Feedback Control and Nonlinear Controllability of Nonholonomic Systems

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(ABSTRACT)

In this thesis we study the methods for motion planning for nonholonomic systems. These systems are characterized by nonholonomic constraints on their generalized velocities. The motion planning problem with constraints on the velocities is transformed into a control problem having fewer control inputs than the degrees of freedom. The main focus of the thesis is on the study of motion planning and design of the feedback control laws for an autonomous underwater vehicle: a nonholonomic system. The nonlinear controllability issues for the system are also studied. For the design of feedback controllers, the system is transformed into chained and power forms. The methods of transforming a nonholonomic system into these forms are discussed. The work presented in this thesis is a step towards the initial study concerning the applicability of kinematic-based control on underwater vehicles.

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Chapter 1

Introduction

This chapter gives the brief overview on the thesis topic and the motivation behind the research work presented here. The chapter also gives a brief overview and the organization of the chapters following in the text.

1.1 Overview

The purpose of this research is to study the issues related to motion planning, nonlinear controllability and design of the feedback controllers for a specific class of nonholonomically constrained mechanical systems. Specifically, differential geometric control theory, nonlinear system analysis and control design techniques, and the results of recent research in the motion planning of nonholonomic systems are used and presented for the support of the current research work.

Finally, a mathematical model of an autonomous underwater vehicle is developed. The kinematic modeling and the feedback controller design for the same are presented in detail and simulation results are obtained. The methods for converting the system into chained and power forms are also discussed. A brief mathematical analysis of the concepts involved in the study of controllability, control design and modeling is presented. The work presented in this thesis is a step towards the initial study concerning the applicability of kinematic- based control on underwater vehicles.

1.2 Motivation

The initial motivation for the thesis came from the need of motion planning of nonholonomic systems. These systems are characterized by the presence of nonholonomic constraints on their generalized velocities. The control model for such systems is drift free, nonlinear and under actuated, given by

$$\dot{q} = g_1(q)v_1 + g_2(q)v_2 + ... + g_m(q)v_m \tag{1.1}$$

Here $q \in M$ is the state of the system, M is the state space and $M \subset \mathbb{R}^n$. Thus q belongs to a configuration space of dimension n. $v \in \mathbb{R}^m$ is the input or the control vector of dimension m. $g_i(q) \in \mathbb{R}^n$; $i = 1,2......,m$ are vector fields on M and are assumed to be smooth and linear time invariant. The system is called drift free, because the system state does not change under zero input conditions. Also the system is under actuated because the dimension of the space spanned by the control vector is less than the dimension of the configuration space.

A special case of (1.1) with two inputs was presented in [1]. In [1] the motion planning tasks for a car-like robot were defined and the feedback control design was studied. The control was achieved using various control strategies for each task. This work is motivated by the desire to extend the similar work to an underwater vehicle. The extended problem is higher dimensional with four inputs. In [1] the design is done using the chained forms. In our work we will make use of chained and power forms to achieve control.

The motivation also comes from the fact that the design of globally asymptotic stabilizing controllers for nonholonomic systems is challenging. The design is difficult in a sense that no time invariant smooth static stabilizing controller exists for such systems [2]. Various control schemes have been adopted for this purpose. One way to deal with this is to use time varying, smooth controllers. This approach has been extensively studied in [3] [4]. In [3] it is shown that, time varying smooth, control laws for driftless systems have necessarily algebraic (not exponential) convergence rates. Another alternative is the use of the nonsmooth feedback controllers which can achieve exponential convergence. These schemes have been proposed in [5], [6]. In our case we will be adopting the former approach. The control design for stabilization used herein is adopted from [7]. In this case global stabilization is achieved.

1.3 Thesis outline and organization of the chapters

Chapter 2 gives an introduction and general overview of the motion planning of the autonomous vehicles. The concepts of nonholonomy, under actuated systems, kinematic model of the nonholonomic systems and some examples are shown. Then the general problem of motion planning and the related issues are formulated for a class of the nonholonomic systems, with a review of some particular applications.

Chapter 3 presents an overview and detailed analysis of the related motion planning tasks of an autonomous underwater vehicle. The chapter presents in detail the derivation of the mathematical modeling of the system. For motion planning tasks, the kinematic model of the system is obtained and the issues related to nonlinear controllability of the system are studied in detail. Finally, for the purpose of control design, the system is converted into chained form. The method of converting a multi input nonholonomic system into a chained form is also discussed.

Chapter 4 presents the control design and the simulation results obtained for the model of an underwater vehicle developed in chapter 3. The feedback control design is developed using the kinematic model of the system. The performance of the controllers using various techniques of control design is obtained and evaluated for different motion planning tasks, such as trajectory tracking, point stabilization and path following. The chapter also presents the simulation results obtained for different controllers. The simulation results are used to compare and evaluate the performance of the various controllers for different path following tasks.

Chapter 5 presents the conclusions of the work. The contributions of the presented research work and the expansion or scope for the future work on the topic are also discussed.

1.4 Previous research and contributions of the thesis

In this thesis we will be studying the motion planning for an example of nonholonomic systems. Our example is the four-input nonholonomic system of an underwater vehicle. The configuration of an underwater vehicle is given by six dimensional special Euclidian group $SE(3)$. If the velocity of the vehicle is constrained so that only the forward velocity component can be non zero, the vehicle has four degrees of freedom and two non holonomic constraints. The control inputs are the linear velocity in x direction and three angular velocities along the x, y and z coordinate axes. The controllability of the system is discussed and proved as related to motion planning. We present feedback control laws which give global stabilization of the vehicle about a desired trajectory and about a point. This is achieved by transforming the kinematic into a canonical chained form. The thesis presents the method of converting the kinematic model into the chained form via state feedback and coordinate transformation.

For trajectory tracking of underwater vehicles [23] proposed a stable tracking control method based on a Lyapunov function. In [23] and [22] Lyapunov like function is used to develop a nonlinear feedback control scheme. The control achieves global stabilization about a desired trajectory. However the system is not point stablizable with the use of the proposed controller. In our case first we will be making use of the full state feedback (approximate linearization) scheme for trajectory tracking. This scheme results in local asymptotic stabilization only. Exact nonlinear control (full state linearization) design is used to achieve the global stabilization. In this case static state feedback fails to achieve the goal. However the dynamic state feedback is used to serve the purpose. Here the control design is done on the chained form system.

The kinematic model of underwater vehicle belongs to a class of systems which cannot be stabilized by a pure state feedback law [2]. Thus to achieve point stabilization different schemes have been implemented. Asymptotic stabilization for underwater vehicles using time varying smooth feedback laws was achieved in [4]. In [25] a discontinuous piecewise smooth control law was proposed and exponential convergence to a constant desired configuration was achieved. In [15] a non smooth time invariant controller was proposed to achieve the exponential convergence with stability to a constant desired configuration. The controller was implemented using chained form. In our case we will be making use of a time varying and smooth feedback. The controller achieves global stabilization to a constant configuration for an underwater vehicle. To this end, a transformation of the kinematic model into power form is derived and the controller proposed in [7] is applied.

Chapter 2

Problem Formulation and Examples

This chapter gives an overview of motion planning and issues related to motion planning tasks of autonomous vehicles. The control or the kinematic model obtained for such vehicles involves the concepts of nonholonomy. It will be seen that the vehicles are nonlinear and under actuated in nature because of nonholonomic constraints on their generalized velocities. Finally some examples will be cited and motion planning problem will be formulated for two specific examples of autonomous vehicles and issues related to the various motion planning tasks and the feedback control design for these examples will be discussed.

2.1 Motion planning of nonholonomic systems

The initial motivation for the work presented here comes from the research work done in order to do the motion planning and control design for the nonholonomically constrained systems. Motion planning for nonholonomic systems has been studied in great detail and a lot of research is being done in this field. This problem has attracted researchers because of its challenging theoretical nature and practical importance. The nonholonomic constraints arise in a number of advanced robotic systems and the application of such systems is numerous. The problem is also interesting because its theoretical behavior presents a number of challenges. Firstly, such systems are under actuated, i.e.; the number of control inputs is less than the number of the states or the variables of the system to be controlled. Thus motion planning implies that the systems can be completely controlled with a fewer number of actuators, thereby improving the overall cost effectiveness of the system. Also under actuation can provide backup control techniques for a fully actuated system. Secondly, both planning and control are more difficult than for holonomic systems. Some of the motion control problems which have been studied in detail are those of regulation (stabilization) and tracking.

The problem of stabilizing such systems is a big issue, as it has been proved by Brockett [2], that the nonholonomic control systems with restricted mobility cannot be stabilized to a desired configuration (equilibrium) using a smooth, time invariant state feedback law. Because of this fact there has been extensive study of this problem. Some authors have proposed non smooth or discontinuous control laws. Others have proposed smooth but time-varying control laws for the purpose of regulation and some have proposed the combination of both i.e. discontinuous time varying control laws [8], [9]. The method of transforming the kinematic model into the chained from model and doing the control on the same was first proposed by [10] for the case of a car like robot. The study of feedback control of a nonholonomic car like robot is done in [1]. Various motion planning tasks such as tracking a time varying reference trajectory, path following and point to point stabilization of a car like robot were presented in [1]. The work presented in chapter 3 is along the same lines as [1], extended and modified for the application of underwater vehicles. The design of feedback controllers will be used for different motion tasks utilizing the kinematic model of the system. The kinematic model will be developed using the definition of nonholonomic constraints. The work presented in this thesis is a step towards the initial study concerning the applicability of kinematic- based control on underwater vehicles.

2.2 *Nonholonomic constraints*

System constraints on the mechanical systems whose expression involves generalized coordinates and velocities are known as kinematic constraints of the system. These are of the following form

$$a_i(q,\dot{q}) = 0, \quad i = 1,2,........,k < n \tag{2.1}$$

where q is the generalized coordinate vector or the state vector. $q \in M \subset \mathbb{R}^n$, where n is the dimension of the configuration space M , to which the vector q belongs. These will limit the admissible motions of the system by restricting the set of generalized velocities that can be attained at a given configuration. Usually such constraints are in mechanics in Pfaffian form

$$a_i^T(q)\dot{q} = 0 \quad i = 1,2,........,k < n \tag{2.2}$$

or

$$C(q)\dot{q} = 0 \tag{2.3}$$

which means they are linear in the generalized velocities. $a_i(q) \in \mathbb{R}^n$, $i = 1,2,...,k$ are row vectors. The vectors $a_i : M \mapsto \mathbb{R}^n$ are assumed to be smooth and linearly independent. The matrix $C(q) \in \mathbb{R}^{n \times n}$ is a constraint matrix.

The kinematic constraints restrict the motion by limiting the set of generalized velocities. The nonholonomic constraints cannot be integrated to the positions. Thus while the instantaneous mobility that a system can perform is restricted to $(n-1)$ dimensional null space of the constraint matrix $C(q)$, we can still say that it is possible that any configuration in state space M can be reached. In general for a system with n

coordinates and k nonholonomic constraints, although the velocities are restricted to $n-k$ dimensional space, the global controllability in the configuration space is still attainable.

These constraints mostly arise due to rolling of two surfaces against one another, roll without the slip condition as in case of a wheel and the road. These can also arise due to conservation laws, applicable to the system or from the nature of the control inputs physically applied to the system [11]. Thus nonholonomic constraints allow the global movement of the system in the configuration space while at the same time restricting or reducing the degrees of freedom or motion performed locally by the system.

The concept of nonholonomy is related to controllability of the corresponding control system. Redefining the constraint specification as the directions or degrees of freedom in which the system can move rather than the direction in which it cannot move, is equivalent to stating the controllability problem of the corresponding control system. Thus we can safely say that if the system is maximally nonholonomic, the system is controllable as any point in the configuration space can be reached. This way a motion problem can be converted into a control problem.

Nonholonomic constraints arise in a number of ways and in various mechanical systems and applications. These can arise because of the reasons already given in the previous paragraph. For more detailed analysis, the reader is referred to [11] and [12]. Some of the typical examples of the nonholonomic systems can be summarized as

- Wheeled mobile robots.
- Space robots.
- Underwater vehicles.

- Satellites.
- Multifingered hands manipulators.
- Hopping robots.

2.3 *Problem description*

The motion planning tasks for nonholonomic systems as pertaining to robots are achieved through the use of the feedback controllers. The basic motion tasks considered for a robot are as follows

Point to point motion Here a desired goal configuration must be reached by a robot starting from a given initial configuration.

Path following Here the robot has to reach a desired final configuration starting from a given initial configuration while at the same time it has to follow a given geometric path. The initial configuration can be considered to be either on or off the path.

Trajectory following Here the robot must reach a final configuration while following a trajectory in the Cartesian space (i.e. a geometric path with an associated timing law) starting from a given initial configuration (either on or off the trajectory). The tasks are assumed here such that the systems work in an obstacle free environment and are shown in Fig. 2.1a, Fig. 2.1b, and Fig. 2.1c for a car- like robot.

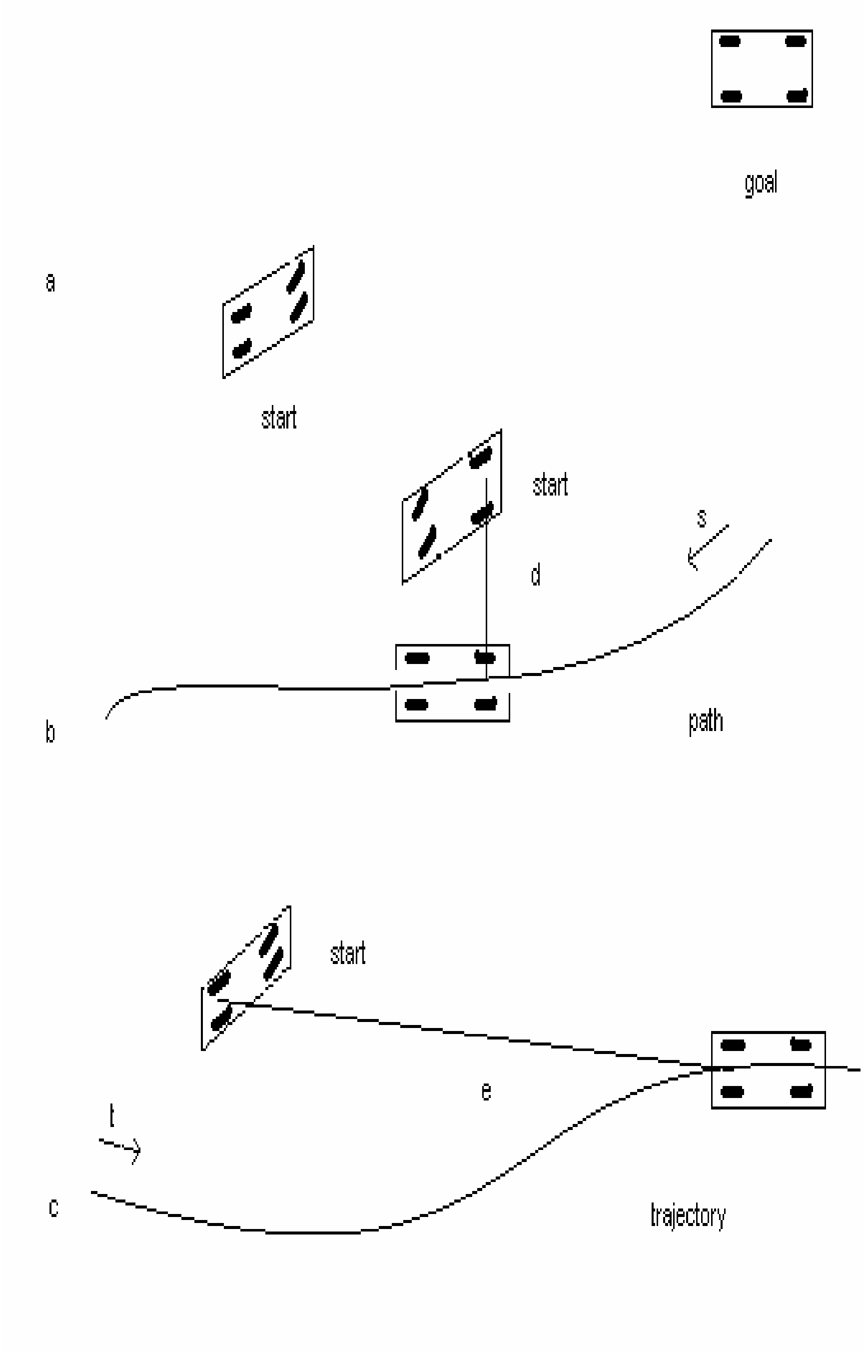

Figure2.1: Motion planning tasks for a car-like robot

The tasks can be obtained using either the feed forward (open loop) or feedback (closed loop) control or a combination of the both. Since the feedback control is generally robust and can work well in presence of disturbances, we will make use of feedback control.

Thinking in terms of controls, point to point task can be thought of as a regulation control problem or a posture stabilization problem for an equilibrium point in the state space. Trajectory following is a tracking problem such that the error between the reference and the desired trajectories asymptotically goes to zero.

For nonholonomic systems, tracking or path following or both is easier than stabilization, whereas usually the reverse is true. This difference can be explained by drawing a comparison between the numbers of inputs and outputs (or states) to be controlled. In case of a regulation problem m inputs (two in case of a car like robot) are required to regulate or control 'n' independent control variables or states (four in case of a car like robot) with m less than n. Thus point to point stabilization is the most difficult of all the three. In case of path following and trajectory tracking the output to be controlled has the dimension (p) equal to that of the input (m). Thus these control problems are square and their difficulty level is similar and less than the stabilization one.

For a car like robot, in case of path following m is one and p is one while for trajectory tracking m is two and p is two, i.e. we have to stabilize to zero the two dimensional error vector associated with the Cartesian trajectory[1].

2.4 Control model formulation

In this section we will be formulating the control model for the nonholonomic systems. For developing the control model consider the first order kinematic constraints on the system. As seen in section 2.2 such constraints are of the following form

$$a_i^T\left(q(t)\right)\dot{q}(t)=0 \qquad\qquad i=1,2,........,k<n$$

or

$$C\left(q(t)\right)\dot{q}(t)=0$$

where q are the generalized coordinates and $\dot{q}$ the first order derivative (velocities) of the coordinates and $C\left(q\right)$ is the constraint matrix.

Let us denote the set of vector fields spanning the m dimensional distribution Δ which is annihilated by the constraints as $g_j{}'s$; $j=1,2,......m$ such that

$$\Delta = span\{g_1, g_2, ..., g_m\}$$

The $g_j{}'s$ are the basis for the $\left(n-k\right)$ right null space of the constraint matrix $C\left(q\right)$ so that we have

$$a_i^T\left(q\right)\, g_j(q)=0; \qquad i=1,2,........,k<n \qquad j=1,2,......\left(n-k\right)=m \quad (2.4)$$

or

$$C\left(q\right)G\left(q\right)=0 \tag{2.5}$$

The vector fields $g_j{}'s$ are assumed to be smooth and linearly independent as a consequence of the assumption on $a_i^T\left(q\right)'s$ being smooth and independent. By expressing all the feasible velocities as a linear combination of these basis vectors, we obtain the first order kinematic model of the system as

$$\dot{q}=g_1(q)v_1+g_2(q)v_2+.........+g_m(q)v_m \tag{2.6}$$

or

$$\dot{q}=\sum_{j=1}^{n-k=m}\left(g_j(q)v_j\right) \tag{2. 7}$$

where v_j's known as psuedovelocities are taken as the control inputs and g_j's are the input vector fields. The model directly shows the presence of k nonholonomic constraints on the system having n states or configuration variables and $m = (n-k)$ control inputs. The control model of equation (1) is known as the kinematic model of the system. The model is a drift less (i.e. no motion takes place under zero input conditions), nonlinear and under actuated (number of control inputs is less than the number of states to be controlled) control system.

2.5 Controllability issues

Since the control model is driftless, the terms local accessibility and controllability can be used interchangeably. Moreover, the controllability of the whole configuration space is the (complete) nonholonomy of the kinematic constraints. The controllability condition can be established using the Chows theorem. According to the theorem, for the driftless control systems, if the accessibility rank condition

$$\dim \Delta_c (q_0) = n \tag{2.8}$$

holds, then the control system is locally accessible (controllable) from q_0. Δ_c is the accessibility distribution of the kinematic model given by equation (2.6) and is defined as the span of all the input vector fields

$$\Delta_c = span\{v | v \in \Delta_i \forall i \geq 1\}$$

with

$$\Delta_i = \Delta_{i-1} + span\{[g, v] | g \in \Delta_1, v \in \Delta_{i-1}\}, i \geq 2$$

$$\Delta_1 = span\{g_1, g_2, ..., g_m\} \tag{2.9}$$

This implies that Δ_c is the involutive closure under lie bracketing of the distribution Δ_1 associated with the input vector fields g_j's. The term $[g,v]$ is the lie bracket of the two vector fields g and v defined as

$$[g,v](q) = \frac{\partial v}{\partial q} g(q) - \frac{\partial g}{\partial q} v(q) \tag{2.10}$$

The Chows theorem provides both necessary and sufficient condition for the controllability [12]. Moreover if the system is controllable then its dynamic extension given by

$$\dot{q} = \sum_{j=1}^{n-k=m} \left(g_j(q) v_j \right)$$

and (2.11)

$$\dot{v}_j = u_j; \qquad j = 1,2,........,m$$

is also controllable. In some cases the use of the nilpotent basis is made, that is the input vector fields g_j's are the nilpotent basis. This eliminates the need for cumbersome computations as we will see that using this concept all higher order lie brackets above some particular order are zero [12].

2.6 Stabilization

The stabilization problem for the control system of (2.7) can be defined as finding the feedback control law of the form $u(q,t)$ in order to make the closed loop system asymptotically stable about an equilibrium point or a reference (feasible) trajectory. In the point stabilization problem we assume equilibrium point for the open loop system i.e. $\dot{q} = f(q_e) = 0$.

2.6.1 Controllability and stabilization at a point

For the driftless control system of (2.7), any configuration (q_e) is an open loop equilibrium point under zero input conditions. For linear systems $\dot{x} = Ax + Bu$, it is a well established fact that if the system satisfies the controllability rank condition given by

$$rank\begin{bmatrix} B & AB & A^2B & \ldots\ldots & A^{n-1}B \end{bmatrix} = n \tag{2.12}$$

then the asymptotic (actually exponential) stabilization by a smooth, time invariant state feedback is guaranteed. In other words we can say that if the controllability condition is satisfied, there exists a feed back law $u = k(x - x_e)$ such that the closed loop system is asymptotically stable about the equilibrium point x_e.

For the control model of (2.7) we would like to make a similar kind of analysis. For this purpose we will look at the approximate linearization of the system at any equilibrium point (q_e).The approximate linearization given by

$$\dot{q} = d\,\dot{q} = G\left(q_e\right)v \tag{2.13}$$

with $d\,q = q - q_e$ is clearly not controllable as the rank of the controllability matrix $G\left(q_e\right)$ is m (which is less than n). Hence we can say that a linear controller can not achieve stabilization, not even locally.

However the controllability of the nonlinear system can be established by using the tools from the differential geometry, i.e. we can make use of the Lie algebra rank condition to prove its controllability. However, even if the system can be proven to be globally controllable (in a nonlinear sense) there is still a severe theoretical limitation on point stabilization. The limitation is in a sense that Lyapunov (asymptotic) stability can not be achieved by means of a smooth, time invariant feedback [13].

The above result can be established on the basis Brockett's theorem [2] which says that the stabilization of a driftless regular system by a smooth time invariant feedback is not possible. For the driftless, under actuated control system (2.7) where vector fields g_j's are linearly independent (regular) at q_e, the theorem implies the number of inputs m is equal to the number of states n as both a necessary and sufficient condition for smooth stabilization. Also it should be noted that if the system can not be stabilized by a smooth feedback, the same negative result is true for its dynamic extension and also the theorem is not applicable to time varying feedback laws.

Thus in order to design the feedback controllers for posture stabilization, it is obligatory either to give up the continuity requirement, i.e. include the non smooth feedback or to apply the time varying laws or apply a combination of both.

2.6.2 Controllability and stabilization about trajectory

In case of trajectory following, for stabilization about a trajectory it is ensured that the reference trajectories are feasible for the system. In other words we should take or generate only those state and input trajectories that satisfy the nonholonomic constraints of the system, i.e. should satisfy (2.2)

2.6.3 Approximate linearization

For approximate linearization of the system (2.7) we take the desired state trajectory as $q_d(t)$ and the input trajectory as $v_d(t)$. It can be easily seen that the linearization about a smooth trajectory results into a linear time varying system. The system can easily be shown to satisfy the controllability condition i.e. the controllability Grammian is nonsingular [21], as long as the input reference trajectory is persistent, i.e. it does not come to a stop. Thus it implies that we can achieve stabilization about the desired trajectory via a smooth, time invariant control law as long as the trajectories do not come to a stop. One observation should be made here. As we know that the control scheme presented here is based on the approximate linearization of the original system in the neighborhood of a reference trajectory, the closed loop system is asymptotically stable only locally. In order to achieve global stabilization for trajectory tracking error, we have to make use of the nonlinear feedback design.

2.6.4 Exact feedback linearization

It is well known in robotics that if the number of generalized coordinates equals the number of control inputs i.e. $n = m$, the system kinematics or dynamics can be transformed into a linear system with the use of a nonlinear static state feedback [14]. The linearity is displayed by the system equations only after a coordinate transformation in the state space.

For exact linearization of nonlinear systems outputs are chosen to which a desired behavior is assigned. Two types of exact linearization are possible. The two schemes are full state feedback linearization and input output linearization. In the first case the feedback transformation is such that the whole set of system equations become linear while as in the second case the transformation is such that the input and output response of the closed loop system is linear. For MIMO systems this transformation results in decoupling of the input and output vectors.

Both the transformations can be achieved through static feedback or the dynamic feedback [14].

2.6.5 Static feedback linearization

For the nonholonomic kinematic model the full state feedback linearization cannot be achieved using a static (time invariant) state feedback. The reason for this is the violation of the necessary condition for the full state feedback linearization according to [14]. The controllability condition for the system derived in section 2.6 requiring that the distribution Δ_o generated by g_js not be involutive violates the necessary condition for static feedback transformation.

However, input output linearization is possible with the use of static feedback. Here m equations are transformed via feedback into simple decoupled integrators. However the choice of outputs which are linearized is not unique. Here it is worth noticing that in the case of $(2, n)$ chained form transformation, the two variables are indeed the examples of linearizing outputs with static feedback given by the input transformation equation. Also it must be noted that in case of input output linearization the internal dynamics may be left in the closed loop system. Thus for the exponential (global) convergence of the trajectory error to go to zero, these internal dynamics should be properly modeled, analyzed and their stability guaranteed.

2.6.6 Dynamic feedback linearization

For exact feedback linearization, if the static feedback design fails, we can make use of dynamic feedback for nonholonomic systems. The use of dynamic feedback can also result in full state feedback linearization. For model (2.7)

$$\dot{q} = G(q)v; \qquad q \in \mathbb{R}^n; v \in \mathbb{R}^m \tag{2.14}$$

the dynamic feedback compensator is of the form

$$v = a(q, z) + b(q, z)r$$

$$\dot{z} = c(q,z) + d(q,z)r \tag{2.15}$$

where $z(t) \in \mathbb{R}^{v}$ is the compensator state vector of dimensions v and $r(t) \in \mathbb{R}^{v}$ (having the same dimensions as the compensator state vector $z(t)$) is the auxiliary input. (2.15) is such that the closed loop system obtained from (2.14) and (2.15) is equivalent to a linear system under a state transformation $z = \mathrm{T}(q,z)$. For the applications to nonholonomic systems, the linearization process involves the following procedure.

Initially we define the output of the system (2.14) as $y = h(q)$. To this output a desired behavior is assigned (track a trajectory). Then the output y is successively differentiated until the system inputs appear explicitly in a nonsingular way. The non singularity is a must for the inversion of the differentiated equations to solve for the inputs. If in a step involving differentiation of system outputs, the decoupling matrix (differential map) of the system is singular (which means that some input is still not appearing), integrators are added on some of the input channels and the process of differentiation is continued. It is also necessary to avoid direct differentiation of the system inputs in the next differentiation. This operation is known as dynamic extension and converts a system input into a state of dynamic compensator. The dynamic compensator has the new auxiliary input r as its input. The process of differentiation continues until at some point the system is invertible (i.e. solution for new inputs can be obtained) from the chosen output vector y and the process terminates. The number of successive addition of integrators gives dimensions of the state z of the dynamic compensator. Also, if the sum of the orders of the output differentiation is equal to the dimensions of the extended state space system (original and dynamic compensator state) which is $n+v$, then the full state linearization of the system is obtained as there are no internal dynamics left in the system.

2.7 Examples of nonholonomic systems

The simplest example of a nonholonomic system can be a wheel that rolls on plane surface, such as a unicycle. The constraints here arise due to the roll without slip condition. The configuration or the generalized coordinate vector is $q=(x, y, q)$. The coordinates x and y are the position coordinates of the wheel and q is the angle which the wheel makes with the x axis. The unicycle is shown in Fig. 2.2. The constraint here is that the wheel cannot slip in the lateral direction.

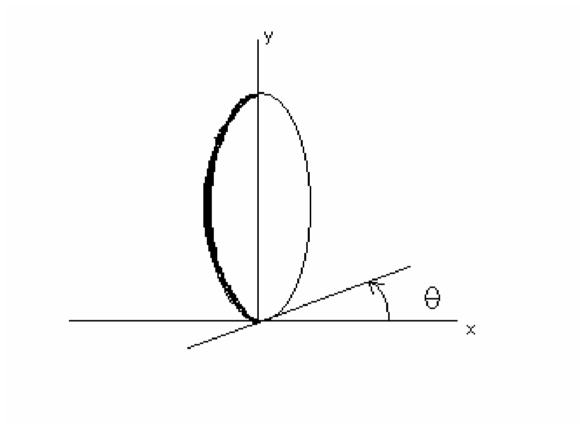

Figure2.2: The nonholonomic constraints on a unicycle.

The generalized velocities are subject to the following kinematic constraint

$$\dot{x}\sin q - \dot{y}\cos q = 0 \qquad (2.16)$$

In other words the velocity along the plane perpendicular to the point of contact between the wheel and the ground is zero. The above equation is of the form $C(q)\dot{q} = 0$ with constraint matrix $C(q) = [\sin q \quad -\cos q \quad 0]$.

Expressing the feasible velocities as a linear combination of vector fields spanning the null space of the matrix $C(q)$, we get the following kinematic model

$$\dot{q} = g_1(q)v_1 + g_2(q)v_2$$

Or (2.17)

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{q} \end{pmatrix} = \begin{pmatrix} \cos q \\ \sin q \\ 0 \end{pmatrix} v_1 + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} v_2$$

where v_1 is the linear velocity of the wheel and v_2 is its angular velocity around the vertical axis. Here we observe that the number of states $n = 3$, number of control inputs $m = 2$ and the number of nonholonomic constraints $k = 1$.

Another example is that of a car like robot shown in Fig.2.3. The robot has two wheels and each wheel is subject to one nonholonomic constraint. The constraint is the same as in the case of unicycle. The generalized coordinate vector is $q = (x, y, q, f)$, with x, y and q same as before. The angle f is the steering angle.

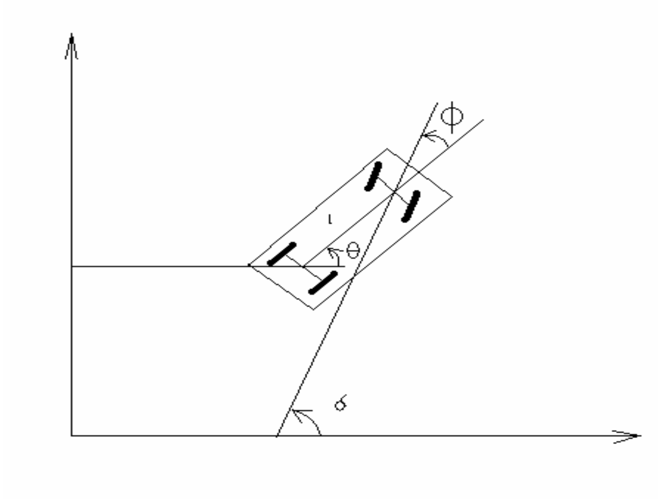

Figure2.3: The nonholonomic constraints on a car-like robot.

The two nonholonomic constraints on the front and the rear wheels respectively are

$$\dot{x}\sin(q+f)-\dot{y}\cos(q+f)-\dot{q}\,l\cos f=0$$

$$\dot{x}\sin q-\dot{y}\cos q=0 \tag{2.18}$$

Here l is the distance between the wheels. Again this is of the form $C(q)\dot{q}=0$ with

$$C(q)=\begin{bmatrix}\sin(q+f) & -\cos(q+f) & -l\cos f & 0\\ \sin q & -\cos q & 0 & 0\end{bmatrix}$$

Choosing the rear wheel drive the kinematic model is obtained as

$$\dot{q}=g_1(q)v_1+g_2(q)v_2$$

$$\begin{pmatrix}\dot{x}\\ \dot{y}\\ \dot{q}\\ \dot{f}\end{pmatrix}=\begin{pmatrix}\cos q\\ \sin q\\ \tan f/l\\ 0\end{pmatrix}v_1+\begin{pmatrix}0\\ 0\\ 0\\ 1\end{pmatrix}v_2 \tag{2.19}$$

Here v_1 is the rear driving velocity input and v_2 is the steering velocity input. The above model is not defined at $f = p/2$, where g_1 is discontinuous. Physically this corresponds to car becoming jammed because of its front wheel being normal to axis of the body.

The feedback control design, controllability analysis and the motion planning for all the three motion tasks are done in [1]. Another example of nonholonomic systems is that of an underwater vehicle which is discussed in next chapter. In next chapter we will be studying the motion planning for the same and controllability of the system is discussed and proved. We also present feedback control laws which give global stabilization of the vehicle about a desired trajectory and about a point.

Chapter 3

Mathematical Modeling and Controllability Analysis of an Underwater Vehicle

In this chapter an overview of an underwater vehicle is given and mathematical model of the vehicle is derived. The chapter presents in detail the derivation of the mathematical modeling of the system. For motion planning tasks, the kinematic model of the system is obtained and the issues related to nonlinear controllability of the system are studied in detail. Finally for the purpose of control design, the system is converted into chained form.

3.1 Mathematical modeling

In this section the mathematical model of the under water vehicle is briefly discussed. An underwater vehicle is generally defined as a six degree of freedom body. It follows the laws of rigid body motion. The dynamics of the system are highly non linear due to rigid body coupling and hydrodynamic forces on the vehicle. The mathematical model of the underwater vehicle is obtained through the following two models.

Dynamic model This type of model allows for the actual forces, causing the motion and the dynamic properties of the vehicle to be taken into account. The equations of translation and rotation are obtained using Newton's law [15].

Kinematic model The kinematic model of the system is the model where actual forces causing the motion and the dynamic properties of the vehicle do not enter the equations of motion. This type of model allows for the decoupling of the vehicle dynamics from its movement. An autonomous underwater vehicle has nonholonomic nature due to its nonlinear kinematic model [22]. In the following section the kinematic model of the vehicle is derived. For the remaining chapter and the chapter following, the kinematic model of the system will be used for analysis and control purposes.

3.1.1 Kinematic modeling and nonholonomic constraints

The kinematic model of the system is obtained by taking into consideration the nonholonomic constraints on the linear velocity. The nonholonomic constraints restrict the velocity of the system to be zero in certain directions but these restrictions do not restrict the global movement of the system. For the development of the kinematic model of the underwater vehicle model we assume two orthogonal coordinate systems [23].

Global coordinates The global or the inertial frame coordinates are denoted by (P, X, Y, Z). The frame remains fixed at the ocean surface with origin P.The unit vector in the Z direction points up into the water while the unit vectors along X and Y direction complete a right handed system.

Local coordinates The local or the body frame coordinates are denoted by (p, x, y, z). The frame remains fixed on the vehicle with origin surface with origin p.

The two coordinate systems are as shown below in Fig. 3.1[23].

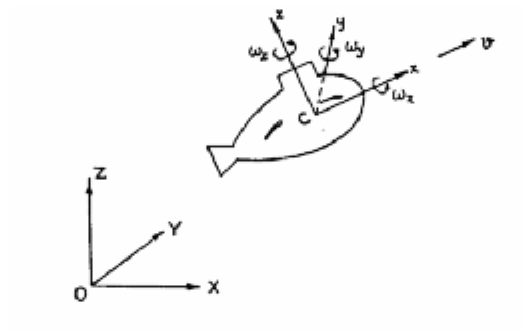

Figure3.1: The coordinate systems of an under water vehicle. (From [23]. Copyright © 1991 IEEE.)

3.1.2 Kinematic model with respect to global coordinates

The kinematics of the vehicle are described by six state variables and four input variables. The kinematic relationships describing the transformations between the two coordinate systems can have a number of parameterizations. The one used here is the Euler angle parameterization [25]. In the Euler angle representation the orientation between the inertial and the local coordinate frame is expressed in terms of a sequence of three rotations: roll (f), pitch (q) and yaw (y) about the axes *x, y* and *z* respectively.

Let q be the vector of six generalized coordinates required to specify the kinematics of the vehicle. The six coordinates are the Cartesian coordinate vector $p=[x,y,z]^T$ of the vehicle in the local frame and the Orientation coordinate vector $h=[f,q,y]^T$. The orientation vector is the vector of Euler angles which give the orientation of the body frame with respect to the inertial frame. The transformation from the local coordinate frame to the global coordinate frame is given by means of a Rotation matrix R . $R\in$ S(O3), where S(O3) is the group of rigid body rotations. R satisfies the relation $RR^T = I$, i.e. $R^T = R^{-1}$ or R is an orthogonal matrix and $\det(R) = 1$[16]. The matrix R is given below as

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} n & s & a \end{bmatrix} \tag{3.1}$$

with

$$r_{11} = \cos q \cos y$$

$$r_{12} = \sin q \sin j \cos y - \cos j \sin y$$

$$r_{21} = \cos q \sin y$$

$$r_{22} = \sin q \sin j \sin y + \cos j \cos y$$

$$r_{23} = \sin q \cos j \sin y - \sin j \cos y$$

$$r_{31} = -\sin q$$

$$r_{32} = \sin j \cos q$$

$$r_{33} = \cos j \cos q \tag{3.2}$$

Let $v = [v_x, 0, 0]^T$ be the linear velocity of the vehicle i.e. the vehicle has linear velocity along the x-axis only and $w = [w_x, w_y, w_z]^T$ be the angular velocity components along x, y and z directions respectively in the body frame. The velocity vector along three coordinate axes and the time derivative of the Euler angles are obtained from the following relations:

$$\dot{p} = Rv = \begin{bmatrix} n & s & a \end{bmatrix} v \tag{3.3}$$

$$\dot{R} = RS(w) \tag{3.4}$$

where $S(w)$ is the skew symmetric matrix given as

$$S(w) = \begin{bmatrix} 0 & -w_z & w_y \\ w_z & 0 & -w_x \\ -w_y & w_x & 0 \end{bmatrix}$$

The above equations give the following on solving

$$\dot{p} = J_1(h)v \tag{3.4}$$

$$\dot{h} = J_2(h)w \tag{3.5}$$

With

$$J_1(h) = [\cos q \cos y \ , \cos q \sin y \ , -\sin q \,]^T$$

$$J_2(h) = \begin{bmatrix} 1 & \sin f \tan q & \cos f \tan q \\ 0 & \cos f & -\sin f \\ 0 & \sin f \sec q & \cos f \sec q \end{bmatrix}$$

The above set of equations can be written as the following equations.

$$\dot{x} = r_{11}v = \cos y \ \cos q\, v$$

$$\dot{y} = r_{21}v = \sin y \ \cos q\, v$$

$$\dot{z} = r_{31}v = -\sin q\, v$$

$$\dot{f} = w_x + \sin f \tan q w_y + \cos f \tan q w_z$$

$$\dot{q} = \cos f w_y - \sin f w_z$$

$$\dot{y} = \sin f \sec q w_y + \cos f \sec q w_z \tag{3.6}$$

This can be written in the matrix form as

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \dot{f} \\ \dot{q} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} \cos q \cos y & 0 & 0 & 0 \\ \cos q \sin y & 0 & 0 & 0 \\ -\sin q & 0 & 0 & 0 \\ 0 & 1 & \sin f \tan q & \cos f \tan q \\ 0 & 0 & \cos f & -\sin f \\ 0 & 0 & \sin f \sec q & \cos f \sec q \end{bmatrix} \begin{bmatrix} v \\ w_x \\ w_y \\ w_z \end{bmatrix} \tag{3.7}$$

The equations can be written in the generalized vector form as

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \dot{f} \\ \dot{q} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} \cos q \cos y \\ \cos q \sin y \\ -\sin q \\ 0 \\ 0 \\ 0 \end{bmatrix} v + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} w_x + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \sin f \tan q \\ \cos f \\ \sin f \sec q \end{bmatrix} w_y + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \cos f \tan q \\ -\sin f \\ \cos f \sec q \end{bmatrix} w_z \tag{3.8}$$

The system here is subject to two non holonomic constraints. The constraints are on the linear velocities along y and z directions. The velocities along these directions are zero. The two constraints are

$$s^T \dot{p} = 0$$

$$a^T \dot{p} = 0 \tag{3.8}$$

This can be written as

$$r_{12}\dot{x} + r_{22}\dot{y} + r_{32}\dot{z} = 0$$

$$r_{13}\dot{x} + r_{23}\dot{y} + r_{33}\dot{z} = 0$$

Or

$$(\cos y \ \sin q \sin f - \sin y \ \cos f)\dot{x} + (\sin y \ \sin q \sin f + \cos y \ \cos f)\dot{y} + (\cos q \ \sin f)\dot{z} = 0$$

$$(\sin y \ \sin q \cos f - \sin y \ \sin f)\dot{x} + (\sin y \ \sin q \cos f - \cos y \ \sin f)\dot{y} + (\cos q \ \cos f)\dot{z} = 0 \tag{3.9}$$

The above equation is of the form

$$A(q)\dot{q} = 0$$

with

$$A(q) = \begin{bmatrix} r_{12} & r_{22} & r_{32} & 0 & 0 & 0 \\ r_{13} & r_{23} & r_{33} & 0 & 0 & 0 \end{bmatrix} \tag{3.10}$$

Expressing the feasible velocities as the linear combination of vector fields $g_1(q)$, $g_2(q)$, $g_3(q)$ and $g_4(q)$ spanning the null space of matrix $A(q)$ we have the following kinematic model

$$\dot{q} = g_1(q)v_1 + g_2(q)v_2 + g_3(q)v_3 + g_4(q)v_4$$

$$\dot{q} = \begin{bmatrix} g_1(q) & g_2(q) & g_3(q) & g_4(q) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} \tag{3.11}$$

where

$$g_1(q) = \begin{pmatrix} \cos q \cos y \\ \cos q \sin y \\ -\sin q \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad g_2(q) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$g_3(q) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \sin f \tan q \\ \cos f \\ \sin f \sec q \end{pmatrix} \qquad g_4(q) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \cos f \tan q \\ -\sin f \\ \cos f \sec q \end{pmatrix} \tag{3.12}$$

and

$$v_1 = v = v_x; v_2 = w_x; v_3 = w_y; v_4 = w_z$$

More generally we can write

$$\dot{q} = G(q)v \tag{3.13}$$

The above equations are the kinematic model of the system. The system is nonlinear and under actuated, which means that the number of inputs to the system is less than its states. The generalized velocity vector $\dot{q}$ cannot assume any independent value unless it satisfies the nonholonomic constraints. The constraints are the examples of the Pfaffian Constraints which are linear in velocities. The admissible generalized velocities as given by (3.1) are contained in the null space of the constraint matrix $A(q)$.

3.2 Controllability analysis

Consider the system $\dot{q} = G(q)v$. The system is nonlinear, under actuated and driftless. Thus, in order to establish the controllability of the vehicle we make use of the mathematical concepts that are involved in Lie algebra rank condition.

3.2.1 Controllability about a point

Consider the linear approximation of the system (3.13) at equilibrium point q_e while setting the input v_e equal to zero. Let the error associated with the equilibrium point be given as

$$\tilde{q} = q - q_e$$

The time derivative of the error is given as

$$\dot{\tilde{q}} = g_1(q_e)v_1 + g_2(q_e)v_2 + g_3(q_e)v_3 + g_4(q_e)v_4$$

$$\dot{\tilde{q}} = G(q_e)v \tag{3.14}$$

Here $G(q_e)$ is the controllability matrix at the equilibrium point. The rank of the controllability matrix is four. Thus if we linearize the system about an equilibrium point the linearized system is not controllable. Hence the linear controller will not work here. To test the controllability of the above system we make use of the lie algebra rank condition and nilpotent basis concepts.

Nilpotent basis The definition of nilpotent basis for a distribution is recalled here. Given a set of generators or basis vector fields $g_{1,}g_{2,}........., g_m$ we define the length of a Lie product recursively as

$$l\{g_i\} = 1; \qquad i = 1,2,.............., m$$

$$l([A,B]) = l[A] + l[B]$$

Where A and B are themselves Lie products. Alternatively, $l[A]$ is the number of generators in the expansion for A. A Lie algebra or basis is nilpotent if there exists an integer k such that all Lie products of length greater than k are zero. The integer k is called the order of nilpotency [17].The use of the nilpotent basis eliminates the need for cumbersome computations as we see all higher order lie brackets above some particular order are zero.

In the light of the above definition and conditions we see that the lie algebra $L\{g_1, g_2, g_3, g_4\}$ is nilpotent algebra of order 2 $(k = 2)$ i.e. the vector fields g_1, g_2, g_3 and g_4 are the nilpotent basis. Thus all lie brackets of order more than two are zero. The only independent Lie brackets computed from the four basis vector fields are $[g_1, g_3]$ and $[g_1, g_4]$. Thus for our system the lie algebra rank condition becomes

$$rank[C_c] = 6$$

$$rank[g_1, g_2, g_3, g_4, [g_1, g_3][g_1, g_4]] = 6 \tag{3.15}$$

where $[g_1, g_3]$ and $[g_1, g_4]$ are the two independent lie brackets computed from the four vector fields (g_1, g_2, g_3, g_4) as per the following definition.

$$[g, h](x) = \frac{\partial h}{\partial x} g - \frac{\partial g}{\partial x} h \tag{3.16}$$

Thus we have

$$[g_1, g_3] = \frac{\partial g_3}{\partial x} g_1 - \frac{\partial g_1}{\partial x} g_3 = \begin{bmatrix} \cos\psi \sin\theta \cos\phi + \sin\psi \sin\phi \\ \sin\psi \sin\theta \cos\phi - \cos\psi \sin\phi \\ \cos\theta \cos\phi \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$[g_1, g_4] = \frac{\partial g_4}{\partial x} g_1 - \frac{\partial g_1}{\partial x} g_4 = \begin{bmatrix} -\cos\psi \sin\theta \sin\phi + \sin\psi \cos\phi \\ -\sin\psi \sin\theta \sin\phi - \cos\psi \cos\phi \\ -\cos\theta \sin\phi \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Using the above expressions for the Lie brackets the controllability matrix C_c becomes.

$$C_c = \begin{bmatrix} \cos\psi \cos\theta & 0 & 0 & 0 & \cos\psi \sin\theta \cos\phi + \sin\psi \sin\phi & -\cos\psi \sin\theta \sin\phi + \sin\psi \cos\phi \\ \sin\psi \cos\theta & 0 & 0 & 0 & \sin\psi \sin\theta \cos\phi - \cos\psi \sin\phi & -\sin\psi \sin\theta \sin\phi - \cos\psi \cos\phi \\ -\sin\theta & 0 & 0 & 0 & \cos\theta \cos\phi & -\cos\theta \sin\phi \\ 0 & 1 & \sin\phi \tan\theta & \cos\phi \tan\theta & 0 & 0 \\ 0 & 0 & \cos\phi & -\sin\phi & 0 & 0 \\ 0 & 0 & \sin\phi \sec\theta & \cos\phi \sec\theta & 0 & 0 \end{bmatrix}$$

The above matrix has one nonzero minor of order 6.Thus the rank of the controllability matrix is full as long as $\theta \neq \pi/2$, which is the singularity of the system. Hence we conclude that the system is controllable locally and also globally as long as it avoids the singularity condition.

3.2.2 Controllability about a trajectory

For the nonlinear system $\dot{q} = G(q)v$ let the reference state trajectory be $q_d(t) = [x_d(t), y_d(t), z_d(t), \phi_d(t), \theta_d(t), \psi_d(t)]^T$ and the reference input trajectory be

$v_d(t) = v_{d1}(t), v_{d2}(t), v_{d3}(t), v_{d4}(t)$. The reference trajectory should satisfy the nonholonomic constraints on the system.

For the linear systems $\dot{x} = Ax + Bu$ controllability implies asymptotic (actually exponential) stabilization by smooth state feedback. Thus if the following accessibility rank condition $rank\left[B, AB, A^2B.............A^{n-1}B\right] = n$ is satisfied, then there exists a feedback gain so that the following control law

$$u = k(xd - x)$$

makes the desired trajectory xd asymptotically stable or in other words the error associated with the desired solution goes to zero exponentially.

For nonlinear systems the condition does not apply as such. But for local accessibility we may look at the approximate linearization of the system in the neighborhood of xd. Thus in particular if the linearized system is controllable, the nonlinear system can be stabilized locally at xd by a smooth feedback $u = kx_e$. The condition is sufficient but not necessary.

Let the errors associated with the desired state trajectory and input trajectory be denoted as $q_e(t) = q(t) - q_d(t)$ and $v_e(t) = v(t) - v_d(t)$ respectively. Linearizing the system about the desired trajectory we obtain the following system

$$\dot{q}(t) = \dot{q}_d(t) + \dot{q}_e(t)$$

$$= \{G(q_d + q_e, t)\}\{v_d(t) + v_e(t)\} \tag{3.17}$$

The Taylor series expansion of $G(q,t)$ about the nominal solution $q_d(t)$ is given as:

$$\dot{q}(t) = \left\{ G(q_d, t) + \left.\frac{\partial G(q)}{\partial q}\right|_{q = q_d} q_e(t) + h.o.t \right\}\{v_d(t) + v_e(t)\}$$

Since the nominal solution satisfies (3.16) we have:

$$\dot{q}_e(t) = \left\{ \left. \frac{\partial G(q)}{\partial q} \right|_{q=q_d} q_e(t) \right\} v_d(t) + G(q_d, t) v_e(t)$$

Or (3.17)

$$\dot{q}_e(t) = A(t) q_e(t) + B(t) v_e(t)$$

With

$$A(t) = \sum_{n=1}^{4} \left. \frac{\partial g_n}{\partial q} \right|_{q=q_d} v_{dn}(t) \qquad B(t) = G(q_d, t) \tag{3.18}$$

Upon computations we get

$$A(t) = \begin{bmatrix} O_{3\times3} & A_1(t) \\ O_{3\times3} & A_2(t) \end{bmatrix}$$

$$A_1(t) = \begin{bmatrix} 0 & -\cos\psi_d(t)\sin\theta_d(t)v_{d1}(t) & -\sin\psi_d(t)\cos\theta_d(t)v_{d1}(t) \\ 0 & -\sin\psi_d(t)\sin\theta_d(t)v_{d1}(t) & \cos\psi_d(t)\cos\theta_d(t)v_{d1}(t) \\ 0 & -\cos\theta_d(t)v_{d1}(t) & 0 \end{bmatrix}$$

$$A_2(t) = \begin{bmatrix} \cos\phi_d(t)\tan\theta_d(t)v_{d3}(t) - \sin\phi_d(t)\tan\theta_d(t)v_{d4}(t) & \sin\phi_d(t)\sec^2\theta_d(t)v_{d3}(t) + \cos\phi_d(t)\sec^2\theta_d(t)v_{d4}(t) & 0 \\ -\sin\phi_d(t)v_{d3}(t) - \cos\phi_d(t)v_{d4}(t) & 0 & 0 \\ \cos\phi_d(t)\sec\theta_d(t)v_{d3}(t) - \sin\phi_d(t)\sec\theta_d(t)v_{d4}(t) & \sin\phi_d(t)\sec\theta_d(t)\tan\theta_d(t)v_{d3}(t) + \cos\phi_d(t)\sec\theta_d(t)\tan\theta_d(t)v_{d4}(t) & 0 \end{bmatrix}$$

$$B(t) = \begin{bmatrix} J_1(t) & O_{3\times3} \\ O_{3\times1} & J_2(t) \end{bmatrix}$$

Here

$$J_1(t) = \begin{bmatrix} \cos\theta_d(t)\cos\psi_d(t) \\ \cos\theta_d(t)\sin\psi_d(t) \\ -\sin\theta_d(t) \end{bmatrix}$$

$$J_2(t) = \begin{bmatrix} 1 & \sin f_d(t)\tan q_d(t) & \cos f_d(t)\tan q_d(t) \\ 0 & \cos f_d(t) & -\sin f_d(t) \\ 0 & \sin f_d(t)\sec q_d(t) & \cos f_d(t)\sec q_d(t) \end{bmatrix}$$

The above system is linear time varying. For a linear reference trajectory with constant velocities $v_{dn}(t) = v_{dn}$ and $q_d(0) = q(0)$ the controllability condition becomes

$$rank\left\{B, AB, A^2B, A^3B, A^4B, A^5B\right\} = 6$$

Upon computations we see that the above matrix has a nonzero minor of order 6 as long as $v_{d1} = v_{d3} = v_{d4} \neq 0$ and $q_d \neq p/2$. Thus the linearized system is controllable along a reference trajectory as long as the trajectory does not collapse to a point.

3.3 Chained forms

The chained form systems were first introduced in [18].The chained form introduced in [18] had one chain i.e. two input chained forms. The method for converting the multi input drift free non holonomic systems into chained forms is given in [19]. Sufficient conditions under which an m input system of (3.13) can be transformed to $m-1$ chain single generator form were given in [19]. The method presented in [19] for transforming is similar to the method of exact linearization of nonlinear systems with drift via state feedback as presented in [14]. The same method will be applied for our system with some modifications.

The following system

$$\dot{q} = g_1(q)v_1 + g_2(q)v_2 + g_3(q)v_3 + g_4(q)v_4$$

can be transformed into the chained form by a feedback transformation. In the above equation g_i being smooth and linearly independent vector fields, there exists a feedback transformation $(x, a, h, g) = \Phi(q)$ and $v = b(q)$ that transforms our system into the following chained form [20].

$$\dot{x}_0 = \dot{x}_1^0 = u_1 \qquad \dot{a}_0 = \dot{x}_2^0 = u_2 \qquad \dot{h}_0 = \dot{x}_3^0 = u_3 \qquad \dot{g}_0 = \dot{x}_4^0 = u_3$$

$$\dot{a}_1 = \dot{x}_{n2}^0 = a_0 u_1 \qquad \dot{h}_1 = \dot{x}_{n3}^0 = h_0 u_1 \tag{3.19}$$

where $\dot{x}_j^k$ is the state for the kth level. We call this as a chained form because the derivative of each state depends on the state directly above it in a chained fashion. The form has input u_1 as the generator for the chains and $\sum_{j=2}^{4} n_j = 2$. Thus for our system $n_2 = n_3 = 1$ and $n_4 = 0$.

There exists a basis function f_1, f_2, f_3, f_4 , for the distribution $\Delta_0 = span\begin{pmatrix} g_1 & g_2 & g_3 & g_4 \end{pmatrix}$ having the form

$$f_1 = \frac{\partial}{\partial q_1} + \sum_{i=2}^{6} f_i(q) \frac{\partial}{\partial q_i}$$

$$f_2 = \sum_{i=2}^{6} f_2^i(q) \frac{\partial}{\partial q_i}$$

$$f_3 = \sum_{i=2}^{6} f_3^i(q) \frac{\partial}{\partial q_i}$$

$$f_4 = \sum_{i=2}^{6} f_4^i(q) \frac{\partial}{\partial q_i} \tag{3.20}$$

The basis function is such that the following distributions

$$G_0 = span\{ f_2, f_3, f_4 \}$$

$$G_1 = span\{ f_2, f_3, f_4, [f_1, f_2], [f_1, f_3], [f_1, f_4]\}$$

:

:

$$G_5 = span\{ad_{f_1}^i f_2, ad_{f_1}^i f_3, ad_{f_1}^i f_4; 0 \le i \le 5\}$$

with

$$ad_{f_1}^k f_2 := \left[f_1, ad_{f_1}^{k-1} f_2 \right] \qquad ad_{f_1}^0 f_2 := f_2$$

have constant dimension on the same open set $U \in R^n$, are all involutive and G_5 has dimension 5 on U.

The vector fields f_1, f_2, f_3 and f_4 which satisfy these conditions are

$$f_1 = \frac{g_1}{\cos y \cos q} = \begin{bmatrix} 1 \\ \tan y \\ -\tan q \sec y \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$f_2 = g_2;\ f_3 = g_3 \text{ and } f_4 = g_4 \tag{3.21}$$

The coordinate transformation for the system thus is

$$x_0 = h_1 \qquad a_0 = L_{f_1}^1 h_1 \qquad h_0 = L_{f_1}^0 h_3 \qquad g_0 = h_4$$

$$a_1 = L_{f_1}^0 h_2 \qquad h_1 = h_3 \tag{3.22}$$

where h_1, h_2, h_3, h_4 are the smooth functions such that the following conditions are met

$$dh_1 \perp G_j; \qquad 0 \leq j \leq 5$$

and the distribution G_0 is annihilated by dh_1, dh_2, $dL_{f_1}^0 h_2$, $dL_{f_1}^1 h_2$, dh_3, $dL_{f_1}^0 h_3$, $dL_{f_1}^1 h_3$ dh_4. Here $L_{f_1} h_3$ is the Lie derivative of h_3 with respect to f_1. The detailed proof of the above conditions is in [19]. Here it should also be noted that the choice of functions h_1, h_2, h_3, h_4 is not unique. Choosing

$$h_1 = x \qquad h_2 = y \qquad h_3 = z$$

$$h_4 = \frac{1}{1 + trace(R)} (r_{32} - r_{23})$$

with R being the rotation matrix and $trace(R) = (r_{11} + r_{22} + r_{33})$. Thus the coordinate transformation for the system becomes

$$x_1 = x_0 = x$$

$$x_2 = a_0 = \tan y$$

$$x_3 = a_1 = y$$

$$x_4 = h_0 = -\tan q \sec y$$

$$x_5 = z$$

$$x_6 = \frac{1}{1 + trace(R)} (r_{32} - r_{23}) \tag{3.23}$$

which gives the following chained form system

$$\dot{x}_1 = u_1$$

$$\dot{x}_2 = u_2$$

$$\dot{x}_3 = x_2 u_1$$

$$\dot{x}_4 = u_3$$

$$\dot{x}_5 = x_4 u_1$$

$$\dot{x}_6 = u_4 \tag{3.24}$$

Solving (3.23) and (3.24) and inserting (2.8) we get the input transformation as

$$u_1 = \dot{x}_1$$

$$u_2 = \dot{x}_2$$

$$u_3 = \dot{x}_4$$

$$u_4 = \dot{x}_6$$

Or (3.25)

$$u_1 = \cos y \ \cos q n_1$$

$$u_2 = \sec^2 y \ \sin f \ \sec q n_3 + \sec^2 y \ \cos f \ \sec q n_4$$

$$u_3 = \frac{(-\sin y \ \sin f \ \sin q - \cos y \ \cos f)}{\cos^2 y \ \cos^2 q} n_3 + \frac{(-\sin y \ \cos f \sin q - \cos y \ \sin f)}{\cos^2 y \ \cos^2 q} n_4$$

$$u_4 = \frac{[(1+\cos y \ \cos q)n_2 + (\cos y \ \sin q \sin f - \sin y \ \cos f)n_3 + (\sin y \ \sin q \cos f + \sin y \ \sin f)n_4]}{1+\cos y \ \cos q + \sin y \ \sin q \sin f + \cos y \ \cos f + \cos q \cos f}$$

(3.25) gives

$$v_x = v_1 = \frac{u_1}{\cos y \ \cos q}$$

$$w_y = v_3 = \cos y \ \cos q \left\{ (\cos y \ \sin f - \sin y \ \sin q \cos f) u_2 - (\cos f \ \cos q) u_3 \right\}$$

$$w_z = v_4 = \cos y \ \cos q \left\{ (\sin y \ \sin q \sin f + \cos y \ \cos f) u_2 + (\sin f \ \cos q) u_3 \right\}$$

$$\begin{aligned} w_x = v_2 = \frac{1}{\cos y \ \cos q} \{ & (1 + \cos y \ \cos q + \sin y \ \sin q \sin f + \cos y \ \cos f + \cos q \cos f) u_4 \\ & - (\cos y \ \sin q \sin f - \sin y \ \cos f) v_3 - (\sin y \ \sin q \cos f + \sin y \ \sin f) v_4 \} \end{aligned}$$

The inputs v_1, v_2, v_3, v_4 can be calculated from the above equations provided $\cos y \ \cos q \neq 0$. Also here it should be noted that the chained form system is completely controllable as the controllability is not affected by state feedback and coordinate transformations i.e. they are invariant under the transformations.

Chapter 4

Control Design and Simulation Results

In this chapter controllers will be designed for the vehicle to track a desired trajectory, follow a path and for point to point stabilization. The chapter presents the control design and the simulation results obtained for the model of an underwater vehicle developed in the previous chapter. The feedback control design is developed using the kinematic model of the system. The performance of the controllers obtained using various techniques of control design is evaluated for different motion planning tasks mentioned above. The chapter also presents the simulation results obtained for different controllers. The simulation results are used to compare and evaluate the performance of the various controllers.

4.1 Trajectory tracking and controller design

The system is supposed to track a given (desired) Cartesian trajectory .The problem is to regulate both the vehicles position and orientation with respect to that of a reference system: the trajectory of which is parameterized by the variable 't'. The goal will be achieved using feedback control law with the following control schemes

- Full state feedback using approximate linearization
- Feedback linearization using input output linearization or full state linearization

Before going for the feedback design the problem of generating the desired output trajectory is discussed both for original system and chained form system.

4.2 Reference trajectory generation

Let the reference state trajectory and reference input trajectory for the system be $q_d(t) = x_d(t) y_d(t), z_d(t), f_d(t), q_d(t), y_d(t)$ and $v_d(t) = v_{d1}(t), v_{d2}(t), v_{d3}(t), v_{d4}(t)$. The desired trajectory is feasible only when it satisfies the nonholonomic constraints on the system.

Assume that a feasible and smooth desired output trajectory for the chained form is given as $x_{d1} = x_{d1}(t), x_{d3} = x_{d3}(t), x_{d5} = x_{d5}(t)$ and $x_{d6} = x_{d6}(t)$. From this information we are able to derive the time evolution of the rest of the coordinates of the state trajectory and the associated input trajectory. In other words we should be able to recover the state trajectory and the input trajectory from the reference output trajectory.

From (3.20) we have

$$\dot{x}_{d1}(t) = u_{d1}(t)$$

$$\dot{x}_{d2}(t) = u_{d2}(t)$$

$$\dot{x}_{d3}(t) = x_{d2}(t) u_{d1}(t)$$

$$\dot{x}_{d4}(t) = u_{d3}(t)$$

$$\dot{x}_{d5}(t) = x_{d4}(t) u_{d1}(t)$$

$$\dot{x}_{d6}(t) = u_{d4}(t) \tag{4.1}$$

with initial conditions of the states as $x_{d1}(t_0), x_{d2}(t_0), x_{d3}(t_0), x_{d4}(t_0), x_{d5}(t_0), x_{d6}(t_0)$ at $t = t_0$.

Solving for the state trajectory from (4.1) we get

$$x_{d2}(t) = \dot{x}_{d3}(t) / \dot{x}_{d1}(t)$$

$$x_{d4}(t) = \dot{x}_{d5}(t) / \dot{x}_{d1}(t) \tag{4.2}$$

The corresponding input trajectory is given as

$$u_{d1}(t) = \dot{x}_{d1}(t)$$

$$u_{d2}(t) = \dot{x}_{d2}(t) = (\dot{x}_{d1}(t)\ddot{x}_{d3}(t) - \dot{x}_{d3}(t)\ddot{x}_{d1}(t)) / \dot{x}_{d1}^{2}(t)$$

$$u_{d3}(t) = \dot{x}_{d4}(t) = (\dot{x}_{d1}(t)\ddot{x}_{d5}(t) - \dot{x}_{d5}(t)\ddot{x}_{d1}(t)) / \dot{x}_{d1}^{2}(t)$$

$$u_{d4}(t) = \dot{x}_{d6}(t) \tag{4.3}$$

(4.2.) and (4.3) gives the unique state and input trajectory, from which the desired output trajectory can be reproduced or generated. As is seen, the values of the trajectories depend upon the values of the output trajectory and its second order derivatives. Thus the output trajectory should be differentiable everywhere. The derivation of the reference input and state trajectory which generates a desired output trajectory can also be performed on the original system. The original state and input trajectories can be derived from the output trajectory as

$$x_d(t) = x_{d1}(t)$$

$$y_d(t) = x_{d3}(t)$$

$$z_d(t) = x_{d5}(t)$$

$$y_d = \tan^{-1}(x_{d2}) = \tan^{-1}(\dot{y}_d / \dot{x}_d)$$

$$q_d = -\tan^{-1}(x_{d4} \cos y_d) = -\tan^{-1}(\dot{z}_d \cos y_d / \dot{x}_d)$$

$$f_d = \cot^{-1}\left(\cot q_d / \sin y_d + \tan y_d / \sin q_d\right) \tag{4.4}$$

Similarly the actual input trajectory is

$$v(d_1) = \sqrt{\dot{x}_d^2 + \dot{y}_d^2 + \dot{z}_d^2} = ud_1 / \cos y_d \cos q_d$$

$$v(d_3) = -r_{d11}\left(r_{d23}ud_2 + r_{d33}ud_3\right)$$

$$v(d_4) = r_{d11}\left(r_{d22}ud_2 + r_{d32}ud_3\right)$$

$$v(d_2) = \frac{1}{1+r_{d11}}\left((1 + r_{d11} + r_{d22} + r_{d33})ud_4 - r_{d12}vd_3 - r_{d13}vd_4\right) \tag{4.5}$$

with

$$r_{d11} = \cos q_d \cos y_d$$

$$r_{d12} = \sin q_d \sin j_d \cos y_d - \cos j_d \sin y_d$$

$$r_{d13} = \sin q_d \cos j_d \cos y_d + \sin j_d \sin y_d$$

$$r_{d21} = \cos q_d \sin y_d \; r_{d22} = \sin q_d \sin j_d \sin y_d + \cos j_d \cos y_d$$

$$r_{d23} = \sin q_d \cos j_d \sin y_d - \sin j_d \cos y_d$$

$$r_{d31} = -\sin q_d$$

$$r_{d32} = \sin j_d \cos q_d$$

$$r_{d33} = \cos j_d \cos q_d \tag{4.6}$$

For the tracking simulation purposes consider the following reference sinusoidal output trajectory

$$x_{d1}(t) = t \qquad x_{d3}(t) = A\sin wt \qquad x_{d5}(t) = 1 \qquad x_{d6}(t) = 0 \tag{4.7}$$

This gives the state trajectory as

$$x_{d2}(t) = Aw\cos wt, \qquad x_{d4}(t) = 0 \tag{4.8}$$

and the input trajectory as

$$u_{d1}(t) = 1 \qquad u_{d2}(t) = -Aw^2 \sin wt$$

$$u_{d3}(t) = 0 \qquad u_{d4}(t) = 0 \tag{4.9}$$

The initial states are

$$x_{d1}(0) = 0 \qquad x_{d2}(0) = Aw \qquad x_{d3}(0) = 0$$

$$x_{d4}(0) = 0 \qquad x_{d5}(0) = 1 \qquad x_{d6}(0) = 0 \tag{4.10}$$

Here again it is to be noted that there is a singularity in the state and input trajectories at $\dot{x}_{d1}(t) = 0$ or $u_{d1}(t) = 0$ as the state and input trajectories are not defined at that point.

4.3 Control using approximate linearization.

The feedback controller for trajectory tracking is based on standard linear control theory. The design makes use of the approximate linearization of the system equations about desired trajectory which leads to a time varying system as seen before. The method here is illustrated for the chained form equations about the desired trajectory. The chained form system is linear under piecewise constant inputs.

For the chained form system the desired state and input trajectory computed in correspondence to the reference cartesian trajectory is

$$x_d(t) = \{x_{d1}(t), x_{d2}(t), x_{d3}(t), x_{d4}(t), x_{d5}(t), x_{d6}(t)\}$$

and (4.11)

$$u_d(t) = \{u_{d1}(t), u_{d2}(t), u_{d3}(t), u_{d4}(t)\}$$

An equivalent way to state the tracking problem is to require the difference between the actual configuration and the desired configuration approach to zero. This difference is denoted as the error. Since the vehicle will not necessarily share the same initial conditions as the desired system, the tracking controller will drive the error to zero and minimize the effect of the disturbances as the vehicle converges to the reference trajectory.

In order for the system to track the trajectory of error should approach to zero with time. Denoting the error variables for states and inputs as the following

$$x_e = x - x_d$$

and (4.12)

$$u_e = u - u_d$$

The error differential equations are written by subtracting the desired equations from the system (actual) equations as the following nonlinear set of equations

$$\dot{x}_{e1} = u_{e1}$$

$$\dot{x}_{e2} = u_{e2}$$

$$\dot{x}_{e3} = x_2 u_1 - x_{d2} u_{d1}$$

$$\dot{x}_{e4} = u_{e3}$$

$$\dot{x}_{e5} = x_4 u_1 - x_{d4} u_{d1}$$

$$\dot{x}_{e6} = u_{e4} \tag{4.13}$$

Now linearizing about the desired trajectory we have the following linear system

$$\dot{x}_e(t) = A(t)x_e(t) + B(t)u_e(t) \tag{4.14}$$

with

$$A(t)=\begin{bmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&u_{d1}(t)&0&0&0&0\\0&0&0&0&0&0\\0&0&0&u_{d1}(t)&0&0\\0&0&0&0&0&0\end{bmatrix}$$

$$B(t)=\begin{bmatrix}1&0&0&0\\0&1&0&0\\x_{d2}(t)&0&0&0\\0&0&1&0\\x_{d4}(t)&0&0&0\\0&0&0&1\end{bmatrix}$$

The system given by (4.14) is linear time varying and can easily be proven to be controllable by checking its Grammian [21] to be nonsingular. For a linear trajectory with constant velocity $u_{d1}(t)=u_{d1}$ the controllability condition is given by

$$rank\left\{B, AB, A^2B, A^3B, A^4B, A^5B\right\}=6 \tag{4.15}$$

The matrix in (4.15) is a nonsingular matrix and has at least one non zero minor of order six. The controllability matrix is nonsingular only as long as the input u_{d1} to the system is nonzero. This corresponds to the singularity in the kinematic model of the system. Thus the system is controllable as long as $u_{d1}\neq 0$.

Choosing the linear time varying feedback law for the system as

$$u_e=-Kx_e \tag{4.16}$$

For the chained form system the control law should be such that feedback law for each chain contains the same number of terms as the number of states in that chain. Thus

$$u_{e1}=-k_1x_{e1}$$

$$u_{e2} = -k_2 x_{e2} - \frac{k_3}{u_{d1}} x_{e3}$$

$$u_{e3} = -k_4 x_{e4} - \frac{k_5}{u_{d1}} x_{e5}$$

$$u_{e4} = -k_6 x_{e6} \tag{4.17}$$

The feedback coefficients k_3 and k_5 are divided by u_{d1} so that the characteristic equation of the closed loop system matrix does not contain u_{d1}, thus making the design global.

Thus the matrix K is given as

$$u_{e4} = \begin{bmatrix} k_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & k_2 & k_3/u_{d1} & 0 & 0 & 0 \\ 0 & 0 & 0 & k_4 & k_5/u_{d1} & 0 \\ 0 & 0 & 0 & 0 & 0 & k_6 \end{bmatrix} \tag{4.18}$$

The $k's$ are chosen such that k_1 and k_6 are positive, and k_2, k_3, k_4 and k_5 are such that $l^2 + k_2 l + k_3$ and $l^2 + k_4 l + k_5$ are Hurwitz. The closed loop system matrix is thus given as

$$A_{cl} = A - BK$$

$$= \begin{bmatrix} k_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -k_2 & -k_3/u_{d1}(t) & 0 & 0 & 0 \\ x_{d2}k_1 & u_{d1}(t) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -k_4 & -k_5/u_{d1}(t) & 0 \\ x_{d4}k_1 & 0 & 0 & u_{d1}(t) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & k_6 \end{bmatrix} \tag{4.19}$$

The closed-loop system matrix has constant eigenvalues with negative real parts. This does not guarantee the asymptotic stability of the closed- loop time varying system [26].However, for specific choices of $u_{d1}(t)$, bounded away from zero and $u_{d2}(t)$,

$u_{d3}(t)$ and $u_{d4}(t)$, it is possible to use results on slowly –varying linear systems in order to prove asymptotic stability. The feedback matrix is obtained by using pole placement technique. Since for stability the eigen values of the closed loop matrix should have negative real parts, the characteristic equation of the system should satisfy the following

$$\det(sI - A_{cl}) = (s - p_1)(s - p_2)(s - p_3)(s - p_4)(s - p_5)(s - p_6)$$

where p_i ; $i = 1,2,......,6$ are the eigen values of the system. The resulting closed loop system (4.19) is controllable with the choice of feedback in (4.17).

4.3.1 Simulation of the controller

For simulation the following sinusoidal trajectory is chosen:

$$x_d(t) = t \qquad y_d(t) = a\sin wt \qquad z_d(t) = 1$$

which gives the following desired values for the chained form states and inputs

$$x_{d1}(t) = t \qquad x_{d2}(t) = aw\cos wt \qquad x_{d3}(t) = a\sin wt \qquad x_{d4}(t) = 0$$

$$x_{d5}(t) = 1 \qquad x_{d6}(t) = 0$$

and

$$u_{d1}(t) = 1 \qquad u_{d2}(t) = -aw^2\sin wt \qquad u_{d3}(t) = 0 \qquad u_{d4}(t) = 0$$

The initial conditions for the states are

$$x_{d1}(0) = 1 \qquad x_{d2}(0) = aw \qquad x_{d3}(0) = 2 \qquad x_{d4}(0) = 0$$

$$x_{d5}(0) = 3 \qquad x_{d6}(0) = 0$$

Choosing the six coincident closed loop poles at -2, i.e.

$$p_1 = p_2 = p_3 = p_4 = p_5 = p_6 = -2$$

we get the feedback matrix coefficients as

$$k_1 = k_6 = 2 \quad \text{and} \quad k_2 = k_3 = k_4 = k_5 = 4$$

Choosing $a = 1$ and $w = p$ we get the simulation results. The results show the tracking errors for chained form states and inputs; and for actual states and inputs. Once the tracking errors go to zero, the actual control inputs as obtained from the chained form variables and inputs are same as the computed desired inputs. The desired inputs are computed from actual system variables from (4.5). Since the control design is based on the linearization of the system, the controller will make the controlled system locally asymptotically stable.

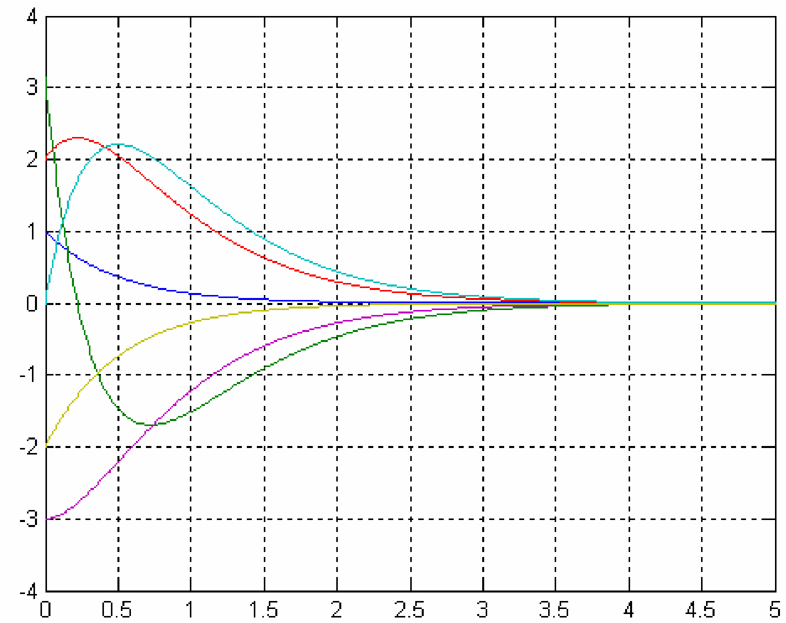

Figure4. 1: The result of approximate linearization: tracking errors in chained form variables x_1 (blue), x_2 (green), x_3 (red), x_4 (cyan), x_5 (magenta), x_6 (yellow), vs. time (sec).

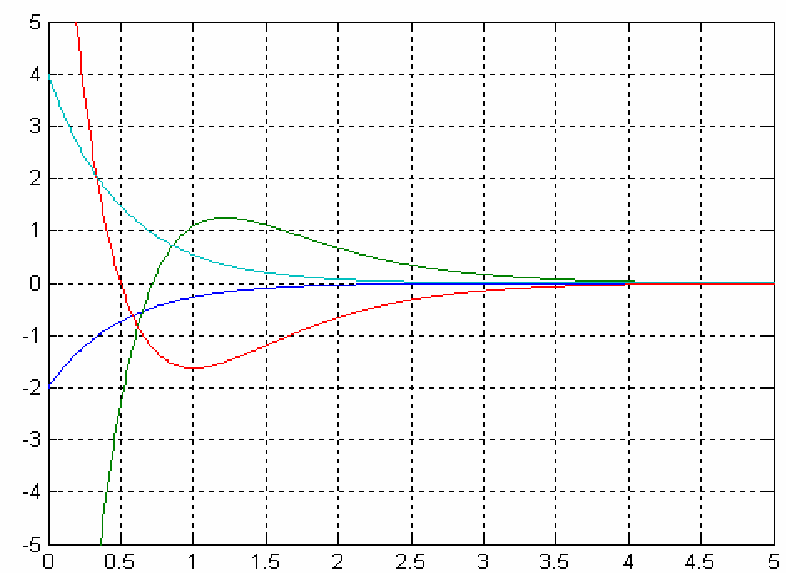

Figure4. 2: The result of approximate linearization: tracking errors (m/sec) in chained form inputs u_1 (blue), u_2 (green), u_3 (red), u_4 (cyan) vs. time (sec).

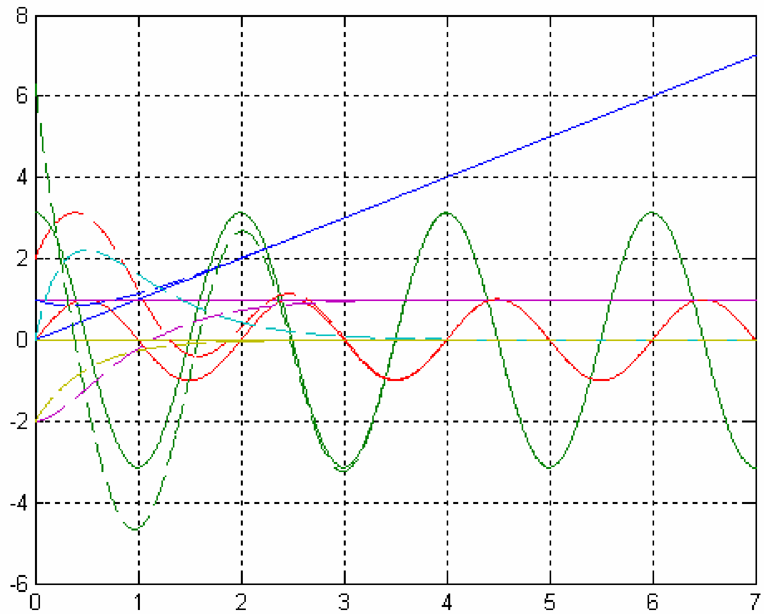

Figure4. 3: The result of approximate linearization: actual (--) and desired ($-$) chained form variables x_1 (blue), x_2 (green), x_3 (red), x_4 (cyan), x_5 (magenta), x_6 (yellow), vs. time (sec).

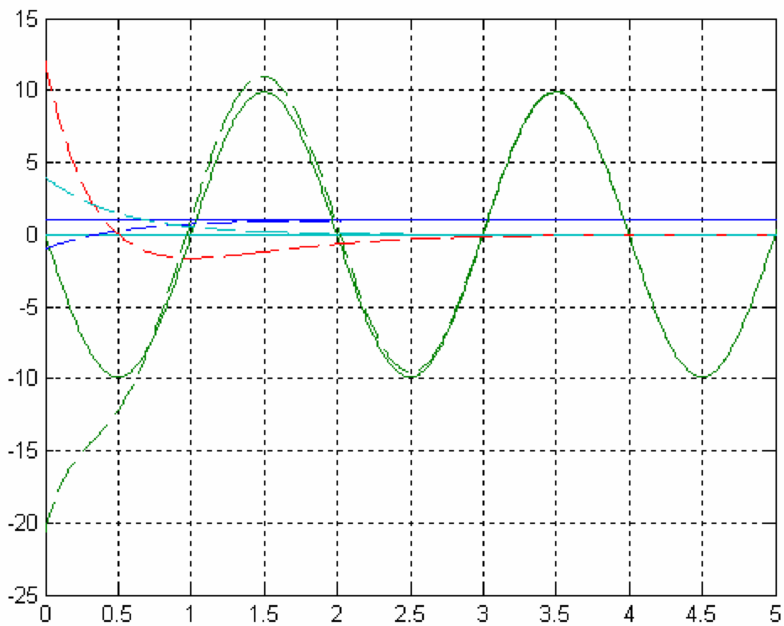

Figure4. 4: The result of approximate linearization: actual (--) and desired ($-$) chained form inputs u_1 (blue), u_2 (green), u_3 (red), u_4 (cyan) vs. time (sec).

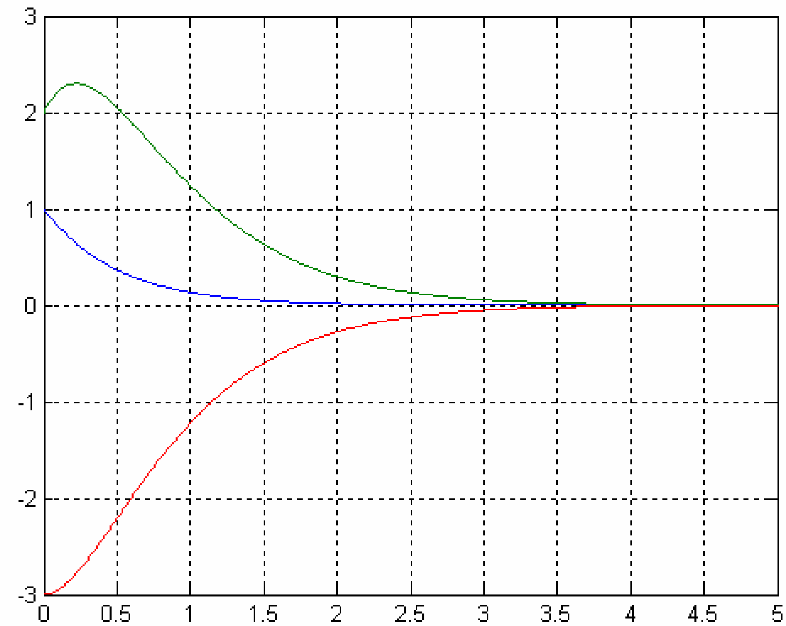

Figure4. 5: The result of approximate linearization: tracking errors (m) in variables x (blue), y (green), z (red) vs. time (sec).

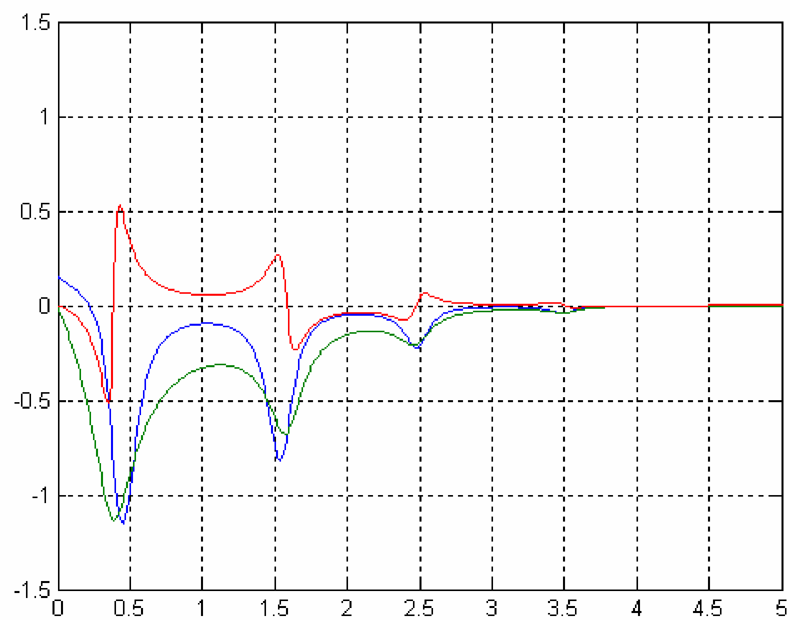

Figure4. 6: The result of approximate linearization: tracking errors (rad) in variables y (blue), q (green), f (red) vs. time (sec).

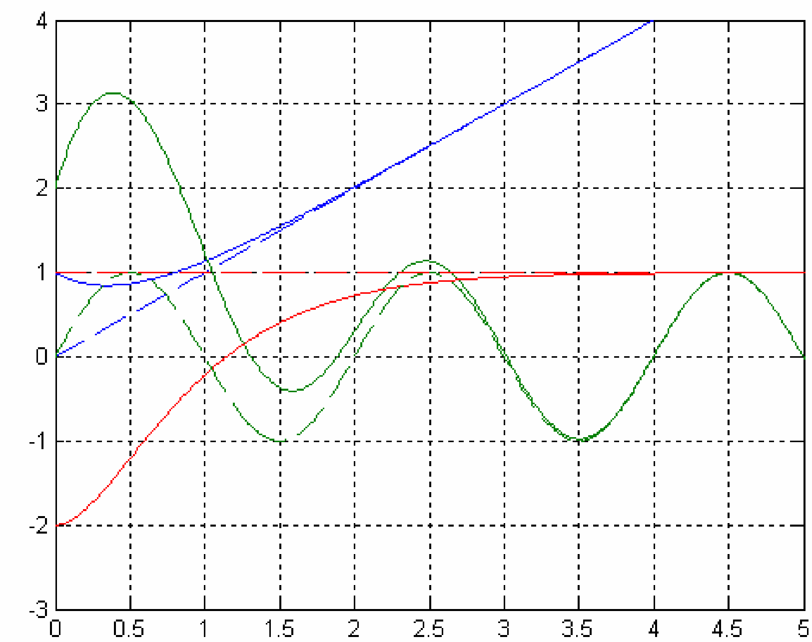

Figure4. 7: The result of approximate linearization: actual (--) and desired (−) original variables x (blue), y (green), z (red) vs. time (sec).

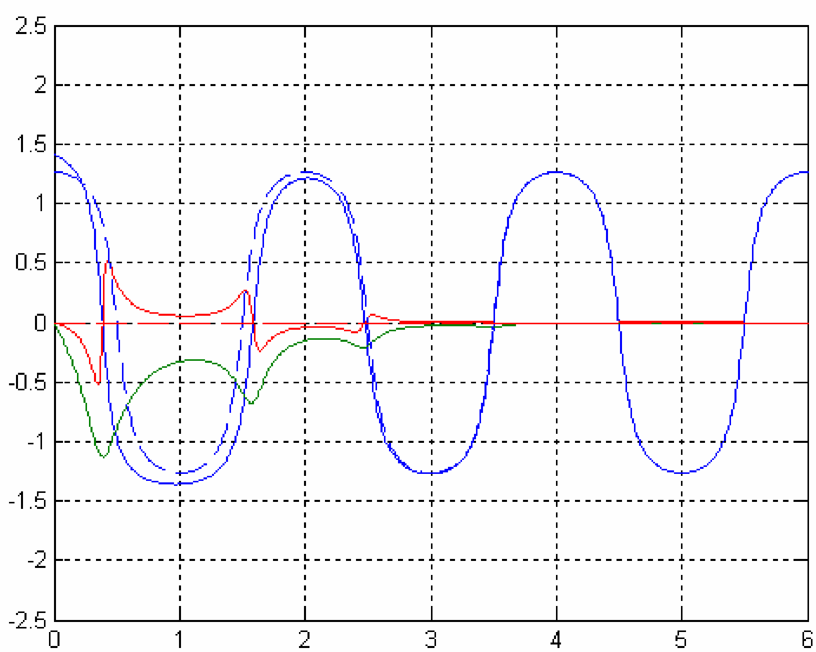

Figure4. 8: The result of approximate linearization: actual (--) and desired (−) original variables y (blue), q (green), f (red) vs. time (sec).

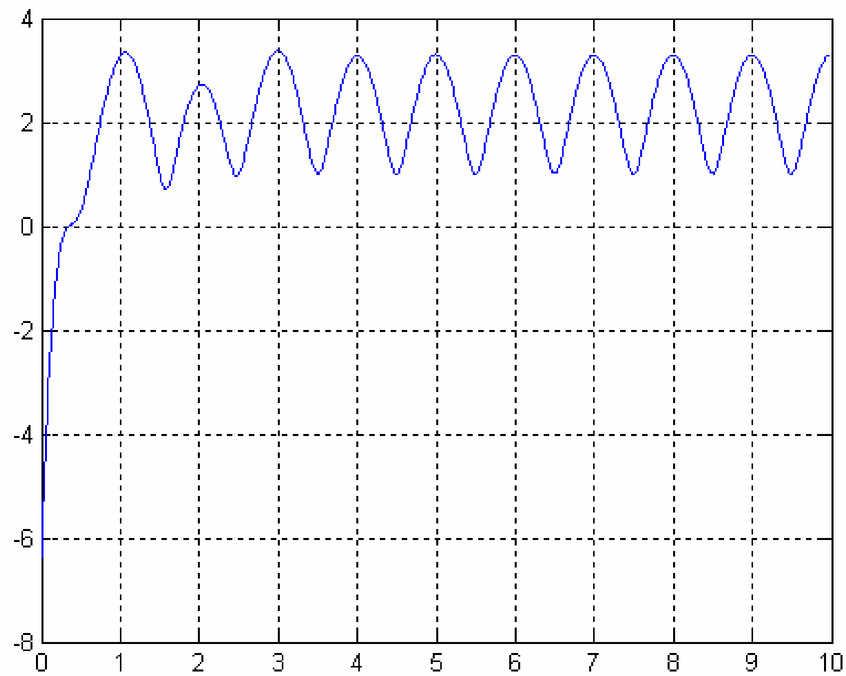

Figure4. 9: The result of approximate linearization: v_1 (m/sec) vs. time (sec).

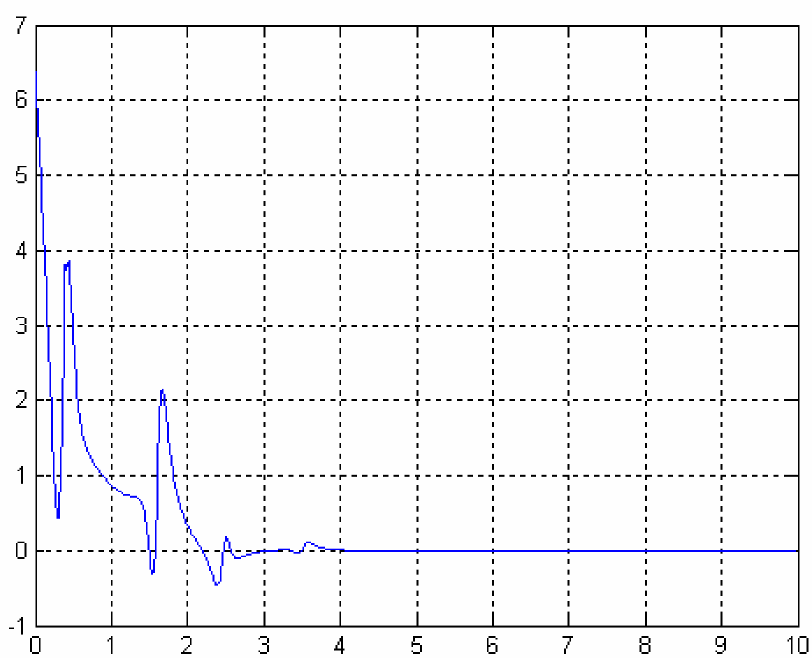

Figure4. 10: The result of approximate linearization: v_2 (rad/sec) vs. time (sec).

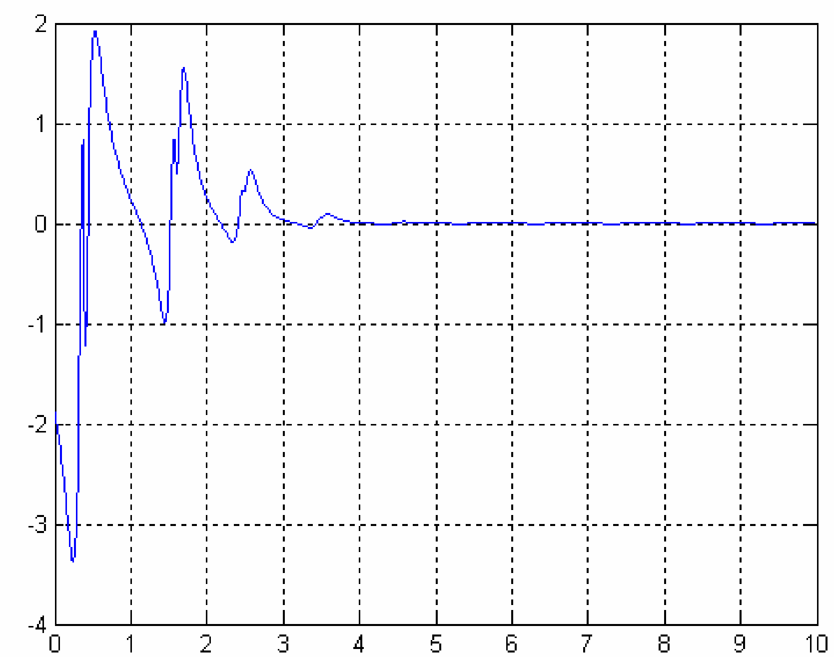

Figure4. 11: The result of approximate linearization: v_3 (rad/sec) vs. time (sec).

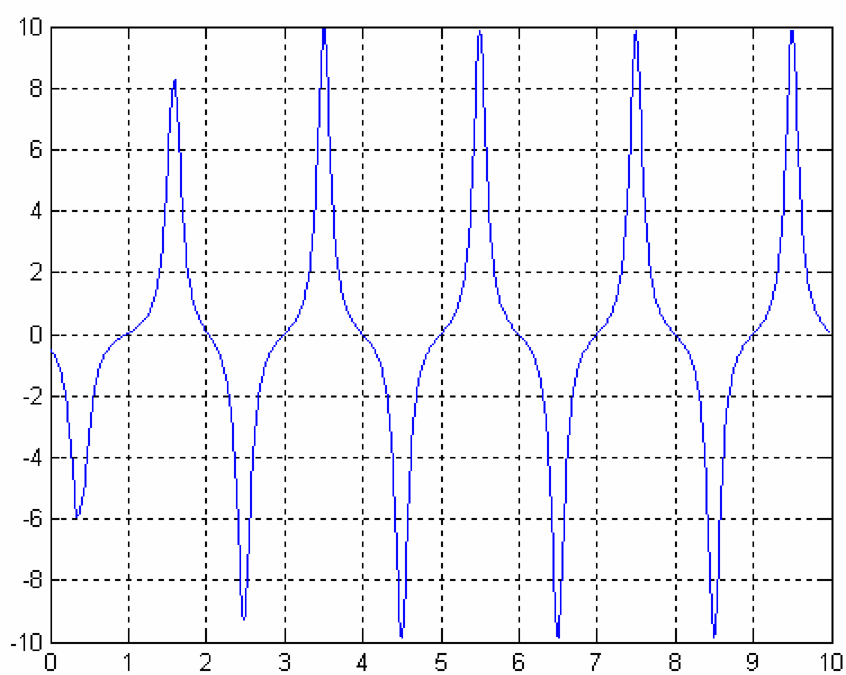

Figure4. 12: The result of approximate linearization: v_4 (rad/sec) vs. time (sec).

4.4 Control using exact feedback linearization

In this section nonlinear feedback design is used for the global stabilization of the tracking error associated with the trajectory. For nonlinear systems two types of exact linearization methods are generally used. One is full state feedback transformation in which the differential equations of the system are transformed into a linear system. Another is the input-output linearization which results in the input-output differential map being linear. Both the feedback problems can either be solved using the static or the dynamic feedback.

For the nonholonomic driftless system $\dot{q} = G(q)v$ the full state linearization of the system can not be achieved by using a smooth static (time invariant) state feedback. The reason for this is the controllability condition given by $rank[g_1, g_2, g_3, g_4, [g_1, g_3][g_1, g_4]] = 6$. This means that the distribution generated by the vector fields g_1, g_2, g_3, g_4 is not involutive which violates the necessary condition for full static state feedback linearization [13]. Thus for exact linearization, the method of dynamic state feedback is used.

For the above non linear system, dynamic feedback linearization consists of finding a dynamic feedback compensator of the form

$$\dot{z} = a(q, z) + b(q, z)r$$

$$u = c(q, z) + d(q, z)r \tag{4.20}$$

The state vector z is the compensator state whose dimensions depend upon the number of integrators added on the input channels. The vector r is the auxiliary input vector which is the new input to the integrators added.

The starting point of the dynamic extension for our problem is to define an m (4) dimensional output $z = h(q)$. A certain desired behavior is assigned to this output vector. The output vector is then successively differentiated until each and every input in the system appears and the invertible map (or matrix) is non singular. During the successive differentiations of the output vector, it becomes necessary to add the chain of integrators on inputs so as to avoid their direct differentiation. The number of integrators results in the compensator state vector z. The inputs to these integrators become the new auxiliary input vector r. The process continues and terminates after a finite number of differentiations if the system is invertible from the chosen output vector z. If the sum of the orders of the output differentiations is equal to the sum of the order of the original system (n) and the dimensions of the compensator, then the full state linearization is achieved in the sense that no internal dynamics are left in the system. The process also results in the decoupling of the output vector from the new auxiliary input.

If at some point of differentiation of output in the algorithm the decoupling matrix of the system is non singular without the addition of any compensator state, the process results in the input output linearization of the system. The static feedback law of the form

$$u = a(q) + b(q)r \tag{4.21}$$

is used to linearize the system.

4.4.1 Control using exact feedback linearization via static feedback

For our system let the output vector in the chained form be defined as

$$z = \begin{pmatrix} x_1 \\ x_3 \\ x_5 \\ x_6 \end{pmatrix} \tag{4.22}$$

The derivative of the output is given as

$$\dot{z} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_3 \\ \dot{x}_5 \\ \dot{x}_6 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ x_2 & 0 & 0 & 0 \\ x_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}$$

or (4.23)

$$\dot{z} = H(q)u$$

The inputs u_2 and u_3 do not appear after differentiation and also the decoupling matrix $H(q)$ is singular and not invertible. Hence the static feedback cannot be applied and the system cannot be linearized by input output linearization. The same result follows if the above procedure is repeated by choosing the actual state variable as the output vector.

4.4.2 Control using exact feedback linearization via dynamic feedback

Since the static feedback fails to solve the problem, we will be making use of dynamic feedback extension. For the linearization via dynamic feedback let us again define the linearizing output vector for the chained form as

$$z = \begin{pmatrix} x_1 \\ x_3 \\ x_5 \\ x_6 \end{pmatrix} \tag{4.24}$$

Differentiating w.r.t time we get

$$\dot{z} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_3 \\ \dot{x}_5 \\ \dot{x}_6 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ x_2 & 0 & 0 & 0 \\ x_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} \tag{4.25}$$

In order for the algorithm to proceed we need to add two integrators whose states are denoted by z_1 and z_2 on the inputs u_1 and u_4 respectively so that their differentiation in the next step is avoided. Thus

$$u_1 = z_1 \qquad u_4 = z_2$$

$$\dot{z}_1 = u_1' \qquad \dot{z}_2 = u_4' \tag{4.26}$$

where u_1' and u_4' are the new auxiliary inputs on the system. Substituting the above values and differentiating the output vector again we obtain

$$\ddot{z} = \begin{pmatrix} \dot{u}_1 \\ \dot{x}_2 u_1 + x_2 \dot{u}_1 \\ \dot{x}_4 u_1 + x_4 \dot{u}_1 \\ \dot{u}_4 \end{pmatrix} = \begin{pmatrix} \dot{z}_1 \\ \dot{x}_2 z_1 + x_2 \dot{z}_1 \\ \dot{x}_4 u_1 + x_4 \dot{z}_1 \\ \dot{z}_2 \end{pmatrix}$$

or (4.27)

$$\ddot{z} = \begin{pmatrix} u_1' \\ z_1 u_2 + x_2 u_1' \\ z_1 u_3 + x_4 u_1' \\ u_4' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ x_2 & z_1 & 0 & 0 \\ x_4 & 0 & z_1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u_1' \\ u_2 \\ u_3 \\ u_4' \end{pmatrix}$$

In the above equations all the inputs appear in a nonsingular way i.e. the decoupling matrix is nonsingular. The value of the determinant of the matrix is z_1^2. Thus the algorithm terminates after two differentiations. The matrix is nonsingular only as long as $z_1 \neq 0$ or $u_1 \neq 0$. Here the order of the compensator is two (*b*=2) and the number of states in the system is six (*n*=6). The sum of the order of differentiations which is eight is equal to *n* + *b*. Thus full state linearization is achieved.

Let the equation (4.27) be rewritten as $\ddot{z} = r$ where r is the auxiliary reference input. Therefore we have the following decoupled chains of integrators

$$\ddot{z}_1 = r_1$$

$$\ddot{z}_2 = r_2$$

$$\ddot{z}_3 = r_3$$

$$\ddot{z}_4 = r_4 \tag{4.28}$$

The resulting nonlinear dynamic feedback controller is

$$u_1 = z_1$$

$$u_2 = (r_2 - x_2 r_1)/z_1$$

$$u_3 = (r_3 - x_4 r_1)/z_1$$

$$u_4 = z_2$$

$$\dot{z}_1 = u_1' = r_1$$

$$\dot{z}_2 = u_4' = r_4 \tag{4.29}$$

Assuming the system follows a smooth desired reference trajectory in chained form coordinates as $z_d(t) = (x_{d1}(t),\, x_{d3}(t),\, x_{d5}(t),\, x_{d6}(t))$, the exponentially stabilizing feedback control law for the linear and decoupled system about this desired trajectory is given as

$$r_i = \ddot{z}_{di}(t) + k_{vi}\left(\dot{z}_{di}(t) - \dot{z}_i(t)\right) + k_{pi}\left(z_{di}(t) - z_i(t)\right); \qquad i = 1,2\ldots,4 \tag{4.30}$$

where k_{vi} and k_{pi} (the PD gains) are chosen such that they are positive and the characteristic polynomials

$$s^2 + k_{vi}s + k_{pi}; \;\; i = 1,2\ldots,4 \tag{4.31}$$

are Hurwitz. The desired values for the variables $\dot{z}_d$ and $\ddot{z}_d$ are obtained from the (4.25) and (4.27). For the simulation the other state variables can be expressed in terms of the desired output trajectory at the initial time $t = t_0$

$$x_1(t_0) = z_{d1}(t_0) = x_d(t_0)$$

$$x_2(t_0) = \dot{z}_{d2}(t_0)/\dot{z}_{d1}(t_0) = y_d(t_0)/\dot{x}_d(t_0)$$

$$x_3(t_0) = z_{d2}(t_0) = y_d(t_0)$$

$$x_4(t_0) = \dot{z}_{d3}(t_0)/\dot{z}_{d1}(t_0) = \dot{z}_d(t_0)/\dot{x}_d(t_0)$$

$$x_{d5}(t_0) = z_{d3}(t_0) = z_d(t_0)$$

$$x_{d6}(t_0) = z_{d4}(t_0)$$

$$z_1(t_0) = \dot{z}_{d1}(t_0)$$

$$z_2(t_0) = \dot{z}_{d4}(t_0) \tag{4.32}$$

From this initialization, the output trajectory is reproducible. Any other initialization gives tracking error which exponentially goes to zero with time. The same results will follow if the dynamic extension is applied to the original kinematic equations.

4.4.3 Simulation of the controller

For simulation again the same desired trajectory is used as was done in the linear case. The trajectory chosen is

$$z_{d1}(t) = x_d(t) = t \qquad z_{d2}(t) = y_d(t) = a\sin wt$$

$$z_{d3}(t) = z_d(t) = 1 \qquad z_{d4}(t) = x_{d6}(t) = 0$$

which gives the desired values for the chained form states and inputs as

$$x_{d1}(t) = t \qquad x_{d2}(t) = aw\cos wt \qquad x_{d3}(t) = a\sin wt$$

$$x_{d4}(t) = 0 \qquad x_{d5}(t) = 1 \qquad x_{d6}(t) = 0$$

and

$$z_{d1}(t) = u_{d1}(t) = 1 \qquad z_{d2}(t) = u_{d4}(t) = 0$$

$$u_{d2}(t) = -aw^2 \sin wt \qquad u_{d3}(t) = 0$$

The initial conditions for the states are

$$x_{d1}(0) = 0 \qquad x_{d2}(0) = aw \qquad x_{d3}(0) = 0$$

$$x_{d4}(0) = 0 \qquad x_{d5}(0) = 1 \qquad x_{d6}(0) = 0$$

$$z_{d1}(0) = u_{d1}(0) = 1 \qquad z_{d2}(0) = u_{d4}(0) = 0$$

Choosing again the six coincident closed loop poles at -2, that is $p_i = -2; i = 1,\ldots,6$ we get the PD gains as $k_{vi} = 4$ and $k_{pi} = 4$.Choosing $a = 1$ and $w = p$ we get the simulation results. The results show tracking errors for the chained form states and inputs; and for actual states and inputs. Once the tracking errors go to zero, the actual inputs (control inputs) are same as the desired inputs.

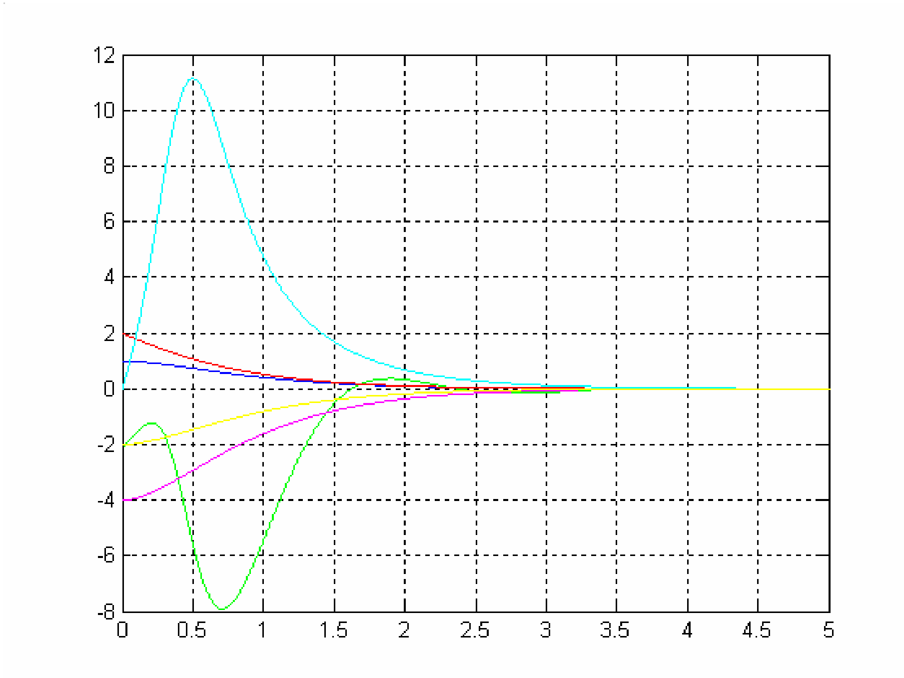

Figure4. 13: The result of dynamic feedback: tracking errors in chained form variables x_1 (blue), x_2 (green), x_3 (red), x_4 (cyan), x_5 (magenta), x_6 (yellow), vs. time (sec).

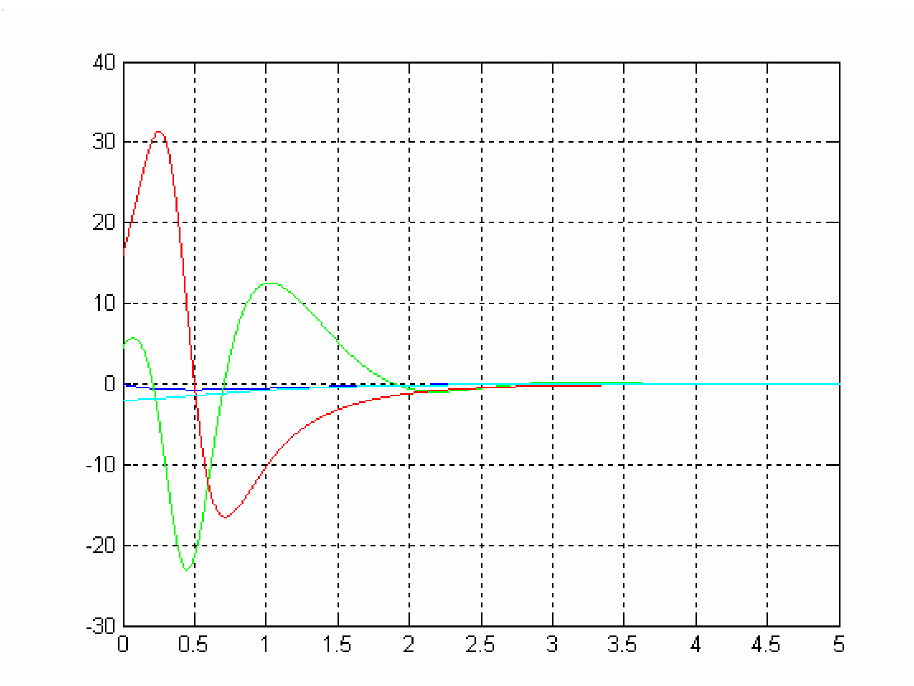

Figure4. 14: The result of dynamic feedback: tracking errors in chained form inputs u_1 (blue), u_2 (green), u_3 (red), u_4 (cyan) vs. time (sec).

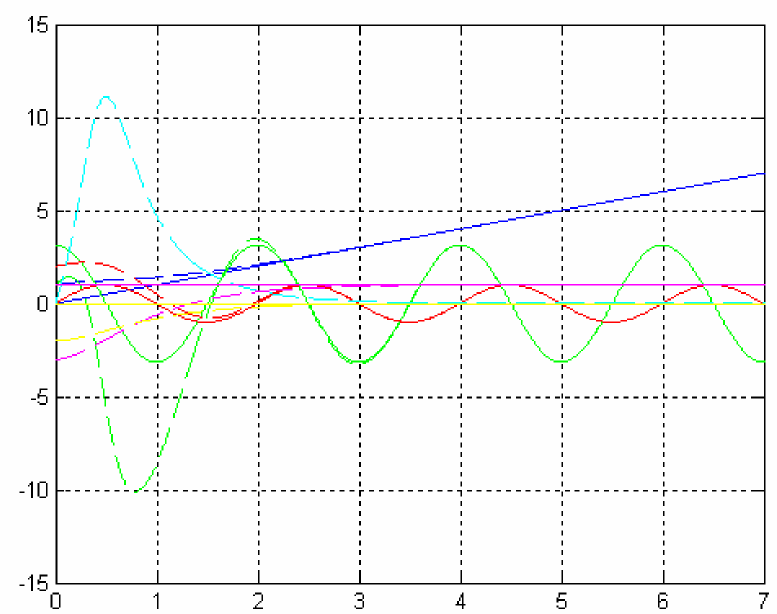

Figure4. 15: The result of dynamic feedback: actual (--) and desired ($-$) chained form variables x_1 (blue), x_2 (green), x_3 (red), x_4 (cyan), x_5 (magenta), x_6 (yellow), vs. time (sec).

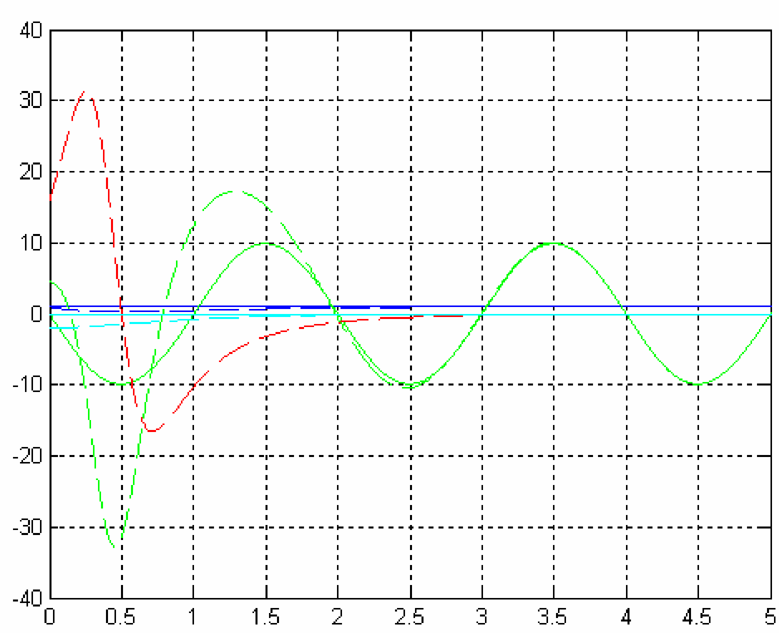

Figure4. 16: The result of dynamic feedback: actual (--) and desired ($-$) chained form inputs u_1 (blue), u_2 (green), u_3 (red), u_4 (cyan) vs. time (sec).

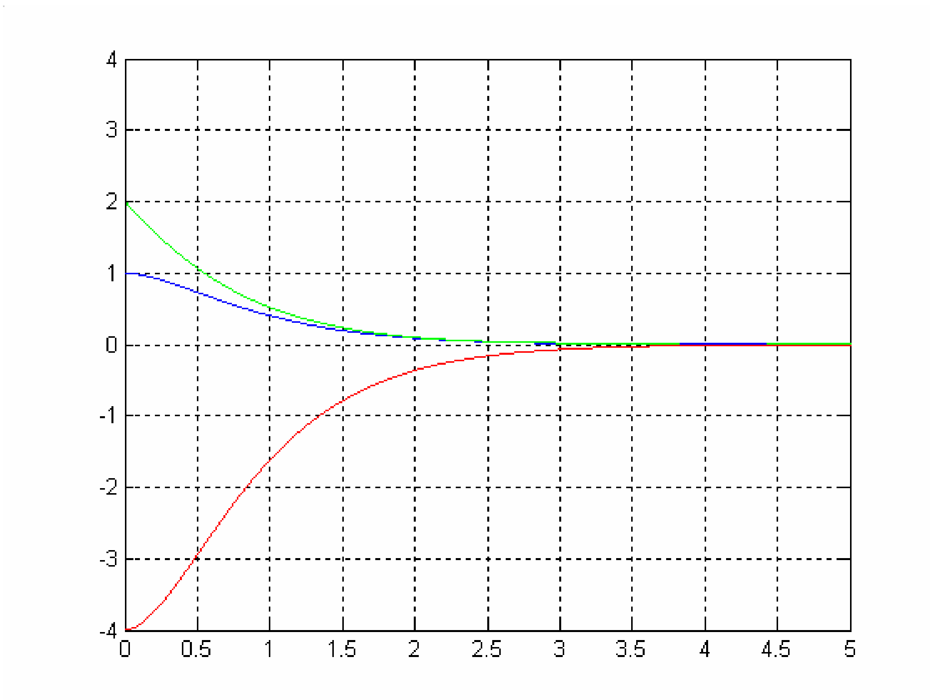

Figure4. 17: The result of dynamic feedback: tracking errors (m) in variables x (blue), y (green), z (red) vs. time (sec).

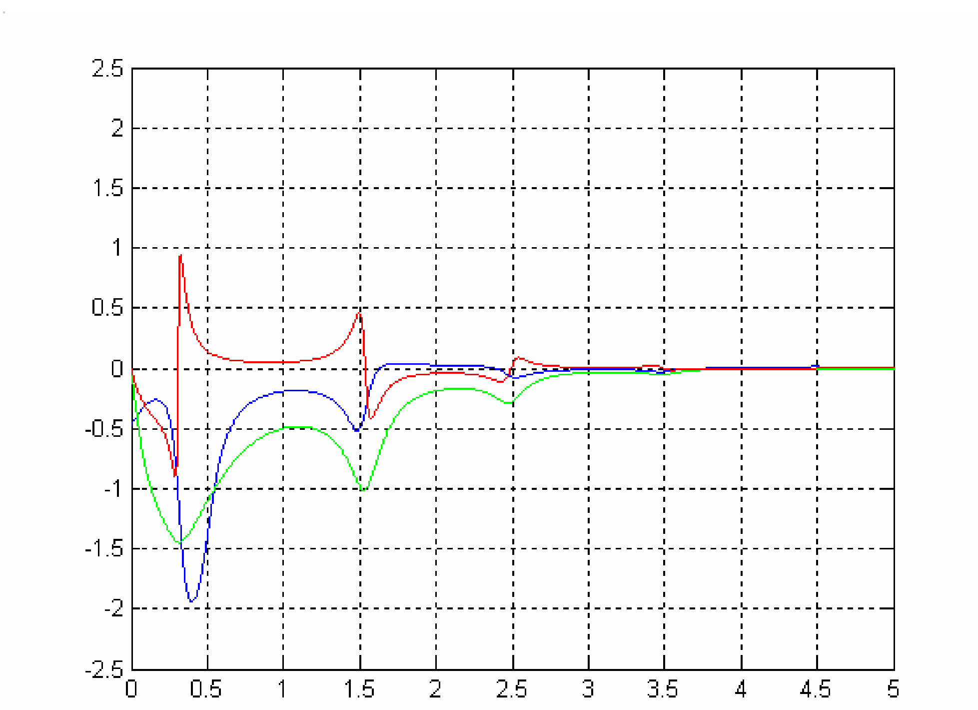

Figure4. 18: The result of dynamic feedback: tracking errors (rad) in variables y (blue), q (green), f (red) vs. time (sec).

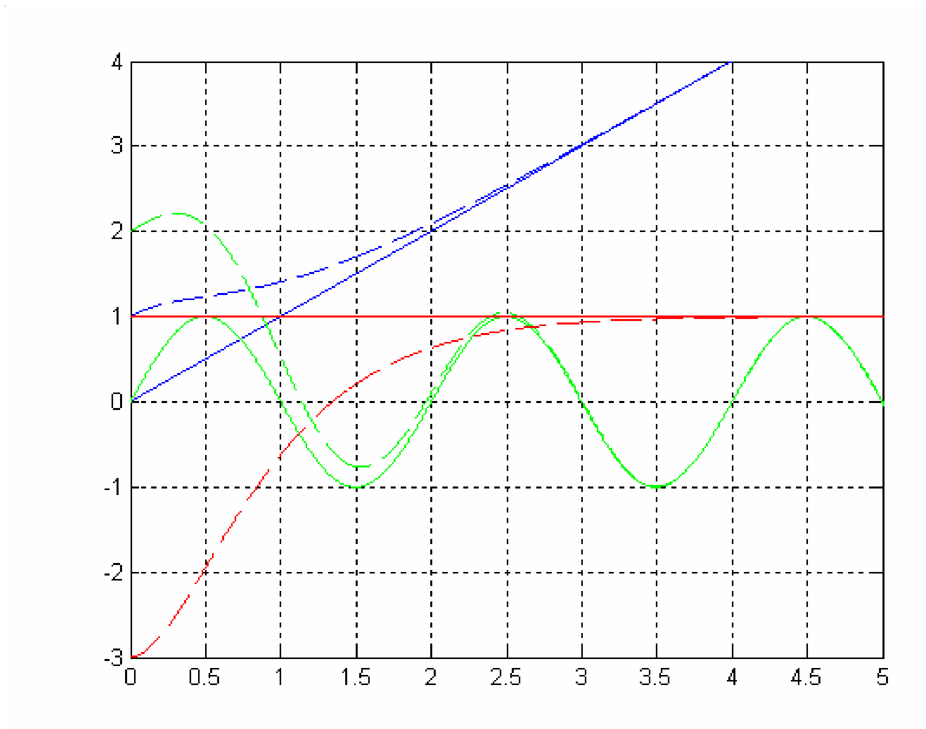

Figure4. 19: The result of dynamic feedback: actual (--) and desired (–) original variables x (blue), y (green), z (red) vs. time

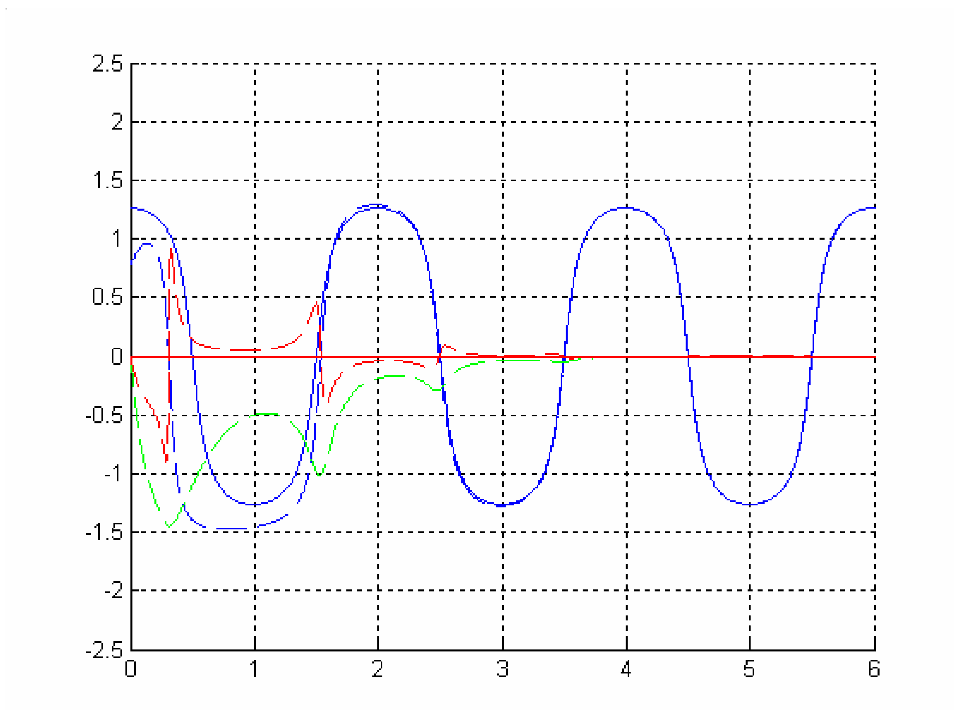

Figure4. 20: The result of dynamic feedback: actual (--) and desired (–) original variables y (blue), q (green), f (red) vs. time

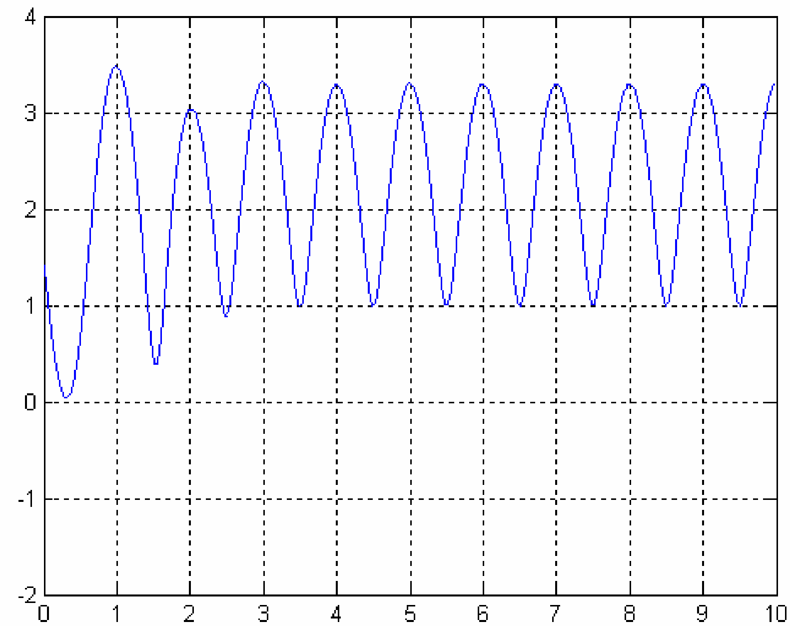

Figure4. 21: The result of dynamic feedback: v_1 (m/sec) vs. time (sec).

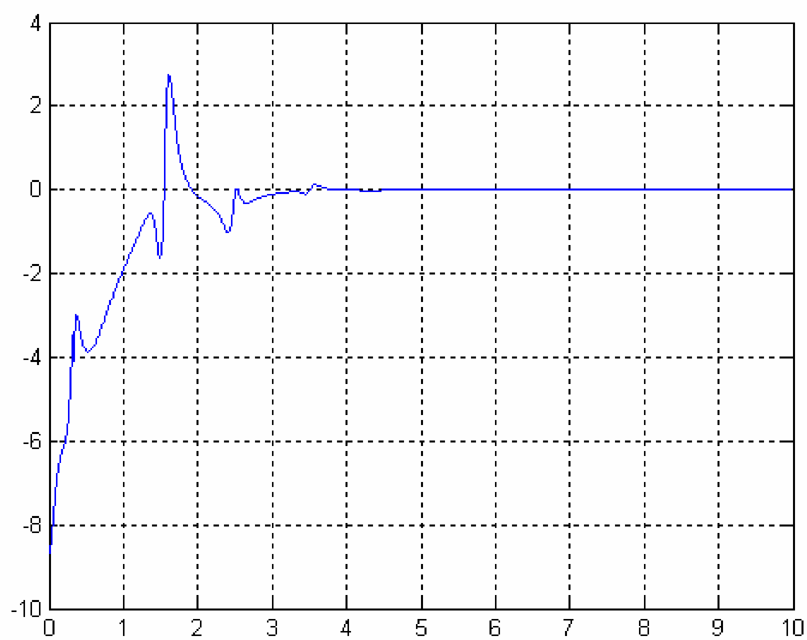

Figure4. 22: The result of dynamic feedback: v_2 (rad/sec) vs. time (sec).

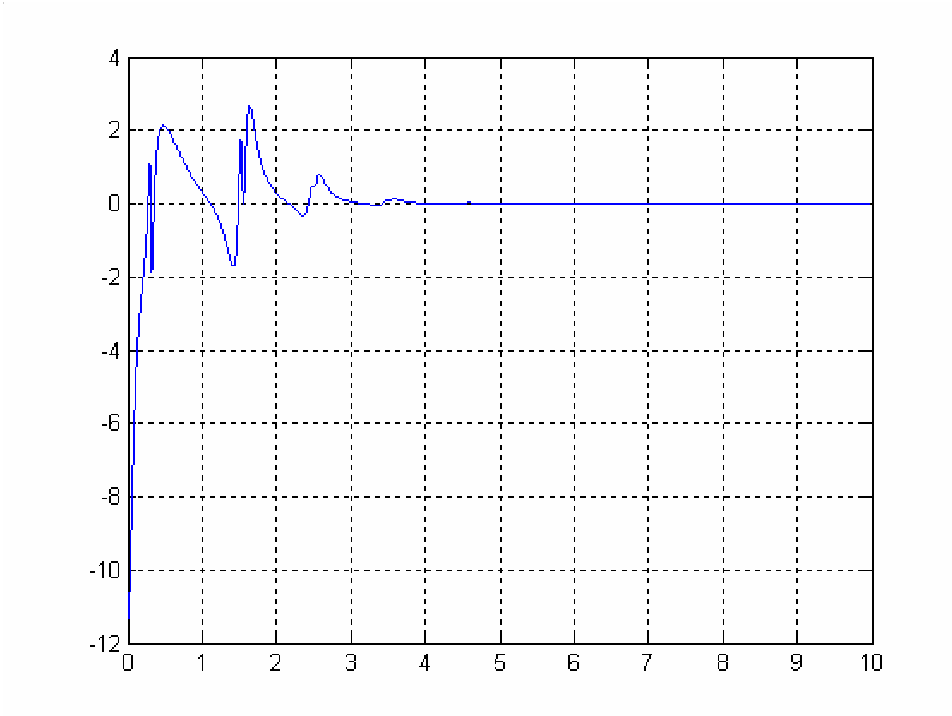

Figure4. 23: The result of dynamic feedback: v_3 (rad/sec) vs. time (sec).

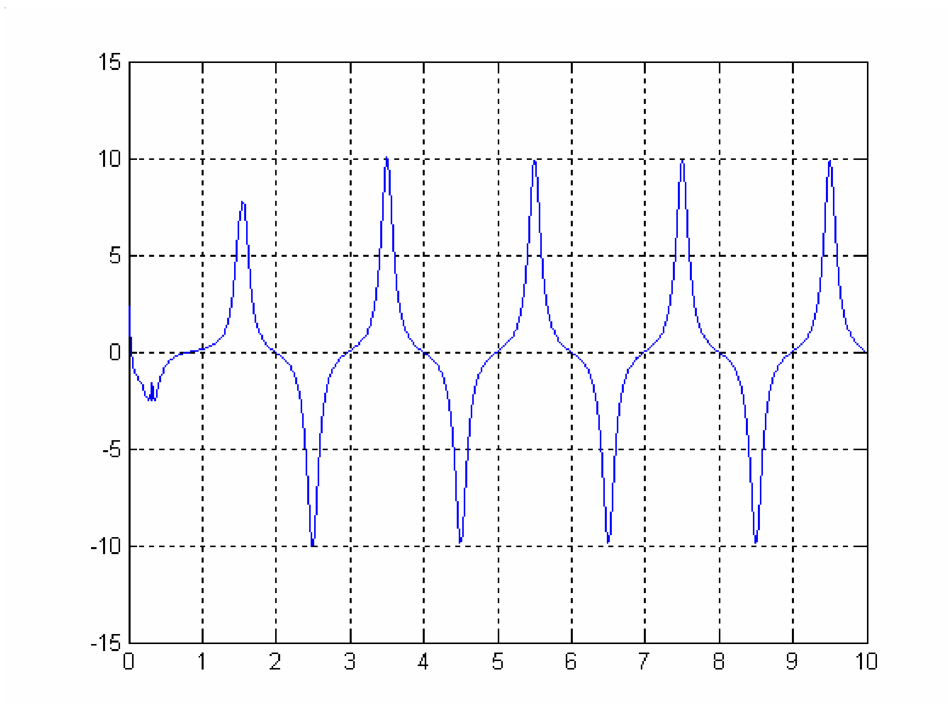

Figure4. 24: The result of dynamic feedback: v_4 (rad/sec) vs. time (sec).

4.5 *Point to point stabilization*

In the following section the problem of point to point stabilization is addressed. The system is supposed to reach a final desired configuration starting from an initial point, without the need to plan a trajectory.

As stated earlier the point stabilization can not be achieved by a smooth time invariant feedback. Only non-smooth or time varying feedback laws are of interest for the task. For our system we will adopt the latter approach.

4.5.1 Control with smooth time varying feedback

The method of designing a stabilizing control law here is the one proposed in [7]. The control law presented there was for a two input nonholonomic system. The controller here is an extension of the same. The statement of the problem is as: given a nonlinear drift free control system (3.11)

$$\dot{q} = g_1(q)v_1 + g_2(q)v_2 + g_3(q)v_3 + g_4(q)v_4 \tag{4.33}$$

we have to find a control law of the form $v(q,t)$ which makes the origin globally stable. In [7] the origin of the control system (4.33) is represented in power form and is then stabilized. Thus before going for the control design we will convert (4.33) into a power form.

4.5.2 Power form

The method of converting (4.33) to power form is as presented in [24]. The transformation is done in two steps. In first step the original system is converted into a three chain, single generator chained form as described in section 3.3. The chained form obtained is given as

$$\dot{x}_1 = x_1^0 = u_1 \qquad \dot{x}_2 = x_2^0 = u_2 \qquad \dot{x}_4 = x_3^0 = u_3 \qquad \dot{x}_6 = x_4^0 = u_4$$

$$\dot{x}_3 = x_{20}^0 = x_2 u_1 \qquad \dot{x}_5 = x_{31}^0 = x_4 u_1 \tag{4.34}$$

In the second step the chained form system (4.34) is converted into power form. For the three chain single generator chained form the global transformation to power form is given below [24]:

$$y_j = x_j^0;\ 1 \le j \le 4$$

$$z_{j0}^k = (-1)^k x_{j0}^k + \sum_{n=0}^{k-1} (-1)^n \frac{1}{(k-n)!} \left(x_1^0\right)^{k-n} x_{j0}^n \ 2 \le j \le 4 \ 1 \le k \le n_j \tag{4.35}$$

which gives the power form as:

$$\dot{y}_j = u_j; \qquad 1 \le j \le 4$$

$$z_{j0}^k = \frac{1}{k!}(y_1)^k u_j \ 2 \le j \le 4 \ 1 \le k \le n_j \tag{4.36}$$

Here we should recall from section 3.3 that $x_{j0}^0 = x_j^0$ which are identified as the top of the chains and $\sum_{n=2}^{4} n_j = 2$. Thus we have $n_2 = n_3 = 1$ and $n_4 = 0$. Using (4.34) and taking the above values, the transformation (4.35) becomes

$$y_1 = x_1$$

$$y_2 = x_2$$

$$y_3 = x_4$$

$$y_4 = x_6$$

$$z_1 = z_{20}^1 = -x_3 + x_1 x_2$$

$$z_2 = z_{30}^1 = -x_5 + x_1 x_4$$

$$z_3 = z_{40}^0 = x_6 \tag{4.37}$$

and the corresponding power form is

$$\dot{y}_1 = u_1$$

$$\dot{y}_2 = u_2$$

$$\dot{y}_3 = u_3$$

$$\dot{y}_4 = u_4$$

$$\dot{z}_{20}^1 = y_1 u_2$$

$$\dot{z}_{30}^1 = y_1 u_3$$

$$\dot{z}_{40}^0 = u_4 \tag{4.38}$$

4.5.3 Control law

The control law for (4.38) from [7] is given here as

$$u_1 = -y_1 + r(z)(\cos t - \sin t)$$

$$u_2 = -y_2 + c_1 z_1 \cos t$$

$$u_3 = -y_3 + c_2 z_2 \cos t$$

$$u_4 = -y_4 \tag{4.39}$$

with c_1 , c_2>0 and $r(z) = (z_1)^2 + (z_2)^2 + (z_3)^2$ The controls (4.39) asymptotically stabilize the origin of (4.38).

For global stabilization the saturation functions are introduced in the control law. These functions eliminate the destabilizing effects away from the origin. The control is thus given as

$$u_1 = -y_1 + s\ (r(z))(\cos t - \sin t)$$

$$u_2 = -y_2 + c_1 s\,(z_1)\cos t$$

$$u_3 = -y_3 + c_2 s\,(z_2)\cos t$$

$$u_4 = -y_4 \tag{4.40}$$

where $c_j > 0$ and $s : R \rightarrow R$ is a non decreasing C^3 saturation function with a magnitude less than some $d > 0$ and is linear between $(-d, d)$ .For global stabilization d should be small enough. The saturation function then satisfies the following [7]:

1. $s\,(z) = z$ when $|z| \leq e$

2. $|s\,(z)| \leq d$ for all z such that $0 < e < d$

The closed loop dynamics are given as:

$$\dot{y}_1 = -y_1 + s\,(r(z))(\cos t - \sin t)$$

$$\dot{y}_2 = -y_2 + c_1 s\,(z_1)\cos t$$

$$\dot{y}_3 = -y_3 + c_2 s\,(z_2)\cos t$$

$$\dot{y}_4 = -y_4 \tag{4.41}$$

For some $0 < e < d$, $\exists d_0$ such that if $e < e_0$, then the closed loop dynamics are globally asymptotically stabilized to zero.

4.5.4 Simulation

For simulation the value of d is chosen to be 0.001 and the saturation functions, $s\,(r(z))$, $s\,(z_1)$ and $s\,(z_2)$ are 0.26, 0.5 and 0.1. The constants c_1 and c_2 are both chosen as 2. The initial values for y_1 , y_2 , y_3 and y_4 are chosen as -5, -2, -7 and -5 respectively.

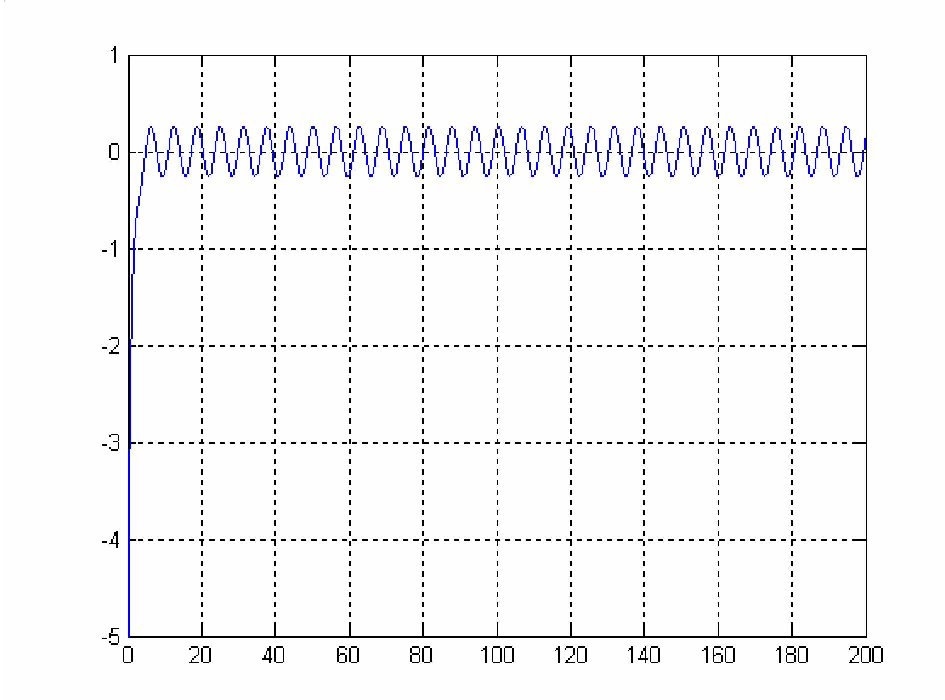

Figure4. 25: Point stabilization using time varying feedback: x (m) vs. time (sec).

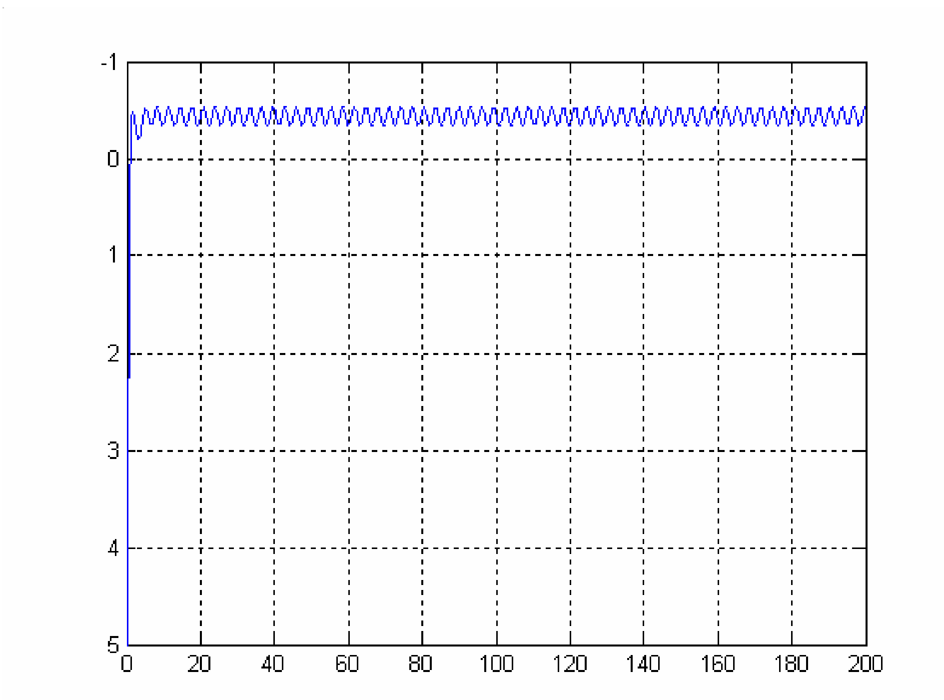

Figure4. 26: Point stabilization using time varying feedback: y (rad) vs. time (sec).

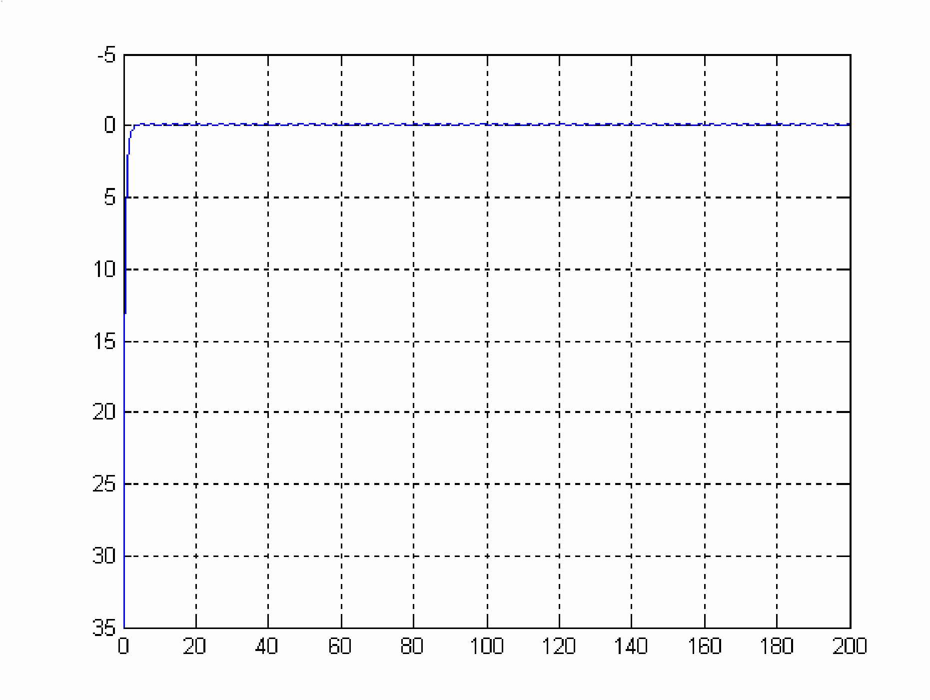

Figure4. 27: Point stabilization using time varying feedback: z (m) vs. time (sec).

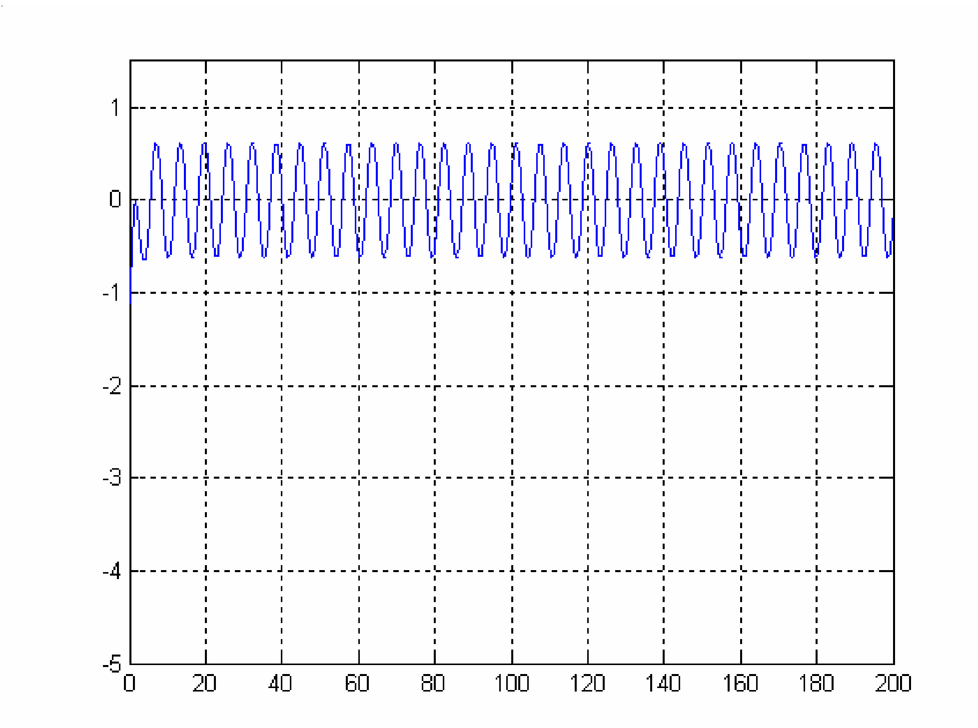

Figure4. 28: Point stabilization using time varying feedback: y (rad) vs. time (sec).

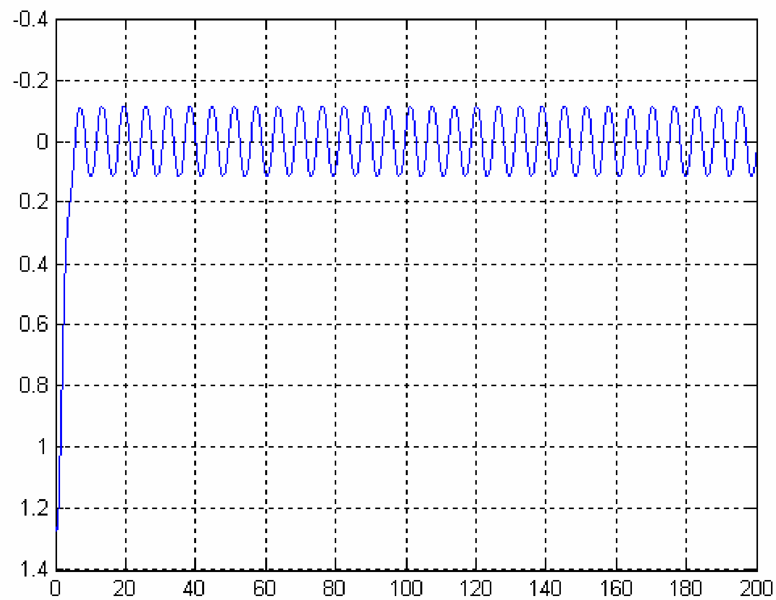

Figure4. 29: Point stabilization using time varying feedback: q (rad) vs. time (sec)**.**

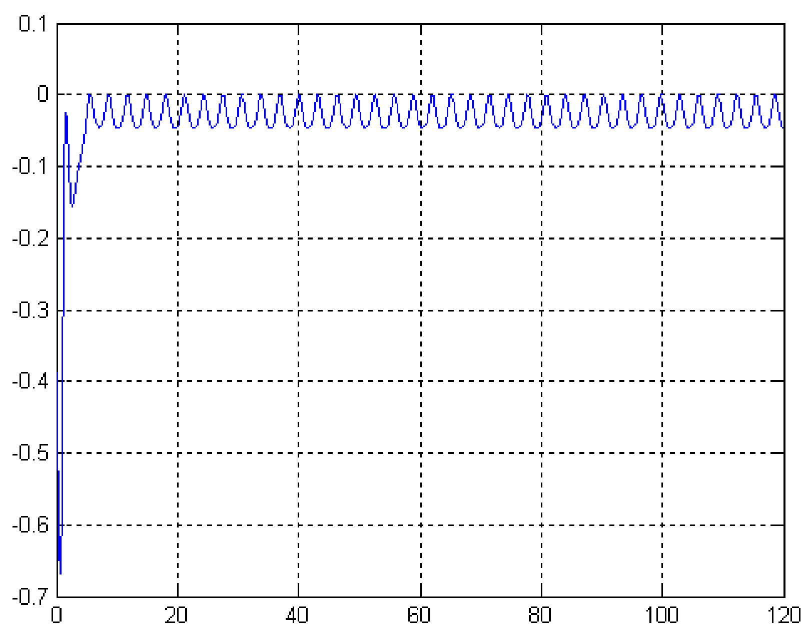

Figure4. 30: Point stabilization using time varying feedback: f (rad) vs. time (sec).

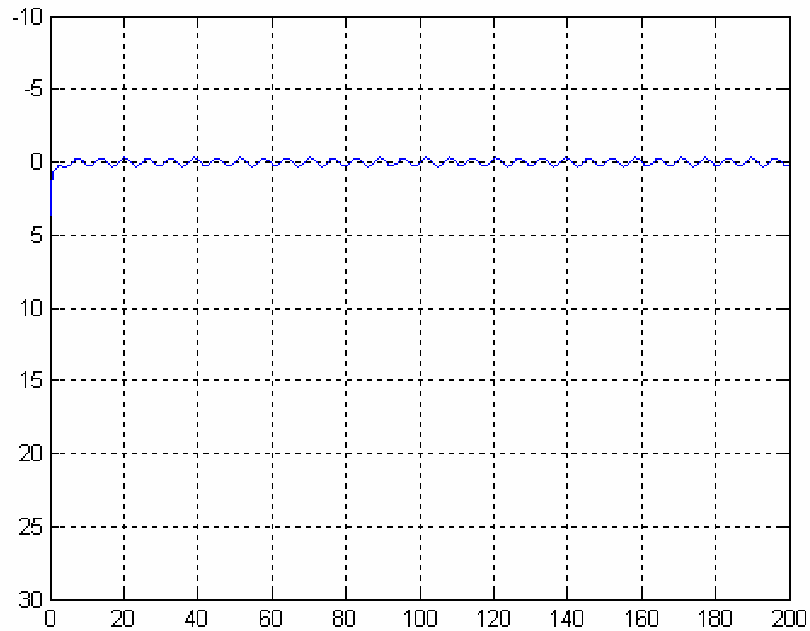

Figure4. 31: Point stabilization using time varying feedback: v_1 (m/sec) vs. time (sec).

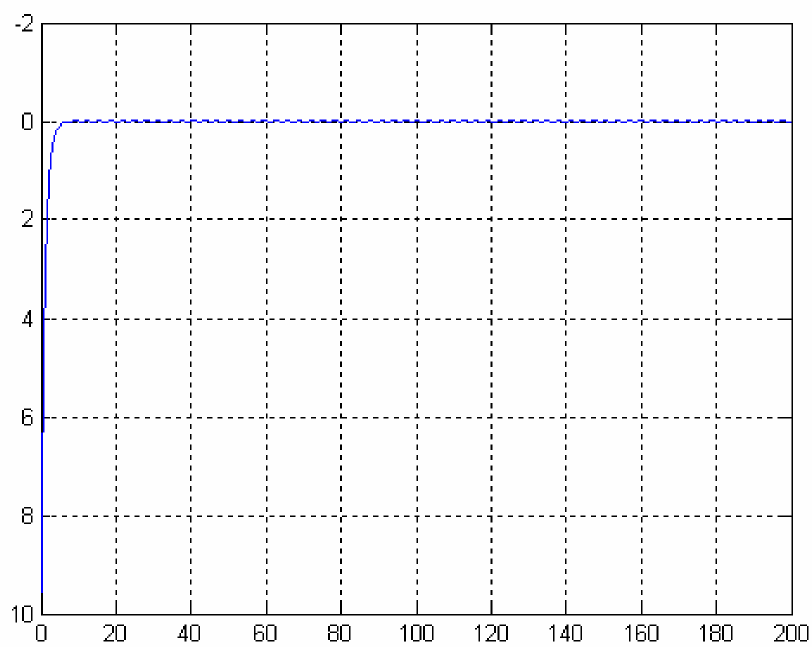

Figure4. 32: Point stabilization using time varying feedback: v_2 (m/sec) vs. time (sec).

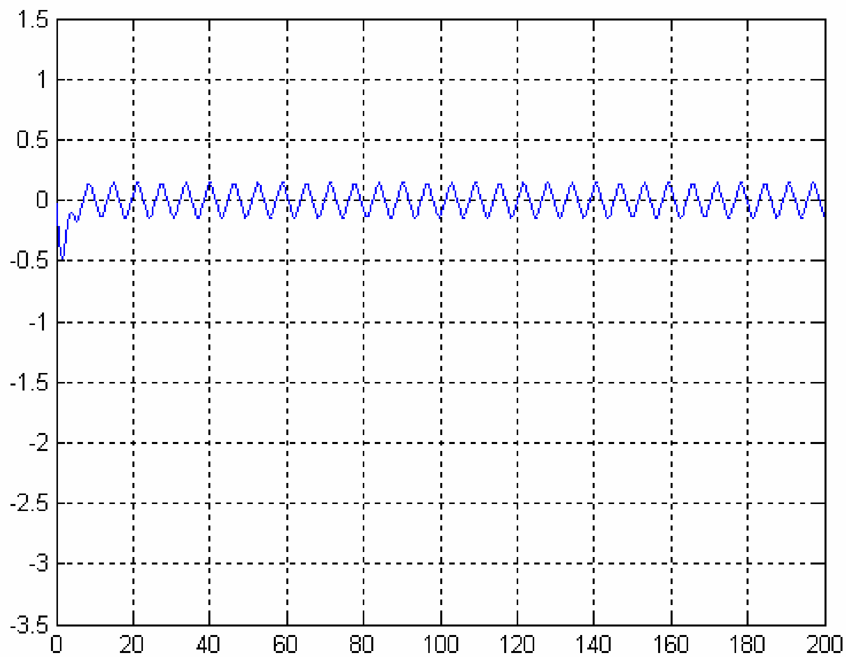

Figure4. 33: Point stabilization using time varying feedback: v_3 (rad/sec) vs. time (sec).

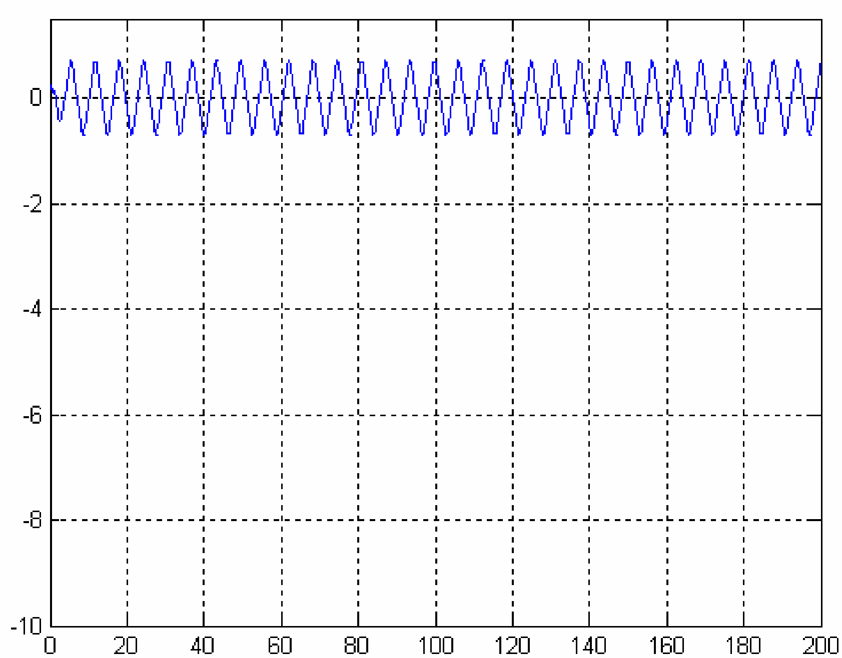

Figure4. 34: Point stabilization using time varying feedback: v_4 (rad/sec) vs. time (sec).

Chapter 5

Conclusions

This chapter presents some final considerations on our work and an outline of future work on the topic.

5.1 Concluding remarks

This thesis described the issues related to the motion planning of nonholonomic systems with application to underwater vehicles. It also described the development of the kinematic control model for the vehicle. The issues related with nonlinear controllability were discussed. The design of the feedback controllers was done.

The thesis used the mathematical concepts involved in Lie algebra for the study of nonlinear controllability. The concept of nilpotent basis was also invoked for establishing the Lie algebra rank condition for the controllability. For design purpose, the system had to be converted into chained form and power form. The thesis discussed the method for transformation into chained and power forms. The method of transformations utilized the concepts of nonlinear feedback transformation.

This thesis discussed the generation of a reference trajectory for an under water vehicle. The controllers were then designed for the trajectory tracking. The control design for trajectory tracking was done using both linear and nonlinear strategies. The stabilization of the system was discussed and the controllers were designed for achieving point-to-point stabilization. Stabilization was achieved using the time varying smooth control law.

5.2 Future work

For future work it is possible to extend similar analysis to a higher dimensional nonholonomic problem. For the same problem as discussed here, the design of stabilizing laws for path following can be done. For path following, the model can be transformed into parametric form in order to apply the control schemes. The method of control with input scaling can also be used.

The stabilization in both point-to-point and path following tasks can also be achieved through other control methods. The use of nonsmooth control and open loop control can be used. Also for underwater vehicle, motion planning can also be done while taking the actual dynamics of the system into consideration.

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