

MODEL ORDER REDUCTION OF LINEAR DYNAMIC SYSTEMS
WITH APPLICATION TO POWER SYSTEM CONTROL

by

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CHAPTER I
INTRODUCTION

1.1 MOTIVATION

The design and analysis of any dynamical system is very dependent upon an adequate representation of the behavior of the system by a mathematical model. Many complex dynamic processes may be modeled by a set of linear time-invariant differential equations in state variable form

$$\dot{\underline{x}}(t) = \underline{A} \underline{x}(t) + \underline{B} \underline{u}(t) \quad (1.1-1)$$

and by a set of output equations

$$\underline{y}(t) = \underline{C} \underline{x}(t), \quad (1.1-2)$$

where

\underline{x} is an n -vector of states,

\underline{u} is an m -vector of inputs,

\underline{y} is an l vector of outputs,

and \underline{A} , \underline{B} , and \underline{C} are constant matrices of appropriate size. Equations (1.1-1) and (1.1-2) are referred to as the system model or, simply, the system. The system is said to be of order n . If n is large, simulating the dynamic performance of the system can be cumbersome and time consuming. Also, control system design is based upon selecting a control $\underline{u}(t)$ such that the system performs in some specified manner. Usually this control is of the closed-loop feedback form $\underline{u}(t) = \underline{F}(\underline{x}(t), t)$. For the case of large n , the synthesis and physical implementation of the feedback law \underline{F} can be quite difficult if not infeasible.

1.2 SIMPLIFICATION IN THE ANALYSIS OF LARGE-SCALE SYSTEMS

The problems associated with the simulation and control of large-scale systems initiated the study of techniques for approximating the system model by simplified models of lower order (cf. references [1]-[21]). Such techniques are referred to as model order reduction or system simplification schemes. The goal of a model order reduction method is the derivation of a model which has approximately the same input-output attributes as the system model, yet with fewer state variables. Furthermore, the study of simplified representations of complex system models is aimed at circumventing some of the computational and economic barriers in the design and implementation of controllers.

1.3 CURRENT MODEL ORDER REDUCTION SCHEMES

Most of the existing schemes for system simplification may be classified as a modal, a frequency domain, or a time-domain approach. The simplest scheme for model order reduction is the dominant mode approach which is the basis for much of classical automatic control theory [1]. The response of the system is assumed to be dominated by two complex-conjugate poles near the origin, and a second order model is derived based on this assumption. Such models are of limited utility for large n . The dominant eigenvalue schemes of Davison [2,3], Davison and Chidambara [4,5], and Kuppurajulu and Elangovan [6] represent natural extensions of the dominant mode concept. These methods suppose that the eigenvalues of \underline{A} may be divided into two classes: eigenvalues which dominate the time response of the system and eigenvalues which have negligible influence upon the time response. The dominant eigenvalues are retained precisely in the simplified

model, while the nondominant eigenvalues are neglected. The applicability of these modal methods clearly depends upon the existence of dominant eigenvalues. This feature is unfortunately not enjoyed by many systems.

Several frequency domain techniques [7,8] involve finding the transfer function of a simplified model such that the frequency response of the model closely matches the frequency response of the system. Chen and Shieh [9], Rothenberg [10], and Chen [11] derive simplified models by truncating a continued fraction expansion of the system transfer function. These methods are only applicable to systems with eigenvalues which have negative real parts and, with the exception of [11], are restricted to single input-single output systems.

Time domain approaches to model order reduction have received considerable attention. Anderson [12], Sinha and Pille [13], Sinha and Bereznaï [14], and Chidambara [15] derive lower order models by a least squares curve fit of the computed time response of the system model to a particular input. These methods require that the input and output be known or be measured at discrete instants of time. Wilson [16], Mitra [17-19], and Galiana [20] propose determining lower order models via the minimization of a quadratic functional of the error between the outputs of the system model and the simplified model. These approaches require that the eigenvalues of \underline{A} have negative real parts. Also, the optimal model parameters satisfy a set of coupled nonlinear matrix equations which can be difficult to solve. However, these formulations are quite general in that they apply to multiple input-multiple output systems.

1.4 SCOPE OF WORK

This thesis considers a twofold theme: (i) the approximation of high order dynamic system models by simplified lower order models and (ii) the application of simplified models in designing controllers for complex systems. In Chapter II, the model order reduction problem is formulated and necessary conditions for an optimum simplified model are presented. Then a reliable numerical algorithm for model order reduction is proposed in Chapter III.

The utilization of simplified system representations in designing suboptimal linear regulators is developed in Chapter IV. Finally, in Chapter V, the application of simplified models in designing suboptimal linear regulators is considered for a complex control problem in electric power systems: synchronous machine stabilization.

CHAPTER II

MODEL ORDER REDUCTION: THEORY

2.1 THE MODEL ORDER REDUCTION PROBLEM

In this section the model order reduction problem is formulated. To facilitate the subsequent discussion, the following definitions are provided.

DEFINITION 1. System S^n

The system model S^n of a given physical system is a linear time-invariant description of the form

$$\begin{aligned}\dot{\underline{x}}(t) &= \underline{A} \underline{x}(t) + \underline{B} \underline{u}(t), \quad t \geq 0 \\ \underline{x}(0) &= \underline{x}_0 \\ \underline{y}(t) &= \underline{C} \underline{x}(t)\end{aligned}\tag{2.1-1}$$

where

\underline{x} is an n -vector of states
 \underline{u} is an m -vector of inputs
 \underline{y} is an l -vector of outputs.

DEFINITION 2. Simplified System Model S^r

The simplified system model S^r of the system S^n is

$$\begin{aligned}\dot{\hat{\underline{x}}}(t) &= \hat{\underline{A}} \hat{\underline{x}}(t) + \hat{\underline{B}} \underline{u}(t), \quad t \geq 0 \\ \hat{\underline{x}}(0) &= \hat{\underline{x}}_0 \\ \hat{\underline{y}}(t) &= \hat{\underline{C}} \hat{\underline{x}}(t)\end{aligned}\tag{2.1-2}$$

where

$\hat{\underline{x}}$ is an r -vector
 $\hat{\underline{y}}$ is an l -vector

and where $r < n$ and \hat{y} approximates y in some "best" sense for some given $\underline{u}(t)$.

In this research, the criteria used to define the "best" approximation of y by \hat{y} is given by the following definition.

DEFINITION 3. Optimum Simplified Model

The reduced model S^r is the optimum reduction of S^n for a given $\underline{u}(t)$ if \hat{A} , \hat{B} , \hat{C} , and \hat{x}_0 are such that the functional

$$E = \frac{1}{2} \int_0^{\infty} [y(t) - \hat{y}(t)]^T Q [y(t) - \hat{y}(t)] dt \quad (2.1-3)$$

is minimum for a given $Q > 0$. E is referred to as the modeling error.

The model order reduction problem may now be stated simply: given a system model S^n and $r < n$, find \hat{A} , \hat{B} , \hat{C} , and \hat{x}_0 such that E is minimum. Note that there are $r(r+m+l+1)$ degrees of freedom in choosing S^r .

Inherent in the model order reduction problem is the dependence of the optimum S^r upon the test function $\underline{u}(t)$ employed in the reduction. The input plays an extremely important role in the reduction process. For the sake of simplicity consider the case of S^n having a single input and a single output. Let the transfer functions of S^n and S^r be $T(s) = Y(s)/U(s)$ and $\hat{T}(s) = \hat{Y}(s)/U(s)$, respectively. Then it can be shown [10] that

$$\int_0^{\infty} [y(t) - \hat{y}(t)]^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |T(j\omega) - \hat{T}(j\omega)|^2 |U(j\omega)|^2 d\omega, \quad (2.1-4)$$

where $j = \sqrt{-1}$. Equation (2.1-4) serves as a unifying link between frequency domain and time domain techniques for model order reduction. Note that the input $u(t)$ acts as a weighting function in the modeling error.

The test input may be chosen to penalize the deviation of \hat{T} from T in a certain frequency range more heavily than at other regions of the spectrum. For example, if $u(t) = u_{-1}(t)$, where $u_{-1}(t)$ is the unit step function, $U(j\omega) = 1/j\omega$ and, consequently, the comparison of T and \hat{T} is stressed at low frequencies.

A relation similar to (2.1-4) can be derived for multiple input-multiple output systems. If $\underline{u}(t) = \underline{w} u(t)$, where \underline{w} is an m -vector and $u(t)$ is a scalar function, then, in addition to the weighting properties of $u(t)$, \underline{w} serves to put a relative weight on the importance of each input.

The application for which a simplified model will be used governs the selection of the test input. In this research, the input $\underline{u}(t) = \underline{w}\delta(t)$, where $\delta(t)$ is the Dirac delta function, was selected as the test function. This choice was made for several reasons. First, this input enables the simple derivation of necessary conditions for an optimum S^r . Secondly, $\underline{u}(t) = \underline{w}\delta(t)$ is general in that all frequencies are weighted equally in the frequency domain comparison of the system and model transfer function matrices. An interesting feature of this particular input is that it yields a S^r equivalent to that produced by minimizing the expected value of E for a white noise input [11,20]. Finally, $\underline{u}(t) = \underline{w}\delta(t)$ yields simplified models which are adequate for the purpose of design of near-optimum output regulators by the method discussed in Chapter IV.

2.2 SIMPLIFICATION OF STABLE SYSTEMS

The necessary conditions for an optimum model order reduction are easily obtainable for an impulse input $\underline{u}(t) = \underline{w}\delta(t)$ and under the assumption that S^n is strictly stable [22]. The results are basically the same as those of Wilson [11]. The proof given here makes use of the concept of a gradient matrix discussed in Appendix A.

THEOREM 2.2.1. Necessary Conditions for Optimum S^T

Let S^n be a strictly stable system, $\underline{x}_0 = \underline{0}$, and $\underline{u}(t) = \underline{w} \delta(t)$. Also, let S^T be strictly stable. Then the necessary condition for E to be minimum is that $\hat{\underline{A}}$, $\hat{\underline{B}}$, and $\hat{\underline{C}}$ satisfy

$$\hat{\underline{G}} \hat{\underline{W}} - \underline{F}^T \underline{V} = \underline{0} \quad (2.2-1)$$

$$(\hat{\underline{G}} \hat{\underline{B}} - \underline{F}^T \underline{B}) \underline{R} = \underline{0} \quad (2.2-2)$$

$$\underline{Q}(\hat{\underline{C}} \hat{\underline{W}} - \underline{C} \underline{V}) = \underline{0} \quad (2.2-3)$$

where $\underline{R} = \underline{w} \underline{w}^T$ and $\hat{\underline{G}}$, $\hat{\underline{W}}$, \underline{V} , and \underline{F} are solutions of the matrix equations

$$\hat{\underline{A}}^T \hat{\underline{G}} + \hat{\underline{G}} \hat{\underline{A}} = -\hat{\underline{C}}^T \underline{Q} \hat{\underline{C}} \quad (2.2-4)$$

$$\underline{A}^T \underline{F} + \underline{F} \hat{\underline{A}} = -\underline{C}^T \underline{Q} \hat{\underline{C}} \quad (2.2-5)$$

$$\hat{\underline{A}} \hat{\underline{W}} + \hat{\underline{W}} \hat{\underline{A}}^T = -\hat{\underline{B}} \underline{R} \hat{\underline{B}}^T \quad (2.2-6)$$

$$\underline{A} \underline{V} + \underline{V} \hat{\underline{A}}^T = -\underline{B} \underline{R} \hat{\underline{B}}^T \quad (2.2-7)$$

proof: With $\underline{u}(t) = \underline{w} \delta(t)$,

$$\underline{y}(t) = \underline{C} \underline{\phi}(t) \underline{B} \underline{w} \quad (2.2-8)$$

and

$$\hat{\underline{y}}(t) = \hat{\underline{C}} \hat{\underline{\phi}}(t) \hat{\underline{B}} \underline{w} \quad (2.2-9)$$

where $\underline{\phi}(t) = e^{\underline{A}t}$ and $\hat{\underline{\phi}}(t) = e^{\hat{\underline{A}}t}$. Define

$$\underline{G} = \int_0^{\infty} \underline{\phi}^T(t) \underline{C}^T \underline{Q} \underline{C} \underline{\phi}(t) dt$$

$$\underline{F} = \int_0^{\infty} \underline{\phi}^T(t) \underline{C}^T \underline{Q} \underline{\hat{C}} \underline{\hat{\phi}}(t) dt \quad (2.2-10)$$

$$\underline{\hat{G}} = \int_0^{\infty} \underline{\hat{\phi}}^T(t) \underline{\hat{C}}^T \underline{Q} \underline{\hat{C}} \underline{\hat{\phi}}(t) dt . \quad (2.2-11)$$

With these definitions, (2.1-5) becomes

$$E = \frac{1}{2} (\underline{w}^T \underline{B}^T \underline{G} \underline{B} \underline{w} - 2 \underline{w}^T \underline{B}^T \underline{F} \underline{\hat{B}} \underline{w} + \underline{w}^T \underline{\hat{B}}^T \underline{\hat{G}} \underline{\hat{B}} \underline{w})$$

or, equivalently,

$$E = \frac{1}{2} \text{Tr} (\underline{G} \underline{B} \underline{R} \underline{B}^T - 2 \underline{F} \underline{\hat{B}} \underline{R} \underline{B}^T + \underline{\hat{G}} \underline{\hat{B}} \underline{R} \underline{\hat{B}}^T) . \quad (2.2-12)$$

From Lemma A.1,

$$\frac{\partial E}{\partial \underline{\hat{A}}} = \underline{G} \underline{W} - \underline{F}^T \underline{V} \quad (2.2-13)$$

where

$$\underline{W} = \int_0^{\infty} \underline{\phi}(t) \underline{B} \underline{R} \underline{B}^T \underline{\phi}^T(t) dt \quad (2.2-14)$$

$$\underline{V} = \int_0^{\infty} \underline{\phi}(t) \underline{B} \underline{R} \underline{\hat{B}}^T \underline{\hat{\phi}}^T(t) dt \quad (2.2-15)$$

$$\underline{\hat{W}} = \int_0^{\infty} \underline{\hat{\phi}}(t) \underline{\hat{B}} \underline{R} \underline{\hat{B}}^T \underline{\hat{\phi}}^T(t) dt . \quad (2.2-16)$$

Applying (A.3) and (A.4) to (2.2-12), it is clear that

$$\frac{\partial E}{\partial \underline{\hat{B}}} = (\underline{\hat{G}} \underline{\hat{B}} - \underline{F}^T \underline{B}) \underline{R} . \quad (2.2-17)$$

Employing (2.2-15)-(2.2-17), (2.1-3) can be written as

$$E = \text{Tr}(\underline{Q} \underline{C} \underline{W} \underline{C}^T - 2 \underline{Q} \underline{C} \underline{V} \underline{\hat{C}}^T + \underline{Q} \underline{\hat{C}} \underline{\hat{W}} \underline{\hat{C}}^T) \quad (2.2-18)$$

and by application of (A.2) and (A.4),

$$\frac{\partial E}{\partial \underline{\hat{C}}} = \underline{Q}(\underline{\hat{C}} \underline{\hat{W}} - \underline{C} \underline{V}^T) . \quad (2.2-19)$$

The necessary condition for E to be minimum is that

$$\frac{\partial E}{\partial \underline{\hat{A}}} = \underline{0}, \quad \frac{\partial E}{\partial \underline{\hat{B}}} = \underline{0}, \quad \text{and} \quad \frac{\partial E}{\partial \underline{\hat{C}}} = \underline{0} .$$

Therefore (2.2-1)-(2.2-3) must hold. In addition, since S^n and S^r are strictly stable (2.2-10), (2.2-11), (2.2-15), and (2.2-16) are equivalent to (2.2-4)-(2.2-7). Q.E.D.

Finding $\underline{\hat{A}}$, $\underline{\hat{B}}$, and $\underline{\hat{C}}$ which satisfy the necessary conditions of Theorem 2.2.1 is clearly a nontrivial task. Equations (2.2-1)-(2.2-7) are nonlinear in the unknowns and no closed form solution is apparently possible. An iterative solution to these equations is presented in Chapter III.

The solution to the model order reduction problem is not complete until $\underline{\hat{x}}_0$ is specified for a given \underline{x}_0 . The initial condition $\underline{\hat{x}}_0$ may be found by applying the following theorem.

THEOREM 2.2.2. Optimum $\underline{\hat{x}}_0$

Let $\underline{\hat{A}}$ and $\underline{\hat{C}}$ be given. If $(\underline{\hat{A}}, \underline{\hat{C}})$ is an observable pair then the optimum $\underline{\hat{x}}_0$ is

$$\underline{\hat{x}}_0 = \underline{\hat{G}}^{-1} \underline{F}^T \underline{x}_0 . \quad (2.2-20)$$

proof: For the unforced S^n and S^r , $y = \underline{C} \underline{\phi}(t) \underline{x}_0$ and $\hat{y} = \underline{\hat{C}} \underline{\hat{\phi}}(t) \hat{\underline{x}}_0$. Then

$$E = \frac{1}{2} (\underline{x}_0^T \underline{G} \underline{x}_0 - 2 \underline{x}_0^T \underline{F} \hat{\underline{x}}_0 + \hat{\underline{x}}_0^T \underline{\hat{G}} \hat{\underline{x}}_0)$$

and

$$\frac{\partial E}{\partial \underline{x}_0} = \underline{\hat{G}} \hat{\underline{x}}_0 - \underline{F}^T \underline{x}_0 .$$

The necessary condition for optimality is

$$\underline{\hat{G}} \hat{\underline{x}}_0 - \underline{F}^T \underline{x}_0 = \underline{0} .$$

It is well known [23] that if $(\underline{\hat{A}}, \underline{\hat{C}})$ is an observable pair then $\underline{\hat{G}}$ is positive definite. Therefore $\underline{\hat{G}}^{-1}$ exists and $\hat{\underline{x}}_0 = \underline{\hat{G}}^{-1} \underline{F}^T \underline{x}_0$. Q.E.D.

2.3 REDUCTION OF UNSTABLE SYSTEMS

The results given in Theorem (2.2.1) depend upon S^n being a strictly stable system. This is a very restrictive condition since many systems of interest will have some eigenvalues with nonnegative real parts. In designing controllers for an unstable system it is usually desirable that any model used for design purposes retain precisely the unstable modes of the system. The retention of the unstable modes in the simplified model is possible if the system model is first algebraically decoupled into its stable and unstable components.

Let P be a linear transformation such that

$$\underline{P}^{-1} \underline{A} \underline{P} = \begin{bmatrix} \underline{A}_1 & \underline{0} \\ \underline{0} & \underline{A}_2 \end{bmatrix} \quad (2.3-1)$$

where \underline{A}_1 possesses all of the eigenvalues of \underline{A} with nonnegative real parts. Then if

$$\underline{x} = \underline{P} \begin{bmatrix} \underline{v}_1 \\ \underline{v}_2 \end{bmatrix},$$

an equivalent representation of S^n is

$$\frac{d}{dt} \begin{bmatrix} \underline{v}_1 \\ \underline{v}_2 \end{bmatrix} = \begin{bmatrix} \underline{A}_1 & \underline{0} \\ \underline{0} & \underline{A}_2 \end{bmatrix} \begin{bmatrix} \underline{v}_1 \\ \underline{v}_2 \end{bmatrix} + \begin{bmatrix} \underline{B}_1 \\ \underline{B}_2 \end{bmatrix} \underline{u} \quad (2.3-2)$$

$$\underline{y} = [\underline{C}_1 \quad \underline{C}_2] \begin{bmatrix} \underline{v}_1 \\ \underline{v}_2 \end{bmatrix}$$

where

$$\begin{bmatrix} \underline{B}_1 \\ \underline{B}_2 \end{bmatrix} = \underline{P}^{-1} \underline{B}$$

and

$$[\underline{C}_1 \quad \underline{C}_2] = \underline{C} \underline{P}.$$

The system model S^n is now decoupled into an unstable subsystem S_1 described by

$$\dot{\underline{v}}_1 = \underline{A}_1 \underline{v}_1 + \underline{B}_1 \underline{u} \quad (2.3-3)$$

$$\underline{y}_1 = \underline{C}_1 \underline{v}_1$$

and a stable subsystem S_2 described by

$$\dot{\underline{v}}_2 = \underline{A}_2 \underline{v}_2 + \underline{B}_2 \underline{u} \quad (2.3-4)$$

$$\underline{y}_2 = \underline{C}_2 \underline{v}_2$$

with $\underline{y} = \underline{y}_1 + \underline{y}_2$. Subsystem S_2 satisfies the conditions of Theorem 2.1.1. If S_2 can now be optimally simplified to a lower order model of the form given by (2.1-2), the complete simplified model for S^n is

$$\frac{d}{dt} \begin{bmatrix} \underline{v}_1 \\ \underline{\hat{x}} \end{bmatrix} = \begin{bmatrix} \underline{A}_1 & \underline{0} \\ \underline{0} & \underline{\hat{A}} \end{bmatrix} \begin{bmatrix} \underline{v}_1 \\ \underline{\hat{x}} \end{bmatrix} + \begin{bmatrix} \underline{B}_1 \\ \underline{\hat{B}} \end{bmatrix} \underline{u} \quad (2.3-5)$$

$$\underline{y} = \begin{bmatrix} \underline{C}_1 & \underline{\hat{C}} \end{bmatrix} \begin{bmatrix} \underline{v}_1 \\ \underline{\hat{x}} \end{bmatrix}.$$

The choice and construction method for a suitable \underline{P} matrix is presented in Chapter III.

2.4 SUMMARY

In this chapter, the model order reduction of linear time-invariant systems has been formulated. The modeling error has been defined, and the role of the test input used in the reduction process has been discussed. Necessary conditions for an optimum simplified model of a stable system have been simply derived utilizing a gradient matrix approach. Finally, the simplification of unstable systems has been considered. This chapter provides the necessary foundation for a computational algorithm for model order reduction to be discussed in Chapter III.

CHAPTER III

COMPUTATION OF SIMPLIFIED SYSTEM MODELS

3.1 INTRODUCTION

The development of a computational scheme for model order reduction is the concern of this chapter. First, under the assumption that the given system model S^n possesses no unstable modes, a canonical form for \hat{A} is selected. An iterative method for minimizing the modeling error is then outlined. The case when S^n has several unstable modes is next considered. A method of retaining the unstable modes of S^n in the simplified model S^r via a suitable transformation of S^n into its stable and unstable subsystems is presented. Finally, a numerical example is given to demonstrate the implementation of the algorithms.

3.2 A CANONICAL STRUCTURE FOR THE SIMPLIFIED MODEL

For a given S^n , the model order reduction process requires the determination of the parameters of the model S^r for which the modeling error is minimum. Suppose the matrix A of the system model S^n given by (2.1-1) has only eigenvalues with negative real parts. The results of Theorem 2.1.1 then apply to S^n . The optimal parameters of the simplified model satisfy the set of coupled matrix equations (2.2-1)-(2.2-7). Since an explicit solution for \hat{A} , \hat{B} , and \hat{C} is not available, an iterative solution technique is necessary for finding a model S^r satisfying the necessary conditions for optimality. One possible iterative scheme is as follows:

- i) Select an initial guess for the simplified model description, say $(\hat{\underline{A}}^{(0)}, \hat{\underline{B}}^{(0)}, \hat{\underline{C}}^{(0)})$.
- ii) Compute $E^{(0)} = E(\hat{\underline{A}}^{(0)}, \hat{\underline{B}}^{(0)}, \hat{\underline{C}}^{(0)})$.
- iii) Set $k = 0$.
- iv) Solve (2.2-4) - (2.2-7) for $\hat{\underline{G}}, \hat{\underline{F}}, \hat{\underline{W}}$ and \underline{V} .
- v) Compute the gradient matrices $\partial E/\partial \hat{\underline{A}}, \partial E/\partial \hat{\underline{B}},$ and $\partial E/\partial \hat{\underline{C}}$ via equations (2.2-13), (2.2-17), and (2.2-19).
- vi) Compute the correction $(\Delta \hat{\underline{A}}^{(k)}, \Delta \hat{\underline{B}}^{(k)}, \Delta \hat{\underline{C}}^{(k)})$ to $(\hat{\underline{A}}^{(k)}, \hat{\underline{B}}^{(k)}, \hat{\underline{C}}^{(k)})$ by a gradient or variable metric method [25]. Compute

$$\hat{\underline{A}}^{(k+1)} = \hat{\underline{A}}^{(k)} + \alpha \Delta \hat{\underline{A}}^{(k)}$$

$$\hat{\underline{B}}^{(k+1)} = \hat{\underline{B}}^{(k)} + \alpha \Delta \hat{\underline{B}}^{(k)}$$

$$\hat{\underline{C}}^{(k+1)} = \hat{\underline{C}}^{(k)} + \alpha \Delta \hat{\underline{C}}^{(k)}$$

where α is selected such that $\hat{\underline{A}}^{(k+1)}$ is a stable matrix and $E^{(k+1)} < E^{(k)}$.

- vii) If $\|\hat{\underline{A}}^{(k+1)} - \hat{\underline{A}}^{(k)}\| \leq \epsilon_1$, $\|\hat{\underline{B}}^{(k+1)} - \hat{\underline{B}}^{(k)}\| \leq \epsilon_2$, and $\|\hat{\underline{C}}^{(k+1)} - \hat{\underline{C}}^{(k)}\| \leq \epsilon_3$, where ϵ_1, ϵ_2 , and ϵ_3 are some prescribed tolerances, stop; otherwise let $k \leftarrow k + 1$ and go to (iv).

Unfortunately this process is quite impractical. Step (iv) is extremely time consuming for a general form of \underline{A} and $\hat{\underline{A}}$. As the number of unknowns $r(r+m+l)$ becomes large, convergence tends to be very slow. If a variable metric method is used in step (vi), a large amount of computer storage is required.

Despite its shortcomings, the above algorithm serves to point out

two features which a good algorithm based on Theorem 2.2.1 should possess. First, the solution of the matrix equations (2.2-4) - (2.2-7) should be efficient and should be required as few times as possible. Second, the number of parameters which must be determined should be kept at a minimum.

Assuming S^r to be in some canonical form is a practical means of reducing the number of unknowns to be determined while simultaneously simplifying the solution of the matrix equations. Wilson [16] has suggested confining S^r to be one of the Luenberger canonical forms [26]. However, these forms require that some scheme be determined for specifying the size of each block. The Luenberger forms do not significantly simplify the solution of (2.2-4) - (2.2-7).

The Jordan canonical form is another possible structure for \hat{A} . This structure permits simple verification that S^r is stable at any step of an iterative process. Moreover, relatively efficient solution of the matrix equations is possible. The Jordan form also has disadvantages. Unless the eigenvalues of S^r are constrained to be real and distinct, some criterion for establishing a priori the block structure of the Jordan form is needed. Also, it has been found [27] that when \hat{A} is a Jordan form the equations used for parameter optimization are numerically ill-conditioned and iterative solution techniques converge only for initial guesses very near the optimal parameters.

The Schwarz form [23] has been selected as a canonical structure for \hat{A} . The matrix \hat{A} is said to be in Schwarz form if

$$\hat{\underline{A}} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ -\hat{a}_r & 0 & 1 & \dots & 0 & 0 \\ 0 & -\hat{a}_{r-1} & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & -\hat{a}_2 & -\hat{a}_1 \end{bmatrix} \quad (3.2-1)$$

Several characteristics make the Schwarz form an attractive choice for $\hat{\underline{A}}$. First, the $\hat{\underline{A}}$ matrix has only r parameters to be determined. Second, it can be shown [23] that $\hat{\underline{A}}$ is strictly stable if, and only if, $\hat{a}_i > 0$ for all i . Thus the stability of S^r during an iterative process can be easily ascertained. Finally, the Schwarz form enables efficient solution of the matrix equations (2.2-4) - (2.2-7) (see Appendix B).

The specification of $\hat{\underline{A}}$ as a Schwarz form substantially decreases the number of parameters to be determined in the reduction process. A further reduction in the number of unknown parameters which must be determined iteratively can be achieved by the following argument. Note that the time response of S^n to an applied input $\underline{u}(t) = \underline{w}\delta(t)$ is equivalent to the response of the single input system

$$\begin{aligned} \dot{\underline{x}}(t) &= \underline{A} \underline{x}(t) + \underline{b} u(t) \\ \underline{y}(t) &= \underline{C} \underline{x}(t) \end{aligned} \quad (3.2-2)$$

to the input $u(t) = \delta(t)$ provided $\underline{b} = \underline{B} \underline{w}$. Now consider the reduction process applied to (3.2-2) with $u(t) = \delta(t)$; a simplified model of the form

$$\begin{aligned}\dot{\hat{\underline{x}}}(t) &= \hat{\underline{A}} \hat{\underline{x}}(t) + \hat{\underline{b}} u(t) \\ \hat{\underline{y}}(t) &= \hat{\underline{C}} \hat{\underline{x}}(t)\end{aligned}\tag{3.2-3}$$

is desired. If $\hat{\underline{b}} = [0, 0, \dots, 1]^T$ and $\hat{\underline{A}}$ is a Schwarz form, (3.2-3) is said to be in Schwarz canonical form. Any single input, controllable system can be transformed to Schwarz canonical form [28]. Thus, without loss of generality S^T can be assumed to be in Schwarz canonical form.

The solution of the model order reduction problem now requires the specification of the $r(\ell+1)$ unknowns contained in $\hat{\underline{A}}$ and $\hat{\underline{C}}$. Given $\hat{\underline{A}}$ and $\hat{\underline{b}}$, the matrix $\hat{\underline{C}}$ is determined algebraically from (2.2-3). Since $(\hat{\underline{A}}, \hat{\underline{b}})$ is a controllable pair, $\hat{\underline{W}}^{-1}$ exists [23] and from (2.2-3)

$$\hat{\underline{C}} = \underline{C} \underline{V} \underline{W}^{-1} .\tag{3.2-4}$$

Once the simplified model (3.2-3) has been found, a suitable $\hat{\underline{B}}$ matrix can be computed from (2.2-2) by requiring

$$\hat{\underline{G}} \hat{\underline{B}} - \underline{F}^T \underline{B} = \underline{0} .$$

If (3.2-3) is completely observable, $\hat{\underline{G}}^{-1}$ exists and

$$\hat{\underline{B}} = \hat{\underline{G}}^{-1} \underline{F}^T \underline{B} .$$

Fixing $\hat{\underline{b}}$ as discussed above essentially means that the $\hat{\underline{B}}$ is fixed in the process of selecting $\hat{\underline{A}}$ and $\hat{\underline{C}}$. In actual test cases using the algorithm discussed in the next section, it has been found that the $r(\ell+1)$ degrees of freedom in $\hat{\underline{A}}$ and $\hat{\underline{C}}$ usually allow for a accurate approximation to S^n . However, the possibility exists that after fixing

$\hat{\underline{B}}$ and finding the $\hat{\underline{A}}$ and $\hat{\underline{C}}$ minimizing E , a further reduction in E can be achieved by varying $\hat{\underline{B}}$.

3.3 MINIMIZATION OF THE MODELING ERROR

For convenience, let the r -vector $\hat{\underline{a}}$ be defined by $\hat{\underline{a}} = [\hat{a}_1, \hat{a}_2, \dots, \hat{a}_r]^T$. The model order reduction solution can now be cast in the form of the solution of the nonlinear programming problem:

$$\begin{aligned} \text{Minimize: } & E(\hat{\underline{a}}, \hat{\underline{C}}) \\ \text{Subject to: } & \hat{a}_i > 0, \quad i = 1, 2, \dots, r. \end{aligned} \quad (3.3-1)$$

The modeling error E as given by (2.2-18) can be written as

$$E = E_S + E_M,$$

where

$$\begin{aligned} E_S &= \text{Tr}(\underline{Q} \underline{C} \underline{W} \underline{C}^T) \\ E_M &= \text{Tr}(\underline{Q} \hat{\underline{C}} \hat{\underline{W}} \hat{\underline{C}}^T) - 2 \text{Tr}(\underline{Q} \underline{C} \underline{V} \hat{\underline{C}}^T) \end{aligned} \quad (3.3-2)$$

The term E_S depends only on the system model description and, consequently, does not need to be computed in the minimization process. The modeling error E is minimized by minimizing E_M .

For a given system S^n , model order r , and input weighting vector \underline{w} , an algorithm for solving (3.3-1) is:

Algorithm I

- i) Choose an initial guess $\hat{\underline{a}}^{(0)}$, $\hat{a}_i^{(0)} > 0$, $i = 1, 2, \dots, r$.
- ii) Compute $\underline{b} = \underline{B} \underline{w}$.
- iii) Set $k = 0$.
- iv) Compute the solutions $\hat{\underline{W}}$ and \underline{V} of the matrix equations

$$\hat{\underline{A}}^{(k)} \hat{\underline{W}} + \hat{\underline{W}} (\hat{\underline{A}}^{(k)})^T = - \hat{\underline{b}} \hat{\underline{b}}^T \quad (3.3-3)$$

$$\underline{A} \underline{V} + \underline{V} (\hat{\underline{A}}^{(k)})^T = -\underline{b} \underline{\hat{b}}^T. \quad (3.3-4)$$

- v) Compute $\hat{\underline{C}}^{(k)} = \underline{C} \underline{V} \underline{\hat{W}}^{-1}$.
- vi) Calculate $E_M^{(k)}$ from equation (3.3-2).
- vii) Compute the solutions $\hat{\underline{G}}, \underline{F}$ of the matrix equations

$$(\hat{\underline{A}}^{(k)})^T \hat{\underline{G}} + \hat{\underline{G}} \underline{A}^{(k)} = -(\hat{\underline{C}}^{(k)})^T \underline{Q} \hat{\underline{C}}^{(k)} \quad (3.3-5)$$

$$\underline{A}^T \underline{F} + \underline{F} \hat{\underline{A}}^{(k)} = -\underline{C}^T \underline{Q} \hat{\underline{C}}^{(k)}. \quad (3.3-6)$$

- viii) Find the gradient $\frac{\partial E}{\partial \hat{\underline{a}}}$ by selecting the appropriate entries from the gradient matrix $\frac{\partial E}{\partial \hat{\underline{A}}}$ given by (2.2-13).
- ix) Using the results from (viii) compute the correction vector $\underline{s}^{(k)}$ by a variable metric method; then

$$\hat{\underline{a}}^{(k+1)} = \hat{\underline{a}}^{(k)} + \alpha^* \underline{s}^{(k)} \quad (3.3-7)$$

where $\alpha^* > 0$ is selected such that $E_M^{(k+1)} < E_M^{(k)}$ and $\hat{a}_i^{(k+1)} > 0, i = 1, 2, \dots, 4$.

- x) If $\|\hat{\underline{a}}^{(k+1)} - \hat{\underline{a}}^{(k)}\| \leq \epsilon \|\hat{\underline{a}}^{(k)}\|$, where ϵ is some prescribed error tolerance, compute $\hat{\underline{C}}^{(k+1)}$ and go to (xi); otherwise, let $k \leftarrow k + 1$ and go to (iv).
- xi) Calculate $\hat{\underline{B}} = \hat{\underline{G}}^{-1} \underline{F}^T \underline{B}$.

The details of step (ix) above require some amplification. An upper bound of α_{\max} on α^* can be easily derived. From (3.3-7) and the condition that $\hat{a}_i^{(k+1)} > 0, i = 1, 2, \dots, r$, it follows that

$$\alpha^* s_i^{(k)} > -\hat{a}_i^{(k)} \quad i = 1, 2, \dots, r.$$

Thus,

$$\alpha_{\max} = \min_{s_i(k) < 0} [-a_i(k)/s_i(k)].$$

The speed of convergence of variable metric methods is very dependent upon finding the appropriate α^* . For most methods, the α^* such that

$$E_M(\hat{\underline{a}}(k) + \alpha^* \underline{s}(k)) = \min_{0 < \alpha < \alpha_{\max}} E_M(\hat{\underline{a}}(k) + \alpha \underline{s}(k))$$

is needed to achieve fast convergence. Finding this α^* requires a linear search technique [25]. However, linear search techniques generally demand that the function E_M and, in some cases, the gradient of E_M be evaluated for many different α 's until α^* is located. As can be seen from (3.2-4), $\hat{\underline{c}}(k)$ is a function of $\hat{\underline{a}}(k)$. Thus each time in the linear search E_M and $\partial E_M / \partial \hat{\underline{a}}$ are evaluated for a new α , the matrix equations (3.3-3)-(3.3-6) must be solved. Vast amounts of computing time can obviously be spent in the linear search stage.

In the actual implementation of the above algorithm the variable metric method of Fletcher [29] has been utilized. Fletcher's method is specifically designed to eliminate the use of the time consuming linear search process. Often it is possible to use $\alpha^* = 1$ provided $\hat{a}_i^{(k+1)} > 0$, $i = 1, 2, \dots, r$, and E_M is reduced sufficiently. In most instances where $\alpha^* = 1$ is not suitable, cubic interpolation between $\alpha = 0$ and $\alpha = \min(1, \alpha_{\max})$ yields an acceptable α^* . In numerical examples worked using Fletcher's method with Algorithm I, it has been found that usually no more than two evaluations of E_M are required per iteration to find a suitable α^* .

3.4 DECOMPOSITION OF UNSTABLE SYSTEMS

A numerical algorithm for constructing the transformation matrix \underline{P} introduced in Section 2.3 is developed in this section. Recall that the matrix \underline{P} is needed to obtain an equivalent representation of the system model S^n given by (2.1-1) in terms of stable and unstable subsystems described by (2.3-3) and (2.3-4).

Suppose the $n \times n$ system matrix \underline{A} has eigenvalues $\{\mu_i\}_{i=1}^q$ such that $\text{Re}(\mu_i) \geq 0$. Let each μ_i have algebraic multiplicity v_i , where $\sum_{i=1}^q v_i = p < n$, and unity geometric multiplicity. Let

$$p_D(\lambda) = \prod_{i=1}^q (\lambda - \mu_i)^{v_i} = \lambda^p + \alpha_1 \lambda + \dots + \alpha_{p-1} \lambda + \alpha_p \quad (3.4-1)$$

and define the $p \times p$ matrix \underline{A}_1 to be in companion form

$$\underline{A}_1 = \begin{bmatrix} 0 & 0 & \dots & 0 & -\alpha_p \\ 1 & 0 & \dots & 0 & -\alpha_{p-1} \\ 0 & 1 & \dots & 0 & -\alpha_{p-2} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 1 & -\alpha_1 \end{bmatrix} \quad (3.4-2)$$

Now let the $n \times p$ matrix \underline{T}_1 be constructed such that

$$\underline{A} \underline{T}_1 = \underline{T}_1 \underline{A}_1 \quad (3.4-3)$$

and $\text{rank}(\underline{T}_1) = p$. If $\underline{T}_1 = [\underline{t}_1 \ \underline{t}_2 \ \dots \ \underline{t}_p]$, then (3.4-3) implies

$$\underline{A} \underline{t}_{k-1} = \underline{t}_k \quad k = 2, \dots, p \quad (3.4-4)$$

and

$$\underline{A} \underline{t}_p = -\alpha_p \underline{t}_1 - \alpha_{p-1} \underline{t}_2 - \dots - \alpha_1 \underline{t}_p. \quad (3.4-5)$$

From (3.4-4) and (3.4-5),

$$\underline{\Gamma} \underline{t}_1 = \underline{0}, \quad (3.4-6)$$

where

$$\underline{\Gamma} = \underline{A}^p + \alpha_1 \underline{A}^{p-1} + \dots + \alpha_{p-1} \underline{A} + \alpha_p \underline{I}. \quad (3.4-7)$$

By Frobenius' Theorem [23], $\underline{\Gamma}$ has eigenvalues $\{p_D(\lambda_i)\}_{i=1}^n$, where λ_i are the eigenvalues of \underline{A} . Since \underline{A} has p eigenvalues which satisfy $p_D(\lambda) = 0$, $\underline{\Gamma}$ has precisely p zero eigenvalues. Therefore, $\text{rank}(\underline{\Gamma}) = n - p$ and (3.4-6) has p linearly independent solutions. In principle, \underline{T}_1 may be constructed by choosing a solution \underline{t}_1 of (3.4-6) such that

$$\text{rank} [\underline{t}_1 \underline{A} \underline{t}_1 \dots \underline{A}^{p-1} \underline{t}_1] = p. \quad (3.4-8)$$

A vector \underline{t}_1 such that (3.4-6) and (3.4-8) are satisfied is given by the following important result which is proved in Appendix C.

THEOREM 3.4.1.

For $i = 1, 2, \dots, q$, let \underline{v}_i be the grade v_i generalized eigenvector of \underline{A} corresponding to the eigenvalue μ_i . If

$$\underline{t}_1 = \sum_{i=1}^q \beta_i \underline{v}_i \quad (3.4-9)$$

for any $\{\beta_i\}_{i=1}^q$, then $\underline{\Gamma} \underline{t}_1 = \underline{0}$. Furthermore, if $\beta_i \neq 0$ for all i , then (3.4-8) is satisfied.

The choice of the β_i 's in (3.4-9) is arbitrary so long as no β_i is

zero. Thus $\beta_i = 1$ is acceptable.

If the $n \times n$ matrix $\underline{T} = [\underline{T}_1 \ \underline{T}_2]$ is constructed with \underline{T}_2 selected to give \underline{T} maximum rank, then

$$\underline{A}' = \underline{T}^{-1} \underline{A} \underline{T} = \begin{bmatrix} \underline{A}_1 & \underline{Z} \\ \underline{0} & \underline{A}_2 \end{bmatrix}$$

where, in general, $\underline{Z} \neq \underline{0}$. Define the $n \times n$ matrix \underline{S} as

$$\underline{S} = \begin{bmatrix} \underline{I}_p & \underline{R} \\ \underline{0} & \underline{I}_{n-p} \end{bmatrix}$$

with the $p \times (n-p)$ matrix \underline{R} unspecified for the moment. Consider the similarity transformation

$$\underline{A}'' = \underline{S}^{-1} \underline{A}' \underline{S} = \begin{bmatrix} \underline{A}_1 & \underline{A}_1 \underline{R} - \underline{R} \underline{A}_2 + \underline{Z} \\ \underline{0} & \underline{A}_2 \end{bmatrix}$$

If \underline{R} is selected to satisfy

$$\underline{A}_1 \underline{R} - \underline{R} \underline{A}_2 = -\underline{Z}, \quad (3.4-10)$$

then

$$\underline{A}'' = \begin{bmatrix} \underline{A}_1 & \underline{0} \\ \underline{0} & \underline{A}_2 \end{bmatrix}$$

Since \underline{A}_1 and \underline{A}_2 have no common eigenvalues, the solution \underline{R} to equation (3.4-10) is unique [48]. By defining the \underline{P} matrix in (2.3-1) to be

$$\underline{P} = \underline{T} \underline{S}$$

the system model S^n may be transformed as depicted by (2.3-2).

The discussion above is summarized by the following computational scheme:

Algorithm II

- i) Compute the eigenvalues of \underline{A} using the QR transformation of Francis [30]. Pick out the set $\{\mu_i\}_{i=1}^q$ and determine v_i by inspection.
- ii) Compute the generalized eigenvector \underline{v}_i of grade v_i associated with μ_i for $i = 1, 2, \dots, q$.
- iii) Let $\underline{t}_1 = \sum_{i=1}^q \underline{v}_i$.
- iv) Form $\underline{T}_1 = [\underline{t}_1 \ \underline{A} \ \underline{t}_1 \ \dots \ \underline{A}^{p-1} \ \underline{t}_1]$.
- v) Construct $\underline{T} = [\underline{T}_1 \ \underline{T}_2]$ by selecting \underline{T}_2 to guarantee $\text{rank}(\underline{T}) = n$.
- vi) Make the transformation

$$\underline{T}^{-1} \underline{A} \underline{T} = \begin{bmatrix} \underline{A}_1 & \underline{Z} \\ \underline{0} & \underline{A}_2 \end{bmatrix}$$

$$\underline{T}^{-1} \underline{B} = \begin{bmatrix} \underline{B}'_1 \\ \underline{B}'_2 \end{bmatrix}$$

$$\underline{C} \underline{T} = \begin{bmatrix} \underline{C}'_1 & \underline{C}'_2 \end{bmatrix}.$$

- vii) Solve the matrix equation $\underline{A}_1 \underline{R} - \underline{R} \underline{A}_2 = -\underline{Z}$ for \underline{R} .

viii) Compute

$$\underline{B}_1 = \underline{B}'_1 - \underline{R} \underline{B}_2$$

$$\underline{C}_2 = \underline{C}_1 \underline{R} + \underline{C}'_2 .$$

Note that this algorithm completely specifies the decomposition of S^n in terms of its stable and unstable components.

At this point several comments on various steps of Algorithm II should be made. First, the QR transformation is an efficient and extremely reliable method for computing the eigenvalues of a general matrix \underline{A} . It requires on the order of $8n^3$ multiplication and division operations to find all of the eigenvalues of an $n \times n$ matrix. In step (ii), the generalized eigenvector \underline{v}_i can be computed by finding a \underline{v}_i such that

$$(\underline{A} - \mu_i \underline{I})^{v_i} \underline{v}_i = \underline{0} \quad (3.4-11)$$

but

$$(\underline{A} - \mu_i \underline{I})^{v_i-1} \underline{v}_i \neq \underline{0} .$$

The accuracy of the \underline{v}_i found will, of course, depend upon the precision

with which $(\underline{A} - \mu_i \underline{I})^{v_i}$ is computed and (3.4-11) is solved. If $v_i = 1$, then a very precise \underline{v}_i can be obtained using the inverse iteration method [31]. Finally, with \underline{A}_1 in companion form the solution for \underline{R} in step (vi) can be obtained by the method devised by Jameson [32].

3.5 EXAMPLE

A digital computer program implementing Algorithm I and Algorithm

II has been written in FORTRAN IV and tested extensively for system models of order up to twelve. As an example of the application of the methods of this chapter consider the 8-th order model [33] of a synchronous machine tied to an infinite bus:

$$\dot{\underline{x}}(t) = \underline{A} \underline{x}(t) + \underline{B} \underline{u}(t),$$

where

$$\underline{A} = \begin{bmatrix} -0.196 & 1.0 & -1.39 & -0.003 & 0 & 0 & 0 & 0 \\ -50.9 & -20 & 87.0 & -2.4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -2.94 & 0 & -22.6 & -0.008 & 0 & 0 & 38.8 & 58.2 \\ 0 & 0 & 0 & -0.133 & -2.5 & -50 & 0 & 0 \\ 0 & 0 & 0 & -0.033 & -0.625 & -12.7 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & -4 & 0 & 4 & -12.5 \end{bmatrix}$$

$$\underline{B} = \begin{bmatrix} 0 & 0 & 0 & 0 & -50 & -12.5 & 0 & 0 \end{bmatrix}^T.$$

Suppose that the states x_2 and x_4 are the outputs of the system which are of interest. Then

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \underline{x}.$$

The variables x_2 and x_4 are actually the incremental changes in terminal voltage and angular velocity, respectively, about a steady state oper-

ating point. The eigenvalues of the above A matrix are:

$$\lambda_{1,2} = 0.231 \pm j4.805$$

$$\lambda_3 = -0.0329$$

$$\lambda_4 = -1.944$$

$$\lambda_5 = -3.636$$

$$\lambda_{6,7} = -13.929 \pm j0.729$$

$$\lambda_8 = -16.894$$

Retaining λ_1 and λ_2 in the simplified model and selecting $r = 4$, the following simplified model is obtained for $u(t) = \delta(t)$:

$$\frac{d}{dt} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \\ \hat{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & -23.140 & 0 & 0 \\ 1 & 0.462 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -0.0855 & -2.786 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \\ \hat{x}_4 \end{bmatrix} + \begin{bmatrix} 37.88 \\ 6.07 \\ 0 \\ 1 \end{bmatrix} u(t)$$

$$\begin{bmatrix} \hat{y}_1 \\ \hat{y}_2 \end{bmatrix} = \begin{bmatrix} 0.0194 & 4.182 & -4.497 & -51.562 \\ 1.0 & 0.231 & 1.118 & -3.386 \end{bmatrix} \underline{\hat{x}}$$

The eigenvalues of the simplified model are:

$$\hat{\lambda}_{1,2} = 0.231 \pm j4.805$$

$$\hat{\lambda}_3 = -0.0310$$

$$\hat{\lambda}_4 = -2.755$$

Figure 3.5.1 gives a comparison between y_1 and \hat{y}_1 .

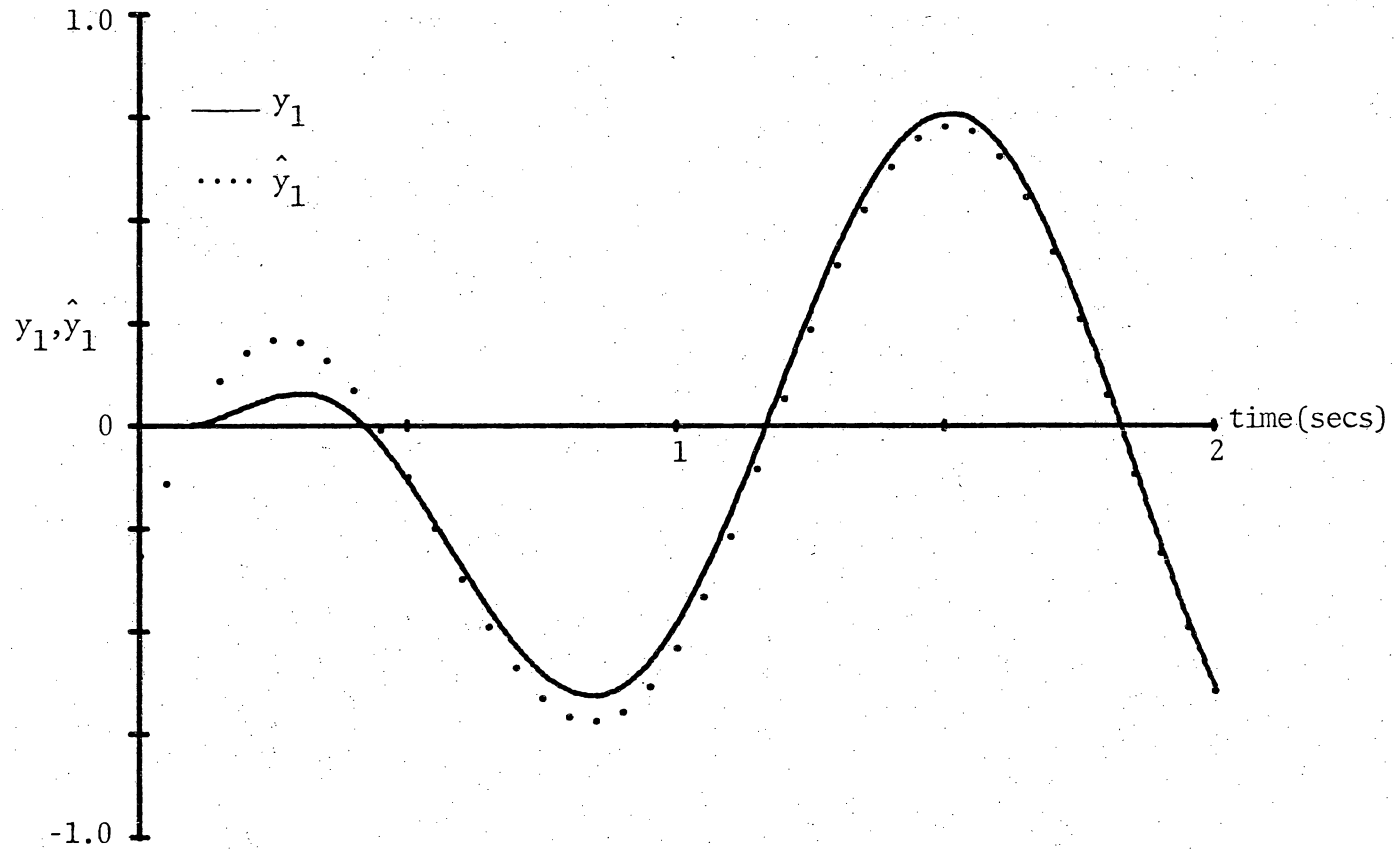


Figure 3.5.1. Response of System Output y_1 and Simplified Model Output \hat{y}_1 to the Input $u(t) = 0.01 \delta(t)$.

3.6 CONCLUDING REMARKS

In summary, two algorithms that enable computation of a simplified representation of a high order system have been presented. Algorithm I applies to the model order reduction of strictly stable systems. The \hat{A} matrix of the simplified model is restricted to be in Schwarz form to reduce the computational complexity of the reduction process. The model parameters which minimize the modeling error E are then determined by an iterative procedure employing Fletcher's variable metric method.

If S^n has any unstable modes, Algorithm II partitions the system into stable and unstable subsystems by way of a linear transformation. An important point concerning this algorithm should be mentioned. Recall that in the development of Algorithm II it was assumed that each of the eigenvalues of \underline{A} with zero or positive real parts had unity geometric multiplicity. If the geometric multiplicity of any of the μ_i 's is greater than one, then it is not possible [49] to transform \underline{A} as indicated in (2.3-2) and still maintain \underline{A}_1 in the form given by (3.4-2). Thus, Algorithm II must be modified for the case when μ_i has larger than unity geometric multiplicity. If the size of each Jordan block associated with μ_i is known, the \underline{A}_1 can be selected to be in the block diagonal form

$$\underline{A}_1 = \text{diag}(\underline{A}_{11}, \underline{A}_{12}, \dots, \underline{A}_{1s}),$$

where \underline{A}_{11} is the companion form associated with the Jordan blocks of highest order for each μ_i , \underline{A}_{12} is the companion form associated with the Jordan blocks of next highest order for each μ_i , and so on. With \underline{A}_1 selected in this manner the development given in Section 3.4 and, conse-

quently, Algorithm II can be modified readily to account for μ_i with geometric multiplicity greater than one.

The procedure for applying Algorithm I and Algorithm II can be summarized:

- i) If S^n possesses any unstable modes, Algorithm II is applied to partition the system into an unstable subsystem S_1 of order p and a stable subsystem S_2 of order $n-p$. Then S_2 is approximated by a model of order $r < n-p$ via Algorithm I.
- ii) If S^n has no unstable modes, then Algorithm I is applied directly to S^n .

CHAPTER IV
OPTIMAL CONTROL AND SYSTEM SIMPLIFICATION

4.1 CONTROL OF COMPLEX SYSTEMS

Modern control theory has provided systematic means for handling many types of control problems associated with complex systems described by (2.1-1). However, in closed-loop feedback configurations, the measurement of all the state variables is usually required. In case all the state variables are not accessible for measurement, the state vector may be reconstructed by an asymptotic observer [34]. Inclusion of an observer in the control system design necessarily increases the equipment needed and, consequently, the cost for controlling complex processes. Computational difficulties are also encountered in designing optimal controllers for high-order systems. Thus, the realization of optimal controllers for large-scale systems can be both expensive and computationally unwieldy.

In an attempt to simplify control strategies for complex systems, research has been directed toward the design of suboptimal controllers. Most of this work has concentrated on the design of suboptimal linear regulators which provide performance commensurate with the optimal linear regulator yet which are simpler to implement. Aoki [35], Chidambara and Schainker [36], and Joyner [37] have proposed designing suboptimal linear regulators with the aid of lower order system models. In these references, the suboptimal controller of the system is merely the optimal controller for the appropriately chosen simplified system model.

In this chapter, the use of simplified system models for designing suboptimal output regulators is discussed.

4.2 REVIEW OF THE OUTPUT REGULATOR PROBLEM

The output regulator problem [22] may be stated as follows: given a plant described by (2.1-1), find the control $\underline{u}(t)$ such that the performance measure,

$$J = \frac{1}{2} \underline{y}(t_f)^T \underline{H} \underline{y}(t_f) + \frac{1}{2} \int_{t_0}^{t_f} [\underline{y}^T(t) \underline{Q} \underline{y}(t) + \underline{u}^T(t) \underline{R} \underline{u}(t)] dt, \quad (4.2-1)$$

where $\underline{H} \geq \underline{0}$, $\underline{Q} > \underline{0}$, $\underline{R} \geq \underline{0}$, and t_f is fixed, is minimum. From a heuristic viewpoint, the purpose of the output regulator is to maintain the system outputs near $\underline{y} = \underline{0}$ with the minimum expenditure of control energy. If S^n is observable and $\underline{u}(t)$ is unconstrained, the unique optimal control \underline{u}^* is

$$\underline{u}^*(t) = \underline{F}(t) \underline{x}(t), \quad (4.2-2)$$

where

$$\underline{F}(t) = -\underline{R}^{-1} \underline{B}^T \underline{K}(t) \underline{x}(t) \quad (4.2-3)$$

and where $\underline{K}(t)$ is the symmetric, positive definite solution of the matrix Riccati equation

$$\dot{\underline{K}}(t) = -\underline{K}(t) \underline{A} - \underline{A}^T \underline{K}(t) - \underline{C}^T \underline{Q} \underline{C} + \underline{K}(t) \underline{B} \underline{R}^{-1} \underline{B}^T \underline{K}(t) \quad (4.2-4)$$

with $\underline{K}(t_f) = \underline{C}^T \underline{H} \underline{C}$. The minimum value of the performance measure is

$$J^* [\underline{x}(t), t] = \frac{1}{2} \underline{x}^T(t) \underline{K}(t) \underline{x}(t). \quad (4.2-5)$$

Difficulties which arise in the implementation of the linear regulator solution deserve mention. First, the solution of (4.2-4) is a nontrivial undertaking. To find $\underline{K}(t)$, $n(n+1)/2$ first-order nonlinear differential equations must be solved. Typically, (4.2-4) is integrated numerically from $t = t_f$ backwards in time to $t = t_0$, and $\underline{K}(t)$ stored at various instants of time. Some type of memory device for storing $\underline{K}(t)$ is therefore required in the controller. This problem is alleviated if $t_f \rightarrow \infty$, $\underline{H} = \underline{0}$, and S^n is controllable, for then \underline{K} is time-invariant.

A second difficulty in implementing the output regulator is apparent from (4.2-2). The optimal control depends upon the entire state vector. Generally the complete state is not available for measurement, and an observer must be included in the feedback loop.

4.3 SUBOPTIMAL CONTROL VIA MODEL ORDER REDUCTION

A suboptimal control law for the output regulator problem is now derived from a simplified model of S^n . Suppose that a simplified model of order ℓ denoted by S^ℓ and described by (2.1-2) has been derived for S^n . Furthermore, assume that $\text{rank } \hat{C} = \ell$.

Consider the linear regulator problem for S^ℓ associated with the performance measure

$$\hat{J} = \frac{1}{2} \hat{y}^T(t_f) \underline{H} \hat{y}(t_f) + \frac{1}{2} \int_{t_0}^{t_f} [\hat{y}^T(t) \underline{Q} \hat{y}(t) + \underline{u}^T(t) \underline{R} u(t)] dt. \quad (4.3-1)$$

The optimal control is

$$\hat{u}^*(t) = \hat{F}(t) \hat{x}(t), \quad (4.3-2)$$

where

$$\underline{\hat{F}}(t) = -\underline{\hat{R}}^{-1} \underline{\hat{B}} \underline{\hat{K}}(t) \quad (4.3-3)$$

and where $\underline{\hat{K}}$ is the solution of the ℓ -th order matrix Riccati equation

$$\dot{\underline{\hat{K}}}(t) = -\underline{\hat{K}}(t) \underline{\hat{A}} - \underline{\hat{A}}^T \underline{\hat{K}}(t) + \underline{\hat{C}}^T \underline{Q} \underline{\hat{C}} - \underline{\hat{K}}(t) \underline{\hat{B}} \underline{\hat{R}}^{-1} \underline{\hat{B}}^T \underline{\hat{K}}(t) \quad (4.3-4)$$

with $\underline{\hat{K}}(t_f) = \underline{\hat{C}}^T \underline{H} \underline{\hat{C}}$.

Since $\text{rank}(\underline{\hat{C}}) = \ell$, $\underline{\hat{C}}^{-1}$ exists and (4.3-2) can be written in terms of the model outputs:

$$\underline{\hat{u}}^*(t) = \underline{\hat{F}}(t) \underline{\hat{C}}^{-1} \underline{\hat{y}}(t). \quad (4.3-5)$$

The goal in constructing S^ℓ is to make

$$\underline{\hat{y}}(t) \approx \underline{y}(t), \quad t \in (t_0, t_f), \quad (4.3-6)$$

for arbitrary \underline{u} . If (4.3-6) is valid, then (4.3-5) immediately suggests the suboptimal control law for S^n given by

$$\underline{u}^\circ(t) = \underline{\hat{F}}(t) \underline{\hat{C}}^{-1} \underline{y}(t). \quad (4.3-7)$$

The suboptimal control law (4.3-7) has several advantages over the optimal control law from the implementation standpoint. First, the matrix Riccati equation (4.3-4) is simpler to solve than (4.2-4). Second, the feedback matrix $\underline{\hat{F}}(t) \underline{\hat{C}}^{-1}$ requires less hardware for implementation than $\underline{F}(t)$. Thus a simplified control strategy is achieved. Third, the suboptimal control law requires only the output vector of the system rather than the entire state vector and thus eliminates the need for a state observer.

It can easily be shown [46] that

$$J^{\circ} [\underline{x}(t), t] = \frac{1}{2} \underline{x}^T(t) \underline{P}(t) \underline{x}(t) \quad (4.3-8)$$

where $\underline{P}(t)$ is the solution of

$$-\dot{\underline{P}}(t) = (\underline{A} + \underline{B} \underline{F}^{\circ}(t))^T \underline{P}(t) + \underline{P}(t) (\underline{A} + \underline{B} \underline{F}^{\circ}(t)) + \underline{C}^T \underline{Q} \underline{C} + [\underline{F}^{\circ}(t)]^T \underline{R} \underline{F}^{\circ}(t) \quad (4.3-9)$$

where

$$\underline{F}^{\circ}(t) = \hat{\underline{F}}(t) \hat{\underline{C}}^{-1} \underline{C} \quad (4.3-10)$$

Thus the performance degradation ΔJ resulting from the use of the sub-optimal control law is

$$\Delta J = \frac{1}{2} \underline{x}^T(t) [\underline{P}(t) - \underline{K}(t)] \underline{x}(t). \quad (4.3-11)$$

It must be emphasized that the development of \underline{u}° is formal in nature. The nearness of the suboptimal performance J° to the optimal performance J^* depends heavily upon the assumption that $\hat{\underline{y}} \approx \underline{y}$. If Algorithm I is employed to obtain S^{ℓ} , then the test function $\underline{u}(t) = \underline{w}\delta(t)$ is used. This by no means insures that $\hat{\underline{y}} \approx \underline{y}$ for arbitrary \underline{u} . However, in particular numerical examples, it has been found that if ℓ is large enough to allow a reasonably good approximation to S^{η} to be made, then very little performance deterioration results in using \underline{u}° instead of \underline{u}^* . This fact is demonstrated by the applications given in the next chapter.

When S^{ℓ} is a sufficiently good approximation to S^{η} then it is reasonable that the suboptimal control produces an asymptotically stable

closed-loop system. However stability of the suboptimal closed-loop system is not guaranteed.

4.4 SUMMARY

A suboptimal control scheme for the output regulator problem has been discussed. This scheme requires that only the system outputs be fed back. The suboptimal control law is easier to compute and to implement than the optimal control law.

The design procedure for the suboptimal controller may be summarized as follows:

- i) Obtain a simplified system model of order ℓ by Algorithm I and, if necessary, Algorithm II.
- ii) Solve the output regulator problem posed by (4.3-1) to obtain $\hat{\underline{F}}(t)$.
- iii) Form the suboptimal control law via (4.3-7).

CHAPTER V

MODEL ORDER REDUCTION AND POWER SYSTEM CONTROL

5.1 INTRODUCTION

Recently much attention has been given to the application of optimal control theory to power system control problems. Fosha and Elgerd [38] and Čalović [39] have applied linear regulator theory to the load-frequency control problem. Power system stabilization via linear regulator theory has been considered by Yu et al. [40,42].

Unfortunately, most realistic power system control design problems inherit the difficulties associated with large-scale systems. For large interconnected power systems, the mathematical model describing the dynamic behavior of the system is of very high order. Thus, the application of system order reduction and suboptimal control methods is receiving attention (cf. references [43], [44]).

In this chapter, the use of simplified models in designing suboptimal controllers for power systems is demonstrated. In particular, the power system stabilization control problem is considered. The model order reduction scheme developed in Chapter III is employed to obtain the suboptimal control law proposed in Chapter IV for two numerical examples.

5.2 STABILIZATION OF MULTI-MACHINE POWER SYSTEMS

Consider a p -machine power system and suppose the system is operating at some desired equilibrium state. The goal of stabilization control is to maintain the system at this equilibrium point for "small-

scale" perturbations in the system and to improve the response of the system to large-scale disturbances. A perturbation in the power system is considered small-scale if linearized differential equations are adequate in representing the dynamic response of the system to the perturbation.

Power system stabilization may be achieved using linear regulator theory. First, the p-machine system is modeled by a set of linearized state and output equations of the form

$$\begin{aligned}\underline{x}(t) &= \underline{A} \underline{x}(t) + \underline{B} \underline{u}(t) \\ \underline{y}(t) &= \underline{C} \underline{x}(t).\end{aligned}\tag{5.2-1}$$

The formulation of mathematical models in the form of (5.2-1) for multi-machine systems has been considered by Undrill [41] and Yu and Moussa [40]. The state vector \underline{x} is the vector of perturbations of the system states from equilibrium values. The control \underline{u} usually consists of exciter-voltage regulator and governor input signals. Next, a quadratic performance measure

$$J = \frac{1}{2} \int_0^{\infty} (\underline{y}^T \underline{Q} \underline{y} + \underline{u}^T \underline{R} \underline{u}) dt\tag{5.2-2}$$

is set up with suitably chosen \underline{Q} and \underline{R} . Finally, the optimal control law \underline{u}^* ,

$$\underline{u}^*(t) = \underline{F} \underline{x}(t),\tag{5.2-3}$$

which minimizes (5.2-2) is found.

The state vector \underline{x} may typically [40] have between 4p and 13p com-

ponents. The exact number of entries in \underline{x} depends upon the detail with which each synchronous machine, governor and voltage regulator unit in the system is modeled. Thus, for large p , the control law given by (5.2-3) is impractical to implement for the reasons discussed in Chapter IV. The use of system simplification and suboptimal control schemes is, therefore, an attractive means of obtaining a practical near-optimum controller design for the stabilization problem.

5.3 SUBOPTIMAL STABILIZATION CONTROL: EXAMPLES

The application of the suboptimal control scheme developed in Chapter IV to the power system stabilization problem is illustrated in this section by way of two simple examples.

Example 5.3.1

This first example has been treated extensively by Elangovan and Kuppurajulu [44]. Consider the power system described by (5.2-1) with

$$\underline{A} = \begin{bmatrix} -0.188 & 0 & 0.277 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -1.815 & -0.57 & -0.5 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & -20 & -12 \end{bmatrix},$$

$$\underline{B} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix}^T,$$

$$\underline{C} = \begin{bmatrix} 0.49 & -0.233 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

This power system consists of a single machine connected to an infinite bus. The system description is comprised of a third order model of a synchronous machine and a second order model of a speed governor. Control action is accomplished via input signals to the excitation circuit. The state and output vectors are

$$\underline{x} = [\Delta E_q \quad \Delta \delta \quad \Delta \omega \quad \Delta p \quad \Delta \dot{p}]^T$$

$$\underline{y} = [\Delta V_T \quad \Delta \delta \quad \Delta \omega]$$

where

ΔE_q is the incremental change in excitation voltage,

$\Delta \delta$ is the rotor angle deviation of the machine,

$\Delta \omega$ is the angular velocity deviation of the machine,

Δp is the incremental change in power input to the machine due to governor action,

ΔV_T is the deviation in terminal voltage.

It is assumed that the outputs are available for measurement. The performance measure to be minimized is

$$J = \frac{1}{2} \int_0^{\infty} (\Delta V_T^2 + \Delta \delta^2 + 2\Delta \omega^2 + u^2) dt.$$

Before the suboptimal controller can be designed, a simplified model of the system must be computed. Upon application of Algorithm I with $r = 3$, the following simplified model is obtained:

$$\dot{\underline{x}} = \begin{bmatrix} 0 & 1 & 0 \\ -0.149 & 0 & 1 \\ 0 & -0.965 & 0.739 \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \quad (5.3-3)$$

$$\hat{\underline{y}} = \begin{bmatrix} 0.648 & 0.268 & 0.489 \\ -1.862 & 0.016 & -0.0036 \\ -0.0023 & -1.860 & 0.0129 \end{bmatrix} \underline{x} \quad (5.3-4)$$

The optimal control law for the simplified model (5.3-3) is

$$\hat{u}^* = [-1.606 \quad -3.123 \quad -1.913] \hat{\underline{x}} \quad (5.3-5)$$

Employing (4.3-7) the suboptimal control for the system becomes

$$u^{\circ} = [-3.953 \quad -0.515 \quad -1.105] \begin{bmatrix} \Delta V_T \\ \Delta \delta \\ \Delta \omega \end{bmatrix} \quad (5.3-6)$$

The performance degradation associated with this suboptimal control as computed by (4.3-11) is given by $\Delta J = (5 \times 10^{-3}) \underline{x}_O^T (\underline{P}-\underline{K}) \underline{x}_O$, where $\underline{P}-\underline{K}$ is

$$\begin{bmatrix} 1.17 \times 10^{-2} & 4.25 \times 10^{-3} & 1.03 \times 10^{-2} & 1.63 \times 10^{-1} & 1.14 \times 10^{-2} \\ 4.25 \times 10^{-3} & 7.00 \times 10^{-4} & 7.12 \times 10^{-3} & 1.23 \times 10^{-1} & 1.15 \times 10^{-2} \\ 1.03 \times 10^{-2} & 7.12 \times 10^{-3} & 3.44 \times 10^{-2} & 5.42 \times 10^{-1} & 5.08 \times 10^{-2} \\ 1.63 \times 10^{-1} & 1.23 \times 10^{-1} & 5.42 \times 10^{-1} & 9.01 & 8.44 \times 10^{-1} \\ 1.14 \times 10^{-2} & 1.15 \times 10^{-2} & 5.08 \times 10^{-2} & 8.44 \times 10^{-1} & 7.90 \times 10^{-2} \end{bmatrix},$$

and indicates that u° gives near-optimum system performance.

The quality of the suboptimal control policy is illustrated in

Table 5.3.1 by a comparison of the pole placement which the suboptimal controller gives with the pole placement given by the optimal control law

$$\underline{u}^* = [-1.910 \quad 0.410 \quad 1.133 \quad 0.569 \quad 0.046] \underline{x}. \quad (5.3-7)$$

As another indication of the quality of \underline{u}^o , Figures 5.3.1, 5.3.2 and 5.3.3 display the optimal and suboptimal responses of the outputs of the system to the initial condition

$$\underline{x}_0 = [0 \quad 0.01 \quad 0 \quad 0.01 \quad 0]^T$$

The suboptimal performance index is $J^o = 1.30 \times 10^{-4}$ which is less than a 4% degradation in the optimal performance index of $J^* = 1.26 \times 10^{-4}$.

Notice that the suboptimal control (5.3-6) requires that only the outputs be fed back. Moreover, the optimal control (5.3-7) requires the determination of $n(n+1)/2 = 15$ unknowns in the steady state matrix Riccati equation while the suboptimal control law requires the determination of $r(r+1)/2 = 6$ unknowns.

Example 5.3.2

This example is also a single machine tied to an infinite bus. The model for the system [45] is given in the form of (5.2-1) with A given by

Table 5.3.1 Eigenvalues of the Optimal and Suboptimal Closed-Loop Systems of Example 5.3.1.

OPTIMAL SYSTEM	SUBOPTIMAL SYSTEM
- 0.6909	- 0.6872
- 0.9701 + j1.3898	- 0.9586 + j1.3791
- 0.9701 - j1.3898	- 0.9586 - j1.3791
- 1.9507	- 2.0080
-10.0130	-10.0126

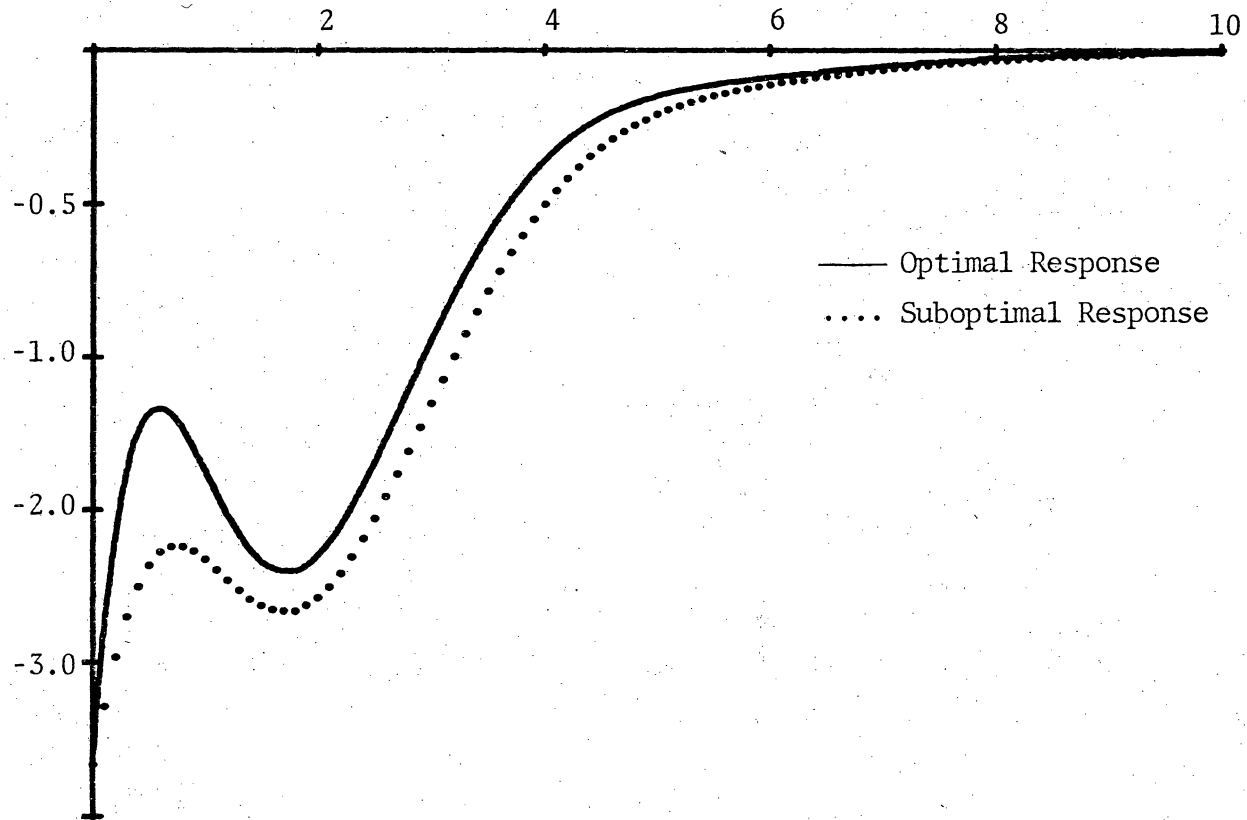


Figure 5.3.1. Optimal and Suboptimal Responses of ΔV_T in Example 5.3.1.

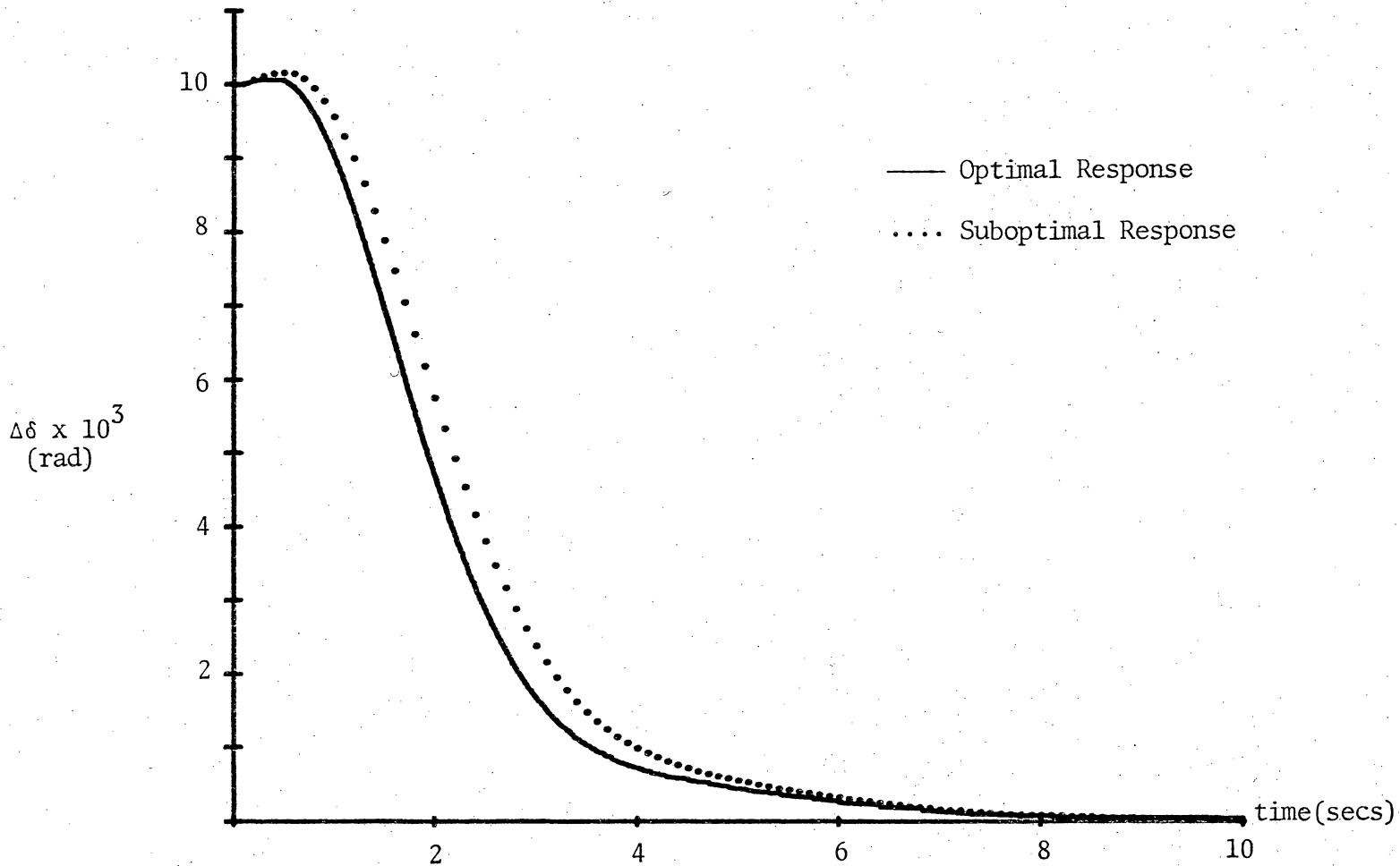


Figure 5.3.2. Optimal and Suboptimal Responses of $\Delta\delta$ in Example 5.3.1.

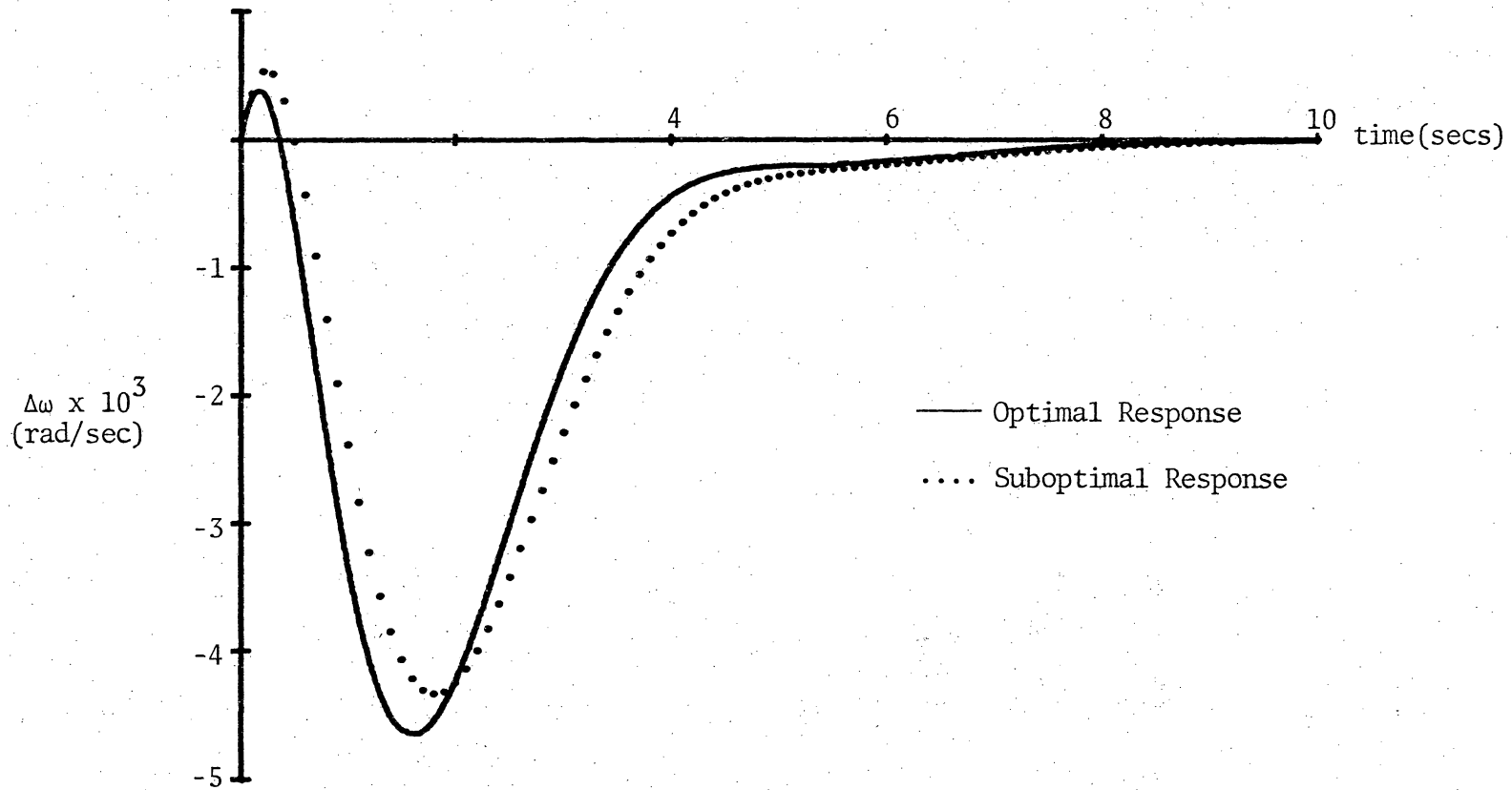


Figure 5.3.3. Optimal and Suboptimal Responses of $\Delta\omega$ in Example 5.3.1.

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -52.08 & -23.54 & 9.415 & -23.54 & -99.2 & 27.88 \\ -541.2 & 0.6953 & -2.126 & -36.23 & 24.15 & 664.2 & -362.3 \\ -1136 & 1.46 & -0.6642 & -76.08 & -12.08 & 1395 & -760.8 \\ -541.2 & 0.6953 & 1.328 & -36.23 & -38.65 & 664.2 & -362.3 \\ 398 & 3.403 & 897.1 & -1346 & 897.1 & -53.83 & -26.91 \\ 298.5 & 2.552 & 672.9 & -1009 & 672.9 & -40.37 & -35.89 \end{bmatrix},$$

$$\underline{B} = \begin{bmatrix} 0 & 0 & 2.121 & 0.6642 & -1.328 & 0 & 0 \\ 0 & 52.08 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T,$$

$$\underline{C} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -0.1699 & 0 & 0 & 0.3018 & 0 & -0.0375 & 0 \end{bmatrix}.$$

The system model consists entirely of a seventh order model of the synchronous machine. Since no voltage regulator or governor subsystems are included in the model, the control action is achieved by changes in the excitation voltage and prime mover torque. The output vector is

$$\underline{y} = [\Delta\delta \quad \Delta\omega \quad \Delta V_T]^T$$

and the input is

$$\underline{u} = [\Delta E_q \quad \Delta T]^T,$$

where ΔT is the incremental change in prime mover torque. A performance measure in the form of (5.2-2) is selected with $\underline{Q} = \text{diag}[10,10,10]$ and $\underline{R} = \text{diag}[1,1]$. The optimal control law is found to be

$$\underline{u}^* = \begin{bmatrix} 0.14 & 0.0022 & -0.61 & 0.55 & -0.56 & -2.6 \times 10^{-4} & 6.5 \times 10^{-5} \\ -1.79 & -2.55 & 1.27 & -1.38 & 1.25 & -0.10 & 0.335 \end{bmatrix} \underline{x}. \quad (5.3-8)$$

With the system and performance measure now specified, the design of a suboptimal controller can proceed. The simplified system representation derived from Algorithm I is

$$\dot{\hat{\underline{x}}} = \begin{bmatrix} 0 & 1 & 0 \\ -0.590 & 0 & 1 \\ 0 & -138.386 & -53.343 \end{bmatrix} \hat{\underline{x}} + \begin{bmatrix} -0.0129 & 0.0129 \\ 6.54 \times 10^{-3} & -6.54 \times 10^{-3} \\ -8.7 \times 10^{-3} & 1.0087 \end{bmatrix} \underline{u}, \quad (5.3-9)$$

$$\hat{\underline{y}} = \begin{bmatrix} 26.733 & 52.492 & -0.0039 \\ -30.975 & 27.275 & 52.283 \\ -7.625 & 7.624 & 0.1196 \end{bmatrix} \hat{\underline{x}}. \quad (5.3-10)$$

The optimal control law computed for the simplified model (5.3-9) is given by

$$\hat{\underline{u}}^* = \begin{bmatrix} 13.231 & -0.651 & -0.183 \\ 6.227 & -149.79 & -120.69 \end{bmatrix} \hat{\underline{x}}, \quad (5.3-11)$$

and the suboptimal control for the original system then becomes via (4.3-7)

$$\underline{u}^o = \begin{bmatrix} 0.155 & -7.67 \times 10^{-4} & -1.189 \\ -1.873 & -2.313 & 2.014 \end{bmatrix} \begin{bmatrix} \Delta \delta \\ \Delta \omega \\ \Delta V_T \end{bmatrix}. \quad (5.3-12)$$

The eigenvalues of the optimal and suboptimal closed-loop systems are presented in Table 5.3.2. Note that the eigenvalues of the suboptimal system are very near the eigenvalues of the optimal system.

Figures 5.3-4, 5.3-5 and 5.3-6 show a comparison of the suboptimal and optimal responses of the outputs of the system to the initial condition

$$\underline{x}_0 = [0.01 \quad 0 \quad 0.02 \quad 0 \quad 0 \quad 0 \quad 0]^T$$

For this initial condition the optimal performance index is $J^* = 0.861 \times 10^{-3}$; the suboptimal performance is $J^o = 0.865 \times 10^{-3}$.

The actual calculation for the performance degradation via (4.3-11) for arbitrary \underline{x}_0 indicates that the suboptimal control law results in system performance reasonably near that achieved by \underline{u}^* . It must be emphasized that this near-optimal performance is achieved by feeding back only measurable outputs. Moreover, the suboptimal control law requires the solution of a third order matrix Riccati equation, while the optimal control law requires the solution of a seventh order matrix Riccati equation.

Table 5.3.2 Eigenvalues of the Optimal and Suboptimal Closed-Loop Systems of Example 5.3.2.

OPTIMAL SYSTEM	SUBOPTIMAL SYSTEM
- 0.8384	- 0.9277
- 1.2131	- 1.1535
- 26.650	- 26.808
- 36.632	- 36.559
-183.941	-171.021
- 71.294 + j636.272	- 71.445 + j636.243
- 71.294 - j636.272	- 71.445 - j636.243

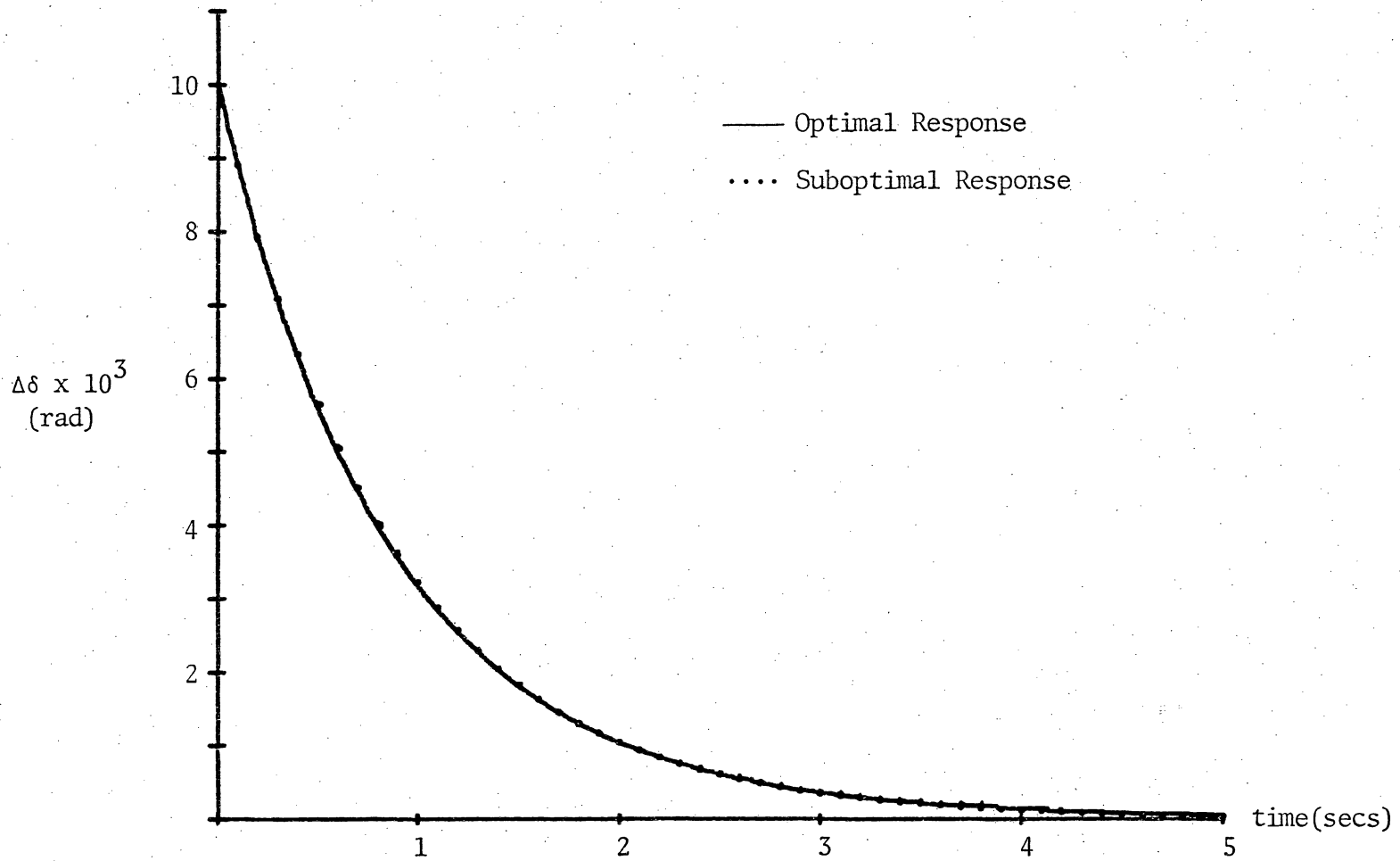


Figure 5.3.4. Optimal and Suboptimal Responses of $\Delta\delta$ in Example 5.3.2.

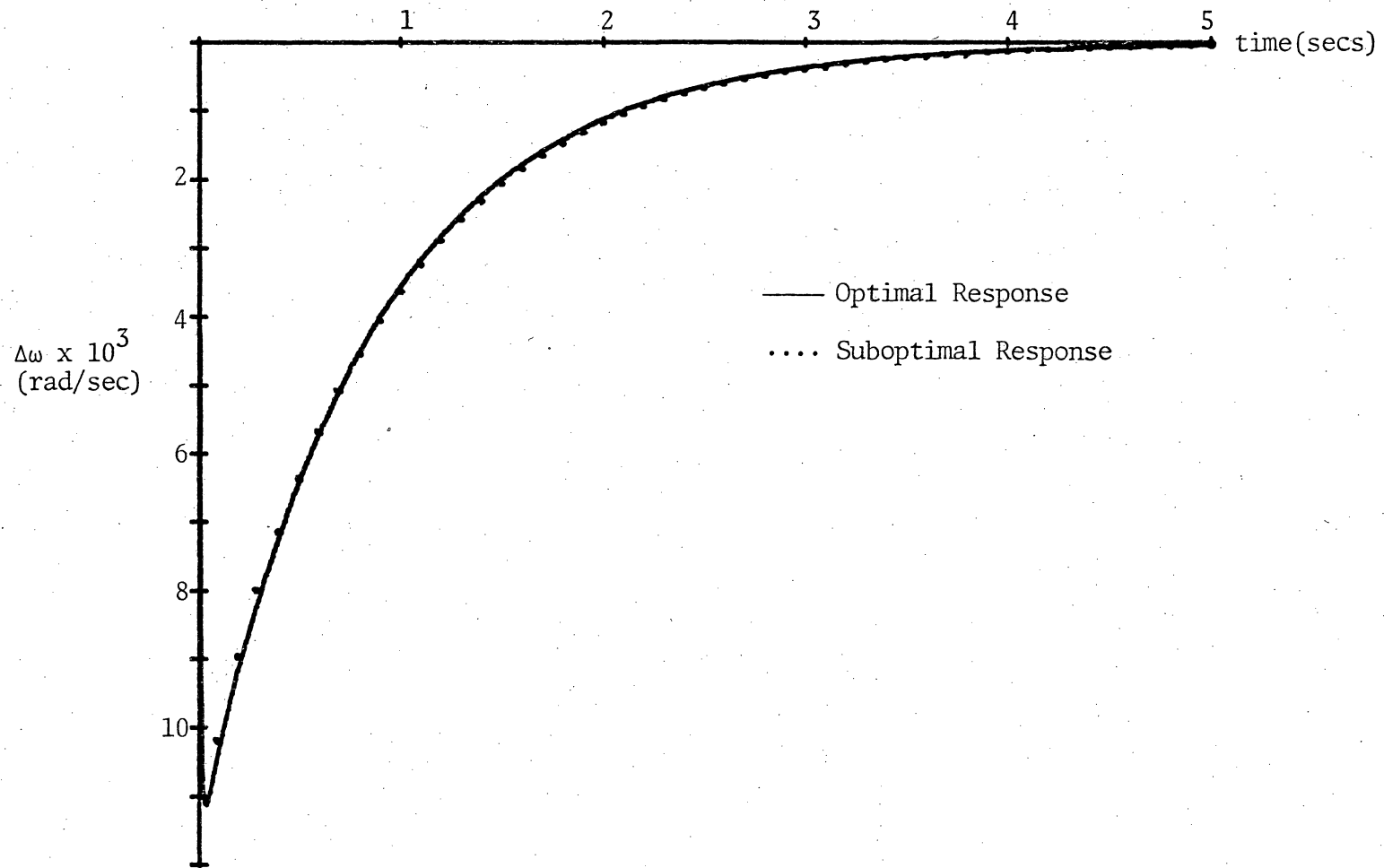


Figure 5.3.5. Optimal and Suboptimal Responses of $\Delta\omega$ in Example 5.3.2.

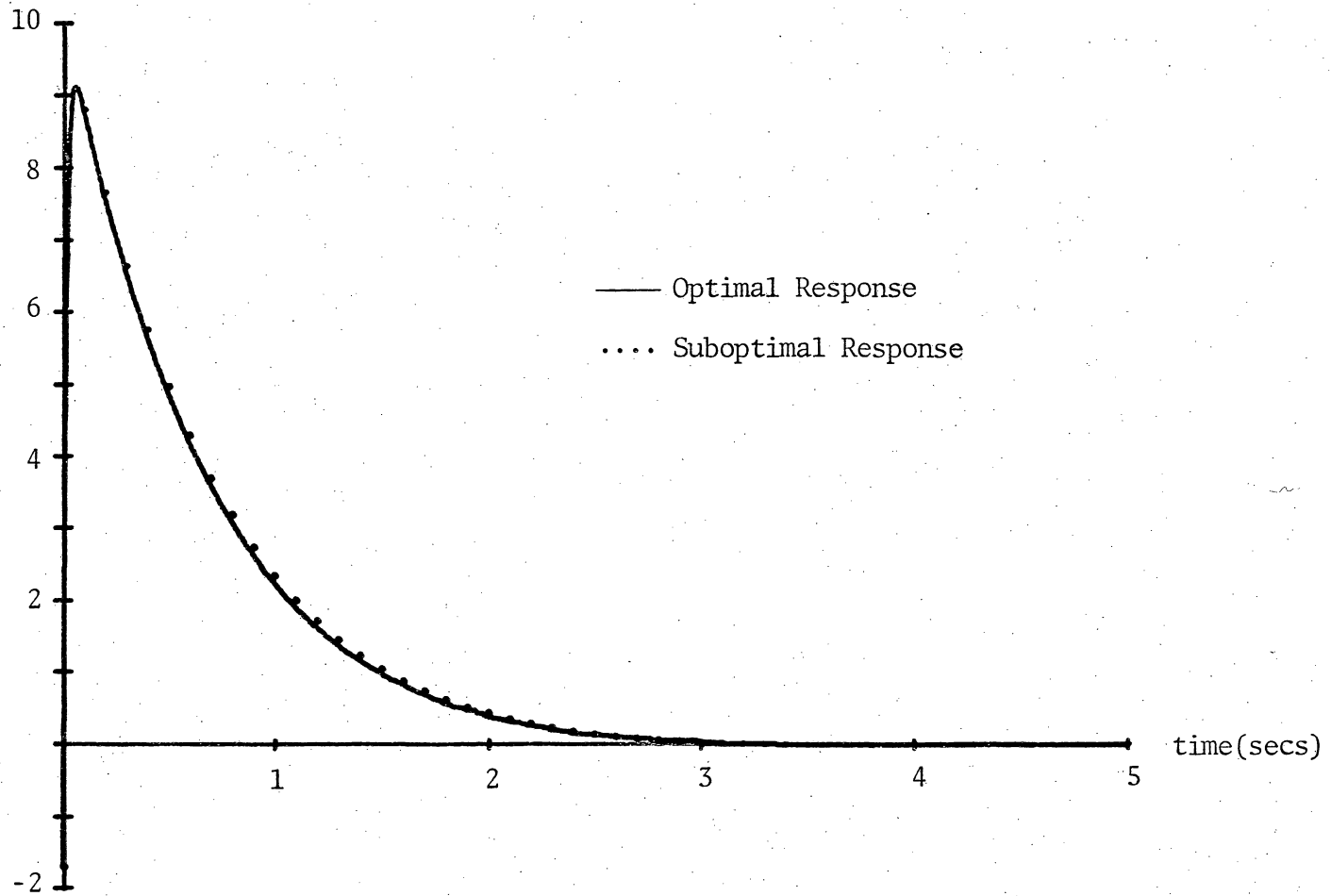


Figure 5.3.6. Optimal and Suboptimal Responses of ΔV_T in Example 5.3.2.

CHAPTER VI

CONCLUSION

6.1 SUMMARY OF THE RESEARCH

This research has been concerned with the approximation of linear time-invariant system models of high order by simplified models of lower order. The utilization of simplified models in designing linear regulators for complex systems, such as an electric power system, has also been under investigation.

The problem of obtaining lower order models for systems having no unstable modes is first considered. The optimum simplified model is obtained by minimizing a quadratic functional of the error between the system outputs and the simplified model outputs. The necessary conditions for a simplified model to be optimum are derived and are then employed to construct an effective iterative scheme for selecting the model parameters. The first main contribution of this thesis is the development of a reliable algorithm for minimizing the modeling error. The multiple input problem is first converted to an equivalent single-input problem. The number of unknown model parameters is then drastically reduced through the choice of a Schwarz canonical form for the structure of the simplified model. An iterative algorithm for determining the model parameters is proposed. This algorithm utilizes special properties of the Schwarz canonical form to lessen the computational complexity of the procedure.

The problem of simplifying systems with several unstable modes is also discussed. The approach taken consists of decomposing the given

system model into stable and unstable subsystems, retaining exactly the unstable modes in the simplified model and obtaining a simplified representation of the stable subsystem by the algorithm developed for simplifying stable systems. Herein lies a second contribution of this research. A linear transformation for decomposing the system into stable and unstable components is proposed. A computational procedure for constructing the transformation matrix is outlined.

A suboptimal output regulator derived from a simplified system model is then proposed. The scheme is shown to result in a much simpler control and observation strategy than the optimal regulator since only the outputs of the system are required in the feedback law. The suboptimal design also reduces the computational complexity of deriving a controller.

Finally, the feasibility of the suboptimal control policy is demonstrated for a power system control problem. The suboptimal control policy derived via a simplified system model is shown to be both effective and practical for solving the power system stabilization problem.

6.2 RECOMMENDATIONS FOR FURTHER STUDY

Several interesting problems and potentially fruitful areas for further study pertaining to the model order reduction problem are briefly mentioned below.

i) A feature of the simplification algorithm which has practical significance is a criterion for selecting the model order r prior to the actual reduction process. An upper bound on the modeling error as a function of r would be extremely useful. The model order r could then

be selected to enable the simplified model to meet the specification of a maximum allowable modeling error.

ii) The test input used in this investigation was $\underline{u}(t) = \underline{w}\delta(t)$. However, examination of the model order reduction process for different test inputs merits attention. Deterministic and stochastic test inputs should be considered. The derivation of necessary conditions for an optimum simplified model for an arbitrary input would be an important addition to the model order reduction theory.

iii) For a very large system, it may often be useful for analysis purposes to subdivide the system into several subsystems. When no natural subdivisions exist in the system, some systematic procedure for breaking the system up is desirable. A matrix transformation constructed in a manner similar to the way \underline{P} in Section 3.4 is constructed can be employed to partition the system into several subsystems. The extension of the decomposition procedure given by Algorithm II to permit decomposing the system model into several component parts is, therefore, suggested. If a simplified system representation is subsequently required, then each component part of the system can be reduced separately.

iv) A better suboptimal output regulator may be possible when the simplified model is derived using a test input other than $\underline{u}(t) = \underline{w}\delta(t)$. A criterion for selecting a test input for the reduction process when the model is to be used for design of a controller would be helpful. Moreover, establishing a connection between the modeling error and the performance degradation of the suboptimal controller may be possible.

v) Recall from Section 4.3 that a simplified model of order ℓ is

needed to construct the suboptimal control policy given by (4.3-7). If ℓ is not sufficiently large to guarantee that the simplified model adequately approximates the system, then the suboptimal system performance may be poor. Thus, a scheme for designing regulators using simplified models of order $r > \ell$ and maintaining a relatively simple control and observation structure should be considered.

vi) Finally, several areas of application for the model order reduction algorithm presented in this thesis should be studied. One significant application is in the construction of simplified models for use in transient stability studies of interconnected power systems [47]. Suboptimal regulation via simplified models may be useful in solving the multi-area power system load-frequency control problem [38]. The model order reduction algorithm may also be applied readily to the network synthesis-optimization problem [27].

BIBLIOGRAPHY

- [1] S. C. Gupta and L. Hasdorff, Fundamentals of Automatic Control, New York: John Wiley and Sons, Inc., 1970.
- [2] E. J. Davison, "A Method of Simplifying Linear Dynamic Systems," IEEE Trans. Auto. Control, Vol. AC-11, pp. 93-101, January 1966.
- [3] E. J. Davison, "A New Method for Simplifying Large Linear Dynamic Systems," IEEE Trans. Auto. Control, Vol. AC-13, pp. 214-215, April 1968.
- [4] E. J. Davison and M. R. Chidambara, "On a Method for Simplifying Linear Dynamic Systems," IEEE Trans. Auto. Control, Vol. AC-12, pp. 119-121, February 1967.
- [5] E. J. Davison and M. R. Chidambara, "Further Remarks on Simplifying Linear Dynamic Systems," IEEE Trans. Auto. Control, Vol. AC-12, pp. 799-800, December 1967.
- [6] A. Kuppurajulu and S. Elangovan, "System Analysis by Simplified Models," IEEE Trans. Auto. Control, Vol. AC-15, pp. 234-237, April 1970.
- [7] C. K. Sanathanan and J. Koerner, "Transfer Function Synthesis as a Ratio of Two Complex Polynomials," IEEE Trans. Auto. Control, Vol. AC-8, pp. 56-58, 1963.
- [8] V. F. Bakalanov, "Lowering the Order of Differential Equations and Transfer Functions," Soviet Automatic Control, No. 5, 1968.
- [9] C. F. Chen and L. S. Shieh, "A Novel Approach to Linear Model Simplification," JACC Preprints, pp. 454-461, 1968.
- [10] D. H. Rothenberg, "Simplification of Linear Stationary Continuous Dynamic Systems," Ph.D. Thesis, Case Western Reserve University, September 1970.
- [11] C. F. Chen, "Model Reduction of Multivariable Control Systems by Means of Matrix Continued Fraction," Proc. IFAC 5-th World Congress (Paris, France), paper 35.1, pp. 1-8, June 1972.
- [12] J. H. Anderson, "Geometrical Approach to Reduction of Dynamic Systems," Proc. IEE, Vol. 114, pp. 1014-1018, July 1967.
- [13] N. K. Sinha and W. Pille, "A New Method for Reduction of Dynamic Systems," Int. J. Control, Vol. 14, No. 1, pp. 111-118, 1971.
- [14] N. K. Sinha and G. T. Bereznoi, "Optimum Approximation of High-Order Systems by Low-Order Models," Int. J. Control, Vol. 14, No. 5, pp. 951-959, 1971.

- [15] M. R. Chidambara, "Two Simple Techniques for the Simplification of Large Dynamic Systems," JACC Preprints, pp. 93-101, 1969.
- [16] D. A. Wilson, "Optimum Solution of Model-Reduction Problem," Proc. IEE, Vol. 117, pp. 1161-1165, June 1970.
- [17] D. Mitra, "On the Reduction of Complexity of Linear Dynamical Systems," Atomic Energy Establishment, Winfrith, England, report R520, 1967.
- [18] D. Mitra, "On the Reduction of Complexity of Linear Dynamical Models-Computer Studies," Atomic Energy Establishment, Winfrith, England, report R353, 1967.
- [19] D. Mitra, "W Matrix and the Geometry of Model Equivalence," Proc. IEE, Vol. 116, pp. 1101-1106, June 1969.
- [20] F. D. Galiana, "On the Approximation of Multiple Input-Multiple Output Constant Linear Systems," Int. J. Control, Vol. 17, No. 6, pp. 1313-1324, 1973.
- [21] J. B. Riggs and T. F. Edgar, "Simplification of Large-Scale Linear Dynamic Systems," JACC Preprints, pp. 491-497, 1973.
- [22] M. Athans and P. L. Falb, Optimal Control - An Introduction to the Theory and Its Applications, New York: McGraw-Hill Book Co., 1966.
- [23] K. Ogata, State Space Analysis of Control Systems, New York: McGraw-Hill Book Co., 1967.
- [24] M. Athans, "The Matrix Minimum Principle," Information and Control, Vol. 11, pp. 592-605, November 1967.
- [25] D. M. Himmelblau, Applied Nonlinear Programming, New York: McGraw-Hill Book Co., 1972.
- [26] D. G. Luenberger, "Canonical Forms for Linear Multivariable Systems," IEEE Trans. Auto. Control, Vol. AC-12, pp. 290-293, June 1967.
- [27] J. Brian Connell, "The Use of the Newton Raphson Method in Time-Domain Synthesis," IEEE Trans. Circuits and Systems, Vol. CAS-21, pp. 67-69, January 1974.
- [28] I. G. Sarma, M. A. Pai, and R. Viswanathan, "On the Transformation to Schwarz Canonical Form," IEEE Trans. Auto. Control, Vol. AC-13, pp. 311-312, June 1968.
- [29] R. Fletcher, "A New Approach to Variable Metric Algorithms," Computer J., Vol. 13, pp. 317-322, August 1970.

- [30] J. G. F. Francis, "The QR Transformation - Part I," Computer J., Vol. 4, pp. 265-271, October 1961. "The QR Transformation - Part II," Computer J., Vol. 4, pp. 332-345, January 1962.
- [31] J. H. Wilkinson, The Algebraic Eigenvalue Problem, London: Oxford University Press, 1965.
- [32] A. Jameson, "Solution of the Equation $AX + XB = C$ by Inversion of an $m \times m$ or $n \times n$ Matrix," SIAM J. Appl. Math., Vol. 16, pp. 1020-1023, September 1968.
- [33] H. A. M. Moussa and Y. N. Yu, "Optimal Power System Stabilization Through Excitation and/or Governor Control," IEEE Trans. Power Apparatus and Systems, Vol. PAS-91, pp. 1166-1173, May/June 1972.
- [34] D. G. Luenberger, "Observers for Multivariable System," IEEE Trans. Auto. Control, Vol. AC-11, pp. 190-197, April 1966.
- [35] M. Aoki, "Control of Large-Scale Dynamic Systems by Aggregation," IEEE Trans. Auto. Control, Vol. AC-13, pp. 246-253, June 1968.
- [36] M. R. Chidambara and R. B. Schainker, "Lower Order Generalized Aggregated Model and Suboptimal Control," IEEE Trans. Auto. Control, Vol. AC-16, pp. 175-180, April 1971.
- [37] L. L. Joyner, "The Design of Suboptimal Linear Regulators Using Reduced Order Aggregated Models," Ph.D. Thesis, Virginia Polytechnic Institute and State University, July 1972.
- [38] C. E. Fosha, Jr. and O. I. Elgerd, "The Megawatt-Frequency Control Problem: A New Approach via Optimal Control Theory," IEEE Trans. Power Apparatus and Systems, Vol. PAS-89, pp. 563-577, April 1970.
- [39] M. Čalović, "Linear Regulator Design for a Load Frequency Control," IEEE Trans. Power Apparatus and Systems, Vol. PAS-91, pp. 2271-2285, November/December 1972.
- [40] Y. N. Yu and H. A. M. Moussa, "Optimal Stabilization of a Multi-Machine System," IEEE Trans. Power Apparatus and Systems, Vol. PAS-91, pp. 1174-1182, May/June 1972.
- [41] J. M. Undrill, "Dynamic Stability Calculations for an Arbitrary Number of Interconnected Synchronous Machines," IEEE Trans. Power Apparatus and Systems, Vol. PAS-87, pp. 835-844, March 1968.
- [42] Y. N. Yu, K. Vongsuriya, and L. N. Wedman, "Application of an Optimal Control Theory to a Power System," IEEE Trans. Power Apparatus and Systems, Vol. PAS-89, pp. 55-62, January 1970.

- [43] A. K. DeSarkar and N. Dharma Rao, "Dynamic-System Simplification and an Application to Power System-Stability Studies," Proc. IEEE, Vol. 119, pp. 904-910, July 1972.
- [44] S. Elangovan and A. Kuppurajulu, "Suboptimal Control of Power Systems Using Simplified Models," IEEE Trans. Power Apparatus and Systems, Vol. PAS-91, pp. 911-919, May/June 1972.
- [45] M. Ramamoorthy and M. Arumugam, "Design of Optimal Regulators for Synchronous Machines," IEEE Trans. Power Apparatus and Systems, Vol. PAS-92, pp. 269-277, January/February 1973.
- [46] D. Mitra, "Analytical Results on the Use of Reduced Models in the Control of Linear Dynamical Systems," Proc. IEE, Vol. 116, pp. 1439-1444, August 1969.
- [47] J. M. Undrill and E. A. Turner, "Construction of Power System Electromechanical Equivalents," IEEE Trans. Power Apparatus and Systems, Vol. PAS-90, pp. 2049-2059, September/October 1971.
- [48] F. R. Gantmacher, The Theory of Matrices, Vol. 1, New York: Chelsea, 1959.
- [49] S. Barnett and C. Storey, Matrix Methods in Stability Theory, New York: Barnes & Noble, Inc., 1970.

APPENDIX A
GRADIENT MATRICES

If $f = f(\underline{X})$ is a scalar function of the $n \times m$ matrix $\underline{X} = [x_{ij}]$, then the gradient matrix of f with respect to \underline{X} is the $n \times m$ matrix

$$\frac{\partial f}{\partial \underline{X}} = \left[\frac{\partial f}{\partial x_{ij}} \right] \begin{matrix} i = 1, 2, \dots, n \\ j = 1, 2, \dots, m \end{matrix} .$$

For the function $f(\cdot) = \text{Tr}(\cdot)$, the following identities [24] are useful for evaluating gradient matrices:

$$\frac{\partial}{\partial \underline{X}} \text{Tr}(\underline{A} \underline{X}) = \underline{A}^T \tag{A.1}$$

$$\frac{\partial}{\partial \underline{X}} \text{Tr}(\underline{A} \underline{X}^T) = \underline{A} \tag{A.2}$$

$$\frac{\partial}{\partial \underline{X}} \text{Tr}(\underline{A} \underline{X} \underline{B}) = \underline{A}^T \underline{B}^T \tag{A.3}$$

$$\frac{\partial}{\partial \underline{X}} \text{Tr}(\underline{A} \underline{X} \underline{B} \underline{X}^T) = \underline{A}^T \underline{X} \underline{B}^T + \underline{A} \underline{X} \underline{B} . \tag{A.4}$$

Another useful result for computing gradient matrices is given below.

LEMMA A.1

Let

$$\underline{X} = \int_0^{\infty} e^{\underline{A}t} \underline{K} e^{\underline{B}t} dt$$

where the eigenvalues of \underline{A} and \underline{B} have negative real parts.

Then

$$\frac{\partial}{\partial \underline{A}} \text{Tr}(\underline{X} \underline{L}) = \underline{Y}^T \underline{X}^T \quad (\text{A.5})$$

where

$$\underline{Y} = \int_0^\infty e^{\underline{B}t} \underline{L} e^{\underline{A}t} dt .$$

Furthermore, if $\underline{B} = \underline{A}^T$ and \underline{L} and \underline{K} are symmetric

$$\frac{\partial}{\partial \underline{A}} \text{Tr}(\underline{X} \underline{L}) = 2 \underline{Y}^T \underline{X}^T . \quad (\text{A.6})$$

proof:

Since the eigenvalues of both \underline{A} and \underline{B} have negative real parts, \underline{X} is the solution of

$$\underline{A} \underline{X} + \underline{X} \underline{B} = -\underline{K} . \quad (\text{A.7})$$

Let $\gamma = 0$ if \underline{B} is not a function of \underline{A} and let $\gamma=1$ for the special case that $\underline{B} = \underline{A}^T$. If $\underline{A} = [a_{ij}]$, then differentiation of (A.7) with respect to a_{ij} gives

$$\underline{A} \frac{\partial \underline{X}}{\partial a_{ij}} + \frac{\partial \underline{X}}{\partial a_{ij}} \underline{B} = \frac{-\partial \underline{A}}{\partial a_{ij}} \underline{X} - \gamma \underline{X} \frac{\partial \underline{A}^T}{\partial a_{ij}} .$$

Hence,

$$\frac{\partial \underline{X}}{\partial a_{ij}} = \int_0^\infty e^{\underline{A}t} \left[\frac{\partial \underline{A}}{\partial a_{ij}} \underline{X} + \gamma \underline{X} \frac{\partial \underline{A}^T}{\partial a_{ij}} \right] e^{\underline{B}t} dt . \quad (\text{A.8})$$

By using (A.8) and the identities $\text{Tr}(\underline{X}_1 \underline{X}_2) = \text{Tr}(\underline{X}_2 \underline{X}_1)$ and

$\text{Tr}(\int \underline{X}_1 dt) = \int \text{Tr}(\underline{X}_1) dt$, it can easily be shown that

$$\frac{\partial}{\partial a_{ij}} \text{Tr}(\underline{X} \underline{L}) = \text{Tr} \left(\underline{X} \underline{Y} \frac{\partial A}{\partial a_{ij}} \right) + \gamma \text{Tr} \left(\underline{Y} \underline{X} \frac{\partial A^T}{\partial a_{ij}} \right).$$

Thus,

$$\frac{\partial}{\partial A} \text{Tr}(\underline{X} \underline{L}) = \frac{\partial}{\partial A} \text{Tr}(\underline{X} \underline{Y} \underline{A}) + \gamma \frac{\partial}{\partial A} \text{Tr}(\underline{Y} \underline{X} \underline{A}^T). \quad (\text{A.9})$$

From (A.1) and (A.2), (A.9) becomes

$$\frac{\partial}{\partial A} \text{Tr}(\underline{X} \underline{L}) = \underline{Y}^T \underline{X}^T + \gamma \underline{Y} \underline{X}. \quad (\text{A.10})$$

Equation (A.5) follows immediately when $\gamma=0$. Also, if $\underline{B} = \underline{A}^T$ and \underline{K} and \underline{L} are symmetric, then \underline{Y} and \underline{X} are symmetric and (A.6) holds.

Q.E.D.

APPENDIX B

ALGORITHMS FOR SOLVING $\underline{Z} \underline{X} + \underline{X} \underline{A} = -\underline{C}$

Algorithms for the efficient solution of the matrix equations (2.2-4)-(2.2-7) are outlined in this appendix.

Consider first a matrix equation of the form

$$\underline{Z} \underline{X} + \underline{X} \underline{A}_{-n}^T = -\underline{C} \quad (\text{B.1})$$

where \underline{Z} is an arbitrary $m \times n$ matrix and \underline{A}_{-n} is an $n \times n$ Schwarz form,

$$\underline{A}_{-n} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ -a_n & 0 & 1 & \dots & 0 & 0 \\ 0 & -a_{n-1} & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & & \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & -a_2 & -a_1 \end{bmatrix}. \quad (\text{B.2})$$

A scheme for computing the solution \underline{X} of (B.1) is suggested by the following result.

THEOREM B.1.

Let \underline{Z} be an $m \times n$ matrix and let \underline{A}_{-n} be given by (B.2). If $\underline{X} = [\underline{x}_1 \ \underline{x}_2 \ \dots \ \underline{x}_n]$ and $\underline{C} = [\underline{c}_1 \ \underline{c}_2 \ \dots \ \underline{c}_n]$, then the solution \underline{X} of (B.1) is given by the following scheme:

Algorithm B.1

- i) \underline{x}_1 is the solution of

$$\underline{D}_n \underline{x}_1 = \sum_{j=0}^{n-1} (-1)^{n+j} \underline{D}_j \underline{c}_{n-j} \quad (\text{B.3})$$

where the $m \times m$ matrices \underline{D}_k , $k = 1, 2, \dots, n$, are defined as

$$\underline{D}_1 = \underline{Z} - a_1 \underline{I}, \quad \underline{D}_0 = \underline{I} \quad (\text{B.4})$$

$$\underline{D}_k = \underline{D}_{k-1} \underline{Z} + a_k \underline{D}_{k-2} \quad k = 2, 3, \dots, n. \quad (\text{B.5})$$

ii) The columns $\underline{x}_2, \underline{x}_3, \dots, \underline{x}_n$ are given by

$$\underline{x}_2 = -\underline{Z} \underline{x}_1 - \underline{c}_1 \quad (\text{B.6})$$

$$\underline{x}_k = -\underline{Z} \underline{x}_{k-1} + a_{n-k+3} \underline{x}_{k-2} - \underline{c}_{k-1} \quad k = 3, \dots, n. \quad (\text{B.7})$$

Moreover, if $-\underline{Z}$ and \underline{A} share no common eigenvalues, \underline{D}_n^{-1} exists and consequently \underline{X} is unique.

proof: With \underline{A}_n given by (B.2), (B.1) implies

$$\underline{Z} \underline{x}_1 + \underline{x}_2 = -\underline{c}_1, \quad (\text{B.8})$$

$$\underline{Z} \underline{x}_{k-1} + \underline{x}_k - a_{n-k+3} \underline{x}_{k-2} = -\underline{c}_{k-1} \quad k = 3, \dots, n, \quad (\text{B.9})$$

$$\underline{Z} \underline{x}_n - a_2 \underline{x}_{n-1} - a_1 \underline{x}_n = -\underline{c}_n. \quad (\text{B.10})$$

Equations (B.9) and (B.10) imply that

$$(-1)^{i+1} \underline{D}_i \underline{x}_{n-i+1} + (-1)^i a_{i+1} \underline{D}_{i-1} \underline{x}_{n-i} = \sum_{j=0}^{i-1} (-1)^{j+1} \underline{D}_j \underline{c}_{n-j} \quad (\text{B.11})$$

for $i = 1, 2, \dots, n-1$. The validity of this statement can be shown by induction. Suppose $i = 1$; then (B.11) becomes

$$\underline{D}_1 x_n - a_2 \underline{D}_0 x_{n-1} = -\underline{D}_0 c_n \quad (\text{B.12})$$

However, from the definitions of \underline{D}_0 and \underline{D}_1 given by (B.4), (B.12) is clearly equivalent to (B.10). Therefore (B.11) holds for $i = 1$.

Now suppose (B.11) is true for $i = \ell$, $\ell < n - 1$. For $i = \ell$, (B.11) becomes

$$(-1)^{\ell+1} \underline{D}_\ell x_{n-\ell+1} + (-1)^\ell a_{\ell+1} \underline{D}_{\ell-1} x_{n-\ell} = \sum_{j=0}^{\ell-1} (-1)^{j+1} \underline{D}_j c_{n-j}. \quad (\text{B.13})$$

With $k = n - \ell + 1$, (B.9) becomes

$$\underline{Z} x_{n-\ell} + x_{n-\ell+1} - a_{\ell+2} x_{n-\ell-1} = -c_{n-\ell}. \quad (\text{B.14})$$

Pre-multiplying (B.14) by $(-1)^{\ell+1} \underline{D}_\ell$ and subtracting the result from (B.13) gives

$$\begin{aligned} (-1)^{\ell+2} \underline{D}_\ell \underline{Z} x_{n-\ell} - (-1)^{\ell+2} a_{\ell+1} \underline{D}_{\ell-1} x_{n-\ell} + (-1)^{\ell+1} a_{\ell+2} \underline{D}_\ell x_{n-\ell-1} = \\ (-1)^{\ell+1} c_{n-\ell} + \sum_{j=1}^{\ell-1} (-1)^{j+1} \underline{D}_j c_{n-j} \end{aligned}$$

which simplifies to

$$(-1)^{\ell+2} \underline{D}_{\ell+1} x_{n-\ell} + (-1)^{\ell+1} a_{\ell+2} \underline{D}_\ell x_{n-\ell-1} = \sum_{j=0}^{\ell} (-1)^{j+1} \underline{D}_j c_{n-j}. \quad (\text{B.15})$$

Observe that (B.15) is simply (B.11) with $i = \ell + 1$. Since (B.13) implies (B.11) holds for $i = \ell + 1$, (B.11) is true for $i = 1, 2, \dots, n-1$.

For the particular case $i = n - 1$, (B.11) is

$$(-1)^n \underline{D}_{n-1} x_2 + (-1)^{n-1} \underline{D}_{n-2} x_1 = \sum_{j=0}^{n-2} (-1)^{j+1} \underline{D}_j c_{n-j}. \quad (\text{B.16})$$

Pre-multiplying (B.6) by $(-1)^n \underline{D}_{n-1}$ and subtracting the result from (B.16) yields

$$(-1)^{n-1} (\underline{D}_{n-1} \underline{Z} + a_n \underline{D}_{n-2}) \underline{x}_1 = (-1)^n \underline{D}_{n-1} \underline{c}_1 + \sum_{j=0}^{n-2} (-1)^{j+1} \underline{D}_j \underline{c}_{n-j}. \quad (\text{B.17})$$

This result simplifies to give (B.3). Equations (B.6) and (B.7) follow from (B.8) and (B.9).

Let

$$d_k(\lambda) = \det (\underline{A}_k - \lambda \underline{I}) \quad k = 1, 2, \dots, n$$

be the characteristic polynomial of the $k \times k$ Schwarz matrix \underline{A}_k . Then it can easily be shown in general that

$$d_k(\lambda) = -\lambda d_{k-1}(\lambda) + a_k d_{k-2}(\lambda) \quad k = 2, \dots, n$$

with $d_0 = 1$. Note that

$$\underline{D}_n = d_n(-\underline{Z}).$$

By Frobenius' Theorem [23], the eigenvalues of \underline{D}_n are $d_n(\lambda_i)$, $i = 1, 2, \dots, m$, where $\{\lambda_i\}_{i=1}^m$ is the set of eigenvalues of $-\underline{Z}$. Therefore, if $-\underline{Z}$ and \underline{A} share no common eigenvalues $d_n(\lambda_i) \neq 0$, $i = 1, 2, \dots, m$. Then \underline{D}_n is non-singular and the solution \underline{x}_1 of (B.3) is unique and consequently $\underline{x}_2, \underline{x}_3, \dots, \underline{x}_n$ are uniquely determined.

Q.E.D.

The number of multiplication and division operations necessary for computing \underline{X} can be broken down as follows:

i) forming \underline{D}_k $k = 2, \dots, n$ -- $(n-1)(m^3 + m^2)$

- ii) Computing $\sum_{j=0}^{n-1} (-1)^{n+j} \frac{D_j}{c_{n-j}} \dots n m^2$
- iii) Solving (B.3) $\dots \frac{1}{3} m^3 + m^2 - \frac{1}{3} m$
- iv) Computing $\underline{x}_2, \dots, \underline{x}_n \dots (n-1)m^2 + (n-2)m.$

Thus, the solution of (B.1) requires on the order of nm^3 operations.

The unique solution of (2.2-7) can always be obtained using the algorithm presented in Theorem B.1. Equation (2.2-5), however, is not in a form suitable for the application of Algorithm B.1. Theorem B.2 gives a transformation which permits the use of Algorithm B.1 in solving equations of the form

$$\underline{Z} \underline{Y} + \underline{Y} \underline{A}_n = -\underline{E}.$$

THEOREM B.2.

Let \underline{A}_n be given by (B.2) and $a_i \neq 0, i = 1, 2, \dots, m.$ If

$$\underline{S} = \text{diag}[1, -a_n, a_n a_{n-1}, \dots, (-1)^{n-1} a_n a_{n-1} \dots a_2], \quad (\text{B.17})$$

then

$$\underline{S}^{-1} \underline{A}_n \underline{S} = \underline{A}_n^T$$

proof: Consider

$$\underline{A}_n \underline{S} = \begin{bmatrix} 0 & -a_n & 0 & \cdot & \cdot & \cdot & 0 & 0 \\ -a_n & 0 & a_n a_{n-1} & \cdot & \cdot & \cdot & 0 & 0 \\ 0 & a_n a_{n-1} & 0 & \cdot & \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 & (-1)^{n-1} a_n \dots a_2 \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & (-1)^{n-1} a_n \dots a_2 & (-1)^n a_n \dots a_1 \end{bmatrix}$$

Note that $\underline{A}_n \underline{S} = (\underline{A}_n \underline{S})^T$; but

$$(\underline{A}_n \underline{S})^T = \underline{S}^T \underline{A}_n^T = \underline{S} \underline{A}_n^T.$$

Thus, $\underline{A}_n \underline{S} = \underline{S} \underline{A}_n^T$. Since $a_i \neq 0$, $i = 1, 2, \dots, n$, \underline{S}^{-1} exists and (B.18) follows. Q.E.D.

Let

$$\underline{Y} = \underline{V} \underline{S}^{-1} \tag{B.19}$$

where \underline{S} is given by (B.17). Then from (B.16), \underline{V} satisfies

$$\underline{Z} \underline{V} + \underline{V} \underline{S}^{-1} \underline{A}_n \underline{S} = -\underline{E} \underline{S} \tag{B.20}$$

Now using (B.18), (B.20) becomes

$$\underline{Z} \underline{V} + \underline{V} \underline{A}_n^T = -\underline{E} \underline{S}. \tag{B.21}$$

Equation (B.21) is of the form such that Theorem B.1 applies. After solving (B.21) for \underline{V} , \underline{Y} is determined by (B.19).

The matrix equations (2.2-4) and (2.2-6) can also be solved by employing Algorithm B.1. However, a much more efficient computational process is possible. Consider the matrix equation

$$\underline{A}_n \underline{X} + \underline{X} \underline{A}_n^T = -\underline{C}, \tag{B.22}$$

where \underline{A}_n is defined by (B.2). Let $\underline{C} = [c_{ij}]$ and $\underline{X} = [x_{ij}]$. By careful consideration of the entries in \underline{C} and the matrix $\underline{A}_n \underline{X} + \underline{X} \underline{A}_n^T$, the following algorithm for solving (B.22) can be derived.

Algorithm B.2

i) $x_{12} = -\frac{1}{2} c_{ij}$

ii) If $n=2$ go to (vii); otherwise, for $i = 2, \dots, n-1$, do:

$$x_{i,i+1} = a_{n+2-i} x_{i-1,i} - \frac{1}{2} c_{ii}.$$

iii) If $n = 3$ go to (v); otherwise, set

$$\ell = \begin{cases} n-2, & \text{if } n \text{ even} \\ n-3, & \text{if } n \text{ odd.} \end{cases}$$

iv) For $k = 2, 4, \dots, \ell$, do:

$$x_{1,k+2} = a_{n-k+1} x_{1,k} - x_{2,k+1} - c_{1,k+1},$$

For $i = 2, \dots, n-k-1$, do:

$$x_{i,i+k+1} = a_{n+2-i} x_{i-1,i+k} + a_{n+2-i-k} x_{i,i+k-1} - x_{i+1,i+k} - c_{i,i+k}.$$

v) If n even, set $\ell = 2$ and go to (vi); otherwise, set $\ell = 3$ and compute

$$x_{1n} = -(a_2 x_{1,n-1} - x_{2n} - c_{1n})/a_1.$$

If $n = 3$, go to (vii).

vi) For $i = \ell, \ell+2, \dots, n-2$, do:

$$x_{in} = -(a_{n+2-i} x_{i-1,n} + a_2 x_{i,n-1} - x_{i+1,n} - c_{in})/a_1.$$

vii) Compute

$$x_{nn} = -(a_2 x_{n-1,n} - \frac{1}{2} c_{nn})/a_1.$$

viii) If n odd, set $\ell = 2$ and go to (ix); otherwise, set $\ell = 3$, and compute

$$x_{1,n-1} = -(a_1 x_{1n} - x_{2n} - c_{1n})/a_2.$$

If $n = 2$, go to (xii).

ix) For $i = \ell, \dots, n-1$, do:

$$x_{i,n-1} = -(a_{n+2-i} x_{i-1,n} + a_1 x_{in} - x_{i+1,n} - c_{in})/a_2.$$

- x) If n even, set $\ell = 2$ and go to (xi); otherwise, set $\ell = 3$ and compute

$$x_{1-n-2} = (x_{2,n-1} + x_{1n} + c_{1,n-1})/a_3.$$

If $n = 3$, go to (xii).

- xi) For $k = \ell, \dots, n-2$, do:
For $i = 1, \dots, k-1$, do:

$$x_{k-i+1, n-i-1} = -(a_{n-k+i+1} x_{k-i, n-i} - x_{k-i+2, n-i} - x_{k-i+1, n-i+1} - c_{k-i+1, n-i})/a_{i+2},$$

$$x_{1, n-k-1} = (x_{2, n-k} + x_{1, n-k+1} + c_{1, n-k})/a_{k+2}.$$

- xii) For $i = 1, \dots, n-1$, do:

$$x_{ji} = x_{ij} \quad j = i+1, \dots, n.$$

The number of multiplications/divisions required by this algorithm varies as n^2 . Thus, a substantial amount of work is saved by solving (B.22) with Algorithm B.2 rather than Algorithm B.1.

Algorithm B.2 may also be used to solve equations of the form

$$\underline{A}_n^T \underline{X} + \underline{X} \underline{A}_n = -\underline{E}. \quad (\text{B.23})$$

Define the matrix \underline{R} by the equation

$$\underline{X} = \underline{S}^{-1} \underline{R} \underline{S}^{-1} \quad (\text{B.24})$$

where \underline{S} is defined by (B.17). Then (B.23) requires that \underline{R} satisfy

$$\underline{A}_n \underline{R} + \underline{R} \underline{A}_n^T = - \underline{S} \underline{E} \underline{S}. \quad (\text{B.25})$$

which can be solved by Algorithm B.2. Once \underline{R} has been found, \underline{X} is computed via (B.24).

APPENDIX C

PROOF OF THEOREM 3.4.1

THEOREM 3.4.1

Let \underline{A} be an $n \times n$ matrix. For $i = 1, 2, \dots, q$, let \underline{v}_i be the grade v_i generalized eigenvector of \underline{A} corresponding to the eigenvalue μ_i . Define $p = \sum_{i=1}^q v_i < n$ and

$$p_D(\lambda) = \prod_{i=1}^q (\underline{A} - \mu_i \underline{I})^{v_i} = \lambda^p + \alpha_1 \lambda^{p-1} + \dots + \alpha_p. \quad (C.1)$$

If

$$\underline{t}_1 = \sum_{i=1}^q \beta_i \underline{v}_i \quad (C.2)$$

for arbitrary $\{\beta_i\}_{i=1}^q$, then

$$p_D(\underline{A}) \underline{t}_1 = \underline{0}. \quad (C.3)$$

Furthermore, if $\beta_i \neq 0$ for all i , then

$$\text{rank} [\underline{t}_1 \ \underline{A} \underline{t}_1 \ \dots \ \underline{A}^{p-1} \underline{t}_1] = p. \quad (C.4)$$

proof:

(i) From (C.2)

$$p_D(\underline{A}) \underline{t}_1 = \sum_{i=1}^q \beta_i p_D(\underline{A}) \underline{v}_i. \quad (C.5)$$

Then

$$p_D(\underline{A}) \underline{v}_i = \left[\prod_{\substack{j=1 \\ j \neq i}}^q (\underline{A} - \mu_j \underline{I})^{v_j} \right] \cdot (\underline{A} - \mu_i \underline{I})^{v_i} \underline{v}_i.$$

By definition, the generalized eigenvector \underline{v}_i satisfies

$$(\underline{A} - \mu_i \underline{I})^{v_i} \underline{v}_i = \underline{0}.$$

Thus $p_D(\underline{A}) \underline{v}_i = \underline{0}$. This implies from (C.5) that (C.3) holds for arbitrary β_i 's.

(ii) Now suppose $\beta_i \neq 0$, $i = 1, 2, \dots, q$. For notational convenience, let $\underline{v}_i^{(1)} = \underline{v}_i$. Define $\underline{v}_i^{(j)}$, $j = 1, 2, \dots, v_i$, to be the grade $(v_i + 1 - j)$ generalized eigenvectors corresponding to μ_i . Note that $\underline{v}_i^{(v_i)}$ is an eigenvector of \underline{A} .

Consider the chain of v_i generalized eigenvectors associated with μ_i :

$$\begin{aligned} \underline{A} \underline{v}_i^{(1)} &= \mu_i \underline{v}_i^{(1)} + \underline{v}_i^{(2)} \\ \underline{A} \underline{v}_i^{(2)} &= \mu_i \underline{v}_i^{(2)} + \underline{v}_i^{(3)} \\ &\vdots \\ \underline{A} \underline{v}_i^{(v_i)} &= \mu_i \underline{v}_i^{(v_i)}. \end{aligned} \quad (C.6)$$

It follows from (C.6) that

$$\begin{aligned} \underline{A}^2 \underline{v}_i^{(1)} &= \mu_i^2 \underline{v}_i^{(1)} + 2 \mu_i \underline{v}_i^{(2)} + \underline{v}_i^{(3)}, \\ \underline{A}^3 \underline{v}_i^{(1)} &= \mu_i^3 \underline{v}_i^{(1)} + 3 \mu_i^2 \underline{v}_i^{(2)} + 3 \mu_i \underline{v}_i^{(3)} + \underline{v}_i^{(4)}, \end{aligned}$$

and it can be easily verified that

$$\underline{A}^{k-1} \underline{v}_i^{(1)} = \sum_{j=1}^k C_{ijk} \underline{v}_i^{(j)} \quad k = 1, 2, \dots, v_i \quad (C.7)$$

and

$$\underline{A}^{k-1} \underline{v}_i^{(1)} = \sum_{j=1}^{v_i} C_{ijk} \underline{v}_i^{(j)} \quad k = v_i+1, \dots, p \quad (\text{C.8})$$

where

$$C_{ijk} = \frac{(k-1)! \mu_i^{k-j}}{(k-j)! (j-1)!} = \binom{k-1}{j-1} \mu_i^{k-j}. \quad (\text{C.9})$$

Now consider the implications of the statement

$$\sum_{k=1}^p \gamma_k \underline{A}^{k-1} \underline{t}_1 = \underline{0}. \quad (\text{C.10})$$

Using (C.2), (C.10) may be written as

$$\sum_{i=1}^q \beta_i \sum_{k=1}^p \gamma_k \underline{A}^{k-1} \underline{v}_i^{(1)} = \underline{0}. \quad (\text{C.11})$$

Substitution of (C.7) and (C.8) into (C.11) yields

$$\sum_{i=1}^q \beta_i \sum_{k=1}^{v_i} \sum_{j=1}^k \gamma_k C_{ijk} \underline{v}_i^{(j)} + \sum_{k=v_i+1}^p \sum_{j=1}^{v_i} \gamma_k C_{ijk} \underline{v}_i^{(j)} = \underline{0}. \quad (\text{C.12})$$

Since

$$\sum_{k=1}^{v_i} \sum_{j=1}^k \gamma_k C_{ijk} \underline{v}_i^{(j)} = \sum_{j=1}^{v_i} \sum_{k=j}^{v_i} \gamma_k C_{ijk} \underline{v}_i^{(j)},$$

(C.12) can be rewritten as

$$\sum_{i=1}^q \sum_{j=1}^{v_i} \beta_i \sum_{k=j}^{v_i} \gamma_k C_{ijk} \underline{v}_i^{(j)} = \underline{0}. \quad (\text{C.13})$$

However, the set $\{\underline{v}_i^{(j)}\}$, $j = 1, 2, \dots, v_i$ and $i = 1, 2, \dots, q$, is linearly independent and $\beta_i \neq 0$. Thus (C.13) requires that

$$\sum_{k=j}^{v_i} \gamma_k C_{ijk} = 0 \quad 1 \leq j \leq v_i, \quad 1 \leq i \leq q. \quad (\text{C.14})$$

Notice that (C.14) can be written in matrix form as

$$\underline{H} \underline{\gamma} = \underline{0}, \quad (\text{C.15})$$

where $\underline{\gamma} = [\gamma_1, \gamma_2, \dots, \gamma_p]^T$, and

$$\underline{H} = \begin{bmatrix} \underline{H}_1 \\ \underline{H}_2 \\ \vdots \\ \underline{H}_q \end{bmatrix}$$

with the $v_i \times p$ matrix \underline{H}_i defined by $\underline{H}_i = [h_{ijk}]$, where

$$h_{ijk} = \begin{cases} C_{ijk} & j \leq k \leq p, 1 \leq j \leq v_i \\ 0 & k < j \end{cases}.$$

\underline{H} is the generalized Vandermonde matrix which is nonsingular for distinct μ_i . Therefore, the only solution to (C.15) is $\underline{\gamma} = \underline{0}$. Since (C.10) requires that $\gamma_k = 0, k = 1, 2, \dots, p$, the set $\{\underline{A}^{k-1} \underline{t}_1\}_{k=1}^p$ is linearly independent. Therefore, (C.4) holds.

Q.E.D.

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MODEL ORDER REDUCTION OF LINEAR DYNAMIC SYSTEMS
WITH APPLICATION TO POWER SYSTEM CONTROL

by

Anthony Neal Payne

(ABSTRACT)

An approach for approximating linear time-invariant system models of high order by simplified models of low order is developed. The problem of approximating systems having no unstable modes is investigated first. An iterative scheme for minimizing a quadratic functional of the error between the system outputs and simplified model outputs is proposed. The computational requirements of the algorithm are reduced by first converting the multiple input system to an equivalent single input system and then choosing a canonical structure for the simplified model. The Schwarz canonical form is selected and advantage is taken of the special properties of the Schwarz form.

The simplification of systems having unstable modes is then considered. A technique for decomposing the system model into stable and unstable subsystems is presented. The unstable modes of the system are retained in the simplified model, and the algorithm for reducing the order of stable systems is applied to the stable subsystem.

Finally, the use of simplified models in designing suboptimal output regulators for complex systems is outlined. The suboptimal control scheme is applied to the power system stabilization problem. Two examples of power system control are given to demonstrate the value of the control scheme.