APPENDIX D COORDINATE MAPPING FOR FINITE ELEMENT ANALYSIS

Appendix D.1 Volume Mapping for Two-Dimensional Meshes

The differential volume is defined as follows:

$$
\begin{aligned}
dV &= \left(d\bar{x} \times d\bar{y}\right) \bullet d\bar{z} \\
&= \left(\begin{Bmatrix} dx \\ 0 \\ 0 \end{Bmatrix} \times \begin{Bmatrix} 0 \\ dy \\ 0 \end{Bmatrix} \right) \bullet \begin{Bmatrix} 0 \\ 0 \\ dz \end{Bmatrix} \\
&= \begin{Bmatrix} 0 \\ 0 \\ dxdy \end{Bmatrix} \bullet \begin{Bmatrix} 0 \\ 0 \\ dz \end{Bmatrix} \\
&= dxdydz
\end{aligned}
$$

While this definition may seem unnecessary for the case above, it serves as an aid in evaluating the transformation of the differential volume to another coordinate system. The differential volume in a natural coordinate system (ξ,η,ζ) is evaluated below. This natural coordinate system may be any general curvilinear coordinate system. In the case where the mesh is defined in only two dimensions, many of the partial derivatives are equal to zero. The third coordinate direction (z for plane strain, c for axisymmetric) is independent of the coordinate directions in the plane of the mesh (x and y, r and z), so the following are true:

$$
\begin{aligned}
\tfrac{\partial x}{\partial \zeta} = \tfrac{\partial y}{\partial \zeta} = 0 \\
\tfrac{\partial z}{\partial \xi} = \tfrac{\partial z}{\partial \eta} = 0
\end{aligned}
$$

The differential volume, dV is determined in the natural coordinate system as follows:

$$
\begin{aligned}
dV &= \left(\begin{Bmatrix} \frac{\partial x}{\partial \xi} \\ \frac{\partial y}{\partial \xi} \\ \frac{\partial z}{\partial \xi} \end{Bmatrix} d\xi \times \begin{Bmatrix} \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \eta} \\ \frac{\partial z}{\partial \eta} \end{Bmatrix} d\eta \right) \bullet \begin{Bmatrix} \frac{\partial x}{\partial \zeta} \\ \frac{\partial y}{\partial \zeta} \\ \frac{\partial z}{\partial \zeta} \end{Bmatrix} d\zeta \\
&= \begin{Bmatrix} \frac{\partial y}{\partial \xi}\frac{\partial z}{\partial \eta} - \frac{\partial y}{\partial \eta}\frac{\partial z}{\partial \xi} \\ \frac{\partial z}{\partial \xi}\frac{\partial x}{\partial \eta} - \frac{\partial z}{\partial \eta}\frac{\partial x}{\partial \xi} \\ \frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta} - \frac{\partial x}{\partial \eta}\frac{\partial y}{\partial \xi} \end{Bmatrix} \bullet \begin{Bmatrix} \frac{\partial x}{\partial \zeta} \\ \frac{\partial y}{\partial \zeta} \\ \frac{\partial z}{\partial \zeta} \end{Bmatrix} d\xi d\eta d\zeta \\
&= \begin{Bmatrix} 0 \\ 0 \\ \frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta} - \frac{\partial x}{\partial \eta}\frac{\partial y}{\partial \xi} \end{Bmatrix} \bullet \begin{Bmatrix} 0 \\ 0 \\ \frac{\partial z}{\partial \zeta} \end{Bmatrix} d\xi d\eta d\zeta \\
&= \left(\frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta} - \frac{\partial x}{\partial \eta}\frac{\partial y}{\partial \xi} \right) \cdot \frac{\partial z}{\partial \zeta} d\xi d\eta d\zeta
\end{aligned}
$$

In developing the requisite equations for finite element analysis, various expressions are integrated about the element volume. It is necessary then to transform dV into the natural coordinate system if the integration is to be performed in this coordinate system. This is accomplished by the use of the determinate of the Jacobian matrix for the transformation (referred to as "$DETJAC$" in the remainder of the equations). The determinant may be thought of as a scale factor for transforming the differential volume into another coordinate system as follows:

$$
\begin{aligned}
\iiint_V dV &= \int_z \int_y \int_x dxdydz \\
&= \int_\zeta \int_\eta \int_\xi \left(\frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta} - \frac{\partial x}{\partial \eta}\frac{\partial y}{\partial \xi} \right) * \frac{\partial z}{\partial \zeta} d\xi d\eta d\zeta \\
&= \int_\eta \int_\xi DETJAC \cdot d\xi d\eta
\end{aligned}
$$

Note that one of the coordinate directions disappears since the mesh is defined by only two dimensions. In addition, since the third coordinate direction in either system is dependent only on the third direction in the other system, the substitution $\frac{\partial z}{\partial \zeta} d\zeta = dz$ (or dc for axisymmetric analysis) can be made.

The determinant of the Jacobian for plane strain analysis is:

$$\begin{aligned} DETJAC &= \int_{\zeta}\left(\tfrac{\partial x}{\partial \xi}\tfrac{\partial y}{\partial \eta} - \tfrac{\partial x}{\partial \eta}\tfrac{\partial y}{\partial \xi}\right) * \tfrac{\partial z}{\partial \zeta} d\zeta \\ &= \int_{-1}^{+1}\left(\tfrac{\partial x}{\partial \xi}\tfrac{\partial y}{\partial \eta} - \tfrac{\partial x}{\partial \eta}\tfrac{\partial y}{\partial \xi}\right) * \tfrac{\partial z}{\partial \zeta} d\zeta \\ &= \int_{0}^{z}\left(\tfrac{\partial x}{\partial \xi}\tfrac{\partial y}{\partial \eta} - \tfrac{\partial x}{\partial \eta}\tfrac{\partial y}{\partial \xi}\right) dz \\ &= \left(\tfrac{\partial x}{\partial \xi}\tfrac{\partial y}{\partial \eta} - \tfrac{\partial x}{\partial \eta}\tfrac{\partial y}{\partial \xi}\right) \end{aligned}$$

The determinant of the Jacobian for axisymmetric analysis is:

$$\begin{aligned} DETJAC &= \int_{\zeta}\left(\tfrac{\partial r}{\partial \xi}\tfrac{\partial z}{\partial \eta} - \tfrac{\partial r}{\partial \eta}\tfrac{\partial z}{\partial \xi}\right) * \tfrac{\partial c}{\partial \zeta} d\zeta \\ &= \int_{-1}^{+1}\left(\tfrac{\partial r}{\partial \xi}\tfrac{\partial z}{\partial \eta} - \tfrac{\partial r}{\partial \eta}\tfrac{\partial z}{\partial \xi}\right) * \tfrac{\partial c}{\partial \zeta} d\zeta \\ &= \int_{0}^{c}\left(\tfrac{\partial r}{\partial \xi}\tfrac{\partial z}{\partial \eta} - \tfrac{\partial r}{\partial \eta}\tfrac{\partial z}{\partial \xi}\right) dc \\ &= 2\pi r\left(\tfrac{\partial r}{\partial \xi}\tfrac{\partial z}{\partial \eta} - \tfrac{\partial r}{\partial \eta}\tfrac{\partial z}{\partial \xi}\right) \end{aligned}$$

So far the transformation is as follows:

$$\iiint_V dV = \iiint_V dxdydz = \iint_A 2\pi r\left(\tfrac{\partial r}{\partial \xi}\tfrac{\partial z}{\partial \eta} - \tfrac{\partial r}{\partial \eta}\tfrac{\partial z}{\partial \xi}\right) d\xi d\eta$$

The partial derivatives are still needed to accurately transform the volume integral. Typically in finite element analysis, the evaluation of coordinates and spatial derivatives is accomplished using Lagrangian interpolation. The basic form of Lagrangian interpolation for coordinate evaluation is:

$$x = \sum_{i=1}^{n} N_i x_i \quad \text{and} \quad y = \sum_{i=1}^{n} N_i y_i$$

where: N_i = interpolation function for node I
x_i and y_i = nodal coordinates for node I
n = number of nodes for the element

From this basic definition, and since the nodal coordinates are constants, it can be seen that:

$$\frac{\partial x}{\partial \xi} = \frac{\partial \sum_{i=1}^{n} N_i x_i}{\partial \xi} \quad \text{and} \quad \frac{\partial y}{\partial \xi} = \frac{\partial \sum_{i=1}^{n} N_i y_i}{\partial \xi}$$

$$\frac{\partial x}{\partial \xi} = \frac{\partial \sum_{i=1}^{n} N_i}{\partial \xi} x_i \quad \text{and} \quad \frac{\partial y}{\partial \xi} = \frac{\partial \sum_{i=1}^{n} N_i}{\partial \xi} y_i$$

$$\frac{\partial x}{\partial \xi} = \sum_{i=1}^{n} \frac{\partial N_i}{\partial \xi} x_i \quad \text{and} \quad \frac{\partial y}{\partial \xi} = \sum_{i=1}^{n} \frac{\partial N_i}{\partial \xi} y_i$$

Assuming that repeated indices imply summation, and returning to the axisymmetric volume transformation results in the following:

$$\begin{aligned}
\iiint_V dV &= \iint_A 2\pi r \left(\frac{\partial r}{\partial \xi}\frac{\partial z}{\partial \eta} - \frac{\partial r}{\partial \eta}\frac{\partial z}{\partial \xi} \right) d\xi d\eta \\
&= \iint_A 2\pi r \left(\frac{\partial r}{\partial \xi}\frac{\partial z}{\partial \eta} - \frac{\partial r}{\partial \eta}\frac{\partial z}{\partial \xi} \right) d\xi d\eta \\
&= \iint_A 2\pi \left(N_i r_i \right) \left(\frac{\partial N_j}{\partial \xi} r_j \frac{\partial N_k}{\partial \eta} z_k - \frac{\partial N_j}{\partial \eta} r_j \frac{\partial N_k}{\partial \xi} z_k \right) d\xi d\eta
\end{aligned}$$

This is the complete transformation necessary to evaluate an axisymmetric volume based on a two dimensional natural coordinate system.

Also of interest is the mapping of spatial derivatives with respect to natural and local coordinates. The basic form of the transformation may be written as a matrix equation as follows:

$$\begin{Bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \\ \frac{\partial}{\partial \zeta} \end{Bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} & \frac{\partial z}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} & \frac{\partial z}{\partial \eta} \\ \frac{\partial x}{\partial \zeta} & \frac{\partial y}{\partial \zeta} & \frac{\partial z}{\partial \zeta} \end{bmatrix} \begin{Bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{Bmatrix}$$

$$= [\mathrm{J}] \begin{Bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{Bmatrix}$$

where the matrix $[\mathrm{J}]$ is the Jacobian matrix.

The Jacobian can be expressed as follows:

$$[\mathrm{J}]=\begin{bmatrix}\frac{\partial x}{\partial\xi} & \frac{\partial y}{\partial\xi} & \frac{\partial z}{\partial\xi}\\ \frac{\partial x}{\partial\eta} & \frac{\partial y}{\partial\eta} & \frac{\partial z}{\partial\eta}\\ \frac{\partial x}{\partial\zeta} & \frac{\partial y}{\partial\zeta} & \frac{\partial z}{\partial\zeta}\end{bmatrix}$$

$$[\mathrm{J}]=\begin{bmatrix}\frac{\partial N_i}{\partial\xi}x_i & \frac{\partial N_i}{\partial\xi}y_i & \frac{\partial N_i}{\partial\xi}z_i\\ \frac{\partial N_i}{\partial\eta}x_i & \frac{\partial N_i}{\partial\eta}y_i & \frac{\partial N_i}{\partial\eta}z_i\\ \frac{\partial N_i}{\partial\zeta}x_i & \frac{\partial N_i}{\partial\zeta}y_i & \frac{\partial N_i}{\partial\zeta}z_i\end{bmatrix}$$

$$[\mathrm{J}]=\begin{bmatrix}\mathrm{J}_{11} & \mathrm{J}_{12} & \mathrm{J}_{13}\\ \mathrm{J}_{21} & \mathrm{J}_{22} & \mathrm{J}_{23}\\ \mathrm{J}_{31} & \mathrm{J}_{32} & \mathrm{J}_{33}\end{bmatrix}$$

The inverse of the Jacobian can be expressed as follows:

$$[\mathrm{J}]^{-1}=\begin{bmatrix}\frac{\partial\xi}{\partial x} & \frac{\partial\eta}{\partial x} & \frac{\partial\zeta}{\partial x}\\ \frac{\partial\xi}{\partial y} & \frac{\partial\eta}{\partial y} & \frac{\partial\zeta}{\partial y}\\ \frac{\partial\xi}{\partial z} & \frac{\partial\eta}{\partial z} & \frac{\partial\zeta}{\partial z}\end{bmatrix}$$

$$[\mathrm{J}]^{-1}=\frac{\begin{bmatrix}\mathrm{J}_{22}\mathrm{J}_{33}-\mathrm{J}_{23}\mathrm{J}_{32} & \mathrm{J}_{23}\mathrm{J}_{31}-\mathrm{J}_{21}\mathrm{J}_{33} & \mathrm{J}_{21}\mathrm{J}_{32}-\mathrm{J}_{22}\mathrm{J}_{31}\\ \mathrm{J}_{32}\mathrm{J}_{13}-\mathrm{J}_{33}\mathrm{J}_{12} & \mathrm{J}_{33}\mathrm{J}_{11}-\mathrm{J}_{31}\mathrm{J}_{13} & \mathrm{J}_{31}\mathrm{J}_{12}-\mathrm{J}_{32}\mathrm{J}_{11}\\ \mathrm{J}_{12}\mathrm{J}_{23}-\mathrm{J}_{13}\mathrm{J}_{22} & \mathrm{J}_{13}\mathrm{J}_{21}-\mathrm{J}_{11}\mathrm{J}_{23} & \mathrm{J}_{11}\mathrm{J}_{22}-\mathrm{J}_{12}\mathrm{J}_{21}\end{bmatrix}}{\mathrm{J}_{11}\left(\mathrm{J}_{22}\mathrm{J}_{33}-\mathrm{J}_{23}\mathrm{J}_{32}\right)+\mathrm{J}_{12}\left(\mathrm{J}_{23}\mathrm{J}_{31}-\mathrm{J}_{21}\mathrm{J}_{33}\right)+\mathrm{J}_{13}\left(\mathrm{J}_{21}\mathrm{J}_{32}-\mathrm{J}_{22}\mathrm{J}_{31}\right)}$$

$$[\mathrm{J}]^{-1}=\frac{1}{DETJAC}\begin{bmatrix}\mathrm{J}_{22}\mathrm{J}_{33}-\mathrm{J}_{23}\mathrm{J}_{32} & \mathrm{J}_{23}\mathrm{J}_{31}-\mathrm{J}_{21}\mathrm{J}_{33} & \mathrm{J}_{21}\mathrm{J}_{32}-\mathrm{J}_{22}\mathrm{J}_{31}\\ \mathrm{J}_{32}\mathrm{J}_{13}-\mathrm{J}_{33}\mathrm{J}_{12} & \mathrm{J}_{33}\mathrm{J}_{11}-\mathrm{J}_{31}\mathrm{J}_{13} & \mathrm{J}_{31}\mathrm{J}_{12}-\mathrm{J}_{32}\mathrm{J}_{11}\\ \mathrm{J}_{12}\mathrm{J}_{23}-\mathrm{J}_{13}\mathrm{J}_{22} & \mathrm{J}_{13}\mathrm{J}_{21}-\mathrm{J}_{11}\mathrm{J}_{23} & \mathrm{J}_{11}\mathrm{J}_{22}-\mathrm{J}_{12}\mathrm{J}_{21}\end{bmatrix}$$

$$[\mathrm{J}]^{-1}=[\Gamma]=\begin{bmatrix}\Gamma_{11} & \Gamma_{12} & \Gamma_{13}\\ \Gamma_{21} & \Gamma_{22} & \Gamma_{23}\\ \Gamma_{31} & \Gamma_{32} & \Gamma_{33}\end{bmatrix}$$

The spatial derivatives with respect to the local coordinates can be mapped using the inverse of the Jacobian as follows:

$$\begin{Bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{Bmatrix} = \begin{bmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \eta}{\partial x} & \frac{\partial \zeta}{\partial x} \\ \frac{\partial \xi}{\partial y} & \frac{\partial \eta}{\partial y} & \frac{\partial \zeta}{\partial y} \\ \frac{\partial \xi}{\partial z} & \frac{\partial \eta}{\partial z} & \frac{\partial \zeta}{\partial z} \end{bmatrix} \begin{Bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \\ \frac{\partial}{\partial \zeta} \end{Bmatrix}$$

$$\begin{Bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{Bmatrix} = [\mathrm{J}]^{-1} \begin{Bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \\ \frac{\partial}{\partial \zeta} \end{Bmatrix}$$

$$\begin{Bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{Bmatrix} = \frac{1}{DETJAC} \begin{bmatrix} J_{22}J_{33} - J_{23}J_{32} & J_{23}J_{31} - J_{21}J_{33} & J_{21}J_{32} - J_{22}J_{31} \\ J_{32}J_{13} - J_{33}J_{12} & J_{33}J_{11} - J_{31}J_{13} & J_{31}J_{12} - J_{32}J_{11} \\ J_{12}J_{23} - J_{13}J_{22} & J_{13}J_{21} - J_{11}J_{23} & J_{11}J_{22} - J_{12}J_{21} \end{bmatrix} \begin{Bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \\ \frac{\partial}{\partial \zeta} \end{Bmatrix}$$

As an example, the three-dimensional Jacobian matrix is defined, but substitutions for zero-valued terms applicable to both axisymmetric and plane analyses are made. The resulting inverse and determinant of the Jacobian matrix is as follows:

$$[\mathrm{J}]^{-1} = \begin{bmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \eta}{\partial x} & 0 \\ \frac{\partial \xi}{\partial y} & \frac{\partial \eta}{\partial y} & 0 \\ 0 & 0 & \frac{\partial \zeta}{\partial z} \end{bmatrix}$$

$$[\mathrm{J}]^{-1} = \frac{1}{J_{11}(J_{22}J_{33} - 0*0) + J_{12}(0*0 - J_{21}J_{33}) + 0(J_{21}*0 - J_{22}*0)} \begin{bmatrix} J_{22}J_{33} - 0*0 & 0*0 - J_{21}J_{33} & J_{21}*0 - J_{22}*0 \\ 0*0 - J_{33}J_{12} & J_{33}J_{11} - 0*0 & 0*J_{12} - 0*J_{11} \\ J_{12}*0 - 0*J_{22} & 0*J_{21} - J_{11}*0 & J_{11}J_{22} - J_{12}J_{21} \end{bmatrix}$$

$$[\mathrm{J}]^{-1} = \frac{1}{J_{11}(J_{22}J_{33}) + J_{12}(-J_{21}J_{33})} \begin{bmatrix} J_{22}J_{33} & -J_{21}J_{33} & 0 \\ -J_{33}J_{12} & J_{33}J_{11} & 0 \\ 0 & 0 & J_{11}J_{22} - J_{12}J_{21} \end{bmatrix}$$

$$[\mathrm{J}]^{-1} = \frac{1}{J_{33}(J_{11}J_{22} - J_{12}J_{21})} \begin{bmatrix} J_{22}J_{33} & -J_{21}J_{33} & 0 \\ -J_{33}J_{12} & J_{33}J_{11} & 0 \\ 0 & 0 & J_{11}J_{22} - J_{12}J_{21} \end{bmatrix}$$

$$[\mathrm{J}]^{-1} = \begin{bmatrix} \frac{J_{22}}{(J_{11}J_{22} - J_{12}J_{21})} & \frac{-J_{21}}{(J_{11}J_{22} - J_{12}J_{21})} & 0 \\ \frac{-J_{12}}{(J_{11}J_{22} - J_{12}J_{21})} & \frac{J_{22}}{(J_{11}J_{22} - J_{12}J_{21})} & 0 \\ 0 & 0 & \frac{1}{J_{33}} \end{bmatrix}$$

$$|\mathrm{J}| = DETJAC = J_{33}(J_{11}J_{22} - J_{12}J_{21})$$

In most ordinary plane problems, the term $J_{33} = \frac{\partial z}{\partial \zeta}$ is equal to 1, so the third row and column of the Jacobian matrix and its inverse are dropped. If $J_{33}=1$, the determinant of the Jacobian is the same as that published in many finite element method texts (Zienkiewicz and Taylor 1989, Cook et al. 1991). For axisymmetric problems, however, the J_{33} term is not equal to 1, but is equal to $2\pi r$. The third row and column are still dropped, since no spatial derivatives with respect to that direction are involved in the development of the finite element equations, but the J_{33} term must be included in the determinant of the Jacobian in order to correctly transform the volume of integration for the finite element equations.

Appendix D.2 Area Mapping in the Plane of the Mesh - Axisymmetric Case

The differential area vector is:

$$
\begin{aligned}
d\vec{A} &= \left(d\vec{r} \times d\vec{z}\right) \\
&= \left(\begin{Bmatrix} dr \\ 0 \\ 0 \end{Bmatrix} \times \begin{Bmatrix} 0 \\ dz \\ 0 \end{Bmatrix} \right) \\
&= \begin{Bmatrix} 0 \\ 0 \\ drdz \end{Bmatrix}
\end{aligned}
$$

The magnitude of the differential area is defined as follows:

$$
\begin{aligned}
dA &= \left\| d\vec{A} \right\| = \sqrt{d\vec{A} \bullet d\vec{A}} \\
&= \sqrt{\begin{Bmatrix} 0 \\ 0 \\ drdz \end{Bmatrix} \bullet \begin{Bmatrix} 0 \\ 0 \\ drdz \end{Bmatrix}} \\
&= \sqrt{\left(drdz\right)^2} \\
&= drdz
\end{aligned}
$$

In the natural coordinate system, the differential area vector is:

$$
\begin{aligned}
d\vec{A} &= \left(\begin{Bmatrix} \frac{\partial r}{\partial \xi} \\ \frac{\partial z}{\partial \xi} \\ \frac{\partial c}{\partial \xi} \end{Bmatrix} d\xi \times \begin{Bmatrix} \frac{\partial r}{\partial \eta} \\ \frac{\partial z}{\partial \eta} \\ \frac{\partial c}{\partial \eta} \end{Bmatrix} d\eta \right) \\
&= \left(\begin{Bmatrix} \frac{\partial r}{\partial \xi} \\ \frac{\partial z}{\partial \xi} \\ 0 \end{Bmatrix} \times \begin{Bmatrix} \frac{\partial r}{\partial \eta} \\ \frac{\partial z}{\partial \eta} \\ 0 \end{Bmatrix} \right) d\xi d\eta \\
&= \begin{Bmatrix} 0 \\ 0 \\ \frac{\partial r}{\partial \xi}\frac{\partial z}{\partial \eta} - \frac{\partial r}{\partial \eta}\frac{\partial z}{\partial \xi} \end{Bmatrix} d\xi d\eta
\end{aligned}
$$

The magnitude of the differential area is:

$$
\begin{aligned}
dA &= \left\| d\bar{A} \right\| \\
&= \left\| \begin{Bmatrix} 0 \\ 0 \\ \frac{\partial r}{\partial \xi}\frac{\partial z}{\partial \eta} - \frac{\partial r}{\partial \eta}\frac{\partial z}{\partial \xi} \end{Bmatrix} d\xi d\eta \right\| \\
&= \left(\frac{\partial r}{\partial \xi}\frac{\partial z}{\partial \eta} - \frac{\partial r}{\partial \eta}\frac{\partial z}{\partial \xi} \right) d\xi d\eta \\
&= DETJAC \cdot d\xi d\eta
\end{aligned}
$$

In developing the requisite equations for finite element analysis, various expressions are integrated about the element area. It is necessary, then to transform dA into the natural coordinate system if the integration is to be performed in this coordinate system. This is accomplished by the use of the determinate of the Jacobian matrix for the transformation (referred to as "$DETJAC$" in the remainder of the equations). The determinant may be thought of as a scale factor for transforming the differential area into another coordinate system as follows:

$$
\begin{aligned}
\iint_A dA &= \int_z \int_r drdz \\
&= \int_\eta \int_\xi \left(\frac{\partial r}{\partial \xi}\frac{\partial z}{\partial \eta} - \frac{\partial r}{\partial \eta}\frac{\partial z}{\partial \xi} \right) d\xi d\eta \\
&= \int_\eta \int_\xi DETJAC \cdot d\xi d\eta
\end{aligned}
$$

Adding the typical interpolation to the area mapping yields the same results as for the volume mapping transformation.

Also of interest is the mapping of spatial derivatives with respect to natural and local coordinates. The basic form of the transformation may be written as a matrix equation as follows:

$$
\begin{aligned}
\begin{Bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{Bmatrix} &= \begin{bmatrix} \frac{\partial r}{\partial \xi} & \frac{\partial z}{\partial \xi} \\ \frac{\partial r}{\partial \eta} & \frac{\partial z}{\partial \eta} \end{bmatrix} \begin{Bmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial z} \end{Bmatrix} \\
&= [\mathrm{J}] \begin{Bmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial z} \end{Bmatrix}
\end{aligned}
$$

where the matrix $[\mathrm{J}]$ is the Jacobian matrix.

$$[\mathrm{J}] = \begin{bmatrix} \frac{\partial r}{\partial \xi} & \frac{\partial z}{\partial \xi} \\ \frac{\partial r}{\partial \eta} & \frac{\partial z}{\partial \eta} \end{bmatrix}$$

$$[\mathrm{J}] = \begin{bmatrix} \frac{\partial N_i}{\partial \xi} r_i & \frac{\partial N_i}{\partial \xi} z_i \\ \frac{\partial N_i}{\partial \eta} r_i & \frac{\partial N_i}{\partial \eta} z_i \end{bmatrix}$$

$$[\mathrm{J}] = \begin{bmatrix} \mathrm{J}_{11} & \mathrm{J}_{12} \\ \mathrm{J}_{21} & \mathrm{J}_{22} \end{bmatrix}$$

and

$$[\mathrm{J}]^{-1} = \begin{bmatrix} \frac{\partial \xi}{\partial r} & \frac{\partial \eta}{\partial r} \\ \frac{\partial \xi}{\partial z} & \frac{\partial \eta}{\partial z} \end{bmatrix}$$

$$[\mathrm{J}]^{-1} = \frac{1}{\mathrm{J}_{11}\mathrm{J}_{22} - \mathrm{J}_{12}\mathrm{J}_{21}} \begin{bmatrix} \mathrm{J}_{22} & -\mathrm{J}_{12} \\ -\mathrm{J}_{21} & \mathrm{J}_{11} \end{bmatrix}$$

$$[\mathrm{J}]^{-1} = \frac{1}{DETJAC} \begin{bmatrix} \mathrm{J}_{22} & -\mathrm{J}_{12} \\ -\mathrm{J}_{21} & \mathrm{J}_{11} \end{bmatrix}$$

$$[\mathrm{J}]^{-1} = [\Gamma] = \begin{bmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{21} & \Gamma_{22} \end{bmatrix}$$

The spatial derivatives with respect to the local coordinates can be mapped using the inverse of the Jacobian as follows:

$$\begin{Bmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial z} \end{Bmatrix} = \begin{bmatrix} \frac{\partial \xi}{\partial r} & \frac{\partial \eta}{\partial r} \\ \frac{\partial \xi}{\partial z} & \frac{\partial \eta}{\partial z} \end{bmatrix} \begin{Bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{Bmatrix}$$

$$\begin{Bmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial z} \end{Bmatrix} = [\mathrm{J}]^{-1} \begin{Bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{Bmatrix}$$

$$\begin{Bmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial z} \end{Bmatrix} = \frac{1}{DETJAC} \begin{bmatrix} \mathrm{J}_{22} & -\mathrm{J}_{12} \\ -\mathrm{J}_{21} & \mathrm{J}_{11} \end{bmatrix} \begin{Bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{Bmatrix}$$

Appendix D.3 Area Mapping Out of the Plane of the Mesh - Axisymmetric Case

The differential area vector is:

$$\begin{aligned} d\vec{A} &= \left(d\vec{s} \times d\vec{c}\right) \\ &= \left(\begin{Bmatrix} \frac{\partial r}{\partial s} \\ \frac{\partial z}{\partial s} \\ \frac{\partial c}{\partial s} \end{Bmatrix} ds \times \begin{Bmatrix} 0 \\ 0 \\ dc \end{Bmatrix} \right) \\ &= \begin{Bmatrix} \frac{\partial z}{\partial s} \\ -\frac{\partial r}{\partial s} \\ 0 \end{Bmatrix} dsdc \end{aligned}$$

The magnitude of the differential area is defined as follows:

$$\begin{aligned} dA &= \left\| d\vec{A} \right\| = \sqrt{d\vec{A} \bullet d\vec{A}} \\ &= \left\| \begin{Bmatrix} \frac{\partial z}{\partial s} \\ -\frac{\partial r}{\partial s} \\ 0 \end{Bmatrix} dsdc \right\| \\ &= \sqrt{\frac{\partial z}{\partial s}^2 + \left(-\frac{\partial r}{\partial s}\right)^2}\, dsdc \\ &= \sqrt{\frac{\partial r}{\partial s}^2 + \frac{\partial z}{\partial s}^2}\, dsdc \end{aligned}$$

In the natural coordinate system, the differential area vector is:

$$
\begin{aligned}
d\bar{A} &= \left(\begin{Bmatrix} \frac{\partial r}{\partial \xi} \\ \frac{\partial z}{\partial \xi} \\ \frac{\partial c}{\partial \xi} \end{Bmatrix} d\xi \times \begin{Bmatrix} \frac{\partial r}{\partial \zeta} \\ \frac{\partial z}{\partial \zeta} \\ \frac{\partial c}{\partial \zeta} \end{Bmatrix} d\zeta \right) \\
&= \left(\begin{Bmatrix} \frac{\partial r}{\partial \xi} \\ \frac{\partial z}{\partial \xi} \\ 0 \end{Bmatrix} \times \begin{Bmatrix} 0 \\ 0 \\ \frac{\partial c}{\partial \zeta} \end{Bmatrix} \right) d\xi d\zeta \\
&= \begin{Bmatrix} \frac{\partial z}{\partial \xi} \\ -\frac{\partial r}{\partial \xi} \\ 0 \end{Bmatrix} \frac{\partial c}{\partial \zeta} d\xi d\zeta \\
&= \begin{Bmatrix} \frac{\partial z}{\partial \xi} \\ -\frac{\partial r}{\partial \xi} \\ 0 \end{Bmatrix} d\xi dc
\end{aligned}
$$

The magnitude of the differential area is:

$$
\begin{aligned}
dA &= \left\| d\bar{A} \right\| \\
&= \left\| \begin{Bmatrix} \frac{\partial z}{\partial \xi} \\ -\frac{\partial r}{\partial \xi} \\ 0 \end{Bmatrix} \frac{\partial c}{\partial \zeta} d\xi d\zeta \right\| \\
&= \sqrt{\frac{\partial z}{\partial \xi}^2 + \left(-\frac{\partial r}{\partial \xi}\right)^2} \frac{\partial c}{\partial \zeta} d\xi d\zeta \\
&= \sqrt{\frac{\partial r}{\partial \xi}^2 + \frac{\partial z}{\partial \xi}^2} d\xi dc
\end{aligned}
$$

In developing the requisite equations for finite element analysis, various expressions are integrated about areas out of the plane of the mesh. Good examples of this are for deriving element stiffnesses for interface elements, and deriving equivalent nodal loads from boundary pressures (surface tractions). It is necessary, then to transform dA into the natural coordinate system if the integration is to be performed in this coordinate system. Since the mapping is from one dimension to two dimensions, the transformation matrix is not square. Thus, it is not a true “Jacobian matrix” there is no determinant or inverse. The basic concept of inverting the transformation, however, is still valid, but some care must be taken in inverting this relationship. The “determinant”, $DETJAC$, may still be regarded as a scale factor for transforming the differential area into another coordinate system, but is not actually the determinant of a matrix in this case, and will be denoted as *DETJAC**.

$$\begin{aligned}\iint_A dA &= \int_c\int_s \sqrt{\tfrac{\partial r}{\partial s}^2 + \tfrac{\partial z}{\partial s}^2}\, ds dc \\ &= \int_c\int_\xi \sqrt{\tfrac{\partial r}{\partial \xi}^2 + \tfrac{\partial z}{\partial \xi}^2}\, d\xi dc \\ &= \int_\xi 2\pi r \sqrt{\tfrac{\partial r}{\partial \xi}^2 + \tfrac{\partial z}{\partial \xi}^2}\, d\xi \\ &= \int_\xi DETJAC^* \cdot d\xi\end{aligned}$$

Here the axisymmetric, out of plane mapping function is:

$$DETJAC^* = 2\pi r \sqrt{\tfrac{\partial r}{\partial \xi}^2 + \tfrac{\partial z}{\partial \xi}^2}$$

Also of interest is the mapping of spatial derivatives with respect to natural and local coordinates. The basic form of the transformation may be written as a matrix equation as follows:

$$\begin{aligned}\tfrac{\partial}{\partial \xi} &= \left\lfloor \tfrac{\partial r}{\partial \xi} \quad \tfrac{\partial z}{\partial \xi} \right\rfloor \begin{Bmatrix} \tfrac{\partial}{\partial r} \\ \tfrac{\partial}{\partial z} \end{Bmatrix} \\ &= \left[\mathrm{J}^* \right] \begin{Bmatrix} \tfrac{\partial}{\partial r} \\ \tfrac{\partial}{\partial z} \end{Bmatrix}\end{aligned}$$

where the matrix $[J^*]$ is the transformation matrix, but is not a true Jacobian matrix since it is not square.

$$\begin{aligned}\left[\mathrm{J}^*\right] &= \left\lfloor \tfrac{\partial r}{\partial \xi} \quad \tfrac{\partial z}{\partial \xi} \right\rfloor \\ \left[\mathrm{J}^*\right] &= \left\lfloor \tfrac{\partial N_i}{\partial \xi} r_i \quad \tfrac{\partial N_i}{\partial \xi} z_i \right\rfloor \\ \left[\mathrm{J}^*\right] &= \left\lfloor \mathrm{J}^*_{11} \quad \mathrm{J}^*_{12} \right\rfloor\end{aligned}$$

and

$$\begin{aligned}\left[\Gamma^*\right] &= \begin{Bmatrix} \tfrac{\partial \xi}{\partial r} \\ \tfrac{\partial \xi}{\partial z} \end{Bmatrix} \\ \left[\Gamma^*\right] &= \begin{Bmatrix} \tfrac{1}{\mathrm{J}^*_{11}} \\ \tfrac{1}{\mathrm{J}^*_{12}} \end{Bmatrix} \\ \left[\Gamma^*\right] &= \begin{Bmatrix} \Gamma^*_{11} \\ \Gamma^*_{21} \end{Bmatrix}\end{aligned}$$

The spatial derivatives with respect to the local coordinates can be mapped using the inverse of the transformation matrix as follows:

$$\begin{Bmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial z} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial \xi}{\partial r} \\ \frac{\partial \xi}{\partial z} \end{Bmatrix} \frac{\partial}{\partial \xi}$$

$$\begin{Bmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial z} \end{Bmatrix} = \left[\Gamma^*\right] \frac{\partial}{\partial \xi}$$

$$\begin{Bmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial z} \end{Bmatrix} = \begin{Bmatrix} \frac{1}{J^*_{11}} \\ \frac{1}{J^*_{12}} \end{Bmatrix} \frac{\partial}{\partial \xi}$$

Appendix D.4 Area Mapping in the Plane of the Mesh - Plane Strain Case

The differential area vector is:

$$
\begin{aligned}
d\bar{A} &= \left(d\bar{x} \times d\bar{y}\right) \\
&= \left(\begin{Bmatrix} dx \\ 0 \\ 0 \end{Bmatrix} \times \begin{Bmatrix} 0 \\ dy \\ 0 \end{Bmatrix} \right) \\
&= \begin{Bmatrix} 0 \\ 0 \\ dxdy \end{Bmatrix}
\end{aligned}
$$

The magnitude of the differential area is defined as follows:

$$
\begin{aligned}
dA = \left\| d\bar{A} \right\| &= \sqrt{d\bar{A} \bullet d\bar{A}} \\
&= \sqrt{\begin{Bmatrix} 0 \\ 0 \\ dxdy \end{Bmatrix} \bullet \begin{Bmatrix} 0 \\ 0 \\ dxdy \end{Bmatrix}} \\
&= \sqrt{\left(dxdy\right)^2} \\
&= dxdy
\end{aligned}
$$

In the natural coordinate system, the differential area vector is:

$$
\begin{aligned}
d\bar{A} &= \left(\begin{Bmatrix} \frac{\partial x}{\partial \xi} \\ \frac{\partial y}{\partial \xi} \\ \frac{\partial z}{\partial \xi} \end{Bmatrix} d\xi \times \begin{Bmatrix} \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \eta} \\ \frac{\partial z}{\partial \eta} \end{Bmatrix} d\eta \right) \\
&= \left(\begin{Bmatrix} \frac{\partial x}{\partial \xi} \\ \frac{\partial y}{\partial \xi} \\ 0 \end{Bmatrix} \times \begin{Bmatrix} \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \eta} \\ 0 \end{Bmatrix} \right) d\xi d\eta \\
&= \begin{Bmatrix} 0 \\ 0 \\ \frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta} - \frac{\partial x}{\partial \eta}\frac{\partial y}{\partial \xi} \end{Bmatrix} d\xi d\eta
\end{aligned}
$$

The magnitude of the differential area is:

$$
\begin{aligned}
dA &= \left\| d\bar{A} \right\| \\
&= \left\| \begin{Bmatrix} 0 \\ 0 \\ \frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta} - \frac{\partial x}{\partial \eta}\frac{\partial y}{\partial \xi} \end{Bmatrix} d\xi d\eta \right\| \\
&= \left(\frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta} - \frac{\partial x}{\partial \eta}\frac{\partial y}{\partial \xi} \right) d\xi d\eta \\
&= DETJAC \cdot d\xi d\eta
\end{aligned}
$$

In developing the requisite equations for finite element analysis, various expressions are integrated about the element area. It is necessary, then to transform dA into the natural coordinate system if the integration is to be performed in this coordinate system. This is accomplished by the use of the determinate of the Jacobian matrix for the transformation (referred to as "$DETJAC$" in the remainder of the equations). The determinant may be thought of as a scale factor for transforming the differential area into another coordinate system as follows:

$$
\begin{aligned}
\iint_A dA &= \int_y \int_x dx dy \\
&= \int_\eta \int_\xi \left(\frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta} - \frac{\partial x}{\partial \eta}\frac{\partial y}{\partial \xi} \right) d\xi d\eta \\
&= \int_\eta \int_\xi DETJAC \cdot d\xi d\eta
\end{aligned}
$$

Adding the typical interpolation to the area mapping yields the same results as for the volume mapping transformation.

Appendix D.5 Area Mapping Out of the Plane of the Mesh - Plane Strain Case

The differential area vector is:

$$
\begin{aligned}
d\vec{A} &= \left(d\vec{s} \times d\vec{z}\right) \\
&= \left(\begin{Bmatrix} \frac{\partial x}{\partial s} \\ \frac{\partial y}{\partial s} \\ \frac{\partial z}{\partial s} \end{Bmatrix} ds \times \begin{Bmatrix} 0 \\ 0 \\ dz \end{Bmatrix} \right) \\
&= \begin{Bmatrix} \frac{\partial y}{\partial s} \\ -\frac{\partial x}{\partial s} \\ 0 \end{Bmatrix} dsdz
\end{aligned}
$$

The magnitude of the differential area is defined as follows:

$$
\begin{aligned}
dA &= \left\| d\vec{A} \right\| = \sqrt{d\vec{A} \bullet d\vec{A}} \\
&= \left\| \begin{Bmatrix} \frac{\partial y}{\partial s} \\ -\frac{\partial x}{\partial s} \\ 0 \end{Bmatrix} dsdz \right\| \\
&= \sqrt{\frac{\partial y}{\partial s}^2 + \left(-\frac{\partial x}{\partial s}\right)^2} dsdz \\
&= \sqrt{\frac{\partial x}{\partial s}^2 + \frac{\partial y}{\partial s}^2} dsdz
\end{aligned}
$$

In the natural coordinate system, the differential area vector is:

$$
\begin{aligned}
d\bar{A} &= \left(\begin{Bmatrix} \frac{\partial x}{\partial \xi} \\ \frac{\partial y}{\partial \xi} \\ \frac{\partial z}{\partial \xi} \end{Bmatrix} d\xi \times \begin{Bmatrix} \frac{\partial x}{\partial \zeta} \\ \frac{\partial y}{\partial \zeta} \\ \frac{\partial z}{\partial \zeta} \end{Bmatrix} d\zeta \right) \\
&= \left(\begin{Bmatrix} \frac{\partial x}{\partial \xi} \\ \frac{\partial y}{\partial \xi} \\ 0 \end{Bmatrix} \times \begin{Bmatrix} 0 \\ 0 \\ \frac{\partial z}{\partial \zeta} \end{Bmatrix} \right) d\xi d\zeta \\
&= \begin{Bmatrix} \frac{\partial y}{\partial \xi} \\ -\frac{\partial x}{\partial \xi} \\ 0 \end{Bmatrix} \frac{\partial z}{\partial \zeta} d\xi d\zeta \\
&= \begin{Bmatrix} \frac{\partial y}{\partial \xi} \\ -\frac{\partial x}{\partial \xi} \\ 0 \end{Bmatrix} d\xi dz
\end{aligned}
$$

The magnitude of the differential area is:

$$
\begin{aligned}
dA &= \left\| d\bar{A} \right\| \\
&= \left\| \begin{Bmatrix} \frac{\partial y}{\partial \xi} \\ -\frac{\partial x}{\partial \xi} \\ 0 \end{Bmatrix} \frac{\partial z}{\partial \zeta} d\xi d\zeta \right\| \\
&= \sqrt{\frac{\partial y}{\partial \xi}^2 + \left(-\frac{\partial x}{\partial \xi}\right)^2} \frac{\partial z}{\partial \zeta} d\xi d\zeta \\
&= \sqrt{\frac{\partial x}{\partial \xi}^2 + \frac{\partial y}{\partial \xi}^2} d\xi dz
\end{aligned}
$$

In developing the requisite equations for finite element analysis, various expressions are integrated about areas out of the plane of the mesh. Good examples of this are for deriving element stiffnesses for interface elements, and deriving equivalent nodal loads from boundary pressures (surface tractions). It is necessary, then to transform dA into the natural coordinate system if the integration is to be performed in this coordinate system. Since the mapping is from one dimension to two dimensions, the transformation matrix is not square. Thus, it is not a true "Jacobian matrix" there is no determinant or inverse. The basic concept of inverting the transformation, however, is still valid, but some care must be taken in inverting this relationship. The "determinant", *DETJAC*, may still be regarded as a scale factor for transforming the differential area into another coordinate system, but is not actually the determinant of a matrix in this case, and will be denoted as *DETJAC**.

$$
\begin{aligned}
\iint_A dA &= \int_z\int_s \sqrt{\frac{\partial x}{\partial s}^2 + \frac{\partial y}{\partial s}^2}\, ds dz \\
&= \int_z\int_\xi \sqrt{\frac{\partial x}{\partial \xi}^2 + \frac{\partial y}{\partial \xi}^2}\, d\xi dz \\
&= \int_\xi \sqrt{\frac{\partial x}{\partial \xi}^2 + \frac{\partial y}{\partial \xi}^2}\, d\xi \\
&= \int_\xi DETJAC^* \cdot d\xi
\end{aligned}
$$

Here the plane strain, out of plane mapping function is:

$$DETJAC^* = \sqrt{\frac{\partial r}{\partial \xi}^2 + \frac{\partial z}{\partial \xi}^2}$$

Also of interest is the mapping of spatial derivatives with respect to natural and local coordinates. The basic form of the transformation may be written as a matrix equation as follows:

$$
\begin{aligned}
\frac{\partial}{\partial \xi} &= \left\lfloor \frac{\partial x}{\partial \xi} \quad \frac{\partial y}{\partial \xi} \right\rfloor \begin{Bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{Bmatrix} \\
&= \left[J^* \right] \begin{Bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{Bmatrix}
\end{aligned}
$$

where the matrix [J] is the transformation matrix, but is not exactly a Jacobian matrix since it is not square.

$$
\begin{aligned}
\left[J^*\right] &= \left\lfloor \frac{\partial x}{\partial \xi} \quad \frac{\partial y}{\partial \xi} \right\rfloor \\
\left[J^*\right] &= \left\lfloor \frac{\partial N_i}{\partial \xi} x_i \quad \frac{\partial N_i}{\partial \xi} y_i \right\rfloor \\
\left[J^*\right] &= \left\lfloor J^*_{11} \quad J^*_{12} \right\rfloor
\end{aligned}
$$

and

$$
\begin{aligned}
\left[\Gamma^*\right] &= \begin{Bmatrix} \frac{\partial \xi}{\partial x} \\ \frac{\partial \xi}{\partial y} \end{Bmatrix} \\
\left[\Gamma^*\right] &= \begin{Bmatrix} \frac{1}{J^*_{11}} \\ \frac{1}{J^*_{12}} \end{Bmatrix} \\
\left[\Gamma^*\right] &= \begin{Bmatrix} \Gamma^*_{11} \\ \Gamma^*_{21} \end{Bmatrix}
\end{aligned}
$$

The spatial derivatives with respect to the local coordinates can be mapped using the inverse of the transformation matrix as follows:

$$\begin{Bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial \xi}{\partial x} \\ \frac{\partial \xi}{\partial y} \end{Bmatrix} \frac{\partial}{\partial \xi}$$

$$\begin{Bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{Bmatrix} = \left[\Gamma^{*} \right] \frac{\partial}{\partial \xi}$$

$$\begin{Bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{Bmatrix} = \begin{Bmatrix} \frac{1}{J^{*}_{11}} \\ \frac{1}{J^{*}_{12}} \end{Bmatrix} \frac{\partial}{\partial \xi}$$