THE RADIATION FIELD PRODUCED BY LONGITUDINAL SLOTS IN A LONG CIRCULAR CYLINDER

by

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LIST OF SYMBOLS

$a_n = a_n' + a_n''$

$a_n' = \dfrac{\epsilon_n J_n(ka)}{J_n'(ka)}$

$a_n'' = \dfrac{-\epsilon_n H_n^{(2)}(ka)}{H_n^{(2)'}(ka)}$

a = Radius of the slot antenna

a_z = Unit vector in Z-direction

A' = Vector potential at a distant point P due to an elemental strip at Z of thickness dZ.

$A_\theta' = \dfrac{e^{-jkr}}{4\pi r} e^{jZd} N_\theta'$

$A_\phi' = \dfrac{e^{-jkr}}{4\pi r} e^{jZd} N_\phi'$

$A_\theta = \int_{-\ell}^{\ell} A_\theta'$

$A_\phi = \int_{-\ell}^{\ell} A_\phi'$

A_u'' = Vector potential at a distant point P due to an elemental surface u.

$\bar{A}$ = Magnetic vector potential.

$\bar{B} = \mu \bar{H}$ = Magnetic flux density web/m^2

$b = \dfrac{2\pi}{\lambda_1}$

B = Total slot distributed susceptance.

B' = Slot distributed susceptance due to conduction currents.

C = ka Sinθ Cos$(\phi - \Phi)$.

C_s = Slot distributed capacitance.

$d = k \cos \theta$

$\bar{D} = \epsilon \bar{E}$ = Vector electric flux density Coulomb/m^2.

$\bar{E}$ = Vector electric intensity volt/m.

$\bar{F}$ = Electric vector potential volt.

$f_n(k_Z)$ = Function to be determined from boundary condition.

$g_n(k\rho)$ = Function to be determined from boundary condition.

$g_n = \dfrac{\sin nx}{nx}$

G = Slot distributed conductance mho/m.

$\bar{H}$ = Vector magnetic intensity.

$\bar{I}(w)$ = Transform of current $I(w)$.

$J(Z, \Phi)$ = Current density distribution about the circumference of a cylinder at (Z, Φ).

$J(Z, \pi)$ = Current density distribution about the circumference of a cylinder at (Z, π).

$J_n = J_n(\frac{1}{2}ka)$ = Bessel function of the first kind.

$J_m(\pi)$ = Maximum value of J(as Z varies) at $\phi = \pi$

J = Current density ampere/m^2.

J_u = Current density due to elemental area u.

J_v = Current density due to elemental area v.

$J_s = J_u + J_v$

$Ja = J_u + J_v$.

$K = J_u e^{jc} - J_v e^{-jc}$

$k_1 = \sqrt{k^2 - w^2}$

k_ρ, k_z = Separation constant $k^2 = k_\rho^2 + k_z^2$

$k = \frac{2\pi}{\lambda}$ = Wave number of the medium.

L = Slot distributed reactance, henry/m

2l = Slot length

M = Magnetic source, volt/m^2

$\bar{N}$ = Radiation Vector.

$N''_x = N''_{u,x} + N''_{v,x}$

$N''_y = N''_{u,y} + N''_{v,y}$

$N'_x = \int_{\phi=0}^{\pi} N''_x$

$N'_y = \int_{\phi=0}^{\pi} N''_y$

$N''_v = J_v dZ \; e^{jc} d\bar{l}'_v$

$N''_u = J_u dZ \; e^{jc} d\bar{l}'_u$

$N'_\theta = (N'_x \cos\phi + N'_y \sin\ ) \cos\theta$.

$N'_\phi = N'_x \sin\phi + N'_y \cos\phi$.

p = A point in the radiation field.

q_v = Charge density amp/m

r' = Distance from u to a distant point p.

t = Slot thickness.

u = Elemental area on the cylinderical surface.

V = Applied voltage

v = Velocity of light 3×10^8 m/sec.

W = Slot width.

X = ωL = Slot distributed reactance ohm/m.

Y = Slot distributed admittance mho/m.

Z = Slot distributed reactance ohm/m

Z = Slot characteristic impedance ohm/m.

ρ, Φ, Z = Cylindrical coordinate symbols.

r, θ, ϕ = Spherical coordinate symbols.

$\epsilon_n = 1$ when $n = 0$, $\epsilon_n = 2$ when $n > 0$.

$\phi_1 = 2\phi_0$ = Slot angle.

σ = Propagation constant.

λ_1 = Wave length of current density distribution about the circumference of the cylinder.

λ_s = Wave length of slot region = $\frac{2\pi}{\beta}$

λ = Wave length of free space = $\frac{2\pi}{k}$

$\phi_1(Z)$, $\phi_2(Z)$ = Slot function.

Φ = Angle measured from X-axis to elemental area u in ϕ direction.

ϵ = Permitivity of free space (8.85×10^{-12} farad/m).

μ = Permeability of free space (4×10^{-7} henry/m).

$\eta = \left(\frac{\mu}{\epsilon}\right)^{\frac{1}{2}} = 120\pi$ = Intrinsic impedance of free space.

ψ = Field function.

$\omega = 2\pi f$ = Angular velocity.

α = Attenuation constant.

INTRODUCTION

An antenna that has important application at very high frequencies consists of a slot or slots cut into a conducting cylinder. For example, a longitudinal slot in a vertical cylinder produces a horizontally polarized signal suitable for FM or television. The basic problem of communicating to and from aircraft, satellites, and submarines has stimulated research work into the theory and application of slotted cylinder antennas. It is the purpose of this paper to aid this work by describing the field patterns produced by the slot antenna.

In 1950 Silver and Saunders developed general expressions for the external field produced by a slot of arbitrary shape, in the wall of an infinite circular cylinder, on the assumption that the tangential electric field in the slot is a prescribed function. In this case it is assumed that the width of the slot is very much smaller than its length and that the excitation is a cosine distribution along its length and is uniform over its width, which is

$$E_{\phi}(a, \phi, Z) = \frac{V}{2\phi_0 a} \operatorname{Cos} \frac{\pi Z}{2l} \qquad \begin{matrix} -l < Z < l \\ -\phi_0 < \phi < \phi_0 \end{matrix}$$

Where V is the excitation source, $2\phi_0$ is the slot angle, a is the radius of the cylinder antenna, $2l$ is the slot length along the axial axis. But the result can only apply to a slot antenna of a fixed dimension for a single frequency which is unknown. Because the relation between the wave length at slot λ_s and the wave length in the radiation field λ is unknown.

It would be more general to assume that the field in the slot is

$$E_\phi \ (a, \phi, Z) = \frac{V}{2\phi_0 a} \ \text{Cos} \ \beta(1 - |Z|)$$

where β is the phase shift constant defined by $\beta = \frac{2\pi}{\lambda_s}$, where λ_s is wave length at slot region. Wave length at the slot region could be found in terms of the dimensions of the cylinder antenna and the wave length λ. Therefore, different radiation fields correspond for different values of λ_s. But Silver and Saunders did not introduce any method to find the wave length at slot.

Another method was suggested by Dr. C. A. Holt in 1950, that the external field could be found by assuming a current distribution around the circumference. Dr. Holt has treated the slot as the loaded transmission line, and the distributed parameters of the slot region can be found, from which the wave-length along the slot can be determined. However, this method of finding the radiation field is restricted by the assumption of a cosinusoidal current density distribution about the circumference used to find the radiated power, while actually the current density distribution is a series of sinusoidal functions. This method is applicable only for diameter – wave length ratio less than about 0.2*.

In this paper improvement is made in finding the radiation field by assuming an electric field at the slot by

$$E_{\phi}(a, \phi, Z) = \frac{V}{2\phi_0 a} \cos\beta(1 - |Z|),$$

and then using Dr. Holt's method to find the wave length at slot to get β. The radiation found by the combination of these two methods is less restrictive than the above two methods. Satisfactory result is anticipated.

* Ref. (2), sec. 1

2. Introduction to the Fundamental Concepts

Maxwell's Equations: Assume the time variation is a sinusoid function; then

$$\nabla \times \bar{E} = -j\omega \bar{B} \quad (1) \qquad \nabla \cdot \bar{B} = 0 \quad (3)$$

$$\nabla \times \bar{H} = j\omega \bar{D} + \bar{J} \quad (2) \qquad \nabla \cdot \bar{D} = q_v \quad (4)$$

Vector potentials:

For a linear antenna energized at the center, carrying a current I (z'), the magnetic vector potential is (see Fig. 1)

$$A_z = \frac{1}{4\pi} \int_{-\ell}^{\ell} \frac{I(z')e^{-jk|r-r'|}}{|\bar{r} - \bar{r}'|} dz'$$

where $|\bar{r} - \bar{r}'| = \sqrt{r^2 + z'^2 - 2rz' \cos\theta}$, A_z is the magnetic potential in z direction, k is the wave number of the free space, and given by $k = \omega(\mu\epsilon)^{\frac{1}{2}}$.

In the far zone

$$|\bar{r} - \bar{r}'| = r - z' \cos\theta$$

and

$$A_z = \frac{e^{-jkr}}{4\pi r} \int_{-\ell}^{\ell} I(z')e^{jkz'\cos\theta} dz' \quad (5)$$

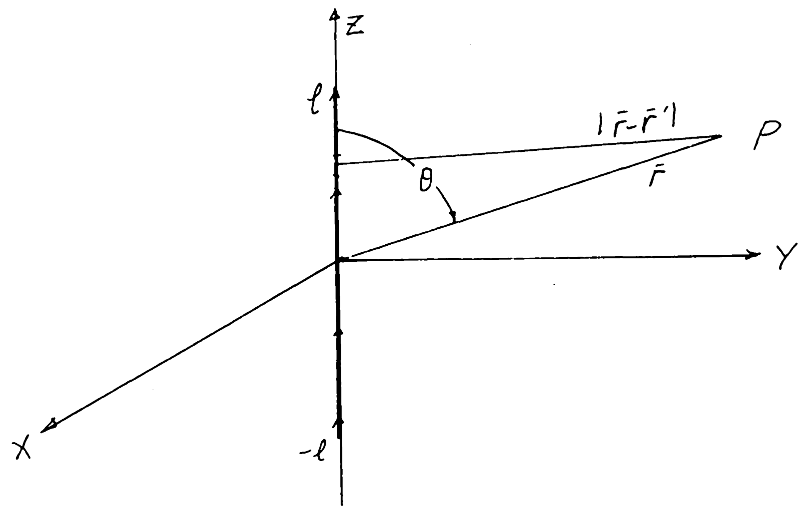

Fig. (1)

Linear antenna energized at the center

Same expression for the electric vector potential can be made by duality between electric source and magnetic source

$$F_z = \frac{e^{-jkr}}{4\pi r} \int_{-\ell}^{\ell} K(z')e^{jkz'\cos\theta}\, dz' \qquad (6)$$

where K (z') is the magnetic current oriented in the Z-direction, and F_z is the electric vector potential in Z-direction.

Construction of solution:

In a homogeneous source-free lossless region, the fields satisfy

$$-\nabla \times \bar{E} = j\omega\mu\bar{H} \qquad \nabla\cdot\bar{H} = 0$$

$$\nabla \times \bar{H} = j\omega\epsilon\bar{E} \qquad \nabla\cdot\bar{E} = 0$$

Expressions for the fields in terms of vector potentials can be obtained by expressing part of the field in terms of $\bar{F}$ and part in terms of $\bar{A}$.

$$\bar{E} = \bar{E}' + \bar{E}''$$

$$\bar{H} = \bar{H}' + \bar{H}''$$

where $\bar{E}'$ and $\bar{H}'$ are due to electric sources and $\bar{E}''$ and $\bar{H}''$ are due to magnetic sources. Then

$$\bar{E} = -\nabla\times\bar{F} + \frac{1}{j\omega\epsilon}\nabla\times\nabla\times\bar{A} \tag{7}$$

$$\bar{H} = \nabla\times\bar{A} + \frac{1}{j\omega\mu}\nabla\times\nabla\times\bar{F} \tag{8}$$

From the vector identity $\nabla\times\nabla\times\bar{A} = \nabla(\nabla\cdot\bar{A}) - \nabla^2\bar{A}$ and Maxwell's equations, Equations (7) and (8) become

$$\bar{E} = -\nabla\times\bar{F} - j\omega\mu\bar{A} + \frac{1}{j\omega\epsilon}\nabla(\nabla\cdot\bar{A}) \tag{9}$$

$$\bar{H} = \nabla\times\bar{A} - j\omega\epsilon\bar{F} + \frac{1}{j\omega\mu}\nabla(\nabla\cdot\bar{F}) \tag{10}$$

Far zone field (radiation field):

The distant field of an electric current element consists essentially of outward traveling plane waves. The same is true of a magnetic current element by duality. Hence, the radiation zone must be characterized by

$$E_\theta = \eta H_\phi \qquad E_\phi = -\eta H_\theta \tag{11}$$

since the field is a superposition of the fields from many current elements by evaluating the partial $\bar{H}$-field due to electric source $\bar{J}$, according to $\bar{H}' = \nabla \times \bar{A}$, and retaining only the dominant terms (r^{-1} variation). It can be shown* that

$$H'_\theta = (\nabla \times \bar{A})_\theta = jkA_\phi$$
$$H'_\phi = (\nabla \times \bar{A})_\phi = - jkA_\theta$$

with $\bar{E}'$ given by Eq. (11). Similarly, for the partial $\bar{E}$ - field due to $\bar{M}$, in the radiation zone

$$E''_\theta = - (\nabla \times \bar{F})_\theta = - jkF_\phi$$
$$E''_\phi = - (\nabla \times \bar{F})_\phi = jkF_\theta$$

with $\bar{H}''$ given by Eq. (11). The total field is the sum of these partial fields, or

$$E_\theta = E'_\theta + E''_\theta = - j\omega\mu A - jkF_\phi \tag{12}$$
$$E_\phi = E'_\phi + E''_\phi = - j\omega\mu A + jkF_\theta \tag{13}$$

in the radiation zone, with $\bar{H}$ given by Eq. (11)

The wave function:

Equations (9) and (10) show how to construct the general solutions to the field equations in homogeneous regions once the general solutions to the scalar Helmholtz equation are obtained. Use is made of the method

*Ref. 1, sec. 13, chap. 3.

'separation of variables'. General solutions to the Helmholtz equation can be constructed in certain coordinate systems. In this case, the cylindrical coordinate system is used.

The scalar Helmholtz equation in cylindrical coordinates is

$$\frac{\partial}{\rho\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2} + k^2\psi = 0 \qquad (14)$$

where ψ is the field function. Let the solutions be the form

$$\psi = R(\rho)\,\Phi(\phi)\,Z(z) \qquad (15)$$

Using the method of separation-of-variables yields

$$\rho\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right) + \left[(k_\rho\rho)^2 - n^2\right]R = 0$$

$$\frac{d^2\Phi}{d\phi^2} + n^2\Phi = 0$$

$$\frac{d^2Z}{dZ^2} + k_z^2 Z = 0$$

where $k^2 = k_\rho^2 + k^2$ and these together with n are separation constants. The Φ and Z equations are harmonic equations, giving rise to harmonic functions, denoted by $h(n,\phi)$ and $h(k_z, z)$. The R equation is Bessel's equation of order n, denoted by $B_n(k_\rho,\rho)$. Commonly used solutions to Bessel's equation are

$$B_n(k_\rho,\rho) \sim J_n(k_\rho,\rho),\; N_n(k_\rho,\rho),\; H_n^{(1)}(k_\rho,\rho),$$

$$H_n^{(2)}(k_\rho,\rho)$$

Any two of the functions are linearly independent solutions; so $B_n(k_\rho,\rho)$ is, in general, a linear combination of any two of them. According to Eq. (15) the solutions to the Helmholtz equation become

$$\psi_{k_\rho,n,k_z} = B_n(k_\rho,\rho)h(n,\phi)h(k_z,Z)$$

where ψ is the elementary wave function.

Linear combinations of the elementary wave functions are also solutions to the Helmholtz equation. Possible values of n and k_ρ, or n and k_z can be summed to get the desired solution, (but not k_ρ and k_z for they are interrelated). For example,

$$\psi = \sum_n \sum_{k_\rho} C_{n,k_\rho}\, \psi_{k_\rho,n,k_z}$$

$$= \sum_n \sum_{k_\rho} C_{n,k_\rho} B_n(k_\rho,\rho)h(n,\phi)h(k_z,Z)$$

is a solution to the Helmholtz equation, where C_{n,k_ρ} are constants. It is also possible to integrate over the separation variable. The possible solutions to the Helmholtz equation are

$$\psi = \sum_n \int_{k_z} f_n(k_z)B_n(k_\rho,\rho)h(n,\phi)h(k_z,Z)\; dk_z$$

$$\psi = \sum_n \int_{k_\rho} g_n(k_\rho) B_n(k_\rho, \rho) h(n, \phi) h(k_z, Z) \quad dk_\rho$$

where the integrations are over any contour in the complex plane and $f_n(k_z)$ and $g_n(k_\rho)$ are functions to be determined from boundary conditions. If ψ is single valued, it is necessary that $\psi(\phi) = \psi(\phi + 2\pi)$. This means that $h(n,\phi)$ must be periodic in ϕ, in which case n must be an integer. In this condition, we choose $e^{jn\phi}$. Thus, the n summations of Eqs. (24) and (25) are usually Fourier series of ϕ. As $h(k_z Z)$ is a harmonic solution to a harmonic function, a possible solution of $e^{jk_z z}$ is taken in this case.

Considering the various solution to Bessel's equation, it is apparent that $H_n^{(2)}(k_\rho, \rho)$ are the only solutions which vanish for large ρ if k_ρ is complex. They represent outward-traveling waves if k_ρ is real. Therefore, if there are no sources at infinity, the $B_n(k_\rho, \rho)$ must be $H_n^{(2)}(k_\rho, \rho)$ if $\rho \to \infty$ is to be included. Hence the elementary wave function becomes

$$\psi_{k_\rho, n, k_z} = H_n^{(2)}(k_\rho, \rho) e^{jn\phi} e^{jk_z Z}$$

and the general solution to the Helmholtz equation becomes

$$\psi = \sum_n e^{jn\phi} \int_{k_z} f_n(k_z) H_n^{(2)}(k_\rho, \rho) e^{jk_z Z} \quad dk_z \quad (16)$$

$$\text{or} \quad \psi = \sum_{n} e^{jn\phi} \int_{k_\rho} g_n(k_\rho) H_n^{(2)}(k_\rho, \rho) e^{jk_z Z} \, dk_\rho \tag{17}$$

where $k^2 = k_\rho^2 + k_z^2$ and n = integer.

3. Three-dimensional radiation

A three-dimensional problem having cylindrical boundaries can be reduced to a two-dimensional problem by applying a Fourier transformation with respect to Z (the cylinder axis).* For example, if ψ (X,Y,Z,) is a solution to the three-dimensional wave equation

$$\left(\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2} + \frac{\partial^2}{\partial Z^2} + K^2\right) \psi = 0$$

then

$$\bar{\psi}(X,Y,w) = \int_{-\infty}^{\infty} \psi(X,Y,Z) e^{-jwZ} \, dz$$

is a solution to the two-dimensional problem wave equation

$$\left(\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2} + k_1^2\right) \bar{\psi} = 0$$

where $k_1^2 = k^2 - w^2$ is the Fourier transform of ψ. Once the two-dimensional problem for $\bar{\psi}$ is solved, the three-dimensional solution is obtained from the inversion

$$\psi(X,Y,Z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{\psi}(X,Y,w) e^{j\omega Z} \, dw$$

*See Ref. 1, sec. 11, chap 5.

This is usually a difficult operation. Fortunately, in the radiation zone the inversion becomes quite simple. This far-zone inversion formula will now be obtained.

Consider the problem of a filament of Z-directed current along the Z axis, as illustrated by Fig. (2). The only restriction placed on the current I(z) is that it be fourier-transformable. In the usual way, we construct a solution

$$\bar{H} = \nabla \times \bar{A} \qquad \bar{A} = \bar{a}_z \psi$$

where ψ is a wave function independent of ϕ and representing out-ward traveling waves at large .

From the general solution of Eq. (16), the solution to this problem would be n(separation variable associated with ϕ) = 0. A harmonic function is given by e^{jwz}, and a traveling wave in the radial direction is given by the Hankel function of the second kind $H_0^{(2)}(k_1\rho)$, where $k_1 = \sqrt{k^2 - w^2}$. Therefore, anticipating the need for Fourier transforms, we write

$$\psi = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(w)\, H_0^{(2)}(\rho\sqrt{k^2 - w^2})e^{jwz}\, dw \tag{18}$$

The Fourier transform of ψ is evidently

$$\bar{\psi} = f\,(w)H_0^{(2)}(\,\rho\sqrt{k^2 - w^2})$$

where f(w) is determined by the nature of the source, according to Ampere's Circuital Law

$$\int_0^{2\pi} \bar{H}_\phi\;\rho\;d\phi \xrightarrow[\rho \to 0]{} \bar{I}(w) \tag{19}$$

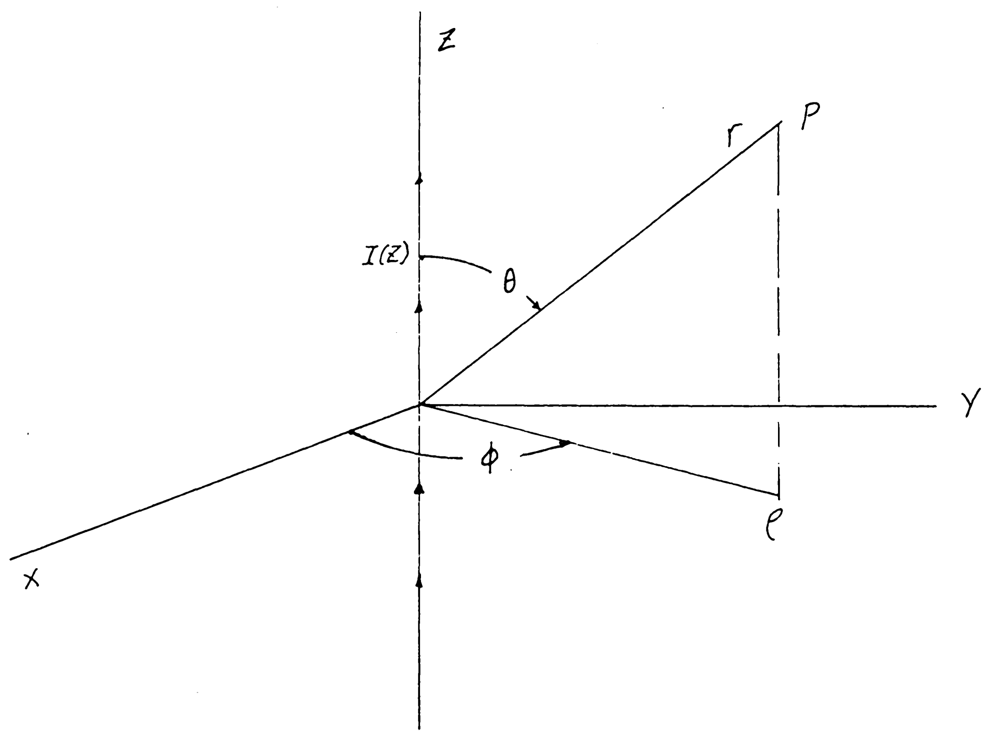

Fig. (2)

A filament of current along the Z-axis

where $\bar{H}_\phi$ and $\bar{I}(w)$ are transforms of H_ϕ and I.

The small-argument formula for $H_0^{(2)}$ yields*

$$H_0^{(2)}(\rho\sqrt{k^2-w^2}) \xrightarrow[\rho \longrightarrow 0]{} (1 - \frac{j2}{\pi} \log \frac{\gamma\rho\sqrt{k^2-w^2}}{2})$$

$\gamma = 1.781$ (Euler's constant)

Therefore,

$$\bar{\psi} = f(w)(1 - \frac{j2}{\pi} \log \frac{\gamma\rho\sqrt{k^2-w^2}}{2})$$

and $\bar{H} = \nabla \times \bar{A}$ therefore $\bar{H}_\phi = -\frac{\partial\bar{\psi}}{\partial\rho}$

$$\bar{H}_\phi \xrightarrow[\rho \longrightarrow 0]{} \frac{2j}{\pi\rho} f(w)$$

Hence equation (19) yields

$$f(w) = \frac{\bar{I}(w)}{4j}$$

Substitution in Eq. (18) gives the transform solution to the problem of Fig. (2)

$$\psi = \frac{1}{j8\pi} \int_{-\infty}^{\infty} \bar{I}(w)\, H_0^{(2)}(\rho\sqrt{k^2-w^2})e^{jwz}\, dw \qquad (20)$$

where $\bar{I}(w) = \int_{-\infty}^{\infty} I(z)e^{-jwz'}\, dz'$

$I(z')$ is the filamentary current along the Z-axis.

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* Ref. 1, appendix D.

Consider the case of the 'linear antenna' carrying a current I(z'). The magnetic vector potential for this problem by Eq. (5), is

$$A_z \doteq \frac{e^{-jkr}}{4\pi r} \int_{-\ell}^{\ell} I(z')e^{jkz'\cos\theta}\, dz' \qquad r >> 1$$

As $\bar{A} = \bar{a}_z \psi$

and $\bar{I}(w) = \int_{-\ell}^{\ell} I(z')e^{-jwz'}\, dz'$

Therefore it can be written

$$\psi = A_z \xrightarrow[r \to \infty]{} \frac{e^{-jkr}}{4\pi r} I(-k\cos\theta) \qquad (21)$$

Eq. (12) yields

$$E_\theta \xrightarrow[r \to \infty]{} -jw\mu A = jw\mu \sin\theta\, \psi$$

$$E_\theta \xrightarrow[r \to \infty]{} jw\mu \frac{e^{-jkr}}{4\pi r} \sin\theta\ \bar{I}(-k\cos\theta)$$

Hence the radiation field is simply related to the transform of the source evaluated at $w = -k\cos\theta$. Comparison of Eq. (20) and Eq. (21) reveals the identity

$$\int_{-\infty}^{\infty} \bar{I}(w)H_0^{(2)}(\rho\sqrt{k^2-w^2})e^{jwz}\, dw \xrightarrow[r \to \infty]{} \frac{j2e^{-jkr}}{r} \bar{I}(-k\cos\theta) \qquad (22)$$

which holds for any function I(w).

The asymptotic expression of Hankel function of arbitrary order is

$$H_n^{(2)}(x) \xrightarrow[x \to \infty]{} \sqrt{\frac{2j}{\pi x}} \; j^n e^{-jx}$$

from which it is evident that

$$H_n^{(2)}(x) \xrightarrow[x \to \infty]{} j^n H_o^{(2)}(x)$$

As long as $\theta \neq 0$ or π, we have $\rho \to \infty$ as $r \to \infty$, because $\rho = r \sin\theta$. Also, if k is complex (some dissipation assumed), then $\sqrt{k^2 - w^2}$ is never zero on the path of integration. We are then justified in using the asymptotic formula for the Hankel function and we can replace the $H_o^{(2)}$ of Eq. (22) by $j^{-n}H_n^{(2)}$. The result is

$$\int_{-\infty}^{\infty} \bar{I}(w) H_n^{(2)}\left(\rho\sqrt{k^2 - w^2}\right) e^{jwz}\, dw \xrightarrow[r \to \infty]{} \frac{2e^{-jkr}}{r} \; j^{n+1} \; \bar{I}(-k\cos\theta) \qquad (23)$$

This formula will be used in the radiation problem that follows.

4. The Tangential field over the cylinder

Consider a conducting cylinder of infinite length in which one or more apertures exist. The geometry is shown in Fig. (3). We seek a solution for the field external to the cylinder in terms of the tangential components of E over the apertures.

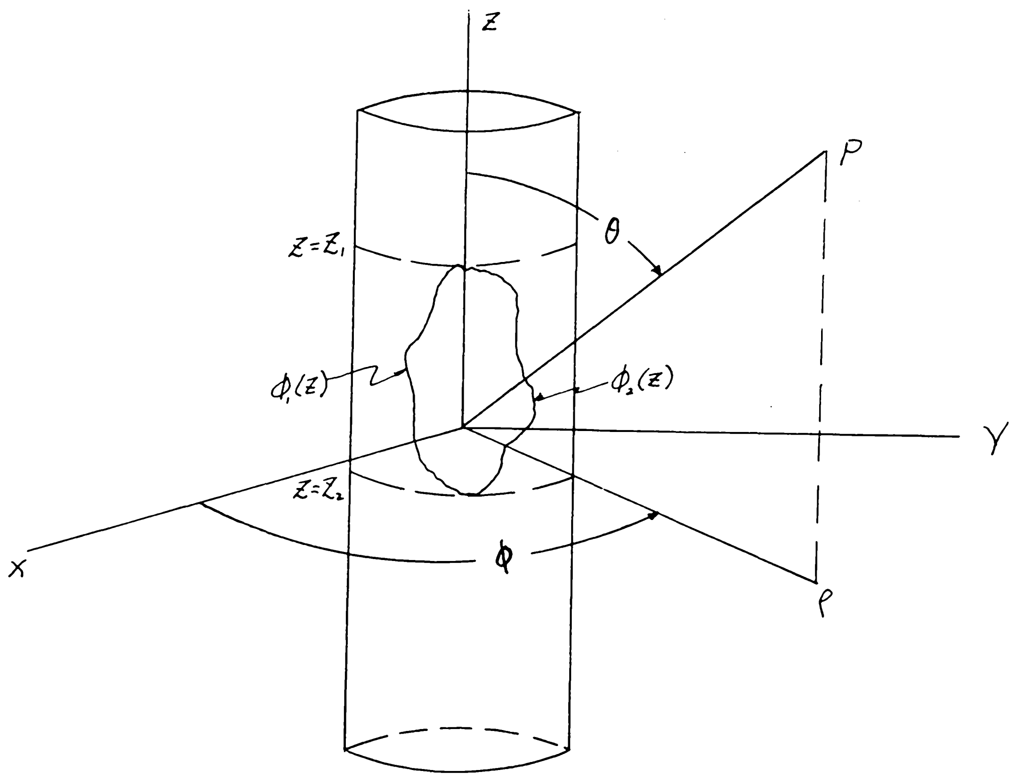

Fig. (3) Slotted cylinder antenna with infinite length

We shall first develop the Fourier expansion for the tangential component of the electric field over the surface of the cylinder. The tangential electric field in the slot in general has both ϕ and Z components, which we consider to be prescribed functions $E_{\phi}(a, \phi, Z)$ and $E_Z(a, \phi, Z)$ respectively.

Now let E_α denote either $E_\phi(a, \phi, Z)$ or $E_z(a, \phi, Z)$. In the ϕ direction E_α is a periodic function and therefore, can be represented by a Fourier series

$$E_\alpha(a, \phi, Z) = \sum_{n=-\infty}^{\infty} A_n(Z) e^{jn\phi}$$

the co-efficient being a function of Z. It is readily evident that

$$A_n(Z) = \frac{1}{2\pi} \int_{\phi_1(z)}^{\phi_2(z)} E_\alpha(a, \phi, Z) e^{-jn\phi}\, d\phi \qquad Z_1 \leqslant Z \leqslant Z_2$$

$$A_n(Z) = 0 \qquad Z < Z_1 \quad Z > Z_2$$

An is thus a piecewise continuous function and its Fourier representation is the Fourier integral

$$A_n(Z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} A_n(Z) e^{-jwZ}\, dZ \right] e^{jwZ}\, dw$$

The previous expression for $A_n(Z)$ yields

$$A_n(Z) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} e^{jwZ}\, dw \int_{-\infty} e^{-jwZ}\, dZ \int_{\phi_1(z)}^{\phi_2(z)} E_\alpha(a, \phi, Z) e^{-jn\phi}\, d\phi$$

Hence the tangential field

$$E_\alpha(a, \phi, Z) = \frac{1}{4\pi^2} \sum_{n=-\infty}^{\infty} e^{jn\phi} \int_{-\infty}^{\infty} e^{jwZ}\, dw \int_{-\infty}^{\infty} e^{-jwZ}\, dZ \int_{\phi_1(z)}^{\phi_2(z)} E_\alpha(a, \phi, Z) e^{-jn\phi}\, d\phi$$

It is more obvious, if it is written

$$\bar{E}_z(n,w) = \frac{1}{2\pi} \int_0^{2\pi} d\phi \int_{-\infty}^{\infty} dZ\, E_z(a, \phi, Z) e^{-jn\phi}\, e^{-jwZ} \tag{24}$$

$$\bar{E}_\phi(n,w) = \frac{1}{2\pi} \int_0^{2\pi} d\phi \int_{-\infty}^{\infty} dZ\, E_\phi(a, \phi, Z) e^{-jn\phi}\, e^{-jwZ} \tag{25}$$

Then the inverse transformation is

$$\bar{E}_z(a, \phi, Z) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} e^{jn\phi} \int_{-\infty}^{\infty} \bar{E}_z(n,w) e^{jwZ}\, dw \tag{26}$$

$$E_\phi(a, \phi, Z) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} e^{jn\phi} \int_{-\infty}^{\infty} \bar{E}_\phi(n,w) e^{jwZ}\, dw \tag{27}$$

The field external to the cylinder can be expressed as the sum of a TE component and TM component.

Inspecting from Eqs. (9) and (10), the wave functions A_z and F_z are constructed in such a manner that ϕ and Z functions possess the same form as Eq. (26) or (27), and they represent outward traveling waves at large ρ. Then it is assumed that

$$A_z = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} e^{jn\phi} \int_{-\infty}^{\infty} f_n(w) H_n^{(2)}(\rho\sqrt{k^2 - w^2}) e^{jwZ}\, dw \tag{28}$$

$$F_z = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} e^{jn\phi} \int_{-\infty}^{\infty} g_n(w) H_n^{(2)}(\rho\sqrt{k^2 - w^2}) e^{jwZ}\, dw \tag{29}$$

E_ϕ and E_z are calculated from Eqs. (9) and (10) and from the boundary condition at $\rho = a$. Then $f_n(w)$ and $g_n(w)$ are determined. The procedure is

$$E_z(\rho, \phi, Z) = -j\omega\mu A_z + \frac{1}{j\omega\epsilon} \quad \frac{\partial^2 A_z}{\partial Z^2}$$

$$= -j\omega\mu A_z + \frac{1}{j\omega\epsilon} \quad (-w^2) \quad A_z$$

$$= \frac{1}{j\omega\epsilon} (k^2 - w^2) \quad A_z$$

where $k^2 = -(j\omega\mu)(j\omega\epsilon) = \omega^2\mu\epsilon$. Therefore,

$$E_\phi \quad (\rho, \phi, Z) = \frac{\partial F_z}{\partial \rho} + \frac{1}{j\omega\epsilon\rho} \quad \frac{\partial^2 A_z}{\partial\phi\partial Z}$$

and

$$E_z(\rho, \phi, Z) = \frac{1}{j2\pi\omega\epsilon} \sum_{n=-\infty}^{\infty} e^{jn\phi} \int_{-\infty}^{\infty} (k^2 - w^2) \, f_n(w) \, H_n^{(2)} \left(\rho\sqrt{k^2 - w^2}\right) e^{jwZ} \, dw$$

Since these equations, specialized to $\rho = a$, must equal Eqs. (24) and (25), comparison yields

$$f_n(w) = \frac{j\omega\epsilon \, \bar{E}_z(n,w)}{(k^2 - w^2) \, H_n^{(2)}\left(a\sqrt{k^2 - w^2}\right)} \tag{30}$$

$$g_n(w) = \frac{1}{\sqrt{k^2 - w^2} \, H_n^{(2)\prime}\left(a\sqrt{k^2 - w^2}\right)} \left[\bar{E}_\phi(n,w) + \frac{n\,\omega\,\bar{E}_z(n,w)}{a(k^2 - w^2)}\right] \tag{31}$$

5. The radiation from a transverse rectangular slot in a circular cylinder

It is shown that the principal transverse plane pattern of such a slot in which the excitation has only a circumferential tangential electric field component is identical to the pattern generated by an infinite axial slot with the same circumferential excitation. The computations have been made for the especially important case of a narrow slot having an axial extent of a half-wave length.

The Field Distribution in the Slot.

At the boundaries of the slot the component of the electric field that is tangent to the boundary must be zero. In general, the excitation of the slot may be conceived as the superposition of many modes of field distribution, each of which satisfies the general excitation of a thin wire antenna, which may be synthesized by superposition of characterstic sinusoidal distributions that satisfy the requirement that the current by zero at the ends.

The geometry of the configuration suggests that the field components are separable functions of ϕ and Z; thus (see Fig. 4)

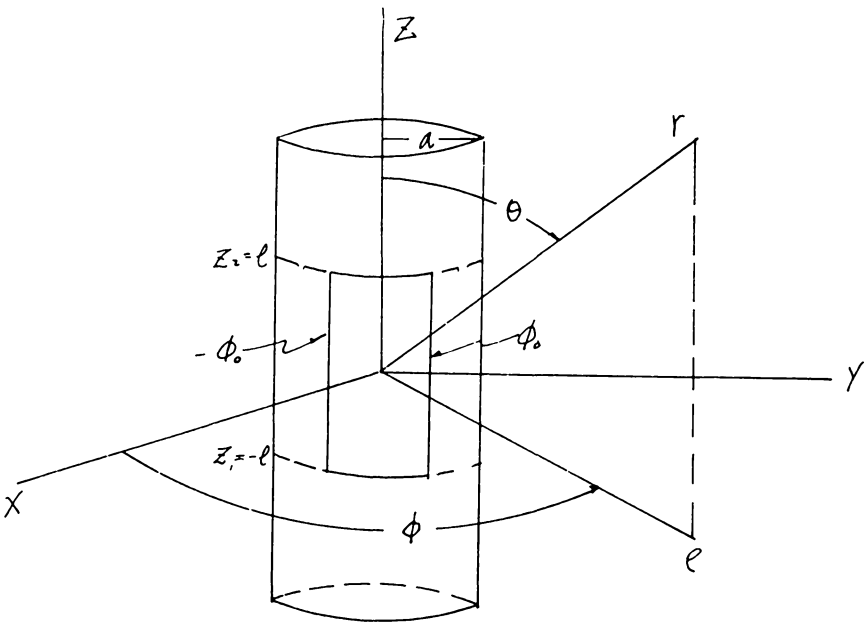

Fig. (4) Radiation from a transverse rectangular slot in a circular cylinder

$$E_\phi(a,\phi,Z) = F_1(\phi)\,G_1(Z)$$
$$E_Z(a,\phi,Z) = F_2(\phi)\,G_2(Z)$$

With the boundary condition at the edge of the slots it is evident that

$$G_1(Z) = \begin{cases} \mathrm{Sin}\,(p\pi Z/2l) & p \ \text{even} \\ \mathrm{Cos}\,(p\pi Z/2l) & p \ \text{odd} \end{cases}$$

$$F_2(\phi) = \begin{cases} \text{Sin } (p\pi\phi/2\phi_0) & p \text{ even} \\ \text{Cos } (p\pi\phi/2\phi_0) & p \text{ odd} \end{cases}$$

From the point of view that the slot is a very shallow section of wave guide in consequence of which the field configuration is like that over the cross section of a wave guide, on this basis, it should be that

$$F_1(\phi) = \begin{cases} \text{Cos } (q\pi\phi/2\phi_0) & q \text{ even} \\ \text{Sin } (q\pi\phi/2\phi_0) & q \text{ odd} \end{cases}$$

$$G_2(Z) = \begin{cases} \text{Cos } (q\pi Z/21) & q \text{ even} \\ \text{Sin } (q\pi Z/21) & q \text{ odd} \end{cases}$$

The narrow rectangular slot.

When the transverse dimension $2\phi_0 a$ of the slot is small compared with the wave length and the slot length, the significant mode of excitation is that in which there is only an E_ϕ -component that is uniform across the slot and has a sinusoidal distribution along its length. Thus

$$F_1(\phi)\, G_1(Z) = (V/2a\,\phi_0)G_1(Z) = E_\phi\,(a, \phi, Z)$$

Radiation Field

Method one:

Assume in the aperture

$$E_\phi (a, \phi, Z) = \frac{V}{2\phi_0 a} \quad \sin\beta (l - |Z|) \quad -\phi_0 < \phi < \phi_0$$

$$E_z(a, \phi, Z) = 0$$

where β is the phase shift constant given by $\beta = \frac{2\pi}{\lambda_s}$, λ_s is the wave length of the slot region, $k = \frac{2\pi}{\lambda}$, and λ = wave length of free space.

For a very narrow slot ($\phi_0 \to o$) the transforms of Eqs. (24) and (25) become

$$\bar{E}_\phi(n,w) = \frac{V}{4\pi\phi_0 a} \int_{-\phi_0}^{\phi_0} \int_{-l}^{l} e^{-jn\phi} \, e^{jwZ} \sin \beta (l - |Z|) \, dZ \, d\phi$$

$$= \frac{2V \sin n\phi_0}{4\pi \phi_0 an} \left[\int_{-l}^{o} \sin\beta (l + Z) \, e^{-jwZ} \, dZ + \int_{o}^{l} \sin\beta (l - Z) \, e^{-jwZ} \, dZ \right]$$

The integrations yield

$$\bar{E}_\phi(n,w) = \frac{V \sin n\phi_0}{2\pi n\phi_0 a} \left[\frac{-2\beta \cos \beta l}{\beta^2 - w^2} + \frac{2\beta \cos wl}{\beta^2 - w^2} \right]$$

$$\doteq \frac{V\beta (\cos wl - \cos \beta l)}{\pi a (\beta^2 - w^2)}$$

$$\bar{E}_z(n,w) = 0$$

From Eqs. (30), (31) we obtain

$$f_n(w) = 0 \qquad (32)$$

$$g_n(w) = \frac{-V\beta\,(\cos\beta l - \cos wl)}{\pi a\,(\beta^2 - w^2)\sqrt{k^2 - w^2}\;H_n^{(2)\prime}(a\sqrt{k^2 - w^2})} \tag{33}$$

A_z and F_z are constructed according to Eqs. (28) and (29). For the radiation field use is made of the Eq. (23).

$$A_z \xrightarrow[r \to \infty]{} \frac{e^{-jkr}}{\pi r} \sum_{n=-\infty}^{\infty} e^{jn\phi}\, j^{n+1}\, f_n(w)$$

$$F_z \xrightarrow[r \to \infty]{} \frac{e^{-jkr}}{\pi r} \sum_{n=-\infty}^{\infty} e^{jn\phi}\, j^{n+1}\, g_n(w)$$

Substitute Eqs. (32) and (33) in A_z. In the radiation zone

$$E_\theta = -j\omega\mu A_\theta - jkF_\phi$$

$$E_\phi = -j\omega\mu A_\phi + jkF_\theta$$

where $A_\theta = -A_z \sin\theta$, $A_\phi = 0$, $F_\theta = -F_z \sin\theta$, $F_\phi = 0$.

Then

$$A_z \xrightarrow[r \to \infty]{} 0 \qquad E_\theta(r, \theta, \phi) = 0$$

$$E_\phi(r, \theta, \phi) \xrightarrow[r \to \infty]{} -jkF_z \sin\theta$$

$$= \frac{-jk \sin\theta\; e^{-jkr}}{\pi r} \sum_{n=-\infty}^{\infty} e^{jn\phi} j^{n+1} g_n(w)$$

$$= \frac{kV\beta \sin\theta\; e^{-jkr}(\cos wl - \cos\beta l)}{\pi^2 ra(\beta^2 - w^2)\sqrt{k^2 - w^2}}$$

$$\sum_{n=-\infty}^{\infty} \frac{e^{jn\phi}\, j^n}{H_n^{(2)\prime}(a\sqrt{k^2 - w^2})}$$

Finally the radiation field $E_\theta = 0$, and, since $-k \cos\theta = w$, $k^2 \cos^2\theta = w^2$, $\sqrt{k^2 - w^2} = k \sin\theta$

$$E = \frac{V\beta e^{-jkr} \cos(kl \cos\theta) - \cos\beta l}{ra(\beta^2 - k^2 \cos^2\theta)}$$

$$\sum_{n=0}^{\infty} \frac{\epsilon_n j^n \cos n\phi}{H_n^{(2)\prime}(ka \sin\theta)} \qquad (34)$$

where $\epsilon_n = 1$ when $n = 0$, $\epsilon_n = 2$ when $n > 0$.

The radiation zone must be characterized by the plane wave relations

$$E_\theta = \eta H_\phi \qquad\qquad E_\phi = -\eta H_\theta$$

therefore, $H_\phi = 0$, and

$$H_\theta = -\frac{E_\phi}{\eta}$$

where E_ϕ is given by Eq. (34)

Method Two: Field in terms of the current distribution

If sinusoidal axial and cosinusoidal circumferential current distributions are assumed around a slotted cylinder antenna, then the radiation field can be obtained in terms of the assumed current distribution.

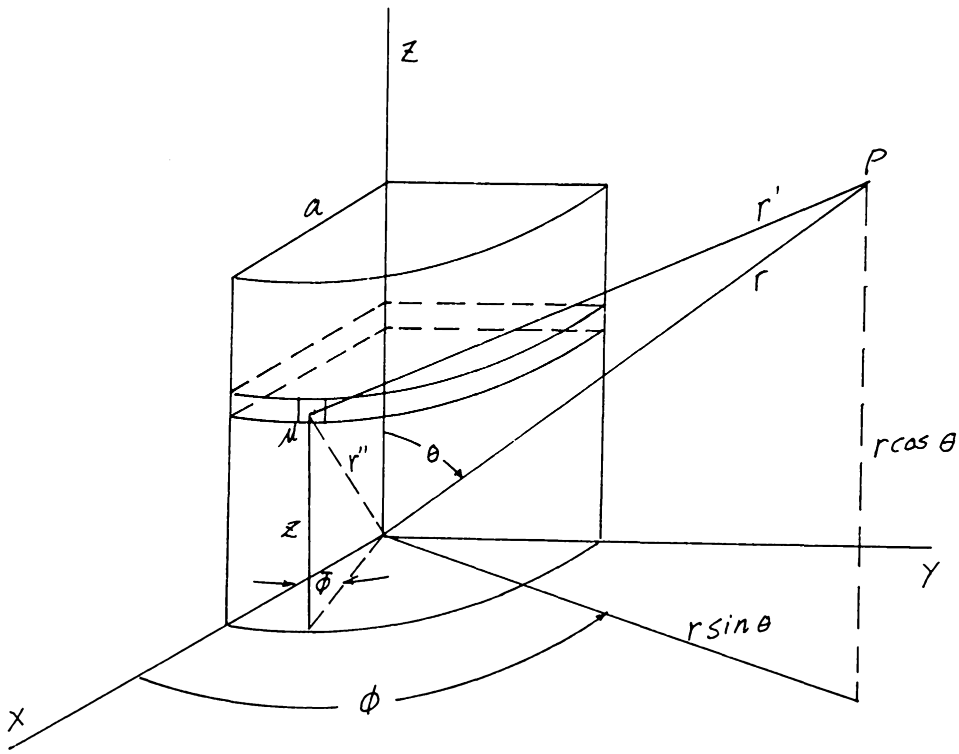

Fig. (5)

Field in terms of Current Distribution

Fig. (5) represents a quadrant of an axial section of the cylinder. μ represents an elemental area on the surface. The point P is at a great distance from the cylinder. An approximate expression for r' in terms of the radius and the coordinates will now be derived.

$$r'^2 = r^2 + r''^2 - 2rr'' \operatorname{Cos} \bar{r}\bar{r}''$$
$$= r^2 + a^2 + z^2 - 2rr'' \operatorname{Cos} \bar{r}\bar{r}''$$
$$r_x = r \operatorname{Sin}\theta \operatorname{Cos}\phi$$
$$r_y = r \operatorname{Sin}\theta \operatorname{Sin}\phi$$
$$r_z = r \operatorname{Cos}\theta$$
$$r_x'' = a \operatorname{Cos}\Phi$$
$$r_y'' = a \operatorname{Sin}\Phi$$
$$r_z'' = Z$$

$$r' = \sqrt{(r_x - r''_x)^2 + (r_y - r_y'')^2 + (r_z - r_z'')^2}$$

$$= r\left[1 + \frac{a^2 + Z^2}{r^2} - \frac{2a}{r}(\operatorname{Sin}\theta \operatorname{Cos}\phi \operatorname{Cos}\Phi) - \frac{2a}{r}(\operatorname{Sin}\theta \operatorname{Sin}\phi \operatorname{Sin}\Phi) - \frac{2Z}{r}\operatorname{Cos}\theta\right]^{\frac{1}{2}}$$

Applying the binomial expansion, neglecting the second and higher orders of r, and simplifying yield

$$r' = r - a \operatorname{Sin}\theta \operatorname{Cos}(\phi - \Phi) - Z \operatorname{Cos}\theta$$

$$\bar{A}''_\mu = J_\mu dZ e^{-jkr'} \frac{d l'_\mu}{4\pi r} \bar{a}_\phi = \frac{e^{-jkr}}{4\pi r} e^{jkZ \operatorname{Cos}\theta} \bar{N}_\mu''$$

In the preceding equations A''_μ is a vector potential at a distant point P due to an elemental surface μ, and J_μ is the current density at the elemental surface μ ; l'_μ = circumferential distance to elemental surface measured from

$\phi = \pi$, and

$$\bar{N}''_u = J_u \, dZ \, e^{jc} \, dl'_u$$

$$c = ka \, \text{Sin}\,\theta \, \text{Cos}(\phi - \Phi)$$

Let v represent a similar elemental surface diametrically opposite u . The X and Y components of $\bar{N}''_u$ and $\bar{N}''_v$ may be readily determined. If N''_x and N''_y are defined by the relations

$$N''_x = N''_{u,x} + N''_{v,x}$$

$$N''_y = N''_{u,y} + N''_{v,y}$$

it is easily shown that

$$N''_x = - K \, dZ \, a \, \text{Sin}\,\phi \, d\phi$$

$$N''_y = K \, dZ \, a \, \text{Cos}\,\phi \, d\phi$$

where $K = J_u \, e^{jc} - J_v e^{-jc}$ K, may be expressed by

$$K = \tfrac{1}{2}(J_s + J_a) \, e^{jc} - \tfrac{1}{2}\,(J_s - J_a) \, e^{-jc}$$

$$= j \, J_s \, \text{Sin}\, c + J_a \, \text{Cos}\, c$$

where $J_s = J_u + J_v$ $J_a = J_u - J_v$

$$\text{Sin}\, c = \text{Sin}\left[\, ka \, \text{Sin}\,\theta \, \text{Cos}\,(\phi - \Phi)\right]$$

$$= \text{Sin}\left[\tfrac{1}{2}\, ka \, \text{Sin}\,(\,\theta + \phi - \Phi\,) + \tfrac{1}{2}\, ka \quad \text{Sin}\,(\,\theta - \phi + \Phi\,)\right]$$

$$= \operatorname{Sin}\left[\tfrac{1}{2}\,ka\,\operatorname{Sin}\,(\theta + \phi - \Phi)\right]\operatorname{Cos}\left[\tfrac{1}{2}\,ka\,\operatorname{Sin}\,(\theta - \phi + \Phi)\right]$$

$$+ \operatorname{Cos}\left[\tfrac{1}{2}\,ka\,\operatorname{Sin}\,(\theta + \phi - \Phi)\right]\operatorname{Sin}\left[\tfrac{1}{2}\,ka\,\operatorname{Sin}(\theta - \phi + \Phi)\right]$$

Similarly

$$\operatorname{Cos}\,c = \operatorname{Cos}\left[\tfrac{1}{2}\,ka\,\operatorname{Sin}\,(\theta + \phi - \Phi)\right]\operatorname{Cos}\left[\tfrac{1}{2}\,ka\,\operatorname{Sin}(\theta - \phi + \Phi)\right]$$

$$- \operatorname{Sin}\left[\tfrac{1}{2}\,ka\,\operatorname{Sin}\,(\theta + \phi - \Phi)\right]\operatorname{Sin}\left[\tfrac{1}{2}\,ka\,\operatorname{Sin}\,(\theta - \phi + \Phi)\right]$$

But $\operatorname{Sin}\,(\tfrac{1}{2}\,ka\,\operatorname{Sin}\phi) = 2\sum_{n=0}^{\infty} J_{2n+1}\operatorname{Sin}\,(2n+1)\phi$

and $\operatorname{Cos}\,(\tfrac{1}{2}\,ka\,\operatorname{Sin}\phi) = J_o + 2\sum_{n=1}^{\infty} J_{2n}\operatorname{Cos}\,2n\phi$ *

The Bessel function argument $\tfrac{1}{2}$ ka is understood. For ka $\leqslant$ 1 an excellent approximation for each of the above expressions is obtained by using only the first term of the infinite series. Using only the first term and substituting into the expressions for Sin c and Cos c yield

$$\operatorname{Sin}\,c = 4\,J_oJ_1\,\operatorname{Sin}\theta\operatorname{Cos}(\phi - \Phi)$$

and

$$\operatorname{Cos}\,c = J_o^2$$

then

$$K = J_o\left[j4\,J_sJ_1\,\operatorname{Sin}\theta\operatorname{Cos}(\phi - \Phi) + J_a\,J_o\right]$$

Let

$$N_x = \int_{\Phi=0}^{\pi} N_x''$$

* Ref. 5, sec. 5, chap 8.

and define N_y' similarly. In order to perform these integrations it is necessary to specify the current-density distribution with respect to Φ . Let

$$J(Z, \Phi) = J(Z, \pi) \operatorname{Cos} bl'$$

with l' = circumferential distance measured from $\phi = \pi$, $b = \frac{2\pi}{\lambda_1}$, and λ_1 = wave length of current density distribution about the circumference of the cylinder. This assumption is based on the current distribution about the circumference of an infinite cylinder, uniformly fed along the slot with equal, in-phase voltages, which has been found* to be

$$J(\phi) = -j \frac{V\, p(\phi)}{2\pi\eta a}$$

where $P(\phi) = a_o + \sum_{n=1}^{\infty} a_n g_n \operatorname{Cos} n\phi$, $g_n = \frac{\operatorname{Sin} nx}{nx}$,

$$a_n' = a_n' + a_n''$$

and $a_n' = \frac{\epsilon_n J_n(ka)}{J_n'(ka)}$, $\quad a_n = \frac{-\epsilon_n H_n^{(2)}(ka)}{H_n^{(2)\prime}(ka)}$. $P(\phi)$ is a series of Cosinusoidal functions, converges more rapidly for small values of ka. Therefore, it is assumed that the current distribution about the circumference is a simple

*Ref. (2), sec. III

cosinusoidal variation, i.e. Cos bl' , provided that ka = $\frac{2\pi a}{\lambda}$ is small, which means 'a' small for a certain frequencies, or the frequencies could not be too higher than its cut-off frequency for a certain dimension of the antenna.

Then

$$J_u = J(Z, \pi)\ \text{Cos}\ ba(\pi - \Phi)$$

And

$$J_v = J(Z, \pi)\ \text{Cos}\ ba\,\Phi$$

Therefore K becomes upon substituting,

$$K = J_o J(Z, \pi)\ \Big[j4J_1\ \text{Sin}\ \theta\ (\text{Cos}\ ba\,\pi + 1)\ \text{Cos}\ ba\,\Phi\ \text{Cos}(\phi - \Phi) + j4J_1\ \text{Sin}\,\theta\ \text{Sin}\ ba\,\Phi\ \text{Sin}\ ba\,\pi\ \text{Cos}\ (\phi - \Phi)\ J_o(\text{Cos}\ ba\,\pi - 1)\ \text{Cos}\ ba\,\Phi + J_o\ \text{Sin}\ ba\,\pi\ \text{Sin}\ ba\,\Phi \Big]$$

Substituting for K into the expressions for N''_x and N''_y, and integration from $\Phi = 0$ to $\Phi = \pi$, give N'_x and N'_y. The simplified results are

$$N'_x = \frac{-j\ 16a\ J_o J_1 J(Z, \pi)\ dZ\ \text{Sin}\ ba\,\pi\ \text{Sin}\,\theta\ \ \text{Sin}\,\phi}{ba(4 - b^2a^2)}$$

$$N'_y = 2a\ J_o J_1(Z, \pi)\ dZ\ \text{Sin}\ ba\,\pi \left[\frac{j4\ J_1(2 - b^2a^2)\ \text{Sin}\,\theta\ \text{Cos}\,\phi}{ba(4 - b^2a^2} - \frac{ba\ J_o}{(1 - b^2a^2\ )} \right]$$

$$N'_\theta = (N'_x \operatorname{Cos}\phi + N'_y \operatorname{Sin}\phi)\operatorname{Cos}\theta$$

$$N'_\phi = N'_x \operatorname{Sin}\phi + N'_y \operatorname{Cos}\phi$$

Substituting for N'_x and N'_y gives

$$N'_\theta = -2ba^2 J_o^2 \operatorname{Sin} ba\pi \operatorname{Sin}\phi \operatorname{Cos}\theta \; J(Z,\pi)\, dZ$$

$$\left[\frac{1}{(1 - b^2a^2)} + \frac{j4J_1 \operatorname{Sin}\theta \operatorname{Cos}\phi}{J_o(4 - b^2a^2)}\right]$$

$$N'_\phi = -2ba^2 J_o^2 \operatorname{Sin}\pi ba \; J(Z,\pi)\, dZ$$

$$\left[\frac{\operatorname{Cos}\phi}{(1 - b^2a^2)} \quad \frac{j4J_1 \operatorname{Sin}\theta (b^2a^2 \operatorname{Cos}\phi - 2)}{J_o b^2a^2 (4 - b^2a^2)}\right]$$

Now

$$A'_\theta = \frac{e^{-jkr}}{4\pi r} e^{jZd} N'_\theta$$

$$A'_\phi = \frac{e^{-jkr}}{4\pi r} e^{jZd} N'_\phi$$

where $d = k \operatorname{Cos}\theta$.

The vector-potential components are

$$A_\theta = \int_{-\ell}^{\ell} A'_\theta$$

$$A_\phi = \int_{-\ell}^{\ell} A'_\phi$$

The total slot length being $2l$. Before these integrations can be performed, a distribution function f or J (Z,π)

must be specified.

Let

$$J\ (Z, \pi\ ) = J_m(\ \pi\ )\ \mathrm{Sin}\ \beta\ (1 - |Z|\ )$$

$J_m(\ \pi\ )$ = Maximum value of J (as Z varies) at $\phi = \pi$

$\beta = \frac{2\pi}{\lambda_s}$, where λ_s is the wave length at slot region. This current density distribution about the circumference of the cylinder in the Z-direction is identical to the method one of a field distribution along a slot in the axial direction. The current density is zero at both ends of the slot. All the current densities are confined in the circumference of the slot region. Substituting N'_θ , A'_θ into the integrating equation of A_θ ; N'_ϕ , A'_ϕ into A_ϕ , and integrating yield* the result

$$A_\theta = \frac{e^{-jkr}}{4\ \pi\ r}\ N_\theta$$

$$A_\phi = \frac{e^{-jkr}}{4\ \pi\ r}\ N_\phi$$

where

$$N_\theta = \left[- \frac{4\ ba^2 J_o^2 J_m(\ \pi)\ \mathrm{Sin}\ ba\,\pi}{(d^2 - \beta^2)} \right] \left[\mathrm{Sin}\,\phi\ \mathrm{Cos}\,\theta \right]$$

$$\left[\mathrm{Cos}\,\beta 1 - \mathrm{Cos}\ d1 \right] \left[\frac{1}{(1-b^2a^2)} + \frac{j4\ J_1 \mathrm{Sin}\,\theta\ \mathrm{Cos}\,\phi}{J_o\ (4 - b^2a^2)} \right]$$

---

*Ref. (2), sec. VI.

$$N_\phi = \left[- \frac{4\beta ba^2 J_o^2 J_m(\pi) \operatorname{Sin} ba\pi}{(d^2 - \beta^2)} \right] \left[\operatorname{Cos}\beta l - \operatorname{Cos} dl \right]$$

$$\left[\frac{\operatorname{Cos}\phi}{(1 - b^2a^2)} + \frac{j4J_1 \operatorname{Sin}\theta \, (b^2a^2\operatorname{Cos}^2\phi - 2)}{J_o \, b^2a^2 \, (4 - b^2a^2)} \right]$$

N_ϕ and N_θ are radiation vectors of ϕ and θ components respectively.

The radiation field is given by Eqs. (12) and (13)

$$E_\theta = - j\,\omega\mu\, A_\theta \tag{35}$$

$$E_\phi = - j\,\omega\mu\, A_\phi \tag{36}$$

while F_ϕ and F_θ are zero in this case.

6. Expression for $J_m(\pi)$, λ_1, β, f_c.

Determination of $J_m(\pi)$

From the circumferential current-density distribution along a uniformly-fed infinite slotted cylinder antenna it was found* that

$$J(\pi) = \frac{-jVP(\pi)}{2\pi\eta a}$$

where $P(\pi) = a_o + \sum_{n=1}^{m} (-1)^n a_n g_n - 2.772\ ka + 4ka \sum_{n=1}^{m} (-1)^{n+1}$

*Ref. (2), sec. III.

$\frac{g_n}{n}$ is the intrinsic impedance of free space = 120π

and

$$a_n = a_n' + a_n''$$

$$a_n' = \frac{2J_n\,(ka)}{J_n'\,(ka)} \qquad a_n'' = \frac{-2\,H_n^{(2)}\,(ka)}{H_n^{(2)\prime}\,(ka)}$$

For n = 0 omit the factor 2.

$$g_n = \frac{\sin n\pi}{n\pi} = 0$$

$$P(\pi) = a_o - 2.772\ ka$$

$$= a_o' + a_o'' - 2.772\ ka$$

$J_m\,(\pi)$ can be expressed in terms of the input voltage. The specified current-density distribution is

$$J\,(Z, \Phi) = J_m\,(\pi)\,\sin\beta(1 - |Z|)\,\cos ab\,(\pi - \Phi)$$

Therefore

$$J\,(0, \pi) = J_m\,(\pi)\,\sin\beta 1$$

The circumferential current density distribution is to be matched as closely as possible with that of the corresponding infinite cylinder, uniformly fed. Therefore,

$$J\,(0, \pi) = \frac{-jVP\,(\pi)}{2\,\pi\,\eta\,a}$$

V being the applied voltage at Z = 0. Equating the above two expressions for $J(0, \pi)$ and solving for $J_m(\pi)$ gives

$$J_m(\pi) = \frac{-j\,VP(\pi)}{2\pi\eta\, a\, \mathrm{Sin}\,\beta l}$$

As Im $P(\pi)$ is negligible compared with Re $P(\pi)$*

$$J_m(\pi) = \frac{-jV\, \mathrm{Re}\, P(\pi)}{2\pi\eta\, a\, \mathrm{Sin}\,\beta l}$$

Determination of the wave length λ_1 around cylinder.

λ_1 is to be found from the current-density distribution of the infinite cylinder. For $ab < 0.5$, satisfactory results are obtained as the distribution is approximately consinusoidal as assumed. In reference (2) is shown that $J(Z,\phi) = J(Z,\pi)\,\mathrm{Cos}\, bl'$. For $\phi = \frac{1}{2}\phi_1 = \phi_0$, $l' \to a\pi$, therefore, $J(0,\frac{1}{2}\phi_1) = J(0,\pi)\mathrm{Cos}\, ab$. As Im $P(\frac{1}{2}\phi_1)$, Im $P(\pi)$ are negligible compared with Re $P(\frac{1}{2}\phi_1)$, Re $P(\pi)$ respectively, therefore,

$$\mathrm{Re}\, P(\tfrac{1}{2}\phi_1) = \mathrm{Re}\, P(\pi)\,\mathrm{Cos}\, ab$$

yields reasonable values of b, where ϕ_1 is the slot angle $= 2\phi_0$ and $b = \frac{2\pi}{\lambda_1}$.

Determination of $\beta = \frac{2\pi}{\lambda_s}$ with λ_s denoting the wave length of slot region.

In order to find the phase constant β it is necessary first to find the input impedance of the slot. The slot is viewed as a loaded transmission line. The slot width is very small compared with the wave length, and the metal is

* Ref. (2), sec. III.

assumed to be perfectly conducting. The assumption is made that Y and Z are independent of the axial coordinate Z, i.e. that the line is uniform. As in ordinary uniform line theory the propagation constant and the characteristic impedance can be expressed in terms of Y and Z as follows.

If

$$\sigma = \alpha + j\beta = \sqrt{YZ}$$

$$Z_0 = \sqrt{Z/Y}$$

where $Y = G + jB$, $Z = jX = j\,\omega L$, it is easily shown* that the phase-shift constant

$$\beta = \frac{X}{2}\left(\frac{G}{\alpha}\right) = \sqrt{\frac{X}{2}}\sqrt{(B^2 + G^2)^{\frac{1}{2}} + B} = \frac{2\pi}{\lambda_s}$$

For frequencies considerably above the cut-off frequency, G become negligible, and

$$\beta = \sqrt{BX}$$

For frequencies near or below the cut-off frequency, the G is found from

$$G = \mathrm{Re}\,\frac{J(\frac{1}{2}\,\phi_1\,)}{V} = \mathrm{Im}\,\frac{P(\frac{1}{2}\,\phi_1\,)}{2\pi\eta\, a}$$

The slot distributed susceptance is

$$B = \omega C_s + B'$$

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* Ref. (2), sec. II.

B' is the susceptance per unit length due to the conduction current density and C_s is the capacitance per unit length between the slot faces.

C_s is determined by the relation

$$C_s = \frac{t}{W} \epsilon$$

where t is the thickness of the cylinder at the slot and W is the slot width, and ϵ is the permittivity of free space.

As B' is evaluated at the slot surface, all displacement current except that flowing directly between the slot faces is accounted for in the evaluation of B'.

From the infinite cylinder, the current distribution around cylinder*

$$J(\phi) = -j \frac{VP(\phi)}{2\pi\eta a}$$

At the slot $\phi = \frac{1}{2}\phi_1$

and $J(\frac{1}{2}\phi_1) = -j\frac{VP(\frac{1}{2}\phi_1)}{2\pi\eta a}$, $\text{Im}(\frac{1}{2}\phi_1) = -\frac{V\text{Re}(\frac{1}{2}\phi_1)}{2\pi\eta a}$

Therefore

$$B' = -\frac{\text{Im}\, J(\frac{1}{2}\phi_1)}{V} = \frac{\text{Re}P(\frac{1}{2}\phi_1)}{2\pi\eta a}$$

---

*Ref. (2) sec. III

Then

$$B = wC_s + \frac{\mathrm{Re}P(\frac{1}{2}\phi_1)}{2\pi\eta a}$$

where

$$C_s = \frac{t}{W}\epsilon$$

The inductance L per unit length is expressed in Ref. 2 Sec. V as

$$L = \frac{1}{v^2 C_s + \frac{v b_1}{2\pi\eta}}$$

where v is velocity of light $3\text{x}10^8$ m/sec.

If P_{ka} denotes $\mathrm{Re}P(\frac{1}{2}\phi_1)$ evaluated at ka,

then

$$b_1 = \frac{5}{6}(3P_1 - 4P_{0.05} - P_{0.2})$$

$$P_1 = \mathrm{Re}\,P(\tfrac{1}{2}\phi_1)\Big|_{ka = 1}$$

$$P_{0.05} = \mathrm{Re}\,P(\tfrac{1}{2}\phi_1)\Big|_{ka = 0.05}$$

$$P_{0.2} = \mathrm{Re}\,P(\tfrac{1}{2}\phi_1)\Big|_{ka = 0.2}$$

where $P(\frac{1}{2}\phi_1) = a_o + \sum_{n=1}^{m} a_n g_{2n} + 4ka \sum_{n=m+1}^{\infty} \frac{g_{2n}}{n}$,

m is a sufficiently large so that $a_n = \frac{4ka}{n}$ for $n > m$

$$X = \omega L$$

From the calculated values of B, G, and X, the β is obtained from

$$\beta = \sqrt{\frac{X}{2}}\sqrt{(B^2 + G^2)^{\frac{1}{2}} + B}$$

Cut-off frequency f_c

The cut-off frequency is defined as that frequency at which the wave length in the slot would become infinite if there were no radiation. If there were no radiation, G would be zero. Then

$$\alpha = \sqrt{\frac{X}{2}}\sqrt{(B^2 + G^2)^{\frac{1}{2}} - B} = 0 \text{*}$$

and $\beta = \sqrt{BX}$. It follows that $\beta = 0$ and λ_s becomes infinite when $B = 0$. In the actual case G will not be zero. Therefore, for $f = f_c$, $\alpha = \beta = \sqrt{\frac{1}{2}GX}$. Thus the cut-off frequency would be defined as that frequency at which $\alpha = \beta$ where $B = 0$

Therefore,

$$\omega_c C_s + \frac{\text{Re } P(\frac{1}{2}\phi_1)}{2\pi\eta a} = 0$$

or

$$\frac{\text{Re } P(\frac{1}{2}\phi_1)}{ka} = -0.711 \times 10^{12} C_s$$

By plotting the curve Re $P(\frac{1}{2}\phi_1)$ versus ka, it is an easy matter to find the value of ka which satisfies the above expression. The cut-off frequency in terms of $k_c a$ is

$$f_c = \frac{47.75}{a} k_c a \quad \text{mc}$$

*Ref. (2) sec. II.

In plotting of the field patterns, the MKS system of units is used. For other variables convention units are used, such as β in neper/m, G in mho/m, etc.

7. Field patterns at f= 900 mc

The field patterns are made under the conditions that $a \ll \lambda$, $\dfrac{\text{Sin } n\phi_0}{n\phi_0} \longrightarrow 1$, and ab < 0.5, which satisfy the assumptions in both methods.

Let

$$f = 900\text{mc}$$

$$2l = 1.41\ \lambda$$

$$W = 0.00111\ \lambda$$

$$a = 0.0318\ \lambda$$

$$t = 0.0075\ \lambda$$

and the calculated values become

$$ab = 0.302$$

$$ka = 0.2$$

$$f_c = 820\text{ mc}$$

$$k = 18.85$$

$$C_s = 5.98\text{x}10^{-11}$$

$$b_1 = 174$$

$$L = 3.7\text{x}10^{-8}$$

$$X = 209$$

$$B = 0.085$$

$G = 0.0119$

$\beta = 4.42$

$\lambda = 33.3$ cm

Principal H-plane

The principal H-plane is obtained by setting $\theta = 90^{o}$ and the given data into the expression for radiation. The field pattern is a variation of the field with different values of ϕ at constant r. From Eq. (34) of method (1) and Eq. (35), (36) of method (2) it is easy to draw the field patterns of the principal H-plane.

Method (1) Eq. (34) becomes

$$\left| E_{\phi} \right| = \left| A \sum_{n=0}^{\infty} \frac{\epsilon_n j^n \operatorname{Cos} n\phi}{H_n^{(2)\prime}(0.2)} \right|$$

where A is a constant.

Method (2) Eqs. (35), (36) become

$$E_{\theta} = 0$$

$$\left| E_{\phi} \right| = \left| B \left[1.1 \operatorname{Cos}\phi - j0.562(2 - 0.091 \operatorname{Cos}^2\phi\ ) \right] \right|$$

where B is a constant.

Principal E-plane

The principal E-plane is obtained by setting $\phi = 0$, or $\phi = 90^{o}$ and the given data into the expression for the radiation. The field pattern is a variation of the field

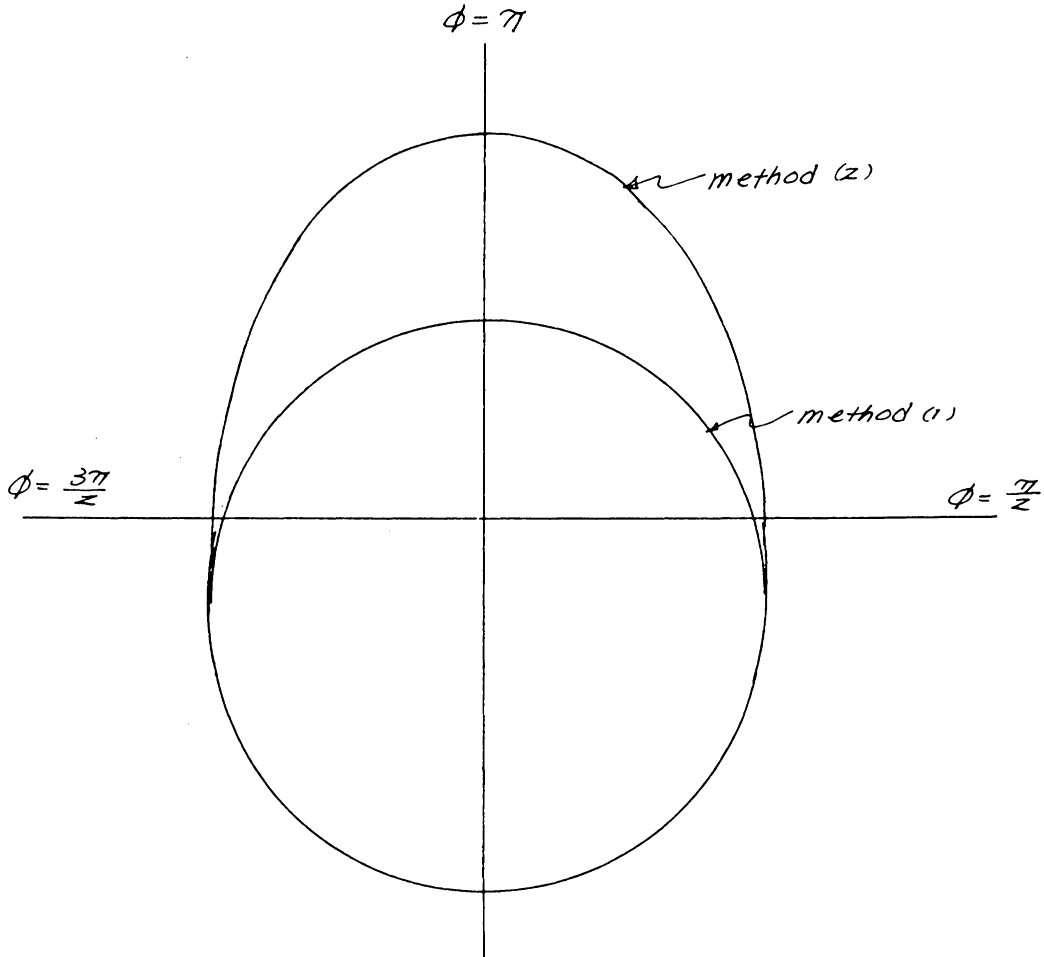

Fig. (6)

Comparison of Principal H-plane

with different values of θ , at constant r. From Eq. (34) of the method (1) and Eqs. (35), (36) of the method (2) it is easy to draw the field pattern of the principal E-plane.

Method (1) Eq. (34) becomes

$$|E_\phi| = \left| A' \frac{\cos(4.52 \cos\theta) - 0.475}{18 - 355 \cos^2\theta} \sum_{n=0}^{\infty} \frac{\epsilon_n j^n \cos n\phi}{H_n^{(2)'}(0.2 \sin\theta)} \right|$$

where A' is a constant.

Method (2) Eqs. (35), (36) become $E_\theta = 0$

$$|E_\phi| = C \left| \frac{0.524 - \cos(4.52 \cos\theta)}{355 \cos^2\theta - 18} \left[1.1 - j0.538(2 - 0.091 \cos^2\theta)\right] \right|$$

where C is a constant. (see Fig, 7)

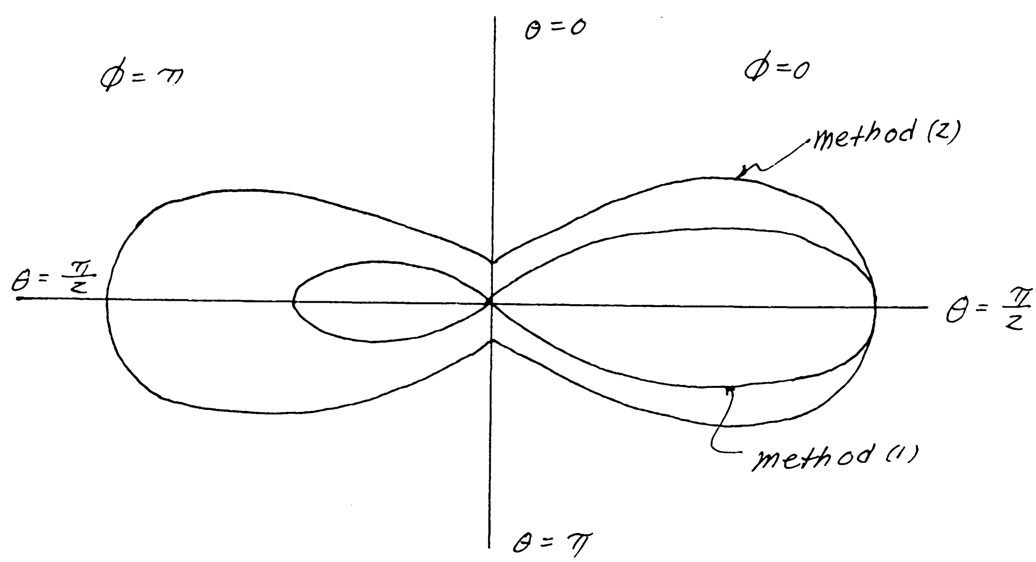

Fig. (7)

Comparison of Principal E-plane

8. Comparison of methods (1) and (2)

The principal H-plane

Method (1)

$$\left| E_\phi \right| = \left| \frac{A'}{a} \sum_{n=0}^{\infty} \frac{\epsilon_n \, j^n \, \mathrm{Cos}\, n\phi}{H_n^{(2)'}\,(ka)} \right| \qquad \text{(a)}$$

where $A'/a = A$

Method (2)

$$\left| E_\phi \right| = aB' \left| \frac{\mathrm{Cos}\,\phi}{1 - b^2a^2} + \frac{j4J(b^2a^2\mathrm{Cos}^2 - 2)}{J_o \, b^2a^2 \, (4 - b^2a^2)} \right| \qquad \text{(b)}$$

where $aB' = B$, as $ka \longrightarrow 0, ab << 0.5$, Eq. (a) becomes

$$\left| E_\phi \right| = \left| \frac{A'}{a} \frac{1}{H_o^{(2)'}\,(ka)} \right| = \left| \frac{A'}{a} \frac{1}{J_o'\,(ka) - jN_o'\,(ka)} \right|$$

$$= \left| \frac{A'}{-j\,\frac{2a}{ka}} \right| = A''k = \text{Constant} \qquad \text{(c)}$$

In Eq. (b), as $ka \longrightarrow 0$, $ab << 0.5$

$$\left| E_\phi \right| = B'a \left| \mathrm{Cos}\,\phi + \frac{jka\,(-2)}{4\,b^2a^2} \right|$$

$$= B'a \left| \mathrm{Cos}\,\phi - \frac{jk}{2b^2a} \right|$$

For $a \longrightarrow 0$, $\frac{k}{2b^2a} >> \mathrm{Cos}\,\phi$,

Therefore,

$$\left| E_\phi \right| = B'a \left| \frac{k}{2b^2a} \right| = B''\,k = \text{Constant} \qquad \text{(d)}$$

It is obvious from Eqs. (c) and (d) that, as $ka \longrightarrow 0$

with ab << 0.5, the field derived from both methods approachs a constant, and the field pattern is a circle. In this case, the slot antenna behaves like a dipole antenna.

The coincidence of these two methods could be anticipated. In method (1), the assumption is made that the slot angle is very small so that $\frac{\operatorname{Sin} n\phi_o}{n\phi_o} \doteq 1$, and a $\longrightarrow$ 0. In method (2) the assumption is made that ab < 0.5, and good results can be obtained for ab << 0.5. Therefore, under these conditions as ka $\longrightarrow$ 0, ab << 0.5, which satisfy both assumptions in these two methods.

The condition for ka $\longrightarrow$ 0, i.e. a $\longrightarrow$ 0 for a certain frequency, reduces the series of the Hankel functions to the n = 0 term. Therefore, for a finite small slot angle $\frac{\operatorname{Sin} n\phi_o}{n\phi_o} \underset{n\phi_o \to 0}{=} 1$. The assumption in method (1) is satisfied under the condition ka $\longrightarrow$ 0.

The principal E-plane ($\phi = 0$, or $\phi = \pi$)

Method (1)

$$|E_\phi| = \left| C'' \frac{\operatorname{Cos}(kl \operatorname{Cos}\theta) - \operatorname{Cos}\beta l}{\beta^2 - k^2 \operatorname{Cos}^2\theta} \sum_{n=0}^{\infty} \frac{\epsilon_n j^n \operatorname{Cos} n\phi}{H_n^{(2)\prime}(ka \operatorname{Sin}\theta)} \right|$$

where C" is a constant.

Method (2)

$$|E_\phi| = \left| D'' \frac{\operatorname{Cos}(kl \operatorname{Cos}\theta) - \operatorname{Cos}\beta l}{\beta^2 - k^2 \operatorname{Cos}^2\theta} \left[\frac{\operatorname{Cos}\phi}{1 - b^2a^2} \quad \frac{j4J_1}{J_o} \quad \frac{(b^2a^2 - 2)}{b^2a^2(4 - b^2a^2)} \right] \right|$$

where D is a constant.

As ka $\longrightarrow$ 0, ab $\ll$ 0.5, the field pattern in method (1) becomes

$$|E_\phi| = \left| A''' \frac{\text{Cos}\,(kl\ \text{Cos}\,\theta) - \text{Cos}\,\beta l}{\beta^2 - k^2 \text{Cos}^2\theta} \right|$$

where $A''' = A''k$, and the field pattern in method (2) becomes

$$|E_\phi| = \left| B''' \frac{\text{Cos}\,(kl\ \text{Cos}\,\theta) - \text{Cos}\,\beta l}{\beta^2 - k^2 \text{Cos}^2\theta} \right|$$

where $B''' = B''k$.

The field patterns derived from these two methods are exactly the same, provided that ka $\rightarrow$ 0, ba $\ll$ 0.5 and the field is much like that of a dipole antenna.

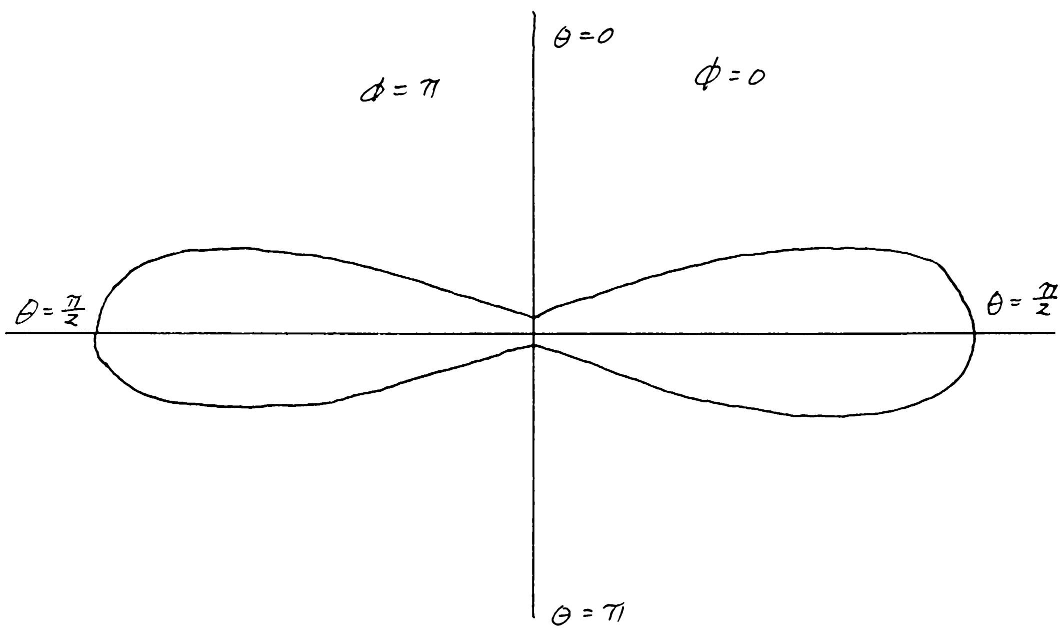

Fig. (8) Principal E-plane as ka $\longrightarrow$ 0

9. Field pattern with change of frequencies

The frequencies ranged from 500 mc to 1500 mc; only 500 mc, 700 mc, 900 mc, 1100 mc, and 1500 mc are used. In all of these cases, same dimensions and the same excitation are assumed. Method one is adopted in these cases.

List of data

$f = 500$ mc

$\lambda = 60$ cm

$a = 0.0178\ \lambda$

$W = 0.000616\ \lambda$

$t = 0.00416\ \lambda$

$2l = 0.8\ \lambda$

The calculated values:

$ka = 0.111$

$C_s = 5.98\text{x}10^{-11}$

$ab = 0.105$

$b_1 = 174$

$L = 3.7\text{x}10^{-8}$

$X = 114$

$B = -\ 0.924$

$G = -\ 0.00715$

$f_c = 820$ mc

$\beta = 0.0397$

The principal H-plane:

$$|E_\phi| = \left| C \sum_{n=0}^{\infty} \frac{\epsilon_n \, j^n \operatorname{Cos} n\phi}{H_n^{(2)'}(0.111)} \right|$$

The principal E-plane:

$$|E_\phi| = \left| C' \frac{\operatorname{Cos}(2.5\operatorname{Cos}\theta)-0.9999973}{(0.00158 - 108 \operatorname{Cos}^2\theta)} \sum_{n=0}^{\infty} \frac{\epsilon_n \, j^n \operatorname{Cos} n\phi}{H_n^{(2)'}(0.111 \operatorname{Sin}\theta)} \right|$$

where C and C' are constants.

List of data:

$f = 7000$ mc

$\lambda = 42.8$ cm

$a = 0.0247\ \lambda$

$t = 0.00584\ \lambda$

$W = 0.000864\ \lambda$

$2l = 1.12\ \lambda$

The calculated values:

$ka = 0.155$

$C_s = 5.98 \times 10^{-11}$

$ab = 0.22$

$b_1 = 174$

$L = 3.7 \times 10^{-8}$

$X = 163$

$B = -0.135$

$G = -0.00955$

$f_c = 820$ mc

$\beta = 0.1435$

The principal H-plane:

$$\left|E_\phi\right| = \left| D \sum_{n=0}^{\infty} \frac{\epsilon_n \, j^n \cos n\phi}{H_n^{(2)'}(0.155)} \right|$$

The principal E-plane:

$$\left|E_\phi\right| = \left| D' \frac{\cos(3.53\cos\theta)-0.999}{0.0206 - 216\cos^2\theta} \sum_{n=0}^{\infty} \frac{\epsilon_n \, j^n \cos n\phi}{H_n^{(2)'}(0.155\sin\theta)} \right|$$

where D and D' are constants.

List of data:

$f = 1100$ mc

$a = 0.0388\ \lambda$

$t = 0.00915\ \lambda$

$W = 0.00135\ \lambda$

$2l = 1.75\ \lambda$

$\lambda = 27.3$ cm

The calculated values:

$ka = 0.244$

$C_s = 5.98\times10^{-11}$

$ab = 0.357$

$b_1 = 174$

$L = 3.7\times10^{-8}$

$X = 256$

$B = 0.27$

$G = -0.016$

$f_c = 820$ mc

$\beta = 8.32$

The principal H-plane:

$$|E_\phi| = \left| E \sum_{n=0}^{\infty} \frac{\epsilon_n \, j^n \, \mathrm{Cos}\, n\phi}{H_n^{(2)'}(0.244)} \right|$$

$$|E_\phi| = \left| E' \, \frac{\mathrm{Cos}(5.55\,\mathrm{Cos}\,\theta) - 0.414}{69.2 - 534\,\mathrm{Cos}^2\theta} \sum_{n=0}^{\infty} \frac{\epsilon_n \, j^n \, \mathrm{Cos}\, n\phi}{H_n^{(2)'}(0.244\,\mathrm{Sin}\,\theta)} \right|$$

where E and E' are constants.

List of data:

$f = 1500$ mc

$\lambda = 20$ cm

$a = 0.053\,\lambda$

$t = 0.0125\,\lambda$

$W = 0.00185\,\lambda$

$2l = 2.4\,\lambda$

The calculated values:

$ka = 0.333$

$C_s = 5.98\mathrm{x}10^{-11}$

$ab = 0.523$

$b_1 = 174$

$L = 3.7\mathrm{x}10^{-8}$

$X = 349$

$B = 0.6$

$G = -\,0.021$

$\beta = 14.3$

The principal H-plane:

$$\left|E_\phi\right| = \left|F \sum_{n=0}^{\infty} \frac{\epsilon n \; j^n \; \text{Cos } n\phi}{H_n^{(2)\prime}(0.333)}\right|$$

The principal E-plane:

$$\left|E_\phi\right| = \left|F' \; \frac{\text{Cos}(7.54\text{Cos}\,\theta\,)-0.956}{205 - 985 \; \text{Cos}^2\theta} \sum_{n=0}^{\infty} \frac{\epsilon n \; j^n \; \text{Cos } n\phi}{H_n^{(2)\prime}(0.333 \; \text{Sin}\,\theta\,)}\right|$$

where F and F' are constants.

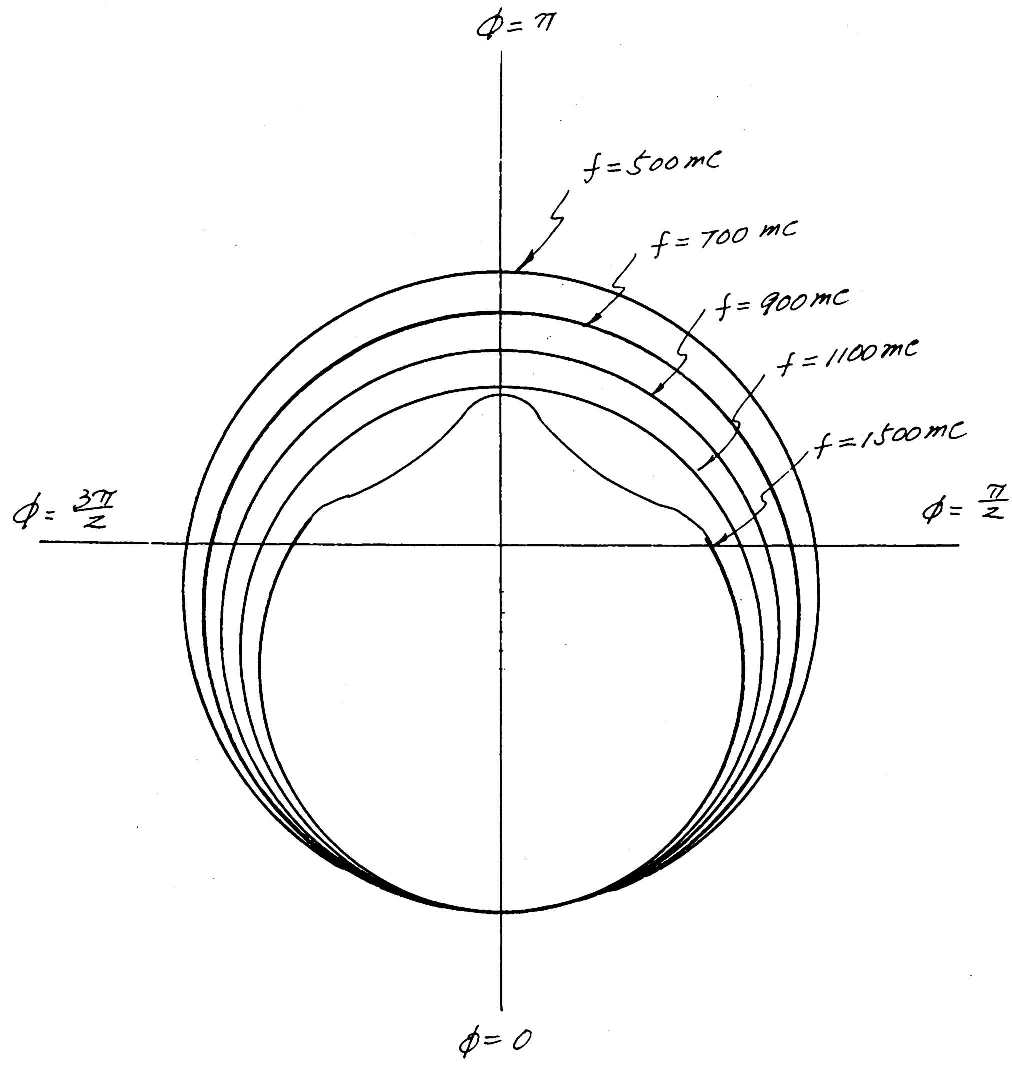

Fig. (9)

Principal H-plane varies with the frequencies f

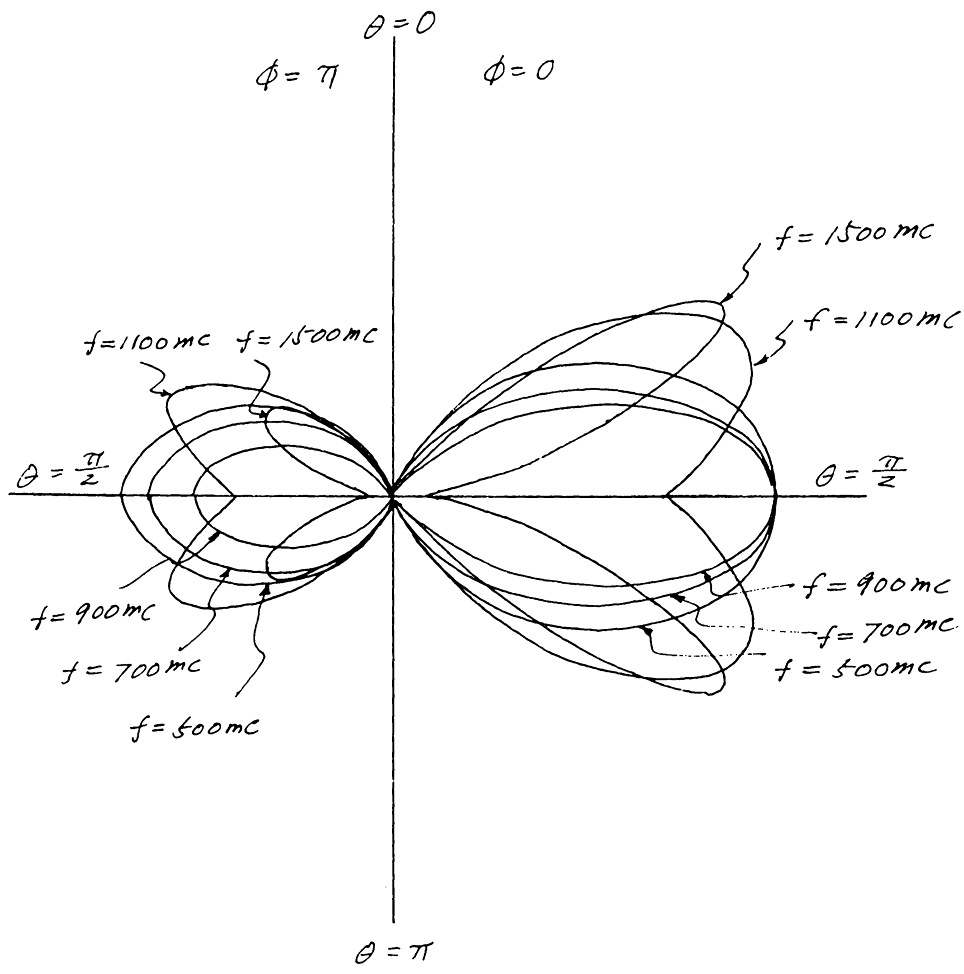

Fig. (10)

Principal E-plane varies with the frequencies f

The cut-off frequency is defined as that frequency at which $\alpha = \beta$. Below cut-off B is negative, rapidly becoming a large negative quantity as the frequency is reduced and α becomes very much greater than β. For frequencies sufficiently above the cut-off frequency B is positive, and β becomes very much greater than α .

The field patterns only change slightly for operating frequencies not far beyond that of the cut-off frequencies. For frequencies much below that of the cut-off frequency, the principal field pattern is approximately circular. For frequencies much higher that the cut-off frequency, errors occur for the assumption that $ka \longrightarrow 0$,
where $k = \frac{2\pi f}{V}$.

When β is comparable with k, the E-field pattern is no longer like that of a dipole antenna, (maximum field occurs not at $\theta = 90^{\circ}$). If this is the case, as the frequencies are higher than the cut-off frequency and ka becomes large, the assumption made before is invalid.

The maximum field of the principal E-field could be found by differentiating the field with respect to θ . As the series form of the Hankel function changes almost linearly with θ, it is convenient to treat the derivative of the series as a constant. From this it is found that maximum field occurs at $\theta = \mathrm{Cos}^{-1} \frac{\beta}{k}$. Unless $\beta \ll k$, the maximum field would not occur at $\theta = 90^{\circ}$.

10. Two slotted antennas

The radiation field produced by two slots in the same direction in a cylinder as shown in Fig. (11), could be obtained by superposition. With the same assumptions made as far a single slot, and the coordinates as shown in Fig. (11), results could be found by the following procedures.

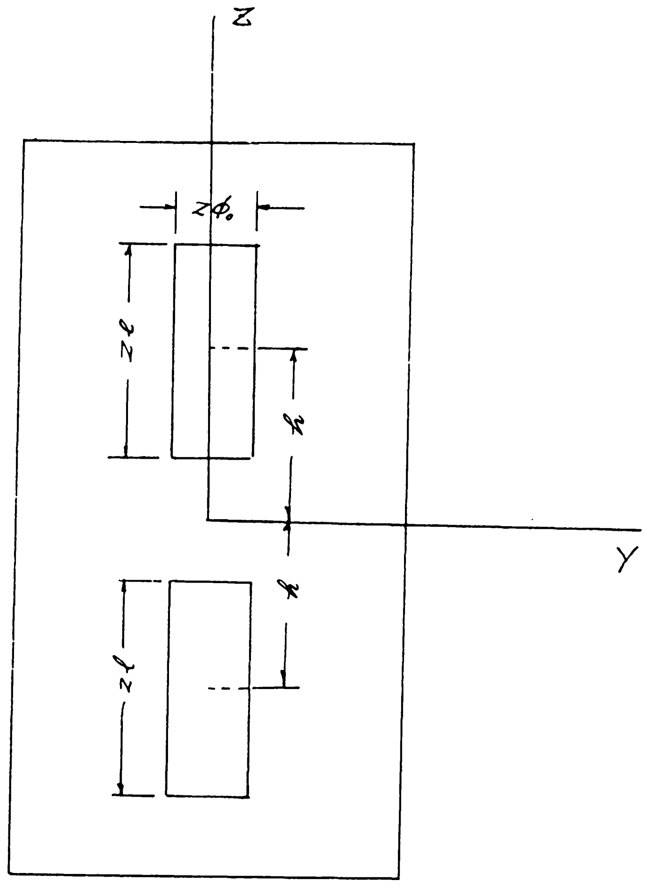

Fig. (11)

Two slotted antenna with slots in the same orientation

It is assumed in the apertures

$$E\,(a,\phi,Z) = \frac{V}{2\,\phi_o\,a}\,\mathrm{Sin}\,\beta\,(h + l - |Z|)$$

under the condition that $2l = \frac{\lambda_s}{2}$, that is, the slot length is one half of the wave length of the slot region, where $\beta = \frac{2\pi}{\lambda_s}$, h is the distance from the center of the slot to the origion.

The transforms of Eqs. (24) and (25) become

$$\bar{E}\,(n,w) = \frac{2V\mathrm{Sin}\,(n\,\phi_o\,)}{a\,n\,\pi\,\phi_o}\,(\mathrm{Cos}\;wh\;\mathrm{Cos}\;wl)\;\frac{\beta}{\beta^2 - w^2}$$

Eqs. (30) and (31) yield

$$f_n(w) = 0$$

$$g_n(w) = \frac{2\beta\,V\mathrm{Cos}\;wh\;\mathrm{Cos}\;wl}{\pi\,a\sqrt{k^2 - w^2}\;H_n^{(2)\prime}(a\sqrt{k^2 - w^2})\;(\beta^2 - w^2\,)}$$

for small slot angles.

Since $-k\mathrm{Cos}\theta = w$, $k^2\mathrm{Cos}^2\theta = w^2$, $\sqrt{k^2 - w^2} = k\mathrm{Sin}\theta$,

the radiation field is

$$E_\phi = \frac{2V\beta\;e^{-jkr}\;\mathrm{Cos}(kh\mathrm{Cos}\,\theta\,)\mathrm{Cos}(kl\mathrm{Cos}\,\theta\,)}{\pi^2 ra\,(\;\beta^2 - k^2\mathrm{Cos}^2\theta\;\;)}$$

$$\sum_{n=0}^{\infty}\frac{\epsilon n\;j^n\;\;\mathrm{Cos}\;n\phi}{H_n^{(2)\prime}(ka\mathrm{Sin}\,\theta\;\;)}$$

This could be checked by setting $h = 0$ and $l = \frac{\lambda_s}{4}$ in this field and the field derived in Eq. (34).

Field pattern

The principal H-plane is the same as for a single slot antenna. This is anticipated from the superposition theory. The principal E-plane is much like that of the pattern of end-fire arrays of point sources. Use the same dimensions as for a single slot antenna. And let

$f = 900$ mc

$b = \lambda = 33.3$ cm

$l = \dfrac{\lambda_s}{4} = 35.6$ cm

The principal E-plane is plotted as

$$\left|E_\phi\right| = \left| \frac{\text{Cos}(6.28\text{Cos}\,\theta)\text{Cos}(6.97\text{Cos}\,\theta)}{18 - 354\text{Cos}^2\theta} \sum_{n=0}^{\infty} \frac{\epsilon_n\, j^n \text{Cos}\, n\phi}{H_n^{(2)\prime}(0.2\text{Sin}\,\theta)} \right|$$

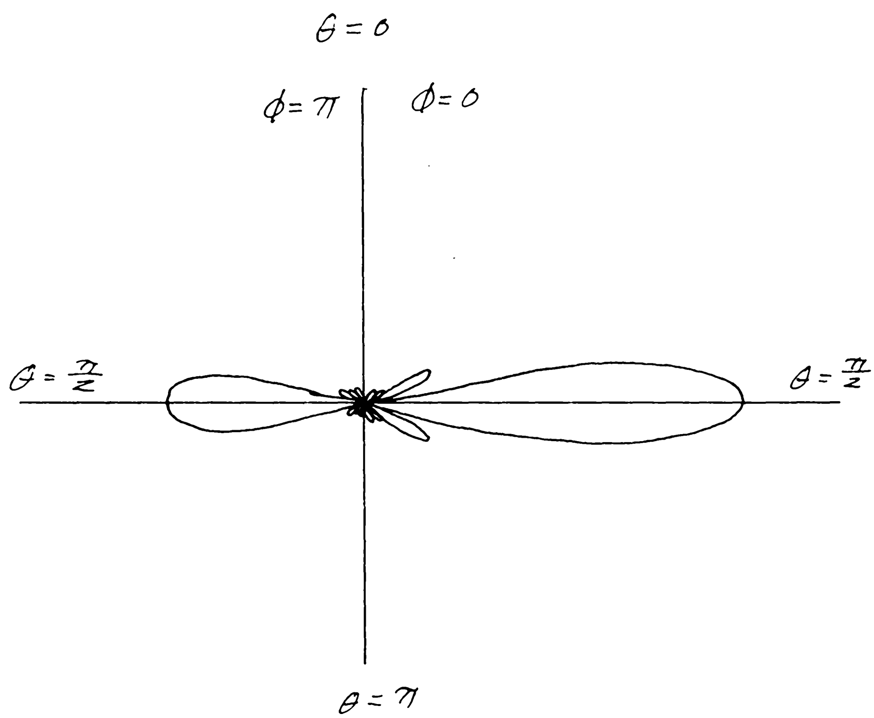

Fig. (12)

Principal E-plane of two slotted antennas with two slots oriented in the same direction

Radiation field produced by two slots on opposite sides of a cylinder could be obtained by the superposition method. With the same assumptions made as for the single slot antenna, and with the coordinates taken as shown in Fig. (13), the result is found by the following procedures.

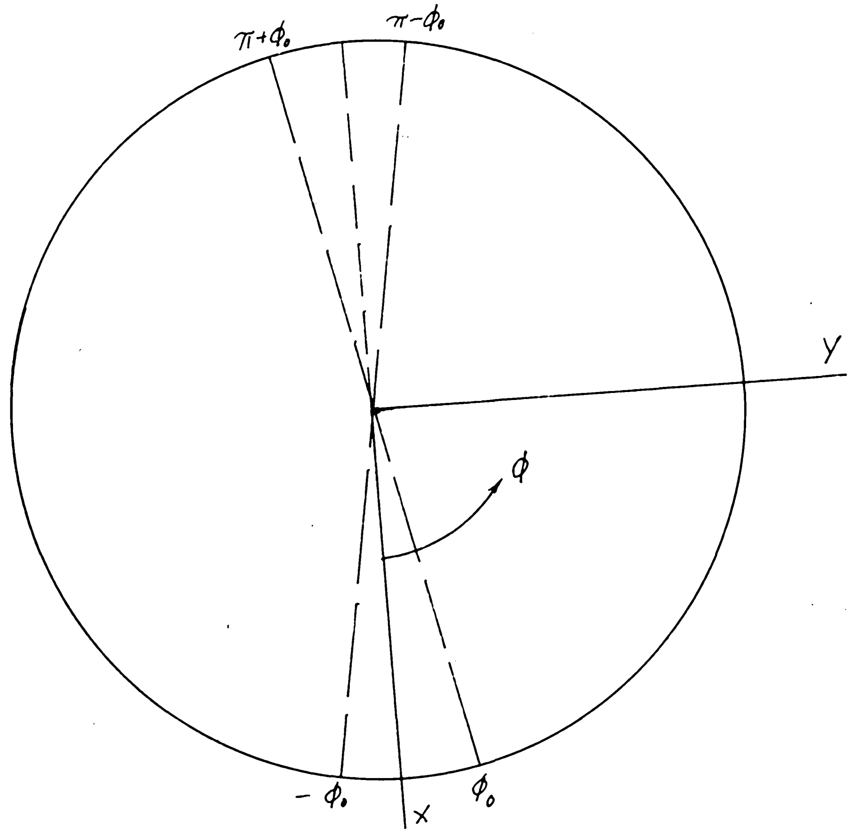

Fig. (13)

Two slotted antennas with slots on opposite sides

It is assumed in the apertures

$$E_\phi(a, \phi, Z) = \frac{V}{2\,\phi_0\, a} \operatorname{Sin} \beta (1 - |Z|) \quad \left[\begin{array}{l} -\phi_0 < \phi < \phi_0 \\ \pi - \phi_0 < \phi < \pi - \phi_0 \end{array} \right.$$

$$E_z(a, \phi, Z) = 0$$

The transforms of Eqs. (24) and (25) become (for small ϕ_0)

$$\bar{E}_\phi(n, w) = \frac{2V\beta\,(\operatorname{Cos} wl - \operatorname{Cos} \beta l)}{\pi a\,(\beta^2 - w^2)} \qquad n \text{ even}$$

$$\bar{E}_\phi(n, w) = 0 \qquad n \text{ odd}$$

Finally the radiation field $E_\theta = 0$, and

$$E_\phi = \frac{2V\beta e^{-jkr} \operatorname{Cos}(kl\operatorname{Cos}\theta) - \operatorname{Cos} \beta l}{\pi^2 ra(\beta^2 - k^2 \operatorname{Cos} \quad)} \sum_{n=0}^{m} \frac{j^n \operatorname{Cos} n\phi}{H_n^{(2)\prime}(ka\operatorname{Sin}\theta)}$$

where m = even.

Field pattern of two slotted antennas

For the same dimensions as the single slot antenna and with f = 900 mc, the principal E-plane is much like that of a dipole antenna for small ka. As it is easily seen for small ka the predominant terms in the series are approximately constants for various value of θ. The principal H-plane is

$$|E_\phi| = \left| A' \sum_{n=0}^{m} \frac{\epsilon_n\, j^n \operatorname{Cos} n\phi}{H_n^{(2)\prime}(0.2)} \right|$$

where A' is a constant. The field pattern is almost a circle. This result might well be expected from the superposition theory. (see Fig. 14)

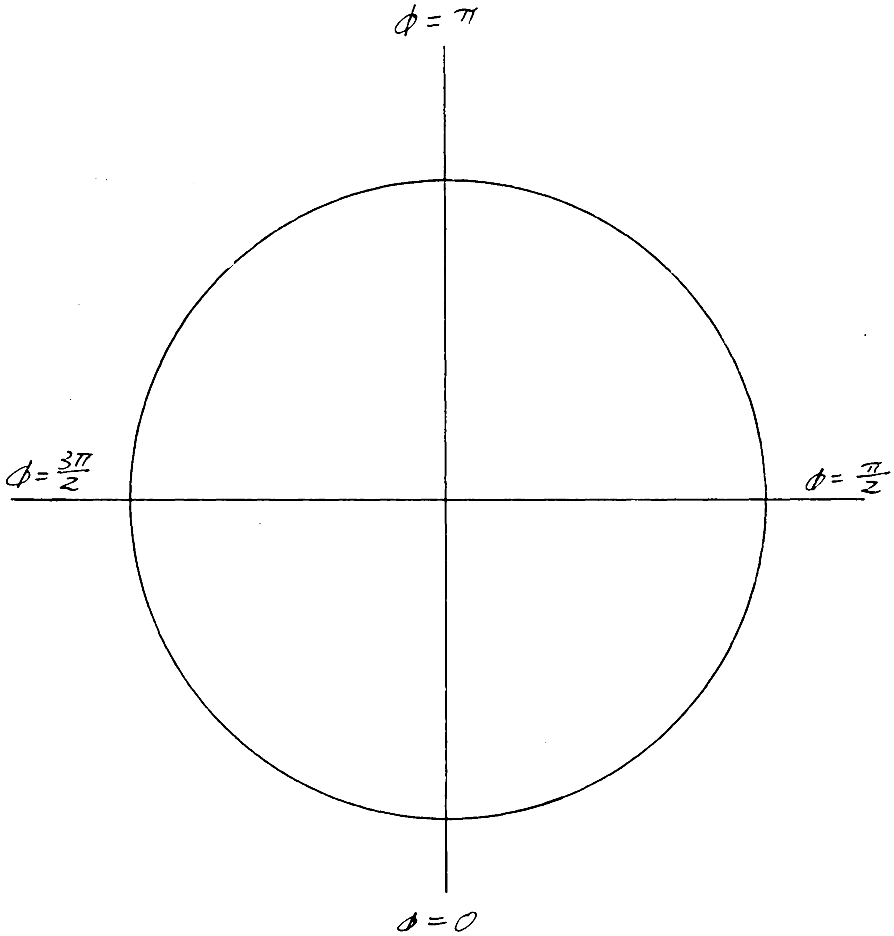

Fig. (14)

Principal H-plane of two slotted antennas with two slots oriented in diametrically opposite directions

11. Discussion

The results given in this paper are restricted to the far-zone field of various slots. The far-zone fields are given in terms of an infinite series of terms involving the reciprocal of the first derivative of the Hankel function where the argument are functions of ka Sin θ, where k = $2\pi/\lambda$ and a is the radius of the cylinder. This series converges rapidly for small value of ka Sin θ (for a certain value of θ this series converges rapidly for small value of ka). However, as the cylinder becomes larger, the series representation f or the far field coverges more slowly and a greater number of terms are required to approximate the sum of the infinite series to a given accuracy. Therefore, in the previous assumption ka is so small that only two terms of the Hankel are taken.

The disagreement between these two calculated field patterns might come from the assumption of a cosinusoidal current density distribution arround the circumference in the second method. Actually it could be imagined from the circumferential current density distribution of an infinite cylinder, uniformly fed, that the current distribution along the circumference is a series of sinusoidal functions. This series converges slowly. It is very difficult to find the vector potential in this series form. But if ab $\ll$ 0.5, the predominant term in the form of the current distribution is just a simple cosine function. With this restriction the radiation is much easier to handle by assuming a cosinusoidal current distribution along the circumference. However, the main lobes in the corresponding field patterns are approximately the same. The assumption of a cosinusoidal circumferential current distribution does not seriously limit the validity to find the radiation field.

The field pattern produced by two slotted antennas with these slots oriented in the same direction does not change greatly with the distance between the slots. The principal E-plane is identically the field pattern of the end-fire arrays of point sources. The field pattern produced by two slotted antennas with these slots diametrically opposite is a circle in the principal H-plane. This result is useful especially for TV radiation.

Method one assumed that ϕ_0 is small; therefore, for large ϕ_0 modification must be applied to the derivation of the radiation field. The same restriction is also presented in method two in finding the wave length λ around the cylinder.

From the agreement of these two methods, it is easy to design a slot antenna with the desired field pattern by assuming the approximate current distribution along the circumference, or a reasonable assumption can be made concerning the field distribution in the slot and the result can be obtained. The slotted cylinder antenna has many desirable properties. Its band width, while not large, is suitable for many communication purposes at very high and ultra-high frequencies. As such slots are easy to construct and excite, they are useful in application to microwave antenna design.

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