Global well-posedness and scattering results for nonlinear waves

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Nonlinear Waves: Introduction

We consider the Cauchy problem for the nonlinear wave equation

$$\begin{aligned} &\partial_t^2 u - \Delta u = \pm |u|^{p-1} u, \qquad (x,t) \in \mathbb{R}^d \times I, \\ &u(0) = u_0,\ \partial_t u(0) = u_1 \end{aligned} \tag{NLW}$$

$+$: focusing case
$-$: defocusing case

$$E(\vec{u}) = \int_{\mathbb{R}^d} \left(\frac{|u_t|^2}{2} + \frac{|\nabla u|^2}{2} \mp \frac{|u|^{p+1}}{p+1} \right) dx = constant.$$

Note: (NLW) is a toy model for particle physics and general relativity.

Scaling

- The Cauchy problem (NLW) is invariant under the scaling
$$u(t,x) \longmapsto \lambda^{2/(p-1)} u(\lambda t, \lambda x).$$
- The only homogeneous L_x^2 based Sobolev norm left invariant by the λ-scaling is given by
$$\|u(t,\cdot)\|_{\dot{H}_x^{s_c}}^2 = \|(-\Delta)^{s_c/2} u(t,\cdot)\|_{L_x^2}^2 = \int_{\mathbb{R}^d} |\xi|^{2s_c} |\hat{u}(t,\xi)|^2 \, d\xi, \qquad s_c = \frac{d}{2} - \frac{2}{p-1}$$
- $s_c = 1$: This is equivalent to $p - 1 = 4/(d-2)$. We get
$$(u_0, u_1) \in \dot{H}^1 \times L^2.$$
The Cauchy problem (NLW) is called **energy-critical**, and the conserved energy is finite.
- $s_c > 1$: **energy-supercritical regime**
$$(u_0, u_1) \in \dot{H}^{s_c} \times \dot{H}^{s_c - 1}, \qquad s_c = \frac{d}{2} - \frac{2}{p-1}$$
- $s_c < 1$: **energy-subcritical regime**

Energy-supercritical Regime

By means of standard techniques based on the Strichartz estimates, we may establish the local well-posedness and small data theory.

Theorem [C.-Kenig, 2023]

1. **Local Wellposedness** For all initial data $(u_0, u_1) \in \dot{H}^{s_p} \times \dot{H}^{s_p - 1}$, there is a unique solution, $\vec{u}(t)$, defined on a maximal interval of existence
$$I_{max}(\vec{u}) := (T_-(\vec{u}), T_+(\vec{u}))$$
with $\vec{u} \in C_t^0(I_{max}; \dot{H}^{s_p} \times \dot{H}^{s_p - 1})$.
2. **Small Data Theory** If $\|(u_0, u_1)\|_{\dot{H}^{s_p} \times \dot{H}^{s_p - 1}}$ is small, the corresponding solution $\vec{u}(t)$ is defined for all time, i.e.,
$$I_{max}(\vec{u}) = \mathbb{R}.$$
Moreover, $\vec{u}(t)$ exhibits linear behavior as $t \to \pm\infty$. In other words, there exists $(u_0^{\pm}, u_1^{\pm})$ so that
$$\|\vec{u}(t) - S(t)(u_0^{\pm}, u_1^{\pm})\|_{\dot{H}^{s_p} \times \dot{H}^{s_p - 1}} \to 0 \qquad t \to \pm\infty.$$

Blow-up Phenomena

Summary

If $\|(u_0, u_1)\|_{\dot{H}^{s_p} \times \dot{H}^{s_p - 1}}$ is sufficiently small, then we have

- global existence
- scattering.

Question: What about large data?

- There are solutions to the focusing problems that blow-up in finite time: for instance
$$\varphi(x,t) = \frac{C(d)}{(1-t)^{\frac{2}{p-1}}}$$
solves the ODE $\quad \partial_{tt}\varphi = |\varphi|^{p-1}\varphi$. Note that $|\varphi(x,t)| \to \infty$ as $t \nearrow 1$.
- More generally, we may find solutions $\vec{u}(t)$ that has unbounded critical Sobolev norm:
$$\lim_{t \nearrow 1} \|\vec{u}(t)\|_{\dot{H}^{s_p} \times \dot{H}^{s_p - 1}} = \infty.$$
This behavior typically known as type-I blow up.
- If $\vec{u}(t)$ obeys
$$\sup_{(T_-, T_+)} \|\vec{u}(t)\|_{\dot{H}^{s_p} \times \dot{H}^{s_p - 1}} < \infty$$
then we say that $\vec{u}(t)$ is a type-II solution.
- In particular, if $\vec{u}(t)$ is a type-II solution with either $T_+ < \infty$ or $T_- > -\infty$, then we say that $\vec{u}(t)$ is a type-II blow-up solution.

Examples of Type-II solutions

- When the size of the initial data $\|(u_0, u_1)\|_{\dot{H}^{s_p} \times \dot{H}^{s_p - 1}}$ is small, the solution $\vec{u}(t)$ is a global and scattering solution with
$$\sup_{(T_-, T_+)} \|\vec{u}(t)\|_{\dot{H}^{s_p} \times \dot{H}^{s_p - 1}} < \infty.$$
- For the energy-critical focusing problem in $\mathbb{R}^3$
$$\begin{aligned} &\partial_t^2 u - \Delta u - u^5 = 0, \quad \mathbb{R}^3 \times I_{max}, \\ &\vec{u}(0) = (u_0, u_1) \in \dot{H}^1 \times L^2, \end{aligned}$$
there are type-II blow-up constructions by Krieger-Schlag-Tataru (2009) and Krieger-Schlag (2014). These are solutions with
$$T_+ < \infty \quad \text{and} \quad \sup_{[0, T_+)} \|\vec{u}(t)\|_{\dot{H}^1 \times L^2} < \infty.$$
- Energy-critical focusing problems also admit stationary solutions that do not scatter: for instance
$$W(x) = \left(1 + \frac{|x|^2}{3}\right)^{-\frac{1}{2}}$$
is in $\dot{H}^1(\mathbb{R}^d)$ and solves the elliptic equation
$$\Delta W + W^5 = 0.$$
$W(x)$ is a global solution that do not scatter. Nevertheless, $(W(x), 0)$ is a type-II solution of the energy-critical focusing wave equation.

Main Result: type-II solutions in the energy-supercritical regime

Theorem [C.-Kenig, 2023]

Let $d \geq 7$ be an odd integer. Assume that $\vec{u}(t)$ is a radial solution of
$$\begin{aligned} &\partial_t^2 u - \Delta u - |u|^{p-1} u = 0, \quad (x,t) \in \mathbb{R}^d \times I_{max}, \\ &\vec{u}(0) = (u_0, u_1) \in \dot{H}^{s_p} \times \dot{H}^{s_p - 1} \end{aligned}$$
where $p \in \mathbb{N}_{\geq 3}$. Suppose
$$\sup_{[0, T_+)} \|\vec{u}(t)\|_{\dot{H}^{s_p} \times \dot{H}^{s_p - 1}} < \infty.$$
Then, we have
$$I_{max} \cap [0, \infty) = [0, \infty)$$
and $\vec{u}(t)$ scatters to a free wave as $t \to \infty$.

History

- Analogous results were shown in dimensions $3, 5$:
$$d = 3: \quad \text{Duyckaerts-Kenig-Merle, 2014}$$
$$d = 5: \quad \text{Dodson-Lawrie, 2015}$$
- Remark 1: Our main result implies that any finite time solution admits
$$\lim_{t_n \to T_+} \|\vec{u}(t)\|_{\dot{H}^{s_p} \times \dot{H}^{s_p - 1}} = \infty$$
along a sequence of times $\{t_n\}$.
- Remark 2: The condition on the critical Sobolev norm is necessary to understand the behavior of type-II solutions. In 2018, Collot provided a construction of a family of blow-up solutions to (NLW) in the energy-supercritical regime.

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