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LINEARLY IMPLICIT GENERAL LINEAR METHODS*

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Abstract. Linearly implicit Runge–Kutta methods provide a fitting balance of implicit treatment of stiff systems and computational cost. In this paper we extend the class of linearly implicit Runge–Kutta methods to include multi-stage and multi-step methods. We provide the order condition theory to achieve high stage order and overall accuracy while admitting arbitrary Jacobians. Several classes of linearly implicit general linear methods (GLMs) are discussed based on existing families such as type 2 and type 4 GLMs, two-step Runge–Kutta methods, parallel IMEX GLMs, and BDF methods. We investigate the stability implications for stiff problems and provide numerical studies for the behavior of our methods compared to linearly implicit Runge–Kutta methods. Our experiments show nominal order of convergence in test cases where Rosenbrock methods suffer from order reduction.

Key words. Time integration, Implicit methods

AMS subject classifications. 68Q25, 68R10, 68U05

1. Introduction. We are concerned with solving stiff initial value problems of the form

$$y' = f(y), \qquad y(t_0) = y_0, \qquad y(t) \in \mathbb{C}^d. \tag{1.1}$$

Implicit methods used to solve (1.1), such as backward differentiation formula (BDF) and diagonally implicit Runge–Kutta (DIRK) methods, require solving nonlinear systems of equations to compute implicitly-defined stage values. The nonlinear solves can be very expensive and often dominate the cost of a step. Linearly implicit methods, on the other hand, reduce the computational cost by posing stages as the solution to a *linear* systems [23, 17, 1]. A limitation of this process is that the accuracy of the method hinges on the availability of the exact Jacobian of the system. In response to this limitation, W-methods have been developed with additional order conditions such that terms involving the Jacobian are eliminated from the error expansion up to a certain order. As a result, these methods provide a flexible choice for the Jacobian of the system: the closer the approximate Jacobian is to the exact one, the better the stability of the method, while the order of convergence is unchanged. Rosenbrock-W (ROW) methods [31] are examples of this class of time stepping methods in the Runge–Kutta framework. [29, 21]

There are a number of incentives to extend W-methods beyond the Runge–Kutta family. ROW methods can encounter error reduction when stages are solved using iterative methods [32]. Furthermore, linearly-implicit methods, in general, are vulnerable to order reduction when used for solving PDEs with time-dependent boundary conditions as studied in [18]. Various improvements for ROW methods applied to parabolic problems have been proposed in [16, 2, 3].

A more systematic approach is to avoid low stage orders. Rosenbrock methods with an explicit first stage can raise the stage order to two. Glandon *et* al. [8]

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propose linearly implicit multi-step methods with order conditions to derive methods both for exact and approximate Jacobians. As A-stability becomes unattainable for high order linear multi-step methods, one can also turn to multi-stage multi-step schemes to build high order, high stage order linearly implicit methods. A family of parallel two-step W-methods is reported in [26, 27, 33] with prominent features such as step-size adaptation using variable coefficients, parallel stage computations, and super-convergence of the output variable.

We are inspired by these contributions to further study new possibilities of creating linearly implicit W-methods based on the general linear methods (GLMs) framework.

We start by reviewing GLMs in section 2. In section 3 we introduce the formulation for linearly implicit general linear methods. Section 4 gives the order conditions for these new methods. Sections 5 and 6 are dedicated to linear and stiff stability analysis for linearly implicit GLMs. In section 7 we show a number of new and existing families that fit in our framework including methods based on type 2 and type 4 DIMSIMS, BDF-W, and parallel methods. Sections 8 and 9 presents the numerical experiments and concluding remarks are presented in section 9. {Steven: broken refs}

2. Traditional General Linear Methods. A GLM with s internal and r external stages advances the numerical solution of (1.1) over the time interval $[t_{n-1}, t_n]$ with $t_n = t_{n-1} + h$ as follows:

$$Y_i = h \sum_{j=1}^{s} a_{i,j}\, f(Y_j) + \sum_{j=1}^{r} u_{i,j}\, \mathbb{y}_j^{[n-1]}, \qquad i = 1, \dots, s, \tag{2.1a}$$

$$\mathbb{y}_i^{[n]} = h \sum_{j=1}^{s} b_{i,j}\, f(Y_j) + \sum_{j=1}^{r} v_{i,j}\, \mathbb{y}_j^{[n-1]}, \qquad i = 1, \dots, r. \tag{2.1b}$$

Provided that the external stages from the previous step have the Taylor series expansion

$$\mathbb{y}_i^{[n-1]} = \sum_{k=0}^{p} w_{i,k}\, h^k\, y^{(k)}(t_{n-1}) + \mathcal{O}\big(h^{p+1}\big), \tag{2.2}$$

a method is said to have stage order q if

$$Y_i = y(t_{n-1} + \mathbf{c}\, h) + \mathcal{O}\big(h^{q+1}\big),$$

and order p if

$$\mathbb{y}_i^{[n]} = \sum_{k=0}^{p} w_{i,k}\, h^k\, y^{(k)}(t_n) + \mathcal{O}\big(h^{p+1}\big).$$

For brevity, the coefficients are represented in matrix form

$$\mathbf{A} \coloneqq [a_{i,j}] \in \mathbb{R}^{s \times s}, \quad \mathbf{U} \coloneqq [u_{i,j}] \in \mathbb{R}^{s \times r}, \quad \mathbf{B} \coloneqq [b_{i,j}] \in \mathbb{R}^{r \times s}, \tag{2.3}$$

$$\mathbf{V} \coloneqq [v_{i,j}] \in \mathbb{R}^{r \times r}, \quad \mathbf{W} \coloneqq [w_{i,j}] = [\mathbf{w}_0 \ \cdots \ \mathbf{w}_p] \in \mathbb{R}^{r \times (p+1)}, \tag{2.4}$$

and the GLM Eq. (2.1) can be represented by the following Butcher tableau:

$$\begin{array}{c|c|c} \mathbf{c} & \mathbf{A} & \mathbf{U} \\ \hline & \mathbf{B} & \mathbf{V} \end{array}.$$

GLM framework is extensive and well-established. We refer readers interested in theoretical foundation of these methods to the literature [14, 6, 7]. Similar to Runge–Kutta schemes, two broad categories of GLMs are of interest. Explicit GLMs computes internal stages using information from incoming external stages and previously computed internal stages. The computational cost is therefore limited to evaluating right hand side functions. Implicit methods on the other hand may compute coupled stages that require nonlinear system solves. methods with lower triangular $\mathbf{A}$ compute stages that are only implicit in the stage being computed. These methods also benefit from excellent linear stability properties. In the following we reiterate an important theorem for the order conditions of GLMs.

Definition 2.1 (GLM preconsistency). *The GLM Eq. (2.1) is said to be preconsistent if the following conditions hold:*

$$\mathbf{U}\,\mathbf{w}_0 = \mathbf{1}_s, \quad \mathbf{V}\,\mathbf{w}_0 = \mathbf{w}_0. \tag{2.5}$$

Theorem 2.2 (GLM order conditions). *Consider a preconsistent GLM with external stages satisfying (2.2). All of the following are equivalent:*

1. *The GLM Eq. (2.1) has order p and stage order $q \in \{p-1, p\}$ for all sufficiently smooth f.*
2. *The method coefficients satisfy*

$$\frac{\mathbf{c}^{\times k}}{k!} - \frac{\mathbf{A}\,\mathbf{c}^{\times(k-1)}}{(k-1)!} - \mathbf{U}\,\mathbf{w}_k = \mathbf{0}_s, \qquad k = 1,\ldots,q,$$
$$\sum_{\ell=0}^{k} \frac{\mathbf{w}_{k-\ell}}{\ell!} - \frac{\mathbf{B}\,\mathbf{c}^{\times(k-1)}}{(k-1)!} - \mathbf{V}\,\mathbf{w}_k = \mathbf{0}_s, \qquad k = 1,\ldots,p,$$

where we have used the notation $\mathbf{c}^{\times k}$ to denote component-wise k-th power of the vector $\mathbf{c}$.

3. *The GLM Eq. (2.1) has order p and stage order $q \in \{p-1, p\}$ for all scalar, linear problems*

$$y' = \lambda\,y, \qquad y(0) = 1, \tag{2.6}$$

where $\lambda \in \mathbb{R}$.

Proof. The equivalence of statements one and two is discussed in [14]. Clearly statement one implies statement three as it uses the special case $f(y) = \lambda\,y$. To complete the proof, we prove the converse.

First, assume a GLM has order p and stage order $q \in \{p-1, p\}$ for (2.6). Therefore, the internal stages satisfy

$$\begin{aligned} Y &= h\,\mathbf{A}\,f(Y) + \mathbf{U}\,\mathbb{y}^{[n-1]}, \\ e^{\mathbf{c}\,z} &= z\,\mathbf{A}\,e^{\mathbf{c}\,z} + \mathbf{U}\sum_{k=0}^{p} \mathbf{w}_k\,h^k\,y^{(k)}(0) + \mathcal{O}\big(h^{q+1}\big), \\ e^{\mathbf{c}\,z} &= z\,\mathbf{A}\,e^{\mathbf{c}\,z} + \mathbf{U}\,\mathbf{w}(z) + \mathcal{O}\big(h^{p+1}\big), \end{aligned} \tag{2.7}$$

where $z = h\,\lambda$, $\mathbf{w}(z) = \sum_{k=0}^{p} \mathbf{w}_k\,z^k$, and the exponential of vectors are performed component-wise. Similarly, the external stages satisfy

$$\begin{aligned} \mathbb{y}^{[n]} &= h\,\mathbf{B}\,f(Y) + \mathbf{V}\,\mathbb{y}^{[n-1]}, \\ e^{\mathbf{c}\,z}\,\mathbf{w}(z) &= z\,\mathbf{B}\,e^{\mathbf{c}\,z} + \mathbf{V}\,\mathbf{w}(z) + \mathcal{O}\big(h^{p+1}\big). \end{aligned} \tag{2.8}$$

Together (2.7) and (2.8) are exactly the GLM order conditions given in [14, Theorems 2.4.1 and 2.4.2]. Thus, statement one is proved, and the proof is complete. ∎

3. Formulation of linearly implicit GLMs.

One step of a linearly implicit GLM applied to (1.1) with a (possibly) different approximate Jacobian matrix $\mathbf{L}_i \approx \frac{\partial f}{\partial y}(t_{n-1} + \mathbf{c}\, h)$ at each stage $i = 1, \dots, s$ reads as

$$K_i = h\, f\left(\sum_{j=1}^{i-1} a_{i,j}\, K_j + \sum_{j=1}^{r} u_{i,j}\, \mathrm{y}_j^{[n-1]}\right) + h\, \mathbf{L}_i \sum_{j=1}^{i} \gamma_{i,j}\, K_j + h\, \mathbf{L}_i \sum_{j=1}^{r} \psi_{i,j}\, \mathrm{y}_j^{[n-1]}, \qquad i = 1, \dots, s, \tag{3.1a}$$

$$\mathrm{y}_i^{[n]} = \sum_{j=1}^{s} b_{i,j}\, K_j + \sum_{j=1}^{r} v_{i,j}\, \mathrm{y}_j^{[n-1]}, \qquad i = 1, \dots, r. \tag{3.1b}$$

The formulation of linearly implicit GLMs for non-autonomous systems will be discussed in subsection 4.3 after order conditions are introduced.

Equation (3.1) can be represented more compactly in matrix form as

$$K = h\, F\Big(\mathbf{A} \circledast K + \mathbf{U} \circledast \mathrm{y}^{[n-1]}\Big) + h\, \mathbf{L}\, \mathbf{\Gamma} \circledast K + h\, \mathbf{L}\, \mathbf{\Psi} \circledast \mathrm{y}^{[n-1]}, \tag{3.2a}$$

$$\mathrm{y}^{[n]} = \mathbf{B} \circledast K + \mathbf{V} \circledast \mathrm{y}^{[n-1]}, \tag{3.2b}$$

where $\mathbf{L} = \mathrm{blkdiag}\,(\mathbf{L}_1, \mathbf{L}_2, \dots, \mathbf{L}_s)$ and we have used the $\circledast$ notation to indicate Kronecker product with the identity matrix such that $\mathbf{A} \circledast K := (\mathbf{A} \otimes \mathbf{I}_{d\times d})\, K$, and

$$K = \begin{bmatrix} K_1 \\ \vdots \\ K_s \end{bmatrix}, \quad \mathrm{y}^{[n]} = \begin{bmatrix} \mathrm{y}_1^{[n]} \\ \vdots \\ \mathrm{y}_s^{[n]} \end{bmatrix}, \quad F(Y) = \begin{bmatrix} f(Y_1) \\ \vdots \\ f(Y_s) \end{bmatrix}. \tag{3.3}$$

We represent a linearly implicit GLM using the Butcher tableau

$$\begin{array}{c|c:c|c:c} \mathbf{c} & \mathbf{A} & \mathbf{\Gamma} & \mathbf{U} & \mathbf{\Psi} \\ \hline & \mathbf{B} & & \mathbf{V} & \end{array}, \tag{3.4}$$

where $\mathbf{\Gamma} \in \mathbb{R}^{s\times s}$ and $\mathbf{\Psi} \in \mathbb{R}^{s\times r}$.

3.1. Efficient implementation.

To build a general and computationally practical formulation of linearly implicit GLMs, we consider a slightly more general form of (1.1):

$$\mathbf{M}\, y' = f(y), \tag{3.5}$$

where $\mathbf{M}$ is a mass matrix. The special case of a singular $\mathbf{M}$ is considered in section 6.

Assuming $\mathbf{\Gamma}$ is lower triangular and invertible, we can formulate the method in the new variable Z_i, where

$$K_i = \sum_{j=1}^{i} \left(\mathbf{\Gamma}^{-1}\right)_{i,j} Z_j - \sum_{j=1}^{r} \left(\mathbf{\Gamma}^{-1}\, \mathbf{\Psi}\right)_{i,j} \mathrm{y}_j^{[n-1]}. \tag{3.6}$$

After some algebraic manipulations, Eq. (3.1) becomes

$$\left(\frac{1}{\gamma_{i,i}}\mathbf{M} - h\,\mathbf{L}_i\right) Z_i = h\,f\left(\sum_{j=1}^{i-1}\left(\mathbf{A}\,\boldsymbol{\Gamma}^{-1}\right)_{i,j} Z_j + \sum_{j=1}^{r}\left(\mathbf{U} - \mathbf{A}\,\boldsymbol{\Gamma}^{-1}\,\boldsymbol{\Psi}\right)_{i,j} \mathbb{y}_j^{[n-1]}\right)$$
$$+\,\mathbf{M}\left(-\sum_{j=1}^{i-1}\left(\boldsymbol{\Gamma}^{-1}\right)_{i,j} Z_j + \sum_{j=1}^{r}\left(\boldsymbol{\Gamma}^{-1}\,\boldsymbol{\Psi}\right)_{i,j} \mathbb{y}_j^{[n-1]}\right),$$
$$\mathbb{y}_i^{[n]} = \sum_{j=1}^{s}\left(\mathbf{B}\,\boldsymbol{\Gamma}^{-1}\right)_{i,j} Z_j + \sum_{j=1}^{r}\left(\mathbf{V} - \mathbf{B}\,\boldsymbol{\Gamma}^{-1}\,\boldsymbol{\Psi}\right)_{i,j} \mathbb{y}_j^{[n-1]}.$$

Note that the approximate Jacobians only appear on the left-hand side (in the linear system matrix), and this formulation avoids the additional matrix vector products present in the right-hand side of Eq. (3.1a).

4. Order conditions for linearly implicit GLMs. In this section we derive the necessary and sufficient order conditions for desired internal and external stage orders and show how the conditions can ensure accuracy of the method with arbitrary Jacobians. We also explore how these conditions are related to IMEX GLM pairs.

4.1. Preconsistency conditions. We apply the method in Eq. (3.2) to the test problem

$$y' = 0, \qquad y(t_0) = \kappa, \tag{4.1}$$

where $\kappa \in \mathbb{R}$ and note that the constant solution is $y(t) = \kappa$. We would like to have all approximations of $y(t)$ and $y'(t)$ to be $\mathcal{O}(h)$ accurate. We therefore require

$$\begin{aligned}\mathbf{A}\,K + \mathbf{U}\,\mathbb{y}^{[n-1]} &= \mathbf{1}_s\,\kappa + \mathcal{O}(h),\\ \mathbf{B}\,K + \mathbf{V}\,\mathbb{y}^{[n-1]} &= \mathbf{w}_0\,\kappa + \mathcal{O}(h),\\ h\,\mathbf{L}_i\,\boldsymbol{\Psi}\,\mathbb{y}^{[n-1]} &= \mathbf{0}_s + \mathcal{O}\big(h^2\big),\end{aligned}$$

which leads to the following preconsistency conditions:

$$\mathbf{U}\,\mathbf{w}_0 = \mathbf{1}_s, \tag{4.2a}$$
$$\boldsymbol{\Psi}\,\mathbf{w}_0 = \mathbf{0}_s, \tag{4.2b}$$
$$\mathbf{V}\,\mathbf{w}_0 = \mathbf{w}_0. \tag{4.2c}$$

4.2. Order conditions.

Definition 4.1 (Linearly implicit GLM internal stage order). *A linearly implicit GLM with incoming external stages satisfying* (2.2) *has stage order q if the following approximation holds:*

$$K_i = h\,y'(t_{n-1} + c_i\,h) + \mathcal{O}\big(h^{q+2}\big). \tag{4.3}$$

Definition 4.2 (Linearly implicit GLM external stage order). *A linearly implicit GLM with incoming external stages satisfying* (2.2) *has external stage order p if*

$$\mathbb{y}_i^{[n]} = \sum_{k=0}^{p} w_{i,j}\,h^k\,y^{(k)}(t_n) + \mathcal{O}\big(h^{p+1}\big). \tag{4.4}$$

DEFINITION 4.3 (Method order). *We say that a linearly implicit GLM is of order (q,p) if it has internal stage order q and external stage order p.*

THEOREM 4.4 (Order conditions). *A linearly implicit GLM that satisfies the preconsistency conditions Eq. (4.2) has order (q,p) with $q \in \{p, p-1\}$ for all $\mathbf{L}_i$ and sufficiently smooth f if and only if its coefficients satisfy the following order conditions:*

$$\frac{\mathbf{c}^{\times k}}{k!} - \frac{\mathbf{A}\,\mathbf{c}^{\times(k-1)}}{(k-1)!} - \mathbf{U}\,\mathbf{w}_k = \mathbf{0}_s, \qquad k = 1, \dots, q, \tag{4.5a}$$

$$\sum_{\ell=0}^{k} \frac{\mathbf{w}_{k-\ell}}{\ell!} - \frac{\mathbf{B}\,\mathbf{c}^{\times(k-1)}}{(k-1)!} - \mathbf{V}\,\mathbf{w}_k = \mathbf{0}_r, \qquad k = 1, \dots, p, \tag{4.5b}$$

$$\frac{\mathbf{\Gamma}\,\mathbf{c}^{\times k-1}}{(k-1)!} + \mathbf{\Psi}\,\mathbf{w}_k = \mathbf{0}_s, \qquad k = 1, \dots, q. \tag{4.5c}$$

REMARK 1 (Implicit and explicit components). *Note that Eqs. (4.5a) and (4.5b) are also the order (q,p) conditions for the explicit traditional GLM with coefficients $(\mathbf{A}, \mathbf{U}, \mathbf{B}, \mathbf{V})$. Next, we introduce the following notation:*

$$\widehat{\mathbf{A}} \coloneqq \mathbf{\Gamma} + \mathbf{A} \quad \text{and} \quad \widehat{\mathbf{U}} \coloneqq \mathbf{\Psi} + \mathbf{U}.$$

An equivalent formulation of Eq. (4.5c) can be given as follows: Subtracting Eq. (4.5c) from Eq. (4.5a) leads to

$$\frac{\mathbf{c}^{\times k}}{k!} - \frac{\widehat{\mathbf{A}}\,\mathbf{c}^{\times(k-1)}}{(k-1)!} - \widehat{\mathbf{U}}\,\mathbf{w}_k = 0, \qquad k = 1, \dots, q, \tag{4.6}$$

and vice-versa, subtracting Eq. (4.5a) from (4.6) restores Eq. (4.5c). Therefore the order conditions (4.5) can be written in an equivalent from by replacing Eq. (4.5c) with (4.6).

We note that (4.6) and Eq. (4.5b) define order (q,p) conditions for a traditional implicit GLM with coefficients $(\widehat{\mathbf{A}}, \widehat{\mathbf{U}}, \mathbf{B}, \mathbf{V})$, having the same weights $\mathbf{W}$ as the explicit method.

REMARK 2 (Abscissa vector). *From Eq. (4.5a) it follows that, for methods of stage order $q \geq 1$:*

$$\mathbf{c} = \mathbf{A}\,\mathbf{1}_s + \mathbf{U}\,\mathbf{w}_1. \tag{4.7}$$

REMARK 3 (IMEX pair). *From (4.6) the abscissa of the implicit method is*

$$\widehat{\mathbf{c}} = \widehat{\mathbf{A}}\,\mathbf{1}_s + \widehat{\mathbf{U}}\,\mathbf{w}_1 = \mathbf{c} + \mathbf{\Gamma}\,\mathbf{1}_s + \mathbf{\Psi}\,\mathbf{w}_1 = \mathbf{c}.$$

As a consequence of (4.5c), the explicit method $(\mathbf{A}, \mathbf{U}, \mathbf{B}, \mathbf{V})$ and the implicit method $(\widehat{\mathbf{A}}, \widehat{\mathbf{U}}, \mathbf{B}, \mathbf{V})$ also share the same abscissa vector. As a result this pair of methods also form an IMEX-GLM of order (q,p) [34].

Proof. First assume a linearly implicit GLM has order (q,p) with $q \in \{p, p-1\}$ for all $\mathbf{L}_i$ and sufficiently smooth f. It must have this order for the particular case $\mathbf{L}_i = \mathbf{0}_{d\times d}$ in which the method degenerates to a traditional, explicit GLM with coefficients $(\mathbf{A}, \mathbf{B}, \mathbf{U}, \mathbf{V})$. The traditional GLM order conditions of Theorem 2.2 imply Eqs. (4.5a) and (4.5b).

Consider also the particular case $f(y) = \mathbf{L}\, y$ and $\mathbf{L}_i = \mathbf{L}$ as in (2.6). For linear problems, a linearly implicit GLM degenerates to the implicit GLM with coefficients $(\widehat{\mathbf{A}}, \widehat{\mathbf{U}}, \mathbf{B}, \mathbf{V})$. By Theorem 2.2, a GLM of order (q, p) for linear problems must have coefficients that satisfy Eq. (4.5b) and (4.6). Based on the discussion in Remark 1, this implies the final order condition given in Eq. (4.5c).

Now assume a linearly implicit GLM has coefficients satisfying Eq. (4.5). In the following analysis, we will use the matrix form of linearly implicit GLMs Eq. (3.2), but drop the Kronecker product for brevity. Substituting the exact solution into the right-hand side of Eq. (3.2a) and using Taylor expansions we get

(4.8)
$$
\begin{aligned}
& h\, F\Big(h\,\mathbf{A} \otimes y'(t_{n-1} + \mathbf{c}\, h) + \mathbf{U} \otimes \mathrm{y}^{[n-1]}\Big) + h\,\mathbf{L}\,\mathbf{\Gamma} \otimes y'(t_{n-1} + \mathbf{c}\, h) + h\,\mathbf{L}\,\mathbf{\Psi} \otimes \mathrm{y}^{[n-1]} \\
&= h\, F\left((\mathbf{U}\,\mathbf{w}_0) \otimes\, y(t_{n-1}) + \sum_{k=1}^{p} \left(\frac{\mathbf{A}\,\mathbf{c}^{\times(k-1)}}{(k-1)!} + \mathbf{U}\,\mathbf{w}_k \right) \otimes h^k y^{(k)}(t_{n-1}) + \mathcal{O}\big(h^{p+1}\big) \right) \\
&\quad + h\,\mathbf{L} \left((\mathbf{\Psi}\,\mathbf{w}_0)\, \otimes\, y(t_{n-1}) + \sum_{k=1}^{p} \left(\frac{\mathbf{\Gamma}\,\mathbf{c}^{\times(k-1)}}{(k-1)!} + \mathbf{\Psi}\,\mathbf{w}_k \right) \otimes h^k y^{(k)}(t_{n-1}) + \mathcal{O}\big(h^{p+1}\big) \right) \\
&= h\, F(y(t_{n-1} + \mathbf{c}\, h)) + h^{p+1}\,\mathbf{L} \left(\frac{\mathbf{\Gamma}\,\mathbf{c}^{\times(p-1)}}{(p-1)!} + \mathbf{\Psi}\,\mathbf{w}_p \right) \otimes y^{(p)}(t_{n-1}) \\
&\quad + h^{p+1}\, \frac{\partial F}{\partial y}(t_{n-1}) \left(\frac{\mathbf{A}\,\mathbf{c}^{\times(p-1)}}{(p-1)!} + \mathbf{U}\,\mathbf{w}_p - \frac{\mathbf{c}^{\times p}}{p!} \right) \otimes y^{(p)}(t_{n-1}) + \mathcal{O}\big(h^{p+2}\big) \\
&= h\, y'(t_{n-1} + \mathbf{c}\, h) + \mathcal{O}\big(h^{q+2}\big).
\end{aligned}
$$

Next, we subtract (4.8) from Eq. (3.2a) to approximate the error of the internal stages:

$$
\begin{aligned}
K - h\, y'(t_{n-1} + \mathbf{c}\, h) = h\, F\Big(\mathbf{A} \otimes\, K + \mathbf{U} \otimes\, \mathrm{y}^{[n-1]}\Big) - h\, F\Big(\mathbf{A} \otimes\, h\, y'(t_{n-1} + \mathbf{c}\, h) + \mathbf{U} \otimes\, \mathrm{y}^{[n-1]}\Big) \\
+ h\,\mathbf{L}\,(K - h\, y'(t_{n-1})) + \mathcal{O}\big(h^{q+2}\big).
\end{aligned}
$$

Applying the 2-norm $\|\cdot\| = \|\cdot\|_2$ yields

$$
\begin{aligned}
\|K - h\, y'(t_{n-1} + \mathbf{c}\, h)\| \le h\, C\, \|\mathbf{A}\|\, \|K - h\, y'(t_{n-1} + \mathbf{c}\, h)\| \\
+ h\, \|\mathbf{L}\|\, \|K - h\, y'(t_{n-1} + \mathbf{c}\, h)\| + \mathcal{O}\big(h^{q+2}\big),
\end{aligned}
$$

where C is the Lipschitz constant of f. Thus,

$$
\|K - h\, y'(t_{n-1} + \mathbf{c}\, h)\| \le (1 - h\, C\, \|\mathbf{A}\| - h\, \|\mathbf{L}\|)^{-1} \cdot \mathcal{O}\big(h^{q+2}\big),
$$

and for sufficiently small h

$$
\|K - h\, y'(t_{n-1} + \mathbf{c}\, h)\| = \mathcal{O}\big(h^{q+2}\big),
$$

which proves the stage order of the method. Next, we will show that the order is p.

Noting that $\min(p+1, q+2) = p+1$ we have:

$$
\begin{aligned}
\mathbb{y}^{[n]} &= \mathbf{B} \otimes_{\mathrm{d}} K + \mathbf{V} \otimes_{\mathrm{d}} \mathbb{y}^{[n-1]} \\
&= \mathbf{B} \otimes_{\mathrm{d}} h\, y'(t_{n-1} + \mathbf{c}\, h) + \mathbf{V} \otimes_{\mathrm{d}} \mathbb{y}^{[n-1]} + \mathcal{O}\big(h^{q+2}\big) \\
&= (\mathbf{V}\, \mathbf{w}_0) \otimes_{\mathrm{d}} y(t_{n-1}) + \sum_{k=1}^{p} \left(\frac{\mathbf{B}\, \mathbf{c}^{\times(k-1)}}{(k-1)!} + \mathbf{V}\, \mathbf{w}_k \right) \otimes_{\mathrm{ds}} h^k y^{(k)}(t_{n-1}) + \mathcal{O}\big(h^{p+1}\big) \\
&= \mathbf{w}_0 \otimes_{\mathrm{ds}} y(t_{n-1}) + \sum_{k=1}^{p} \left(\sum_{\ell=0}^{k} \frac{\mathbf{w}_{k-\ell}}{\ell!} \right) \otimes_{\mathrm{ds}} h^k y^{(k)}(t_{n-1}) + \mathcal{O}\Big(h^{\min(p+1,q+2)}\Big) \\
&= \sum_{k=0}^{p} \mathbf{w}_k\, h^k\, y^{(k)}(t_n) + \mathcal{O}\big(h^{p+1}\big).
\end{aligned}
$$

∎

Corollary 4.5. *Suppose a linearly implicit GLM has stage order $q = p-1$, but the underlying implicit method $(\widehat{\mathbf{A}}, \widehat{\mathbf{U}}, \mathbf{B}, \mathbf{V})$ has stage order $q = p$. If $\mathbf{L}_i = \frac{\partial f}{\partial y}(t_{n-1}) + \mathcal{O}(h)$ for $i = 1, \dots s$, then $K_i = h\, y'(t_{n-1} + \mathbf{c}\, h) + \mathcal{O}\big(h^{p+2}\big)$. That is, the stage order is one order higher than Theorem 4.4 predicts.*

Proof. If the assumptions of Corollary 4.5 hold and $\mathbb{L}_i = \frac{\partial f}{\partial y}(t_{n-1}) + \mathcal{O}(h)$ the local truncation error in (4.8) can be sharpened. We have that

$$
\begin{aligned}
& h\, F\Big(\mathbf{A}\, h\, y'(t_{n-1} + \mathbf{c}\, h) + \mathbf{U}\, \mathbb{y}^{[n-1]}\Big) + h\, \mathbf{L}\, \boldsymbol{\Gamma}\, h\, y'(t_{n-1} + \mathbf{c}\, h) + h\, \mathbf{L}\, \boldsymbol{\Psi}\, \mathbb{y}^{[n-1]} \\
&= h\, F(y(t_{n-1} + \mathbf{c}\, h)) + h^{p+1}\, \mathbf{L} \left(\frac{\boldsymbol{\Gamma}\, \mathbf{c}^{\times(p-1)}}{(p-1)!} + \boldsymbol{\Psi}\, \mathbf{w}_p \right) y^{(p)}(t_{n-1}) \\
&\quad + h^{p+1} \frac{\partial f}{\partial y}(t_{n-1}) \left(\frac{\mathbf{A}\, \mathbf{c}^{\times(p-1)}}{(p-1)!} + \mathbf{U}\, \mathbf{w}_p - \frac{\mathbf{c}^{\times p}}{p!} \right) y^{(p)}(t_{n-1}) + \mathcal{O}\big(h^{p+2}\big) \\
&= h\, F(y(t_{n-1} + \mathbf{c}\, h)) + h^{p+1} \frac{\partial f}{\partial y}(t_{n-1}) \left(\frac{\widehat{\mathbf{A}}\, \mathbf{c}^{\times(p-1)}}{(p-1)!} + \widehat{\mathbf{U}}\, \mathbf{w}_p - \frac{\mathbf{c}^{\times p}}{p!} \right) y^{(p)}(t_{n-1}) \\
&\quad + \mathcal{O}\big(h^{p+2}\big) \\
&= h\, y'(t_{n-1} + \mathbf{c}\, h) + \mathcal{O}\big(h^{p+2}\big).
\end{aligned}
$$

Proceeding exactly as in the proof of Theorem 4.4 gives the desired result of

$$
\| K - h\, y'(t_{n-1} + \mathbf{c}\, h) \| = \mathcal{O}\big(h^{p+2}\big).
$$

∎

4.3. Non-autonomous formulation. Using the method in Eq. (3.1) on the non-autonomous system

$$
\begin{bmatrix} y \\ t \end{bmatrix}' = \begin{bmatrix} f(t,y) \\ 1 \end{bmatrix}, \tag{4.9}
$$

we consider the approximate Jacobian of (4.9)

$$
\begin{bmatrix} \mathbf{L}_i & g_i \\ 0 & 0 \end{bmatrix} \approx \begin{bmatrix} \frac{\partial f}{\partial y}(t_{n-1} + c_i\, h) & \frac{\partial f}{\partial t}(t_{n-1} + c_i\, h) \\ 0 & 0 \end{bmatrix}, \tag{4.10}
$$

to get

$$\begin{bmatrix} K_i \\ \kappa_i \end{bmatrix} = h \begin{bmatrix} f(\tau_i, Y_i) \\ 1 \end{bmatrix} + h \begin{bmatrix} \mathbf{L}_i & g_i \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \sum_{i=1}^{s} \gamma_{i,j}\, K_j \\ \sum_{i=1}^{s} \gamma_{i,j}\, \kappa_j \end{bmatrix} + h \begin{bmatrix} \mathbf{L}_i & g_i \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \sum_{i=1}^{r} \psi_{i,j}\, \mathbb{y}_j^{[n-1]} \\ \sum_{i=1}^{r} \psi_{i,j}\, \zeta_j^{[n-1]} \end{bmatrix},$$

$$\begin{bmatrix} \mathbb{y}_i^{[n]} \\ \zeta_i^{[n]} \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{s} b_{i,j}\, K_j \\ \sum_{i=1}^{s} b_{i,j}\, \kappa_j \end{bmatrix} + \begin{bmatrix} \sum_{i=1}^{r} v_{i,j}\, \mathbb{y}_j^{[n-1]} \\ \sum_{i=1}^{r} v_{i,j}\, \zeta_j^{[n-1]} \end{bmatrix}.$$

Assuming an exact starting procedure for the time variable, $\kappa_i = h$ and $\zeta_i^{[n]} = t_n\, w_{i,0} + h\, w_{i,1}$. The time argument for f satisfies

$$\tau_i = \sum_{j=1}^{s} a_{i,j}\, \kappa_j + \sum_{j=1}^{r} u_{i,j}\, \zeta_j^{[n-1]} = t_{n-1} + c_i\, h, \tag{4.11}$$

by the preconsistency condition Eq. (4.2a) and stage order one condition Eq. (4.5a). Furthermore,

$$\begin{aligned} K_i &= h\, f(t_{n-1} + c_i\, h, Y_i) + h\, \mathbf{L}_i \sum_{j=1}^{i} \gamma_{i,j}\, K_j + h\, \mathbf{L}_i \sum_{j=1}^{r} \psi_{i,j}\, \mathbb{y}_j^{[n-1]} \\ &\quad + g_i \left(\sum_{j=1}^{i} \gamma_{i,j}\, \kappa_j + \psi_{i,j}\, \zeta_j^{[n-1]} \right) \\ &= h\, f(t_{n-1} + c_i\, h, Y_i) + h\, \mathbf{L}_i \sum_{j=1}^{i} \gamma_{i,j}\, K_j + h\, \mathbf{L}_i \sum_{j=1}^{r} \psi_{i,j}\, \mathbb{y}_j^{[n-1]} \end{aligned} \tag{4.12}$$

by the preconsistency condition Eq. (4.2c) and stage order one condition Eq. (4.5c). Note, the only difference between the autonomous form Eq. (3.1) and the non-autonomous form (4.12) is the time argument in the right hand side function f and that all the time-derivative terms have vanished.

5. Stability analysis for linearly implicit GLMs. We will now discuss the stability properties of linearly implicit GLMs when applied to linear, Prothero-Robinson, index-1 DAEs and singularly perturbed systems.

5.1. Linear stability. Applying Eq. (3.2) to the scalar Dahlquist test problem

$$y' = \lambda\, y, \tag{5.1}$$

and using the Jacobian approximations $\mathbf{L}_i = \widehat{\lambda}$ reveals that the stability matrix of the linearly implicit GLM is

$$\begin{aligned} \mathbb{y}^{[n]} &= \mathbf{M}(z, \widehat{z})\, \mathbb{y}^{[n-1]}, \\ \mathbf{M}(z, \widehat{z}) &= \mathbf{V} + \mathbf{B}\left(\mathbf{I}_{s\times s} - z\,\mathbf{A} - \widehat{z}\,\mathbf{\Gamma}\right)^{-1}\left(z\,\mathbf{U} + \widehat{z}\,\mathbf{\Psi}\right), \end{aligned}$$

where $z = h\,\lambda$ and $\widehat{z} = h\,\widehat{\lambda}$.

In the case the exact Jacobian is used, $\widehat{z} = z$ and, as expected, the stability function of the underlying implicit method is recovered:

$$\widehat{\mathbf{M}}(z) = \mathbf{M}(z, z) = \mathbf{V} + z\,\mathbf{B}\left(\mathbf{I}_{s\times s} - z\,\widehat{\mathbf{A}}\right)^{-1}\widehat{\mathbf{U}}.$$

In the case where the arbitrary Jacobian is chosen as $\mathbf{L}_i = 0$ the stability matrix of linearly implicit GLM is the same as that of the expilcit GLM:

$$\widetilde{\mathbf{M}}(z) = \mathbf{M}(z,0) = \mathbf{V} + z\,\mathbf{B}\,(\mathbf{I}_{s\times s} - z\,\mathbf{A})^{-1}\,\mathbf{U}.$$

REMARK 4. *We can extend this result to the linear, nonautonomous, nonhomogeneous test problem*

$$\mathbf{M}(t)\,y' = \lambda(t)\,y + \phi(t). \tag{5.2}$$

Similarly, the linearly implicit GLM Eq. (3.1) applied to (5.2) using the exact Jacobians $\mathbf{L}_i = \lambda(t_{n-1} + \mathbf{c}\,h)$, computes stages as:

$$\begin{aligned} \mathbf{M}(t)\,K &= h\,\lambda(t_{n-1} + \mathbf{c}\,h)\left(\widehat{\mathbf{A}}\,K + \widehat{\mathbf{U}}\,\xi^{[n-1]}\right) + \phi(t_{n-1} + \mathbf{c}\,h), \\ \mathrm{y}^{[n]} &= \mathbf{B}\,K + \mathbf{V}\,\mathrm{y}^{[n-1]}, \end{aligned} \tag{5.3}$$

where (5.3) is exactly the stages of the underlying implicit method applied to (5.2). As a result the stability properties of linearly implicit GLM for this problem is the same as its underlying implicit GLM.

6. Stiff accuracy of linearly implicit GLMs.

6.1. Convergence of linearly implicit GLMs applied to the Prothero–Robinson problem. The Prothero–Robinson problem [20] is a linear problem of the form

$$y' = \mu\left(y - \phi\,(t)\right) + \phi'\,(t)\,. \tag{6.1}$$

To show convergence of linearly implicit GLMs we rely on convergence results for traditional GLMs as discussed in [34, Theroem 3] and [5, Section 4].

THEOREM 6.1. *The linearly implicit GLM Eq. (3.2) satisfying stage order conditions (q,p) with $q \in \{p, p-1\}$ and $L_i = \mu$ applied to the Prothero-Robinson problem has error $y - \phi(t_n) = \mathcal{O}\big(h^{\min(q+1,p)}\big)$.*

Proof. The stages for the method in Eq. (3.2) applied to (6.1) are

$$\begin{aligned} K &= z\left(\mathbf{A}\,K + \mathbf{U}\mathrm{y}^{[n-1]} - \phi\,(t_{n-1} + \mathbf{c}\,h)\right) \\ &\quad + h\,\phi'\,(t_{n-1} + \mathbf{c}\,h) + z\,\mathbf{\Psi}\,\mathrm{y}^{[n-1]} + z\,\mathbf{\Gamma}\,K, \\ &= z\left(\widehat{\mathbf{A}}\,K + \widehat{\mathbf{U}}\mathrm{y}^{[n-1]} - \phi\,(t_{n-1} + \mathbf{c}\,h)\right) + h\,\phi'\,(t_{n-1} + \mathbf{c}\,h)\,, \qquad (6.2a) \\ \mathrm{y}^{[n]} &= \mathbf{B}\,K + \mathbf{V}\,\mathrm{y}^{[n-1]}, \qquad (6.2b) \end{aligned}$$

where $z = h\mu$. Denote the global internal and external stage errors by

$$e_K := K - h\phi'\,(t_{n-1} + \mathbf{c}h)\,, \tag{6.3}$$

$$e_\xi^{[n-1]} := \mathrm{y}^{[n-1]} - \mathbf{W}\,\eta\,(t_{n-1})\,. \tag{6.4}$$

Note that $Y = \widehat{\mathbf{A}}\,K + \widehat{\mathbf{U}}\mathrm{y}^{[n-1]} - \phi\,(t_{n-1} + \mathbf{c}\,h)$ are the stages of an implicit GLM as discussed in Remark 3. Following the results in [34, 5],

$$Y = \widehat{\mathbf{A}}\,K + \widehat{\mathbf{U}}\mathrm{y}^{[n-1]} - \phi\,(t_{n-1} + \mathbf{c}\,h) = \mathcal{O}\big(h^{q+1}\big),$$

and from Eq. (6.2a) we have

$$e_K = h\mu \left(\widehat{\mathbf{A}}\, K + \widehat{\mathbf{U}} \mathbb{y}^{[n-1]} - \phi\left(t_{n-1} + \mathbf{c}\, h\right) \right) = \mathcal{O}\big(h^{q+2}\big).$$

For the external stage error from Eq. (6.2) we have

$$\begin{aligned} e_K &= z \left(\mathbf{I}_{s\times s} - z\widehat{\mathbf{A}} \right)^{-1} \left(\widehat{\mathbf{U}} \mathbb{y}^{[n-1]} - \phi\left(t_{n-1} + \mathbf{c}\, h\right) + h\phi'\left(t_{n-1} + \mathbf{c}\, h\right) \right), \\ e_\xi^{[n]} + \mathbf{W}\, \eta\left(t_n\right) &= \widehat{\mathbf{M}}\left(z\right) e_\xi^{[n-1]} + h\mathbf{B}\phi'\left(t_{n-1} + \mathbf{c}\, h\right) + \mathbf{V}\mathbf{W}\, \eta\left(t_{n-1}\right) + \mathcal{O}\big(h^{q+1}\big) \\ &\quad + z\mathbf{B} \left(\mathbf{I}_{s\times s} - z\widehat{A} \right)^{-1} \left(h\widehat{A}\phi'\left(t_{n-1} + \mathbf{c}\, h\right) - \phi\left(t_{n-1} + \mathbf{c}\, h\right) \right). \end{aligned}$$

where $\eta\left(t_n\right)$ is the Nordsieck vector at time t_n defined by

$$\eta\left(t_n\right) = \left[\phi\left(t_n\right), h\phi'(t_n), \ldots, h^p\phi^{(p)}(t_n)\right]^T.$$

Now simplifying the terms involving ϕ and ϕ' using the Taylor expansion and order condition Eq. (4.5b) similar to [34, Equation 52] results in error recurrence

$$e_\xi^{[n]} = \widehat{\mathbf{M}}\left(z\right) e_\xi^{[n-1]} + \mathcal{O}\Big(h^{\min\left(q+1,p\right)}\Big)$$

The convergence of the global error is determined by the eigenvalues of the stability matrix $\widehat{\mathbf{M}}$ as discussion in detail in [24, Lemma 2]. Here, we consider one specific case:

Remark 5. *If the starting procedure is exact such that* $e_\xi^{[n]} = 0$, $\widehat{\mathbf{A}}$ *is non-singular and if*

$$\rho\left(\widehat{\mathbf{M}}\left(z\right)\right) < \alpha < 1, \forall z \in \mathcal{D},$$

where $\mathcal{D}$ *is the region of absolute stability for the implicit underlying method, then linearly implicit GLM Eq. (3.2) applied to Prothero-Robinson problem (6.1) is PR-Convergent with order* $\min\left(q+1,p\right)$

$$\left\| e_\xi^{[n]} \right\| = \mathcal{O}\Big(h^{\min\left(q+1,p\right)}\Big).$$

6.2. Convergence of linearly implicit GLMs applied to singularly perturbed problems. Consider the singular perturbation problem

$$\begin{bmatrix} y' \\ z' \end{bmatrix} = \begin{bmatrix} f(y,z) \\ \varepsilon^{-1} g(y,z) \end{bmatrix}. \tag{6.7}$$

We apply the linearly implicit GLM (3.2) to (6.7). Using the notation

$$Y := \mathbf{A} \circledast K_y + \mathbf{U} \circledast \mathbb{y}^{[n-1]}, \tag{6.8a}$$

$$Z := \mathbf{A} \circledast K_z + \mathbf{U} \circledast \mathbb{z}^{[n-1]}, \tag{6.8b}$$

the numerical method reads as

$$\begin{aligned} \begin{bmatrix} K_y \\ \varepsilon\, K_z \end{bmatrix} &= h \begin{bmatrix} F\left(Y,Z\right) \\ G\left(Y,Z\right) \end{bmatrix} + h \begin{bmatrix} F_y\, \mathbf{\Gamma} \circledast K_y + F_z\, \mathbf{\Gamma} \circledast K_z \\ G_y\, \mathbf{\Gamma} \circledast K_y + G_z\, \mathbf{\Gamma} \circledast K_z \end{bmatrix} \\ &\quad + h \begin{bmatrix} F_y\, \mathbf{\Psi} \circledast \mathbb{y}^{[n]} + F_z\, \mathbf{\Psi} \circledast \mathbb{y}^{[n-1]} \\ G_y\, \mathbf{\Psi} \circledast \mathbb{z}^{[n]} + G_z\, \mathbf{\Psi} \circledast \mathbb{z}^{[n-1]} \end{bmatrix}, \end{aligned} \tag{6.8c}$$

$$\begin{bmatrix} \mathbb{y}^{[n]} \\ \mathbb{z}^{[n]} \end{bmatrix} = \begin{bmatrix} \mathbf{B} \circledast K_y + \mathbf{V} \circledast \mathbb{y}^{[n-1]} \\ \mathbf{B} \circledast K_z + \mathbf{V} \circledast \mathbb{z}^{[n-1]} \end{bmatrix}, \tag{6.8d}$$

where we use $F_y = f_y(Y,Z) \otimes \mathbf{I}_{s\times s}$ to denote the sub-Jacobian evaluation and similar notation for F_z, G_y, and G_z in relation to (3.3) is adopted. We first consider convergence of linearly implicit GLMs for index-1 DAEs. Later, we will extend the result to singularly perturbed problems.

6.2.1. Index-1 DAEs. When $\varepsilon \to 0$ in Eq. (6.8) retreives the index-1 DAE:

$$\begin{aligned} y' &= f\,(y,z)\,, \\ 0 &= g\,(y,z)\,. \end{aligned}$$

We will consider the numerical values of the stages of a linearly implicit GLM satisfying order (q,p) conditions with $q \in \{p, p-1\}$ applied to the index-1 DAE and compare them to the same stage equations when exact values are substituted. Then we will derive an error recurrence that will determine the convergence rates for the differential and algebraic variables. Let us start with Eq. (6.8) and take the limit $\varepsilon \to 0$:

$$\begin{aligned} K_y &= h\,F\,(Y,Z) + h\,\left(F_y\,\mathbf{\Gamma} \circledast K_y + F_z\,\mathbf{\Gamma} \circledast K_z\right) \\ &\quad + h\,\left(F_y\,\mathbf{\Psi} \circledast \mathbb{y}^{[n-1]} + F_z\,\mathbf{\Psi} \circledast \mathbb{z}^{[n-1]}\right), \end{aligned} \tag{6.9a}$$

$$\begin{aligned} 0 &= G\,(Y,Z) + G_y\,\mathbf{\Gamma} \circledast K_y + G_z\,\mathbf{\Gamma} \circledast K_z \\ &\quad + G_y\,\mathbf{\Psi} \circledast \mathbb{y}^{[n-1]} + G_z\,\mathbf{\Psi} \circledast \mathbb{z}^{[n-1]}. \end{aligned} \tag{6.9b}$$

Define $\widetilde{Y}$ and $\widetilde{Z}$ as function arguments when exact values are inserted into Eqs. (6.8a) and (6.8b). Considering the internal stage order conditions for the underlying explicit method we have that:

$$\begin{aligned} \widetilde{Y} &= h\mathbf{A} \circledast y'\,(t_{n-1} + \mathbf{c}\,h) + (\mathbf{U}\mathbf{W}) \circledast \eta_y\,(t_{n-1}) \\ &= y(t_{n-1} + \mathbf{c}h) + \mathcal{O}\big(h^{q+1}\big), \end{aligned} \tag{6.10a}$$

$$\begin{aligned} \widetilde{Z} &= h\mathbf{A} \circledast z'\,(t_{n-1} + \mathbf{c}\,h) + (\mathbf{U}\mathbf{W}) \circledast \eta_z\,(t_{n-1}) \\ &= z(t_{n-1} + \mathbf{c}h) + \mathcal{O}\big(h^{q+1}\big). \end{aligned} \tag{6.10b}$$

Furthermore, define the global errors as:

$$\begin{aligned} \Delta K_y &:= K_y - hy'\,(t_{n-1} + \mathbf{c}\,h)\,, \\ \Delta \mathbb{y}^{[n-1]} &:= \mathbb{y}^{[n-1]} - \mathbf{W} \circledast \eta_y\,(t_{n-1})\,, \\ \Delta Y := Y - \widetilde{Y} &= \mathbf{A} \circledast\ \left(K_y - hy'\,(t_{n-1} + \mathbf{c}\,h)\right) \\ &\quad + \mathbf{U} \circledast\ \mathbb{y}^{[n-1]} - (\mathbf{U}\mathbf{W}) \circledast \eta_y\,(t_{n-1}) \\ &= \mathbf{A} \circledast\ \Delta K_y + \mathbf{U} \circledast\ \Delta \mathbb{y}^{[n-1]}, \end{aligned}$$

where $\eta_y\,(t_{n-1})$ is the Nordsieck vector corresponding to the derivatives of $y(t)$ at time t_{n-1}. We assume similar notation for the errors in the algebraic variable $\Delta Z, \Delta K_z$ and $\Delta \mathbb{z}^{[n-1]}$.

Now, we insert exact values in Eq. (6.9a) and using the fact that the linearly implicit GLM satisfies the internal and external stage order conditions Eq. (4.5) to write:

(6.12)
$$hy'\left(t_{n-1}+\mathbf{c}\,h\right)=hF\left(\widetilde{Y},\widetilde{Z}\right)+h\left(hF_y\mathbf{\Gamma}\circledast y'\left(t_{n-1}+\mathbf{c}\,h\right)+hF_z\mathbf{\Gamma}\circledast z'\left(t_{n-1}+\mathbf{c}\,h\right)\right)$$
$$+\,h\left(F_y(\mathbf{\Psi W})\circledast\eta_y(t_{n-1})+F_z(\mathbf{\Psi W})\circledast\eta_z(t_{n-1})\right)$$
$$+\,\mathcal{O}\big(h^{q+2}\big)+\mathcal{O}\big(h^{p+1}\big).$$

Subtracting (6.12) from Eq. (6.9a)

$$\begin{aligned}\Delta K_y&=hF\left(Y,Z\right)-hF\left(\widetilde{Y},\widetilde{Z}\right)\\&\quad+h\left(F_y\mathbf{\Gamma}\circledast\Delta K_y+F_z\mathbf{\Gamma}\circledast\Delta K_z+F_y\mathbf{\Psi}\circledast\Delta\mathbb{y}^{[n-1]}+F_z\mathbf{\Psi}\circledast\Delta\mathbb{z}^{[n-1]}\right)\\&\quad+\mathcal{O}\big(h^{\nu+1}\big),\end{aligned}\tag{6.13}$$

where $\nu=\min\left(q+1,p\right).$

We expand $hF\left(Y,Z\right)$ and $hF\left(\widetilde{Y},\widetilde{Z}\right)$ around their exact values

$$\begin{aligned}hF\left(Y,Z\right)&=hF\left(y\left(t_{n-1}+\mathbf{c}\,h\right),z\left(t_{n-1}+\mathbf{c}\,h\right)\right)\\&\quad+h\,F_y\,\left(\mathbf{A}\circledast K_y+\mathbf{U}\circledast\mathbb{y}^{[n-1]}-y\left(t_{n-1}+\mathbf{c}\,h\right)\right)\\&\quad+h\,F_z\,\left(\mathbf{A}\circledast K_z+\mathbf{U}\circledast\mathbb{z}^{[n-1]}-z\left(t_{n-1}+\mathbf{c}\,h\right)\right)\\&\quad+\mathcal{O}\big(h^{2q+3}\big),\end{aligned}\tag{6.14}$$

where the order of the residual term is determined by the internal stage order conditions of the explicit underlying method. Similarly,

(6.15)
$$\begin{aligned}hF\left(\widetilde{Y},\widetilde{Z}\right)&=hF\left(y\left(t_{n-1}+\mathbf{c}\,h\right),z\left(t_{n-1}+\mathbf{c}\,h\right)\right)\\&\quad+h\,F_y\,\left(h\mathbf{A}\circledast y'\left(t_{n-1}+\mathbf{c}\,h\right)+(\mathbf{UW})\circledast\eta_y\left(t_{n-1}\right)-y\left(t_{n-1}+\mathbf{c}\,h\right)\right)\\&\quad+h\,F_z\,\left(h\mathbf{A}\circledast z'\left(t_{n-1}+\mathbf{c}\,h\right)+(\mathbf{UW})\circledast\eta_z\left(t_{n-1}\right)-z\left(t_{n-1}+\mathbf{c}\,h\right)\right)\\&\quad+\mathcal{O}\big(h^{2q+3}\big),\end{aligned}$$

where the order of the residual is due to Eq. (6.10). The Jacobians F_y and F_z in (6.14) and (6.15) are evaluated at (Y,Z) and $(y(t_{n-1}+\mathbf{c}\,h),z(t_{n-1}+\mathbf{c}\,h))$ respectively, however, we note that changing the argument to the Jacobian does not change the accuracy of these equations due to Eq. (6.10). Using (6.14) and (6.15) in (6.13) we have:

$$\begin{aligned}\Delta K_y&=h\left(F_y\left(\mathbf{\Gamma}+\mathbf{A}\right)\circledast\Delta K_y+F_z\left(\mathbf{\Gamma}+\mathbf{A}\right)\circledast\Delta K_z\right)\\&\quad+h\left(F_y\left(\mathbf{\Psi}+\mathbf{U}\right)\circledast\Delta\mathbb{y}^{[n-1]}+F_z\left(\mathbf{\Psi}+\mathbf{U}\right)\circledast\Delta\mathbb{z}^{[n-1]}\right)\\&\quad+\mathcal{O}\big(h^{\nu+1}\big).\end{aligned}\tag{6.16}$$

Repeating the same steps for the algebraic stages we get:

$$\begin{aligned}0&=G_y\left(\mathbf{\Gamma}+\mathbf{A}\right)\circledast\Delta K_y+G_z\left(\mathbf{\Gamma}+\mathbf{A}\right)\circledast\Delta K_z\\&\quad+G_y\left(\mathbf{\Psi}+\mathbf{U}\right)\circledast\Delta\mathbb{y}^{[n-1]}+G_z\left(\mathbf{\Psi}+\mathbf{U}\right)\circledast\Delta\mathbb{z}^{[n-1]}\\&\quad+\mathcal{O}(h^{\nu}),\end{aligned}$$

$$\Delta K_z=-\left(G_z\widehat{\mathbf{A}}\right)^{-1}\left(G_y\widehat{\mathbf{A}}\,\Delta K_y+G_y\widehat{\mathbf{U}}\,\Delta\mathbb{y}^{[n-1]}+G_z\widehat{\mathbf{U}}\,\Delta\mathbb{z}^{[n-1]}\right)+\mathcal{O}(h^{\nu}),\tag{6.17}$$

where $\widehat{\mathbf{A}} = (\mathbf{A} + \mathbf{\Gamma}) \otimes_{d}$ and $\widehat{\mathbf{U}} = (\mathbf{U} + \mathbf{\Psi}) \otimes_{d}$.

Assuming G_z is invertible, we define some new notations that will become useful:

$$\begin{aligned} \mathbf{J}^{\text{red}} &= F_y - F_z\, G_z^{-1}\, G_y, \\ F^{\text{red}}(y,z) &= F(y,z) - F_z\, G_z^{-1}\, G(y,z), \end{aligned}$$

noting that these are the right hand side and Jacobian of the index-reduced system obtained by differentiating the DAE. Rearranging (6.16) and (6.17) we have

$$\begin{bmatrix} \Delta K_y \\ \Delta K_z \end{bmatrix} = \mathbf{S}(h)^{-1} \left(\begin{bmatrix} hF_y\widehat{\mathbf{U}} & hF_z\widehat{\mathbf{U}} \\ G_y\widehat{\mathbf{U}} & G_z\widehat{\mathbf{U}} \end{bmatrix} \begin{bmatrix} \Delta\mathbb{y}^{[n-1]} \\ \Delta\mathbb{z}^{[n-1]} \end{bmatrix} + \begin{bmatrix} \mathcal{O}\big(h^{\nu+1}\big) \\ \mathcal{O}(h^{\nu}) \end{bmatrix} \right), \tag{6.18}$$

where

$$\mathbf{S}(h) = \begin{bmatrix} \mathbf{I} - hF_y\widehat{\mathbf{A}} & -hF_z\widehat{\mathbf{A}} \\ -G_y\widehat{\mathbf{A}} & -G_z\widehat{\mathbf{A}} \end{bmatrix},$$

with the inverse,

$$\mathbf{S}(h)^{-1} = \begin{bmatrix} \left(\mathbf{I} - h\mathbf{J}^{\text{red}}\widehat{\mathbf{A}}\right)^{-1} & -h\left(\mathbf{I} - h\mathbf{J}^{\text{red}}\widehat{\mathbf{A}}\right)^{-1} F_z\, G_z^{-1} \\ -\widehat{\mathbf{A}}^{-1}G_z^{-1}G_y\widehat{\mathbf{A}}\left(\mathbf{I} - h\mathbf{J}^{\text{red}}\widehat{\mathbf{A}}\right)^{-1} & -\widehat{\mathbf{A}}^{-1}G_z^{-1}\left(\mathbf{I} - hG_y\widehat{\mathbf{A}}\left(\mathbf{I} - h\mathbf{J}^{\text{red}}\widehat{\mathbf{A}}\right)^{-1} F_zG_z^{-1}\right) \end{bmatrix}. \qquad \blacksquare$$

Inserting (6.18) into Eq. (6.8d) we can get the error recurrence:

(6.19)

$$\begin{aligned} \begin{bmatrix} \Delta\mathbb{y}^{[n]} \\ \Delta\mathbb{z}^{[n]} \end{bmatrix} &= \begin{bmatrix} \mathbf{B} & \mathbf{0} \\ \mathbf{0} & \mathbf{B} \end{bmatrix} \begin{bmatrix} \Delta K_y \\ \Delta K_z \end{bmatrix} + \begin{bmatrix} \mathbf{V} & \mathbf{0} \\ \mathbf{0} & \mathbf{V} \end{bmatrix} \begin{bmatrix} \Delta\mathbb{y}^{[n-1]} \\ \Delta\mathbb{z}^{[n-1]} \end{bmatrix} \\ &= \left(\begin{bmatrix} \mathbf{B} & \mathbf{0} \\ \mathbf{0} & \mathbf{B} \end{bmatrix} \mathbf{S}(h)^{-1} \begin{bmatrix} hF_y\widehat{\mathbf{U}} & hF_z\widehat{\mathbf{U}} \\ G_y\widehat{\mathbf{U}} & G_z\widehat{\mathbf{U}} \end{bmatrix} + \begin{bmatrix} \mathbf{V} & \mathbf{0} \\ \mathbf{0} & \mathbf{V} \end{bmatrix} \right) \begin{bmatrix} \Delta\mathbb{y}^{[n-1]} \\ \Delta\mathbb{z}^{[n-1]} \end{bmatrix} \\ &\quad + \begin{bmatrix} \mathcal{O}(1) & \mathcal{O}(h) \\ \mathcal{O}(1) & \mathcal{O}(1) \end{bmatrix} \begin{bmatrix} \mathcal{O}\big(h^{\nu+1}\big) \\ \mathcal{O}(h^{\nu}) \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{V} + h\mathbf{B}\left(\mathbf{I} - h\mathbf{J}^{\text{red}}\widehat{\mathbf{A}}\right)^{-1}\mathbf{J}^{\text{red}}\widehat{\mathbf{U}} & \mathbf{0} \\ \mathcal{R}\left(h\mathbf{J}^{\text{red}}\right) & \mathbf{V} - \mathbf{B}\widehat{\mathbf{A}}^{-1}\widehat{\mathbf{U}} \end{bmatrix} \begin{bmatrix} \Delta\mathbb{y}^{[n-1]} \\ \Delta\mathbb{z}^{[n-1]} \end{bmatrix} + \begin{bmatrix} \mathcal{O}\big(h^{\nu+1}\big) \\ \mathcal{O}(h^{\nu}) \end{bmatrix} \\ &= \begin{bmatrix} \widehat{\mathbf{M}}\left(\mathcal{Z}\right) & \mathbf{0} \\ \mathcal{R}\left(\mathcal{Z}\right) & \widehat{\mathbf{M}}\left(\infty\right) \end{bmatrix} \begin{bmatrix} \Delta\mathbb{y}^{[n-1]} \\ \Delta\mathbb{z}^{[n-1]} \end{bmatrix} + \begin{bmatrix} \mathcal{O}\big(h^{\nu+1}\big) \\ \mathcal{O}(h^{\nu}) \end{bmatrix}. \end{aligned}$$

where $\mathcal{Z} = h\mathbf{J}^{\text{red}}$ and $\widehat{\mathbf{M}}\left(\mathcal{Z}\right) = \mathbf{V} + \mathbf{B}\left(\mathbf{I} - \mathcal{Z}\widehat{\mathbf{A}}\right)^{-1}\mathcal{Z}\widehat{\mathbf{U}}$. Convergence of the error recurrence in (6.19) depends on the eigenvalues of the error amplification matrix which in this case has lower block-triangular structure. As a result convergence is determined by the eigenvalues of $\widehat{\mathbf{M}}\left(\mathcal{Z}\right)$ including at infinity. A detailed discussion of various cases can be found in [24, Lemma 2]. Here, we use the following corollary.

Corollary 6.2. *Assume g_z is invertible in a neighborhood of the solution, and also assume we start from an algebraically consistent initial condition $g(y_0, z_0) = 0$. If the underlying implicit method satisfies $\rho(\widehat{\mathbf{M}}\left(\infty\right)) < 1$ and is power bounded* *elsewhere then the global error of linearly implicit GLM* (3.2) *applied to the index-1 DAE problem satisfies:*

$$\mathbb{y}^{[n]} - y(t_n) = O(h^\nu), \qquad K_y - hy'(t_{n-1} + \mathbf{c}\, h) = O(h^{\nu+1}),$$
$$\mathbb{z}^{[n]} - z(t_n) = O(h^\nu), \qquad K_z - hz'(t_{n-1} + \mathbf{c}\, h) = O(h^{\nu+1}).$$

Proof. Simple application of [24, Lemma 2] to estimate the global errors leads to $\mathbb{y}^{[n]} - y(t_n) = O(h^\nu)$ and $\mathbb{z}^{[n]} - z(t_n) = O(h^\nu)$. For internal stages we use (6.18) to get:

$$\begin{bmatrix} \Delta K_y \\ \Delta K_z \end{bmatrix} = \begin{bmatrix} \mathcal{O}(h) & \mathcal{O}(h) \\ \mathcal{O}(1) & \mathcal{O}(1) \end{bmatrix} \begin{bmatrix} \Delta \mathbb{y}^{[n-1]} \\ \Delta \mathbb{z}^{[n-1]} \end{bmatrix} + \begin{bmatrix} \mathcal{O}\big(h^{\nu+1}\big) \\ \mathcal{O}(h^\nu) \end{bmatrix}.$$

If global error estimates for the external stages are substituted in this equation we arrive at $\Delta K_y = \mathcal{O}\big(h^{\nu+1}\big)$ and $\Delta K_z = \mathcal{O}\big(h^{\nu+1}\big)$.

Alternatively, we can investigate the convergence for the differential variable with a different approach. Solving for $\widehat{\mathbf{A}} \circledast \Delta K_z + \widehat{\mathbf{U}} \circledast \Delta \mathbb{z}^{[n-1]}$ in (6.17) and inserting into (6.16) we get

$$\begin{aligned} \Delta K_y &= h\mathbf{J}^{\text{red}} \widehat{\mathbf{A}} \circledast \Delta K_y + h\mathbf{J}^{\text{red}} \widehat{\mathbf{U}} \circledast \Delta \mathbb{y}^{[n-1]} + \mathcal{O}\big(h^{\nu+1}\big), \\ \Delta \mathbb{y}^{[n]} &= \mathbf{B} \Delta K_y + \widehat{\mathbf{U}} \Delta \mathbb{y}^{[n-1]}, \end{aligned} \tag{6.20}$$

which is an error recurrence for the underlying implicit method applied to linear ODE $y' = \mathbf{J}^{\text{red}} y$ with local truncation error of $\mathcal{O}\big(h^{\nu+1}\big)$. Therefore, convergence of traditional GLMs also leads to the same result. ◻

6.2.2. $\varepsilon-$expansion of smooth solutions. We are now interested in smooth solutions of the singular perturbation problem (6.7) that can be expanded according to

$$y(t) = \sum_{k \ge 0} y^k(t) \varepsilon^k, \qquad z(t) = \sum_{k \ge 0} z^k(t) \varepsilon^k, \tag{6.21}$$

where solution components $y^k(t)$ and $z^k(t)$ are not dependent on ε. We assume that the initial values $\mathbb{y}^{[0]}, \mathbb{z}^{[0]}$ are on the smooth solution manifold. We similarly expand the function arguments and internal and external stages of the numerical solution in powers of ε:

$$\begin{aligned} Y &= \sum_{i \ge 0} Y^i\, \varepsilon^i, & Z &= \sum_{i \ge 0} Z^i \varepsilon^i, \\ K_y &= \sum_{i \ge 0} K_y^i\, \varepsilon^i, & K_z &= \sum_{i \ge 0} K_z^i \varepsilon^i, \\ \mathbb{y}^{[n]} &= \sum_{i \ge 0} \mathbb{y}^{[n],i}\, \varepsilon^i, & \mathbb{z}^{[n]} &= \sum_{i \ge 0} \mathbb{z}^{[n],i}\, \varepsilon^i. \end{aligned}$$

Inserting (6.21) into (6.7) and separating $\mathcal{O}(1)$ terms leads to the index-1 DAE discussed in section 6.2.1:

$$\begin{aligned} y^{0'} &= f\left(y^0, z^0\right), \\ 0 &= g\left(y^0, z^0\right), \end{aligned}$$

for which the convergence of linearly implicit GLMs is already discussed. If $\mathcal{O}(\varepsilon^1)$ terms are considered, in addition to Eq. (6.22) we get the index-2 DAE:

$$y^{1'} = f_y\left(y^0, z^0\right) y^1 + f_z\left(y^0, z^0\right) z^1, \tag{6.23a}$$
$$z^{0'} = g_y\left(y^0, z^0\right) y^1 + g_z\left(y^0, z^0\right) z^1. \tag{6.23b}$$

Assuming g_z is invertible we can insert z^1 from Eq. (6.23b) into Eq. (6.23a) to get the reduced ODE:

$$z^1 = -\left(g_z^{-1} g_y\right) y^1 + g_z^{-1} z^{0'}, \tag{6.24a}$$
$$y^{1'} = \left(f_y - f_z g_z^{-1} g_y\right) y^1 + \left(f_z g_z^{-1}\right) z^{0'}. \tag{6.24b}$$

All Jacobians in Eq. (6.24) are evaluated at $(y^0(t), z^0(t))$ and we have dropped their argument for brevity. We note that Eq. (6.24b) is a linear ODE in $y^1(t)$ with a time-dependent term on the right-hand side function. In general the differential-algebraic equation for component α of the SPP problem can be represented by [10, Section VI.3]

$$y^{\nu'} = f_y y^\nu + f_z z^\nu + \phi^\nu\left(y^0, z^0, \ldots, y^{\nu-1}, z^{\nu-1}\right), \tag{6.25a}$$
$$z^{\nu-1'} = g_y y^\nu + g_z z^\nu + \psi^\nu\left(y^0, z^0, \ldots, y^{\nu-1}, z^{\nu-1}\right). \tag{6.25b}$$

In [28] Schneider shows the convergence of SPP components for general linear methods. Here, we use the linearity of the systems in Eq. (6.25) to show that the stages of linearly implicit GLM applied to these problems are within the local truncation errors of the stages of the underlying implicit method applied to the same system when the exact Jacobian is used. Convergence of the SPP problem, therefore, follows the same asymptotic as the implicit GLM.

THEOREM 6.3 (Convergence of linearly implicit GLMs for the $\varepsilon-$expansion terms of SPP). *For an internally consistent linearly implicit GLM of order $(p, q), q \in \{p-1, p\}$, with the underlying implicit method satisfying $\rho(\mathbf{M}(\infty)) < 1$, invertible $\widehat{\mathbf{A}}$ and starting with consistent initial values*

$$\begin{aligned} \Delta \mathbb{y}^{\nu,[n]} &= \mathcal{O}\big(h^{q+2-\nu}\big), \quad \Delta Y^{\nu,[n]} = \mathcal{O}\big(h^{q+2-\nu}\big), \\ \Delta \mathbb{z}^{\nu,[n]} &= \mathcal{O}\big(h^{q+1-\nu}\big), \quad \Delta Z^{\nu,[n]} = \mathcal{O}\big(h^{q+1-\nu}\big), \end{aligned} \tag{6.26}$$

for $1 \le \nu \le q+1$.

Proof. We will present the proof for $\nu = 1$. For $\nu > 1$ an induction argument can be constructed. Starting at $\nu = 1$ the index-2 problem reads as:

$$y^{1'} = f_y\left(y^0, z^0\right) y^1 + f_z\left(y^0, z^0\right) z^1 = \mathcal{F}(y^1, z^1), \tag{6.27a}$$
$$z^{0'} = g_y\left(y^0, z^0\right) y^1 + g_z\left(y^0, z^0\right) z^1 = \mathcal{G}(y^1, z^1). \tag{6.27b}$$

We like to show that the difference between the linearly implicit GLM and the underlying implicit stages remain bounded. Consider the implicit method's stages:

$$\begin{aligned} \widehat{Y}^1 &= \widehat{\mathbf{A}} \widehat{K}_y^1 + \widehat{\mathbf{U}} \widehat{\mathbb{y}}^{1,[n-1]}, \quad & \widehat{Z}^1 &= \widehat{\mathbf{A}} \widehat{K}_z^1 + \widehat{\mathbf{U}} \widehat{\mathbb{z}}^{1,[n-1]}, \\ \widehat{K}_y^1 &= h\mathcal{F}\left(\widehat{Y}^1, \widehat{Z}^1\right), & \widehat{K}_z^0 &= h\mathcal{G}\left(\widehat{Y}^1, \widehat{Z}^1\right). \end{aligned}$$

where the stage values according to Eq. (6.27) are

$$\widehat{K}_y^1 = hF_y\left(\widehat{Y}^0, \widehat{Z}^0\right)\left(\widehat{\mathbf{A}}\widehat{K}_y^1 + \widehat{\mathbf{U}}\widehat{\mathbb{y}}^{1,[n-1]}\right) + hF_z\left(\widehat{Y}^0, \widehat{Z}^0\right)\left(\widehat{\mathbf{A}}\widehat{K}_z^1 + \widehat{\mathbf{U}}\widehat{\mathbb{y}}^{1,[n-1]}\right),$$
$$\widehat{K}_z^0 = hG_y\left(\widehat{Y}^0, \widehat{Z}^0\right)\left(\widehat{\mathbf{A}}\widehat{K}_y^1 + \widehat{\mathbf{U}}\widehat{\mathbb{y}}^{1,[n-1]}\right) + hG_z\left(\widehat{Y}^0, \widehat{Z}^0\right)\left(\widehat{\mathbf{A}}\widehat{K}_z^1 + \widehat{\mathbf{U}}\widehat{\mathbb{z}}^{1,[n-1]}\right),$$

where $\widehat{Y}^0$ and $\widehat{Z}^0$ are the index-1 solutions computed by the implicit method. Similarly, the linearly implicit GLM stages read as

$$\begin{aligned}
Y^1 &= \mathbf{A}K_y^1 + \mathbf{U}\mathbb{y}^{1,[n-1]}, \quad Z^1 = \mathbf{A}K_z^1 + \mathbf{U}\mathbb{z}^{1,[n-1]},\\
K_y^1 &= h\mathcal{F}\left(Y^1, Z^1\right) + \mathcal{F}_y\left(Y^1, Z^1\right)\mathbf{\Gamma}K_y^1\\
&\quad + \mathcal{F}_z\left(Y^1, Z^1\right)\mathbf{\Gamma}K_z^1 + \mathcal{F}_y\left(Y^1, Z^1\right)\mathbf{\Psi}\mathbb{y}^{1,[n-1]} + \mathcal{F}_z\left(Y^1, Z^1\right)\mathbf{\Psi}\mathbb{z}^{1,[n-1]}\\
&= hF_y\left(Y^0, Z^0\right)\left(\widehat{\mathbf{A}}K_y^1 + \widehat{\mathbf{U}}\mathbb{y}^{1,[n-1]}\right) + hF_z\left(Y^0, Z^0\right)\left(\widehat{\mathbf{A}}K_z^1 + \widehat{\mathbf{U}}\mathbb{z}^{1,[n-1]}\right),\\
K_z^0 &= hG_y\left(Y^0, Z^0\right)\left(\widehat{\mathbf{A}}K_y^1 + \widehat{\mathbf{U}}\mathbb{y}^{1,[n-1]}\right) + hG_z\left(Y^0, Z^0\right)\left(\widehat{\mathbf{A}}K_z^1 + \widehat{\mathbf{U}}\mathbb{z}^{1,[n-1]}\right),
\end{aligned}$$

Convergence of index-1 problems for linearly implicit GLMs is discussed in section 6.2.1. ∎ *Since we arrive at the same rates as the underlying implicit method [28, Section 4], we can change the argument of the sub-Jacobians such that*

$$Y^0 = \widehat{Y}^0 + \mathcal{O}\big(h^{q+1}\big) \quad \text{and} \quad Z^0 = \widehat{Z}^0 + \mathcal{O}\big(h^{q+1}\big).$$

Defining $\delta K_y^1 := K_y^1 - \widehat{K}_y^1$ and similar notation for the error between stages of the underlying implicit GLM and the linearly implicit GLM we have

$$\delta K_y^1 = hF_y\left(\widehat{Y}^0, \widehat{Z}^0\right)\left(\widehat{\mathbf{A}}\delta K_y^1 + \widehat{\mathbf{U}}\delta\mathbb{y}^{1,[n-1]}\right) + hF_z\left(\widehat{Y}^0, \widehat{Z}^0\right)\left(\widehat{\mathbf{A}}\widehat{K}_z^1 + \widehat{\mathbf{U}}\delta\mathbb{y}^{1,[n-1]}\right) + \mathcal{O}\big(h^{q+2}\big), \tag{6.30a}$$

$$\delta K_z^0 = hG_y\left(\widehat{Y}^0, \widehat{Z}^0\right)\left(\widehat{\mathbf{A}}\delta K_y^1 + \widehat{\mathbf{U}}\delta\mathbb{y}^{1,[n-1]}\right) + hG_z\left(\widehat{Y}^0, \widehat{Z}^0\right)\left(\widehat{\mathbf{A}}\delta K_z^1 + \widehat{\mathbf{U}}\delta\mathbb{z}^{1,[n-1]}\right) + \mathcal{O}\big(h^{q+2}\big), \tag{6.30b}$$

$$\delta\mathbb{y}^1 = h\widehat{\mathbf{A}}\delta K_y^1 + \widehat{\mathbf{U}}\delta\mathbb{y}^{1,[n-1]}, \tag{6.30c}$$

$$\delta\mathbb{z}^1 = h\widehat{\mathbf{A}}\delta K_z^1 + \widehat{\mathbf{U}}\delta\mathbb{z}^{1,[n-1]}. \tag{6.30d}$$

Note that in Eq. (6.30) the error between the two methods has the same structure as the local truncation error of the implicit GLM [28, Therorem 2.1] and [10, Theorem 3.4]. Therefore under the same assumptions, we have the following result:

$$\Delta K_y^1 = \delta K_y^1 + \Delta\widehat{K}_y^1 = \mathcal{O}\big(h^{q+2}\big), \quad \Delta K_z^1 = \delta K_z^1 + \Delta\widehat{K}_z^1 = \mathcal{O}\big(h^{q+1}\big), \tag{6.31a}$$

$$\Delta\mathbb{y}^1 = \delta\mathbb{y}^1 + \Delta\widehat{\mathbb{y}}^1 = \mathcal{O}\big(h^{q+1}\big), \quad \Delta\mathbb{z}^1 = \delta\mathbb{z}^1 + \Delta\widehat{\mathbb{z}}^1 = \mathcal{O}\big(h^{q}\big). \tag{6.31b}$$

Corollary 6.4 (Estimation of the remainder in the numerical solution). *Under the same assumptions as in Theorem 6.3, and assuming eigenvalues of $\widehat{A}$ have positive real parts, the global error of linearly implicit GLM applied to the SPP problem (6.7) satisfies:*

$$\begin{aligned}
y_n - y(t_n) &= \mathcal{O}\big(h^p + \varepsilon^2 h^q\big),\\
z_n - z(t_n) &= \mathcal{O}\big(h^p + \varepsilon h^q\big),
\end{aligned} \tag{6.32}$$

for some $h/\varepsilon \geq D, h \leq h_0$ with h_0 independent of ε.

Proof. *This result directly follows from [28, Theorem 2.4], having shown the $\varepsilon-$expansion terms in the stages of linearly implicit GLM have the same error convergence rate as the underlying implicit method.*

7. Special families of linearly implicit GLMs.

7.1. Two-step Runge Kutta and Peer methods. A general formulation of Two-step Runge–Kutta methods and Peer methods [15, 4] may treat both families as special cases of GLMs. Let us consider a method that advance the solution y_{n+1} as:

$$
\begin{aligned}
Y^{[n]} &= \mathbf{u}^{[n-2]} \circledast y_{n-2} + \mathbf{u}^{[n-1]} \circledast y_{n-1} + \mathbf{D}^{[n-1]} \circledast Y^{[n-1]} \\
&\quad + h\, \mathbf{A}^{[n]} \circledast F\left(Y^{[n]}\right) + h\, \mathbf{A}^{[n-1]} \circledast F\left(Y^{[n-1]}\right), \qquad (7.1a)\\
y_n &= \vartheta^{[n-2]}\, y_{n-2} + \vartheta^{[n-1]}\, y_{n-1} + \boldsymbol{\theta}^{[n-1]\,T} \circledast Y^{[n-1]} \\
&\quad + h\, \mathbf{v}^{[n]\,T} \circledast F\left(Y^{[n]}\right) + h\, \mathbf{v}^{[n-1]\,T} \circledast F\left(Y^{[n-1]}\right). \qquad (7.1b)
\end{aligned}
$$

The method (7.1) is a GLM with the tableau of coefficients:

$$
\begin{array}{c|c} \mathbf{A} & \mathbf{U} \\ \hline \mathbf{B} & \mathbf{V} \end{array}
=
\begin{array}{c|cccc}
\mathbf{A}^{[n]} & \mathbf{u}^{[n-2]} & \mathbf{u}^{[n-1]} & \mathbf{D}^{[n-1]} & \mathbf{A}^{[n-1]} \\ \hline
\mathbf{0}_{s\times s} & \mathbf{0}_{s\times 1} & \mathbf{1}_s & \mathbf{0}_{s\times s} & \mathbf{0}_{s\times s} \\
\mathbf{v}^{[n]\,T} & \vartheta^{[n-2]} & \vartheta^{[n-1]} & \boldsymbol{\theta}^{[n-1]\,T} & \mathbf{v}^{[n-1]\,T} \\
\mathbf{A}^{[n]} & \mathbf{u}^{[n-2]} & \mathbf{u}^{[n-1]} & \mathbf{D}^{[n-1]} & \mathbf{A}^{[n-1]} \\
\mathbf{I}_{d\times d} & \mathbf{0}_{s\times 1} & \mathbf{0}_{s\times 1} & \mathbf{0}_{s\times s} & \mathbf{0}_{s\times s}
\end{array}, \qquad (7.2)
$$

The external stages contain solution and function values:

$$
\mathbb{y}^{[n-1]} = \begin{bmatrix} y_{n-2} \\ y_{n-1} \\ Y^{[n-1]} \\ hF\left(Y^{[n-1]}\right) \end{bmatrix} \in \mathbb{R}^{(2s+2)d\times 1},
$$

and using Taylor series expansion of the external stages we have the following for the columns of the $\mathbf{W}$ matrix:

$$
\mathbf{w}_0 = \begin{bmatrix} 1 & 1 & \mathbf{1}_{s\times 1} & \mathbf{0}_{s\times 1} \end{bmatrix}^T, \qquad (7.3a)
$$

$$
\mathbf{w}_k = \begin{bmatrix} \frac{(-1)^k}{k!} & 0 & \frac{(\mathbf{c}-\mathbf{1}_s)^k}{k!} & \frac{(\mathbf{c}-\mathbf{1}_s)^{k-1}}{(k-1)!} \end{bmatrix}^T, \qquad k = 1, \dots, p. \qquad (7.3b)
$$

The pre-consistency conditions for the method in (7.2) is

$$
\left(\mathbf{u}^{[n-2]} + \mathbf{u}^{[n-1]} + \mathbf{D}^{[n-1]}\right) \mathbf{1}_s = \mathbf{1}_s, \qquad (7.4a)
$$

$$
\vartheta^{[n-2]} + \vartheta^{[n-1]} + \boldsymbol{\theta}^{[n-1]\,T} \mathbf{1}_s = 1, \qquad (7.4b)
$$

Using (7.3) in the GLM order conditions we get the following internal stage order conditions:

$$
\begin{aligned}
&\frac{\mathbf{c}^{\times k}}{k!} - \frac{\mathbf{A}^{[n]} \mathbf{c}^{\times (k-1)}}{(k-1)!} - \frac{\mathbf{A}^{[n-1]} (\mathbf{c} - \mathbf{1}_s)^{\times (k-1)}}{(k-1)!} \\
&- \frac{\mathbf{D}^{[n-1]} (\mathbf{c} - \mathbf{1}_s)^{\times k}}{k!} - \mathbf{u}^{[n-2]} \frac{(-1)^k}{k!} = 0, \qquad k = 1, \dots, q, \qquad (7.5)
\end{aligned}
$$

Similarly, using the structure of the tableau (7.2) and Eq. (7.3) in the GLM external stage order conditions Theorem 2.2 we have the following identities:

$$\sum_{\ell=0}^{k} \frac{(-1)^{k-\ell}}{(k-\ell)!\,\ell!} = 0, \qquad k = 1,\dots,p, \tag{7.6a}$$

$$\begin{aligned} &\frac{1}{k!} - \frac{\mathbf{v}^{[n]\,T}\mathbf{c}^{\times(k-1)}}{(k-1)!} - \frac{\mathbf{v}^{[n-1]\,T}(\mathbf{c}-\mathbf{1}_s)^{\times(k-1)}}{(k-1)!} \\ &\quad - \vartheta^{[n-2]}\frac{(-1)^k}{k!} - \frac{\boldsymbol{\theta}^{[n-1]\,T}(\mathbf{c}-\mathbf{1}_s)^{\times k}}{k!} = 0, \qquad k = 1,\dots,p, \end{aligned} \tag{7.6b}$$

$$\begin{aligned} &\sum_{\ell=0}^{k} \frac{(\mathbf{c}-\mathbf{1}_s)^{\times(k-\ell)}}{(k-\ell)!\,\ell!} - \frac{\mathbf{A}^{[n]}\mathbf{c}^{\times(k-1)}}{(k-1)!} - \frac{\mathbf{A}^{[n-1]}(\mathbf{c}-\mathbf{1}_s)^{\times(k-1)}}{(k-1)!} \\ &\quad - \frac{\mathbf{D}^{[n-1]}\mathbf{c}^{\times k}}{k!} - \mathbf{u}^{[n-2]}\frac{(-1)^k}{k!} = 0, \qquad k = 1,\dots,q, \end{aligned} \tag{7.6c}$$

$$\sum_{\ell=0}^{k-1} \frac{(\mathbf{c}-\mathbf{1}_s)^{\times(k-\ell-1)}}{(k-\ell-1)!\,\ell!} - \frac{\mathbf{c}^{\times(k-1)}}{(k-1)!} = 0, \qquad k = 1,\dots,p. \tag{7.6d}$$

Equations (7.6a) and (7.6d) are trivially satisfied. Equation (7.6c) is equivalent to the stage order equation (7.5) since

$$\sum_{\ell=0}^{k} \frac{(\mathbf{c}-\mathbf{1}_s)^{\times(k-\ell)}}{(k-\ell)!\,\ell!} = \frac{\mathbf{c}^{\times k}}{k!}, \quad k = 1,\dots,q.$$

We have the following result:

Remark 6 (TSRK/Peer order conditions). *A two-step Runge-Kutta method or a GLM peer method represented by formulation Eq. (7.1) is order (q,p) if the coefficients satisfy the preconsistency conditions Eq. (7.4a) and*

$$\begin{aligned} &\frac{\mathbf{c}^{\times k}}{k!} - \frac{\mathbf{A}^{[n]}\mathbf{c}^{\times(k-1)}}{(k-1)!} - \frac{\mathbf{A}^{[n-1]}(\mathbf{c}-\mathbf{1}_s)^{\times(k-1)}}{(k-1)!} \\ &\quad - \frac{\mathbf{D}^{[n-1]}(\mathbf{c}-\mathbf{1}_s)^{\times k}}{k!} - \mathbf{u}^{[n-2]}\frac{(-1)^k}{k!} = 0, \qquad k = 1,\dots,q, \end{aligned} \tag{7.7a}$$

$$\begin{aligned} &\frac{1}{k!} - \frac{\mathbf{v}^{[n]\,T}\mathbf{c}^{\times(k-1)}}{(k-1)!} - \frac{\mathbf{v}^{[n-1]\,T}(\mathbf{c}-\mathbf{1}_s)^{\times(k-1)}}{(k-1)!} \\ &\quad - \vartheta^{[n-2]}\frac{(-1)^k}{k!} - \frac{\boldsymbol{\theta}^{[n-1]\,T}(\mathbf{c}-\mathbf{1}_s)^{\times k}}{k!} = 0, \qquad k = 1,\dots,p. \end{aligned} \tag{7.7b}$$

7.1.1. Linearly implicit TSRK/Peer methods. Once the TSRK/Peer methods are considered as GLMs, extension to linearly-implicit version is a straight-forward process. We consider the linearly-implicit method

$$
\begin{aligned}
Y^{[n]} &= \mathbf{u}^{[n-2]} \circledast y_{n-2} + \mathbf{u}^{[n-1]} \circledast y_{n-1} + \mathbf{A}^{[n]} \circledast K^{[n]} + \mathbf{A}^{[n-1]} \circledast K^{[n-1]}, \\
K^{[n]} &= hF\left(Y^{[n]}\right) + h\mathbf{L}\left(\mathbf{\Gamma}^{[n]} \circledast K^{[n]} + \mathbf{\Gamma}^{[n-1]} \circledast K^{[n-1]}\right) \\
&\quad + h\mathbf{L}\left(\widetilde{\mathbf{u}}^{[n-2]} \circledast y_{n-2} + \widetilde{\mathbf{u}}^{[n-1]} \circledast y_{n-1}\right), \\
y_n &= \vartheta^{[n-2]}\, y_{n-2} + \vartheta^{[n-1]}\, y_{n-1} \\
&\quad + \mathbf{v}^{[n-1]\,T} \circledast K^{[n-1]} + \mathbf{v}^{[n]\,T} \circledast K^{[n]}.
\end{aligned}
\tag{7.8}
$$

The order conditions are:

$$
\frac{\mathbf{c}^{\times k}}{k!} - \frac{\mathbf{A}^{[n]}\mathbf{c}^{\times(k-1)}}{(k-1)!} - \frac{\mathbf{A}^{[n-1]}(\mathbf{c}-\mathbf{1}_s)^{\times(k-1)}}{(k-1)!} - \mathbf{u}^{[n-2]}\frac{(-1)^k}{k!} = 0, \quad k = 1, \dots, q, \tag{7.9a}
$$

$$
\frac{1}{k!} - \frac{\mathbf{v}^{[n]\,T}\mathbf{c}^{\times(k-1)}}{(k-1)!} - \frac{\mathbf{v}^{[n-1]\,T}(\mathbf{c}-\mathbf{1}_s)^{\times(k-1)}}{(k-1)!} - \vartheta^{[n-2]}\frac{(-1)^k}{k!} = 0, \quad k = 1, \dots, p, \tag{7.9b}
$$

$$
\frac{\mathbf{\Gamma}^{[n]}\mathbf{c}^{\times(k-1)}}{(k-1)!} + \frac{\mathbf{\Gamma}^{[n-1]}(\mathbf{c}-\mathbf{1}_s)^{\times(k-1)}}{(k-1)!} + \widetilde{\mathbf{u}}^{[n-2]}\frac{(-1)^k}{k!} = 0, \quad k = 1, \dots, q. \tag{7.9c}
$$

7.1.2. Example. The sequential peer two-step method `PEER4A` described in [19] is a special case of (7.8) with order (2,3) conditions. Indeed in [19, Example 4], the "Nordsieck form" of the peer method is the underlying implicit and the "predictor" coefficients give the underlying explicit method.

7.2. Parallel Methods. The family of parallel linearly implicit GLMs allows independent and parallel computation of internal stages. In the case of linearly implicit GLMs choosing $\mathbf{A} = \mathbf{0}$ and $\mathbf{\Gamma} = \lambda\,\mathbf{I}_{s\times s}$ leads to a family of parallel methods . Similarly, the two-step linearly implicit Runge–Kutta methods become parallel when $\mathbf{A}^{[n]} = 0$ and $\mathbf{\Gamma}^{[n]} = \lambda\,\mathbf{I}_{s\times s}$.

7.3. Parallel Ensemble Methods.

$$
\mathbf{B} = \mathbf{C}_s\,\mathbf{F}_s\,(\mathbf{I}_{s\times s} - \lambda\,\mathbf{K}_s)\,\mathbf{C}_s^{-1}, \quad \mathbf{V} = \mathbf{I}_{s\times s}, \quad \mathbf{W} = \mathbf{C}_{s+1} - \lambda\,\mathbf{C}_{s+1}\,K_{s+1},
$$

where the Toeplitz matrices

$$
\mathbf{K}_n = \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ & & & & 0 \end{bmatrix} \in \mathbb{R}^{n\times n}, \qquad
\mathbf{F}_n = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{6} & \cdots & \frac{1}{n!} \\ & 1 & \frac{1}{2} & \cdots & \frac{1}{(n-1)!} \\ & & \ddots & \ddots & \vdots \\ & & & 1 & \frac{1}{2} \\ & & & & 1 \end{bmatrix} \in \mathbb{R}^{n\times n}
$$

and scaled Vandermone matrix

$$
\mathbf{C}_n = \begin{bmatrix} \mathbf{1}_s & \mathbf{c} & \frac{\mathbf{c}^2}{2} & \cdots & \frac{\mathbf{c}^{n-1}}{(n-1)!} \end{bmatrix} \in \mathbb{R}^{s\times n}
$$

are used.

$$\boldsymbol{\Psi} = -\lambda\, \mathbf{C}_s\, \mathbf{K}_s\, (\mathbf{I}_{s\times s} - \lambda\, \mathbf{K}_s)^{-1}\, \mathbf{C}_s^{-1}$$

The stability essentially matches that of the one stage Rosenbrock W-method:

$$y_{n+1} = y_n + h\, (\mathbf{I}_{d\times d} - h\, \lambda\, \mathbf{L}_1)\, f(y_n).$$

7.4. Backward Differentiation Formulae. Backward differentiation formula (BDF) methods are a popular family of implicit linear multistep methods. The general form for a k-step BDF method is

$$y_n = \beta_0 h f(y_n) + \sum_{i=1}^{k} \alpha_i y_{n-i}. \tag{7.10}$$

These can be cast into the framework of GLMs and have a tableau of the form

$$\begin{array}{c|c|ccccc} 1 & \beta_0 & \alpha_1 & \dots & \alpha_{k-1} & \alpha_k \\ \hline & \beta_0 & \alpha_1 & \dots & \alpha_{k-1} & \alpha_k \\ & 0 & 1 & \dots & 0 & 0 \\ & \vdots & \vdots & \ddots & \vdots & \vdots \\ & 0 & 0 & \dots & 1 & 0 \end{array}.$$

Now we use this GLM to construct the linearly implicit GLM

$$K_1 = h\, f\left(\sum_{i=1}^{k} \widehat{\alpha}_i\, y_{n-i}\right) + \beta_0\, h\, \mathbf{L}\, K_1 + h\, \mathbf{L}\, \sum_{i=1}^{k} \psi_i\, y_{n-i}, \tag{7.11a}$$

$$y_n = \beta_0\, K_1 + \sum_{i=1}^{k} \alpha_i\, y_{n-i}, \tag{7.11b}$$

where $\alpha_i = \widehat{\alpha}_i + \psi_i$ to preserve (7.10) as the underlying implicit method. The tableau for Eq. (7.11a) is

$$\begin{array}{c|c:c|ccccc:ccc} 1 & 0 & \beta_0 & \widehat{\alpha}_1 & \dots & \widehat{\alpha}_{k-1} & \widehat{\alpha}_k & \psi_1 & \dots & \psi_k \\ \hline & \beta_0 & & \alpha_1 & \dots & \alpha_{k-1} & \alpha_k & & & \\ & 0 & & 1 & \dots & 0 & 0 & & & \\ & \vdots & & \vdots & \ddots & \vdots & \vdots & & & \\ & 0 & & 0 & \dots & 1 & 0 & & & \end{array} \tag{7.12}$$

We use this structure to derive order conditions for linearly implicit BDF methods with $p = q + 1 = k$. Solving Eq. (4.5) leads to the method coefficients

$$\beta_0 = \left(\sum_{j=0}^{k} \frac{1}{j}\right)^{-1}, \quad \widehat{\alpha}_i = (-1)^{i+1} \binom{k}{i}, \quad \psi_i = \widehat{\alpha}_i \left(\frac{\beta_0}{i} - 1\right), \quad i = 1, \dots, k. \tag{7.13}$$

By applying the transformation (3.6) and some algebraic simplifications, we arrive at the following compact form of linearly-implicit BDF methods:

$$y_n = (\mathbf{I}_{s\times s} - \beta_0\, h\, \mathbf{L})^{-1} \left(\beta_0\, h\, f\Big(\sum_{i=1}^{k} \widehat{\alpha}_i\, y_{n-i}\Big) + \sum_{i=1}^{k} \psi_i\, y_{n-i} \right) + \sum_{i=1}^{k} \widehat{\alpha}_i\, y_{n-i}. \tag{7.14}$$

8. Numerical experiments. In this section we provide numerical experiments using various linearly implicit GLMs derived from the families introduced in section 7 applied to a number of test problems. Test problems are chosen to highlight features of this class of linearly implicit methods. We present convergence results on a DAE problem, a PDE with time-dependent boundary conditions, Euler equations of gas dynamics with approximate Jacobian and the Van der Pol oscillator for a stiff system.

8.1. The transistor-amplifier test problem. The transistor-amplifier test problem is an index-1 DAE describing various voltages across a two-transistor amplifier circuit. The derivation of the equations is documented in [30] and the Matlab implementation of this problem can be found in `ODE Test Problems` suit. [22]. We perform the experiments for a range of time-steps using the exact Jacbobian over a timespan $t = [0, 0.2]$ and report the l_2 error between the final solution and a reference solution computed with tight tolerances. We observe that our methods behave at their nominal order of convergence as shown in Figure 8.1.

8.2. Convergence study with the Hundsdorfer problem problem. In the next set of experiments we use the Hundsdorfer PDE with time-dependent boundary conditions [12, Sec. 6.3]. The PDE is described as

$$\begin{aligned} u_t + u_x &= -k_1 u + k_2 v, \\ v_t &= k_1 u - k_2 v + 1, \end{aligned} \tag{8.1}$$

where $x, t \in [0, 1]$ and the parameters are chosen as $k_1 = 10^6$ and $k_2 = 2k_1$. Fourth-order central finite differences are used for spatial discretization and the time-dependent Dirichlet boundary condition is $u(0,t) = 1-\sin{(12t)}^4$. This linear advection-reaction problem causes order reduction in Runge-Kutta methods due to the influences of the time-dependent boundary conditions [13, Chapter 2]. In our experiments we compare the convergence rates of linearly implicit GLMs with a number of Rosenbrock and Rosenbrock-W methods from the literature: RODAS3 and RODAS4 are Rosenbrock methods of orders 3 and 4 [25]. ROS34PW1b, and ROS34PW2, are ROW methods of order 3 and ROS34PW3 is an order 4 method.[21]. 6S4O(C)-W is a Rosenbrock-W method of order 4 from [**?**]. We note that among Rosenbrock and ROW methods tested on this problem only ROS34PW1b and ROS34PW3 retain their nominal convergence rates as shown in Figure 8.2a. On the other hand as Figure 8.2b indicates, the linearly implicit GLMs perform close their theoretical order of convergence. We note the slight fluctuation in the convergence plot of LIMSIM5 method is due to numerical errors in the starting procedure. The error is also exacerbated by accumulation of round-off errors due to large coefficients of this method.

8.3. Euler equations with approximate Jacobian . In this set of experiments we use the 2D Euler equations for compressible gas dynamics summarized in (8.2) and discretized using Discrete Galerkin (DG) finite element method with degree 5 nodal polynomial basis. We have used the `Matlab` implementation of the Isentropic Vortex test provided in [11, Section 6.6]. The system is integrated over timespan

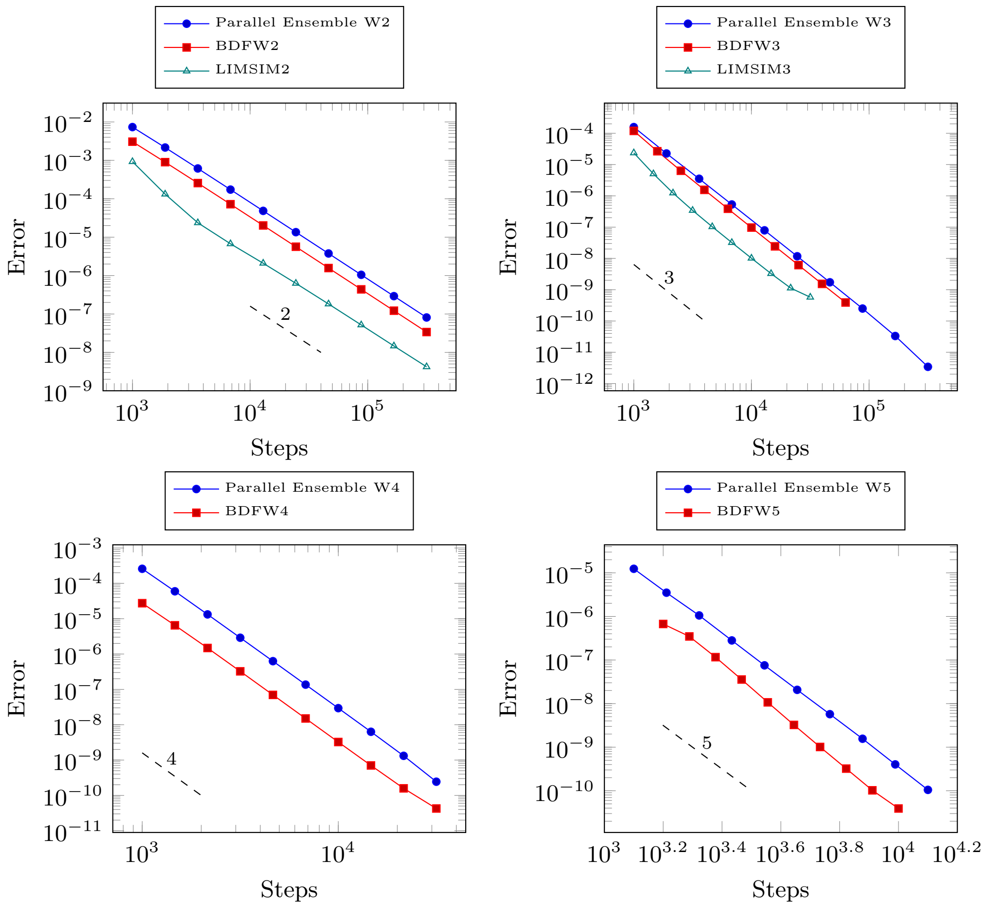
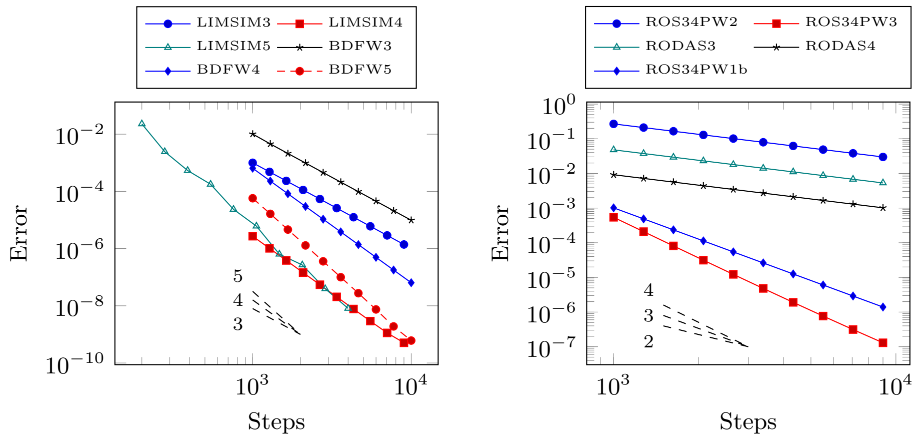


Fig. 8.1: Convergence study with the Transistor-Amplifier test problem

$t = [0, 10]$ with parameter $\gamma = 1.4$ and $E = \frac{p}{\gamma-1} + \frac{\rho}{2}\left(u^2 + v^2\right)$.

$$
\begin{aligned}
&\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} = 0,\\
&\frac{\partial \rho u}{\partial t} + \frac{\partial \rho u^2 + p}{\partial x} + \frac{\partial \rho u v}{\partial y} = 0,\\
&\frac{\partial \rho v}{\partial t} + \frac{\partial \rho u v}{\partial x} + \frac{\partial \rho v^2 + p}{\partial y} = 0,\\
&\frac{\partial E}{\partial t} + \frac{\partial u(E+p)}{\partial x} + \frac{\partial v(E+p)}{\partial y} = 0.
\end{aligned}
\tag{8.2}
$$

In many practical applications the exact Jacobian of a complex discretization is not available. Here, we use the `numjac` function to create an approximate sparse Jacobian at the beginning of the integration and employ it in our computations. Furthermore, the stage equations are solved with `GMRES`. As a result , the final error is affected both by the local truncation error of the method as well as the convergence of the iterative solver (i.e. the iterative solver may not converge within specified tolerance



(a) linearly implicit GLMs (b) Rosernbrock and ROW methods

Fig. 8.2: Convergence plots for the Hundsdorfer problem

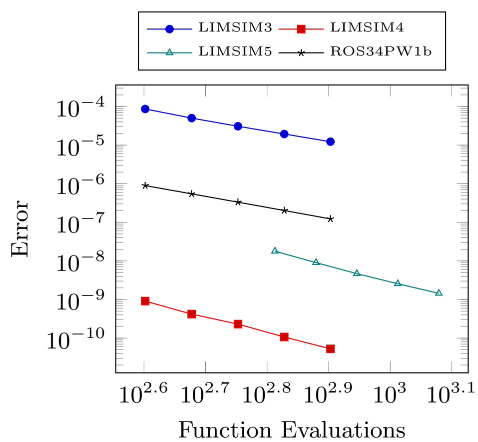
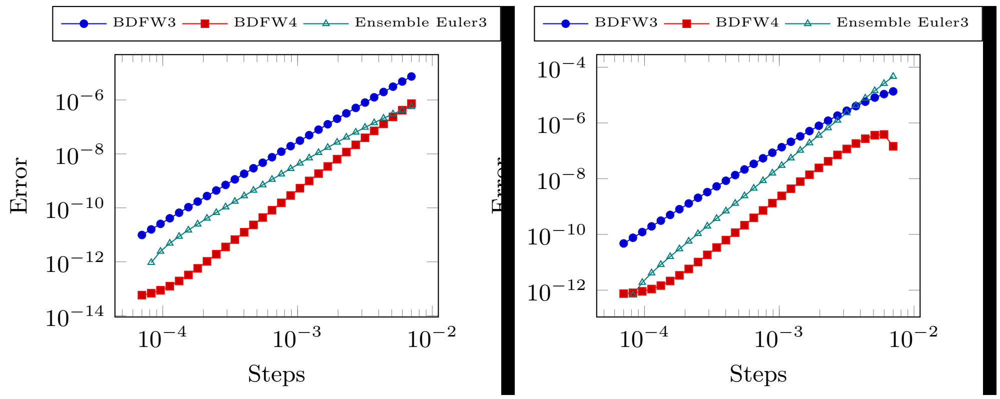


Fig. 8.3: Error versus number of function evaluations for the Isentropic vortex test problem

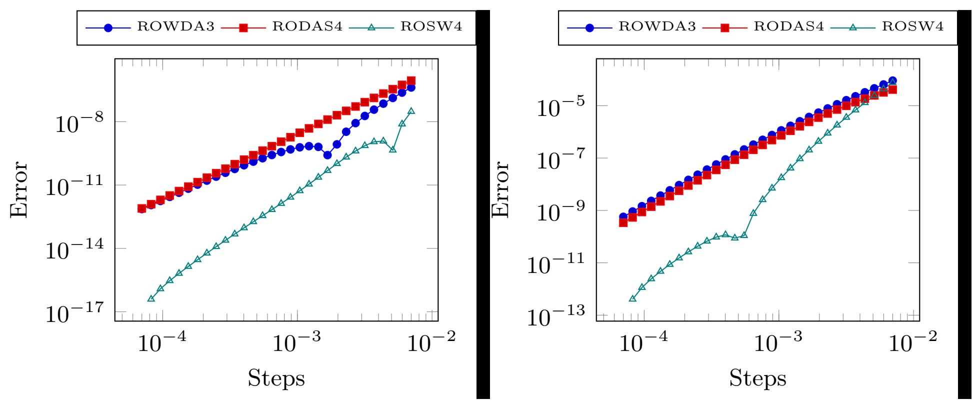
for some methods). Figure 8.3 shows the final error for a number of methods plotted against the number of function evaluations for a small subset of linearly implicit GLMs and a ROSW method. This plot signifies the many different factors in the overall performance of linearly-implicit methods. We note that the order four method outperforms others in this case. This is due to optimized coefficients and striking a balance between number of stages and the overall order of the method, as well having a simpler starting procedure compared to the order 5 method which is prone to approximation errors.

8.4. Van der Pol oscillator. The Van der Pol problem is commonly used in the literature for studying the stiff behavior of time-stepping methods. Equation (8.3) describes a two variable system where z is singularly perturbed and therefore constitutes



(a) $y-$component (b) $z-$component

Fig. 8.4: Convergence plots for linearly implicit GLMs applied to the Van der Pol oscillator



(a) $y-$component (b) $z-$component

Fig. 8.5: Convergence plots for linearly implicit GLMs applied to the Van der Pol oscillator

the stiff component of the system for small positive values of ε.

$$\begin{aligned} y' &= z, \\ \varepsilon z' &= (1 - y^2)z - y. \end{aligned} \tag{8.3}$$

For the complete problem definition and a highly accurate initial condition please refer to [9, Section 5]. Here, we provide convergence rates of the $y-$ and $z-$ components of the system as we integrate (8.3) with $\varepsilon = 10^{-3}$ over timespan $t = [0, 0.4]$.

9. Conclusion. We presented the class of linearly implicit GLMs for linearly implicit integration of systems with high external and internal stage orders. The order conditions show a close relation between coefficients of linearly implicit GLMs and IMEX-GLM pairs that can be leveraged to design methods with preferable stability and error properties. We provide convergence results for Prothero-Robinson problem, index-1 DAEs and singularly perturbed problems to investigate the stability of these methods. Adaptivity strategies are briefly discussed based on re-scaling of the Nordsieck forms of the solution. We provide a wide range of methods that can be chosen based on the specific application and implementation factors such as available parallel processors, storage, and stability requirements. Our numerical experiments highlight a number of advantages for linearly implicit GLMs when competing methods encounter order reduction due to approximate Jacobians, stiffness, or larger residuals.

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Appendix A. linearly implicit GLM methods.

LIMSIM 3.

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \frac{78251}{2544264} & 0 & 0 & 0 \\ \frac{272815}{3146256} & -\frac{657}{2612} & 0 & 0 \\ -\frac{58763}{417408} & \frac{411789}{404672} & -\frac{162911}{350432} & 0 \end{pmatrix},$$

$$\mathbf{U} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & \frac{1617925}{2544264} & \frac{487141}{1272132} & \frac{519103}{2544264} \\ 1 & \frac{1023459409}{2054505168} & \frac{280506013}{1027252584} & \frac{230678263}{2054505168} \\ 1 & \frac{16997068281757}{28902654355584} & \frac{3391766958863}{14451327177792} & \frac{6373655464681}{28902654355584} \end{pmatrix},$$

$$\mathbf{B} = \begin{pmatrix} -\frac{39}{128} & \frac{63}{64} & -\frac{9}{32} & \frac{1}{4} \\ 0 & 0 & 0 & 1 \\ \frac{3}{4} & -\frac{9}{2} & \frac{9}{4} & 2 \\ \frac{1}{3} & -\frac{15}{2} & 6 & \frac{8}{3} \end{pmatrix},$$

$$\mathbf{V} = \begin{pmatrix} 1 & \frac{45}{128} & -\frac{1}{64} & -\frac{7}{128} \\ 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 \\ 0 & -\frac{3}{2} & 0 & 0 \end{pmatrix},$$

$$\mathbf{\Gamma} = \begin{pmatrix} \frac{1}{4} & 0 & 0 & 0 \\ -\frac{744}{11779} & \frac{1}{4} & 0 & 0 \\ \frac{262}{7283} & -\frac{645}{5224} & \frac{1}{4} & 0 \\ -\frac{1069}{6522} & -\frac{210}{6323} & \frac{2011}{10951} & \frac{1}{4} \end{pmatrix},$$

$$\mathbf{\Psi} = \begin{pmatrix} 0 & -\frac{1}{4} & -\frac{1}{2} & -\frac{3}{4} \\ 0 & -\frac{8803}{47116} & -\frac{7315}{35337} & -\frac{5083}{35337} \\ 0 & -\frac{6182751}{38046392} & -\frac{1055648}{14267397} & -\frac{1519825}{57069588} \\ 0 & -\frac{213624339371}{903207948612} & -\frac{113049029563}{451603974306} & -\frac{248570917961}{903207948612} \end{pmatrix},$$

$$\mathbf{c} = \begin{pmatrix} 1 & \frac{2}{3} & \frac{1}{3} & 1 \end{pmatrix}^T.$$

LIMSIM 4.

$$
\mathbf{A} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
\frac{3367}{2911232} & 0 & 0 & 0 & 0 \\
\frac{1533}{20320} & -\frac{19423}{70722} & 0 & 0 & 0 \\
\frac{1234803}{7851008} & -\frac{1849}{3200} & -\frac{607}{4996} & 0 & 0 \\
-\frac{85487}{260640} & \frac{339968}{183699} & -\frac{10059}{5756} & \frac{12238}{30819} & 0
\end{pmatrix},
$$

$$
\mathbf{U} = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 \\
1 & \frac{2180057}{2911232} & \frac{815417}{1455616} & \frac{1218075}{2911232} & \frac{56729}{181952} \\
1 & \frac{502397027}{718535520} & \frac{61203929}{119755920} & \frac{7227751}{19959320} & \frac{17898017}{79837280} \\
1 & \frac{194164388861}{245147724800} & \frac{90233588327}{122573862400} & \frac{299064458359}{490295449600} & \frac{201312980409}{490295449600} \\
1 & \frac{65099936835498259}{78643608352094880} & \frac{16868249117333519}{39321804176047440} & \frac{2547146550312569}{26214536117364960} & \frac{745125993133019}{19660902088023720}
\end{pmatrix},
$$

$$
\mathbf{B} = \begin{pmatrix}
-\frac{29}{96} & \frac{7}{9} & -\frac{1}{4} & \frac{1}{3} & \frac{1}{4} \\
0 & 0 & 0 & 0 & 1 \\
\frac{13}{6} & -8 & 6 & -\frac{8}{3} & 2 \\
\frac{43}{9} & -\frac{64}{3} & \frac{64}{3} & -\frac{64}{9} & \frac{8}{3} \\
\frac{21}{4} & -\frac{800}{27} & 40 & -\frac{32}{3} & \frac{8}{3}
\end{pmatrix},
$$

$$
\mathbf{V} = \begin{pmatrix}
1 & \frac{55}{288} & \frac{1}{48} & -\frac{1}{32} & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{2} & 0 & 0 & 0 \\
0 & -\frac{1}{3} & -\frac{2}{3} & 0 & 0 \\
0 & -\frac{823}{108} & -\frac{109}{18} & -\frac{7}{4} & 0
\end{pmatrix},
$$

$$
\mathbf{\Gamma} = \begin{pmatrix}
\frac{1}{4} & 0 & 0 & 0 & 0 \\
-\frac{313}{11372} & \frac{1}{4} & 0 & 0 & 0 \\
-\frac{449}{10160} & \frac{1285}{7858} & \frac{1}{4} & 0 & 0 \\
-\frac{240}{7667} & \frac{1547}{9600} & \frac{4961}{9992} & \frac{1}{4} & 0 \\
\frac{211}{8145} & -\frac{21899}{20411} & \frac{2155}{1439} & -\frac{655}{10273} & \frac{1}{4}
\end{pmatrix},
$$

$$
\mathbf{\Psi} = \begin{pmatrix}
0 & -\frac{1}{4} & -\frac{1}{2} & -\frac{3}{4} & -1 \\
0 & -\frac{1265}{5686} & -\frac{7277}{22744} & -\frac{61737}{181952} & -\frac{56729}{181952} \\
0 & -\frac{14743339}{39918640} & -\frac{8121559}{19959320} & -\frac{26416089}{79837280} & -\frac{17898017}{79837280} \\
0 & -\frac{80562288001}{91930396800} & -\frac{49066928401}{61286931200} & -\frac{292840004409}{490295449600} & -\frac{201312980409}{490295449600} \\
0 & -\frac{6260149023115573}{9830451044011860} & -\frac{2006130607958233}{4915225522005930} & -\frac{105198462624382}{819204253667655} & -\frac{745125993133019}{19660902088023720}
\end{pmatrix},
$$

$$
\mathbf{c} = \begin{pmatrix} 1 & \frac{3}{4} & \frac{1}{2} & \frac{1}{4} & 1 \end{pmatrix}^T. \qquad \blacksquare
$$

LIMSIM 5.

$$
\mathbf{A} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{580528}{7115625} & 0 & 0 & 0 & 0 & 0 \\
-\frac{7453409}{27040625} & \frac{131337}{229024} & 0 & 0 & 0 & 0 \\
-\frac{8759}{21875} & \frac{982283}{915824} & -\frac{141571}{589095} & 0 & 0 & 0 \\
-\frac{8507513057749212485}{11600146740199555072} & \frac{186780365}{60984144} & -\frac{2434553}{476775} & \frac{213337}{126738} & 0 & 0 \\
\frac{7351771}{7102404} & -\frac{86029409}{19994688} & \frac{85782343}{11488878} & -\frac{1015549}{353484} & -\frac{179407}{198324} & 0
\end{pmatrix},
$$

$$
\mathbf{U} = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
1 & \frac{6273028}{7115625} & \frac{5715056}{7115625} & \frac{1794928}{2371875} & \frac{5236672}{7115625} & \frac{5234288}{7115625} \\
1 & \frac{2974515605821612037214 7}{984373297760588141690 88} & -\frac{285393113}{45536412500} & -\frac{1323706191}{22768206250} & \frac{38636113}{669653125} & \frac{3204547204}{11384103125} \\
1 & -\frac{11442903308130768694988473}{35942678286169463901388800 0} & -\frac{38815470851}{83134143750} & -\frac{16010970402221724302637301}{29952231905141219917824000} & -\frac{23217453584028088827078569}{64183354082445471252480000} & -\frac{32116971744795597901645 99}{11232086964427957469184000 0} \\
1 & \frac{187018763188389454083940198896 23}{144569029262773130719798493184 00} & \frac{200557999599611479634752826034551}{144569029262773130719798493184000} & \frac{623147234955823923510736559239 91}{602370955261554711332493721600 00} & \frac{4651879541232367359765539590629311}{7228451463138656535989924659200000} & \frac{176435767656468955572026148382875 1}{361422573156932826799496232960000 0} \\
1 & \frac{154406525226491360708 47}{266892936412199106119 04} & -\frac{8101588847938942221961}{16680808525762444132440} & -\frac{5846810144763092165951}{13900673771468703443700} & -\frac{800925190567442308351}{521275266430076379138 75} & \frac{180204893529420689141 27}{104255053286015275827750}
\end{pmatrix},
$$

$$
\mathbf{B} = \begin{pmatrix}
\frac{689}{972} & -\frac{22225}{5184} & \frac{1175}{162} & -\frac{1025}{324} & -\frac{25}{36} & \frac{1}{3} \\
0 & 0 & 0 & 0 & 0 & 1 \\
\frac{101}{24} & -\frac{25}{2} & \frac{25}{2} & -\frac{25}{3} & \frac{25}{8} & \frac{3}{2} \\
\frac{487}{36} & -\frac{4775}{96} & \frac{3475}{54} & -\frac{2825}{72} & \frac{25}{4} & \frac{3}{2} \\
\frac{589}{32} & -\frac{19575}{256} & \frac{16175}{144} & -\frac{2225}{32} & \frac{225}{32} & \frac{9}{8} \\
15 & -\frac{36095}{512} & \frac{355}{3} & -\frac{1215}{16} & \frac{45}{8} & \frac{27}{40}
\end{pmatrix},
$$

$$
\mathbf{V} = \begin{pmatrix}
1 & \frac{13219}{15552} & -\frac{233}{1944} & -\frac{41}{324} & \frac{43}{243} & \frac{68}{243} \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{1}{2} & 0 & 0 & 0 & 0 \\
0 & \frac{2891}{864} & \frac{43}{36} & 0 & 0 & 0 \\
0 & \frac{16375}{2304} & \frac{125}{96} & -\frac{9}{16} & 0 & 0 \\
0 & \frac{52241}{7680} & -\frac{657}{320} & -\frac{591}{160} & -\frac{13}{10} & 0
\end{pmatrix},
$$

$$
\mathbf{\Gamma} = \begin{pmatrix}
\frac{1}{3} & 0 & 0 & 0 & 0 & 0 \\
\frac{16}{3795} & \frac{1}{3} & 0 & 0 & 0 & 0 \\
\frac{2077}{25959} & -\frac{2986}{7157} & \frac{1}{3} & 0 & 0 & 0 \\
\frac{869}{7875} & -\frac{9193}{13209} & \frac{2339}{15105} & \frac{1}{3} & 0 & 0 \\
-\frac{3670}{10303} & \frac{7685}{10859} & \frac{259}{4075} & \frac{3399}{4694} & \frac{1}{3} & 0 \\
-\frac{2384}{7307} & \frac{178}{11571} & -\frac{15139}{70919} & -\frac{5707}{19638} & \frac{6947}{33054} & \frac{1}{3}
\end{pmatrix},
$$

$$
\mathbf{\Psi} = \begin{pmatrix}
0 & -\frac{1}{3} & -\frac{2}{3} & -1 & -\frac{4}{3} & -\frac{5}{3} \\
0 & -\frac{427}{1265} & -\frac{2056}{3795} & -\frac{4128}{6325} & -\frac{66368}{94875} & -\frac{22256}{31625} \\
0 & \frac{42292}{10928739} & \frac{5875328}{54643695} & \frac{3661462}{18214565} & \frac{19801184}{80358375} & \frac{325676654}{1366092375} \\
0 & \frac{486000268}{4988048625} & \frac{2196534356}{4988048625} & \frac{1127245114}{1662682875} & \frac{2724956816}{3562891875} & \frac{18220866694}{24940243125} \\
0 & -\frac{9453732024817201}{6420154597409550} & -\frac{19401943990497478}{16050386493523875} & -\frac{19966598832966692}{26750644155873125} & -\frac{11055214656972550 4}{40125966233809687 5} & \frac{782896199776839 22}{401259662338096875} \\
0 & \frac{41926526885850186 22}{15445193079409670493} & \frac{28251363011285089441}{77225965397048352465} & \frac{75699523449532751519}{257419884656827841550} & \frac{371301397189028640638}{1930649134926208811625} & \frac{413102379314756288699}{3861298269852417623250}
\end{pmatrix},
$$

$$
\mathbf{c} = \begin{pmatrix} 1 & \frac{4}{5} & \frac{50665495807918080047}{84442493013196800000} & \frac{35128077093489869 28277}{87820192733724672000 00} & \frac{105384231280469641}{526921156402348032} & 1 \end{pmatrix}.
$$

∎