PROBABILITY-ONE HOMOTOPY ALGORITHMS FOR FULL AND REDUCED ORDER H^2/H^∞ CONTROLLER SYNTHESIS

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Summary

Homotopy algorithms for both full- and reduced-order LQG controller design problems with an H^∞ constraint on disturbance attenuation are developed. The H^∞ constraint is enforced by replacing the covariance Lyapunov equation by a Riccati equation whose solution gives an upper bound on H^2 performance. The numerical algorithm, based on homotopy theory, solves the necessary conditions for a minimum of the upper bound on H^2 performance. The algorithms are based on two minimal parameter formulations: Ly, Bryson, and Cannon's 2×2 block parametrization and the input normal Riccati form parametrization. An over-parametrization formulation is also proposed. Numerical experiments suggest that the combination of a globally convergent homotopy method and a minimal parameter formulation applied to the upper bound minimization gives excellent results for mixed-norm H^2/H^∞ synthesis. The nonmonotonicity of homotopy zero curves is demonstrated, proving that algorithms more sophisticated than standard continuation are necessary.

Key words Homotopy method Input normal Riccati form LQG controller synthesis H^∞ performance

1. Introduction

The H^2/H^∞ mixed-norm controller synthesis problem provides the means for simultaneously addressing H^2 and H^∞ performance objectives. In practice such controllers provide both robust performance (via suboptimal H^2) and robust stability (via H^∞). Hence mixed-norm synthesis provides a technique for trading off performance and robustness, a fundamental objective in control design.

The H^2/H^∞ mixed-norm problem has been addressed in a variety of settings. The treatment in [2], [12] utilized an H^2 cost bound as the basis for an auxiliary minimization problem. Necessary conditions for optimality within a full- and reduced-order fixed-structure setting were then used to characterize feedback control gains. These necessary conditions have the form of coupled Riccati equations in both the full- and reduced-order cases. In related work [10], [17], the H^2 cost bound in the case of equalized H^2 and H^∞ performance weights was

shown to be equal to an entropy cost functional. The centralized controller was then shown to yield a full-order controller that optimizes the entropy cost.

An additional treatment in [22] using a bounded power characterization of the H^2 norm obtained both necessary and sufficient conditions for optimality. Finally, a convex optimization approach was developed in [15] for the full-order problem.

The purpose of the present paper is to develop numerical algorithms for solving the mixed-norm H^2/H^∞ problem addressed in [2], [12]. The approach here is based upon homotopy methods which have been applied to related fixed-structure problems in [23], [7]–[9]. Using globally convergent homotopy techniques similar to those applied to the combined H^2/H^∞ model reduction problem in [7]–[9], and using a controller parametrization suggested by Ly, Bryson, and Cannon and the input normal Riccati form [6], results are obtained for the combined H^2/H^∞ full and reduced-order controller synthesis problems. However, such controller parametrizations, which use the minimum possible number of parameters, make structural assumptions about the optimal controller which may not be valid in a particular case. Invalidity of these assumptions manifests itself in numerical instability, and failure to converge. An over-parametrization formulation which does not make structural assumptions is also proposed. However, over-parametrization introduces singularity of the homotopy map at the solution and the algorithm may fail for a high dimensional system.

These homotopy methods utilize the solution of a related easily solved problem as the starting point. In the case of full-order H^2/H^∞ control with unequalized weights, the starting point is provided by the standard LQG solution. For the reduced-order problem, the starting point is obtained by constructing a low authority, nearly nonminimal LQG compensator [4].

There is considerable confusion in the control literature over the terms *continuation*, *homotopy*, and *globally convergent*. A careful discussion of the distinct meanings of these terms can be found in [20]. *Continuation* refers to the standard classical technique of solving $\rho(\theta, \bar{\lambda} + \Delta\lambda) = 0$ with fixed $\Delta\lambda > 0$, given a solution $(\bar{\theta}, \bar{\lambda})$: $\rho(\bar{\theta}, \bar{\lambda}) = 0$. It is implicitly assumed that $\theta = \theta(\lambda)$, i.e., the zero curve γ of $\rho(\theta, \lambda)$ being tracked in (θ, λ) space is monotone in λ. Other tacit assumptions are that γ does not bifurcate or otherwise contain singularities. The more general *homotopy* methods make no such assumptions, and include mechanisms to deal with bifurcations and turning points. In particular, homotopy methods do not assume that the zero curve γ is monotone in λ, i.e., $\theta = \theta(\lambda)$. *Globally convergent* means that the zero curve γ reaches a solution $\bar{\theta}$, $\rho(\bar{\theta}, 1) = 0$, from an *arbitrary* starting point θ_0, $\rho(\theta_0, 0) = 0$. A continuation or homotopy algorithm is not *a priori* globally convergent. A particular class of homotopy methods, known as *probability-one homotopy methods*, are provably globally convergent under mild assumptions [19], and their zero curve γ is guaranteed to contain no singularities with probability one. The homotopy algorithms proposed here are examples of probability-one globally convergent homotopy methods; the matrices A_0, B_0, ..., and the starting point θ_0 defined in later sections play the role of the parameter vector a in the probability-one homotopy theory [20]. Computational results for the example in [2] clearly demonstrate the nonmonotonicity in λ and that standard continuation in λ would fail.

The LQG controller synthesis problem with an H^∞ performance bound can be stated as follows: given the n-th order stabilizable and detectable plant

$$\begin{aligned} \dot{x}(t) &= A\,x(t) + B\,u(t) + D_1\,w(t), \\ y(t) &= C\,x(t) + D_2\,w(t), \end{aligned} \tag{1}$$

where $A \in \mathbf{R}^{n \times n}$, $B \in \mathbf{R}^{n \times m}$, $C \in \mathbf{R}^{l \times n}$, $D_1 \in \mathbf{R}^{n \times p}$, $D_2 \in \mathbf{R}^{l \times p}$, $D_1 D_2^T = 0$, and $w(t)$ is p-dimensional white noise, find a n_c-th order dynamic compensator

$$\begin{aligned} \dot{x}_c(t) &= A_c\, x_c(t) + B_c\, y(t), \\ u(t) &= C_c\, x_c(t), \end{aligned} \tag{2}$$

where $A_c \in \mathbf{R}^{n_c \times n_c}$, $B_c \in \mathbf{R}^{n_c \times l}$, $C_c \in \mathbf{R}^{m \times n_c}$, and $n_c \leq n$, which satisfies the following criteria:

(i) the closed-loop system (1)–(2) is asymptotically stable, i.e., $\tilde{A} = \begin{pmatrix} A & B\,C_c \\ B_c\,C & A_c \end{pmatrix}$ is asymptotically stable;

(ii) the $q_\infty \times p$ closed-loop transfer function

$$H(s) \equiv \tilde{E}_\infty (s I_{\tilde{n}} - \tilde{A})^{-1} \tilde{D},$$

from $w(t)$ to

$$z(t) = E_{1\infty} x(t) + E_{2\infty} u(t), \tag{3}$$

where $\tilde{E}_\infty = (\, E_{1\infty} \quad E_{2\infty} C_c \,)$ ($E_{1\infty} \in \mathbf{R}^{q_\infty \times n}$, $E_{2\infty} \in \mathbf{R}^{q_\infty \times m}$, $E_{1\infty}^T E_{2\infty} = 0$), $\tilde{n} = n + n_c$, and $\tilde{D} = \begin{pmatrix} D_1 \\ B_c D_2 \end{pmatrix}$, satisfies the constraint

$$\|H(s)\|_\infty \leq \gamma \tag{4}$$

where $\gamma > 0$ is a given constant; and

(iii) the performance functional

$$J(A_c, B_c, C_c) \equiv \lim_{t \to \infty} \mathcal{E}\left[x^T(t) R_1 x(t) + u^T(t) R_2 u(t)\right] \tag{5}$$

is minimized, where $\mathcal{E}$ is the expected value, $R_1 = E_1^T E_1 \in \mathbf{R}^{n \times n}$ and $R_2 = E_2^T E_2 \in \mathbf{R}^{m \times m}$ ($E_1 \in \mathbf{R}^{q \times n}$, $E_2 \in \mathbf{R}^{q \times m}$, $E_1^T E_2 = 0$) are, respectively, symmetric positive semidefinite and symmetric positive definite weighting matrices.

The closed-loop system (1)–(3) can be written as the augmented system

$$\begin{aligned} \dot{\tilde{x}}(t) &= \tilde{A}\, \tilde{x}(t) + \tilde{D}\, w(t), \\ z(t) &= \tilde{E}_\infty \tilde{x} \end{aligned} \tag{6}$$

where $\tilde{x} = \begin{pmatrix} x \\ x_c \end{pmatrix}$.

Using this notation and under the condition that $\tilde{A}$ is asymptotically stable, for a given compensator the performance (5) is given by

$$J(A_c, B_c, C_c) = \mathrm{tr}\ (\tilde{Q}\, \tilde{R}), \tag{7}$$

where $\tilde{R} = \begin{pmatrix} R_1 & 0 \\ 0 & C_c^T R_2 C_c \end{pmatrix}$ and $\tilde{Q}$ satisfies the Lyapunov equation

$$\tilde{A}\, \tilde{Q} + \tilde{Q}\, \tilde{A}^T + \tilde{V} = 0, \tag{8}$$

with symmetric positive semidefinite $V_1 = D_1 D_1^T$, symmetric positive definite $V_2 = D_2 D_2^T$, and

$$\tilde{V} = \begin{pmatrix} V_1 & 0 \\ 0 & B_c V_2 B_c^T \end{pmatrix}.$$

LEMMA 1 [2]. *Let (A_c, B_c, C_c) be given and assume there exists $\mathcal{Q} \in \mathbf{R}^{\tilde{n} \times \tilde{n}}$ satisfying*

$$\mathcal{Q} \quad \text{is symmetric and nonnegative definite} \tag{9}$$

and

$$\tilde{A}\mathcal{Q} + \mathcal{Q}\tilde{A}^T + \gamma^{-2}\mathcal{Q}\tilde{R}_\infty\mathcal{Q} + \tilde{V} = 0, \tag{10}$$

where $\tilde{R}_\infty = \begin{pmatrix} R_{1\infty} & 0 \\ 0 & C_c^T R_{2\infty} C_c \end{pmatrix}$, $R_{1\infty} = E_{1\infty}^T E_{1\infty}$, *and* $R_{2\infty} = E_{2\infty}^T E_{2\infty}$ *are symmetric positive semidefinite matrices. Then*

$$(\tilde{A}, \tilde{D}) \quad \text{is stabilizable} \tag{11}$$

if and only if

$$\tilde{A} \quad \text{is asymptotically stable.}$$

In this case

$$\|H(s)\|_\infty \leq \gamma, \tag{12}$$

$$\tilde{Q} \leq \mathcal{Q} \qquad (\mathcal{Q} - \tilde{Q} \text{ is nonnegative definite}),$$

and

$$\text{tr } \tilde{Q}\tilde{R} \equiv J(A_c, B_c, C_c) \leq \mathcal{J}(A_c, B_c, C_c) \equiv \text{ tr } \mathcal{Q}\tilde{R}.$$

Hence the satisfaction of (9) and (10) along with the generic condition (11) leads to: i) closed-loop stability; 2) prespecified H^∞ attenuation; and 3) an upper bound for the H^2 performance criterion.

The auxiliary minimization problem is to determine (A_c, B_c, C_c) that minimizes $\mathcal{J}$ (A_c, B_c, C_c) and thus provides a bound for the actual H^2 criterion $J(A_c, B_c, C_c)$.

$(A_c, B_c, C_c, \mathcal{Q})$ is restricted to the open set

$$\mathcal{S} \equiv \{(A_c, B_c, C_c, \mathcal{Q}) : \tilde{A} \text{ and } \tilde{A} + \gamma^{-2}\mathcal{Q}\tilde{R} \text{ are asymptotically stable,}$$
$$\mathcal{Q} \text{ is symmetric positive definite, and } (A_c, B_c, C_c) \text{ is controllable and observable } \}.$$

2. Homotopy algorithm based on Ly's formulation

In optimizing performance with respect to stabilizing controllers of fixed order n_c, it is desirable to consider controller realizations of a specified structure. In this regard there exist a variety of realizations that involve fewer than the $n_c(n_c+m+l)$ parameters appearing in a fully populated parameterization [13], [11], [1], [5], [14], [18], [6], [16]. However, as discussed in [13], [11], [5], realizations that involve a minimal number of independent parameters cannot provide a smooth, global parameterization of all MIMO systems. Specialized parameterizations are also useful for realizing transfer functions of specified classes [18].

In the present paper we employ the Ly, Bryson, and Cannon parameterization proposed in [16]. The following result characterizes the particular class of transfer functions realized by the Ly, Bryson, and Cannon parameterization.

PROPOSITION 1. Suppose that G has the minimal realization (A_c, B_c, C_c), where the matrix A_c is similar to a 2×2 block-diagonal matrix where each 2×2 block has distinct eigenvalues.

There is an additional 1×1 block if n_c is odd. Then there exists a state space basis with respect to which G has a Ly-Bryson-Cannon realization.

Although this parameterization does not provide a global representation of all transfer functions even in the SISO case, it does provide a generic representation which is particularly suited for parametric optimization. Although the Ly-Bryson-Cannon form implicitly assumes somewhat more than that A_c be diagonalizable, computing the two-dimensional invariant subspaces is better conditioned than computing the eigenvectors, which algorithms that assume diagonalizability attempt to do. When the transformation to Ly-Bryson-Cannon form is not ill conditioned, this particular representation turns out to be very efficient computationally.

Ly et al. [16] introduced a canonical form with $n_c m + n_c l$ parameters. The compensator is represented with respect to a basis such that A_c is a 2×2 block-diagonal matrix (2×2 blocks with an additional 1×1 block if n_c is odd) with 2×2 blocks in the form

$$\begin{pmatrix} 0 & 1 \\ * & * \end{pmatrix},$$

B_c is a full matrix, and

$$C_c = \begin{pmatrix} (C_c)_1 & (C_c)_2 & \cdots & (C_c)_r \end{pmatrix}$$

where

$$(C_c)_i = \begin{pmatrix} 1 & * & \cdots & * \\ 0 & * & \cdots & * \end{pmatrix}^T.$$

$(C_c)_r = \begin{pmatrix} 1 & * & \cdots & * \end{pmatrix}^T$ if n_c is odd.

It is assumed that (A_c, B_c, C_c) is in Ly's form. Let $\mathcal{I}$ be the set of indices of those elements of A_c which are parameters, i.e.,

$$\mathcal{I} \equiv \{(2,1),(2,2),\ldots,(n_c,n_c)\}.$$

To optimize $\mathcal{J}(A_c, B_c, C_c)$ over the open set $\mathcal{S}$ under the constraint that symmetric positive definite $\mathcal{Q}$ satisfies (10), and (A_c, B_c, C_c) is in Ly's form, the following Lagrangian is formed:

$$\mathcal{L}(A_c, B_c, C_c, \mathcal{P}, \mathcal{Q}) \equiv \operatorname{tr}\left[\mathcal{Q}\tilde{R} + (\tilde{A}\mathcal{Q} + \mathcal{Q}\tilde{A}^T + \gamma^{-2}\mathcal{Q}\tilde{R}_\infty\mathcal{Q} + \tilde{V})\mathcal{P}\right],$$

where $\mathcal{P} \in \mathbf{R}^{\tilde{n}\times\tilde{n}}$ is a Lagrange multiplier. Setting $\partial\mathcal{L}/\partial\mathcal{Q} = 0$ yields

$$0 = \left(\tilde{A} + \gamma^{-2}\mathcal{Q}\tilde{R}_\infty\right)^T\mathcal{P} + \mathcal{P}\left(\tilde{A} + \gamma^{-2}\mathcal{Q}\tilde{R}_\infty\right) + \tilde{R}. \tag{13}$$

Partition $\mathcal{Q}$, $\mathcal{P} \in \mathbf{R}^{\tilde{n}\times\tilde{n}}$ as

$$\mathcal{Q} = \begin{pmatrix} \mathcal{Q}_1 & \mathcal{Q}_{12} \\ \mathcal{Q}_{12}^T & \mathcal{Q}_2 \end{pmatrix}, \qquad \mathcal{P} = \begin{pmatrix} \mathcal{P}_1 & \mathcal{P}_{12} \\ \mathcal{P}_{12}^T & \mathcal{P}_2 \end{pmatrix}. \tag{14}$$

The partial derivatives of $\mathcal{L}$ can be computed as

$$\begin{aligned}
\frac{\partial\mathcal{L}}{\partial(A_c)_{ij}} &= 2\left(\mathcal{P}_{12}^T\,\mathcal{Q}_{12} + \mathcal{P}_2\,\mathcal{Q}_2\right)_{ij}, \quad (i,j) \in \mathcal{I}, \\
\frac{\partial\mathcal{L}}{\partial B_c} &= 2\mathcal{P}_2 B_c V_2 + 2(\mathcal{P}_{12}^T\mathcal{Q}_1 + \mathcal{P}_2\mathcal{Q}_{12}^T)C^T, \\
\frac{\partial\mathcal{L}}{\partial(C_c)_{ij}} &= \Big(2R_2C_c\mathcal{Q}_2 + 2B^T(\mathcal{P}_1\mathcal{Q}_{12} + \mathcal{P}_{12}\mathcal{Q}_2) \\
&\qquad + \gamma^{-2}R_{2\infty}C_c\left[(\mathcal{Q}_{12}^T\mathcal{P}_1 + \mathcal{Q}_2\mathcal{P}_{12}^T)\mathcal{Q}_{12}(\mathcal{Q}_{12}^T\mathcal{P}_{12} + \mathcal{Q}_2\mathcal{P}_2)\mathcal{Q}_2\right]\Big)_{ij}, \qquad i > 1.
\end{aligned}$$

Let A_f, B_f, C_f, γ_f, R_{1f}, R_{2f}, $R_{1\infty f}$, $R_{2\infty f}$, V_{1f}, and V_{2f} denote A, B, C, λ, R_1, R_2, $R_{1\infty}$, $R_{2\infty}$, V_1, and V_2 in the above and define $A(\lambda), B(\lambda), C(\lambda), \gamma(\lambda), R_1(\lambda), R_2(\lambda), R_{1\infty}(\lambda)$, $R_{2\infty}(\lambda), V_1(\lambda), V_2(\lambda)$ as

$$\begin{aligned}
A(\lambda) &= A_0 + \lambda(A_f - A_0), \quad B(\lambda) = B_0 + \lambda(B_f - B_0),\\
C(\lambda) &= C_0 + \lambda(C_f - C_0), \quad \gamma(\lambda) = \gamma_0 + \lambda(\gamma_f - \gamma_0),\\
R_1(\lambda) &= R_{1,0} + \lambda(R_{1f} - R_{1,0}), \quad R_2(\lambda) = R_{2,0} + \lambda(R_{2f} - R_{2,0}),\\
R_{1\infty}(\lambda) &= R_{1\infty,0} + \lambda(R_{1\infty f} - R_{1\infty,0}), \quad R_{2\infty}(\lambda) = R_{2\infty,0} + \lambda(R_{2\infty f} - R_{2\infty,0}),\\
V_1(\lambda) &= V_{1,0} + \lambda(V_{1f} - V_{1,0}), \quad V_2(\lambda) = V_{2,0} + \lambda(V_{2f} - V_{2,0}),
\end{aligned} \tag{15}$$

and denote them by A, B, C, γ, R_1, R_2, $R_{1\infty}$, $R_{2\infty}$, V_1, and V_2 respectively in the following. Let

$$\begin{aligned}
H_{A_c}(\theta,\lambda) &= \frac{1}{2}\frac{\partial \mathcal{L}}{\partial A_c} = \mathcal{P}_{12}^T \mathcal{Q}_{12} + \mathcal{P}_2 \mathcal{Q}_2,\\
H_{B_c}(\theta,\lambda) &= \frac{1}{2}\frac{\partial \mathcal{L}}{\partial B_c} = \mathcal{P}_2 B_c V_2 + (\mathcal{P}_{12}^T \mathcal{Q}_1 + \mathcal{P}_2 \mathcal{Q}_{12}^T) C^T,\\
H_{C_c}(\theta,\lambda) &= \frac{1}{2}\frac{\partial \mathcal{L}}{\partial C_c} = R_2 C_c \mathcal{Q}_2 + B^T(\mathcal{P}_1 \mathcal{Q}_{12} + \mathcal{P}_{12}\mathcal{Q}_2)\\
&\quad + \frac{1}{2}\gamma^{-2} R_{2\infty} C_c \big[(\mathcal{Q}_{12}^T \mathcal{P}_1 + \mathcal{Q}_2 \mathcal{P}_{12}^T)\mathcal{Q}_{12} + (\mathcal{Q}_{12}^T \mathcal{P}_{12} + \mathcal{Q}_2 \mathcal{P}_2)\mathcal{Q}_2\big],
\end{aligned} \tag{16}$$

where in H_{A_c} only those elements corresponding to the parameter elements of A_c are of interest and

$$\theta \equiv \begin{pmatrix} (A_c)_{\mathcal{I}} \\ \text{Vec } (B_c) \\ \text{Vec } (C_c)_{\mathcal{T}.} \end{pmatrix} \tag{17}$$

denotes the independent variables, $\mathcal{Q}$ and $\mathcal{P}$ satisfy respectively (10) and (13), $(A_c)_{\mathcal{I}}$ is a vector consisting of those elements in A_c with indices in the set $\mathcal{I}$, i.e.,

$$(A_c)_{\mathcal{I}} = \big((A_c)_{21}, (A_c)_{22}, \cdots, (A_c)_{n_c n_c}\big)^T,$$

and $(C_c)_{\mathcal{T}.}$ is the matrix obtained from rows $\mathcal{T} = \{2, \ldots, m\}$ of C_c. Vec(P) for a matrix $P \in \mathbf{R}^{p \times q}$ is the concatenation of its columns:

$$\text{Vec}(P) \equiv \begin{pmatrix} P_{\cdot 1} \\ P_{\cdot 2} \\ \vdots \\ P_{\cdot q} \end{pmatrix} \in \mathbf{R}^{pq}.$$

The most natural homotopy map, given the embeddings in (15), is defined as

$$\rho(\theta,\lambda) = \begin{pmatrix} \big[H_{A_c}(\theta,\lambda)\big]_{\mathcal{I}} \\ \text{Vec } \big[H_{B_c}(\theta,\lambda)\big] \\ \text{Vec } \big[H_{C_c}(\theta,\lambda)\big]_{\mathcal{T}.} \end{pmatrix}, \tag{18}$$

and its Jacobian matrix is

$$D\rho(\theta,\lambda) = \big(D_\theta \rho(\theta,\lambda), D_\lambda \rho(\theta,\lambda)\big).$$

In practice it may be difficult to find the initial point θ_0 such that $\rho(\theta_0, 0) = 0$. A somewhat more artificial (lacking a physical interpretation) homotopy then, letting θ_0 be the chosen initial point, is the Newton homotopy map defined as

$$\bar{\rho}(\theta, \lambda) = \rho(\theta, \lambda) - (1 - \lambda)\rho(\theta_0, 0),$$

which will give rise to an extra term $\rho(\theta_0, 0)$ in $D_\lambda \bar{\rho}(\theta, \lambda)$. To guarantee a full rank Jacobian matrix along the whole homotopy zero curve except possibly at the solution corresponding to $\lambda = 1$, define the homotopy map to be

$$\hat{\rho}(\theta, \lambda) = \lambda\rho(\theta, \lambda) + (1 - \lambda)(\theta - \theta_0). \tag{19}$$

The Jacobian matrix of $\hat{\rho}$ is given by

$$D\hat{\rho}(\theta, \lambda) = \big(\lambda D_\theta \rho(\theta, \lambda) + (1 - \lambda)I, \rho(\theta, \lambda) + \lambda D_\lambda \rho(\theta, \lambda) - (\theta - \theta_0)\big).$$

In the following, the homotopy map (18) is assumed for the full-order problem and (19) is assumed for the reduced-order case since the reduced-order initialization scheme produces a singular starting point if (18) is used. Define

$$\begin{aligned}
\hat{H}_{A_c}\big(\mathcal{P}^{(j)}, \mathcal{Q}^{(j)}\big) &= \mathcal{P}_{12}^{T(j)}\mathcal{Q}_{12} + \mathcal{P}_{12}^{T}\mathcal{Q}_{12}^{(j)} + \mathcal{P}_2^{(j)}\mathcal{Q}_2 + \mathcal{P}_2\mathcal{Q}_2^{(j)}, \\
\hat{H}_{B_c}\big(\mathcal{P}^{(j)}, \mathcal{Q}^{(j)}\big) &= \mathcal{P}_2^{(j)}B_c V_2 + \big(\mathcal{P}_{12}^{T(j)}\mathcal{Q}_1 + \mathcal{P}_{12}^{T}\mathcal{Q}_1^{(j)} + \mathcal{P}_2^{(j)}\mathcal{Q}_{12}^{T} + \mathcal{P}_2\mathcal{Q}_{12}^{T(j)}\big)C^T, \\
\hat{H}_{C_c}(\mathcal{P}^{(j)}, \mathcal{Q}^{(j)}) &= R_2 C_c \mathcal{Q}_2^{(j)} + B^T\big(\mathcal{P}_1^{(j)}\mathcal{Q}_{12} + \mathcal{P}_1\mathcal{Q}_{12}^{(j)} + \mathcal{P}_{12}^{(j)}\mathcal{Q}_2 + \mathcal{P}_{12}\mathcal{Q}_2^{(j)}\big) \\
&\quad + \frac{\gamma^{-2}}{2} R_{2\infty} C_c \Big[(\mathcal{Q}_{12}^{T(j)}\mathcal{P}_1 + \mathcal{Q}_{12}^{T}\mathcal{P}_1^{(j)} + \mathcal{Q}_2^{(j)}\mathcal{P}_{12}^{T} + \mathcal{Q}_2\mathcal{P}_{12}^{T(j)})\mathcal{Q}_{12} + (\mathcal{Q}_{12}^{T(j)}\mathcal{P}_{12} + \mathcal{Q}_{12}^{T}\mathcal{P}_{12}^{(j)} \\
&\quad + \mathcal{Q}_2^{(j)}\mathcal{P}_2 + \mathcal{Q}_2\mathcal{P}_2^{(j)})\mathcal{Q}_2 + (\mathcal{Q}_{12}^{T}\mathcal{P}_1 + \mathcal{Q}_2\mathcal{P}_{12}^{T})\mathcal{Q}_{12}^{(j)} + (\mathcal{Q}_{12}^{T}\mathcal{P}_{12} + \mathcal{Q}_2\mathcal{P}_2)\mathcal{Q}_2^{(j)}\Big],
\end{aligned} \tag{20}$$

where the superscript (j) means $\partial/\partial\theta_j$. Using the above definitions, we have for $\theta_j = \big(A_c\big)_{kl}$, where $(k, l) \in \mathcal{I}$,

$$\begin{aligned}
\frac{\partial H_{A_c}}{\partial (A_c)_{kl}} &= \hat{H}_{A_c}\big(\mathcal{P}^{(j)}, \mathcal{Q}^{(j)}\big), \\
\frac{\partial H_{B_c}}{\partial (A_c)_{kl}} &= \hat{H}_{B_c}\big(\mathcal{P}^{(j)}, \mathcal{Q}^{(j)}\big), \\
\frac{\partial H_{C_c}}{\partial (A_c)_{kl}} &= \hat{H}_{C_c}\big(\mathcal{P}^{(j)}, \mathcal{Q}^{(j)}\big),
\end{aligned} \tag{21}$$

for $\theta_j = \big(B_c\big)_{kl}$,

$$\begin{aligned}
\frac{\partial H_{A_c}}{\partial (B_c)_{kl}} &= \hat{H}_{A_c}\big(\mathcal{P}^{(j)}, \mathcal{Q}^{(j)}\big), \\
\frac{\partial H_{B_c}}{\partial (B_c)_{kl}} &= \hat{H}_{B_c}\big(\mathcal{P}^{(j)}, \mathcal{Q}^{(j)}\big) + \mathcal{P}_2 E^{(k,l)} V_2, \\
\frac{\partial H_{C_c}}{\partial (B_c)_{kl}} &= \hat{H}_{C_c}\big(\mathcal{P}^{(j)}, \mathcal{Q}^{(j)}\big),
\end{aligned} \tag{22}$$

and for $\theta_j = (C_c)_{kl}$, where $k > 1$,

$$\begin{aligned}
\frac{\partial H_{A_c}}{\partial (C_c)_{kl}} &= \hat{H}_{A_c}\big(\mathcal{P}^{(j)}, \mathcal{Q}^{(j)}\big), \\
\frac{\partial H_{B_c}}{\partial (C_c)_{kl}} &= \hat{H}_{B_c}\big(\mathcal{P}^{(j)}, \mathcal{Q}^{(j)}\big), \\
\frac{\partial H_{C_c}}{\partial (C_c)_{kl}} &= \hat{H}_{C_c}\big(\mathcal{P}^{(j)}, \mathcal{Q}^{(j)}\big) + R_2 E^{(k,l)} \mathcal{Q}_2 + \frac{\gamma^{-2}}{2} R_{2\infty} E^{(k,l)} Y,
\end{aligned} \tag{23}$$

where

$$Y = \big(\mathcal{Q}_{12}^T \mathcal{P}_1 + \mathcal{Q}_2 \mathcal{P}_{12}^T\big)\mathcal{Q}_{12} + \big(\mathcal{Q}_{12}^T \mathcal{P}_{12} + \mathcal{Q}_2 \mathcal{P}_2\big)\mathcal{Q}_2 \tag{24}$$

and $E^{(k,l)}$ is a matrix of the appropriate dimension whose only nonzero element is $e_{kl} = 1$. $\mathcal{P}^{(j)}$ and $\mathcal{Q}^{(j)}$ can be obtained by solving the Lyapunov equation

$$\begin{aligned}
0 &= \big(\tilde{A} + \gamma^{-2}\mathcal{Q}\tilde{R}_\infty\big)\mathcal{Q}^{(j)} + \mathcal{Q}^{(j)}\big(\tilde{A} + \gamma^{-2}\mathcal{Q}\tilde{R}_\infty\big)^T + \tilde{V}^{(j)} + \tilde{A}^{(j)}\,\mathcal{Q} + \mathcal{Q}\tilde{A}^{T\,(j)} + \gamma^{-2}\mathcal{Q}\tilde{R}_\infty^{(j)}\mathcal{Q}, \\
0 &= \big(\tilde{A} + \gamma^{-2}\mathcal{Q}\tilde{R}_\infty\big)^T\mathcal{P}^{(j)} + \mathcal{P}^{(j)}\big(\tilde{A} + \gamma^{-2}\mathcal{Q}\tilde{R}_\infty\big) + \tilde{R}^{(j)} \\
&\quad + \big(\tilde{A}^{(j)} + \gamma^{-2}\mathcal{Q}^{(j)}\tilde{R}_\infty + \gamma^{-2}\mathcal{Q}\tilde{R}_\infty^{(j)}\big)^T\mathcal{P} + \mathcal{P}\big(\tilde{A}^{(j)} + \gamma^{-2}\mathcal{Q}^{(j)}\tilde{R}_\infty + \gamma^{-2}\mathcal{Q}\tilde{R}_\infty^{(j)}\big).
\end{aligned} \tag{25}$$

Similarly for λ, using a dot to denote $\partial/\partial\lambda$,

$$\begin{aligned}
\frac{\partial H_{A_c}}{\partial \lambda} &= \hat{H}_{A_c}\big(\dot{\mathcal{P}}, \dot{\mathcal{Q}}\big), \\
\frac{\partial H_{B_c}}{\partial \lambda} &= \hat{H}_{B_c}\big(\dot{\mathcal{P}}, \dot{\mathcal{Q}}\big) + \mathcal{P}_2 B_c \dot{V}_2 + (\mathcal{P}_{12}^T\mathcal{Q}_1 + \mathcal{P}_2\mathcal{Q}_{12}^T)\dot{C}^T, \\
\frac{\partial H_{C_c}}{\partial \lambda} &= \hat{H}_{C_c}\big(\dot{\mathcal{P}}, \dot{\mathcal{Q}}\big) + \dot{R}_2 C_c \mathcal{Q}_2 + \dot{B}^T\big(\mathcal{P}_1\mathcal{Q}_{12} + \mathcal{P}_{12}\mathcal{Q}_2\big) + \frac{1}{2}\gamma^{-2}\dot{R}_{2\infty} C_c Y - \gamma^{-3}\dot{\gamma} R_{2\infty} C_c Y,
\end{aligned} \tag{26}$$

where $\dot{\mathcal{P}}$ and $\dot{\mathcal{Q}}$ are obtained by solving

$$\begin{aligned}
0 &= \big(\tilde{A} + \gamma^{-2}\mathcal{Q}\tilde{R}_\infty\big)\dot{\mathcal{Q}} + \dot{\mathcal{Q}}\big(\tilde{A} + \gamma^{-2}\mathcal{Q}\tilde{R}_\infty\big)^T + \dot{\tilde{V}} + \dot{\tilde{A}}\,\mathcal{Q} + \mathcal{Q}\dot{\tilde{A}}^T + \gamma^{-2}\mathcal{Q}\dot{\tilde{R}}_\infty\mathcal{Q} \\
&\quad - 2\gamma^{-3}\dot{\gamma}\mathcal{Q}\tilde{R}_\infty\mathcal{Q}, \\
0 &= \big(\tilde{A} + \gamma^{-2}\mathcal{Q}\tilde{R}_\infty\big)^T\dot{\mathcal{P}} + \dot{\mathcal{P}}\big(\tilde{A} + \gamma^{-2}\mathcal{Q}\tilde{R}_\infty\big) + \dot{\tilde{R}} + \big(\dot{\tilde{A}} + \gamma^{-2}\dot{\mathcal{Q}}\tilde{R}_\infty + \gamma^{-2}\mathcal{Q}\dot{\tilde{R}}_\infty \\
&\quad - 2\gamma^{-3}\dot{\gamma}\mathcal{Q}\tilde{R}_\infty\big)^T\mathcal{P} + \mathcal{P}\big(\dot{\tilde{A}} + \gamma^{-2}\dot{\mathcal{Q}}\tilde{R}_\infty + \gamma^{-2}\mathcal{Q}\dot{\tilde{R}}_\infty - 2\gamma^{-3}\dot{\gamma}\mathcal{Q}\tilde{R}_\infty\big).
\end{aligned} \tag{27}$$

3. Homotopy algorithm based on input normal Riccati form

Theorem 1 [6]. *For every minimal $(\bar{A}_c, \bar{B}_c, \bar{C}_c)$, i.e., $(\bar{A}_c, \bar{B}_c)$ is controllable and $(\bar{A}_c, \bar{C}_c)$ is observable, there exist a similarity transformation U and a positive definite matrix $\Omega = \operatorname{diag}(\omega_1, \ldots, \omega_{n_c})$ such that $A_c = U^{-1}\bar{A}_c U$, $B_c = U^{-1}\bar{B}_c$, and $C_c = \bar{C}_c U$ satisfy*

$$\begin{aligned}
0 &= A_c + A_c^T + B_c B_c^T - C_c^T C_c, \\
0 &= A_c^T\Omega + \Omega A_c + C_c^T C_c - \Omega B_c B_c^T \Omega.
\end{aligned} \tag{28}$$

In addition,

$$\begin{aligned}
(A_c)_{ii} &= -\frac{1}{2}\left[(C_c^T C_c)_{ii} - (B_c B_c^T)_{ii}\right], \\
\omega_i &= \frac{(C_c^T C_c)_{ii}}{(B_c B_c^T)_{ii}}, \\
(A_c)_{ij} &= \frac{\omega_j(1+\omega_i)(B_c B_c^T)_{ij} - (C_c^T C_c)_{ij}(1+\omega_j)}{\omega_i - \omega_j}, \quad \text{if } \omega_i \neq \omega_j.
\end{aligned} \tag{29}$$

DEFINITION 1. The triple (A_c, B_c, C_c) satisfying (28) or (29) is said to be in *input normal Riccati form.*

The following algorithm will transform a given triple $(\bar{A}_c, \bar{B}_c, \bar{C}_c)$ to the input normal Riccati form (A_c, B_c, C_c).

1) Find symmetric, positive definite P and Q satisfying

$$\begin{aligned}
0 &= \bar{A}_c^T P + P\bar{A}_c + \bar{C}_c^T \bar{C}_c - P\bar{B}_c\bar{B}_c^T P, \\
0 &= \bar{A}_c Q + Q\bar{A}_c^T + \bar{B}_c\bar{B}_c^T - Q\bar{C}_c^T\bar{C}_c Q.
\end{aligned}$$

2) Compute the Cholesky decompositions of P and Q, i.e.,

$$P = L_P L_P^T, \quad Q = L_Q L_Q^T,$$

where L_P and L_Q are invertible lower triangular matrices.

3) Compute the singular value decomposition of $L_P^T L_Q$, i.e., $L_P^T L_Q = U\Omega V^T$.

4) Let

$$T = L_Q V, \quad T^{-1} = \Omega^{-1} U^T L_P^T.$$

5) Then

$$A_c = T^{-1}\bar{A}_c T, \quad , B_c = T^{-1}\bar{B}_c, \quad C_c = \bar{C}_c T.$$

To optimize $\mathcal{J}(A_c, B_c, C_c)$ over the open set $\mathcal{S}$ under the constraints that symmetric positive definite $\mathcal{Q}$ satisfies (10), and (A_c, B_c, C_c) is in input normal Riccati form, the following Lagrangian is formed:

$$\begin{aligned}
\mathcal{L}(A_c, B_c, C_c, \Omega, \mathcal{Q}, \mathcal{P}, M_c, M_o) \equiv \operatorname{tr}\big[&\mathcal{Q}\tilde{R} + (\tilde{A}\mathcal{Q} + \mathcal{Q}\tilde{A}^T + \gamma^{-2}\mathcal{Q}\tilde{R}_\infty\mathcal{Q} + \tilde{V})\mathcal{P} \\
&+ (A_c + A_c^T + B_c B_c^T - C_c^T C_c)M_c + (A_c^T\Omega + \Omega A_c + C_c^T C_c - \Omega B_c B_c^T\Omega)M_o\big],
\end{aligned}$$

where the symmetric matrices M_c, M_o, and $\mathcal{P} \in \mathbf{R}^{\tilde{n}\times\tilde{n}}$ are Lagrange multipliers. $\Omega = \operatorname{diag}(\omega_1, \ldots, \omega_{n_c})$ is related to the input normal Riccati form constraint. Setting $\partial\mathcal{L}/\partial\mathcal{Q} = 0$ yields (13). Partition $\mathcal{Q}, \mathcal{P} \in \mathbf{R}^{\tilde{n}\times\tilde{n}}$ as in (14). $\partial L/\partial A_c = 0$ and $\partial L/\partial\Omega = 0$ yield, respectively,

$$\begin{aligned}
0 &= 2M_c + 2\Omega M_o + 2(\mathcal{P}_{12}^T\mathcal{Q}_{12} + \mathcal{P}_2\mathcal{Q}_2), \\
0 &= (A_c M_o - B_c B_c^T\Omega M_o)_{ii}, \quad 1 \leq i \leq n_c.
\end{aligned}$$

A straightforward calculation shows

$$\begin{aligned}
\frac{\partial\mathcal{L}}{\partial B_c} &= 2\mathcal{P}_2 B_c V_2 + 2(\mathcal{P}_{12}^T\mathcal{Q}_1 + \mathcal{P}_2\mathcal{Q}_{12}^T)C^T + 2M_c B_c - 2\Omega M_o\Omega B_c, \\
\frac{\partial\mathcal{L}}{\partial C_c} &= 2R_2 C_c\mathcal{Q}_2 + 2B^T(\mathcal{P}_1\mathcal{Q}_{12} + \mathcal{P}_{12}\mathcal{Q}_2) + 2C_c M_o - 2C_c M_c \\
&\quad + \gamma^{-2}R_{2\infty}C_c\big[(\mathcal{Q}_{12}^T\mathcal{P}_1 + \mathcal{Q}_2\mathcal{P}_{12}^T)\mathcal{Q}_{12} + (\mathcal{Q}_{12}^T\mathcal{P}_{12} + \mathcal{Q}_2\mathcal{P}_2)\mathcal{Q}_2\big].
\end{aligned} \tag{30}$$

THEOREM 2 [6]. *The matrices M_c and M_o in* (30) *satisfy*

$$\begin{aligned} M_c &= -\big(S + \Omega M_o\big), \\ \big(M_o\big)_{ii} &= -\frac{1}{\big(A_c + (B_c B_c^T \Omega)\big)_{ii}} \sum_{\substack{j=1 \\ j\neq i}}^{n_c} \big[\big(A_c\big)_{ij}\big(M_o\big)_{ji} + \big(B_c B_c^T \Omega\big)_{ij}\big(M_o\big)_{ji}\big], \\ \big(M_o\big)_{ij} &= \frac{(S)_{ij} - (S)_{ji}}{\omega_j - \omega_i}, \quad \text{if } \omega_j \neq \omega_i, \end{aligned} \tag{31}$$

where

$$S = \mathcal{P}_{12}^T \mathcal{Q}_{12} + \mathcal{P}_2 \mathcal{Q}_2. \tag{32}$$

A homotopy approach based on the input normal Riccati form is now described. Let A_f, B_f, C_f, γ_f, R_{1f}, R_{2f}, $R_{1\infty f}$, $R_{2\infty f}$, V_{1f}, and V_{2f} denote A, B, C, λ, R_1, R_2, $R_{1\infty}$, $R_{2\infty}$, V_1, and V_2 in the above and define $A(\lambda)$, $B(\lambda)$, $C(\lambda)$, $\gamma(\lambda)$, $R_1(\lambda)$, $R_2(\lambda)$, $R_{1\infty}(\lambda)$, $R_{2\infty}(\lambda)$, $V_1(\lambda)$, and $V_2(\lambda)$ as in (15) and denote them by A, B, C, γ, R_1, R_2, $R_{1\infty}$, $R_{2\infty}$, V_1, and V_2 respectively in the following. Let

$$\begin{aligned} H_{B_c}(\theta,\lambda) &= \frac{1}{2}\frac{\partial \mathcal{L}}{\partial B_c} = \mathcal{P}_2 B_c V_2 + (\mathcal{P}_{12}^T \mathcal{Q}_1 + \mathcal{P}_2 \mathcal{Q}_{12}^T) C^T + M_c B_c - \Omega M_o \Omega B_c, \\ H_{C_c}(\theta,\lambda) &= \frac{1}{2}\frac{\partial \mathcal{L}}{\partial C_c} = R_2 C_c \mathcal{Q}_2 + B^T(\mathcal{P}_1 \mathcal{Q}_{12} + \mathcal{P}_{12}\mathcal{Q}_2) + C_c M_o - C_c M_c \\ &\quad + \frac{1}{2}\gamma^{-2} R_{2\infty} C_c \big[(\mathcal{Q}_{12}^T \mathcal{P}_1 + \mathcal{Q}_2 \mathcal{P}_{12}^T)\mathcal{Q}_{12} + (\mathcal{Q}_{12}^T \mathcal{P}_{12} + \mathcal{Q}_2 \mathcal{P}_2)\mathcal{Q}_2\big], \end{aligned}$$

where

$$\theta \equiv \begin{pmatrix} \text{Vec } (B_c) \\ \text{Vec } (C_c) \end{pmatrix} \tag{33}$$

denotes the independent variables B_c and C_c, M_o and M_c satisfy (31), and $\mathcal{Q}$ and $\mathcal{P}$ satisfy respectively (10) and (13) with partitioned forms (14).

There are several ways to construct a homotopy map. The most natural homotopy map is defined as

$$\rho(\theta,\lambda) = \begin{pmatrix} \text{Vec } \big[H_{B_c}(\theta,\lambda)\big] \\ \text{Vec } \big[H_{C_c}(\theta,\lambda)\big] \end{pmatrix}, \tag{34}$$

and its Jacobian matrix is

$$D\rho(\theta,\lambda) = \big(D_\theta \rho(\theta,\lambda), D_\lambda \rho(\theta,\lambda)\big).$$

For the reduced-order problem, both to avoid singular starting points and to guarantee a full rank Jacobian matrix along the homotopy path (except possibly at $\lambda = 1$), the homotopy map

$$\hat{\rho}(\theta,\lambda) = \lambda \rho(\theta,\lambda) + (1-\lambda)(\theta - \theta_0) \tag{35}$$

is used. The Jacobian matrix of $\hat{\rho}$ is given by

$$D\hat{\rho}(\theta,\lambda) = \big(\lambda D_\theta \rho(\theta,\lambda) + (1-\lambda)I, \rho(\theta,\lambda) + \lambda D_\lambda \rho(\theta,\lambda) - (\theta - \theta_0)\big).$$

Define

$$\hat{H}_{B_c}\big(\mathcal{P}^{(j)},\mathcal{Q}^{(j)},M_c^{(j)},M_o^{(j)}\big)=\mathcal{P}_2^{(j)}B_cV_2+\big(\mathcal{P}_{12}^{T\,(j)}\mathcal{Q}_1+\mathcal{P}_{12}^T\mathcal{Q}_1^{(j)}+\mathcal{P}_2^{(j)}\mathcal{Q}_{12}^T+\mathcal{P}_2\mathcal{Q}_{12}^{T\,(j)}\big)C^T$$
$$+M_c^{(j)}B_c-\Omega M_o^{(j)}\Omega B_c,-\Omega^{(j)}M_o\Omega B_c-\Omega M_o\Omega^{(j)}B_c,$$
$$\hat{H}_{C_c}(\mathcal{P}^{(j)},\mathcal{Q}^{(j)},M_c^{(j)},M_o^{(j)})=R_2C_c\mathcal{Q}_2^{(j)}+B^T\big(\mathcal{P}_1^{(j)}\mathcal{Q}_{12}+\mathcal{P}_1\mathcal{Q}_{12}^{(j)}+\mathcal{P}_{12}^{(j)}\mathcal{Q}_2+\mathcal{P}_{12}\mathcal{Q}_2^{(j)}\big)$$
$$+C_cM_o^{(j)}-C_cM_c^{(j)}+\frac{1}{2}\gamma^{-2}R_{2\infty}C_c\Big[(\mathcal{Q}_{12}^{T\,(j)}\mathcal{P}_1+\mathcal{Q}_{12}^T\mathcal{P}_1^{(j)}+\mathcal{Q}_2^{(j)}\mathcal{P}_{12}^T+\mathcal{Q}_2\mathcal{P}_{12}^{T\,(j)})\mathcal{Q}_{12}$$
$$+(\mathcal{Q}_{12}^{T\,(j)}\mathcal{P}_{12}+\mathcal{Q}_{12}^T\mathcal{P}_{12}^{(j)}+\mathcal{Q}_2^{(j)}\mathcal{P}_2+\mathcal{Q}_2\mathcal{P}_2^{(j)})\mathcal{Q}_2+(\mathcal{Q}_{12}^T\mathcal{P}_1+\mathcal{Q}_2\mathcal{P}_{12}^T)\mathcal{Q}_{12}^{(j)}+(\mathcal{Q}_{12}^T\mathcal{P}_{12}+\mathcal{Q}_2\mathcal{P}_2)\mathcal{Q}_2^{(j)}\Big],$$

where the superscript (j) means $\partial/\partial\theta_j$: $Y^{(j)}\equiv\partial Y/\partial\theta_j$. Using the above definitions, we have for $\theta_j=\big(B_c\big)_{kl}$,

$$\frac{\partial H_{B_c}}{\partial(B_c)_{kl}}=\hat{H}_{B_c}\big(\mathcal{P}^{(j)},\mathcal{Q}^{(j)},M_c^{(j)},M_o^{(j)}\big)+\mathcal{P}_2E^{(k,l)}V_2+M_cE^{(k,l)}-\Omega M_o\Omega E^{(k,l)},$$
$$\frac{\partial H_{C_c}}{\partial(B_c)_{kl}}=\hat{H}_{C_c}\big(\mathcal{P}^{(j)},\mathcal{Q}^{(j)},M_c^{(j)},M_o^{(j)}\big),$$

and for $\theta_j=\big(C_c\big)_{kl}$,

$$\frac{\partial H_{B_c}}{\partial(C_c)_{kl}}=\hat{H}_{B_c}\big(\mathcal{P}^{(j)},\mathcal{Q}^{(j)},M_c^{(j)},M_o^{(j)}\big),$$
$$\frac{\partial H_{C_c}}{\partial(C_c)_{kl}}=\hat{H}_{C_c}\big(\mathcal{P}^{(j)},\mathcal{Q}^{(j)},M_c^{(j)},M_o^{(j)}\big)+E^{(k,l)}\big(M_o-M_c\big)+R_2E^{(k,l)}\mathcal{Q}_2+\frac{1}{2}\gamma^{-2}R_{2\infty}E^{(k,l)}Y,$$

where Y is given by (24). $\mathcal{P}^{(j)}$ and $\mathcal{Q}^{(j)}$ can be obtained by solving the Lyapunov equations (25). Similarly for λ, using a dot to denote $\partial/\partial\lambda$,

$$\frac{\partial H_{B_c}}{\partial\lambda}=\hat{H}_{B_c}\big(\dot{\mathcal{P}},\dot{\mathcal{Q}},\dot{M}_c,\dot{M}_o\big)+\mathcal{P}_2B_c\dot{V}_2+(\mathcal{P}_{12}^T\mathcal{Q}_1+\mathcal{Q}_2\mathcal{Q}_{12}^T)\dot{C}^T,$$
$$\frac{\partial H_{C_c}}{\partial\lambda}=\hat{H}_{C_c}\big(\dot{\mathcal{P}},\dot{\mathcal{Q}},\dot{M}_c,\dot{M}_o\big)+\dot{R}_2C_c\mathcal{Q}_2+\dot{B}^T(\mathcal{P}_1\mathcal{Q}_{12}+\mathcal{P}_{12}\mathcal{Q}_2)-\gamma^{-3}\dot{\gamma}R_{2\infty}C_cY$$
$$+\frac{1}{2}\gamma^{-2}\dot{R}_{2\infty}C_c,$$

and $\dot{\mathcal{P}}$ and $\dot{\mathcal{Q}}$ are obtained by solving the Lyapunov equations (27).

4. Homotopy algorithm based on over-parametrization formulation

The parametrizations in the two previous sections, the Ly form and the input normal Riccati form, are both minimal in the sense that they use the minimum possible number of parameters, $n_c(m+l)$, to describe the controller. However, both also assume a particular structure for the controller that may not be satisfied by the optimal controller, or even if it is, that structure may be ill conditioned near the optimum. The formulation in this section makes no assumptions whatsoever on the controller structure, treating all the components of (A_c,B_c,C_c) as independent variables.

To optimize $\mathcal{J}(A_c,B_c,C_c)$ over the open set $\mathcal{S}$ under the constraint that symmetric positive definite $\mathcal{Q}$ satisfies (10), the following Lagrangian is formed:

$$\mathcal{L}(A_c,B_c,C_c,\mathcal{P},\mathcal{Q})\equiv\text{tr}\,\big[\mathcal{Q}\tilde{R}+(\tilde{A}\mathcal{Q}+\mathcal{Q}\tilde{A}^T+\gamma^{-2}\mathcal{Q}\tilde{R}_\infty\mathcal{Q}+\tilde{V})\mathcal{P}\big],$$

where $\mathcal{P} \in \mathbf{R}^{\tilde{n}\times\tilde{n}}$ is a Lagrange multiplier. Setting $\partial\mathcal{L}/\partial\mathcal{Q} = 0$ yields (13). Partition $\mathcal{Q}$, $\mathcal{P} \in \mathbf{R}^{\tilde{n}\times\tilde{n}}$ as in (14). The partial derivatives of $\mathcal{L}$ can be computed as

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial A_c} &= 2\big(\mathcal{P}_{12}^T\, \mathcal{Q}_{12} + \mathcal{P}_2\, \mathcal{Q}_2\big), \\
\frac{\partial \mathcal{L}}{\partial B_c} &= 2\mathcal{P}_2 B_c V_2 + 2(\mathcal{P}_{12}^T \mathcal{Q}_1 + \mathcal{P}_2 \mathcal{Q}_{12}^T) C^T, \\
\frac{\partial \mathcal{L}}{\partial C_c} &= 2R_2 C_c \mathcal{Q}_2 + 2B^T(\mathcal{P}_1 \mathcal{Q}_{12} + \mathcal{P}_{12}\mathcal{Q}_2) \\
&\quad + \gamma^{-2} R_{2\infty} C_c \big[(\mathcal{Q}_{12}^T \mathcal{P}_1 + \mathcal{Q}_2 \mathcal{P}_{12}^T)\mathcal{Q}_{12} + (\mathcal{Q}_{12}^T \mathcal{P}_{12} + \mathcal{Q}_2 \mathcal{P}_2)\mathcal{Q}_2\big].
\end{aligned}$$

Let A_f, B_f, C_f, γ_f, R_{1f}, R_{2f}, $R_{1\infty f}$, $R_{2\infty f}$, V_{1f}, and V_{2f} denote A, B, C, λ, R_1, R_2, $R_{1\infty}$, $R_{2\infty}$, V_1, and V_2 in the above and define $A(\lambda)$, $B(\lambda)$, $C(\lambda)$, $\gamma(\lambda)$, $R_1(\lambda)$, $R_2(\lambda)$, $R_{1\infty}(\lambda)$, $R_{2\infty}(\lambda)$, $V_1(\lambda)$, and $V_2(\lambda)$ as in (15) and denote them by A, B, C, γ, R_1, R_2, $R_{1\infty}$, $R_{2\infty}$, V_1, and V_2 respectively in the following.

Define $H_{A_c}(\theta,\lambda)$, $H_{B_c}(\theta,\lambda)$, and $H_{C_c}(\theta,\lambda)$ as in (16) where

$$\theta \equiv \begin{pmatrix} \text{Vec } (A_c) \\ \text{Vec } (B_c) \\ \text{Vec } (C_c) \end{pmatrix} \tag{36}$$

denotes the independent variables, and $\mathcal{Q}$ and $\mathcal{P}$ satisfy respectively (10) and (13). Vec applied to a matrix is a column vector obtained by concatenating the columns of the matrix.

Define

$$\rho(\theta,\lambda) = \begin{pmatrix} \text{Vec } \big[H_{A_c}(\theta,\lambda)\big] \\ \text{Vec } \big[H_{B_c}(\theta,\lambda)\big] \\ \text{Vec } \big[H_{C_c}(\theta,\lambda)\big] \end{pmatrix}, \tag{37}$$

whose Jacobian matrix is

$$D\rho(\theta,\lambda) = \big(D_\theta \rho(\theta,\lambda), D_\lambda \rho(\theta,\lambda)\big).$$

Note that θ in (37) has $n_c^2 + n_c m + n_c l$ components, more than the minimal number $n_c m + n_c l$ needed. Because of this over-parametrization, the Jacobian matrix of ρ is seriously rank deficient. To remedy this severe rank deficiency, the homotopy map is defined as

$$\hat{\rho}(\theta,\lambda) = \lambda\rho(\theta,\lambda) + (1-\lambda)(\theta - \theta_0), \tag{38}$$

which guarantees a full rank Jacobian matrix along the entire homotopy zero curve except possibly at the solution (corresponding to $\lambda = 1$). The Jacobian matrix of $\hat{\rho}$ is given by

$$D\hat{\rho}(\theta,\lambda) = \big(\lambda D_\theta \rho(\theta,\lambda) + (1-\lambda)I, \rho(\theta,\lambda) + \lambda D_\lambda \rho(\theta,\lambda) - (\theta - \theta_0)\big). \tag{39}$$

To find $D_\theta\rho(\theta,\lambda)$, define the auxiliary matrices $\hat{H}_{A_c}\big(\mathcal{P}^{(j)}, \mathcal{Q}^{(j)}\big)$, $\hat{H}_{B_c}\big(\mathcal{P}^{(j)}, \mathcal{Q}^{(j)}\big)$, and $\hat{H}_{C_c}(\mathcal{P}^{(j)}, \mathcal{Q}^{(j)})$ as in (20). Using the above definitions, we have (21), (22), (23) for $\theta_j = \big(A_c\big)_{kl}$, $\theta_j = \big(B_c\big)_{kl}$, and $\theta_j = \big(C_c\big)_{kl}$ respectively. $\mathcal{P}^{(j)}$ and $\mathcal{Q}^{(j)}$ can be obtained by solving the Lyapunov equation (25). Similarly we have (26) for λ and $\dot{\mathcal{P}}$ where $\dot{\mathcal{Q}}$ are obtained by solving (27).

5. Numerical Algorithms

Choose the initial γ so that γ_0^{-2} is approximately zero. The initial point $(\theta, \lambda) = (\theta_0, 0)$ is chosen so that it satisfies $\rho(\theta_0, 0) = 0$ and the triple $\big((A_c)_0, (B_c)_0, (C_c)_0\big)$ is in the respective form for each homotopy.

It is well known that the full-order LQG compensator (2) for the plant (1) minimizing the steady-state quadratic performance functional (5) is given by:

$$A_c = A - \Sigma P - Q\bar{\Sigma} \tag{40}$$

$$B_c = QC^T V_2^{-1}, \quad C_c = -R_2^{-1} B^T P, \tag{41}$$

where $\Sigma \equiv BR_2^{-1}B^T$, $\bar{\Sigma} \equiv C^T V_2^{-1} C$, and P and Q are the unique, symmetric, positive semidefinite solutions respectively, of

$$\begin{aligned} 0 &= A^T P + PA + R_1 - P\Sigma P, \\ 0 &= AQ + QA^T + V_1 - Q\bar{\Sigma}Q. \end{aligned} \tag{42}$$

A. Full-order Initialization

The initial point for the full-order problem can be chosen as follows:

1) Solve for Q and P from (42) and obtain $\big((\hat{A}_c)_0, (\hat{B}_c)_0, (\hat{C}_c)_0\big)$ from (40) and (41).
2) Transform the triple $\big((\hat{A}_c)_0, (\hat{B}_c)_0, (\hat{C}_c)_0\big)$ to Ly's form for the Ly form homotopy, to input normal Riccati form for the input normal form homotopy, and build θ_0 as described in (17), (33), and (36) for the respective homotopies.

B. Reduced-order Initialization

The initialization scheme for the reduced-order problem is more complicated since a closed form expression for the reduced-order H^2 LQG compensator does not exist. For a given system $(\bar{A}, \bar{B}, \bar{C})$, and matrices $\bar{R}_1$, $\bar{R}_2$, $\bar{R}_{1\infty}$, $\bar{R}_{2\infty}$, $\bar{V}_1$ and $\bar{V}_2$, the reduced order starting point is chosen using a method in [4] which can be summarized as:

1) Compute the real Schur decomposition of $\bar{A}$ so that $\bar{A} = UAU^t, A \equiv \begin{pmatrix} A_1 & 0 \\ A_{21} & A_2 \end{pmatrix}$, where $A_1 \in \mathbf{R}^{n_c \times n_c}$, and transform $\bar{B}$, $\bar{C}$, $\bar{R}_1$, $\bar{V}_1$, $\bar{R}_{1\infty}$ so that $B = U\bar{B}$, $C = \bar{C}U^t$, $R_1 = U\bar{R}_1 U^t$, $V_1 = U\bar{V}_1 U^t$, $R_{1\infty} = U\bar{R}_{1\infty}U^t$ and let $R_2 = \bar{R}_2$, $V_2 = \bar{V}_2$, $R_{2\infty} = \bar{R}_{2\infty}$.
2) If A is not asymptotically stable, modify either diagonal elements or 2×2 diagonal blocks of A so that it is asymptotically stable and call this modified matrix A_0.
3) Take $B_0 = B$, $C_0 = C$, $R_{2,0} = R_{2f} \equiv R_2$, $R_{2\infty,0} = R_{2\infty f} \equiv R_{2\infty}$, $V_{1,0} = V_{1f} \equiv V_1$, $V_{2,0} = \beta V_{2f} \equiv \beta V_2$, $\beta \gg 0$, and $R_{1,0} = \begin{pmatrix} (R_1)_1 & 0 \\ 0 & 0 \end{pmatrix}$ where $(R_1)_1$ is the leading $n_c \times n_c$ block of $R_{1f} \equiv R_1$.
4) Solve

$$\begin{aligned} 0 &= A_0^T P + PA_0 + R_{1,0} - P\Sigma_0 P, \\ 0 &= A_0 Q + QA_0^T + V_{1,0} - Q\bar{\Sigma}_0 Q. \end{aligned}$$

for symmetric, positive semidefinite P and Q, where $\Sigma_0 \equiv B_0 R_{2,0}^{-1} B_0^T$, and $\bar{\Sigma}_0 \equiv C_0^T V_{2,0}^{-1} C_0$.
5) Obtain (A_c, B_c, C_c) from $A_c = A_0 - \Sigma_0 P - Q\bar{\Sigma}_0$, $B_c = QC_0^T V_{2,0}^{-1}$, $C_c = -R_{2,0}^{-1} B_0^T P$.

6) Solve

$$0 = (A_0 - Q\bar{\Sigma}_0)^T \hat{P} + \hat{P}(A_0 - Q\bar{\Sigma}_0) + P\Sigma_0 P,$$
$$0 = (A_0 - \Sigma_0 P)\hat{Q} + \hat{Q}(A_0 - \Sigma_0 P)^T + Q\bar{\Sigma}_0 Q,$$

for symmetric, positive semidefinite $\hat{P}$ and $\hat{Q}$.

7) Following [21], obtain the reduced-order compensator starting point from (A_c, B_c, C_c), $\hat{P}$, $\hat{Q}$, and n_c, as follows:

i) Compute the Cholesky decomposition of (assumed positive definite) $\hat{P}$ and $\hat{Q}$, i.e., $\hat{P} = L_{\hat{P}} L_{\hat{P}}^T$, $\hat{Q} = L_{\hat{Q}} L_{\hat{Q}}^T$.

ii) Compute the singular value decomposition of $L_{\hat{P}}^T L_{\hat{Q}}$, i.e., $L_{\hat{P}}^T L_{\hat{Q}} = U\Omega V^T$.

iii) Let $T = L_{\hat{Q}} V \Omega^{-\frac{1}{2}}$, $T^{-1} = \Omega^{-\frac{1}{2}} U^T L_{\hat{P}}^T$.

iv) Let $\bar{A}_c = T^{-1} A_c T$, $\bar{B}_c = T^{-1} B_c$, and $\bar{C}_c = C_c T$ so that

$$\bar{A}_c = {}^{n_c}\left\{ \begin{pmatrix} \overbrace{(\bar{A}_c)_1}^{n_c} & (\bar{A}_c)_{12} \\ (\bar{A}_c)_{21} & (\bar{A}_c)_2 \end{pmatrix} \right., \quad \bar{B}_c = {}^{n_c}\left\{ \begin{pmatrix} (\bar{B}_c)_1 \\ (\bar{B}_c)_2 \end{pmatrix} \right., \quad \bar{C}_c = \begin{pmatrix} \overbrace{(\bar{C}_c)_1}^{n_c} & (\bar{C}_c)_2 \end{pmatrix}.$$

The starting point θ_0 for the reduced order problem is chosen using $\big((\bar{A}_c)_1, (\bar{B}_c)_1, (\bar{C}_c)_1\big)$, with the construction in (17), (33), and (36) for the respective homotopies.

C. Homotopy Zero Curve Tracking

Once the initial point is chosen, the rest of the computation is as follows:

1) Set $\lambda := 0$, $\theta := \theta_0$.
2) Calculate $\tilde{R}$, $\tilde{R}_\infty$, $\tilde{V}$, and compute $\mathcal{Q}$ and $\mathcal{P}$ according to (10) and (13).
3) Evaluate the homotopy map $\rho(\theta, \lambda)$ or $\hat{\rho}(\theta, \lambda)$ and the Jacobian of the homotopy map $D\rho(\theta, \lambda)$ or $D\hat{\rho}(\theta, \lambda)$.
4) Predict the next point $Z^{(0)} = \big(\theta^{(0)}, \lambda^{(0)}\big)$ on the homotopy zero curve using, e.g., a Hermite cubic interpolant.
5) For $k := 0, 1, 2, \cdots$ until convergence do

$$Z^{(k+1)} = Z^{(k)} - \big[D\rho(Z^{(k)})\big]^\dagger \rho\big(Z^{(k)}\big),$$

where $\big[D\rho(Z)\big]^\dagger$ is the Moore-Penrose inverse of $D\rho(Z)$. Let $\big(\theta_1, \lambda_1\big) = \lim_{k\to\infty} Z^{(k)}$.

6) If $\lambda_1 < 1$, then set $\theta := \theta_1$, $\lambda := \lambda_1$, and go to step 2).
7) If $\lambda_1 \geq 1$, compute the solution $\bar{\theta}$ at $\lambda = 1$.

For the over-parametrization formulation homotopy, because of the singularity at $\lambda = 1$, step 7) is replaced by:

7) If $\lambda_1 \geq 1$, use the last point $\big(\tilde{\theta}, \tilde{\lambda}\big)$ with $\tilde{\lambda} < 1$ to redefine the homotopy map with $\theta_0 = \tilde{\theta}$.
8) Redo steps 1)–6) until $\lambda \geq 1$.
9) Use Hermite polynomial interpolation to obtain the solution at $\lambda = 1$.

6. Numerical Results and Discussion

The following systems are solved by the homotopy algorithms discussed in the previous sections. The homotopy curve tracking was done with HOMPACK [19].

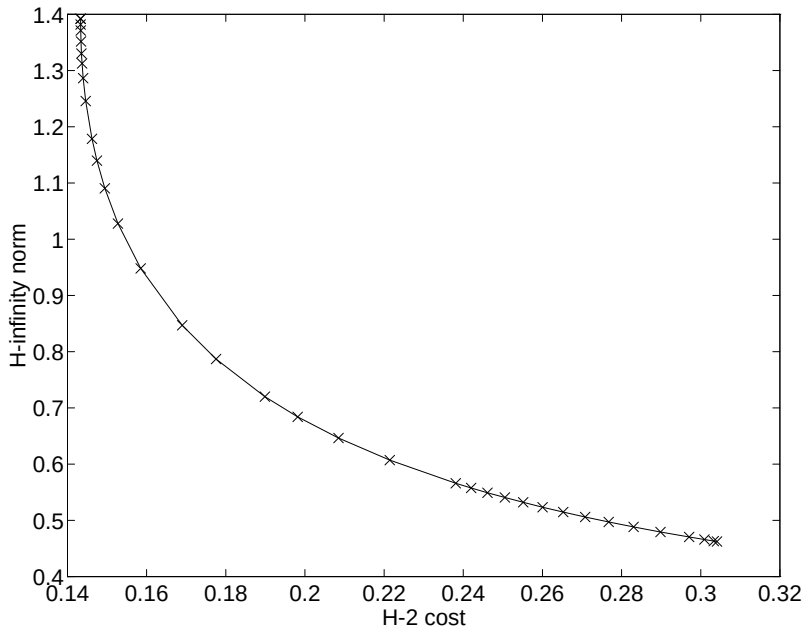

Fig. 1. $\|H(s)\|_\infty$ versus J for $n_c = n$.

The first system, formulated in [3] and studied in [2], is given by

$$A = \begin{pmatrix} -0.161 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -6.004 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -0.5822 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -9.9835 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -0.4073 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -3.982 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 0 \\ 0.0064 \\ 0.00235 \\ 0.0713 \\ 1.0002 \\ 0.1045 \\ 0.9955 \end{pmatrix}, \quad C^T = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

$$E_1 = 10^{-3} \begin{pmatrix} 0 & 0 & 0 & 0 & 0.55 & 11 & 1.32 & 18 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad E_{1\infty} = E_1,$$

$$E_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad E_{2\infty} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad D_1 = \begin{pmatrix} B & 0 \end{pmatrix}, \quad D_2 = \begin{pmatrix} 0 & 1 \end{pmatrix}.$$

For the full-order (i.e., 8th order) problem, the solutions of the auxiliary minimization problem are obtained for $\gamma \geq \gamma_{min} \equiv 0.481$ using the Ly form homotopy approach. For $\gamma < \gamma_{min} = 0.481$, the Riccati equation solver fails and therefore no solution can be found. In Fig. 1, $\|H(s)\|_\infty$ is plotted against J. The ratio of $\|H(s)\|_\infty$ at $\gamma = \gamma_{min}$ to that at $\gamma = \infty$ is 0.33, which indicates that there is about 67% improvement in the H^∞ performance of the compensator with $\gamma = \gamma_{min}$ over the compensator without the H^∞ constraint.

For $n_c = 4, 6$, the solutions of the auxiliary minimization problem are obtained for $\gamma \geq 2.55$ using the Ly form homotopy approach. In Fig. 2, $\|H(s)\|_\infty$ is plotted against J for $n_c = 4$ (solid line with "x" indicating the data points) and $n_c = 6$ (dashed line with "o" indicating the data points). For both $n_c = 4$ and $n_c = 6$, the ratio of $\|H(s)\|_\infty$ at $\gamma = 2.55$ to that at $\gamma = \infty$ is 0.49, which indicates that there is about 51% improvement in the H^∞ performance of the compensator with $\gamma = 2.55$ over the compensator without the H^∞ constraint. For

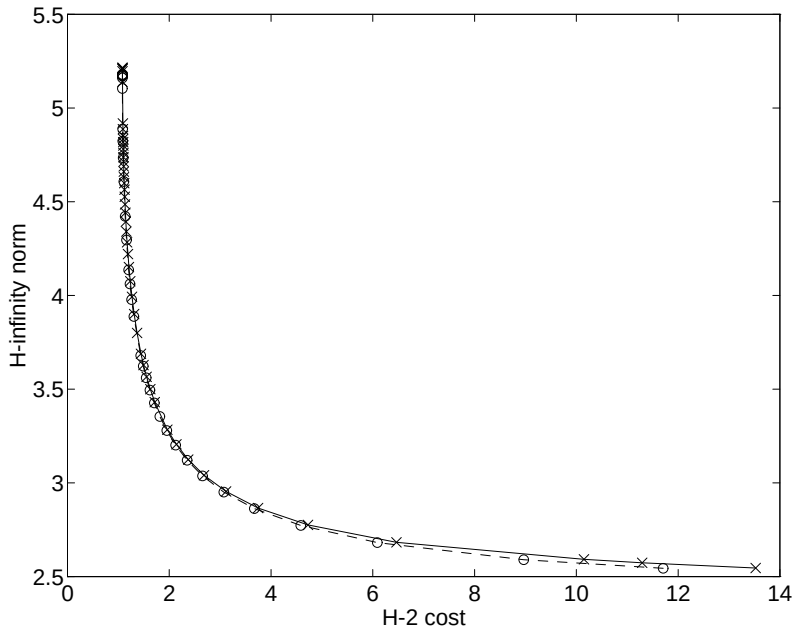

Fig. 2. $\|H(s)\|_\infty$ versus J for $n_c = 4, 6$.

this example, the 4th-order and 6th-order compensators have almost the same H^2 and H^∞ performance.

For $n_c = 2$, two different sets of solutions are obtained by varying β in the initialization step. The trade-off curves are shown in Figs. 3 ($\beta = 100$) and 4 ($\beta = 1$). The first set of solutions (shown in Fig. 3) is obtained for $\gamma \geq 2.54$, while the second set (shown in Fig. 4) is obtained for $\gamma \geq 9.5$. It can be seen that the first set of solutions has lower H^2 cost and better H^∞ performance. It was verified by sampling in a neighborhood that all the points in both Figs. 3 and 4 are local minima of the auxiliary cost $\mathcal{J}$.

The homotopy algorithms proposed here are examples of probability-one globally convergent homotopy methods; the matrices A_0, B_0, ..., and the starting point θ_0 here play the role of the parameter vector a in the probability-one homotopy theory [20]. Fig. 5 shows a portion of γ for the previous example, clearly demonstrating the nonmonotonicity in λ and that standard continuation in λ would fail.

For this problem, with the input normal Riccati form, initial ωs consist of very close pairs, which leads to a significant numerical error in computing M_o and therefore the slow convergence or failure to converge of the homotopy algorithm. It can be verified that the solutions, obtained by the Ly form homotopy approach or over-parametrization approach, also have ωs in close pairs, which implies that the input normal Riccati form homotopy approach leads to intrinsic ill conditioning for this problem.

As a second example, consider the system given by

$$A = \begin{pmatrix} 0 & 1.0000 \\ -9.8696 & -0.0001 \end{pmatrix}, \quad C = \begin{pmatrix} 10.0637 & -9.9363 \end{pmatrix}, \quad B = C^T,$$

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad E_{1\infty} = E_1, \quad E_{2\infty} = E_2$$

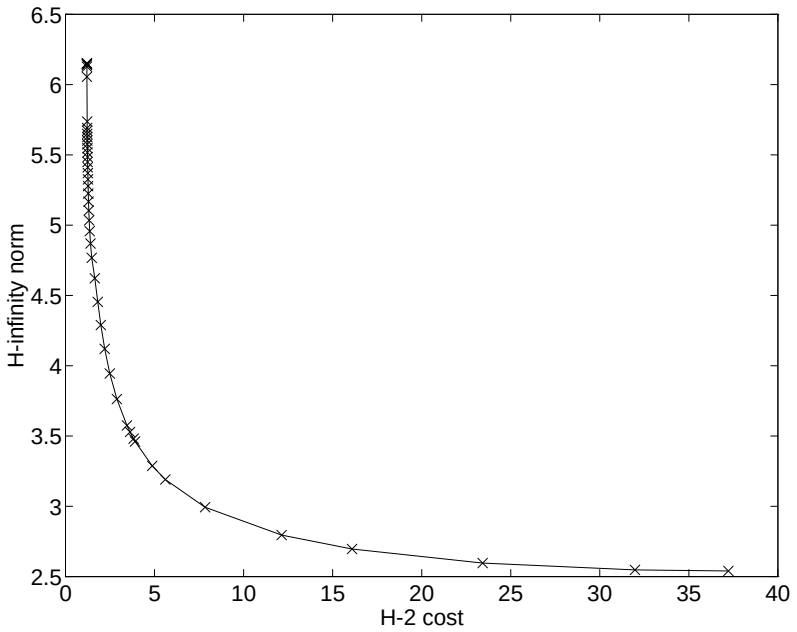

Fig. 3. $\|H(s)\|_\infty$ versus J for $n_c = 2$.

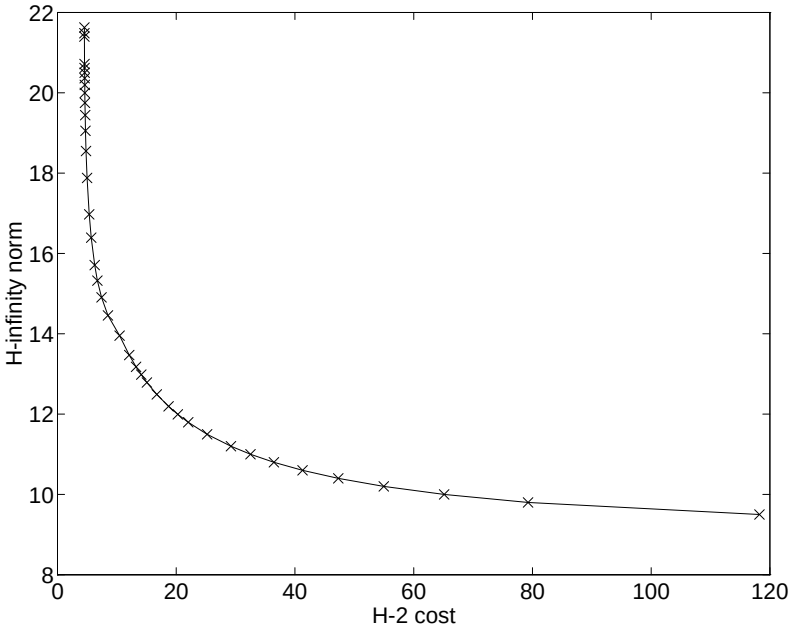

Fig. 4. $\|H(s)\|_\infty$ versus J for $n_c = 2$.

$$D_1 = \begin{pmatrix} 0 & 0 \\ 0.0482 & 0 \end{pmatrix}, \quad D_2 = \begin{pmatrix} 0 & 1 \end{pmatrix}.$$

The input normal Riccati form is used to solve the full-order problem. The trade-off curve is shown in Fig. 6. The solutions of the auxiliary minimization problem are obtained for

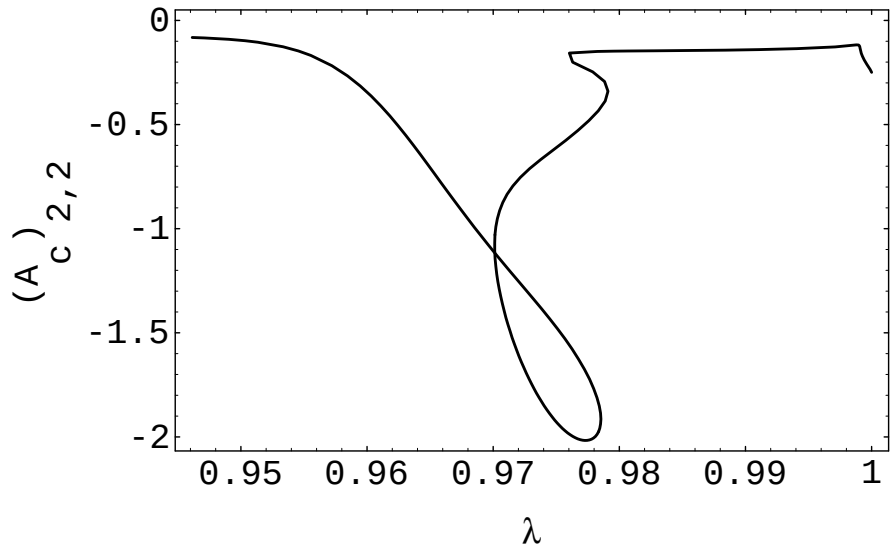

Fig. 5. $(A_c)_{2,2}$ versus λ for $\gamma = 3.8$ and $n_c = 2$.

$\gamma \geq 0.032$. The ratio of $\|H(s)\|_\infty$ at $\gamma = 0.032$ to that at $\gamma = \infty$ is 0.69, which indicates that there is about 31% improvement in the H^∞ performance of the compensator with $\gamma = 0.032$ over the compensator without the H^∞ constraint.

The relative performance of homotopies based on the Ly parametrization, input normal form parametrization, and over-parametrization for the combined H^2/H^∞ model order reduction problem was reported in detail in [7]–[9]. For the reduced-order LQG controller problem, the failure of the input normal Riccati form homotopy on the first example here reinforces a point made in [7]–[8] for the model order reduction problem, namely that making structural assumptions about the solution is both indefensible theoretically and frequently the source of numerical failure (due to extreme ill conditioning). Both the input normal Riccati form formulation and Ly's formulation are very efficient but can fail to exist or lead to ill conditioning and it is conceivable that both of these formulations will fail for some problems. This failure of existence in general is related to the insistence on using the minimal number of parameters $n_c m + n_c l$.

By using $n_c(n_c + m + l)$ parameters, the over-parametrization formulation solves the ill conditioning issue related to existence, but introduces a very high order singularity at the solution. It is doubtful whether the Hermite interpolation used here can handle a large problem with a singularity of order 100. A pragmatic suggestion is to try in order Ly's form, the input normal Riccati form, and the over-parametrization form, switching if ill conditioning or failure occurs. The optimal projection equations formulation of [2] does not make structural assumptions (in fact is completely basis independent), but the optimal projection equations are very difficult and expensive to solve numerically. This cost can be reduced by exploiting tensor product structure and assuming monotonicity in λ of the homotopy zero curves, but Fig. 5 here shows that assumption is not tenable. The over-parameterization formulation makes no structural assumptions and is cheaper computationally than the optimal projection equations,

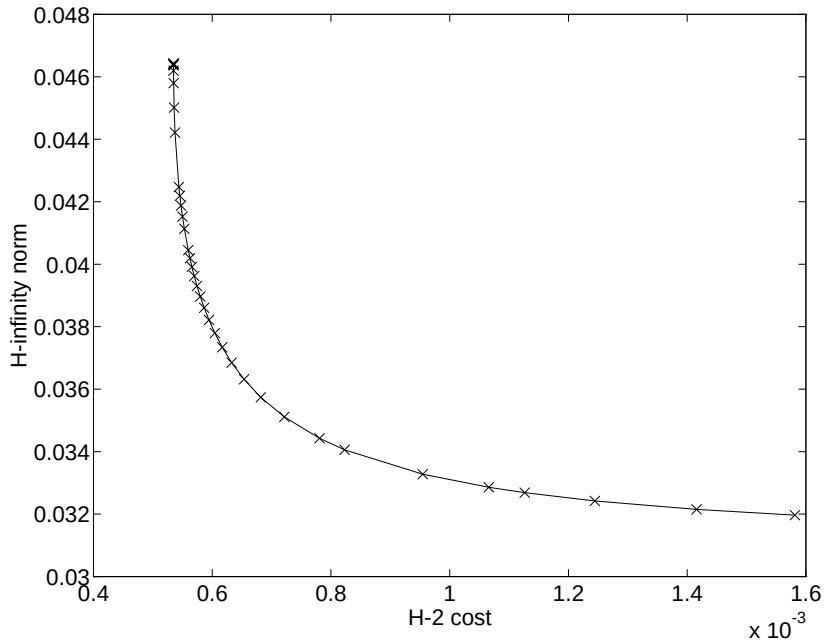

Fig. 6. $\|H(s)\|_\infty$ versus J for the second example.

but it is inherently singular at the solution with rank deficiency n_c^2, which will ultimately overwhelm the numerical linear algebra [7], [9].

ACKNOWLEDGEMENT

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