| dA(\zeta) \right)^p dA(z) \\
&\le \left(2\sqrt{\pi m_2 S} \right)^{p-1} \iint\limits_S \left(\iint\limits_S \frac{|f(\zeta)|^p}{|\zeta - z|} dA(\zeta) \right) dA(z) \\
&\le \left(2\sqrt{\pi m_2 S} \right)^p \|f\|_p^p.
\end{aligned}$$

For the $p = \infty$ case, note $\|\hat{f}\|_\infty \le \|f\|_\infty \sup\limits_{z \in S} C_z$. ■

Proposition 512 *Let $S \subseteq \mathbb{C}$ be measurable, with $m_2 S < \infty$. Let $1 \le p \le \infty$ and assume $r\mathbb{D} \subseteq$ S. Set*

$$\alpha_p = \begin{cases} \left[\frac{p+2}{2(p+1)}\right]^{\frac{1}{p}} & \text{if } 1 \le p < \infty, \\ 1 & \text{if } p = \infty. \end{cases}$$

Think of $\hat{}: L^p(S) \longrightarrow L^p(S)$. Then $\alpha_p \pi r \le \|\hat{}\| \le 2\sqrt{\pi m_2 S}$.

Proof. By Theorem 511, we note $\|\hat{}\| \le 2\sqrt{\pi m_2 S}$. Set $f = z\chi_{r\mathbb{D}}$. Note

$$\|f\|_p = \begin{cases} \left(\frac{2\pi}{p+2} r^{p+2}\right)^{\frac{1}{p}} & \text{if } 1 \le p < \infty, \\ r & \text{if } p = \infty. \end{cases}$$

For $1 \le p < \infty$,

$$\|\hat{f}\|_p^p \ge \iint\limits_{r\mathbb{D}} \left| \iint\limits_{r\mathbb{D}} \frac{\zeta}{\zeta - z} dA(\zeta) \right|^p dA(z) = \iint\limits_{r\mathbb{D}} \left[\pi\left(r^2 - |z|^2\right)\right]^p dA(z) = \frac{(\pi r^2)^{p+1}}{p+1}$$

$$\text{and hence,} \quad \|\hat{f}\|_p^p \ge \frac{(\pi r)^p}{2} \frac{p+2}{p+1} \|f\|_p^p.$$

For $p = \infty$, we get $\|\hat{f}\|_\infty \ge \left|\hat{f}(0)\right| = \pi r^2 = \pi r \|f\|_\infty$. ■

Section 513 By Corollary 121 and Lemma 134, the differential operator $L_1 = -\frac{1}{\pi}\overline{\partial}$ is the algebraic inverse of $^\wedge$: For $\phi \in \mathcal{D}(\mathbb{C})$, $\phi = -\frac{1}{\pi}\left(\overline{\partial}\phi\right)^\wedge$ and $\overline{\partial}\hat{\phi} = -\pi\phi$. That is,

$$(^\wedge \circ L_1)\phi = \phi = (L_1 \circ {}^\wedge)\phi.$$

Note that $L_1 : \mathcal{D}(\mathbb{C}) \longrightarrow \mathcal{D}(\mathbb{C})$ and $L_1 : C^\infty(\mathbb{C}) \longrightarrow C^\infty(\mathbb{C})$, while $^\wedge : \mathcal{D}(\mathbb{C}) \longrightarrow C^\infty(\mathbb{C})$.

While the inverse of a differential operator is not necessarily compact, the above suggests that in the correct setting, the Cauchy transform operator may be compact (see Theorem 515).

Lemma 514 *Let $R > 0$. Then there exist $C, \delta > 0$ satisfying:*

$$\iint_{R\mathbb{D}} \frac{dA}{|z(z-a)|} \leq C\,|ln|a|| \quad \textit{if} \;\; 0 < |a| < \delta.$$

Remark: The method of proof is that of [10].

Proof. Set $D_1 = \frac{|a|}{2}\mathbb{D}$ and $D_2 = D\left(a; \frac{|a|}{2}\right)$. For $z \in (R\mathbb{D}) \setminus (D_1 \bigcup D_2)$, $|z-a| \geq \frac{1}{3}|z|$ and hence $|z(z-a)| \geq \frac{|z|^2}{3}$. Note $[(R\mathbb{D}) \setminus (D_1 \bigcup D_2)] \subseteq A$ and $\iint_A \frac{dA}{|z|^2} = 2\pi\left[lnR - ln\left(\frac{|a|}{2}\right)\right]$, where $A \equiv \left\{z \in \mathbb{C} : \frac{|a|}{2} < |z| < \mathrm{R}\right\}$. Therefore,

$$\begin{aligned} \iint_{R\mathbb{D}} \frac{dA}{|z(z-a)|} &\leq \frac{2}{|a|}\iint_{D_1} \frac{dA}{|z|} + \frac{2}{|a|}\iint_{D_2} \frac{dA}{|z-a|} + 3\iint_A \frac{dA}{|z|^2} \\ &= 2\pi\left\{2 + 3\left[lnR - ln\left(\frac{|a|}{2}\right)\right]\right\}. \end{aligned}$$

■

Theorem 515 *Let $1 \leq p < \infty$ and $\Omega \subseteq \mathbb{C}$ be open and bounded. Then $^\wedge : L^p(\Omega) \longrightarrow L^p(\Omega)$ is compact.*

Proof. Set $B = \{f \in L^p(\Omega) : \|f\|_p \leq 1\}$. Let $f \in B$ and $a \in \mathbb{C}$. Assume $[\Omega \bigcup (a+\Omega)] \subseteq R\mathbb{D}$. As usual, we set $f \equiv 0$ in $\mathbb{C} \setminus \Omega$.

$$\iint_\Omega \left|\hat{f}(z+a) - \hat{f}(z)\right|^p dA(z)$$
$$= \iint_\Omega \left| \iint_\Omega f(\zeta)\left[\frac{1}{\zeta-(z+a)} - \frac{1}{\zeta - z}\right] dA(\zeta)\right|^p dA(z)$$
$$= |a|^p \iint \left| \iint \frac{f(\zeta)}{[(\zeta - z) - a](\zeta - z)} dA(\zeta)\right|^p dA(z)$$
$$= |a|^p \iint |(h * f)(z)|^p \, dA(z) \leq (|a|\|h\|_1\|f\|_p)^p \leq (|a|\|h\|_1)^p,$$

where we have set $h(\lambda) = \frac{1}{\lambda(\lambda+a)}\chi_{(3R)\mathbb{D}}$. By Lemma 514, $\|h\|_1 \leq C\,|ln|a||$ for $a \in \mathbb{C} \setminus \{0\}$ small enough. Hence,

$$\iint_{R\mathbb{D}} \left|\hat{f}(z+a) - \hat{f}(z)\right|^p dA(z) \leq (C\,|a ln|a||)^p$$

and so

$$\lim_{a \longrightarrow 0} \sup_{f \in B} \iint_\Omega \left|\hat{f}(z+a) - \hat{f}(z)\right|^p dA(z) = 0.$$

By Theorem 2.21 [1], ^ is compact. ■

Corollary 516 *Let Ω be open and bounded. Let $1 \leq p < \infty$ and $g \in L^\infty(\Omega)$. Define C_g on $L^p(\Omega)$ by $C_g f = \widehat{gf}$.*

Then $C_g : L^p(\Omega) \longrightarrow L^p(\Omega)$ is compact.

Proof. $C_g = {}^\wedge \circ M_g$, where $M_g f = gf$. Now use Theorem 515. ■

Remark: By Theorem 511, we have $\|C_g\| \leq 2\sqrt{\pi m_2 \Omega}\,\|g\|_\infty$. ■

Proposition 517 *Let $\Omega \subseteq \mathbb{C}$ be open, with $m_2\Omega < \infty$. Let $g \in L^\infty(\Omega)$. Think of $C_g : L^2(\Omega) \longrightarrow L^2(\Omega)$. Then $C_g^* f = -\overline{g\hat{\overline{f}}}$.*

Remark: By Theorem 511, we know $C_g \in L(L^2(\Omega))$, with $\|C_g\| \le 2\sqrt{\pi m_2\Omega}\|g\|_\infty$. We compute its adjoint C_g^*.

Proof. Let $f \in L^2(\Omega)$ and $\phi \in \mathcal{D}(\Omega)$.

$$\begin{aligned}
\left(\phi, C_g^* f\right) &= (C_g\phi, f) \\
&= \iint\limits_{\Omega} \left[\iint\limits_{\Omega} \frac{(g\phi)(\zeta)}{\zeta - z}\, dA(\zeta)\right] \overline{f(z)}\, dA(z) \\
&= \iint\limits_{\Omega} (g\phi)(\zeta) \left[\iint\limits_{\Omega} \frac{\overline{f(z)}}{\zeta - z}\, dA(z)\right] dA(\zeta) \\
&= -\iint\limits_{\Omega} \left(g\phi\overline{\hat{f}}\right)(\zeta)\, dA(\zeta) = -\left(\phi, \overline{g\hat{\overline{f}}}\right).
\end{aligned}$$

Now use: $\mathcal{D}(\Omega)$ is dense in $L^2(\Omega)$. ∎

Proposition 518 *Let $\Omega \subseteq \mathbb{C}$ be open, with $m_2\Omega < \infty$. Let $p \in [1,2) \cup (2,\infty)$ and $g \in L^\infty(\Omega)$. Then $C_g^* h = g\hat{h}$, where we regard $C_g^* : L^q(\Omega) \longrightarrow L^q(\Omega)$ and where $\frac{1}{p} + \frac{1}{q} = 1$.*

Proof. Fix $h \in L^q(\Omega)$. For $f \in L^p(\Omega)$,

$$\begin{aligned}
\left(C_g^* h\right) f = h\,(C_g f) &= \iint\limits_{\Omega} h(z)\,(C_g f)(z)\, dA(z) \\
&= \iint\limits_{\Omega} h(z) \left[\iint\limits_{\Omega} \frac{(gf)(\zeta)}{\zeta - z}\, dA(\zeta)\right] dA(z) \\
&= \iint\limits_{\Omega} (gf)(\zeta) \left[\iint\limits_{\Omega} \frac{h(z)}{\zeta - z}\, dA(z)\right] dA(\zeta) \\
&= -\iint\limits_{\Omega} \left(gf\hat{h}\right)(\zeta)\, dA(\zeta) = -\left(g\hat{h}\right) f.
\end{aligned}$$

Our use of Fubini's Theorem is justified: If $p > 2$, then $gf \in L^p$; if $p < 2$, then $h \in L^q$, where $q > 2$. ∎

Important: We now restrict our attention to the open unit disc $\mathbb{D}$. In the following, the unmentioned measure space will be $(\mathbb{D}, dA)$. So L^p denotes $L^p(\mathbb{D}, dA)$; C and H denote the functions which are continuous and analytic in $\mathbb{D}$, respectively. ■

Notation: For $1 \le p \le \infty$, set $L^p_a = L^p \bigcap H$. ■

Theorem 519 *Let $g \in L^\infty$ and $f \in L^2$. Then*

$$PC_g f = \pi \sum_{n=0}^{\infty} \frac{1}{\sqrt{(n+1)(n+2)}} (fg, e_{n+1})\, e_n,$$

where P is the orthogonal projection of L^2 onto L^2_a, $\left\{e_n = \sqrt{\frac{n+1}{\pi}} z^n : n \in \mathbb{N}^*\right\}$ is the standard orthonormal basis for L^2_a and $(\ ,\ )$ denotes the inner product in L^2_a.

Proof. Fix $n \in \mathbb{N}^*$.

$$\begin{aligned}(C_g f, z^n) &= \iint_{\mathbb{D}} \left[\iint \frac{(gf)(\zeta)}{\zeta - z}\, dA(\zeta) \right] \overline{z}^n\, dA(z) \\ &= \iint_{\mathbb{D}} \left[\iint_{\mathbb{D}} \frac{\overline{z}^n}{\zeta - z}\, dA(z) \right] (gf)(\zeta)\, dA(\zeta).\end{aligned}$$

Fix $\zeta \in \mathbb{D}$. For $0 < \epsilon < 1 - |\zeta|$, set $\Omega_\epsilon(\zeta) = \{z \in \mathbb{D} : |z - \zeta| > \epsilon\}$ and note

$$\begin{aligned}\iint_{\mathbb{D}} \frac{\overline{z}^n}{\zeta - z}\, dA(z) &= \frac{1}{n+1} \lim_{\epsilon \searrow 0} \iint_{\Omega_\epsilon(\zeta)} \overline{\partial} \left(\frac{\overline{z}^{\,n+1}}{\zeta - z} \right) dA(z) \\ &= \frac{1}{2(n+1)i} \lim_{\epsilon \searrow 0} \int_{\partial \Omega_\epsilon(\zeta)} \frac{\overline{z}^{\,n+1}}{\zeta - z}\, dz.\end{aligned}$$

An application of the Residue Theorem yields

$$\int_{\mathbb{T}} \frac{\overline{z}^{\,n+1}}{\zeta - z}\, dz = -\int_{\mathbb{T}} \frac{1}{z} \frac{\overline{z}^{\,n+1}}{1 - \frac{\zeta}{z}}\, dz = -\sum_{m=0}^{\infty} \int_{\mathbb{T}} \frac{\overline{z}^{\,n+1}}{z} \left(\frac{\zeta}{z} \right)^m dz = -\sum_{m=0}^{\infty} \zeta^m \int_{\mathbb{T}} \frac{dz}{z^{m+n+2}} = 0.$$

For $0 < \epsilon < 1 - |\zeta|$, we note

$$\int_{|\zeta - z| = \epsilon} \frac{\overline{z}^{\,n+1}}{\zeta - z}\, dz = -2\pi i \overline{\zeta}^{\,n+1}.$$

So

$$(C_gf, e_n) = \sqrt{\frac{n+1}{\pi}}(C_gf, z^n) = \sqrt{\frac{\pi}{n+1}} \iint_{\mathbb{D}} \overline{\zeta}^{n+1}(gf)(\zeta)\, dA(\zeta)$$
$$= \sqrt{\frac{\pi}{n+1}}\sqrt{\frac{\pi}{n+2}}(gf, e_{n+1})$$

and hence,

$$PC_gf = \sum_{n=0}^{\infty}(C_gf, e_n)\, e_n = \pi \sum_{n=0}^{\infty} \frac{1}{\sqrt{(n+1)(n+2)}}(gf, e_{n+1})\, e_n.$$

■

It is often of interest to know if either $C_gf \in L^2_a$ or $C_gf \in (L^2_a)^{\perp}$.

Corollary 520 *Set $L^2_{a,0} = \{h \in L^2_a : (h, e_0) = 0\}$. Let $g \in L^{\infty}$ and $f \in L^2$. Then*

$$C_gf \perp L^2_a \text{ iff } gf \perp L^2_{a,0}.$$

Proof. Use Theorem 519. ■

Proposition 521 *Let $B : L^2_a \longrightarrow L^2_a$ denote the backward weighted shift having weight sequence $\left(\frac{\pi}{\sqrt{n(n+1)}}\right)_{n=1}^{\infty}$.*

$$\textit{That is,} \quad Be_n = \begin{cases} \frac{\pi}{\sqrt{n(n+1)}} e_{n-1} & \textit{if } n \geq 1, \\ 0 & \textit{if } n = 0. \end{cases}$$

Let $g \in L^{\infty}$. Then $PC_g = BPM_g$ in $L(L^2, L^2_a)$.

Proof. By Theorem 519, for $f \in L^2$,

$$BPM_gf = B\left[\sum_{n=-1}^{\infty}(gf, e_{n+1})\, e_{n+1}\right]$$
$$= \pi \sum_{n=0}^{\infty} \frac{1}{\sqrt{(n+1)(n+2)}}(gf, e_{n+1})\, e_n = PC_gf.$$

■

Proposition 522 *Define* $C : L^\infty \longrightarrow \mathcal{K}(L^2)$ *and* $M : L^\infty \longrightarrow L(L^2)$ *by* $C(g) = C_g$ *and* $M(g) = M_g$*, respectively. Then* $PC = BPM$ *in* $L(L^\infty, L(L^2, L_a^2))$.

Remark: $\mathcal{K}(L^2)$ denotes the compact (linear) operators from L^2 into L^2

Proof. Use Proposition 521. ∎

Notation: Let $0 < p < \infty$ and H be a Hilbert space.. We let $\mathcal{K}^p(H)$ denote those compact operators K (on H) satisfying the following condition: The sequence of (nonzero) eigenvalues of $\sqrt{K^*K}$ belong to ℓ^p. [See [4, Exercise 2(e), p. 18] for the definition of $\sqrt{K^*K}$.]

Lemma 523 *Let* $B : \ell^2(\mathbb{N}^*) \longrightarrow \ell^2(\mathbb{N}^*)$ *be a backwards weighted shift having weight sequence* $b = (b_n)_{n=1}^\infty$*, with each* $b_n \geq 0$*. Let* $0 < p < \infty$.
Then $B \in \mathcal{K}^p$ *iff* $b \in \ell^p$.

Proof. If we think of B as an infinite matrix, then $\sqrt{B^*B} = \operatorname{diag}[0, b_1, b_2, \cdots]$. Hence the sequence of eigenvalues is (b_n). ∎

Theorem 524 *Let* $g \in L^\infty$ *and* $1 < p < \infty$*. Think of* $PC_g : L_a^2 \longrightarrow L_a^2$*.*
Then $PC_g \in \mathcal{K}^p$.

Proof. Use Proposition 521, Lemma 523, and the following fact:
$\mathcal{K}^p$ is an ideal (in $L(L_a^2)$). ∎

Recall: ([4, Chapter I]) Let (e_n) be an orthonormal basis for a (separable) Hilbert space H. For $T \in L(H)$, define $A_T : \mathcal{K}^1(H) \longrightarrow \mathbb{C}$ by
$A_T(K) = tr(TK) \equiv \sum (TKe_n, e_n)$. By Chapter I, Theorem 4.5 [4], $T \mapsto A_T$ is an isometry from $L(H)$ onto $[\mathcal{K}^1(H)]^*$. Hence (with the above identification), we may think of $L(H) = [\mathcal{K}^1(H)]^*$. ∎

Notation: For a normed vector space X, let $(X^*, w-*)$ denote the weak-star topology on its dual X^*.

Lemma 525 *Let H be a (separable) Hilbert space and let $T_\alpha \longrightarrow 0 \;\; w-*$ in $L(H)$. Let $S \in L(H)$. Then $ST_\alpha \longrightarrow 0 \;\; w-*$ in $L(H)$.*

Proof. Fix sequences (x_n) and (y_n) in H, with $\sum(\|x_n\|^2 + \|y_n\|^2) < \infty$. By Chapter I, Proposition 5.5 [4],

$$\sum_n (ST_\alpha x_n, y_n) = \sum (T_\alpha x_n, S^* y_n) \longrightarrow 0$$

and hence, $ST_\alpha \longrightarrow 0 \;\; w-*$ in $L(H)$.

Theorem 526 *Using the notation of Proposition 522, we have that PC is continuous from $(L^\infty, \;\; w-*)$ into $(L(L_a^2), w-*)$.*

Proof. Let $g_\alpha \longrightarrow 0 \;\; w-*$ in L^∞. Fix sequences (f_n) and (h_n) satisfying :

$\sum \left(\|f_n\|_2^2 + \|h_n\|_2^2\right) < \infty.$

$$\begin{aligned} \sum_n (PM_{g_\alpha} f_n, h_n) &= \sum (g_\alpha f_n, h_n) \\ &= \sum \iint_{\mathbb{D}} g_\alpha f_n \bar{h}_n \, dA = \iint_{\mathbb{D}} g_\alpha \left(\sum f_n \bar{h}_n\right) dA \longrightarrow 0 \end{aligned}$$

since the series is an L^1-function.

So $PM_{g_\alpha} \longrightarrow 0 \;\; w-*$ in $L(L_a^2)$. By Proposition 522 and Lemma 525, $PC_{g_\alpha} = BPM_{g_\alpha} \longrightarrow 0 \;\; w-*$ in $L(L_a^2)$. ■

Section 527 Fix $z_0 \in \mathbb{D}$.

Note $f \mapsto f(z_0)$ is a bounded linear functional on L_a^2. Let $k_{z_0} \in L_a^2$ represent this functional: For $f \in L_a^2$, we have $f(z_0) = (f, k_{z_0})$. Note $k_{z_0}(z) = \frac{1}{\pi}\frac{1}{(1-\bar{z}_0 z)^2}$.

Define $\tau_{z_0} : L^\infty \longrightarrow \mathbb{C}$ by $\tau_{z_0}(g) = (PC_g e_0)(z_0)$. By Theorem 511,

$$|\tau_{z_0}(g)| = |(C_g e_0, k_{z_0})| \leq \|C_g e_0\|_2 \|k_{z_0}\|_2 \leq 2\pi \|g\|_\infty \|k_{z_0}\|_2.$$

Hence, $\tau_{z_0} \in (L^\infty)^*$, with $\|\tau_{z_0}\| \leq 2\pi \|k_{z_0}\|_2$.

Set $X = \mathcal{K}^1(L^2_a)$ and $Y = L^1$. By Theorem 526, $PC : (Y^*, w - *) \longrightarrow (X^*, w - *)$ is continuous. Hence, there exists $S \in L(X, Y)$ satisfying $S^* = PC$ (see Exercise 6, Chapter 4 [11]). We have the following:

$$\mathcal{K}^1\left(L^2_a\right) \xrightarrow{S} L^1$$

$$L\left(L^2_a\right) \xleftarrow{S^* = PC} L^\infty$$

Define $\lambda_{z_0} \in \mathcal{K}^1(L^2_a)$ by $\lambda_{z_0}(f) = f(z_0)e_0$. Let $J_1 : \mathcal{K}^1(L^2_a) \longrightarrow [L(L^2_a)]^*$ and $J_2 : L^1 \longrightarrow (L^\infty)^*$ denote the respective canonical isometries (see Section 4.5 [11]).

■

Theorem 528 *$J_2 S \lambda_{z_0} = \tau_{z_0}$ in $(L^\infty)^*$, where S is as in Section 527.*

Proof. Fix $g \in L^\infty$.

$[(PC)^* J_1 \lambda_{z_0}] g = (J_1 \lambda_{z_0})(PC_g) = (PC_g)\lambda_{z_0} = tr(\lambda_{z_0} PC_g) = \sum (\lambda_{z_0} PC_g e_n, e_n)$ $= \sum (PC_g e_n, k_{z_0})(e_0, e_n) = (PC_g e_0, k_{z_0})(e_0, e_0) = \tau_{z_0}(g)$.

So $\tau_{z_0} = (PC)^* J_1 \lambda_{z_0} = S^{**} J_1 \lambda_{z_0} = J_2 S \lambda_{z_0}$ in $(L^\infty)^*$. ■

We simplify the formula of Theorem 519 by eliminating the infinite series.

Proposition 529 *Let $g \in L^\infty$ and $f \in L^2$. Let $z_0 \in \mathbb{D}$ and set $L_{z_0}(z) = \frac{z}{1 - \bar{z}_0 z} \in H^\infty$. Then $(PC_g f)(z_0) = (gf, L_{z_0})$.*

Proof. Note

$$\begin{aligned} K_{z_0}(z) &= \frac{1}{\sqrt{\pi}} \sum_{n=0}^{\infty} \sqrt{n+1}\ \ \overline{z}_0{}^n e_n \quad \text{and} \\ L_{z_0}(z) &= \sqrt{\pi} \sum_{n=0}^{\infty} \frac{\overline{z}_0{}^n}{\sqrt{n+2}} e_{n+1}. \end{aligned}$$

Hence, by Theorem 519,

$$(PC_g f)(z_0) = (C_g f, K_{z_0}) = \sqrt{\pi} \sum_{n=0}^{\infty} \frac{z_0^n}{\sqrt{n+2}} (gf, e_{n+1}) = (gf, L_{z_0}).$$

■

Proposition 530 *Define T on L^2 by $(Tf)(z) = (f, L_z)$. Then $T : L^2 \longrightarrow L_a^2$ is continuous, with $\|T\| = \frac{\pi}{\sqrt{2}}$.*

Proof. Fix $f \in L^2$. Set $Pf = \sum_{n=0}^{\infty} a_n e_n$. Since

$$L_z = \sqrt{\pi} \sum_{n=1}^{\infty} \frac{\overline{z}^{\,n-1}}{\sqrt{n+1}} e_n,$$

we have

$$(Tf)(z) = (Pf, L_z) = \pi \sum_{n=1}^{\infty} \frac{a_n}{\sqrt{n(n+1)}} e_{n-1}(z),$$

which shows $Tf \in L_a^1$.

Note

$$\|Tf\|_2^2 = \pi^2 \sum_{n=1}^{\infty} \frac{|a_n|^2}{n(n+1)} \leq \frac{\pi^2}{2} \sum_{n=1}^{\infty} |a_n|^2 \leq \frac{\pi^2}{2} \|Pf\|_2^2 \leq \frac{\pi^2}{2} \|f\|_2^2.$$

So $\|T\| \leq \frac{\pi}{\sqrt{2}}$.

Since $Te_1 = \frac{\pi}{\sqrt{2}} e_0$, we see $\|T\| = \frac{\pi}{\sqrt{2}}$. ■

Section 531 Fix $g \in L^\infty$. By Proposition 529 and Proposition 530, $PC_g = TM_g$ in $L(L^2, L_a^2)$. Using the notation of Proposition 522,

$PC = TM$ in $L(L^\infty, L(L^2, L_a^2))$. By Proposition 530, $\|PC\| = \|TM\| \leq \|T\| = \frac{\pi}{\sqrt{2}}$. The formula for Tf in the proof of Proposition 530 shows: $T = BP$ in $L(L^2, L_a^2)$.

■

Chapter VI
The T_ϕ Operator

Abstract

For $\phi \in C_c^1(\mathbb{C})$, set $T_\phi f = \phi f + \frac{1}{\pi}\left(f\overline{\partial}\phi\right)^\wedge$, where we assume f is such that the right-hand side makes sense. T_ϕ is an essential tool for studying uniform algebras (see [6]).

The following (Theorem 604) is our most useful result concerning the T_ϕ operator: let $\Omega \subseteq \mathbb{C}$ be open and $f \in L_a^1(\Omega)$. Then for all $z_0 \in \mathbb{C}$,

$$(T_\phi f)(z_0) = \frac{1}{2\pi i}\int_{\partial\Omega} \frac{\phi f}{z - z_0}\, dz.$$

■

Important: We will again let $S \subseteq \mathbb{C}$ denote an m_2-measurable set; m_2 is the measure under discussion. ■

Definition: For $\phi \in C^\infty(\mathbb{C})$, define $T_\phi : \mathcal{D}'(\mathbb{C}) \longrightarrow \mathcal{D}'(\mathbb{C})$ by $T_\phi \Lambda = \phi\Lambda + \frac{1}{\pi}\left(\left(\overline{\partial}\phi\right)\Lambda\right)^\wedge$.

Remark: We have stated the above definition in only one of its possible forms. For example:

Fix $n \in N$. If $K \subseteq \mathbb{C}$ is compact and if $\tau \in \mathcal{D}(\mathbb{C})$, then by Theorem 511,

$$\max\left\{\left|\frac{\partial^{j+\ell}\,\hat{\tau}}{\partial x^j \partial y^\ell}(z)\right| : j+\ell \le n-1 \text{ and } z \in K\right\} \le$$

$$2\sqrt{\pi m_2 K}\max\left\{\left|\frac{\partial^{j+\ell}\,\tau}{\partial x^j \partial y^\ell}(z)\right| : j+\ell \le n-1 \text{ and } z \in K\right\}.$$

So if $\phi \in C_c^n(\mathbb{C})$, then we may define T_ϕ on $\mathcal{D}'_{n-1}(\Omega)$, as Leibniz's Rule and the above inequality shows. ■

Proposition 600 *Let $\phi \in \mathcal{D}(\mathbb{C})$. Then $T_\phi : \mathcal{D}'(\mathbb{C}) \longrightarrow \mathcal{D}'(\mathbb{C})$ is continuous.*

Proof. Let $\Lambda_\alpha \longrightarrow 0$ in $\mathcal{D}'(\mathbb{C})$. Let $\psi \in \mathcal{D}(\mathbb{C})$. By Proposition 143, $\left(\overline{\partial}\phi\right)\hat{\psi} \in \mathcal{D}(\mathbb{C})$. Hence $(T_\phi \Lambda_\alpha)\psi = \Lambda_\alpha(\phi\psi) - \frac{1}{\pi}\Lambda_\alpha\left[\left(\overline{\partial}\phi\right)\hat{\psi}\right] \longrightarrow 0$. ■

Remark: Using the same proof, we see: If $\phi \in C^\infty(\mathbb{C})$, then $T_\phi : \mathcal{E}'(\mathbb{C}) \longrightarrow \mathcal{D}'(\mathbb{C})$ is continuous. ■

Definition: For (m_2)-measurable $S \subseteq \mathbb{C}$, set $(H\mathcal{D}')(S) = \{\Lambda :$there exists an open set $\Omega \subseteq \mathbb{C}$ with $S \subseteq \Omega$, $\Lambda \in \mathcal{D}'(\Omega)$, and $\overline{\partial}\Lambda = 0$ in $\mathcal{D}'(\Omega)\}$.

Proposition 601 *Let $\phi \in \mathcal{D}(\mathbb{C})$ and $K \subseteq \mathbb{C}$ be compact. If $\Lambda \in (H\mathcal{D}')(K)$, then $T_\phi\Lambda \in (H\mathcal{D}')(K)$.*

Remark: During the proof, we must avoid the problems apparent in the above statement.

Proof. Assume $\Omega \subseteq \mathbb{C}$, K $\subseteq \Omega$, $\Lambda \in \mathcal{D}'(\Omega)$, and $\overline{\partial}\Lambda = 0$. Choose a bounded open set $\Omega_1 \subseteq \mathbb{C}$ satisfying: $K \subseteq \Omega_1 \subseteq \overline{\Omega_1} \subseteq \Omega$. Choose $\psi \in \mathcal{D}(\Omega)$, with $\psi \equiv 1$ in $\overline{\Omega_1}$. Then $\psi\Lambda \in \mathcal{D}'(\mathbb{C})$, $\psi\Lambda = \Lambda$ in $\mathcal{D}'(\Omega_1)$, and $\overline{\partial}(\psi\Lambda) = 0$ in $\mathcal{D}'(\Omega_1)$. We show $\overline{\partial}[T_\phi(\psi\Lambda)] = 0$ in $\mathcal{D}'(\Omega_1)$. [Note $T_\phi(\psi\Lambda) \in \mathcal{D}'(\mathbb{C})$.]

Let $\tau \in \mathcal{D}(\Omega_1)$. Note $\phi\tau \in \mathcal{D}(\Omega_1)$.

$$\begin{aligned}
\left\{\overline{\partial}[T_\phi(\psi\Lambda)]\right\}\tau &= -\left[\phi\psi\Lambda + \frac{1}{\pi}\left(\left(\overline{\partial}\phi\right)\psi\Lambda\right)^\wedge\right]\left(\overline{\partial}\tau\right) \\
&= -(\psi\Lambda)\left[\phi\overline{\partial}\tau - \frac{1}{\pi}\left(\overline{\partial}\phi\right)\left(\overline{\partial}\tau\right)^\wedge\right] \\
&= (\psi\Lambda)\left[\phi\overline{\partial}\tau + \left(\overline{\partial}\phi\right)\tau\right] = -(\psi\Lambda)\left[\overline{\partial}(\phi\tau)\right] \\
&= \left[\overline{\partial}(\psi\Lambda)\right](\phi\tau) = 0.
\end{aligned}$$

■

Proposition 602 *Let $\mathcal{D}_0'(\mathbb{C})$ denote the distributions of order zero (see page 15 of the "Preliminary" Section). Let $\phi \in C_c^1(\mathbb{C})$. Then $T_\phi : \mathcal{D}_0'(\mathbb{C}) \longrightarrow \mathcal{D}_0'(\mathbb{C})$.*

Proof. Let $\Lambda \in \mathcal{D}_0'(\mathbb{C})$. Clearly $\phi\Lambda \in \mathcal{D}_0'(\mathbb{C})$. For $\psi \in \mathcal{D}(\mathbb{C})$, $\left(\left(\overline{\partial}\phi\right)\Lambda\right)^\wedge \psi = -\Lambda\left[\left(\overline{\partial}\phi\right)\hat{\psi}\right]$ and hence, $\left(\left(\overline{\partial}\phi\right)\Lambda\right)^\wedge \in \mathcal{D}_0'(\mathbb{C})$ by Theorem 511. ■

Proposition 603 *Let $1 \leq p \leq \infty$ and $\phi \in C_c^1(\mathbb{C})$. Let $S \subseteq \mathbb{C}$ be measurable, with $m_2 S < \infty$. Then $T_\phi : L^p(S) \longrightarrow L^p(S)$ is bounded.*

Remark: See the remark preceding Proposition 600.

Proof. By Theorem 511,

$$\|T_\phi\| \leq \|M_\phi\| + \|^\wedge \circ M_{\overline{\partial}\phi}\| \leq \|\phi\|_\infty + 2\sqrt{\pi m_2 S}\ \|\overline{\partial}\phi\|_\infty.$$

■

Notation: Let $\Omega \subseteq \mathbb{C}$ be open and $1 \leq p \leq \infty$. Set $L_a^p(\Omega) = (L^p \bigcap H)(\Omega)$. Such spaces are called Bergman spaces.

Theorem 604 *Let $\phi \in C_c^1(\mathbb{C})$. Let $\Omega \subseteq \mathbb{C}$ be open and $f \in L_a^1(\Omega)$. Then for all $z_0 \in \mathbb{C}$,*

$$(T_\phi f)(z_0) = \frac{1}{2\pi i}\int_{\partial\Omega} \frac{\phi f}{z - z_0}\, dz.$$

Remark: We set $f \equiv 0$ in $\mathbb{C} \setminus \Omega$. Again we warn: The integral must be interpreted as a limit (see Section 122).

In many applications (see [6]), $\phi \in C_c^1(D(\xi;\delta))$ and $\phi \equiv 1$ in $D(\xi; 2\delta)$, where $\delta > 0$ is small. The only interesting case is $\xi \in \partial\Omega$.

Proof. By Proposition 126, let $K_n \longrightarrow \Omega$ inner interior f.p.a.. By the proof (of Proposition 126), we may assume each $K_n \subseteq \text{int}K_{n+1}$ and $\chi_{\text{int}K_n} \longrightarrow \chi_\Omega$ everywhere.

Fix n. Note K_n is a compact subset of Ω; with n fixed, we may extend $f|_{K_n}$ to be in $\mathcal{D}(\Omega)$ while retaining the (original) values of f in a neighborhood of K_n. By Lemma 106, we may write $(\phi f, K_n, E_n)$. Since $\phi f \in C^1(\mathbb{C})$, we see $E_n \subseteq \partial K_n$.

Let $z_0 \in \Omega$. Assume $z_0 \in \text{int}K_n$. By Lemma 106,

$$\frac{1}{\pi}\iint_{K_n} \frac{f\overline{\partial}\phi}{z - z_0}\, dA = \frac{1}{\pi}\iint_{K_n} \frac{\overline{\partial}(f\phi)}{z - z_0}\, dA = \frac{1}{2\pi i}\int_{B_{K_n}} \frac{\overline{\partial} f}{z - z_0}\, dz - (\phi f)(z_0),$$

hence,

$$\begin{aligned}
(T_\phi f)(z_0) &= (\phi f)(z_0)\, \frac{1}{\pi}\iint_{\Omega} \frac{f\overline{\partial}\phi}{z - z_0}\, dA \\
&= (\phi f)(z_0) + \frac{1}{\pi}\lim \iint_{K_n} \frac{f\overline{\partial}\phi}{z - z_0}\, dA \\
&= \frac{1}{2\pi i}\lim \int_{B_{K_n}} \frac{\phi f}{z - z_0}\, dz = \frac{1}{2\pi i}\int_{\partial\Omega} \frac{\phi f}{z - z_0}\, dz.
\end{aligned}$$

For $z_0 \in \mathbb{C} \setminus \Omega$,

$$\begin{aligned}
\frac{1}{\pi}\iint_{\Omega} \frac{f\overline{\partial}\phi}{z - z_0}\, dA &= \frac{1}{\pi}\lim \iint_{K_n} \overline{\partial}\left(\frac{\phi f}{z - z_0}\right) dA \\
&= \frac{1}{2\pi i}\lim \int_{B_{K_n}} \frac{\phi f}{z - z_0}\, dz = \frac{1}{2\pi i}\int_{\partial\Omega} \frac{\phi f}{z - z_0}\, dz.
\end{aligned}$$

Since $(\phi f)(z_0) = 0$, we are done. ∎

Important: We again restrict our attention to $\mathbb{D}$. So again, L^p, C and H denote $L^p(\mathbb{D}, \text{dA})$, $C(\mathbb{D})$, and $H(\mathbb{D})$, respectively.

Notation: For $f : \mathbb{D} \longrightarrow \mathbb{C}$ and $n \in \mathbb{Z}$, define $\hat{f}(n) = \frac{1}{2\pi i}\lim_{r \nearrow 1} \int_{r\mathbf{T}} f(z)\overline{z}^{n+1}\, dz$ (provided the limit exists), the n^{th} Fourier coefficient of f.

Remark: It is unfortunate that $\hat{f}(n)$ could be used to denote the Cauchy transform of the function f evaluated at $z_0 = n$. We will avoid such confusion: For $n \in \mathbb{Z}$, $\hat{f}(n)$ will denote the n^{th} Fourier coefficient of f.

Proposition 605 *Let $f \in C$, with $\overline{\partial} f \in L^1$. Let $n \in \mathbb{Z}$. Then $\hat{f}(n)$ exists.*

Proof. Fix $0 < r < 1$. WLOG, we may assume $f \in C_c = C_c(\mathbb{D})$ while retaining the values of f on $r_1\mathbb{D}$, where $r < r_1 < 1$. So $\frac{1}{2i} \int\limits_{r\mathbf{T}} f(z)\overline{z}^{n+1}\, dz = \iint\limits_{r\mathbb{D}} \overline{\partial}(\overline{z}^{n+1} f)\, dA$.

Letting $r \nearrow 1$ gives: $\hat{f}(n) = \pi \iint\limits_{\mathbb{D}} \overline{\partial}(\overline{z}^{n+1} f)\, dA$.

Theorem 606 *Let $\phi \in C_c^1(\mathbb{C})$ and $f \in L_a^1$. Then*
$T_\phi f = \sum\limits_{n=0}^{\infty} \widehat{\phi f}(n) z^n$, the Taylor series expansion of $T_\phi f$ in $\mathbb{D}$.

Proof. Fix $z_0 \in \mathbb{D}$. Let $|z_0| < r < 1$. Note

$$\begin{aligned}
\frac{1}{2\pi i} \int\limits_{r\mathbf{T}} \frac{\phi f}{z - z_0}\, dz &= \frac{1}{2\pi i} \int\limits_{r\mathbf{T}} \frac{\phi f}{z(1 - \frac{z_0}{z})}\, dz = \frac{1}{2\pi i} \sum_{n=0}^{\infty} z_0^n \int\limits_{r\mathbf{T}} \frac{\phi f}{z^{n+1}}\, dz \\
&= \frac{1}{2\pi i} \sum_{n=0}^{\infty} \frac{z_0^n}{r^{2(n+1)}} \int\limits_{r\mathbf{T}} \overline{z}^{n+1} \phi f\, dz \\
&= \frac{1}{\pi} \sum_{n=0}^{\infty} \frac{z_0^n}{r^{2(n+1)}} \iint\limits_{r\mathbb{D}} \left[(n+1)\,\overline{z}^n \phi f + \overline{z}^{n+1} \overline{\partial}(\phi f)\right] dA
\end{aligned}$$

and

$$\sum_{n=0}^{\infty} \left| \frac{z_0^n}{r^{2(n+1)}} \int\limits_{r\mathbf{T}} \overline{z}^{n+1} \phi f\, dz \right| \leq 2 \sum_{n=0}^{\infty} \left[\frac{|z_0|^n}{r^{n+2}} (n+1)\, \|\phi f\|_1 + \frac{|z_0|^n}{r^{n+1}} \|\overline{\partial}(\phi f)\|_1 \right] < \infty.$$

Let $r \nearrow 1$. By the Lebesgue Dominated Convergence Theorem and Theorem 604, we get $(T_\phi f)(z_0) = \sum\limits_{n=0}^{\infty} z_0^n \widehat{\phi f}(n)$. ■

Corollary 607 *Let $1 \leq p \leq \infty$ and $\phi \in C_c^1(\mathbb{C})$. Think of $T_\phi \in L^1(L_a^p)$. Let $\lambda \in \mathbb{C}$. Then $\ker(T_\phi - \lambda) = \left\{ f \in L_a^p : ((\phi - \lambda) f)^\wedge (n) = 0 \quad (n \in \mathbb{N}^*) \right\}$.*

Remark: Let $f \in L^p_a$. Thinking of f as being the distribution Λ_f, we see $\Lambda_f \in (H\mathcal{D}')(K)$, for all compact sets $K \subseteq \mathbb{D}$. By Proposition 601, for all such K, $T_\phi \Lambda_f \in (H\mathcal{D}')(K)$. That is, $T_\phi : L^p_a \longrightarrow L^p_a$. By Proposition 603, we see T_ϕ is bounded.

That is, T_ϕ *is* an element of $L(L^p_a)$.

Proof. Let $f \in L^p_a$. Choose $\phi_\lambda \in C^1_c(\mathbb{C})$ satisfying: $\phi_\lambda = \phi - \lambda$ in $\mathbb{D}$. For $n \in \mathbb{N}^*$,

$$\begin{aligned}
\widehat{\phi_\lambda f}(n) &= \lim_{r \nearrow 1} \frac{1}{2\pi i} \int_{r\mathbf{T}} \overline{z}^{n+1} \phi_\lambda f \, dz \\
&= \lim_{r \nearrow 1} \frac{1}{2\pi i} \int_{r\mathbf{T}} \overline{z}^{n+1} (\phi - \lambda) f \, dz \\
&= [(\phi - \lambda) f]^\wedge (n).
\end{aligned}$$

By Theorem 606,

$$(T_\phi - \lambda) f = T_{\phi_\lambda} f = \sum_{n=0}^{\infty} z^n (\phi_\lambda f)^\wedge (n) = \sum_{n=0}^{\infty} z^n ((\phi - \lambda) f)^\wedge (n).$$

■

Proposition 608 *Let* $1 \le p < \infty$ *and* $\phi \in C^1_c(\mathbb{C})$. *For* $S \subseteq L^q$, *write* $S^\perp = \left\{ f \in L^p_a : \iint_{\mathbb{D}} fg \, dA = 0 \quad (g \in S) \right\}$. *Think of* $T_\phi \in L(L^p_a)$. *Let* $\lambda \in \mathbb{C}$. *Then*

$$Ker\ (T_\phi - \lambda) = \bigcap_{n=1}^{\infty} \left\{ \overline{\partial} [\overline{z}^n (\phi - \lambda)] \right\}^\perp.$$

Proof. Let $f \in L^p_a$ and $n \in \mathbb{N}^*$. Then

$$\begin{aligned}
((\phi - \lambda) f)^\wedge (n) &= \lim_{r \nearrow 1} \frac{1}{2\pi i} \int_{r\mathbf{T}} \overline{z}^{n+1} ((\phi - \lambda) f \, dz \\
&= \frac{1}{\pi} \iint_{\mathbb{D}} f \overline{\partial} \left[\overline{z}^{n+1} (\phi - \lambda) \right] dA.
\end{aligned}$$

Now use Corollary 607. ■

Proposition 609 *Let $\phi \in C_c^1$ and $\lambda \in \mathbb{C}$. Think of $T_\phi \in L(L_a^2)$. Then $Ker\,(T_\phi - \lambda) = \{f \in L_a^2 : P\partial[(\phi-\lambda)f] = 0$ and $[(\phi-\lambda)f]^\wedge(0) = 0\}$.*

Proof. Let $f \in L_a^2$ and $n \in \mathbb{N}^*$.

$$\begin{aligned}(z^n, \partial[(\phi-\lambda)f]) &= \iint_{\mathbb{D}} z^n \overline{\partial}\,\overline{[(\phi-\lambda)f]}\,dA \\ &= \lim_{r\nearrow 1} \iint_{r\mathbb{D}} \overline{\partial}\left[z^n\overline{(\phi-\lambda)f}\right] dA = \frac{1}{2i}\lim_{r\nearrow 1}\int_{r\mathbf{T}} z^n\overline{(\phi-\lambda)f}\,dz \\ &= \frac{1}{2i}\lim_{r\nearrow 1} r^{2n}\int_{r\mathbf{T}} \overline{z}^{\,-n}\overline{(\phi-\lambda)f}\,dz = \pi\left(\overline{(\phi-\lambda)f}\right)^\wedge[-(n+1)] \\ &= \pi\overline{[(\phi-\lambda)f]^\wedge(n+1)}.\end{aligned}$$

Now use Corollary 607. ■

Theorem 610 *Let $\phi \in C_c^1(\mathbb{C})$ and $\lambda \in \mathbb{C}$, with $\frac{1}{\phi-\lambda} \in C(\overline{\mathbb{D}})$. Think of $T_\phi \in L(L_a^2)$. TFAE:*

(1) *$Ker\,(T_\phi - \lambda) \neq (0)$*

(2) *There exists $g \oplus h \in L_a^2 \oplus (L_a^2)^\perp$ satisfying:*

(a) $0 \not\equiv \left(\frac{1}{\phi-\lambda}\right)\overline{\left(g - \frac{1}{\pi}\hat{\overline{h}}\right)} \in L_a^2$ *and*

(b) $\left(g - \frac{1}{\pi}\hat{\overline{h}}\right)^\wedge(0) = 0$

Proof. Let $f \in L_a^2$. If $P\partial[(\phi-\lambda)f] = 0$, then $\partial[(\phi-\lambda)f] = h$ for some $h \in (L_a^2)^\perp$. Writing $\overline{\partial}\,\overline{(\phi-\lambda)f} = \overline{h}$, we see $\overline{(\phi-\lambda)f} = -\frac{1}{\pi}\hat{\overline{h}} + g$ for some $g \in L_a^2$.

<u>(1) ⟹ (2)</u> Use the above and Proposition 609.

<u>(2) ⟹ (1)</u> By Proposition 609, we see $f = \frac{1}{\phi-\lambda}\overline{g - \frac{1}{\pi}\hat{\overline{h}}} \in \ker(T_\phi - \lambda)$. ■

Let $f \in L^2$. What is the relationship between Pf and $\sum\limits_{n=0}^{\infty} \hat{f}(n) z^n$?

Proposition 611 *Let $f \in C \cap L^2$, with $\overline{\partial} f \in L^1$. Then*

$$Pf = \sum_{n=0}^{\infty} \left[\hat{f}(n) - \frac{1}{\pi} \left(\overline{\partial} f, z^{n+1} \right) \right] z^n .$$

Proof. Note $Pf = \sum\limits_{n=0}^{\infty} (f, e_n)\, e_n = \sum\limits_{n=0}^{\infty} \frac{n+1}{\pi} (f, z^n)\, z^n$. For each n,

$$\begin{aligned}
(f, z^n) &= \frac{1}{n+1} \lim_{r \nearrow 1} \iint_{r\mathbb{D}} f \overline{\partial} \overline{z}^{n+1}\, dA \\
&= \frac{1}{n+1} \left[\frac{1}{2i} \lim_{r \nearrow 1} \int_{r\mathbf{T}} f \overline{z}^{n+1}\, dz - \iint_{\mathbb{D}} \left(\overline{\partial} f \right) \overline{z}^{n+1} \right] dA \\
&= \frac{1}{n+1} \left[\pi \hat{f}(n) - \left(\overline{\partial} f, z^{n+1} \right) \right].
\end{aligned}$$

■

Notation: Define $B : L_a^2 \longrightarrow L_a^2$ by $Be_n = \frac{e_{n-1}}{\sqrt{n(n-1)}}$, where we set $e_{-1} = 0$. Set $X = \left\{ f \in C \cap L^2 : \overline{\partial} f \in L^2 \right\}$ and define $\mathcal{T}$ on X by $\mathcal{T} f = \sum\limits_{n=0}^{\infty} \hat{f}(n) z^n$.

Proposition 612 $P = \mathcal{T} - BP\overline{\partial}$ *in* X.

Proof. By Proposition 611, for $f \in X$,

$$\begin{aligned}
Pf &= \mathcal{T} f - \frac{1}{\pi} \sum_{n=0}^{\infty} \left(\overline{\partial} f, z^{n+1} \right) z^n \\
&= \mathcal{T} f - \sum_{n=0}^{\infty} \frac{1}{\sqrt{(n+2)(n+1)}} \left(\overline{\partial} f, e_{n+1} \right) e_n \\
&= \mathcal{T} f - B \sum_{n=0}^{\infty} \left(\overline{\partial} f, e_{n+1} \right) e_{n+1} = \mathcal{T} f - BP\overline{\partial} f.
\end{aligned}$$

■

Proposition 613 $\mathcal{T} : X \longrightarrow L_a^2$ *is bounded, where we define* $\|f\|_X^2 = \|f\|_2^2 + \|\overline{\partial} f\|_2^2$ *to be the norm on* X.

Proof. For $f \in X$, Proposition 612 gives: $\|\mathcal{T}f\|_2 \leq \|f\|_2 + \|B\|\|\overline{\partial}f\|_2$. ■

As mentioned in the remark following the statement of Theorem 604, T_ϕ is an "interesting" operator (on L^2) provided both $\mathbb{D} \bigcap \operatorname{supp} \phi$ and $\mathbb{D} \bigcap (\mathbb{C} \setminus \operatorname{supp} \phi)$ are nonempty.

Construction 614 Fix $0 < \delta < \frac{\pi}{4}$. Define $\tau_\delta : \mathbb{R} \longrightarrow [0, 1]$ by

$$\tau(x) = \begin{cases} 1 & \text{if } 0 \leq |x| \leq 2\delta \\ 1 - \frac{(|x|-2\delta)^2}{2\delta^2} & \text{if } 2\delta \leq |x| \leq 3\delta \\ \frac{(|x|-4\delta)^2}{2\delta^2} & \text{if } 3\delta \leq |x| \leq 4\delta \\ 0 & \text{if } 4\delta \leq |x|. \end{cases}$$

Note $\tau_\delta \in C_c^1(\mathbb{R})$ and $0 \leq \tau_\delta \leq 1$.

Define $\phi_\delta : \mathbb{C} \longrightarrow [0, 1]$ as follows: Set $\phi(1) = 1$ and $R = \left|1 - e^{4\delta i}\right|$. Let $z \in \mathbb{C} \setminus \{1\}$ and set $r = |z - 1|$. If $r \geq R$, then set $\phi_\delta(z) = 0$. If $r < R$, then $\mathbf{T} \bigcap \partial D(1; r) = \left\{e^{\theta}, e^{-i\theta}\right\}$ for some $0 < \theta < 4\delta$. Set $\phi_\delta(z) = \tau_\delta(\theta)$.

Note $\phi_\delta \in C_c^1(\mathbb{C})$, $0 \leq \phi_\delta \leq 1$, $\operatorname{supp}\phi_\delta = \overline{\mathrm{D}(1; \mathrm{R})}$, and $\operatorname{supp} \overline{\partial}\phi_\delta = \mathrm{A}$, where $A = \left\{z \in \mathbb{C} : \frac{R}{2} \leq \left| z - 1 \right| \leq R\right\}$.

Calculation 615 With ϕ_δ as above,

$$\hat{\phi}_\delta(n) = \begin{cases} -\frac{1}{\pi n}\left\{\sin 3n\delta + \frac{2}{n^2\delta^2}\left[\sin 2n\delta - 2\sin 3n\delta + \sin 4n\delta\right]\right\} & \text{if } n \neq 0, \\ \frac{3\delta}{\pi} & \text{if } n = 0. \end{cases}$$

Note that for $n \neq 0$,

$$\hat{\phi}_\delta(n) = -\frac{1}{\pi n} \operatorname{Im} \left\{e^{3in\delta} + \frac{2}{n^2\delta^2}\left[e^{2in\delta}\left(1 - e^{in\delta}\right)^2\right]\right\}.$$

■

Theorem 616 *Let $0 < \delta < \frac{\pi}{4}$. Let $\phi \in C_c^1(D(1;4\delta))$, with $\phi \equiv 1$ in $D(1;2\delta)$. Think of $T_\phi \in L(L_a^2)$. Then Ker $T_\phi = (0)$.*

Remark: Note such $\phi (= \phi_\delta)$ were given in Construction 614.

Proof. Let $f \in \ker T_\phi$. Set $\mu = \frac{1}{\pi} f \overline{\partial} \phi m_2$, $\quad K = \left\{ z \in \overline{\mathbb{D}} : 2\delta \leq |z-1| \leq 4\delta \right\}$, $\Omega_1 = \mathbb{D} \setminus \overline{D(1;4\delta)}$, and $\Omega_2 = \mathbb{D} \bigcap D(1;2\delta)$.

Note supp$\mu \subseteq$ K. Now $0 = T_\phi f = \phi f + \hat{\mu}$, hence $\hat{\mu} \equiv 0$ in Ω_1. Since $S^2 \setminus K$ is connected, $\hat{\mu} \equiv 0$ in Ω_2. But $f = \phi f - \hat{\mu}$ in Ω_2 and $f \in H$. Hence $f \equiv 0$ in $\mathbb{D}$. ■

Section 617 Let $\phi \in C_c^1(\mathbb{C})$. Think of $T_\phi \in L(L_a^2)$. We compute T_ϕ^* in two different manners.

(1) Using the notation of Chapter V, we have $T_\phi = M_\phi + \frac{1}{\pi} C_{\overline{\partial}\phi}$. As operators on L^2, we have $M_\phi^* = M_{\bar{\phi}}$ and (by Proposition 517) $C_{\overline{\partial}\phi}^* f = -\overline{\hat{\bar{f}}\ \overline{\partial}\phi}$. Hence $T_\phi^* f = P\left[\bar{\phi} f - \frac{1}{\pi}\overline{\hat{\bar{f}}\ \overline{\partial}\phi}\ \right]$.

(2) Let $f \in L_a^2$ and $g \in H\left(\overline{\mathbb{D}}\right)$. By Proposition 611 and Theorem 606, respectively, we may write

$$f = \sqrt{\pi} \sum_{n=0}^{\infty} \frac{\hat{f}(n)}{\sqrt{n+1}} e_n \quad \text{and}$$

$$T_\phi g = \sqrt{\pi} \sum_{n=0}^{\infty} \frac{\widehat{\phi g}(n)}{\sqrt{n+1}} e_n.$$

Letting (,) denote the inner product on L_a^2,

$$\left(T_\phi^* f, g\right) = (f, T_\phi f) = \pi \sum_{n=0}^{\infty} \frac{\hat{f}(n)\overline{\widehat{\phi g}(n)}}{n+1}.$$

For each $n \in \mathbb{N}^*$,

$$\begin{aligned} \widehat{\phi g}(n) &= \frac{1}{2\pi i} \int_{\mathbf{T}} (\phi g)(z) \overline{z}^{n+1}\, dz \\ &= \frac{1}{2\pi i} \sum_{j=0}^{\infty} \hat{g}(j) \int_{\mathbf{T}} \phi(z) z^j \overline{z}^{n+1}\, dz = \sum_{j=0}^{\infty} \hat{g}(j) \hat{\phi}(n-j). \end{aligned}$$

For $m \in \mathbb{Z}$, we note $\overline{\hat{\phi}(m)} = \hat{\bar{\phi}}(-m)$. Since $\hat{\phi} \in \ell^\infty(\mathbb{Z}\ )$ and $\hat{g} \in \ell^1(\mathbb{N}^*)$, we may use Fubini:

$$\begin{aligned}\left(T_\phi^* f, g\right) &= \pi \sum_{n=0}^{\infty} \frac{\hat{f}(n+1)}{n+1} \left(\sum_{j=0}^{\infty} \overline{\hat{g}(j)\hat{\phi}(n-j)} \right) \\ &= \sum_{j=0}^{\infty} \left[\sqrt{\frac{\pi}{(j+1)}}(j+1) \sum_{n=0}^{\infty} \frac{\hat{f}(n)\hat{\bar{\phi}}(j-n)}{n+1} \right] \left[\sqrt{\frac{\pi}{(j+1)}}\, \overline{\hat{g}(j)} \right].\end{aligned}$$

Using the density of $H\left(\overline{\mathbb{D}}\right)$ in L_a^2 gives:

$$T_\phi^* f = \sum_{j=0}^{\infty} (j+1) \left[\sum_{n=0}^{\infty} \frac{\hat{f}(n)\hat{\bar{\phi}}(j-n)}{n+1} \right] z^j.$$

■

Notation. Set $Y = \left\{ f \in L_a^2 : \left(\frac{\hat{f}(n)}{n+1} \right)_{n=0}^{\infty} \in \ell^1 \right\}$.

Remark. Y is "almost" L_a^2. That is, if $f \in L_a^2$ and $\epsilon > 0$, then by Cauchy-Schwarz,

$$\sum_{n=0}^{\infty} \frac{|\hat{f}(n)|}{(n+1)^{1+\epsilon}} \leq \left(\sum_{n=0}^{\infty} \frac{|\hat{f}(n)|^2}{(n+1)} \right)^{\frac{1}{2}} \left(\sum_{n=0}^{\infty} \frac{1}{(n+1)^{1+2\epsilon}} \right)^{\frac{1}{2}} < \infty,$$

since

$$\|f\|_2^2 = \sum_{n=0}^{\infty} \frac{|\hat{f}(n)|^2}{(n+1)}.$$

Theorem 618 *Let $\phi \in C_c^1(\mathbb{C})$. Think of $T_\phi \in L\left(L_a^2\right)$. Let $f \in Y$. Then $T_\phi^* f = \sum_{j=0}^{\infty} (j+1) \left(\bar{\phi} F\right)^{\wedge}(j) z^j$, where $F = \sum_{n=0}^{\infty} \frac{\hat{f}(n)}{n+1} z^n$.*

Proof. By Section 617,

$$T_\phi^* f = \sum_{j=0}^{\infty} (j+1) \left[\sum_{n=0}^{\infty} \frac{\hat{f}(n)\hat{\bar{\phi}}(j-n)}{n+1} \right] z^j.$$

For $j \in \mathbb{N}^*$,

$$\sum_{n=0}^{\infty} \frac{\hat{f}(n)\hat{\bar{\phi}}(j-n)}{n+1} = \sum_{n=0}^{\infty} \frac{\hat{f}(n)}{n+1} \frac{1}{2\pi i} \int_{\mathbf{T}} \overline{\phi(z)}\, \overline{z}^{j-n+1}\, dz$$

$$= \frac{1}{2\pi i} \int_{\mathbf{T}} \overline{\phi(z)}\, \overline{z}^{j+1} \left(\sum_{n=0}^{\infty} \frac{\hat{f}(n)}{n+1} z^n \right) dz$$

$$= \left(\overline{\phi} F \right)^{\wedge}(j).$$

■

Section 619 Define $\tau \in \ell^\infty(\mathbb{N}^*)$ by $\tau(n) = \sqrt{\frac{\pi}{n+1}}$. Set $\ell_a^2 = \{a = (a_n)_n^\infty : \tau a \in \ell^2(\mathbb{N}^*)\}$, where $(\tau a)(n) = \tau(n) a_n$. Define an inner product on ℓ_a^2 by $(a,b)_{\ell_a^2} = (\tau a, \tau b)_{\ell^2(\mathbb{N}^*)}$.

Define $U : L_a^2 \longrightarrow \ell_a^2$ as follows: Let $f \in L_a^2$ and write $f = \sum_{n=0}^{\infty} \sqrt{\frac{\pi}{n+1}} \hat{f}(n) e_n$. Set $Uf = \hat{f}$, where we write $\hat{f} = \left(\hat{f}(n)\right)_{n=0}^{\infty}$. [We will only use this notation in this section and in Proposition 620.] Note U is linear, $1-1$, and onto. Note U also preserves inner products: For $f, g \in L_a^2$,

$$(Uf, Ug)_{\ell_a^2} = \left(\hat{f}, \hat{g}\right)_{\ell_a^2} = \left(\tau \hat{f}, \tau \hat{g}\right)_{\ell^2(\mathbb{N}^*)} = \sum_{n=0}^{\infty} \frac{\pi}{n+1} \hat{f}(n) \overline{\hat{g}(n)} = (f,g)_{L_a^2}.$$

□

Let $\phi \in C_c^1(\mathbb{C})$. As in Section 617, for $f \in H\left(\overline{\mathbb{D}}\right)$,

$$T_\phi f = \sum_{n=0}^{\infty} \sqrt{\frac{\pi}{n+1}} \widehat{\phi f}(n) e_n = \sum_{n=0}^{\infty} \sqrt{\frac{\pi}{n+1}} \left(\hat{\phi} * \hat{f}\right)(n) e_n,$$

where $\left(\hat{\phi} * \hat{f}\right)(n) = \sum_{n=0}^{\infty} \hat{\phi}(n-j) \hat{f}(j)$.

Define $t_\phi : \ell^1(\mathbb{N}^*) \longrightarrow \ell_a^2$ by $t_\phi a = \hat{\phi} * a$, where we think of $\ell^1(\mathbb{N}^*)$ as sitting inside ℓ_a^2.

$$\text{Set} \quad Z = \left\{ f = \sum a_n z^n : \sum |a_n| < \infty \right\} \quad \text{and}$$

note that $U^*(\ell^1(\mathbb{N}^*)) = Z$ and that $t_\phi = UT_\phi U^*$ in $\ell^1(\mathbb{N}^*)$. Now $UT_\phi U^*$ is defined on all of ℓ_a^2. Since Z is dense in L_a^2, we see $\ell^1(\mathbb{N}^*)$ is dense in L_a^2. Hence we may extend t_ϕ to all of ℓ_a^2 by $t_\phi = UT_\phi U^*$. ■

Proposition 620 *Let $\phi \in C_c^1(\mathbb{C})$ and $b \in \ell_a^2$. Then $t_\phi^* b = \frac{1}{\tau^2}\left[\bar{\hat{\phi}} * (\tau^2 b)\right]$.*

Proof. For $a \in \ell^1(\mathbb{N}^*)$,

$$\begin{aligned}
\left(a, t_\phi^* b\right)_{\ell_a^2} &= (t_\phi a, b)_{\ell_a^2} \\
&= \left(\hat{\phi} * a, b\right)_{\ell_a^2} = \sum_{n=0}^{\infty} \frac{\pi}{n+1}\left(\hat{\phi} * a\right)(n)\,\bar{b}_n \\
&= \sum_{n=0}^{\infty} \frac{\pi}{n+1}\left(\sum_{j=0}^{\infty} \hat{\phi}(n-j)\,a_j\right)\bar{b}_n = \sum_{j=0}^{\infty} a_j \left(\sum_{n=0}^{\infty} \overline{\bar{\hat{\phi}}(j-n)\,\frac{\pi}{n+1}\bar{b}_n}\right) \\
&= \overline{\sum_{j=0}^{\infty}\left[\sqrt{\frac{\pi}{j+1}}a_j\right]\left[\sqrt{\frac{j+1}{\pi}}\sum_{n=0}^{\infty}\bar{\hat{\phi}}(j-n)\,(\tau^2 b)_n\right]} \\
&= \left(\tau a, \frac{1}{\tau}\left[\bar{\hat{\phi}} * \left(\tau^2 b\right)\right]\right)_{\ell^2(\mathbb{N}^*)} \\
&= \left(a, \frac{1}{\tau^2}\left[\bar{\hat{\phi}} * \left(\tau^2 b\right)\right]\right)_{\ell_a^2}.
\end{aligned}$$

Now use the density of $\ell^1(\mathbb{N}^*)$ in ℓ_a^2. ■

Section 621 Let $1 \le p \le \infty$ and $\phi \in C_c^1(\mathbb{C})$. Let $S \subseteq \mathbb{C}$ be bounded. In the next few sections, we consider the following questions:

(a) By Theorem 511, we know: $M_z T_\phi - T_\phi M_z \in L(L^p(S))$. What else can we say about $M_z T_\phi - T_\phi M_z$?

(b) How close is T_ϕ to a multiplication operator?

(c) Let $N \le L^p(S)$. Is $M_z N \subseteq N$? Is $T_\phi N \subseteq N$? ■

Theorem 622 *Let* $1 \le p \le \infty$ *and* $\phi \in C_c^1(\mathbb{C})$. *Let* $S \subseteq \mathbb{C}$ *be bounded (and measurable). Then* $M_z T_\phi - T_\phi M_z : L^p(S) \longrightarrow \mathbb{C}$, *where we identify* $\mathbb{C}$ *with the constant functions (in* $L^p(S)$*). In fact, for* $f \in L^p(S)$,

$$(M_z T_\phi - T_\phi M_z) f = -\frac{1}{\pi} \iint_S f \overline{\partial} \phi \, dA.$$

So if $N \le L^p(S)$, *then* $M_z T_\phi = T_\phi M_z$ *in* N *iff* $\overline{\partial}\phi \perp N$, *by which we mean*

$$\iint_S f \overline{\partial} \phi \, dA = 0 \qquad (f \in N).$$

Proof. Let $f \in L^p(S)$ and $z \in S$. Note

$[(M_z T_\phi - T_\phi M_z) f](z) =$

$$z \left[(\phi f)(z) + \frac{1}{\pi} \iint_S \frac{(f\overline{\partial}\phi)(\zeta)}{\zeta - z} \, dA(\zeta) \right] - \left[\phi(z) z f(z) + \frac{1}{\pi} \iint_S \frac{\zeta (f\overline{\partial}\phi)(\zeta)}{\zeta - z} \, dA(\zeta) \right].$$

Writing $z = (z - \zeta) + \zeta$, we get

$$[(M_z T_\phi - T_\phi M_z) f](z) = -\frac{1}{\pi} \iint_S \left(f \overline{\partial} \phi \right)(\zeta) \, dA(\zeta).$$

■

Lemma 623 $1 \le p \le \infty$ *and* $\phi \in C_c^1(\mathbb{C})$. *Let* S *be bounded and* $N \le L^p(S)$. *If* $\overline{\partial}\phi \notin N^\perp$, *then* $1 \in (M_z T_\phi - T_\phi M_z) N$, *where* 1 *denotes the corresponding constant function.*

Proof. By Theorem 622, for $f \in L^p(S)$,

$$(M_z T_\phi - T_\phi M_z) f = -\frac{1}{\pi} \iint_S f \overline{\partial} \phi \, dA.$$

■

Theorem 624 *Let $1 \leq p \leq \infty$ and $\phi \in C_c^1(\mathbb{C})$. Let S be bounded and $N \leq L^p(S)$. Assume $\overline{\partial}\phi \notin N^{\perp}$, $M_z N \subseteq N$, and $T_\phi N \subseteq N$. Then $\mathcal{P} \subseteq N$, where $\mathcal{P}$ denotes the polynomials (in z).*

Proof. By Lemma 623, we note $1 \in N$. ∎

Corollary 625 *Let $1 \leq p \leq \infty$ and $\phi \in C_c^1(\mathbb{C})$. Let S be bounded and $N \leq L^p(S)$. Assume $M_z N \subseteq N$ and $T_\phi N \subseteq N$. Then either*

(i) *$\mathcal{P} \subseteq N$ or*

(ii) *$M_z T_\phi = T_\phi M_z$ in N.*

Proof. Assume **(i)** does not hold. By Theorem 624, we get $\overline{\partial}\phi \perp N$. Hence, by Theorem 622, **(ii)** holds. ∎

Notation: For $z_0 \in \mathbb{C}$ and f analytic in a neighborhood of z_0, let $n(f; z_0)$ denote the order of the zero of f at z_0.

Proposition 626 *Let $1 \leq p \leq \infty$ and $\phi \in C_c^1(\mathbb{C})$. Let $\Omega \subseteq \mathbb{C}$ be open and bounded. Let $N \leq L_a^p(\Omega)$, with $N \neq L_a^p(\Omega)$, $M_z N \subseteq N$, and $T_\phi N \subseteq N$.*

Let $f \in N$ and $z_0 \in \Omega$. Then $n(T_\phi f; z_0) \geq n(f; z_0)$.

Proof. Set $m = n(f; z_0)$. Write $f = (z - z_0)^m g$, where $g \in L_a^p(\Omega)$ and $g(z_0) \neq 0$. By Corollary 625,

$$T_\phi f = T_\phi (M_z - z_0)^m g = (M_z - z_0)^m T_\phi g = (z - z_0)^m T_\phi g.$$

So $\quad n(T_\phi f; z_0) = n(f; z_0) + n(T_\phi g; z_0).$ ∎

Theorem 627 *Let $\phi \in C_c^1(\mathbb{C})$.*

(a) *Let* $1 \le p \le \infty$, $K \subseteq \mathbb{C}$ *be compact, and* $g \in L^\infty(K)$.

If $T_\phi = M_g$ *in* $L^p(K)$, *then* $\chi_K \left(\overline{\partial}\phi\right) m_2 \perp R(K)$.

(b) *Let* $1 \le p < \infty$ *and* $\Omega \subseteq \mathbb{C}$ *be open and bounded. If* $\overline{\partial}\phi \perp L^p_a(\Omega)$, *then* $T_\phi = M_g$ *in* $L^p_a(\Omega)$ *for some* $g \in H^\infty(\Omega)$.

Proof.

(a) Set $\psi(z) = -\frac{1}{\pi}\iint\limits_{\mathbb{C}\setminus K} \frac{(\overline{\partial}\phi)(\zeta)}{\zeta - z}\, dA(\zeta)$. For $f \in L^p(K)$ and $z \in K$, Corollary 121 shows:

$$\begin{aligned}
(fg)(z) &= (M_g f)(z) = (T_\phi f)(z) \\
&= -\frac{f(z)}{\pi}\iint\limits_{\mathbb{C}} \frac{(\overline{\partial}\phi)(\zeta)}{\zeta - z}\, dA(\zeta) + \frac{1}{\pi}\iint\limits_{K} \frac{(f\overline{\partial}\phi)(\zeta)}{\zeta - z}\, dA(\zeta) \\
&= (f\psi)(z) + \frac{1}{\pi}\iint\limits_{K} \frac{f(\zeta) - f(z)}{\zeta - z}\left(\overline{\partial}\phi\right)(\zeta)\, dA(\zeta).
\end{aligned}$$

Setting $f \equiv 1$ in the above gives: $g = \psi$ in $L^\infty(K)$. So for $f \in L^p(K)$,

$$\iint\limits_{K} \frac{f(\zeta) - f(z)}{\zeta - z}\left(\overline{\partial}\phi\right)(\zeta)\, dA(\zeta) = 0 \quad \text{for a.e.} \quad z \in K.$$

Fix $z_0 \in \mathbb{C} \setminus$ K. Setting $f = \frac{1}{z - z_0}$ gives:

$$\begin{aligned}
0 &= \iint\limits_{K} \frac{\frac{1}{\zeta - z_0} - \frac{1}{z - z_0}}{\zeta - z}\left(\overline{\partial}\phi\right)(\zeta)\, dA(\zeta) \\
&= -\frac{1}{z - z_0}\iint\limits_{K} \frac{(\overline{\partial}\phi)(\zeta)}{\zeta - z_0}\, dA(\zeta) \qquad \text{for a.e.} \quad z \in K.
\end{aligned}$$

(b) Set $g(z) = -\frac{1}{\pi}\iint\limits_{\mathbb{C}\setminus\Omega} \frac{(\overline{\partial}\phi)(\zeta)}{\zeta - z}\, dA(\zeta)$. Note $g \in H^\infty(\Omega)$. Let $f \in L^p_a(\Omega)$ and $z \in \Omega$. Then

$$(T_\phi f)(z) = (fg)(z) + \frac{1}{\pi}\iint\limits_{\Omega} \frac{f(\zeta) - f(z)}{\zeta - z}\left(\overline{\partial}\phi\right)(\zeta)\, dA(\zeta)$$

$$= \quad (fg)(z) = (M_g f)(z),$$

$$\text{since} \quad h(\zeta) \quad = \quad \frac{f(\zeta) - f(z)}{\zeta - z} \in L^p_a(\Omega).$$

■

Remark: The proof of (**a**) shows: Let $1 \leq p \leq \infty$ and $\Omega \subseteq \mathbb{C}$ be open and bounded. If $g \in L^\infty(\Omega)$ and if $T_\phi = M_g$ in $L^p(\Omega)$, then in fact $g \in H^\infty(\Omega)$. ■

Theorem 628 *Let $1 \leq p \leq \infty$ and $\phi \in C^1_c(\mathbb{C})$. Let $S \subseteq \mathbb{C}$ be measurable, with $m_2 S < \infty$. Set*

$$N = \left\{ f \in L^p(S) : \frac{f - f(z_0)}{z - z_0} \perp \overline{\partial}\phi \ \text{ for a.e. } z_0 \in S \right\}.$$

Then $T_\phi = M_g$ in N, where $g = -\frac{1}{\pi}\left(\chi_{\mathbb{C}\setminus S}\overline{\partial}\phi\right)^\wedge$.

Proof. As in the proof of Theorem 627, for $f \in N$ and a.e. $z \in S$,

$$(T_\phi f)(z) = (fg)(z) + \frac{1}{\pi}\iint \frac{f(\zeta) - f(z)}{\zeta - z}\left(\overline{\partial}\phi\right)(\zeta)\, dA(\zeta) = (fg)(z) + (M_g f)(z).$$

■

Remark: Note $g \in H^\infty(\text{int } S)$. ■

Theorem 629 *Let $1 \leq p < \infty$ and S be measurable, with $m_2 S < \infty$. Let $N \leq L^p(S)$ satisfy:*

(i) $\frac{f}{z - z_0} \in N$ *whenever $f \in N$ and for a.e. $z_0 \in \left(\text{supp } \overline{\partial}\phi\right) \setminus S$ and*

(ii) $\frac{f - f(z_0)}{z - z_0} \in N$ *whenever $f \in N$ and for a.e. $z_0 \in S$.* □

Then $T_\phi : N \longrightarrow N$.

Proof. Fix $f \in N$ and $g \perp N$. [Here we think of $g \in (L^p(S))^*$: So $g \in L^q(S)$ and $\iint_S gh\,dA = 0 \quad (h \in N)$.] As in the proof of Theorem 627,

$$\begin{aligned}
g(T_\phi f) &= \iint_S gT_\phi f\,dA \\
&= \frac{1}{\pi}\iint_S \left\{ -f(z)\iint_{\mathbb{C}\setminus S} \frac{(\overline{\partial}\phi)(\zeta)}{\zeta - z}\,dA(\zeta) \right. \\
&\qquad \left. + \iint_S \frac{f(\zeta) - f(z)}{\zeta - z}\left(\overline{\partial}\phi\right)(\zeta)\,dA(\zeta) \right\} g(z)\,dA(z) \\
&= \frac{1}{\pi}\left\{ \iint_{\mathbb{C}\setminus S}\left(\iint_S \frac{(fg)(z)}{z-\zeta}\,dA(z)\right)\left(\overline{\partial}\phi\right)(\zeta)\,dA(\zeta) \right. \\
&\qquad \left. + \iint_S\left(\iint_S \frac{f(\zeta)-f(z)}{\zeta - z} g(z)\,dA(z)\right)\left(\overline{\partial}\phi\right)(\zeta)\,dA(\zeta) \right\} \\
&= 0 + 0
\end{aligned}$$

by (i) and (ii), respectively.

The use of Fubini's Theorem is justified for the following reason:

$$\iint_S |(fg)(z)| \left(\iint_{\mathbb{C}} \left|\frac{(\overline{\partial}\phi)(\zeta)}{\zeta - z}\right| dA(\zeta)\right) dA(z) < \infty$$

since $fg \in L^1(S)$ and the inner integral is a bounded (continuous) function (of z);

$$\iint_S \left(\iint_S \left|\frac{(f\overline{\partial}\phi)(\zeta)}{\zeta - z}\right| dA(\zeta)\right) |g(z)|\,dA(z) < \infty$$

since $\frac{1}{|z|} * \left|\chi_S f\overline{\partial}\phi\right| \in L^p(S)$. ■

Theorem 630 *Let $1 \le p < \infty$ and $\phi \in C_c^1(\mathbb{C})$. Let S be bounded and let $N_1 \le L^p(S)$. Let N_2 denote the closure of $T_\phi N_1$ in $L^p(S)$. If $M_z N_1 \subseteq N_1$ and $\overline{\partial}\phi \perp N_1$, then $M_z N_2 \subseteq N_2$.*

Proof. Fix $f \in N_1$ and $g \perp N_2$. Now $M_z f \in N_1$, hence, $T_\phi M_z f \in N_2$ and so $g(T_\phi M_z f) = 0$. By Theorem 622,

$$g(M_z T_\phi f) = g(T_\phi M_z f) - \frac{1}{\pi}\left(\iint_S f\overline{\partial}\phi \, dA\right) g(1) = 0.$$

■

Section 631 Let $1 \leq p < \infty$ and $\phi \in C_c^1(\mathbb{C})$. Let S be bounded, let $N_1 \leq L^p(S)$, and let N_2 denote the closure of $T_\phi N_1$ in $L^p(S)$. Assume $M_z N_1 \subseteq N_1$.

The proof of Theorem 630 shows the following: For $f \in N_1$ and $g \perp N_2$,

$$g(M_z T_\phi f) = -\frac{1}{\pi}\left(\iint_S f\overline{\partial}\phi \, dA\right)\left(\iint_S g \, dA\right).$$

So if we want $M_z N_2 \subseteq N_2$, then we need either

(i) $1 \in N_2$ or

(ii) $\overline{\partial}\phi \perp N_1$. ■

Important: We restrict our attention to the open unit disc for the next few results.

Lemma 632 *Let $1 \leq p < \infty$ and $\phi \in C_c^1(\mathbb{C})$. Let $N \leq L^p$, with $M_z N \subseteq N$. Then for $f \in N$ and $g \perp N$,*

$$g(T_\phi f) = \frac{1}{\pi}\iint_{\mathbb{D}}\left[\iint_{\mathbb{D}} \frac{f(\zeta) - f(z)}{\zeta - z}\left(\overline{\partial}\phi\right)(\zeta)\, dA(\zeta)\right] g(z)\, dA(z).$$

Proof. Fix $\zeta \in \mathbb{C} \setminus \overline{\mathbb{D}}$. Then $|\zeta| > 1$, hence, for $z \in \mathbb{D}$,

$$\frac{f(z)}{\zeta - z} = f(z)\sum_{n=0}^{\infty}\frac{z^n}{\zeta^{n+1}}.$$

This gives

$$\iint_{\mathbb{D}} \frac{(fg)(z)}{\zeta - z}\, dA(z) = \sum_{n=0}^{\infty}\frac{1}{\zeta^{n+1}}\iint_{\mathbb{D}} z^n fg \, dA = 0$$

since each $z^n f \in N$. Since $m_2\mathbf{T} = 0$, we get

$$\iint_{\mathbb{D}} (fg)(z) \left[\iint_{\mathbb{C}\setminus\mathbb{D}} \frac{(\overline{\partial}\phi)(\zeta)}{\zeta - z} \, dA(\zeta) \right] dA(z)$$

$$= \iint_{\mathbb{C}\setminus\mathbb{D}} \left(\overline{\partial}\phi\right)(\zeta) \left[\iint_{\mathbb{D}} \frac{(fg)(z)}{\zeta - z} \, dA(z) \right] dA(\zeta) = 0.$$

By the proof of Theorem 629,

$$\begin{aligned} g(T_\phi f) &= -\frac{1}{\pi} \iint_{\mathbb{D}} (fg)(z) \left[\iint_{\mathbb{C}\setminus\mathbb{D}} \frac{(\overline{\partial}\phi)(\zeta)}{\zeta - z} \, dA(\zeta) \right] dA(z) \\ &\quad + \frac{1}{\pi} \iint_{\mathbb{D}} \left[\iint_{\mathbb{D}} \frac{f(\zeta) - f(z)}{\zeta - z} \left(\overline{\partial}\phi\right)(\zeta) \, dA(\zeta) \right] g(z) \, dA(z) \end{aligned}$$

and the first term on the right is zero. ■

Theorem 633 *Let* $1 \le p < \infty$ *and* $\phi \in C_c^1(\mathbb{C})$. *Let* $N \le L^p$, *with* $M_z N \subseteq N$. *Assume for* $f \in N$ *and for a.e.* $z_0 \in \mathbb{D} \bigcap \operatorname{supp} \overline{\partial}\phi$ *that* $\frac{f - f(z_0)}{z - z_0} \perp \overline{\partial}\phi$.

Then $T_\phi N \subseteq N$.

Proof. For $f \in N$ and $g \perp N$, Lemma 632 shows: $g(T_\phi f) = 0$. ■

We finish this chapter with a collection of odds and ends (relating to the T_ϕ operator).

Section 634 Let $\phi \in C_c^1(\mathbb{C})$. Let $K \subseteq \mathbb{C}$ be compact and $z_0 \in \mathbb{C} \setminus \mathrm{K}$. Set $f_n(z) = z^n$ and $r_n(z) = \frac{1}{(z - z_0)^n}$. We calculate $T_\phi f_n$ and $T_\phi r_n$.

Set $g = -\frac{1}{\pi} \left(\chi_{\mathbb{C}\setminus\mathrm{K}} \overline{\partial}\phi\right)^{\wedge}$. As in the proof of Theorem 627,

$$\left[(T_\phi - M_g) h\right](z) = \iint_K \frac{h(\zeta) - h(z)}{\zeta - z} \left(\overline{\partial}\phi\right)(\zeta) \, dA(\zeta).$$

Note

$$\begin{aligned}(z-z_0)^n-(\zeta-z_0)^n &= \sum_{j=0}^{n}\binom{n}{j}(-z_0)^{n-j}\left(z^j-\zeta^j\right)\\ &= \sum_{j=0}^{n}\binom{n}{j}(-z_0)^{n-j}(z-\zeta)\sum_{\ell=0}^{j-1}z^{j-1-\ell}\zeta^{\ell}.\end{aligned}$$

Hence,

$$\begin{aligned}[(T_\phi-M_g)r_n](z) &= \frac{1}{\pi}\iint_K\left[\frac{1}{(\zeta-z_0)^n}-\frac{1}{(z-z_0)^n}\right]\frac{(\overline{\partial}\phi)(\zeta)}{\zeta-z}\,dA\\ &= -\frac{1}{\pi(z-z_0)^n}\sum_{j=0}^{n}\binom{n}{j}(-z_0)^{n-j}\sum_{\ell=0}^{j-1}z^{j-1-\ell}\iint_K\frac{\zeta^\ell(\overline{\partial}\phi)(\zeta)}{(\zeta-z_0)^n}\,dA\\ &= -\frac{r_n(z)}{\pi}\sum_{j=0}^{n}\binom{n}{j}(-1)^{n-j}f_{n-j}(z_0)\sum_{\ell=1}^{j-1}f_{j-1-\ell}(z)\iint_K f_\ell r_n\overline{\partial}\phi\,dA.\end{aligned}$$

Similarly,

$$[(T_\phi-M_g)f_n](z)=\frac{1}{\pi}\sum_{\ell=1}^{n-1}z^{n-1-\ell}\iint_K f_\ell\overline{\partial}\phi\,dA.$$

Note $(T_\phi-M_g)r_n=p_nr_n$ and $(T_\phi-M_g)f_n=q_n$, where p_n and q_n are polynomials of degree at most $n-1$. ■

Notation: For $N\in\mathbb{N}^*$, let $\mathcal{P}_N$ denote the polynomials having degree $\le N$, the linear span of $\{z^n:0\le n\le N\}$.

Proposition 635 *Let $1\le p\le\infty$ and $\phi\in C_c^1(\mathbb{C})$. Let $S\subseteq\mathbb{C}$ be measurable, with $m_2S<\infty$. Assume $\overline{\partial}\phi\perp\mathcal{P}_N$ and think of $T_\phi\in L(L^p(S))$.*

Then $\mathcal{P}_{N+1}\subseteq Ker\,(T_\phi-M_g)$, where $g=-\frac{1}{\pi}\left(\chi_{\mathbb{C}\setminus S}\overline{\partial}\phi\right)^\wedge$.

Proof. Use Section 634. ■

Notation: For $N\subseteq\mathbb{N}^*$, let A_N denote the linear span of

$$\left\{\frac{z^j}{(z-z_0)^n}:0\le n\le N\quad\text{and}\quad 0\le j\le n-1\right\}.$$

Proposition 636 *Let $1 \leq p \leq \infty$. Let $\phi \in C_c^1(\mathbb{C})$ and let $K \subseteq \mathbb{C}$ be compact. Assume $\overline{\partial}\phi \perp A_N$ and think of $T_\phi \in L\left(L^p(K)\right)$. Then $A_N \subseteq Ker\left(T_\phi - M_g\right)$, where $g = -\frac{1}{\pi}\left(\chi_{\mathbb{C}\setminus K}\overline{\partial}\phi\right)^\wedge$.*

Proof. Use Section 634. ■

Important: For the remainder of this chapter, we (again) restrict our attention to $\mathbb{D}$.

Proposition 637 *Let $1 \leq p \leq \infty$ and $\phi \in C_c^1(\mathbb{C})$. Let $N \leq L_a^p$. TFAE:*

(i) $\overline{\partial}\phi \perp N$.

(ii) $\widehat{\phi N}(-1) = 0$.

Remark: By (ii), we mean: For $f \in N$, we have $\widehat{\phi f}(-1) = 0$.

Proof. Let $f \in N$. Then

$$\iint_{\mathbb{D}} f\overline{\partial}\phi\, dA = \lim_{r \nearrow 1} \iint_{r\mathbb{D}} \overline{\partial}(f\phi)\; dA = \frac{1}{2i}\lim_{r \nearrow 1}\int_{r\mathbb{T}} f\phi\, dz = \pi\widehat{f\phi}(-1).$$

■

Proposition 638 *Let $1 \leq p \leq \infty$ and $\phi \in C_c^1(\mathbb{C})$. Let $N \leq L_a^p$, where $\overline{\partial}\phi \perp N$ and $M_z N \subseteq N$. Then $\widehat{\phi N}\left(-\mathbb{N}\right) = 0$.*

Remark: The conclusion states: For $f \in N$ and $n \in \mathbb{N}$, we have $\widehat{\phi f}(-n) = 0$.

Proof. Let $f \in N$ and $n \in \mathbb{N}$. Note $z^{n-1}f \in N$, hence,

$$\begin{aligned} 0 = \iint_{\mathbb{D}} z^{n-1} f\overline{\partial}\phi\, dA &= \frac{1}{2i}\lim_{r \nearrow 1}\int_{r\mathbb{T}} z^{n-1} f\phi\, dz \\ &= \frac{1}{2i}\lim_{r \nearrow 1} r^{2(n-1)}\int_{r\mathbb{T}} \overline{z}^{1-n} f\phi\, dz = \pi\widehat{f\phi}(-n). \end{aligned}$$

■

Notation: (Proposition 1.7 [2]) Let $1 \le p < \infty$. Let P denote the projection of L^p onto L^p_a. For $z_0 \in \mathbb{D}$, let $k_{z_0}(z_0) = \frac{1}{\pi}\frac{1}{(1-\overline{z}_0 z)^2}$ denote the L^p_a-reproducing kernel at z_0. That is,

$$(Pf)(z_0) = \iint\limits_{\mathbb{D}} f\overline{k}_{z_0}\, dA \quad (f \in L^p).$$

Theorem 639 *Let $1 \le p \le \infty$ and $\phi \in C^1_c(\mathbb{C})$. Let $z_0 \in \mathbb{D}$. Then there exist constants $C_1, C_2 \in \mathbb{C}$ and $g \in H^\infty$ satisfying: $T_\phi k_{z_0} = (C_1 + C_2 z + g)\, k_{z_0}$.*

Proof. For $z \in \mathbb{D}$,

$$[(T_\phi - M_g)\, k_{z_0}](z) = \frac{1}{\pi}\iint\limits_{\mathbb{D}} \frac{k_{z_0}(\zeta) - k_{z_0}(z)}{\zeta - z}\left(\overline{\partial}\phi\right)(\zeta)\, dA(\zeta),$$

where $g = -\frac{1}{\pi}\left(\chi_{\mathbb{C}\setminus K}\overline{\partial}\phi\right)^\wedge$. For $\zeta \in \mathbb{D}$,

$$\frac{k_{z_0}(\zeta) - k_{z_0}(z)}{\zeta - z} = \frac{1}{\pi}\frac{2\overline{z}_0 - \overline{z}_0^2(z+\zeta)}{(1-\zeta\,\overline{z}_0)^2(1-z\overline{z}_0)^2},$$

hence,

$$\begin{aligned}[(T_\phi - M_g)\, k_{z_0}](z) &= \frac{i}{\pi}\left(2\overline{z}_0 - \overline{z}_0^2 z\right) k_{z_0}(z) \iint\limits_{\mathbb{D}} \frac{(\overline{\partial}\phi)(\zeta)}{(1-\zeta\,\overline{z}_0)^2}\, dA(\zeta) \\ &- \frac{1}{\pi}\overline{z}_0^2 k_{z_0}(z) \iint\limits_{\mathbb{D}} \frac{\zeta(\overline{\partial}\phi)(\zeta)}{(1-\zeta\,\overline{z}_0)^2}\, dA(\zeta)\end{aligned}$$

Since

$$\begin{aligned}\frac{1}{\pi}\iint\limits_{\mathbb{D}} \frac{(\overline{\partial}\phi)(\zeta)}{(1-\zeta\,\overline{z}_0)^2}\, dA(\zeta) &= \overline{\iint\limits_{\mathbb{D}} (\overline{\partial}\phi)(\zeta)\overline{k_{z_0}(\zeta)}\, dA(\zeta)} = \overline{\left[P\left(\partial\overline{\phi}\right)\right](z_0)} \quad \text{and} \\ \frac{1}{\pi}\int\limits_{\mathbb{D}} \frac{\zeta(\overline{\partial}\phi)(\zeta)}{(\zeta - z)}\, dA(\zeta) &= \overline{\left[P\left(\overline{z}\partial\overline{\phi}\right)\right](z_0)},\end{aligned}$$

we are done if we set

$\overline{C}_1 = 2z_0 \left[P\left(\partial\bar{\phi}\right)\right](z_0) - z_0^2\left[P\left(\overline{z}\partial\bar{\phi}\right)\right](z_0)$ and $\overline{C}_2 = -z_0^2\left[P\left(\partial\bar{\phi}\right)\right](z_0)$. ■

We now present a (Banach Space) basis for $L^2 = L^2(\mathbb{D})$. We will use inner product notation: For $f, g \in L^2$, we write $(f,g) = \iint_{\mathbb{D}} f\overline{g}\, dA$.

Section 640 By Stone-Weierstrass, the linear span of $\{z^j\overline{z}^n : j, n \in \mathbb{N}^*\}$ is dense in L^2. Think of $(z^j\overline{z}^n)_{j,n=0}^{\infty}$ as an infinite matrix. Note the diagonals are pairwise orthogonal: If $j - n \neq \ell - m$, then $\left(z^j\overline{z}^n, z^\ell\overline{z}^m\right) = \left(z^{j+m}, z^{\ell+n}\right) = 0$. Note that entries from the same diagonal are never orthogonal: If $j - n = \ell - m$, then $\left(z^j\overline{z}^n, z^\ell\overline{z}^m\right) = \frac{\pi}{j+m+1}$.

Claim: $\{z^j\overline{z}^n : j, n \in \mathbb{N}^*\}$ is a linearly independent set in L^2.

Proof of Claim. By the above, we may assume $\sum_{j=0}^{\ell} a_j z^j \overline{z}^{j+n} = 0$, where $\ell, n \in \mathbb{N}^*$. Factoring out $\overline{z}^n$, we have $\sum_{j=0}^{\ell} a_j |z|^{2j} = 0$. Since $\{x^{2j} : 0 \leq j \leq \ell\}$ is linearly independent in $L^2(0,1)$, we conclude: $a_0 = \cdots = a_\ell = 0$.

[Note we chose to work above the main diagonal: for the "below" case, assume $\sum_{j=0}^{\ell} a_j z^{j+n}\overline{z}^j = 0$ and factor out z^n.] □

Hence $\{z^j\overline{z}^n : j, n \in \mathbb{N}^*\}$ is a (Banach space) basis for L^2. ■

Lemma 641 *Let H be a Hilbert space, $\{a_n : n \in \mathbb{N}\} \subseteq \mathbb{C}$ and $\{x_n : n \in \mathbb{N}\} \subseteq H$. If $\sum_{j=1}^{\infty}\sum_{n=1}^{\infty} |a_j a_n|\,|(x_j, x_n)| < \infty$, then $\sum_{n=1}^{\infty} a_n x_n$ converges (in H).*

Proof. For $M < N$, note $\|\sum_{n=M}^{N} a_n x_n\|^2 = \sum_{j=M}^{N}\sum_{n=M}^{N} a_j \overline{a}_n (x_j, x_n)$. ■

Theorem 642 *Let* $(a_{jn})_{j,n=0}^{\infty}$ *be a sequence of complex numbers. Assume*

$$\sum_{N=-\infty}^{\infty} \sum_{j=\max\{0,-N\}}^{\infty} \sum_{n=\max\{0,-N\}}^{\infty} \frac{|a_{j,j+N}||a_{n,n+N}|}{j+n+N+1} < \infty.$$

Then $\sum_{j=0}^{\infty} \sum_{n=0}^{\infty} a_{jn} z^j \overline{z}^n$ *converges in* L^2.

Proof. We "look down" the diagonals.

$$\sum_{N=-\infty}^{0} \sum_{j=-N}^{\infty} \sum_{n=-N}^{\infty} |a_{j,j+N}|\,|a_{n,n+N}|\left|\left(z^j\overline{z}^{j+N}, z^n\overline{z}^{n+N}\right)\right|$$
$$+\sum_{N=1}^{\infty} \sum_{j=0}^{\infty} \sum_{n=0}^{\infty} |a_{j,j+N}|\,|a_{n,n+N}|\left|\left(z^j\overline{z}^{j+N}, z^n\overline{z}^{n+N}\right)\right|$$
$$= \pi \sum_{N=-\infty}^{\infty} \sum_{j=\max\{0,-N\}}^{\infty} \sum_{n=\max\{0,-N\}}^{\infty} \frac{|a_{j,j+N}||a_{n,n+N}|}{j+n+N+1} < \infty.$$

Now use Lemma 641. ■

Remark: Note in the above proof we used the observation made in Section 640: The "diagonals" are pairwise orthogonal. ■

Theorem 643 *Assume*

$$\sum_{N=-\infty}^{\infty} \left\{ \sum_{j=\max\{1,-N\}}^{\infty} \sum_{n=\max\{1,-N\}}^{\infty} \frac{jn}{j+n+N+1} |a_{j,j+N}|\,|a_{n,n+N}| \right.$$
$$\left. + \sum_{j=\max\{1,-N+1\}}^{\infty} \sum_{n=\max\{1,-N+1\}}^{\infty} \frac{(j+N)(n+n)}{j+n+N+1} |a_{j,j+N}|\,|a_{n,n+N}| \right\} < \infty.$$

Set $f = \sum_{j=0}^{\infty} \sum_{n=0}^{\infty} a_{jn} z^j \overline{z}^n$. *By Theorem 642, we see* $f \in L^2$.

Think of $f \in \mathcal{D}' = \mathcal{D}'(\mathbb{D})$. *Then in* $\mathcal{D}'$,

$$\begin{cases} \partial f = \sum_{j=1}^{\infty} \sum_{n=0}^{\infty} j a_{jn} z^{j-1} \overline{z}^n \\ \overline{\partial} f = \sum_{j=0}^{\infty} \sum_{n=1}^{\infty} n a_{jn} z^j \overline{z}^{n-1}. \end{cases}$$

Proof. Let $\psi \in \mathcal{D} = \mathcal{D}(\mathbb{D})$.

$$\begin{aligned}
(\partial f)\,\psi &= -f\,(\partial\psi) \\
&= -\sum_{j=0}^{\infty}\sum_{n=0}^{\infty} a_{jn} \iint_{\mathbb{D}} z^j \overline{z}^n \partial\psi \, dA \\
&= \sum_{j=1}^{\infty}\sum_{n=0}^{\infty} j a_{jn} \iint_{\mathbb{D}} z^{j-1} \overline{z}^n \psi \, dA \\
&= \left(\sum_{j=1}^{\infty}\sum_{n=0}^{\infty} j a_{jn} z^{j-1}\overline{z}^n\right)\psi,
\end{aligned}$$

where convergence is guaranteed by Lemma 641 and where we used the first "half" of the hypotheses above.

Similarly for $\overline{\partial} f$ (where we use the second "half"). ∎

Lemma 644 *Let $(a_{jn})_{j,n=0}^{\infty}$ satisfy the hypotheses of Theorem 643. Set $f = \sum_{j=0}^{\infty}\sum_{n=0}^{\infty} a_{jn} z^j \overline{z}^n$. Then for $k, \ell \in \mathbb{N}^*$,*

$$\left(z^k \overline{\partial} f, z^\ell\right) = \pi \sum_{n=\max\{1,k-\ell+1\}}^{\infty} \frac{n a_{n-1+\ell-k,n}}{\ell + n}.$$

Proof. By Theorem 643,
$\left(z^k\overline{\partial} f, z^\ell\right) = \left(\overline{\partial} f\right)\left(z^k \overline{z}^\ell\right) = \sum_{j=0}^{\infty}\sum_{n=1}^{\infty} n a_{jn}\left(z^{j+k}, z^{\ell+n-1}\right)$. Note $\left(z^{j+k}, z^{\ell+n-1}\right) = 0$ unless $j = \ell + n - 1 - k$ and $\ell + n - 1 - k \geq 0$. ∎

Theorem 645 *Let $(a_{jn})_{j,n=0}^{\infty}$ satisfy the hypotheses of Theorem 643. Assume $\phi \in C_c^1(\mathbb{C})$ and write $\phi|_{\mathbb{D}} = \sum_{j=0}^{\infty}\sum_{n=0}^{\infty} a_{jn} z^j \overline{z^n}$ in L^2. TFAE:*

(a) *$T_\phi = PM_\phi$ in $L(L_a^2)$.*

(b) *For all $k \in \mathbb{N}^*$ and for all $\ell \in \mathbb{N}$,*

$$\sum_{n=\max\{1,k-\ell+1\}}^{\infty} \frac{n a_{n-1+k-\ell,n}}{\ell + n} = 0.$$

Proof. By Theorem 519, if $g = \sum_{k=0}^{\infty} b_k z^k \in L_a^2$, then

$$PC_{\overline{\partial}\phi} g = \sum_{\ell=0}^{\infty} \left(g\overline{\partial}\phi, z^{\ell+1} \right) z^\ell = \sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} b_k \left(z^k \overline{\partial}\phi, z^{\ell+1} \right) z^\ell.$$

Recall: $T_\phi = PT_\phi = PM_\phi + \frac{1}{\pi} PC_{\overline{\partial}\phi}$ in $L(L_a^2)$.

(a) $\Longrightarrow$ (b): Let $k \in \mathbb{N}^*$. Taking $g = z^k$ in the above gives:
$0 = PC_{\overline{\partial}\phi} g = \sum_{\ell=0}^{\infty} \left(z^k \overline{\partial}\phi, z^{\ell+1} \right) z^\ell$. Hence for each $\ell \in \mathbb{N}$,
$\left(z^k \overline{\partial}\phi, z^\ell \right) = 0$. Now use Lemma 644.

(b) $\Longrightarrow$ (a): By (b) and Lemma 644, we see $\left(z^k \overline{\partial}\phi, z^{\ell+1} \right) = 0 \qquad (k, \ell \in \mathbb{N}^*)$.
By the above calculation, $PC_{\overline{\partial}\phi} = 0$ in $L\left(L_a^2\right)$. Hence $T_\phi = PM_\phi$ in $L\left(L_a^2\right)$. ■

Lemma 646 *Let $j, n \in \mathbb{N}^*$. Then*

$$\widehat{z^j \overline{z}^n} = \begin{cases} \frac{\pi}{n+1} \left(z^{j-n-1} - z^j \overline{z}^{n+1} \right) & \text{if } j - n - 1 \geq 0, \\ -\frac{\pi}{n+1} z^j \overline{z}^{n+1} & \text{if } j - n - 1 < 0. \end{cases}$$

Proof. Fix $z \in \mathbb{D}$. For $\epsilon > 0$ small enough, set $\Omega_\epsilon = \mathbb{D} \setminus D(z; \epsilon)$.

$$\begin{aligned} \widehat{z^j \overline{z}^n} = \iint_{\mathbb{D}} \frac{\zeta^j \overline{\zeta^n}}{\zeta - z} \, dA &= \frac{1}{n+1} \lim_{\epsilon \searrow 0} \iint_{\Omega_\epsilon} \overline{\partial} \left(\frac{\zeta^j \overline{\zeta^{n+1}}}{\zeta - z} \right) dA \\ &= \frac{1}{2(n+1)i} \lim_{\epsilon \searrow 0} \iint_{\partial\Omega_\epsilon} \frac{\zeta^j \overline{\zeta^{n+1}}}{\zeta - z} \, d\zeta. \end{aligned}$$

Now use

$$\begin{aligned} \int_{\mathbb{T}} \frac{\zeta^j \overline{\zeta^{n+1}}}{\zeta - z} \, d\zeta &= \sum_{\ell=0}^{\infty} z^\ell \int_{\mathbb{T}} \frac{\zeta^j \overline{\zeta^{n+1}}}{\zeta^{\ell+1}} \, d\zeta \\ &= \begin{cases} 2\pi i z^{j-n-1} & \text{if } j - n - 1 \geq 0, \\ 0 & \text{if } j - n - 1 < 0, \end{cases} \end{aligned}$$

and

$$\lim_{\epsilon\searrow 0}\int\limits_{|\zeta-z|=\epsilon}\frac{\zeta^j\overline{\zeta}^n}{\zeta-z}\,d\zeta = \lim_{\epsilon\searrow 0}\int_0^{2\pi}\frac{(z+\epsilon e^{i\theta})^j(\overline{z}+\epsilon e^{-i\theta})^n}{\epsilon e^{i\theta}}i\epsilon e^{i\theta}\,d\theta$$
$$= 2\pi i z^j\overline{z}^n.$$

■

Theorem 647 *Let* $f=\sum_{j=0}^{\infty}\sum_{n=0}^{\infty}a_{jn}z^j\overline{z}^n$, *where the series converges in* L^2. *Then*

$$\hat{f}=\pi\sum_{j=0}^{\infty}\left[\sum_{n=0}^{j-1}\frac{a_{jn}}{n+1}z^{j-n-1}-\sum_{n=0}^{\infty}\frac{a_{jn}}{n+1}z^j\overline{z}^{n+1}\right].$$

Remark: If $j=0$, then we assume the first term inside the brackets does not appear.

Proof. By Theorem 511, we may take the Cauchy transform of each summand separately. Now use Lemma 646. ■

Corollary 648 *Let* $f=\sum_{j=0}^{\infty}a_jz^j\in L_a^2$. *Then*

$$\hat{f}=\pi\left[\sum_{j=1}^{\infty}a_jz^{j-1}-a_0\overline{z}-\left(\sum_{j=1}^{\infty}a_jz^j\right)\overline{z}^2\right].$$

Proof. Use Theorem 647. ■

Section 649 Let $(e_n)_{n=0}^{\infty}$ denote the standard orthonormal basis on L_a^2, where $e_n=\sqrt{\frac{n+1}{\pi}}z^n$. Let B denote the following backwards shift: $Be_n=\sqrt{\frac{n+1}{n}}e_{n-1}$. Let P_0 denote the following projection: $P_0\left(\sum_{n=0}^{\infty}a_ne_n\right)=a_0e_0$.

By Corollary 648, for $f\in L_a^2$,

$$\hat{f}=\pi\left\{Bf-\frac{1}{\sqrt{\pi}}(P_0f)\overline{z}-[(I-P_0)f]\overline{z}^2\right\}.$$

Hence,

$$\wedge = \pi \left[B - \frac{1}{\sqrt{\pi}} M_{\overline{z}} P_0 - M_{\overline{z}^2} \left(I - P_0 \right) \right]$$

in $L(L_a^2, L^2)$. □

Once more, let's think of $(z^j \overline{z}^n)_{j,n=0}^{\infty}$ as an infinite matrix. Note the $n = 0$ column is L_a^2. By Corollary 648, $\wedge$ maps L_a^2 into the columns corresponding to $n = 0, 1, 2$. Note that only the $(0, 1)$-entry is "involved" in the $n = 1$ column. ■

Chapter VII
A Generalization of the T_ϕ Operator

Abstract

When we defined T_ϕ in Chapter VI, we required $\phi \in C_c^1(\mathbb{C})$. Applications aside, if we wish to study T_ϕ as an operator, we do not need ϕ to satisfy such a strong condition. Using the notation of Chapter V, we see $T_\phi = M_\phi + \frac{1}{\pi} C_{\overline{\partial}\phi}$. For many of our results, we only need $\phi \in L^\infty$ and $\overline{\partial}\phi \in L^\infty$. (This discussion is included as motivation: hence the underlying set will be omitted.)

We can also consider a further generalization: $M_g + \frac{1}{\pi} C_h$, where $g, h \in L^\infty$. A special case is the above one: $h = \overline{\partial} g$ in $\mathcal{D}'(\mathbb{D})$.

Section 700 Let $1 \le p_1, p_2 \le \infty$ and $S \subseteq \mathbb{C}$ be m_2-measurable.

Set $[X(p_1, p_2)](S) = (L^{p_1} \oplus L^{p_2})(S)$, a Banach space with norm given by

$$\|f_1 \oplus f_2\|_{[X(p_1,p_2)](S)} = \begin{cases} \left(\|f_1\|_{p_1}^{p_1} + \|f_2\|_{p_2}^{p_2}\right)^{\frac{2}{p_1+p_2}} & \text{if } 1 \le p_1 + p_2 < \infty \\ \max\{\|f\|_{p_1}, \|f\|_{p_2}\} & \text{if } p_1 + p_2 = \infty, \end{cases}$$

$$\text{where} \quad \|f\|_p = \begin{cases} \left(\iint_S |f|^p \, dA\right)^{\frac{1}{p}} & \text{if } 1 \le p < \infty \\ \|f\|_{L^\infty(S, dA)} & \text{if } p = \infty. \end{cases}$$

We will write $X^p(S)$ to denote $[X(p,p)](S)$.

Writing $\frac{1}{p_j} + \frac{1}{q_j} = 1 \quad (j = 1, 2)$, we will think of

$$[X(q_1, q_2)](S) \subseteq \{[X(p_1, p_2)](S)\}^*.$$

Given $g_1 \oplus g_2 \in [X(q_1, q_2)](S)$, we will identify it with the following element of $\{[X(p_1, p_2)](S)\}^*$: For $f_1 \oplus f_2 \in [X(p_1, p_2)](S)$, $(g_1 \oplus g_2)(f_1 \oplus f_2) = \sum_{j=1}^{2} \iint_S f_j g_j \, dA$. [If $p_j = 2$, we will write $\overline{g}_j$ instead of g_j.]

Let $\Omega \subseteq \mathbb{C}$ be open. Set $[V(p_1, p_2)](\Omega) = \left\{f \in L^{p_1}(\Omega) : \overline{\partial} f \in L^{p_2}(\Omega)\right\}$. We will think of $[V(p_1, p_2)](\Omega) \subseteq [X(p_1, p_2)](\Omega)$: We will identify $f \in [V(p_1, p_2)](\Omega)$ with $f \oplus \overline{\partial} f \in [X(p_1, p_2)](\Omega)$. ■

Lemma 701 *Let $\Omega \subseteq \mathbb{C}$ be open, (Λ_α) be a net in $\mathcal{D}'(\Omega)$, and $\Gamma_1, \Gamma_2 \in \mathcal{D}'(\Omega)$. Assume $\Lambda_\alpha \longrightarrow \Gamma_1$ in $\mathcal{D}'(\Omega)$ and $\overline{\partial}\Lambda_\alpha \longrightarrow \Gamma_2$ in $\mathcal{D}'(\Omega)$.*

Then $\Gamma_2 = \overline{\partial}\Gamma_1$ in $\mathcal{D}'(\Omega)$.

Proof. For $\phi \in \mathcal{D}(\Omega)$,

$$\Gamma_2\phi = \lim \left(\overline{\partial}\Lambda_\alpha\right)\phi = -\lim \Lambda_\alpha\left(\overline{\partial}\phi\right) = -\Gamma_1\left(\overline{\partial}\phi\right) = \left(\overline{\partial}\Gamma_1\right)\phi.$$

■

Theorem 702 *Let $\Omega \subseteq \mathbb{C}$ be open.*

(a) *Let $1 \le p_1, p_2 \le \infty$. Then $[V(p_1, p_2)](\Omega)$ is a closed subspace of $[X(p_1, p_2)](\Omega)$.*

(b) *Let $1 < p_1, p_2 \le \infty$. Then $[V(p_1, p_2)](\Omega)$ is a $w - *$ closed subspace of $([X(p_1, p_2)](\Omega), w - *)$.*

Proof. Use Lemma 701 and note (in **(b)**) that

$$([X(p_1, p_2)](\Omega), w - *) = (L^{p_1}(\Omega), w - *) \oplus (L^{p_2}(\Omega), w - *).$$

■

Definition: Let $1 \le p \le \infty$ and $S \subseteq \mathbb{C}$ be measurable, with $m_2 S < \infty$. For $g_1 \oplus g_2 \in X_\infty(S)$, define $\Phi(g_1 \oplus g_2)$ on $L^p(S)$ by $[\Phi(g_1 \oplus g_2)] f = g_1 f + \frac{1}{\pi}\widehat{g_2 f}$.

Remark: If $\phi \in C_c^1(\mathbb{C})$, then $\Phi\left(\phi \oplus \overline{\partial}\phi\right) = T_\phi$. ■

Theorem 703 *Let $1 \le p \le \infty$ and $S \subseteq \mathbb{C}$ be measurable, with $m_2 S < \infty$. Then $\Phi : X_\infty(S) \longrightarrow L(L^p(S))$ is continuous.*

Proof. Fix $g_1 \oplus g_2 \in X_\infty(S)$. For $f \in L^p(S)$,

$$\begin{aligned}
\| [\Phi(g_1 \oplus g_2)] f\|_p &\leq \|g_1 f\|_p + \frac{1}{\pi}\|\widehat{g_2 f}\|_p \\
&\leq \|g_1\|_\infty \|f\|_p + \frac{1}{\pi}\|g_2\|_\infty \|f\|_p \left(2\sqrt{\pi m_2 S}\right) \\
&\leq \left(1 + 2\sqrt{\frac{m_2 S}{\pi}}\right) \|g_1 \oplus g_2\|_{X_\infty(S)} \|f\|_p,
\end{aligned}$$

where we have used Theorem 511.

So $\Phi(g_1 \oplus g_2) \in L(L^p(S))$ and $\|\Phi(g_1 \oplus g_2)\| \leq \left(1 + 2\sqrt{\frac{m_2 S}{\pi}}\right) \|g_1 \oplus g_2\|_{X_\infty(S)}$. Hence, Φ is continuous and $\|\Phi\| \leq 1 + 2\sqrt{\frac{m_2 S}{\pi}}$. ■

Corollary 704 *Let $\Omega \subseteq \mathbb{C}$ be open and bounded. Let $g_1 \oplus g_2 \in X_\infty(\Omega)$. Then $\Phi(g_1 \oplus g_2) \in (\mathcal{N} + \mathcal{K})(L^2(\Omega))$.*

Remark: $\mathcal{N}(H)$ and $\mathcal{K}(H)$ denote the normal and compact operators on a Hilbert space H, respectively.

Proof. Use Corollary 516. ■

Theorem 705 *Let $S \subseteq \mathbb{C}$ be bounded. Then Φ is continuous from $(X_\infty(S), w-*)$ into $(L(L^2(S)), w-*)$.*

Remark: See the Recall section preceeding Lemma 525.

Proof. Let $g_\alpha = g_{\alpha 1} \oplus g_{\alpha 2} \longrightarrow 0 \quad w-*$ in $X_\infty(S)$. Fix sequences (f_n) and (h_n) in $L^2(S)$ satisfying: $\sum(\|f_n\|_2^2 + \|h_n\|_2^2) < \infty$.

$$\begin{aligned}
&\sum_n (\Phi(g_\alpha) f_n, h_n) \\
&= \sum \left\{ \iint_S g_{\alpha 1} f_n \overline{h}_n \, dA + \frac{1}{\pi} \iint_S \left(\iint_S \frac{(g_{\alpha 2} f_n)(\zeta)}{\zeta - z} \, dA(\zeta) \right) \overline{h_n(z)} \, dA(z) \right\} \\
&= \iint_S g_{\alpha 1} \sum f_n \overline{h}_n \, dA + \frac{1}{\pi} \iint_S g_{\alpha 2} \sum f_n \left[\left(\frac{1}{z} \chi_{(2R)\mathbb{D}} \right) * h_n \right] dA \longrightarrow 0
\end{aligned}$$

since each series belongs to $L^1(S)$. We have assumed $S \subseteq R\mathbb{D}$.

By Chapter I, Proposition 5.5 [4], we see $\Phi(g_\alpha) \longrightarrow 0 \quad w - *$ in $L(L^2(S))$. ■

Theorem 706 *Let $1 \le p \le \infty$ and $\Omega \subseteq \mathbb{C}$ be open, with $m_2\Omega < \infty$. Then $\Phi(V_\infty(\Omega)) \subseteq L(L^p_a(\Omega))$.*

Proof. We already know (Theorem 703): $\Phi(V_\infty(\Omega)) \subseteq L(L^p(\Omega))$. So let $g \in V_\infty(\Omega)$ and $f \in L^p_a(\Omega)$. By Lemma 134,

$$\begin{aligned} \overline{\partial}[\Phi(g)f] &= \overline{\partial}\left[gf + \frac{1}{\pi}\left(f\overline{\partial}g\right)^\wedge\right] \\ &= g\overline{\partial}f + f\overline{\partial}g - f\overline{\partial}g = g\overline{\partial}f = 0 \quad \text{in} \quad \mathcal{D}'(\Omega). \end{aligned}$$

■

Theorem 707 *Let $1 \le p_1, p_2 \le \infty$ and $\Omega \subseteq \mathbb{C}$ be open. Let $\Omega_1 \subseteq \mathbb{C}$ be open, with $\overline{\Omega}_1 \subseteq \Omega$ compact. Then $\mathcal{D}(\Omega)$ is dense in $[V(p_1, p_2)](\Omega_1)$.*

Proof. Fix $f \in [V(p_1,p_2)](\Omega_1)$. Let (h_n) be an approximate identity on $\mathbb{C}$. By Proposition 111, for n large enough, $h_n * f \in \mathcal{D}(\Omega)$. Note
$h_n * f \longrightarrow f$ in $L^{p_1}(\Omega_1)$ and $\overline{\partial}(h_n * f) = h_n * \left(\overline{\partial}f\right) \longrightarrow \overline{\partial}f$ in $L^{p_2}(\Omega_1)$. ■

Section 708 Let $\Omega \subseteq \mathbb{C}$ be open and bounded. Fix $z_0 \in \Omega$. Let k_{z_0} denote the reproducing kernel (at z_0) for $L^2_a(\Omega) : (f, k_{z_0}) = f(z_0) \quad (f \in L^2_a(\Omega))$.

Choose an orthonormal basis $(e_n)_{n=0}^\infty$ for $L^2_a(\Omega)$, with $e_0 = \frac{1}{\sqrt{m_2\Omega}}$.

Define $\tau_{z_0} : V_\infty(\Omega) \longrightarrow \mathbb{C}$ by

$$\begin{aligned} \tau_{z_0} &= \{[\Phi(g)]\,1\}(z_0) = ([\Phi(g)]\,1, k_{z_0}) \\ &= \left(g + \frac{1}{\pi}\widehat{\overline{\partial}g}, k_{z_0}\right). \end{aligned}$$

By Theorem 511, for $g \in V_\infty$,

$$\begin{aligned} |\tau_{z_0}(g)| &\leq \|g + \frac{1}{\pi}\widehat{\bar{\partial} g}\|_\infty \|k_{z_0}\|_1 \\ &\leq \left(1 + 2\sqrt{\frac{m_2 \Omega}{\pi}}\right) \|g\|_{X_\infty(\Omega)} \|k_{z_0}\|_1. \end{aligned}$$

So $\tau_{z_0} \in [V_\infty(\Omega)]^*$, with $\|\tau_{z_0}\| \leq \left(1 + 2\sqrt{\frac{m_2\Omega}{\pi}}\right) \|k_{z_0}\|_1$.

By Theorem 702, we note $V_\infty(\Omega) \subseteq X_\infty(\Omega)$ is a $w - *$ closed subspace. Hence, there exists a closed subspace M of $X_1(\Omega)$ with $M^\perp = V_\infty(\Omega)$ (see [11, Theorem 4.7 (b)]). Note $[X_1(\Omega)/M]^* = M^\perp = V_\infty(\Omega)$.

By Theorem 706, we know $\Phi : V_\infty(\Omega) \longrightarrow L\left(L^2_a(\Omega)\right)$.

Hence, by the Recall section preceding Lemma 525, there exists $\Psi : \mathcal{K}^1\left(L^2_a(\Omega)\right) \longrightarrow X_1(\Omega)/M$ with $\Phi = \Psi^*$. We have:

$$\mathcal{K}^1\left(L^2_a(\Omega)\right) \xrightarrow{\Psi} X_1(\Omega)/M$$

$$L\left(L^2_a(\Omega)\right) \xleftarrow{\Phi=\Psi^*} V_\infty(\Omega)$$

□

Define $\lambda_{z_0} : L^2_a(\Omega) \longrightarrow L^2_a(\Omega)$ by $\lambda_{z_0}(f) = (f, K_{z_0})\,1$. Note λ_{z_0} is a rank one operator, hence, $\lambda_{z_0} \in \mathcal{K}^1\left(L^2_a(\Omega)\right)$.

Let J_1 and J_2 denote the canonical isometries from $\mathcal{K}^1\left(L^2_a(\Omega)\right)$ into $\left[L\left(L^2_a(\Omega)\right)\right]^*$ and from $X_1(\Omega)/M$ into $[V_\infty(\Omega)]^*$, respectively.

Theorem 709 $J_2\Psi\lambda_{z_0} = \tau_{z_0}$ *in* $[V_\infty(\Omega)]^*$.

Proof. Let $g \in V_\infty(\Omega)$.

$$\begin{aligned} (\Phi^* J_1 \lambda_{z_0})\, g &= (J_1\lambda_{z_0})(\Phi(g)) \\ &= [\Phi(g)]\,\lambda_{z_0} = tr\left(\lambda_{z_0}\Phi(g)\right) = \sum_{n=0}^{\infty} (\lambda_{z_0}\Phi(g)e_n, e_n) \\ &= \sum_{n=0}^{\infty} (\Phi(g)e_n, K_{z_0})(1, e_0) = (\Phi(g)e_0, K_{z_0})(1, e_0) = \tau_{z_0}(g). \end{aligned}$$

Hence, $\tau_{z_0} = \Phi^* J_1 \lambda_{z_0} = J_2 \Psi \lambda_{z_0}$ in $[V_\infty(\Omega)]^*$. ■

Theorem 710 *Let $\Omega \subseteq \mathbb{C}$ be open and bounded. Let M be as in Section 708. Then* $M = \left\{\overline{\partial} f \oplus f \in X_1(\Omega) : f \in V_1(\Omega) \text{ and } g\left(\overline{\partial} f \oplus f\right) = 0 \quad (g \in V_\infty(\Omega))\right\}$.

Proof.

"$\subseteq$" Fix $f_1 \oplus f_2 \in M$. Let $\phi \in \mathcal{D}(\Omega)$. Since $\phi \in V_\infty(\Omega)$, we get

$$0 = \phi(f_1 \oplus f_2) = \iint_\Omega \left(f_1\phi + f_2\overline{\partial}\phi\right) dA = \left(f_1 - \overline{\partial} f_2\right)\phi.$$

So $\overline{\partial} f_2 \in L^1(\Omega)$ and $f_1 = \overline{\partial} f_2$ in $\mathcal{D}'(\Omega)$.

So we may write $f_1 \oplus f_2 = \overline{\partial} f \oplus f$, where $f \in V_1(\Omega)$.

So for $g \in V_\infty(\Omega) = M^\perp$, we have $g\left(\overline{\partial} f \oplus f\right) = 0$.

"$\supseteq$" Clear. ■

We will extend the generalization of the T_ϕ operator to $\mathcal{E}' \oplus \mathcal{E}'$, but first a Lemma.

Lemma 711 *Let $\Lambda \in \mathcal{E}'(\mathbb{C})$. Then $\overline{\partial}\hat{\Lambda} = -\pi\Lambda$ in $\mathcal{D}'(\mathbb{C})$.*

Proof. Let $\phi \in \mathcal{D}(\mathbb{C})$. By Corollary 121,

$$\left(\overline{\partial}\hat{\Lambda}\right)\phi = -\hat{\Lambda}\left(\overline{\partial}\phi\right) = \Lambda\left(\widehat{\overline{\partial}\phi}\right) = -\pi\Lambda\phi.$$

■

Definition: Let $\Omega \subseteq \mathbb{C}$ be open and $\Lambda \in \mathcal{D}'(\Omega)$. We write $\Lambda \in (H\mathcal{D}')(\Omega)$ if $\overline{\partial}\Lambda = 0$ in $\mathcal{D}'(\Omega)$.

Theorem 712 *Let* $K \subseteq \mathbb{C}$ *be compact and* $\Lambda_1, \Lambda_2 \in \mathcal{E}'(\mathbb{C})$, *with* $\text{supp}\Lambda_1 \subseteq K$. *Define* $\Phi(\Lambda_1 \oplus \Lambda_2) : C^\infty(\mathbb{C}) \longrightarrow \mathcal{D}'(\mathbb{C})$ *by* $[\Phi(\Lambda_1 \oplus \Lambda_2)]\psi = \psi\Lambda_1 + \frac{1}{\pi}\widehat{\psi\Lambda_2}$.

TFAE:

(a) $\Phi(\Lambda_1 \oplus \Lambda_2) : (H\mathcal{D})(K) \longrightarrow (H\mathcal{D}')(\mathbb{C})$

(b) $[(H\mathcal{D})(K)]\left(\overline{\partial}\Lambda_1 - \Lambda_2\right) = 0$ *in* $\mathcal{D}'(\mathbb{C})$.

Remarks: By **(b)**, we mean: $\psi\left(\overline{\partial}\Lambda_1 - \Lambda_2\right) = 0$ in $\mathcal{D}'(\mathbb{C})$ $\quad (\psi \in (\text{H}\mathcal{D})(\text{K}))$. Recall: $(H\mathcal{D})(K)$ was defined prior to Proposition 153. Note $\Phi(\Lambda_1 \oplus \Lambda_2)$ is a generalization of the (distribution version of the) T_ϕ operator defined at the beginning of Chapter VI.

Proof. Let $\psi \in C^\infty(\mathbb{C})$. By Lemma 711,

$$\overline{\partial}\{[\Phi(\Lambda_1 \oplus \Lambda_2)]\psi\} = \psi\overline{\partial}\Lambda_1 + \left(\overline{\partial}\psi\right)\Lambda_1 - \psi\Lambda.$$

<u>**(a) $\Longrightarrow$ (b):**</u> For $\psi \in (H\mathcal{D})(K)$, the above calculation shows:

$0 = \psi\left(\overline{\partial}\Lambda_1 - \Lambda_2\right)$ in $\mathcal{D}'(\mathbb{C})$.

<u>**(b) $\Longrightarrow$ (a):**</u> Use the above calculation. ■

Chapter VIII
The h_ϕ Operator

Abstract

A peek at [6] shows the T_ϕ operator is an essential tool in rational approximation. Which properties of this localization operator contribute to its usefulness?

(i) $T_\phi f$ is continuous wherever f is continuous.

(ii) $T_\phi f$ is analytic wherever f is analytic. □

We present the operator h_ϕ, the harmonic (approximation) analogue of T_ϕ. The analogue of (ii) holds:

$$h_\phi f \quad \text{is harmonic wherever} \quad f \quad \text{is harmonic.}$$

Unfortunately, (i) does not quite hold. What can be said is the following:

$$\text{If} \quad 1 < p \leq \infty \quad \text{and} \quad f \in W^{1,p}(\Omega), \quad \text{then} \quad h_\phi f \in C(\Omega).$$

■

Definition: For $\phi \in \mathcal{D}'(\mathbb{C})$, define $h_\phi : \mathcal{D}'(\mathbb{C}) \longrightarrow \mathcal{D}'(\mathbb{C})$ by $h_\phi \Lambda = \phi\Lambda - \frac{1}{2\pi} \log * \left\{(\triangle\phi)\Lambda + 4\left[\left(\overline{\partial}\phi\right)(\partial\Lambda) + (\partial\phi)\left(\overline{\partial}\Lambda\right)\right]\right\}$.

Proposition 800 *Let $\phi \in \mathcal{D}(\mathbb{C})$. Then $h_\phi : \mathcal{D}'(\mathbb{C}) \longrightarrow \mathcal{D}'(\mathbb{C})$ is continuous.*

Proof. Let $\Lambda_\alpha \longrightarrow 0$ in $\mathcal{D}'(\mathbb{C})$. Let $\psi \in \mathcal{D}(\mathbb{C})$.

$$\begin{aligned}(h_\phi \Lambda_\alpha)\psi = \Lambda_\alpha(\phi\psi) - \frac{1}{2\pi}&\{\Lambda_\alpha[(\triangle\phi)(\log * \psi)] \\ &-4\left[\Lambda_\alpha\left\{\partial\left[\left(\overline{\partial}\phi\right)(\log * \psi)\right]\right\} + \Lambda_\alpha\left\{\overline{\partial}[(\partial\phi)(\log * \psi)]\right\}\right] \\ &\longrightarrow 0.\end{aligned}$$

■

Notation: For (m_2 -) measurable $S \subseteq \mathbb{C}$, set

$$(h\mathcal{D}')(S) = \{\Lambda \;:\; \text{there exists an open set } \Omega \subseteq \mathbb{C} \text{ with } S \subseteq \Omega,\; \Lambda \in \mathcal{D}'(\Omega), \text{ and } \triangle\Lambda = 0 \text{ in } \mathcal{D}'(\Omega)\}.$$

Proposition 801 *Let* $\phi \in \mathcal{D}(\mathbb{C})$ *and* $K \subseteq \mathbb{C}$ *be compact.*

Then $h_\phi : (h\mathcal{D}')(K) \longrightarrow (h\mathcal{D}')(K)$.

Remarks: This is the harmonic analogue of property **(ii)** (from the abstract).

We will not explicitly write out the details of the "cutoff" technique which enables us to think of $(h\mathcal{D}')(K)$ as being contained in $\mathcal{D}'(\mathbb{C})$ (as in Proposition 601).

Proof. Let $\Lambda \in (h\mathcal{D}')(K)$. Assume $\Omega \subseteq \mathbb{C}$ is open, $K \subseteq \Omega$, $\Lambda \in \mathcal{D}'(\Omega)$,

and $\triangle\Lambda = 0$ in $\mathcal{D}'(\Omega)$. We show: $\triangle(h_\phi\Lambda) = 0$ in $\mathcal{D}'(\Omega)$. By Corollary 204,

$$\begin{aligned}\triangle(h_\phi\Lambda) &= 4\overline{\partial}\partial(\phi\Lambda) - \left\{(\triangle\phi)\Lambda + 4\left[\left(\overline{\partial}\phi\right)(\partial\Lambda) + (\partial\phi)\left(\overline{\partial}\Lambda\right)\right]\right\} \\ &= \phi\triangle\Lambda = 0 \quad \text{in} \quad \mathcal{D}'(\Omega).\end{aligned}$$

■

Proposition 802 *Let* $1 \le p \le \infty$ *and* $S \subseteq \mathbb{C}$ *be bounded.*

Then $\log * (\ ) : L^p(S) \longrightarrow L^p(S)$ *is bounded.*

Proof. First assume $1 \le p < \infty$. For $z \in S$, set $C_z = \iint_S |\log|z-\zeta||\, dA(\zeta)$. Set $C = \sup_{z\in S} C_z < \infty$. By Jensen's, if $f \in L^p(S)$, then

$$\begin{aligned}\|\log * f\|_p^p &\le \iint_S C_z^p \left[\frac{1}{C_z}\iint_S |f(\zeta)|\,|\log|z-\zeta||\, dA(\zeta)\right]^p dA(z) \\ &\le \iint_S C_z^{p-1}\left[\iint_S |f(\zeta)|^p\,|\log|z-\zeta||\right] dA(z) \\ &\le (C\|f\|_p)^p .\end{aligned}$$

For $p = \infty$, we see $\|\log * \phi\|_\infty \le C\|f\|_\infty$. ■

Proposition 803 *Let* $\mathcal{D}_1'(\mathbb{C})$ *denote the distributions of order 1 (see the "Preliminaries"). Let* $\phi \in C_c^2(\mathbb{C})$. *Then* $h_\phi : \mathcal{D}_1'(\mathbb{C}) \longrightarrow \mathcal{D}_1'(\mathbb{C})$.

Proof. Let $\Lambda \in \mathcal{D}_1'(\mathbb{C})$. Then $\phi\Lambda \in \mathcal{D}_1'(\mathbb{C})$. Let $\psi \in \mathcal{D}(\mathbb{C})$. Note

$$\left(\log * \left\{(\triangle\phi) + 4\left[\left(\overline{\partial}\phi\right)\ (\partial\Lambda)\ + (\partial\Lambda)\left(\overline{\partial}\Lambda\right)\right]\right\}\right)\psi$$
$$= \Lambda\left[(\triangle\phi)(\log * \psi)\right] - 4\left[\Lambda\left\{\partial\left[\left(\overline{\partial}\phi\right)(\log * \psi)\right]\right\} + \Lambda\left\{\overline{\partial}\left[(\partial\phi)(\log * \psi)\right]\right\}\right].$$

Combined with Proposition 802 ($p = \infty$), we see $h_\phi\Lambda \in \mathcal{D}_1'(\mathbb{C})$. ■

Proposition 804 *Let* $1 \leq p \leq \infty$ *and* $\phi \in C_c^2(\mathbb{C})$. *Let* $\Omega \subseteq \mathbb{C}$ *be open and bounded. Then* $h_\phi : W^{1,p}(\Omega) \longrightarrow L^p(\Omega)$ *is bounded.*

Proof Use Proposition 802. ■

Theorem 805 *Let* $1 < p \leq \infty$ *and* $\phi \in C_c^2(\mathbb{C})$. *Let* $\Omega \subseteq \mathbb{C}$ *be open and bounded. Then* $h_\phi : W^{1,p}(\mathbb{C}) \longrightarrow \mathrm{C}(\mathbb{C})$.

Proof. Using Proposition 200 and preceeding as in the proof of Lemma 132, we see that if $f \in L^p(\Omega)$, then $\log * f \in C(\Omega)$. ■

Notation: Let $\Omega \subseteq \mathbb{C}$ be open and $1 \leq p \leq \infty$. Set
$L_h^p(\Omega) = \{f \in L^p(\Omega) : f \text{ is harmonic in } \Omega\}$.

Remark: $L_h^p(\Omega)$ is the (harmonic) analogue of a Bergman space. ■

Theorem 806 *Let* $\phi \in C_c^2(\mathbb{C})$ *and* $\Omega \subseteq \mathbb{C}$ *be open. Let* $f \in L_h^1(\Omega)$. *Then for all* $z_0 \in \mathbb{C}$,

$$(h_\phi f)(z_0) = \frac{1}{2\pi i}\left[\int_{\partial\Omega} \frac{\phi f}{z - z_0}\, dz + 2\int_{\partial\Omega} \left[\overline{\partial}(\phi f)\right] \log|z_0 - z|\, d\overline{z}.\right]$$

Remark: We set $f \equiv 0$ in $\mathbb{C} \setminus \Omega$.

Proof By [12, p. 255], we know $f \in C^\infty(\Omega)$.

Let $K_n \longrightarrow \Omega$ inner interior f.p.a.. As in the first half of the proof of Theorem 604, we may write $(\phi f, K_n, E_n)$, where $E_n \subsetneq \partial K_n$.

Let $z_0 \in \Omega$. Assume $z_0 \in \operatorname{int} K_n$. As in the proof of Theorem 604, with n fixed, we may extend $f \mid_{K_n}$ to be in $\mathcal{D}(\Omega)$ while retaining the original values of f in a neighborhood of K_n. By Theorem 208,

$$\begin{aligned}
-\frac{1}{2\pi} \iint\limits_{K_n} & \left\{ f \triangle \phi + 4\left[\left(\overline{\partial} f\right)(\partial \phi) + (\partial f)\left(\overline{\partial}\phi\right)\right]\right\} \log|z_0 - z| \, dA \\
&= -\frac{1}{2\pi} \iint\limits_{K_n} [\triangle (f\phi)] \log|z_0 - z| \, dA \\
&= -(\phi f)(z_0) + \frac{1}{2\pi i}\left[\int\limits_{B_{K_n}} \frac{\phi f}{z - z_0} dz + 2 \int\limits_{B_{K_n}} \left[\overline{\partial}(\phi f)\right] \log|z_0 - z| \, d\overline{z}\right]
\end{aligned}$$

Taking limits (on n) gives the result for $z_0 \in \Omega$.

Similarly for $z_0 \in \mathbb{C} \setminus \Omega$: Use $(\phi f)(z_0) = 0$ in this case. ■

Section 807 Set $L = \frac{1}{2\pi}\triangle$. By Lemma 214, the differential operator L is the (algebraic) inverse of $\log * (\ )$: For $\Lambda \in \mathcal{E}'$,

$$L(\log * \Lambda) = \Lambda = \log * (L\Lambda) \quad \text{in} \quad \mathcal{D}'.$$

Note $L : \mathcal{E}' \longrightarrow \mathcal{E}'$ and $\log * (\ ) : \mathcal{E}' \longrightarrow \mathcal{D}'$.

As mentioned in Section 513, this suggests that $\log * (\ )$ is compact (in the correct setting). ■

Lemma 808 *For* $z_j \in \mathbb{C}$,

$$\left|\log|z_1| - \log|z_2|\right| \le \frac{|z_1 - z_2|}{|z_1 z_2|^{\frac{1}{2}}}.$$

Proof. Using Cauchy-Schwarz, we see

$$\begin{aligned}
|\log|z_1| - \log|z_2|| &= \left| \int_{|z_2|}^{|z_1|} \frac{dt}{t} \right| \\
&\leq |z_1 - z_2|^{\frac{1}{2}} \left| \int_{|z_2|}^{|z_1|} \frac{dt}{t^2} \right|^{\frac{1}{2}} \\
&= |z_1 - z_2|^{\frac{1}{2}} \left| \frac{1}{|z_1|} - \frac{1}{|z_2|} \right|^{\frac{1}{2}} \\
&\leq \frac{|z_1 - z_2|}{|z_1 z_2|^{\frac{1}{2}}}.
\end{aligned}$$

■

The following is the harmonic analogue of Theorem 515.

Theorem 809 *Let $1 \leq p < \infty$ and $\Omega \subseteq \mathbb{C}$ be open and bounded. Then $\log * (\ \ ) : L^p(\Omega) \longrightarrow L^p(\Omega)$ is compact.*

Proof. We proceed as in the proof of Theorem 515. If $a \in \mathbb{C}$ and $f \in L^p(\Omega)$, with $\|f\|_p \leq 1$, then

$$\begin{aligned}
&\iint_\Omega |(\log * f)(z+a) - (\log * f)(z)|\, dA(z) \\
&\quad = \iint_\Omega \left| \iint_\Omega f(\zeta)\left[\log|\zeta - (z+a)| - \log|\zeta - z|\right] dA(\zeta) \right|^p dA(z) \\
&\quad = \iint_\Omega |(f * h_a)(z)|^p\, dA(z) \leq (\|f\|_p \|h_a\|_1)^p \leq \|h_a\|_1^p,
\end{aligned}$$

where $h_a(z) = (\log|z+a| - \log|z|)\, \chi_{(2R)\mathbb{D}}(z)$ and where we have assumed $\Omega \bigcup (a + \Omega) \subseteq R\mathbb{D}$. Note

$$\|h_a\|_1 \leq \int\int_{(2R)\mathbb{D}} |\log|z+a| - \log|z||\, dA(z) \leq |a| \int\int_{(2R)\mathbb{D}} \frac{dA}{|(z+a)z|^{\frac{1}{2}}},$$

where we have used Lemma 808. That is, $\lim_{a\to 0} \|h_a\|_1 = 0$.

Finishing up as in the proof of Theorem 515, we see $\log * (\ )$ is compact. ■

Lemma 810 *Let $z_0 \in \mathbb{C}$ and $r > 0$. If $f : D(z_0; r) \longrightarrow \mathbb{C}$, then*

$$(\partial f)(z_0) = \lim_{\substack{\alpha\in\mathbb{R} \\ \alpha\longrightarrow 0}} \frac{f(z_0+\alpha) - f(z_0+i\alpha)}{2\alpha}$$

and

$$\left(\overline{\partial} f\right)(z_0) = \lim_{\substack{\alpha\in\mathbb{R} \\ \alpha\longrightarrow 0}} \frac{f(z_0+\alpha) + f(z_0+i\alpha) - 2f(z_0)}{2\alpha},$$

provided these limits exist.

Proof:

$$\begin{aligned} (\partial f)(z_0) &= \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)(z_0) \\ &= \frac{1}{2} \lim_{\substack{\alpha\in\mathbb{R} \\ \alpha\longrightarrow 0}} \left[\frac{f(z_0+\alpha) - f(z_0)}{\alpha} - i\frac{f(z_0+i\alpha) - f(z_0)}{i\alpha}\right] \\ \text{and} \quad \left(\overline{\partial} f\right)(z_0) &= \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)(z_0). \end{aligned}$$

■

We now compute the pointwise (partial) derivatives of $\log * f$. Note Theorem 811 is the (log transform) analogue of Corollary 502.

Theorem 811 *Let $f \in L^1_c$ and $z_0 \in \mathbb{C}$. If $\iint\limits_{\mathbb{C}} \left|\frac{f}{z-z_0}\right| dA < \infty$, then*

$$[\partial(\log * f)](z_0) = -\frac{1}{2}\iint\limits_{\mathbb{C}} \frac{f}{z-z_0}\, dA = -\frac{1}{2}\hat{f}(z_0)$$

$$\text{and} \quad \left[\overline{\partial}(\log * f)\right](z_0) = -\frac{1}{2}\iint\limits_{\mathbb{C}} \frac{f}{\overline{z}-\overline{z}_0}\, dA = -\frac{1}{2}\overline{\hat{\overline{f}}(z_0)}.$$

Proof. For $\alpha \in \mathbb{R} \setminus \{0\}$.

$$\frac{(\log * f)(z_0+\alpha) - (\log * f)(z_0+i\alpha)}{2\alpha} + \frac{1}{2}\iint_{\mathbb{C}} \frac{f}{z-z_0}\,dA$$
$$= \iint_{\mathbb{C}} f(z)\left[\frac{\log|z-(z_0+\alpha)| - \log|z-(z_0+i\alpha)|}{2\alpha} + \frac{1}{2(z-z_0)}\right] dA.$$

Our hypothesis justifies the use of the Lebesgue Dominated Convergence Theorem. From Lemma 202 and Lemma 810, we see

$$[\partial(\log * f)](z_0) = \frac{1}{2}\iint_{\mathbb{C}} \frac{f}{z-z_0}\,dA.$$

Hence,

$$\begin{aligned}\left[\overline{\partial}(\log * f)\right](z_0) &= \overline{\left[\partial\left(\overline{\log * f}\right)\right](z_0)} \\ &= \overline{\left[\partial\left(\log * \overline{f}\right)\right](z_0)} = -\frac{1}{2}\iint_{\mathbb{C}} \frac{f}{\overline{z}-\overline{z}_0}\,dA.\end{aligned}$$

■

Motivation 812 Let $\Omega \subseteq \mathbb{C}$ be open and $f \in L^1_{\text{loc}}(\mathbb{C})$. For $\phi \in \mathcal{D}(\Omega)$,

$$\Lambda_{\overline{f}}\phi = \iint_{\Omega} \phi\overline{f}\,dA = \overline{\iint_{\Omega} \overline{\phi} f\,dA} = \overline{\Lambda_f\overline{\phi}}.$$

■

Definition: Let $\Omega \subseteq \mathbb{C}$ be open and $\Lambda \in \mathcal{D}'(\Omega)$. Define $\overline{\Lambda} : \mathcal{D}(\Omega) \longrightarrow \mathbb{C}$ by $\overline{\Lambda}\phi = \overline{\Lambda\overline{\phi}}$.

Remark: If $\phi_\alpha \longrightarrow 0$ in $\mathcal{D}(\Omega)$, then $\overline{\phi}_\alpha \longrightarrow 0$ in $\mathcal{D}(\Omega)$ and hence, $\Lambda\overline{\phi}_\alpha \longrightarrow 0$. That is, $\overline{\Lambda} \in \mathcal{D}'(\Omega)$, ■

For $\Lambda \in \mathcal{E}'$, we compute the (distributional) partial derivatives of $\log * \Lambda$.

Theorem 813 *Let $\Lambda \in \mathcal{E}'$. Then*

$$\partial(\log * \Lambda) = -\frac{1}{2}\hat{\Lambda} \quad \text{and} \quad \overline{\partial}(\log * \Lambda) = -\frac{1}{2}\overline{\hat{\overline{\Lambda}}}.$$

Proof. Let $\phi \in \mathcal{D}$. By Theorem 811,

$$\begin{aligned} [\partial(\log * \Lambda)]\,\phi &= -(\log * \Lambda)(\partial\phi) = -\Lambda\,[\log * (\partial\phi)] \\ &= -\Lambda\,[\partial(\log * \phi)] = \frac{1}{2}\Lambda\hat{\phi} = -\frac{1}{2}\hat{\Lambda}\phi \end{aligned}$$

and

$$\begin{aligned} \left[\overline{\partial}(\log * \Lambda)\right]\phi &= -\Lambda\left[\overline{\partial}(\log * \phi)\right] = \frac{1}{2}\Lambda\overline{\hat{\overline{\phi}}} \\ &= \frac{1}{2}\overline{\overline{\Lambda}\,\hat{\overline{\phi}}} = -\frac{1}{2}\overline{\hat{\overline{\Lambda}}\,\overline{\phi}} = -\frac{1}{2}\overline{\hat{\overline{\Lambda}}}\phi. \end{aligned}$$

■

Section 814 Let $\Lambda \in \mathcal{E}'$. By Lemma 214, we know $\triangle(\log * \Lambda) = 2\pi\Lambda$. □

For $\phi \in \mathcal{D}$, Corollary 121 gives

$$\left(\overline{\partial}\hat{\Lambda}\right)\phi = -\hat{\Lambda}\left(\overline{\partial}\phi\right) = \Lambda\left[\left(\overline{\partial}\phi\right)^{\wedge}\right] = -\pi\Lambda\phi.$$

That is, $\overline{\partial}\hat{\Lambda} = -\pi\Lambda$.

As a check, note that by Theorem 813,

$$\triangle(\log * \Lambda) = 4\overline{\partial}\partial(\log * \Lambda) = -2\overline{\partial}\hat{\Lambda} = 2\pi\Lambda.$$

■

Section 815 We digress for a moment to mention an important result which has applications in rational approximation (see [6]).

Set $\hat{M}_c = \{\hat{\mu} : \mu \in M_c\}$. By Theorem 2.9 [6], $\hat{M}_c$ is a module over C_c^1. That is, if $\mu \in M_c$ and $\phi \in C_c^1$, and if we set $\nu = \phi\mu - \frac{1}{\pi}\left(\overline{\partial}\phi\right)\hat{\mu}m_2$, then $\hat{\nu} = \phi\hat{\mu}$. □

How should we proceed in order to get the (log transform) analogue of this result? We start by analyzing the above result.

Set $L = -\frac{1}{\pi}\overline{\partial}$ and recall (Section 513) that L is the inverse of the Cauchy transform operator. By Lemma 134, we note $\nu = L(\phi\hat{\mu})$, where we have equality as elements of $\mathcal{D}'$. Hence, $\hat{\nu} = [L(\phi\hat{\mu})]^{\wedge} = \phi\hat{\mu}$. □

This motivates the following theorem

Theorem 816 *Let* $\Lambda \in \mathcal{E}'$ *and* $\phi \in \mathcal{D}$. *Set*

$$\Gamma = \phi\Lambda + \frac{1}{2\pi i}(\Delta\phi)(\log * \Lambda) - \frac{1}{\pi}\left[\left(\overline{\partial}\phi\right)\hat{\Lambda} + \left(\overline{\partial}\phi\right)\overline{\hat{\overline{\Lambda}}}\right].$$

Note $\Gamma \in \mathcal{E}'$. *Then* $\log * \Gamma = \phi(\log * \Lambda)$.

Remark: The hypotheses $\Lambda \in \mathcal{E}'$ and $\phi \in \mathcal{D}$ were stated for convenience (only). Many other combinations are possible: For example, if $\Lambda = \mu \in M_c$ and $\phi \in C_c^2$, then the conclusion (still) holds. It is in this form that Theorem 816 is the direct analogue of Theorem 2.9 [6]. In this case, Γ can be identified with an element of M_c, as the above formula shows.

Proof. Note Lemma 214 and Theorem 813 show $\Gamma = L[\phi(\log * \Lambda)]$, where $L = \frac{1}{2\pi}\Delta$. By Section 807, we see $\log * \Gamma = \phi(\log * \Lambda)$. ■

Section 817 One of the important concepts in [6] is the definition of the T- invariant algebra.

Let $K \subseteq \mathbb{C}$ be compact and let A be a closed subalgebra of $C(K)$. We say A is *T-invariant* if $R(K) \subseteq A$ and if $T_\phi A \subseteq A$ whenever $\phi \in C_c^1$. □

If we use properties (i) and (ii) listed in the abstract (of the present chapter), we see that $R(K)$ and $C(K)$ are T-invariant, as is $A(K) \equiv H(\text{ int } K) \bigcap C(K)$.

The hypotheses of many of the important results in [6] begin with the following: Let A be T-invariant. That is, in the proofs of these results, one need not distinguish between the above algebras.

As we attempt to obtain analogus harmonic results, we note one roadblock: Unlike Rat$K$, harm$K$ is not an algebra.

As we present our (new) results, we will inform the reader of the analogous T-invariant results. □

If we attempted to "copy" the above definition, we would be led to the following (incorrect) definition:

Let $K \subseteq \mathbb{C}$ be compact and $N \leq C(K)$. We say N is h-invariant if harm$K \subseteq N$ and if $h_\phi N \subseteq N$ for all $\phi \in C_c^2$. □

Unfortunately, this would present a regularity problem. If we look at the definition of h_ϕ and recall (our best regularity result) Theorem 805, we see that we would probably require $N \subseteq C^\infty$.

Fortunately, the following result ([6, Lemma 4.5]) gives us an alternative.

Let $K \subseteq \mathbb{C}$ be compact and $A \subseteq C(K)$ be a closed subalgebra. Then the following statements **(a)** and **(b)** are equivalent:

(a) A is T-invariant.

(b) Both **(i)** and **(ii)** hold:

- **(i)** $\hat{\mu} \equiv 0$ in $\mathbb{C} \setminus$ K whenever $\mu \in A^\perp \subseteq M(K)$.
- **(ii)** If $\mu \in A^\perp$, if $\phi \in C_c^1$, and if $\nu \in M(K)$ satisfies $\hat{\nu} = \phi\hat{\mu}$, then $\nu \in A^\perp$.

□

This above result motivates the appropriate definition. ■

Definition: Let $K \subseteq \mathbb{C}$ be compact and $N \leq C(K)$. We say N is *h -invariant* if both of the following hold:

(i) If $\mu \in N^{\perp} \subseteq M(K)$, then $\log * \mu \equiv 0$ in $\mathbb{C} \setminus K$.

(ii) If $\mu \in N^{\perp}$, if $\phi \in C_c^2$, and if $\nu \in M(K)$ satisfies $\log * \nu = \phi(\log * \mu)$, then $\nu \in N^{\perp}$. ■

Remark: Clearly $C(K)$ is h-invariant. The following is not as clear.

Theorem 818 *Let $K \subseteq \mathbb{C}$ be compact. Then $h(K)$ is h-invariant.*

Proof.

(i) Use Theorem 206.

(ii) Let μ, ϕ, and ν be as above. Let $h \in \text{harm}K$. By Theorem 203,

$$\begin{aligned}\int h\, d\nu &= \frac{1}{2\pi} \iint\limits_{\mathbb{C}} (\triangle h)(\log * \nu)\, dA \\ &= \frac{1}{2\pi} \iint\limits_{\mathbb{C}} (\triangle h)\, \phi(\log * \mu)\, dA = 0.\end{aligned}$$

■

Theorem 819 *Let $K \subseteq \mathbb{C}$ be compact and $N \leq C(K)$ be h-invariant. Let $\mu \in N^{\perp}$ and let $\{U_j : 1 \leq j \leq n\}$ be an open cover of K (in $\mathbb{C}$).*

Then there exist $\mu_1, \cdots, \mu_n \in N^{\perp}$ satisfying each $\operatorname{supp} \mu_j \subseteq U_j$ is compact and $\mu = \sum_{j=1}^{n} \mu_j$.

Remark: For each $1 \le j \le n$, there exists $\phi_j \in \mathcal{D}(U_j)$ with $\sum_{j=1}^{n} \phi_j \equiv 1$ in a neighborhood of K [11, Theorem 6.20]. For each $1 \le j \le n$, set

$$\mu_j = \phi_j \mu + \frac{1}{2\pi} (\Delta \phi_j)(\log * \mu) - \frac{1}{\pi} \left[\left(\overline{\partial} \phi_j\right) \hat{\mu} + (\partial \phi) \overline{\hat{\overline{\mu}}} \right].$$

By Theorem 816,

$$\log * \mu_j = \phi_j (\log * \mu) \equiv 0$$

in $\mathbb{C} \setminus \mathrm{K}$ and in $\mathbb{C} \setminus \mathrm{supp}\ \phi_j$. Theorem 205 shows $\mu_j \in M(K)$ and supp $\mu_j \subseteq U_j$ is compact. By **(ii)** of the definition (of h-invariance), $\mu_j \in N^\perp$. Note

$$\begin{aligned} \sum_{j=1}^{n} \mu_j &= \left(\sum_{j=1}^{n} \phi_j\right) \mu + \frac{1}{2\pi} \left(\Delta \sum_{j=1}^{n} \phi_j\right) (\log * \mu) \\ &\quad - \frac{1}{\pi} \left[\left(\overline{\partial} \sum_{j=1}^{n} \phi_j\right) \hat{\mu} + \left(\partial \sum_{j=1}^{n} \phi_j\right) \overline{\hat{\overline{\mu}}} \right] \\ &= \mu. \end{aligned}$$

■

Theorem 820 *Let $K \subseteq \mathbb{C}$ be compact and $N \le C(K)$ be h-invariant. Let $f \in C(K)$ and suppose f is locally an element of N. That is, for each $z_0 \in K$, there exists a neighborhood $U(z_0)$ of z_0 (in $\mathbb{C}$) satisfying: f can be uniformly approximated on $K \bigcap U(z_0)$ by functions in N.*

Then $f \in N$.

Remark: Gamelin [6, Corollary 4.10] proves the above with a T-invariant algebra replacing N.

Proof. Let $\{U_j : 1 \le j \le n\}$ be an open cover of K, where each $U_j = U(z_j)$ is as above. Fix $\mu \in N^\perp$ and write $\mu = \sum_{j=1}^{n} \mu_j$ as in Theorem 819.

Let $1 \leq j \leq n$. By hypothesis, there exists a sequence $(f_{j\ell})_{\ell=1}^{\infty}$ in N with $f_{j\ell} \rightrightarrows f$ in $U_j \bigcap K$. Recall (Theorem 819) that supp $\mu_j \subseteq U_j \bigcap K$ and $\mu_j \in N^{\perp}$. Hence,

$$\int_K f \, d\mu_j = \int_{U_j \bigcap K} f \, d\mu_j = \lim_{\ell} \int_{U_j \bigcap K} f_{j\ell} \, d\mu_j$$

$$= \lim_{\ell} \int_K f_{j\ell} \, d\mu_j = 0.$$

Therefore,

$$\int f \, d\mu = \sum_{j=1}^{n} \int f \, d\mu_j = 0.$$

■

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Index

Index of Symbols

Norms

Sets

Transforms

Vector Spaces

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