# ON THE VIBRATION ANALYSIS OF A COMPLEX FOUNDATION

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## II. Nomenclature.

$A_a^o$ = matrix of shape coefficients for pipe segment "a"

$\underline{B}_{oi}$ = matrix which transfers forces from mass "i" to origin

$E$ = modulus of elasticity

$F_i$ = force in i direction

$F_{ip}$ = force applied at point P in the i-direction

$F_{i,j}$ = force (or moment) on mass "i" in the j-direction (about j-axis)

$\underline{f}_i$ = column matrix of forces at mass "i"

$I$ = section moment of inertia

$\underline{I}$ = unit matrix

$I_{ii}$ = mass moment of inertia about i-axis

$I_{ij}$ = mass product of inertia

$I_{i,j}$ = principal moment of inertia for mass "i" about j-axis

$J$ = section polar moment of inertia

$\underline{K}$ = total stiffness matrix

$\underline{K}_a^o$ = $(\underline{A}_a^o)^{-1}$ = stiffness matrix for pipe segment "a" with forces and deflections referred to origin

KE = total kinetic energy of system

$\underline{K}_{ij}$ = $6 \times 6$ stiffness matrix containing forces and moments at mass "i" due to deflections and rotations at mass "j"

| | |
|---|---|
| $k$ | = flexibility factor |
| $k_{ij}$ | = elements of stiffness matrix |
| $k_{ij,rs}$ | = force on mass "i" in r-direction due to deflection of mass "j" in s-direction |
| $L$ | = Lagrangian( KE - U) |
| $L_i$ | = angular momentum about i-axis |
| $\underline{L}_{oj}$ | = rotational transformation relating components measured in inertial reference frame to a frame parallel to frame at origin |
| $\ell$ | = length of pipe segment |
| $\underline{M}$ | = mass matrix |
| $\underline{M}_i$ | = 6 × 6 inertial matrix for mass i |
| $m$ | = mass |
| $m_i$ | = principal moment of inertia for i = 1, 2, 3<br>= m for i = 4, 5, 6 |
| $m_{ij}$ | = elements of mass matrix |
| $N$ | = number of masses |
| PE | = total potential energy of system |
| $Q$ | = ratio of section moduli |
| $q_i$ | = generalized coordinate |
| $q_{j,s}$ | = generalized deflection of mass "j" in s-direction |
| $\underline{R}_{oj}$ | = matrix of direction cosines describing rotation of axes at mass "j" |
| $T_b$ | = in-plane bending moment |
| $T'_b$ | = transverse bending moment |

$T_i$ = moment about i-axis

$T_t$ = torsional moment

U = strain energy

$\underline{x}$ = column matrix of displacements

$\underline{x}_j$ = column matrix of deflections at mass "j"

$x_n, y_n, z_n$ = coordinates of point N

$\underline{\Delta} = \underline{K}^{-1}$ = total flexibility matrix

$\delta_i^j$ = linear deflection of mass "i" in j-direction

$\theta_i^j$ = rotation of mass "i" about j-axis

$\lambda_n$ = nth eigenvalue

$\psi$ = angle pipe segment makes with horizontal

$\omega$ = circular frequency (radians/second)

## III. Introduction.

The subject of structural vibrations is an imposing one, and several approaches to reducing the complexities involved in solving real problems have been presented in the literature. Probably the most practical of the various methods utilizes a mathematical model which represents the structure as a system of discrete masses interconnected by a network of either flexibilities or stiffnesses. The best method for formulating such a model for a given structure has not been clearly established and will not be discussed in detail here, but studies have been made concerning this subject (8).

The purpose of this paper is to present a detailed analysis of such a model utilizing matrix methods. Utilization of an automatic computer is mandatory for a problem of this type because of the large number of degrees of freedom necessary to describe a vibrating piping system. The pipe will be divided into a number of discrete masses, and each mass will be allowed six degrees of freedom. Thus, for a fifteen-mass structure, one must possess the ability to multiply, invert, and find the eigenvalues and eigenvectors of a $90 \times 90$ matrix. It is easily seen that for a large number of masses, even the largest digital computers presently available can be overburdened. Nevertheless, many important structures can be approximated within the capabilities of these machines. Furthermore, by neglecting such effects as rotatory inertia the size of the eigenvalue—eigenvector problem can be reduced.

## IV. The Review of Literature.

An excellent discussion of the formulation of vibration problems in matrix form is presented in Shock and Vibration Handbook (3), Volume 2, edited by Cyril M. Harris and Charles E. Crede. This discussion is contained in chapter twenty-eight, "Numerical Methods of Analysis" by Stephen H. Crandall and Robert B. McCalley, Jr. Topics treated therein are mass and stiffness coupling, discrete approximations of continuous systems, the matrix eigenvalue problem, and many other related subjects.

The solution to the problem of formulating the stiffness matrix was accomplished primarily with the aid of the following three publications: Design of Piping Systems by the M. W. Kellogg Co. (5); "A Matrix Method for Flexibility Analysis of Piping Systems" by J. E. Brock (1); and "Piping Flexibility Analysis by Stiffness Method" by L. H. Chen (2). In Design of Piping Systems, a set of coefficients is derived with the aid of Castigliano's Theorem, which relates the internal forces in a section of pipe to the relative end deflections of the pipe, all forces and deflections being referred to a common origin. Use of these coefficients requires only that the geometrical configuration of the pipe and its orientation in space be known. The paper by Brock contains an excellent discussion of matrix transformations as they deal with pipe stress problems. Transformations of influence coefficient matrices as presented therein were found to be directly applicable. "Piping Flexibility Analysis By Stiffness Method" introduced the idea of working with stiffnesses of individual pipe sections rather than flexibilities. The formulation of a total stiffness matrix for a vibration problem is found to be much easier to accomplish than the formulation of a total flexibility matrix. This is especially true for complex systems containing many branches and closed loops.

"Dynamic Behavior of a Foundation-Like Structure" by V. H. Neubert and W. H. Ezell (9) contains theoretical and experimental results obtained from shaking a foundation-like structure constructed from segments of pipe which provides a model for checking the work presented in this paper.

## V. The Investigation.

### A. Formulation of the Problem.

The problem to be investigated in this paper is that of finding the natural frequencies and mode shapes of a freely vibrating three-dimensional piping network. The method of attack to be followed was indicated in the introduction; that is, the physical system will be represented by a model consisting of inertial elements interconnected by a network of flexible elements. The assumption of small oscillations will be made and damping will not be considered here. For small oscillations, it has been shown (4), (7), that if generalized coordinates are taken such that they vanish in the equilibrium position, then the kinetic energy and the potential energy of the system can be expressed as quadratics such as

$$KE = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} m_{ij} \dot{q}_i \dot{q}_j \qquad (1)$$

and

$$PE = \frac{1}{2} \sum_{i=1}^{n} \sum_{J=1}^{n} k_{ij} q_i q_j \qquad (2)$$

Here n is the number of generalized coordinates necessary to describe the motion of the system. The $m_{ij}$ are functions of the $q_i$ but for small oscillations they may be taken as the constant values at the equilibrium position (7). Rigid-body translations will not be considered here; hence, both of the above quadratics are positive definite. This insures the invertability of the corresponding mass and stiffness matrices. To obtain the equations of motion for the system, Lagrange's equations may now be used.

The Lagrangian is

$$L = KE-PE = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} (m_{ij}\, \dot{q}_i\, \dot{q}_j - k_{ij}\, q_i\, q_j) \tag{3}$$

and the equations of motion are given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \tag{4}$$

for free vibrations. Substitution of equation (3) into equation (4) yields the equations

$$\sum_{j=1}^{n} m_{ij}\, \ddot{q}_j + \sum_{j=1}^{n} k_{ij}\, q_j = 0 \; (i=1, ---, n) \tag{5}$$

Making the definitions

$$\underline{M} = \begin{bmatrix} m_{11} & m_{12} & \text{---} & m_{in} \\ m_{21} & m_{22} & \diagdown & | \\ m_{ni} & \text{---} & \text{---} & m_{nn} \end{bmatrix}$$

$$\underline{K} = \begin{bmatrix} k_{11} & k_{12} & \text{---} & k_{1n} \\ k_{21} & k_{22} & \ddots & | \\ | & & & \\ k_{n1} & \text{---} & & k_{nn} \end{bmatrix}$$

and

$$\underline{x} = \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{bmatrix}$$

permits equations (5) to be expressed in matrix form as

$$\underline{M}\,\underline{\ddot{x}} + \underline{K}\,\underline{x} = 0 \qquad (6)$$

Note that an underlined upper case letter denotes a square matrix and an underlined lower case letter represents a column matrix.

If the motion is assumed to be sinusoidal, the relation between the displacements and accelerations becomes

$$\underline{\ddot{x}} = -\omega^2 \underline{x} \qquad (7)$$

which upon substitution into (6) yields

$$-\omega^2\,\underline{M}\,\underline{x} + \underline{K}\,\underline{x} = 0 \qquad (8)$$

or, upon rearranging,

$$(\underline{K} - \omega^2\,\underline{M})\,\underline{x} = 0 \qquad (9)$$

Multiplying through by $\underline{M}^{-1}$ gives

$$(\underline{M}^{-1}\ \underline{K} - \omega^2\ \underline{I})\ \underline{x} = 0 \qquad (10)$$

which is the standard form of a matrix eigenvalue problem. Thus, it appears that if the $\underline{M}$ and $\underline{K}$ matrices can be found, a standard mathematical procedure may be followed which will yield the frequencies (square roots of eigenvalues) and mode shapes (eigenvectors). The mathematical model of inertial elements interconnected by springs enables us to find the $k_{ij}$ and $m_{ij}$ from a physical viewpoint.

For any given rigid body the components of angular momentum may be written as

$$\left.\begin{aligned} L_1 &= I_{11}\ \dot{q}_1 - I_{12}\ \dot{q}_2 - I_{13}\ \dot{q}_3 \\ L_2 &= -I_{21}\ \dot{q}_1 + I_{22}\ \dot{q}_2 - I_{23}\ \dot{q}_3 \\ L_3 &= -I_{31}\ \dot{q}_1 - I_{32}\ \dot{q}_2 + I_{33}\ \dot{q}_3 \end{aligned}\right\} \qquad (11)$$

and the linear momentum components are

$$m\ \dot{q}_4,\ m\ \dot{q}_5,\ \text{and}\ m\ \dot{q}_6 \qquad (12)$$

where $q_1$, $q_2$, and $q_3$ are rotations about the 1, 2, and 3 axes, and $q_4$, $q_5$, and $q_6$ are displacements in the 1, 2, and 3 directions.

Under the assumption of small oscillations, neglecting $\dot{q}_i \dot{q}_j$ products, we may take the moments and products of inertia as constants so that the inertia forces on the body would be

$$
\begin{aligned}
F_1 &= I_{11} \ddot{q}_1 - I_{12} \ddot{q}_2 - I_{13} \ddot{q}_3 \\
F_2 &= -I_{21} \ddot{q}_1 + I_{22} \ddot{q}_2 - I_{23} \ddot{q}_3 \\
F_3 &= -I_{31} \ddot{q}_1 - I_{32} \ddot{q}_2 + I_{33} \ddot{q}_3 \\
F_4 &= m \ddot{q}_4 \\
F_5 &= m \ddot{q}_5 \\
F_6 &= m \ddot{q}_6
\end{aligned}
\qquad (13)
$$

Once again the subscripts 1, 2, and 3 refer to rotations and moments about the 1, 2, and 3 axes and the subscripts 4, 5, and 6 refer to displacements and forces in the 1, 2, and 3 directions. If the requirement is made that the 1, 2, and 3 axes be the principal axes of inertia of the mass, then the above equations will be simplified since the products of inertia are zero for this case. Imposing this restriction, equations (13) may be expressed as

$$F_i = m_i \ddot{q}_i \quad (i = 1, 2, ---, 6) \qquad (14)$$

where $m_1 = I_1$

$m_2 = I_2$

$m_3 = I_3$

$m_4 = m$

$m_5 = m$

$m_6 = m$

If the only other forces acting on the mass are considered to be linear functions of deflections of the system, the forces on mass i in the r-direction due to these deflections can be expressed as

$$F_{i,r} = - \sum_{s=1}^{6} \sum_{j=1}^{N} k_{ij,rs} \, q_{j,s} \qquad (15)$$

where N is the number of masses in the system. In this equation subscripts preceding the comma refer to mass locations and subscripts following the comma represent directions. For instance, the term

$$-k_{12,34} \, q_{2,4}$$

would indicate the moment about the 3-axis acting on mass 1 due to a translation of mass 2 in the 1-direction. Combining equations (15) and (14) yields

$$m_{i,r}\,\ddot{q}_{i,r} + \sum_{s=1}^{6}\sum_{j=1}^{N} k_{ij,rs}\,q_{j,s} = 0 \qquad (16)$$

which is found to be of the same form as equation (5) if the restriction is made in equation (5) that $m_{ij} = 0$ except when $i=j$. Hence we may write equation (16) in matrix form (6). Separation of the generalized coordinates into translations and rotations for each mass in the above manner makes possible a meaningful partitioning of the matrices as follows:

$$\begin{bmatrix} \underline{M}_1 & 0 & 0 & \cdots & 0 \\ 0 & \underline{M}_2 & 0 & & \vdots \\ \vdots & & \underline{M}_3 & & \vdots \\ & & & \ddots & \\ 0 & \cdots & & & \underline{M}_n \end{bmatrix} \begin{bmatrix} \ddot{\underline{x}}_1 \\ - \\ - \\ \ddot{\underline{x}}_n \end{bmatrix} + \begin{bmatrix} \underline{K}_{11} & \underline{K}_{12} & \cdots & \underline{K}_{1n} \\ \vdots & \underline{K}_{22} & & \vdots \\ & & \ddots & \\ \underline{K}_{n1} & \cdots & & \underline{K}_{nn} \end{bmatrix} \begin{bmatrix} \underline{x}_1 \\ - \\ - \\ \underline{x}_n \end{bmatrix} = 0 \qquad (17)$$

where

$$\underline{M}_j = \begin{bmatrix} I_{j,1} & 0 & 0 & 0 & 0 & 0 \\ 0 & I_{j,2} & 0 & 0 & 0 & 0 \\ 0 & 0 & I_{j,3} & 0 & 0 & 0 \\ 0 & 0 & 0 & m_j & 0 & 0 \\ 0 & 0 & 0 & 0 & m_j & 0 \\ 0 & 0 & 0 & 0 & 0 & m_j \end{bmatrix}$$

and $$\underline{K}_{ij} = \begin{bmatrix} k_{ij,11} & k_{ij,12} & \text{---} & k_{ij,16} \\ k_{ij,21} & k_{ij,22} & \text{---} & | \\ | & & & | \\ k_{ij,61} & \text{---} & \text{---} & k_{ij,66} \end{bmatrix}$$

and $$\underline{x}_j = \begin{bmatrix} q_{j,1} \\ q_{j,2} \\ q_{j,3} \\ q_{j,4} \\ q_{j,5} \\ q_{j,6} \end{bmatrix}$$

Thus it is seen that the matrix $\underline{M}$ is easily obtained for known masses and moments of inertia. In practice, however, the matter of deciding what values to use for these quantities is not obvious, particularly for the moments of inertia.

B. Derivation of the Stiffness Matrix.

Consider a segment of pipe arbitrarily oriented in space, with a rigid bar connected to each end of the segment and the opposite ends of the rigid bars terminating at a common point (see figure 1). As a matter of

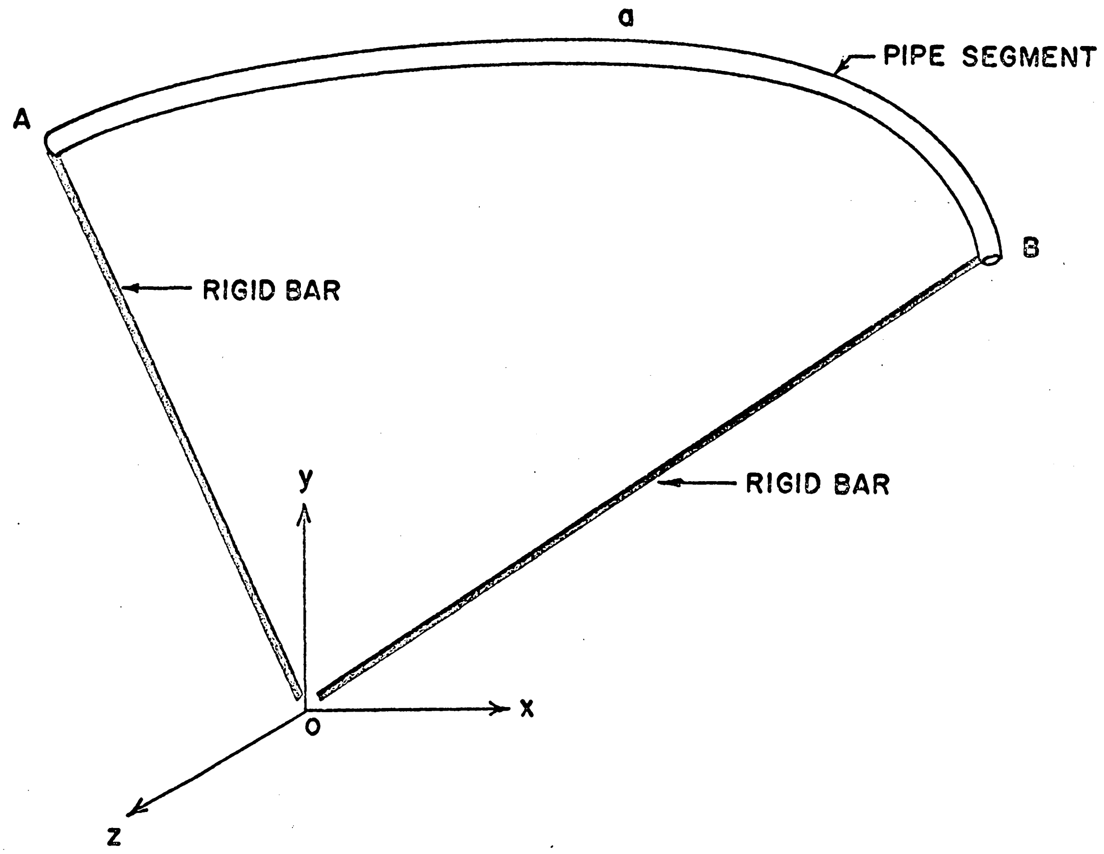



FIGURE I PIPE SEGMENT IN SPACE

convenience, this common point will be located at the origin of a set of rectangular cartesian coordinates. Obviously, any force or set of forces applied at end O of rigid bar OB will cause an equal and opposite force or set of forces at end O of rigid bar OA. Likewise, the accompanying deflections as viewed from end O of OA will be equal and opposite to the deflections as viewed from end O of OB. Thus it follows that if segment AB is considered to be a cantilever beam with one end fixed and a force is applied at the origin to a rigid bar extended from the other end, the resulting deflection at the point of application of the force will be the same no matter which end of the segment is considered to be fixed. Since the deflection at this point is a linear function of the applied force, the following matrix equation may be written:

$$\underline{x}_B^o - \underline{x}_A^o = \underline{A}_a^o \, \underline{f}_B^o \qquad (18)$$

Here the superscript (o) means the forces and deflections are measured at the origin end of a rigid bar extended from the point indicated by the subscript. It is seen that the matrix $\underline{A}_a^o$ is dependent only on the physical characteristics of pipe segment (a) and its geometrical orientation in space. A derivation of coefficients of the above type is presented in detail in reference (5) and has been partially presented in the appendix for the sake of completeness. The matrix $\underline{A}_a^o$ will be called the matrix of "shape coefficients" for segment a in the remainder of this paper.

Multiplication of equation (18) by the inverse of the shape coefficient matrix yields

$$\underline{f}_B^o = \underline{K}_a^o \; (\underline{x}_B^o - \underline{x}_A^o) \tag{19}$$

where

$$\underline{K}_a^o = (\underline{A}_a^o)^{-1}$$

The forces $F_1$ and moments $T_1$ applied at point B can be transferred to the origin with the following equations:

$$\begin{aligned}
T_x^o &= T_x^B - F_y^B \; (z_B - z_o) + F_z^B \; (y_B - y_o) \\
T_y^o &= T_y^B + F_x^B \; (z_B - z_o) - F_z^B \; (x_B - x_o) \\
T_z^o &= T_z^B - F_x^B \; (y_B - y_o) + F_y^B \; (x_B - x_o) \\
F_x^o &= F_x^B \\
F_y^o &= F_y^B \\
F_z^o &= F_z^B
\end{aligned} \tag{20}$$

which can be written in matrix form as

$$\underline{f}_B^o = \underline{B}_{oB}\, \underline{f}_B \tag{21}$$

where $\underline{B}_{oB}$ is defined as

$$\underline{B}_{oB} = \begin{bmatrix} 1 & 0 & 0 & 0 & -(z_B-z_o) & (y_B-y_o) \\ 0 & 1 & 0 & (z_B-z_o) & 0 & -(x_B-x_o) \\ 0 & 0 & 1 & -(y_B-y_o) & (x_B-x_o) & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} . \tag{22}$$

Multiplication of (21) by $(\underline{B}_{oB})^{-1}$ gives

$$\underline{f}_B = (\underline{B}_{oB})^{-1}\, \underline{f}_B^o \tag{23}$$

Note, however, that the transformation in this direction can be accomplished by interchanging the indices of the coordinates in $\underline{B}_{oB}$. In other words,

$$(\underline{B}_{oB})^{-1} = \underline{B}_{Bo}. \tag{24}$$

In a similar manner, the deflections of the rigid bar oB at the origin can be related to the deflections at B as follows:

$$\left.\begin{aligned}
\theta_x^o &= \theta_x^B \\
\theta_y^o &= \theta_y^B \\
\theta_z^o &= \theta_z^B \\
\delta_x^o &= \delta_x^B - \theta_y\ (z_B - z_o) + \theta_z\ (y_B - y_o) \\
\delta_y^o &= \delta_y^B + \theta_x\ (z_B - z_o) - \theta_z\ (x_B - x_o) \\
\delta_z^o &= \delta_z^B - \theta_x\ (y_B - y_o) + \theta_y\ (x_B - x_o)
\end{aligned}\right\} \quad (25)$$

or in matrix form

$$\underline{x}_B^o = \underline{B}_{Bo}^T\ \underline{x}_B \quad (26)$$

In equation (26) $\underline{B}_{Bo}^T$ indicates the transpose of $\underline{B}_{Bo}$.

The above transformations together with the shape coefficient matrix for each segment of pipe make it possible to write equations of equilibrium at each mass of a vibrating system. Consider masses to be located at points A, B, and C in figure 2.

If the principal axes of inertia are the x, y, and z axes and the only external loads are inertia forces, then equilibrium conditions require that

$$\underline{K}_a^o\ (\underline{x}_B^o - \underline{x}_A^o) + \underline{K}_b^o\ (\underline{x}_B^o - \underline{x}_C^o) + \underline{f}_{B(inertia)}^o = 0 \quad (27)$$

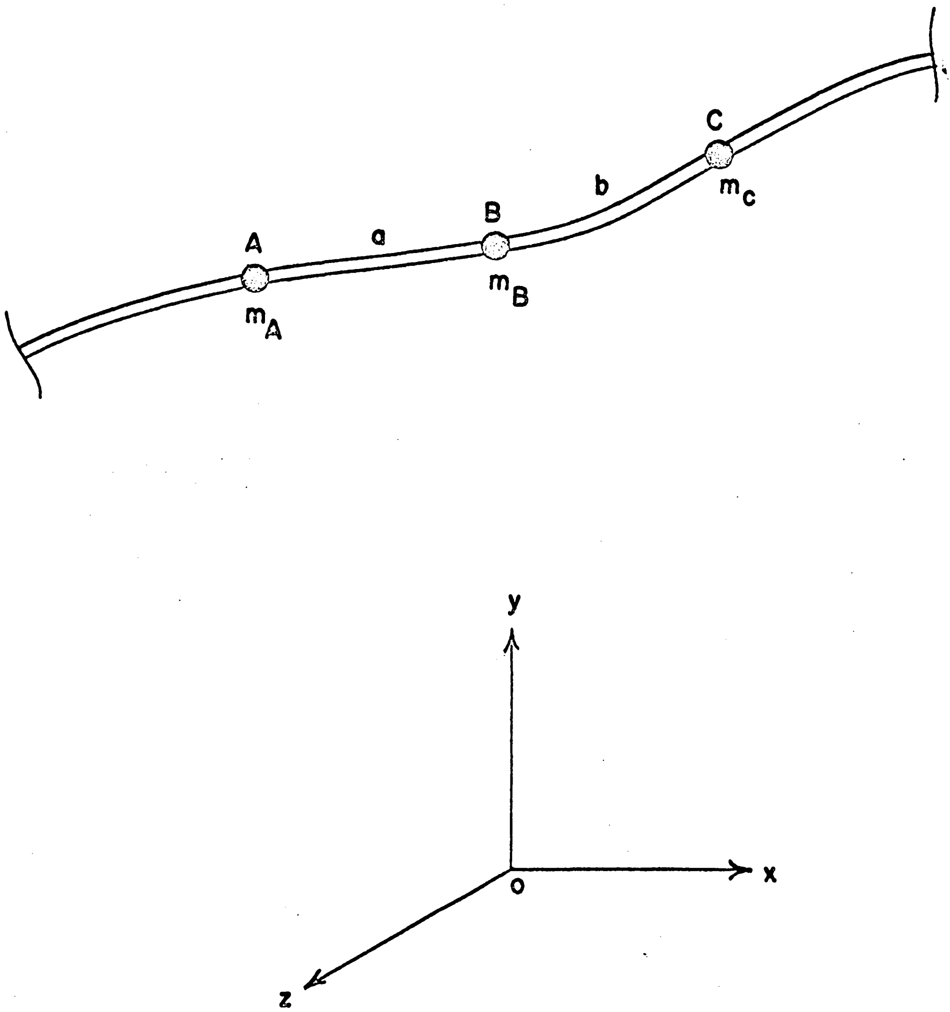



FIGURE 2 PARTIAL PIPING SYSTEM

at mass B. Similar equations may be written for each remaining mass in the system. As an example, consider the system shown in figure 3. The equations of equilibrium are as follows:

$$\underline{K}_1^o (\underline{x}_1^o) + \underline{K}_2^o (\underline{x}_1^o - \underline{x}_3^o) = - \underline{f}_{1(inertia)}^o$$

$$\underline{K}_3^o (\underline{x}_2^o) + \underline{K}_4^o (\underline{x}_2^o - \underline{x}_3^o) = - \underline{f}_{2(inertia)}^o$$

$$\underline{K}_2^o (\underline{x}_3^o - \underline{x}_1^o) + \underline{K}_4^o (\underline{x}_3^o - \underline{x}_2^o) + \underline{K}_5^o (\underline{x}_3^o - \underline{x}_4^o)$$

$$+ \underline{K}_7^o (\underline{x}_3^o - \underline{x}_5^o) = - \underline{f}_{3(inertia)}^o$$

$$\underline{K}_5^o (\underline{x}_4^o - \underline{x}_3^o) + \underline{K}_6^o (\underline{x}_4^o) = - \underline{f}_{4(inertia)}^o$$

$$\underline{K}_7^o (\underline{x}_5^o - \underline{x}_3^o) + \underline{K}_8^o (\underline{x}_5^o) = - \underline{f}_{5(inertia)}^o$$

Which may be written in matrix form as

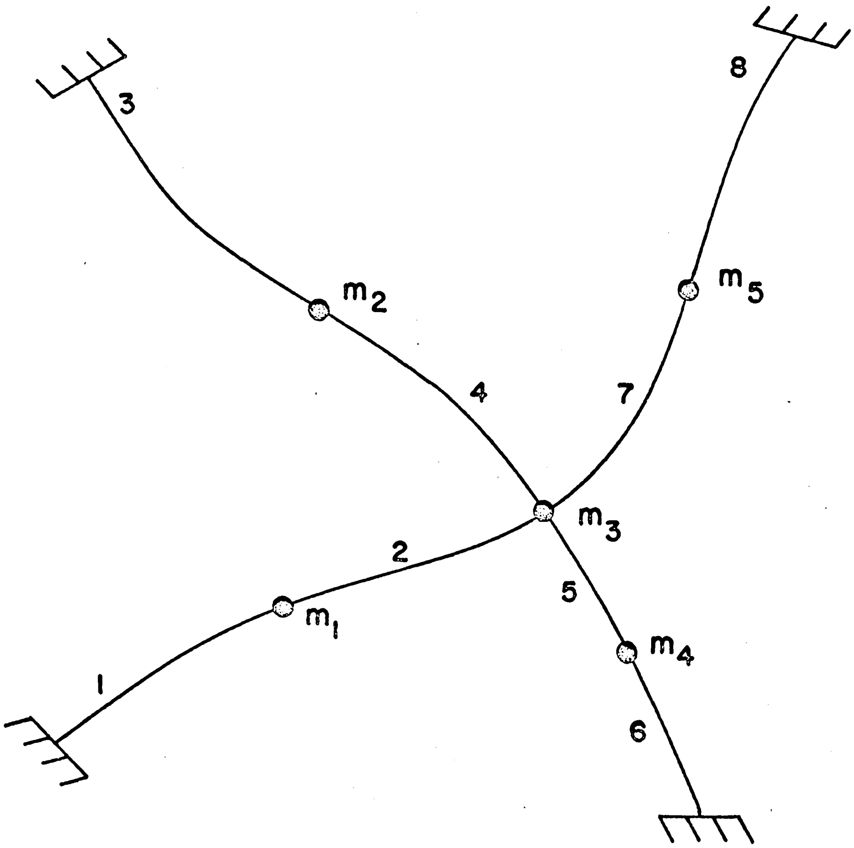



FIGURE 3 EXAMPLE 5-MASS PIPING SYSTEM

$$\begin{bmatrix} \underline{K}_1^o+\underline{K}_2^o & \underline{0} & -\underline{K}_2^o & \underline{0} & \underline{0} \\ \underline{0} & \underline{K}_3^o+\underline{K}_4^o & -\underline{K}_4^o & \underline{0} & \underline{0} \\ -\underline{K}_2^o & -\underline{K}_4^o & \begin{matrix}\underline{K}_2^o+\underline{K}_4^o \\ +\underline{K}_5^o+\underline{K}_7^o\end{matrix} & -\underline{K}_5^o & -\underline{K}_7^o \\ \underline{0} & \underline{0} & -\underline{K}_5^o & \underline{K}_5^o+\underline{K}_6^o & \underline{0} \\ \underline{0} & \underline{0} & -\underline{K}_7^o & \underline{0} & \underline{K}_7^o+\underline{K}_8^o \end{bmatrix} \begin{bmatrix} \underline{x}_1^o \\ \underline{x}_2^o \\ \underline{x}_3^o \\ \underline{x}_4^o \\ \underline{x}_5^o \end{bmatrix} = - \begin{bmatrix} \underline{f}_1^o \\ \underline{f}_2^o \\ \underline{f}_3^o \\ \underline{f}_4^o \\ \underline{f}_5^o \end{bmatrix}_{\text{(inertia)}} \qquad (28)$$

or in more compact form

$$\underline{K}^o \, \underline{x}^o = \underline{f}^o \qquad (29)$$

Note that a mass has been placed at the intersection of the two pipe lines. This simplifies writing the equations since it assures that there is only a simple pipe line connecting adjacent masses. Inherent in this procedure is the assumption that all joints are perfectly rigid.

In this manner a stiffness matrix may be constructed for any given piping structure which relates the inertia forces (transferred to the origin) to the deflections (transferred to the origin). In order to find the absolute stiffness matrix which relates the actual inertia forces and deflections, B-transformation matrices of the type presented in equation (21) and (26) must be constructed for the entire system. This is easily done, and may be written for the problem at hand as

$$\begin{bmatrix} \underline{B}_{o1} & \underline{0} & \underline{0} & \underline{0} & \underline{0} \\ \underline{0} & \underline{B}_{o2} & \underline{0} & \underline{0} & \underline{0} \\ \underline{0} & \underline{0} & \underline{B}_{o3} & \underline{0} & \underline{0} \\ \underline{0} & \underline{0} & \underline{0} & \underline{B}_{o4} & \underline{0} \\ \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{B}_{o5} \end{bmatrix} \begin{bmatrix} \underline{f}_1 \\ \underline{f}_2 \\ \underline{f}_3 \\ \underline{f}_4 \\ \underline{f}_5 \end{bmatrix} = \begin{bmatrix} \underline{f}_1^o \\ \underline{f}_2^o \\ \underline{f}_3^o \\ \underline{f}_4^o \\ \underline{f}_5^o \end{bmatrix} \qquad (30)$$

and

$$\begin{bmatrix} \underline{B}_{1o}^T & \underline{0} & \underline{0} & \underline{0} & \underline{0} \\ \underline{0} & \underline{B}_{2o}^T & \underline{0} & \underline{0} & \underline{0} \\ \underline{0} & \underline{0} & \underline{B}_{3o}^T & \underline{0} & \underline{0} \\ \underline{0} & \underline{0} & \underline{0} & \underline{B}_{4o}^T & \underline{0} \\ \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{B}_{5o}^T \end{bmatrix} \begin{bmatrix} \underline{x}_1 \\ \underline{x}_2 \\ \underline{x}_3 \\ \underline{x}_4 \\ \underline{x}_5 \end{bmatrix} = \begin{bmatrix} \underline{x}_1^o \\ \underline{x}_2^o \\ \underline{x}_3^o \\ \underline{x}_4^o \\ \underline{x}_5^o \end{bmatrix} \qquad (31)$$

or

$$\underline{Bf} = \underline{f}^o \qquad (32)$$

and

$$(\underline{B}^{-1})^T \, \underline{x} = \underline{x}^o \qquad (33)$$

Substituting these transformations into equation (29) yields

$$\underline{K}^{o}(\underline{B}^{-1})^{T}\ \underline{x} = \underline{B}\ \underline{f}$$

or

$$\underline{f} = \underline{B}^{-1}\ \underline{K}^{o}\ (\underline{B}^{-1})^{T}\ \underline{x} \qquad (34)$$

If the principal axes of inertia at each mass are not parallel with the axes of the origin, still another transformation is necessary. This transformation will be a simple rotation of the local coordinate system at each mass. Consider the forces and moments to be measured with respect to the principal axes of inertia.

The following equation will transform these forces and moments to a coordinate system which is parallel to the axes at the origin.

$$\underline{f}_{B}' = \underline{L}_{oB}\ \underline{f}_{B} \qquad (35)$$

where $\underline{f}_{B}$ are the forces at (B) described in the inertial frame of reference, $\underline{f}'_{B}$ are the forces at B described in a frame with axes parallel to those at the origin. The matrix $\underline{L}_{oB}$ is a rotational transformation defined by

$$\underline{L}_{oB} = \begin{bmatrix} \underline{R}_{oB} & \underline{0} \\ \underline{0} & \underline{R}_{oB} \end{bmatrix} \qquad (36)$$

$$\text{where } \underline{R}_{oB} = \begin{bmatrix} \cos(x_o,x_B) & \cos(x_o,y_B) & \cos(x_o,z_B) \\ \cos(y_o,x_B) & \cos(y_o,y_B) & \cos(y_o,z_B) \\ \cos(z_o,x_B) & \cos(z_o,y_B) & \cos(z_o,z_B) \end{bmatrix} \qquad (37)$$

Here the elements of $\underline{R}_{oB}$ are the cosines of the angles between the principal axes and the axes at the origin. It can also be shown that

$$\underline{x}'_B = \underline{L}^T_{Bo} \ \underline{x}_B \qquad (38)$$

Equations (35) and (38) may be written for the entire system in the same manner as were the B-transformations as follows:

$$\begin{bmatrix} \underline{L}_{o1} & \underline{0} & \underline{0} & \underline{0} & \underline{0} \\ \underline{0} & \underline{L}_{o2} & \underline{0} & \underline{0} & \underline{0} \\ \underline{0} & \underline{0} & \underline{L}_{o3} & \underline{0} & \underline{0} \\ \underline{0} & \underline{0} & \underline{0} & \underline{L}_{o4} & \underline{0} \\ \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{L}_{o5} \end{bmatrix} \begin{bmatrix} \underline{f}_1 \\ \underline{f}_2 \\ \underline{f}_3 \\ \underline{f}_4 \\ \underline{f}_5 \end{bmatrix} = \begin{bmatrix} \underline{f}'_1 \\ \underline{f}'_2 \\ \underline{f}'_3 \\ \underline{f}'_4 \\ \underline{f}'_5 \end{bmatrix} \qquad (39)$$

and

$$\begin{bmatrix} \underline{L}^T_{1o} & \underline{0} & \underline{0} & \underline{0} & \underline{0} \\ \underline{0} & \underline{L}^T_{2o} & \underline{0} & \underline{0} & \underline{0} \\ \underline{0} & \underline{0} & \underline{L}^T_{3o} & \underline{0} & \underline{0} \\ \underline{0} & \underline{0} & \underline{0} & \underline{L}^T_{4o} & \underline{0} \\ \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{L}^T_{5o} \end{bmatrix} \begin{bmatrix} \underline{x}_1 \\ \underline{x}_2 \\ \underline{x}_3 \\ \underline{x}_4 \\ \underline{x}_5 \end{bmatrix} = \begin{bmatrix} \underline{x}'_1 \\ \underline{x}'_2 \\ \underline{x}'_3 \\ \underline{x}'_4 \\ \underline{x}'_5 \end{bmatrix} \qquad (40)$$

or, rewriting,

$$\underline{L}\ \underline{f} = \underline{f}' \qquad (41)$$

and

$$(\underline{L}^{-1})^T\ \underline{x} = \underline{x}' \qquad (42)$$

For this more general case in which the principal axes of inertia are not parallel with the axes at the origin, the above transformations must be applied to the results of the transformations given by equations (32) and (33). Performing these successive transformations yields

$$\underline{f}^o = \underline{B}\ \underline{L}\ \underline{f} \qquad (43)$$

and

$$\underline{x}^o = (\underline{B}^{-1})^T\ (\underline{L}^{-1})^T\ \underline{x}. \qquad (44)$$

Defining $\underline{C}$ as

$$\underline{C} = \underline{L}^{-1}\ \underline{B}^{-1} \qquad (45)$$

and substituting for $\underline{f}^{o}$ and $\underline{x}^{o}$ in equation (29) gives

$$\underline{K}^{o} \underline{C}^{T} \underline{x} = \underline{C}^{-1} \underline{f} \qquad (46)$$

or

$$\underline{f} = \underline{C} \underline{K}^{o} \underline{C}^{T} \underline{x} \qquad (47)$$

which relates the forces as seen in the inertial frames of reference to the deflections as seen in the same frames. Hence the matrix

$$\underline{K} = \underline{C} \underline{K}^{o} \underline{C}^{T} \qquad (48)$$

may be thought of as the total stiffness matrix for the system and applied as in equation (10) to the formulation of the eigenvalue problem for the vibrating system.

C. Remarks.

No mention has been made in the preceding developments that the eigenvalue problem as presented in equation (10) could just as easily have been written in terms of the total flexibility matrix $\underline{\Delta} = \underline{K}^{-1}$. To accomplish this, multiply equation (8) by

$$- \frac{1}{\omega^{2}} \underline{K}^{-1}$$

to get $$\underline{K}^{-1}\,\underline{M}\,\underline{x} - \frac{1}{\omega^2}\,\underline{x} = 0$$

or $$(\underline{\Delta}\,\underline{M} - \frac{1}{\omega^2}\,\underline{I})\,\underline{x} = 0 \qquad (49)$$

Hence, the reciprocals of the square roots of the eigenvalues of this equation would be the natural frequencies of the system and the eigenvectors would again give the mode shapes. The best method to use depends on the individual problem. Structures for which the deflection curve can be easily found are usually solved more readily by formulating the flexibility matrix, whose elements are the influence coefficients of the structure. Another advantage of this approach is that the total flexibility matrix need not be found when one is not interested in all possible mode shapes. However, the structures which are complicated by many branches and closed loops are best solved through the formulation of the stiffness matrix because of the ease of understanding how to write the equations of this method, no matter how complicated the structure.

## D. Application of Procedure.

A foundation-like structure made up of straight pipe segments has been analyzed experimentally and theoretically in reference (8), and this structure will be used in this paper to demonstrate the application of the procedure presented herein and will also serve as a check for errors in the computer programs. The structure is assumed to be comprised of 14 movable lumped masses and 4 other masses lumped at each of the anchor points. Consideration of six degrees of freedom at each of the movable mass points yields a total stiffness matrix of size $84 \times 84$. It should be mentioned here that no satisfactory method of lumping the rotational inertia properties of a prismatic bar was found, so the problem was essentially reduced to one of 3N degrees of freedom by inverting the total stiffness matrix and selecting a $42 \times 42$ translatory influence coefficient matrix from the resulting total flexibility matrix. Rotatory inertia has been shown to have a negligible effect for prismatic beams (10). The structure is pictured in figure 4 and the mass distribution and orientation of the local coordinate systems at each mass is shown in figures 5 and 6. These coordinate systems were chosen so that the x-axis would always be along the longitudinal axis of a pipe segment. This actually has no effect on the answers for the problem at hand since rotatory inertia is not considered here, but was done this way as a check

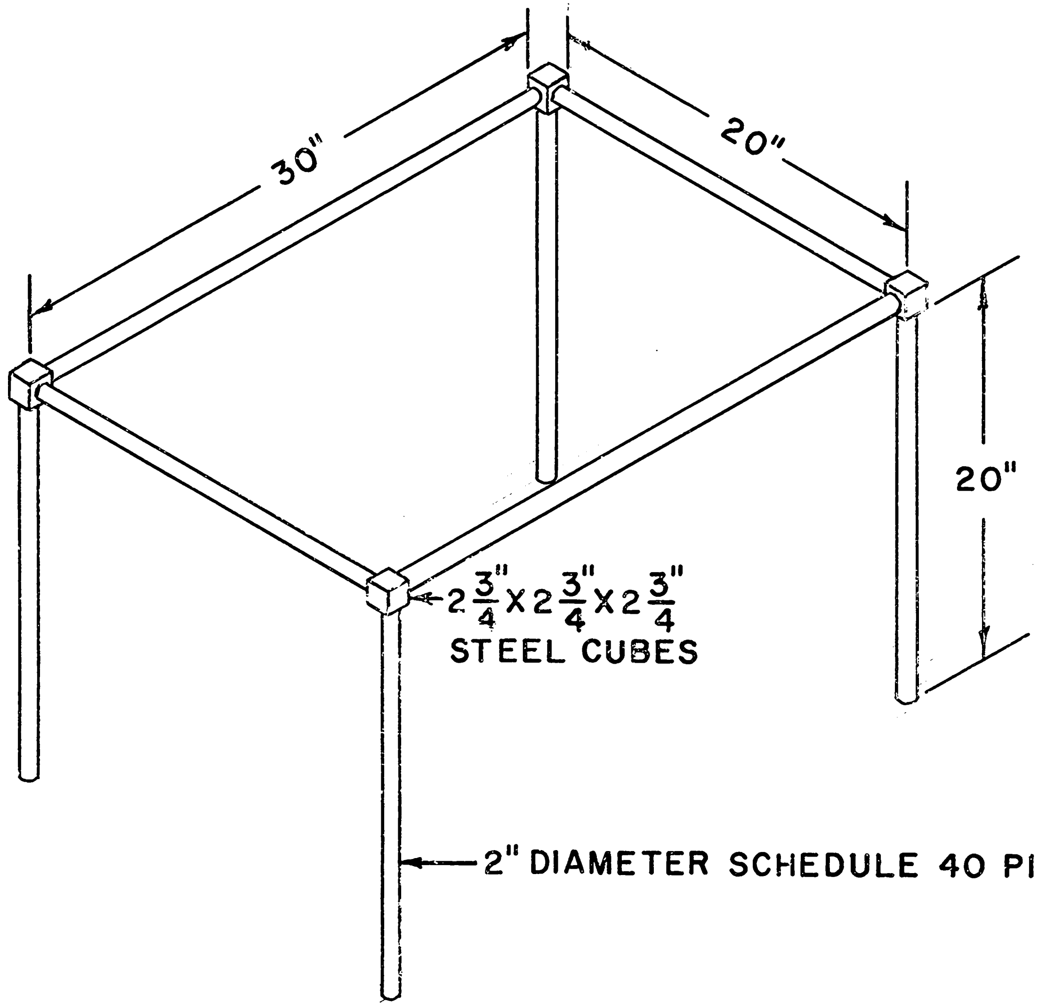



FIGURE 4. PIPE STRUCTURE USED IN SAMPLE CALCULATIONS

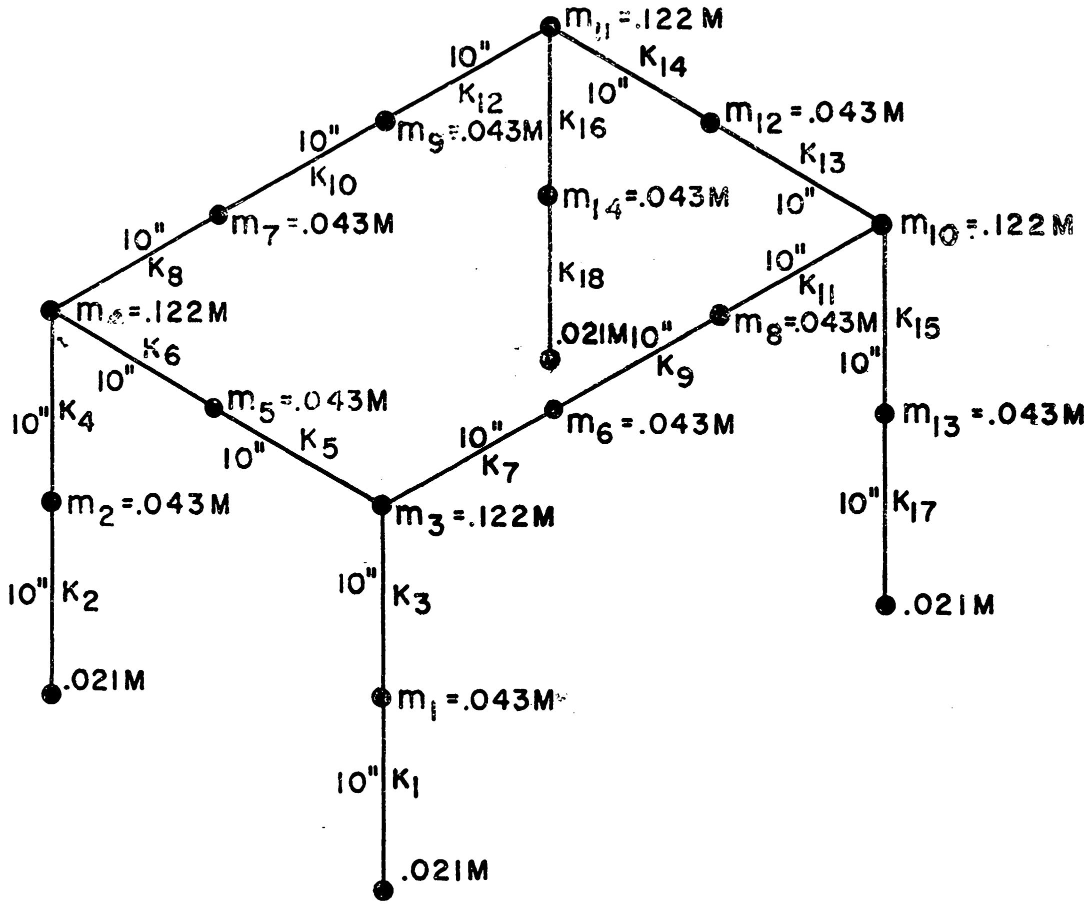



FIGURE 5 MASS LOCATIONS AND DISTRIBUTION FOR SAMPLE PROBLEM

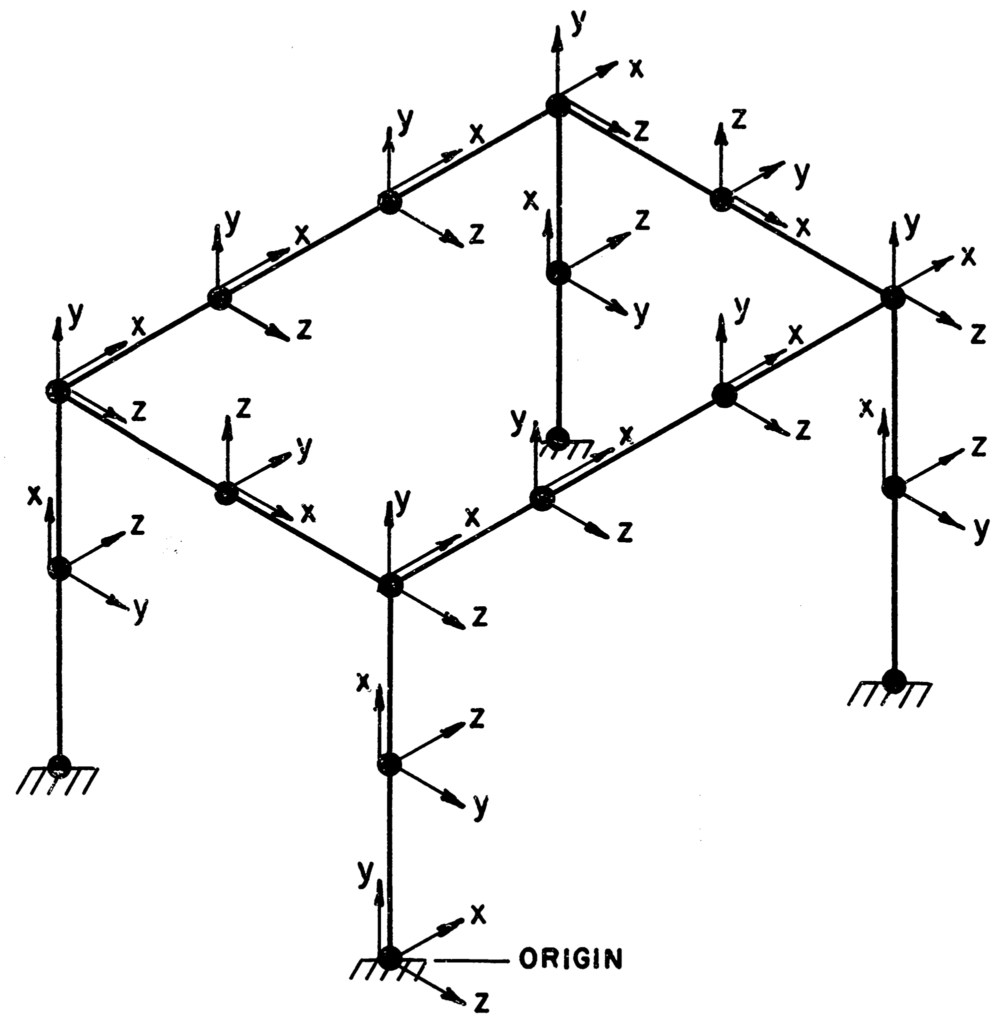



FIGURE 6 ORIENTATION OF LOCAL COORDINATE SYSTEMS FOR SAMPLE PROBLEM

on the portion of the computer program which performs the rotation of axes.

The system for numbering the masses may appear unusual, but there was a reason for doing it this way. It was found (6) that if the numbers of adjacent masses are kept as close in magnitude as possible, the stiffness matrix will appear as a band matrix, possessing properties which may be exploited to reduce the number of arithmetic operations necessary for a given matrix operation. This was not done in this paper, but is a possibility for future work which may reduce the computer storage capacity required to analyze a particular structure.

The stiffness matrix was completed in the foregoing manner and was found to take the following form:

$$
\underline{K} = \left[\begin{array}{c|c|c|c|c|c|c|c|c|c|c|c|c|c}
\underline{K}_{11} & \underline{0} & \underline{K}_{13} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} \\
\underline{0} & \underline{K}_{22} & \underline{0} & \underline{K}_{24} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} \\
\underline{K}_{31} & \underline{0} & \underline{K}_{33} & \underline{0} & \underline{K}_{35} & \underline{K}_{36} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} \\
\underline{0} & \underline{K}_{42} & \underline{0} & \underline{K}_{44} & \underline{K}_{45} & \underline{0} & \underline{K}_{47} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} \\
\underline{0} & \underline{0} & \underline{K}_{53} & \underline{K}_{54} & \underline{K}_{55} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} \\
\underline{0} & \underline{0} & \underline{K}_{63} & \underline{0} & \underline{0} & \underline{K}_{66} & \underline{0} & \underline{K}_{68} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} \\
\underline{0} & \underline{0} & \underline{0} & \underline{K}_{74} & \underline{0} & \underline{0} & \underline{K}_{77} & \underline{0} & \underline{K}_{79} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} \\
\underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{K}_{86} & \underline{0} & \underline{K}_{88} & \underline{0} & \underline{K}_{8-10} & \underline{0} & \underline{0} & \underline{0} & \underline{0} \\
\underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{K}_{97} & \underline{0} & \underline{K}_{99} & \underline{0} & \underline{K}_{9-11} & \underline{0} & \underline{0} & \underline{0} \\
\underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{K}_{10-8} & \underline{0} & \underline{K}_{10-10} & \underline{0} & \underline{K}_{10-12} & \underline{K}_{10-13} & \underline{0} \\
\underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{K}_{11-4} & \underline{0} & \underline{K}_{11-11} & \underline{K}_{11-12} & \underline{0} & \underline{K}_{11-14} \\
\underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{K}_{12-10} & \underline{K}_{12-11} & \underline{K}_{12-12} & \underline{0} & \underline{0} \\
\underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{K}_{13-10} & \underline{0} & \underline{0} & \underline{K}_{13-13} & \underline{0} \\
\underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{K}_{14-11} & \underline{0} & \underline{0} & \underline{K}_{14-14}
\end{array}\right]
$$

where the $\underline{K}_{ij}$ are the $6 \times 6$ matrices of spring constants relating forces at i to deflections at j.

Since the published results for this problem are presented in terms of influence coefficients, the inverted stiffness matrix presented a basis for comparison as shown in Table I. The differences in the values of the influence coefficients presented in reference (9) and those computed here are believed to be accounted for as the shear deflections which were included in this analysis.

In order to solve the eigenvalue-eigenvector problem on a digital computer, some additional adjustments can be made for the sake of reducing necessary storage capacity. Methods have been developed for the solution of symmetrical matrices which reduce the storage required by about 50%. In equation (10), although $\underline{M}^{-1}$ and $\underline{K}$ are symmetric, this is not usually true for the product $\underline{M}^{-1}\ \underline{K}$. Therefore, the following operations were performed to yield a symmetric matrix,

$$(\underline{M}^{-1}\ \underline{K} - \omega^2\ \underline{I})\ \underline{x} = 0 \qquad (10)$$

Making the transformation

$$\underline{P}\ \underline{y} = \underline{x} \qquad (50)$$

where $\underline{P}\ \underline{P}^T = \underline{M}^{-1}$,

TABLE 1 INFLUENCE COEFFICIENTS (x $10^5$) $\frac{in}{lb}$ FOR SAMPLE PROBLEM

| Deflections At | Reference (9) Calculated | Reference (9) Measured | Present Calculations |
|---|---|---|---|
| **Force Location and Direction — 11X** | | | |
| 1Z | 0.311 | 0.295 | 0.307 |
| 2Z | 0.820 | 0.822 | 0.859 |
| 3X | 0.740 | 0.740 | 0.774 |
| 4X | 1.861 | — | 2.072 |
| 5Y | 1.304 | — | 1.423 |
| 6X | 0.740 | — | 0.774 |
| 7X | 1.874 | 1.83 | 2.087 |
| 8X | 0.740 | — | 0.774 |
| 9X | 1.891 | — | 2.103 |
| 10X | 0.740 | 0.655 | 0.774 |
| 11X | 1.906 | 1.88 | 2.117 |
| 12Y | 1.327 | — | 1.446 |
| 13Z | 0.310 | 0.330 | 0.306 |
| 14Z | 0.835 | 0.810 | 0.876 |
| **Force Location and Direction — 11Z** | | | |
| 1Y | 0.134 | 0.124 | 0.143 |
| 2Y | 0.133 | 0.115 | 0.142 |
| 3Z | 0.305 | 0.260 | 0.355 |
| 4Z | 0.305 | 0.280 | 0.355 |
| 5X | 0.305 | — | 0.355 |
| 6Z | 0.802 | — | 0.938 |
| 7Z | 0.810 | — | 0.947 |
| 8Z | 1.466 | — | 1.668 |
| 9Z | 1.486 | — | 1.69 |
| 10Z | 1.967 | 2.05 | 2.251 |
| 11Z | 1.994 | 2.08 | 2.28 |
| 12X | 1.981 | — | 2.26 |
| 13Y | 0.914 | 0.947 | 0.963 |
| 14Y | 0.925 | 0.89 | 0.975 |

equation (10) becomes

$$(\underline{P}\ \underline{P}^T\ \underline{K}\ \underline{P} - \omega^2\ \underline{P})\ y = 0$$

Multiplying by $\underline{P}^{-1}$ yields the equation

$$(\underline{P}^T\ \underline{K}\ \underline{P} - \omega^2\ \underline{I})\ \underline{y} = 0 \qquad (51)$$

Thus, the characteristic roots of the real symmetric matrix $\underline{P}^T\ \underline{K}\ \underline{P}$ are also the squares of the natural frequencies. In order to obtain the desired eigenvectors $\underline{x}$ from the solution of this equation, equation (50) must be employed. The matrix $\underline{P}$ for this problem is easily obtained since $\underline{M}^{-1}$ is a diagonal matrix. $\underline{M}^{-1}$ may be factored as

$$M^{-1} = \begin{bmatrix} \frac{1}{\sqrt{m_{11}}} & 0 & \cdots & 0 \\ 0 & \frac{1}{\sqrt{m_{22}}} & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & \frac{1}{\sqrt{m_{rr}}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{m_{11}}} & 0 & \cdots & 0 \\ 0 & \frac{1}{\sqrt{m_{22}}} & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & \frac{1}{\sqrt{m_{rr}}} \end{bmatrix} = \underline{P}\ \underline{P}^T$$

where r is the order of the matrix $\underline{M}^{-1}$.

If it is desired to obtain the natural frequencies and mode shapes utilizing the flexibility matrix, equation (10) may be modified as follows:

$$(\underline{M}^{-1}\ \underline{K} - \omega^2\ \underline{I})\ \underline{x} = 0$$

$$(\underline{K}^{-1}\ \underline{M} - \frac{1}{\omega^2}\ \underline{I})\ \underline{x} = 0$$

$$(\underline{R}\ \underline{K}^{-1}\ \underline{R}^T\ \underline{R} - \frac{1}{\omega^2}\ \underline{R})\ \underline{x} = 0 \qquad (52)$$

where $\underline{R}^T\ \underline{R} = \underline{M}$

In this case

$$\underline{R} = \underline{R}^T = \begin{bmatrix} \sqrt{m_{11}} & 0 & \cdots & 0 \\ 0 & \sqrt{m_{22}} & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & \sqrt{m_{rr}} \end{bmatrix}$$

Factoring $\underline{R}$ from equation (52) yields

$$(\underline{R}\ \underline{K}^{-1}\ \underline{R} - \frac{1}{\omega^2}\ \underline{I})\ \underline{R}\ \underline{x} = 0 \qquad (53)$$

where $\underline{K}^{-1}$ is the flexibility matrix of the system. It is seen from equation (53) that if the matrix $\underline{R}\ \underline{K}^{-1}\ \underline{R} - \frac{1}{\omega^2}\ \underline{I}$ is considered to be the characteristic matrix, then the eigenvalues are

$$\lambda_n = \frac{1}{\omega_n^2}$$

and the eigenvectors may be multiplied by $\underline{R}^{-1}$ to obtain the corresponding mode shapes. This is the method which was used in the calculations for the sample problem.

E. Results.

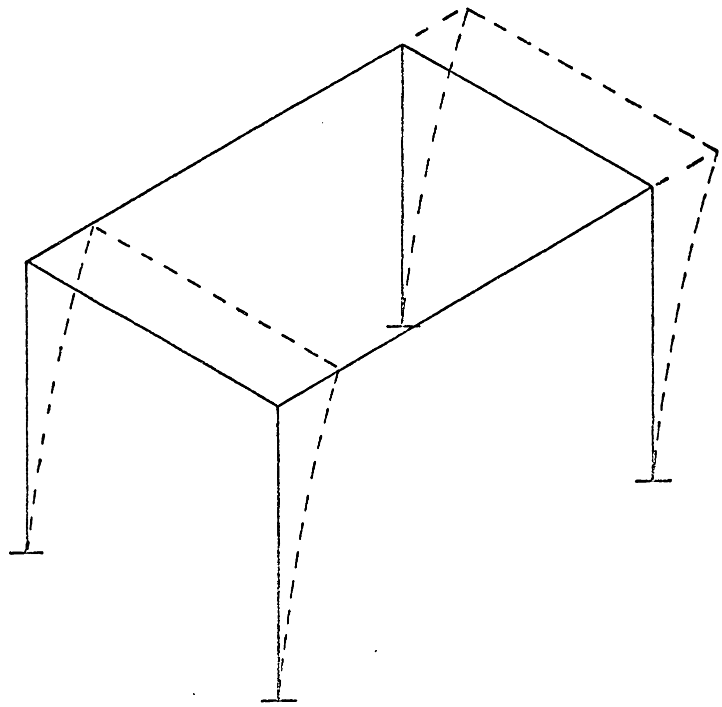
The natural frequencies and corresponding mode shapes for a sample piping system were calculated with the analytical procedures developed in this study and are shown in Table 2 and Figures 7-10. Table 2 shows the natural frequencies computed in the present work as well as computed and measured natural frequencies previously published for the sample system. Figures 7-10 show the computed mode shapes for the lowest eight natural frequencies of the sample system.

VI. Discussion of Results.

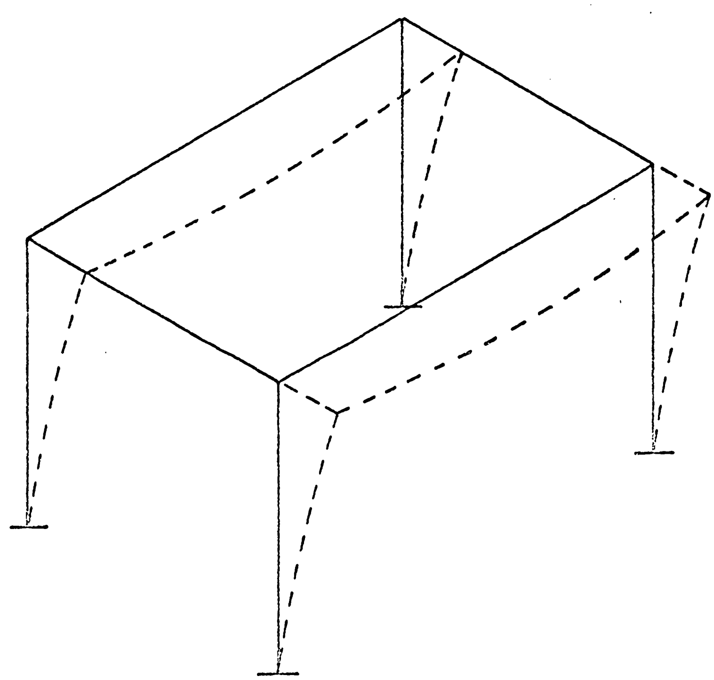
The results are in excellent agreement with the theoretical and experimental results reported in reference (9) for the model investigated therein. The differences in the influence coefficients contained in Table I are believed to be the result of including shear deflections in the present investigation. The resulting differences in the frequencies were not as great since the frequencies are a function of the square root of the influence coefficients. There may have also been slight differences in the computation of the mass of the system and in the value of the modulus of elasticity used. The value used in this paper was $E = 30 \times 10^6$ psi.

TABLE 2 FREQUENCIES FOR SAMPLE PROBLEM

| Reference (9) Calculated | Reference (9) Measured | Laboratory Direction of Excitation | Present Calculations |
|---|---|---|---|
| 109.0 | 110 | X | 105.8 |
| 115.0 | 117 | Z | 109.9 |
| 135.0 | 134 | X,Z | 130.1 |
| 212.5 | 214 | X,Z | 195.9 |
| 352.4 | 359 | X | 355.3 |
| 394.6 | 382 | Y | 380.4 |
| 422.2 | 416 | Y | 404.1 |
| 532.1 | 553 | Z | 486.9 |
| 655.8 | | | 672.1 |
| 684.9 | 697 | Y | 697.1 |
| 760.7 | | | 772.2 |
| 822.1 | 821 | X,Y | 817.5 |
| 849.3 | 853 | Y | 857.2 |
| 873.2 | 885 | X | 880.1 |
| 903.5 | 989 | X,Y | 905.4 |
| 928.2 | 927 | X | 906.5 |
| 935.6 | | | 927.9 |
| 939.0 | | | 929.5 |

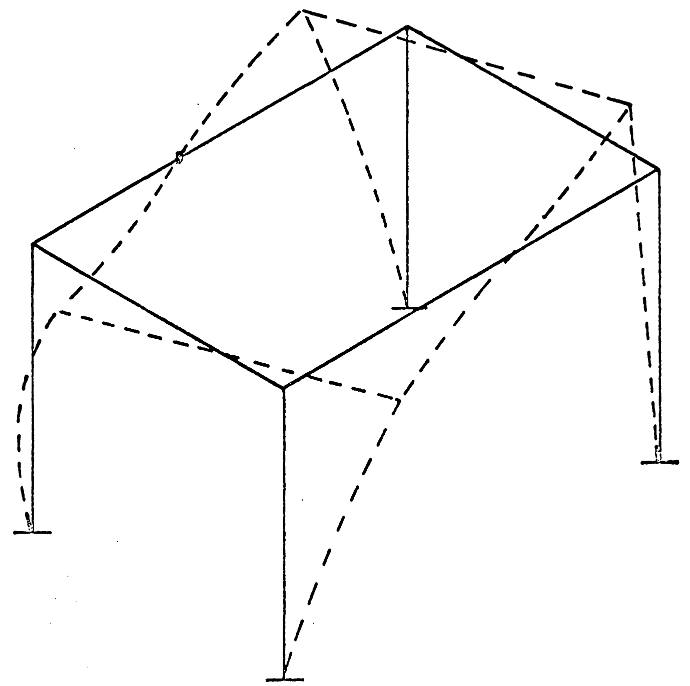


1st MODE - 105.8 CPS

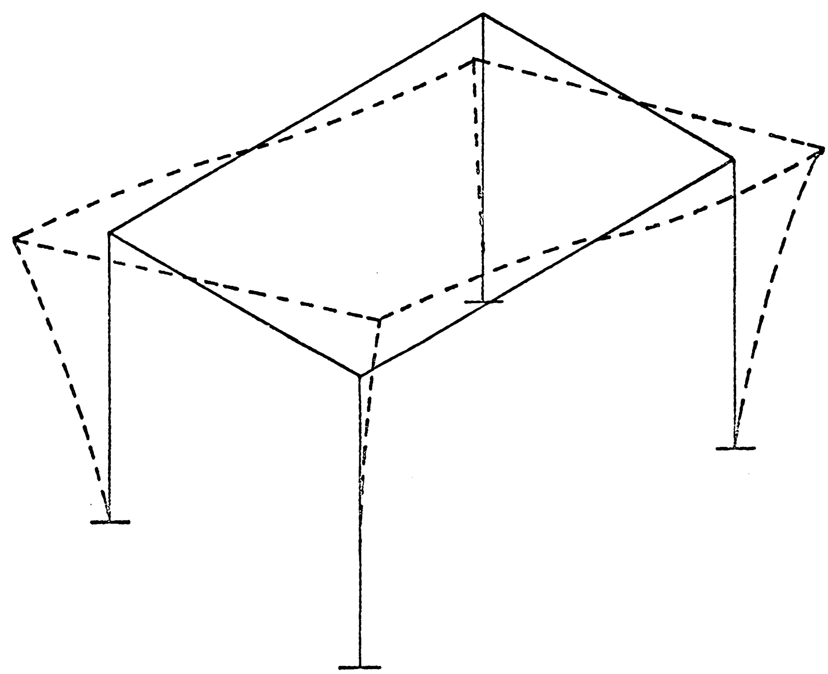


2nd MODE - 109.9 CPS

FIGURE 7 FIRST AND SECOND MODE SHAPES

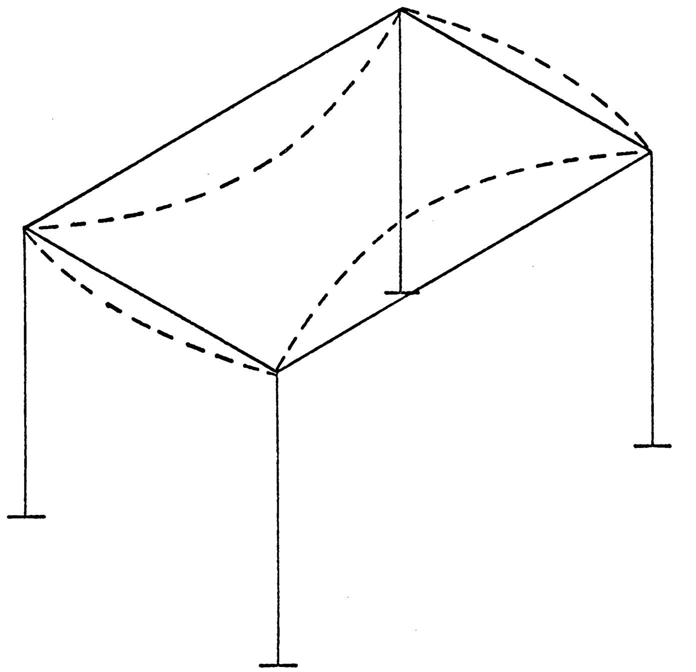


3rd MODE - 130.1 CPS

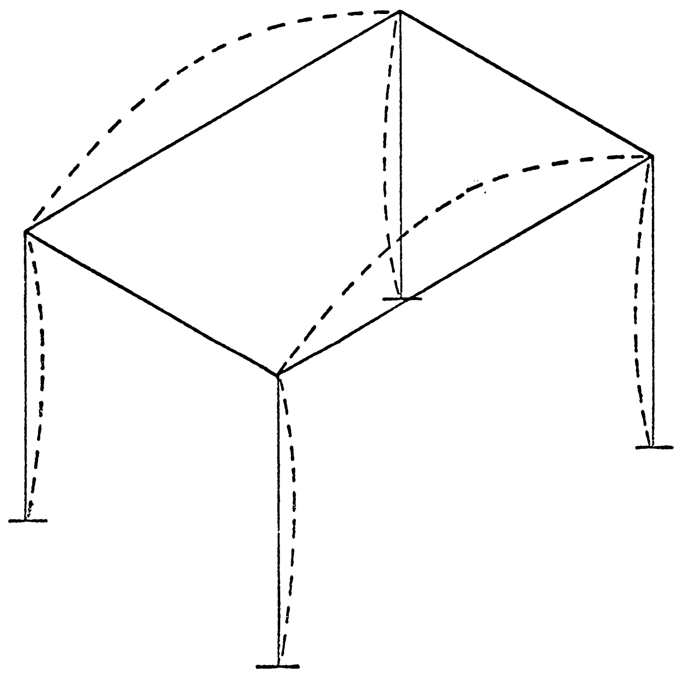


4th MODE - 195.9 CPS

FIGURE 8 THIRD AND FOURTH MODE SHAPES

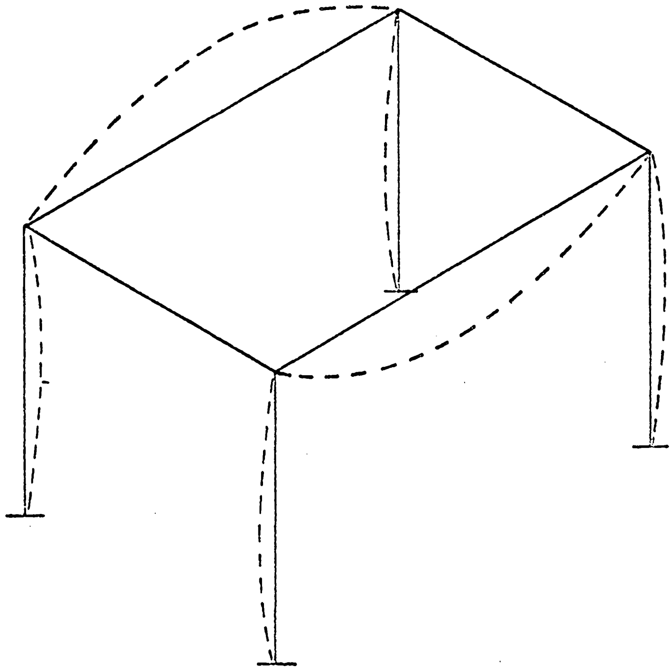


5th MODE-355.3 CPS

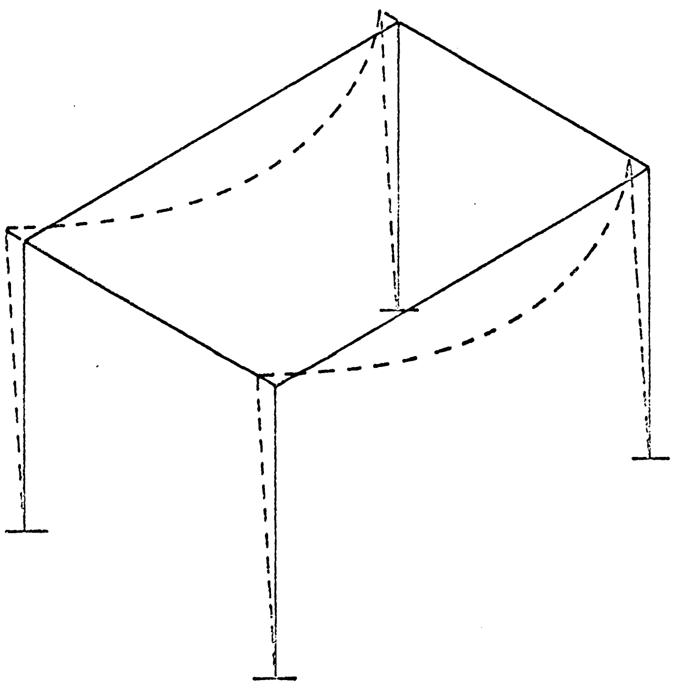


6th MODE-180.4 CPS

FIGURE 9 FIFTH AND SIXTH MODE SHAPES



7th MODE - 404.1 CPS



8th MODE - 486.7 CPS

FIGURE 10 SEVENTH AND EIGHTH MODE SHAPES

A word of caution should be noted at this point, however, lest the belief be conveyed that the process presented here will always yield results of such excellent accuracy. The choice of the mass distribution for a system to be analyzed may exert a large influence on the results, so the matter of good judgment has not been eliminated.

An IBM 1620 digital computer was used for all the computations in the sample problem except for the inversion of the $84 \times 84$ stiffness matrix which was performed by an IBM 7090 at a computer service bureau.

The assumption that rotatory inertia is negligible appears to have been upheld by the good agreement with the measured results shown in reference (9).

## VII. Conclusions.

The approach of developing a total stiffness matrix for the system was found to have the advantage of simplicity. The use of stiffnesses rather than flexibilities facilitated the formulation of a set of equilibrium equations which were readily coded for solution with a digital computer. The equations are just as easily written for a system containing many closed loops, branch points, and anchor points as for a simple two-anchor system. The only limitation on the size of the system which can be analyzed is imposed by the memory storage in the computer the analyst utilizes.

Judgment of the analyst in choosing the number, size, and placement of masses necessary to represent a physical system plays an important role in obtaining good results. This should be further investigated.

If the system to be analyzed is composed of prismatic beams, the size of the eigenvalue-eigenvector problem can be reduced by omitting terms related to rotatory inertia without significant error, as was done for the sample problem.

## VIII. Summary.

This paper presents a complete outline of the steps to be taken for the free vibration analysis of a multi-anchor complex piping system. Although the size of the problem which can be worked is limited by the size of the digital computer available to the analyst, a large number of practical problems can be solved. A lumped mass approach is employed, and the analysis employs matrix methods, with the natural frequencies and mode shapes emerging in the form of eigenvalues and eigenvectors of a symmetric matrix. The most difficult tasks to be performed in solving a vibration problem using lumped masses is actually the static problem of formulating the stiffness matrix or the matrix of influence coefficients. This paper presents a straightforward method for accomplishing this.

It is believed that the procedures presented in this paper can be extended to allow the analysis of foundation-like structures composed of segments with cross-sections different from pipes by changing certain of the pipe-stress shape coefficients which formed a basis for computation of the stiffness matrix. The work required to make this extension should be within reason.

IX. Acknowledgments.

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Appreciation is also extended to the author's immediate supervisor, , for his helpful suggestions and discussions; Professor F. J. Maher who served as thesis advisor; for preparing the digital computer programs; for preparing the figures; and for typing the manuscript.

## X. Bibliography.

### A. Literature Cited:


1. Brock, J. E., "A Matrix Method for Flexibility Analysis of Piping Systems," Journal of Applied Mechanics, Trans. ASME, Vol. 19, 1952, pp. 501-516.

2. Chen, L. H., "Piping Flexibility Analysis by Stiffness Method," Journal of Applied Mechanics, Trans. ASME, Vol. 81, 1959, pp. 608-612.

3. Crandall, S. H., and McCalley, R. B., Jr., "Numerical Methods of Analysis," in Shock and Vibration Handbook, Vol. 2, Harris, C. M., and Crede, C. E. (eds.), McGraw-Hill Book Co., New York, 1961, ch. 28.

4. Goldstein, H., "Classical Mechanics," Addison-Wesley Publishing Co., Inc., Reading, Mass., 1959, pp. 318-321.

5. The M. W. Kellogg Co., "Design of Piping Systems," John Wiley and Sons, Inc., 2nd edition, New York, 1956.

6. Martin, D. W., "Matrices," in "International Science and Technology," No. 34, Conover-Mast Publications, New York, October, 1964, pp. 58-70.

7. McCuskey, S. W., "An Introduction to Advanced Dynamics," Addison-Wesley Publishing Co., Inc., Reading, Mass., 1959, pp. 155-156.

8. Neubert, V. H., "Shock Analysis of Structural Networks," Office of Naval Research Contract No. Nonr-656 (28) (X) Interim Report No. 1, Feb., 1964.

9. Neubert, V. H., and Ezell, W. H., "Dynamic Behavior of a Foundation-Like Structure," in "Colloqium on Mechanical Impedance Methods for Mechanical Vibrations," R. Plunkett (ed.), ASME, December 1958.

10. Timoshenko, S., "Vibration Problems in Engineering," D. Van Nostrand Company, Inc., Princeton, New Jersey, 1955, pp. 329-330.

B. Literature Examined:

1. Belsheim, R. O., and O'Hara, G. J., "Shock Design of Shipboard Equipment, Part 1 - Dynamic Design - Analysis Method," U. S. Naval Research Laboratory Report 5545, 1960.

2. Borg, M. F., "Nobs 66253 Final Technical Report, Phases III, IV, and V, Dynamic Behavior of Foundations and Hull-Like Structures," General Dynamics Corporation, Electric Boat Division, 1960.

3. Crawford, L., "Piping Under Dynamic Loads," Journal of the American Society of Naval Engineers, Vol. 68, 1956, p. 345.

4. McMinn, S. J., "Matrices for Structural Analysis," John Wiley and Sons, Inc., New York, 1962.

5. Soule, J. W., "The Solution of Multiple Branch Piping - Flexibility Analysis by Tensor Analysis," Trans. ASME, Vol. 78, 1956, pp. 176-180.

6. Thomson, W. T., "Mechanical Vibrations," 2nd ed., Prentice-Hall, Inc., Englewood Cliffs, N. J., 1953, pp. 169-170.

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## XII. Appendix. Development of Shape Coefficients.

The following derivation will closely parallel the work presented in Design of Piping Systems by the M. W. Kellogg Company. Essentially, it represents the development of a set of six simultaneous linear equations which apply to various segments of the pipe line. Distributed loads will not be accounted for here, and to keep the presentation as brief as possible, only one of the equations will be derived in its entirety.

Consider a section of pipe line fixed at one end and loaded with various forces and moments at the other end, which is free. If the pipe is thought of as a prismatic bar under the influence of a variable bending moment, T, and a torsional moment $T_t$, the strain energy, U, absorbed by the pipe is

$$U = \int_0^{\ell} \frac{T^2}{2EI}\, d\ell + \int_0^{\ell} \frac{{T_t}^2}{2GJ}\, d\ell \qquad (A.1)$$

where the effects of axial forces and shear deformation have been neglected in comparison with bending and torsional effects. (These coefficients were later added to the shape coefficient matrix in order to make inversion possible.) For curved sections of pipe, correction factors are introduced to account for the variation of the actual stress distribution from that predicted by elementary theory. These correction terms are called flexibility factors and are denoted by the letter k. They are based on theoretical formulas as outlined in Chapter III of the reference (5). Thus, for an arbitrary configuration,

$$U = \int_0^{\ell} k \frac{T_b^2}{2EI} d\ell + \int_0^{\ell} k \frac{T'_b{}^2}{2EI} + \int_0^{\ell} \frac{T_t^2}{2GJ} d\ell \qquad (A.2)$$

where the bending moment has been divided into its two components: $T_b$, in the plane of the configuration, and $T'_b$, transverse to the plane of the configuration; and it is assumed that k is unity for torsion. If the ratio of the elastic moduli is taken as $E/G = 2(\mu+1) = 2.6$ (using Poisson's ratio, $\mu = 0.3$) and it is noted that $J/I = 2$ for an annular cross-section, then U reduces to

$$U = \int_0^{\ell} k \frac{T_b^2}{2EI} d\ell + \int_0^{\ell} k \frac{T'_b{}^2}{2EI} d\ell + \int_0^{\ell} 1.3 \frac{T_t^2}{2EI} d\ell \quad (A.3)$$

The displacement under external loads of any point in an elastic body can be found with the use of Castigliano's Theorem. This theorem states that the deflection, $\delta_p$, in the direction of and at the point of application of a force, $F_p$, is given by the partial derivative of the strain energy with respect to the force $F_p$. Moments and rotations may be thought of as forces and displacements, respectively, in the application of this theorem. Thus

$$\left.\begin{aligned} \delta &= \frac{\partial U}{\partial F_p} \\ \text{and} \qquad \theta_p &= \frac{\partial U}{\partial T_p} \end{aligned}\right\} \qquad (A.4)$$

With these relations the general expressions for the rotations and displacements can be written,

$$\delta p = \int_o^{\ell} k \frac{T_b \, \partial T_b}{EI \, \partial F_p} \, d\ell + \int_o^{\ell} k \frac{T'_b \, \partial T'_b}{EI \, \partial F_p} \, d\ell + \int_o^{\ell} 1.3 \frac{T_t \, \partial T_t}{EI \, \partial F_p} \, d\ell$$

and (A.5)

$$\theta p = \int_o^{\ell} k \frac{T_b \, \partial T_b}{EI \, \partial T_p} \, d\ell + \int_o^{\ell} k \frac{T'_b \partial T'_b}{EI \, \partial T_p} \, d\ell + \int_o^{\ell} 1.3 \frac{T_t \, \partial T_t}{EI \, \partial T_p} \, d\ell$$

These equations can be slightly modified by letting EI be the flexural rigidity of some reference section of the configuration, $E_nI_n$ be the flexural rigidity of the section under consideration, and making the definition

$$Q = \frac{EI}{E_nI_n}$$

Substituting this into equations (5) yields

$$\delta_p = \frac{1}{EI}\left[\int_0^{\ell} kT_b \frac{\partial T_b}{\partial F_p} Q\,d\ell + \int_0^{\ell} kT'_b \frac{\partial T_b}{\partial F_p} Q\,d\ell + \int_0^{\ell} 1.3T_t \frac{\partial T_t}{\partial F_p} Q\,d\ell\right] \tag{A.6}$$

$$\Theta_p = \frac{1}{EI}\left[\int_0^{\ell} kT_b \frac{\partial T_b}{\partial T_p} Q\,d\ell + \int_0^{\ell} kT'_b \frac{\partial T'_b}{\partial T_p} Q\,d\ell + \int_0^{\ell} 1.3T_t \frac{\partial T_t}{\partial T_p} Q\,d\ell\right] \tag{A.6}$$

Figure 1 represents a segment of a pipe line in space assumed completely fixed at one end, point O', and free to rotate and deflect at the other end, point A.

The origin of the standard coordinate system is located at an arbitrary point O. The line is considered weightless. For simplicity, the figure will be confined to one plane, in this case the plane defined by z = constant, which will be called a z-plane. The inclination that any part of the member makes with the horizontal is designated by the angle $\psi$, and in general $\psi$ varies along the member. The angle $\psi$ is considered positive when measured from the horizontal in a counter-clockwise direction as viewed from a positive axis. By applying moments $T_{xA}$, $T_{yA}$, $T_{zA}$, and forces $F_{xA}$, $F_{yA}$, $F_{zA}$, deflections $\delta_{xA}$, $\delta_{yA}$, $\delta_{zA}$, and rotations $\theta_{xA}$, $\theta_{yA}$, $\theta_{zA}$ are imposed. These deformations are related to the applied forces and moments through equations (A.6). The right-hand sign rule for moments will be adopted. For an arbitrary point on the pipe section,

$$T_b = T_{zA} + F_{yA}\,(x_A - x) - F_{xA}\,(y_A - y) \qquad \text{(A.7a)}$$

$$T'_b = \left[T_{yA} + F_{xA}\,(z_A - z) - F_{zA}\,(x_A - x)\right]\cos\psi$$

$$-\left[T_{xA} + F_{zA}\,(y_A - y) - F_{yA}\,(z_A - z)\right]\sin\psi \qquad \text{(A.7b)}$$

$$T_t = T_{xA} + F_{zA}\,(y_A - y) - F_{yA}\,(z_A - z)\cos\psi \qquad \text{(A.7c)}$$

$$+T_{yA} + F_{xA}\,(z_A - z) - F_{zA}\,(x_A - x)\,\sin\psi \tag{A.7d}$$

Writing equation (A.6) for the rotation about the x-axis gives

$$EI\theta_{xA} = \int_0^{\ell} kT_b \frac{\partial T_b}{\partial T_{xA}}\,Qd\ell + \int_0^{\ell} kT'_b \frac{\partial T'_b}{\partial T_{xA}}\,Qd\ell + \int_0^{\ell} 1.3T_t \frac{\partial T_t}{\partial T_{xA}}\,Qd\ell \tag{A.8}$$

which for this presentation is the only deflection equation to be developed in its entirety. The procedure for the remaining five equations will be exactly the same. The partial derivatives appearing in equation (A.8) are given by

$$\frac{\partial T_b}{\partial T_{xA}} = 0$$

$$\frac{\partial T'_b}{\partial T_{xA}} = \sin\psi \tag{A.9}$$

$$\frac{\partial T_t}{\partial T_{xA}} = \cos\psi$$

Substituting equations (A.7) and (A.9) into equation (A.8) and rearranging gives the following equation for $\theta_{xA}$:

$$EI\theta_{xA} = T_{xA}\int_{A}^{O'} (1.3\ \cos^2\psi + k\ \sin^2\psi)\ Q d\ell$$

$$+ T_{yA}\int_{A}^{O'} (1.3\ -k)\ \sin\ \psi\ \cos\ \psi\ Q d\ell$$

$$+ F_{xA}\int_{A}^{O'} (z_A - z)\ (1.3\ -\ k)\ \sin\psi\ \cos\psi\ Q d\ell$$

$$-\ F_{yA}\int_{A}^{O'} (z_A - z)\ (1.3\ \cos^2\psi + k\ \sin^2\psi)\ Q d\ell$$

$$+ F_{zA}\int_{A}^{O'} (y_A - y)\ (1.3\ \cos^2\psi + k\ \sin^2\psi)$$

$$-\ (x_A - x)\ (1.3 - k)\ \sin\psi \cos\psi Q d\ell \qquad (A.10)$$

Making the definitions

$$s = \int k Q d\ell$$

$$u = \int (k\ \cos^2\psi + 1.3\ \sin^2\psi)\ Q d\ell$$

$$v = \int (1.3\ \cos^2\psi + k\ \sin^2\psi)\ Q d\ell$$

$$q = \int (1.3\ -\ k)\ \sin\ \psi\ \cos\ \psi\ Q d\ell$$

$$v_o = \int y dv\ -\ \int x dq$$

makes it possible to write equation (A.10) in a more compact form:

$$EI\theta_{xA} = T_{xA} \int_{A}^{O'} dv + T_{yA} \int_{A}^{O'} dq + F_{xA} \int_{A}^{O'} C_z \, dq$$

$$- F_{yA} \int_{A}^{O'} C_z \, dv - F_{zA} \int_{A}^{O'} (dv_o - y_A \, dv + x_A \, dq)$$

(A.11

where $C_z$ is the constant value for $(z_A - z)$. It is easily seen that the coefficients of the applied forces and moments are dependent only on the shape and orientation of the pipe section under consideration. In the same manner, equations for the remaining rotations and deflections may be developed, yielding the following matrix equation:

$$\underline{x}_A = \underline{G}_A \, \underline{f}_A \qquad \text{(A.12)}$$

where $\underline{G}_A$ is the matrix of coefficients derived in the above manner for pipe segment (a). This derivation has been limited to a z-plane; however, it may be easily extended to the x and y planes by cyclic permutation of the coordinates. Skewed members have also been successfully analyzed, but this will not be considered here. To obtain the shape coefficient matrix from the above coefficients, it is merely necessary to apply equations (21) and (26) to equation (A.12) to obtain

$$\underline{B}^T_{oA}\ \underline{x}^o_A = \underline{G}_A\ \underline{B}_{Ao}\ \underline{f}^o_A \tag{A.13}$$

or

$$\underline{x}^o_A = \underline{B}^T_{Ao}\ \underline{G}_A \underline{B}_{Ao}\ \underline{f}^o_A \tag{A.14}$$

From a comparison of equations (18) and (A.14) it is seen that the shape coefficients for pipe section (a) are given by

$$\underline{A}^o_A = \underline{B}^T_{Ao}\ \underline{G}_A\ \underline{B}_{Ao}$$

A complete list of the various shape coefficients for concentrated loads is presented on pages 310-313 of the Kellogg book.

ON THE VIBRATION ANALYSIS OF A COMPLEX FOUNDATION

By

WILLIAM N. KRAUSE

ABSTRACT

This paper presents a detailed discussion of the development of the matrix equations used in the vibration analysis of a lumped-mass approximation of a multi-anchor piping system or foundation structure. The investigation is presented in two main sections, the first of which presents a formulation of the matrix eigenvalue problem for small oscillations. The development of the stiffness matrix is presented in the second section. The coefficients derived by the M. W. Kellogg Co. for the solution of pipe stress problems are utilized here, as well as the matrix transformation methods developed by J. E. Brock in "A Matrix Method For Flexibility Analysis of Piping Systems".

A sample four-anchor foundation was analyzed and the results were in close agreement with measured results published by V. H. Neubert and W. H. Ezell in "Dynamic Behavior of a Foundation-Like Structure".

The procedures presented in this paper will theoretically apply to any piping system of any degree of complexity, but practical limitations are imposed by the size of presently available digital computers.