Probabilistic Vibration Analysis of Nearly Periodic Structures

by

Karen F. Studebaker

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APPROVED:

Dr. Efstratios Nikolaidis, Chairman

Dr. Raphael T. Haftka

Dr. Rakesh K. Kapania

Dr. Mahendra P. Singh

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(ABSTRACT)

The localization of modes of nearly periodic structures causes a concentration of energy in one part of the mode shape. It occurs for a disordered structure with weakly coupled subsystems. The forced excitation of localized modes may affect the maximum response amplitude of the system. Two nearly periodic structures are analyzed herein: a two span beam and a pair of coupled pendula. Results show that the sensitivity of the forced response to the degree of localization depends on a combination of the symmetry of the mode which is excited and the phase difference between the forces acting on each substructure. These results attempt to explain the range of contrasting conclusions of previous research on the effects of forced response on mistuned structures. Furthermore, a theoretical explanation of the results is given in terms of transfer admittance.

A probabilistic analysis of the free and forced response of a nearly periodic structure is shown to be useful in the design of such structures which are sensitive to the degree of localization. The second moment method is used in the analysis with results verified by Monte Carlo simulation. The probabilities of localization and failure are calculated given the statistics of the system parameters, and the localization and failure tolerances respectively

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Table of Contents

Chapter 1. Introduction 1

1.1: Characteristics of Nearly Periodic Structures 1

1.2: Literature Review 5

1.2.1: Stabilization of Flutter 5

1.2.2: Localization of Bladed Disk Assemblies 7

1.2.3: Localization of Other Nearly Periodic Structures 8

1.3: Objectives 11

1.4: Summary of Present Research 13

Chapter 2. Localization of Modes in Nearly Periodic Structures 15

2.1: Analysis of Two Span Beam 16

2.1.1: Calculation of the Localized Modes 19

2.1.2: Sensitivity of Localization to System Parameters 23

2.2: Analysis of Coupled Pendula 24

2.2.1: Calculation of the Localized Modes 25

2.2.2: Analysis of Localization 27

Chapter 3. Dynamic Response of Nearly Periodic Structures 29

3.1: Analysis of Two Span Beam 31

3.1.1: Rayleigh-Ritz Method 31

3.1.2: Mode Superposition Method 37

3.1.3: Results 40

3.2: Analysis of Coupled Pendula 47

3.2.1: Calculation of the Forced Response 48

3.2.2: Results 50

3.3: Discussion of Results 54

3.3.1: Comparison with Previous Work 54

3.3.2: Transfer Admittance 55

Chapter 4. Probabilistic Analysis of Nearly Periodic Structures 61

4.1: Reliability 62

4.2: Second Moment Method 64

4.3: Probability of Localization 69

4.4: Monte Carlo Simulation 73

4.5: Probability of Failure 75

Chapter 5. Conclusions and Recommendations for Future Work 79

5.1: Conclusions 79

5.2: Recommendations for Future Work 83

References 85

Tables 88

Figures 93

List of Tables

Table 1	Sensitivity of 2nd Mode of Two Span Beam to Number of Modes, NM, Used in Solution	89
Table 2	Probability of Localization of 1st Mode of the Two Span Beam for Varying Localization Factors	90
Table 3	Probability of Survival of the Two Span Beam with Failure Defined by the Maximum Response Due to In Phase Forces Exciting the First Mode	91
Table 4	Probability of Failure of the Two Span Beam	92

List of Figures

Figure 1	Disordered Two Span Beam	94
Figure 2	Nearly Periodic Pair of Coupled Pendula	95
Figure 3	Bladed Disk Assembly	96
Figure 4	1st and 2nd Tuned Modes of the Two Span Beam, $\bar{c} = 1000, \overline{\Delta L} = 0$	97
Figure 5	1st and 2nd Localized Modes of Mistuned Two Span Beam, $\bar{c} = 400, \overline{\Delta L} = 40$	98
Figure 6	Localization Factors of 1st Mode of the Two Span Beam	99
Figure 7	Localization Factors of 2nd Mode of the Two Span Beam	100
Figure 8	1st and 2nd Modes of Ordered, Coupled Pendula, $m_1 = m_2 = m$ and $k = 200, \overline{\Delta L} = 0$	101
Figure 9	1st and 2nd Localized Modes of Disordered, Coupled Pendula, $k = 200$ lbf/in, $\overline{\Delta L} = 0.044$	102
Figure 10	Localization Factors of 1st and 2nd Modes of Disordered Pendula	103
Figure 11	Forced Excitation of the Two Span Beam	104

Figure 12	In Phase Forced Response of the Beam Exciting the 1st Mode	105
Figure 13	Out of Phase Forced Response of the Beam Exciting the 1st Mode	106
Figure 14	In Phase Forced Response of the Beam Exciting the 2nd Mode	107
Figure 15	Out of Phase Forced Response of the Beam Exciting the 2nd Mode	108
Figure 16	Forced Excitation of Disordered, Coupled Pendula	109
Figure 17	In Phase Forced Response Exciting the 1st Mode of the Pendula	110
Figure 18	Out of Phase Forced Response Exciting the 1st Mode of the Pendula	111
Figure 19	In Phase Forced Response Exciting the 2nd Mode the of the Pendula	112
Figure 20	Out of Phase Forced Response Exciting the 2nd Mode of the Pendula	113
Figure 21	Probability Distribution Functions	114
Figure 22	Linear Approximation of Failure Surface in Reduced Space	115
Figure 23	Probability of Localization of the First Beam Mode	116
Figure 24	Probability of Survival for Varying Forced Response Limits	117
Figure 25	Probability of Failure for Varying Mean Stiffness, $\mu_{\bar{c}}$	118

Nomenclature

$a_i(t)$	generalized coordinate
$\bar{a}_i$	ith modal amplitude
a	larger localized modal amplitude
a^*	smaller localized modal amplitude
A	localization factor
c	stiffness of beam's torsional spring
$\bar{c}$	nondimensionalized spring stiffness
$\bar{c}'$	nondimensionalized stiffness in reduced space
[C]	global pendula damping matrix
EI	beam flexural rigidity
$f_{1,2}$	beam constraints
$F_{1,2}$	applied harmonic forces
f_X, f_Y	probability density function
F_X, F_Y	probability distribution function
g	acceleration due to gravity
g(X)	performance function

k	stiffness of linear spring coupling pendula
[K]	global pendula stiffness matrix
L	beam length
L	Lagrangian
$L_{1,2}$	pendula support lengths
m	Mass of single span beam
[M]	global pendula mass matrix
NM	number of tuned modes in Rayleigh-Ritz method
P(t)	generalized force
P_F	probability of failure
P_L	probability of localization
P_S	probability of survival
$P_o(t)$	generalized force amplitude
$q_i(t)$	generalized coordinate
T	kinetic energy of the beam
U	strain energy of the beam
w(x,t)	beam vertical displacement
$\overline{w}(x,t)$	nondimensional beam displacement
W	potential energy of the beam due to applied forces
x	position along beam length
$\overline{x}$	nondimensional position along beam length
X	array of design variables
$x_{1,2}$	pendula displacements
x_c	position of beam's center support
$Y(x_1, x_2, \omega)$	transfer admittance

Greek:

α	phase difference between applied forces
$\beta_{1,2}$	Lagrange multipliers
$\overline{\beta}_{1,2}$	harmonic amplitude of Lagrange multipliers
$\Delta_{1,2}$	damping factor between substructures
ΔL	length difference between beam spans, or mistuning
$\overline{\Delta L}$	nondimensional mistuning
$\overline{\Delta L}'$	nondimensional mistuning in reduced space
$\mu_{1,2}$	eigenvalues of the coupled pendula
$\mu_{\bar{c}}$	mean of nondimensional spring stiffness of beam
$\mu_{\overline{\Delta L}}$	mean of nondimensional mistuning
$\psi_i(t)$	modes of the single span beam
$\psi_{1,2}(t)$	smaller amplitudes of normalized pendulum modes
$\overline{\Psi}(t)$	normalized modes of the two span beam
$\Psi_{1,2}(t)$	normalized pendula modes
$\sigma_{\bar{c}}$	mean of nondimensional spring stiffness of beam
$\sigma_{\overline{\Delta L}}$	mean of nondimensional mistuning
ω_i	frequencies of the single span beam,
$\omega_{1,2}$	tuned frequencies of coupled pendula
Ω	frequency of forced excitation of beam and pendula
$\overline{\Omega}$	nondimensional forcing frequency
$\Omega_{1,2}$	modal frequencies of disordered beam and pendula systems
$\overline{\Omega}_{1,2}$	nondimensional modal frequencies

Chapter 1. Introduction

1.1 CHARACTERISTICS OF NEARLY PERIODIC STRUCTURES

A periodic structure exhibits cyclic symmetry or some other form of periodicity. Such a structure is periodic in the sense that it has a number of substructures with repeating characteristics. A nearly periodic structure, however, is one which has a slightly altered property between its substructures. An example is a two span beam with the two span lengths differing by a small fraction of the beam length (Figure 1). Another example is a system of coupled pendula with slightly varied supporting lengths (Figure 2). In addition to these systems being nearly periodic, the subsystems are coupled together either by the continuity of the beam length or the linear spring between the two pendula.

A bladed disk assembly is also a periodic structure which may have a small deviation causing it to be nearly periodic. A bladed disk assembly is an array of blades around a central disk such as the turbine or compressor in aircraft turbofan engines (Figure 3). Each blade of an assembly is designed to have the same individual natural frequencies. However, in real life the natural frequencies of the blades are not equal.

This frequency difference, called mistuning, is the result of circumferential asymmetries such as slight variations in the individual blade properties and mass or stiffness variations around the disk. The first discussion of structural mistuning was given in a paper by Whitehead in 1965 [1] where he discussed the beneficial and adverse effects of mistuning on bladed disk assemblies.

The property of mistuning in a bladed disk assembly is paralleled in the two span beam and pair of coupled pendula. The disorder in these systems similarly can be called structural mistuning. In the case of the multispan beam, the mistuning is the small difference in length of the two beams (Figure 1). The mistuning in the coupled pendula example is the length differential between the two supports (Figure 2). A series of N coupled pendula models the characteristics of an assembly with N number of blades such that each pendula models a blade attached to a central disk [2]. The natural frequency of each pendulum is equal to the first natural frequencies of a blade. Therefore, the system of two coupled pendula model the limiting case of a bladed disk assembly with just two blades. It has been shown that a two span beam also characterizes the vibration of a bladed disk assembly [3]. In this case the difference in the span lengths of a two span beam is parallel to the difference in blade lengths of a disk with two blades. Furthermore, a multispan beam with N number of spans models a bladed disk assembly with N blades.

In general, mistuning is the small variation in the dynamic properties of a system. The mistuning in a bladed disk assembly is caused by manufacturing imperfections in the blade lengths or general degradation due to aging. When mistuning is introduced into an assembly, the result is a concentration of vibrational energy in one part of the assembly's mode shapes. This phenomenon is called localization of vibrational modes and may occur in any of the structure's modes. Many turbomachinery components have some degree of mistuning and localized vibration. Despite the fact that localization is known to affect the aeroelastic stability and forced response of mistuned structures, the vibrational analysis of rotors is most often performed for the tuned condition. In addition to the mistuned bladed disk assembly, localization occurs in other nearly periodic structures such as the two span beam and coupled pendula discussed in this paper.

Coupling is another important parameter in nearly periodic structures. As the subsystems become weakly coupled, the degree of mode localization increases [3]. Therefore, mistuned structures with nearly independent substructures are more susceptible to a high concentration of energy in one of the substructures. In a bladed disk assembly the coupling between the blades is due to the connection of the blades to the disk, a shroud linking the blades, and also to the gas flowing between the blades. In order to effectively model a bladed disk assembly, the substructures within the nearly periodic beam and pendula systems are coupled to one another. In the case of two span beam, the individual lengths of the beam are coupled since the beam is continous across its length. A torsional torsional spring at the position of the middle support is adjusted to change to the amount of coupling in the system. Similarly, the two pendula are connected by a linear spring.

Localization occurs more readily in a system consisting of disordered substructures which are weakly coupled. This effect of weak coupling can be explained by first analyzing its effect on a tuned system. A tuned system with weakly coupled substructures has closely spaced eigenvalues which are clustered in a small passband [4,5]. A passband is a grouping of the natural frequencies of a system. For example, the first and second modal frequences as well as higher pairs of frequencies become closer in value so that they form what is termed a 'band' of frequencies over the spectral range. When mistuning is introduced, the passbands separate above and below the original frequency. As a result, the localization of the mode shapes becomes more pronounced than the case of strongly coupled substructures due to the closeness of the eigenvalues. For more strongly coupled substructures, localization will still occur; however, an increase in mistuning results in only a slight increase in the degree of localization. Finally, when coupling between subsystems is very strong, there is no amount of mistuning which can cause the structure to have a concentration of energy large enough to be considered localized.

Localization of modes is important because the dynamic response of a mistuned system may be considerably higher than that of a tuned system, which leads to higher vibration levels and larger stresses. It has been found that the small differences in the structural or inertial properties of the blades can affect the amplitudes of individual blades by several hundred percent [1,6]. Ewins [7] showed response levels to be up to twenty percent higher than that of the comparable tuned system. Such large responses could result in structural failure. The importance in predicting the response of a mistuned structure is that localization occurs for even small deviations of periodicity in the structure. In fact, mistuning played an important role in several costly failures in the development and production of modern aircraft turbofan engines [8]. However, the analysis of the forced response of a mistuned bladed disk assembly is difficult and subject to much research and debate. In the next section it will be discussed how most research indicates that mistuning increases the forced response of nearly periodic structures, although different conclusions have been made concerning the degree of this increase and the conditions leading to it. Furthermore, it will be shown that other research has concluded that the forced response of a localized system is less than that of the tuned structure. This leads to the opposite conclusion: mistuning can be beneficial by reducing the vibrational amplitudes of forced response. These conflicting conclusions are due to a lack of complete understanding of the phenomenon of localization, and as a result it limits the incorporation of mistuning in present design of bladed disk assemblies.

1.2 LITERATURE REVIEW

Research dealing with nearly periodic structures covers a range of topics from mistuned bladed disk assemblies, multispan beams and pendula systems to large space structures. Many researchers have studied localization in mistuned bladed disk assemblies as a way to reduce flutter and therefore improve performance in turbomachinery. Other work has been done to study the vibrational characteristics of mistuned bladed disk assemblies. Investigations have been made concerning the free and forced vibrational response of assemblies due to forces caused by engine excitation or incident gas flows. As previously discussed, the importance of these studies lies in the high magnitude of the resulting stress in the blades and the possibility of failure. A newer area of research is the study of localized vibrations in large space reflectors. Also, a limited amount of work towards the understanding of the localization phenomenon has been done using simpler nearly periodic structures such as a series of coupled pendula and continuous beams with multiple supports.

1.2.1 Stabilization of Flutter

It has been shown that mistuning in bladed disk assemblies helps to limit the aerodynamic instabilities in the rotor blades caused by unsteady aerodynamic forces and moments of incident gas flow [5,9,10,11]. This instability is called flutter, and it restricts the high speed operating range of the engine. Compressor blade flutter originates from the aerodynamic interference on a blade caused by the motion of its neighboring blades. Since mistuning is beneficial in limiting flutter, it is sometimes suggested that mistuning be deliberately introduced. However, the practical application of using mistuning to reduce flutter is limited because it may also introduce a high structural response in the system [5]. Therefore, there is a danger in introducing intentional mistune into highly stressed airfoils that are susceptible to high cycle fatigue. Kaza and Kielb [10], however, illustrated that it may be feasible to utilize mistuning as a passive control to increase flutter speed while maintaining forced response at an acceptable level.

Mistuning in bladed disk assemblies falls under two categories: structural mistuning and aerodynamic mistuning. Structural mistuning is the blade to blade variation in structural properties such as blade length, mass, and stiffness. Aerodynamic mistuning is the difference in the unsteady aerodynamic flow field passing through a rotor blade row. Hoyniak and Fleeter [11] have shown that aerodynamic and structural mistuning individually enhance the stability of a rotor with supersonic unstalled flutter. Furthermore, they showed that the combination of both types of disorder gives the greatest reduction in flutter over a complete frequency range. The deliberate introduction of structural mistuning is, however, not practical in terms of manufacturing, material inventory, engine maintenance control, and cost problems.

Crawley and Hall [12] studied the idea of an optimum increase in aeroelastic stability of a high bypass shroudless fan with a minimum amount of mistuning. They found that structural mistuning is more cost effective than aerodynamic mistuning and that it does not depend on an even number of blades being present. However, they concluded that small deviations from the optimum mistuning pattern severely reduces the expected gains in the stability margin. These errors are incurred by manufacturing inaccuracies and approximations in the optimization calculations. On the other hand, aerodynamic mistuning is relatively insensitive to these errors.

Nissim [13] studied the affects of mistuning on the eigenvalues and eigenvectors of a compressor blade system using an aerodynamic energy approach. He showed that the sum of the real parts of the eigenvalues is independent of the amount of mistuning in the system. From this result he concluded that the upper bound for stabilization through mistuning is reached when all real parts of the eigenvalues are equal to the invariant mean of their sum. In addition, any aerodynamic stabilization of a mode through mistuning must be accompanied by a deterioration of stability of another mode. It was also found that a shrouded bladed disk assembly is strongly coupled, and thus, any mistuning in the system results in a small amount of localization. Therefore, the stabilization of flutter using mistuning is less effective for a shrouded rotor.

1.2.2 Localization of Bladed Disk Assemblies

The effect of localized vibration on the free and forced response of bladed disk assemblies has been studied extensively because it has a large influence on the dynamic response of the blades. Some of the earliest work was done in 1965 when Whitehead [1] studied the upper limit of the increase in vibrational amplitude of mistuned blades subject to the wakes of an incident gas flow. Later, in 1976 Srinivasan and Frye [14] developed a method to predict the vibratory response of mistuned rotor blades.

Less work has been done in the area of forced response of mistuned bladed disk assemblies. Sinha [15] calculated the statistics of forced response of an assembly. However, the accuracy and applicability of his method is limited by the assumtion that the amount of damping and mistuning in the system is small. Huang [2] also studied the forced response of a mistuned bladed disk assembly. The author analytically calculated the mean and variance of the statistical distribution of the blades' amplitudes. However, he did not find the probability density functions of the amplitudes nor the probability that the maximum amplitude of a disk assembly be below some critical value. Furthermore, Huang's analysis is restricted to rotors with closely spaced blades. Monte Carlo simulation has also been used to evaluate the forced response of mistuned bladed disk assemblies [16]. As a result of the complexity of the forced response analysis due to the large number of bldes on the disk, it has not been possible to obtain sufficiently large samples for the response characteristics. Therefore, it has not been possible to make conclusions about the statistics of the dynamic response of the blades.

Conflicting results have been found in the area of forced response of mistuned bladed disk assemblies. Most research has shown that an increase in mistuning in an assembly causes an increase in the response amplitude [1,16]; however, there have also been conclusions drawn to the contrary. For example, Sogliero and Srinivasan [17] found that an increase in mistuning caused a decrease in the forced response and a increase in fatigue life of the blades. The conclusions of the present study provide an explanation of the aforementioned contradiction in the forced response predictions.

1.2.3 Localization of Other Nearly Periodic Structures

Hodges [18] illustrated the phenomenon of localization in nearly periodic structures using two models: a vibrating string with attached masses and a system of coupled pendula. Mistuning was introduced in the systems by randomly varying the natural frequencies of the coupled pendula and by irregularly spacing the masses along the vibrating string. His results were obtained for the vibrating string since the solution of the pendulum model is more difficult in the weak disorder limit. He analyzed the degree of localization in the string by calculating the attenuation of vibration in the structure as the incident waves are transmitted and reflected from the masses along the string.

Hodges concluded that the degree of localization is dependent on the ratio of disorder strength to coupling strength. Therefore, if the coupling between masses is weak compared to the length difference between them, then all of the system's modes will become localized. This ratio is an indication of the spread in natural frequencies of the subsystems compared to the width of the passband of the associated tuned system. Furthermore, Hodges found that the modes of a one-dimensional system are all localized even for a small amount of disorder although this localization might be weak. He stipulated that this is true as long as the scale of the system is large. Pierre and Dowell [19] used a system of eight coupled pendula and came to the same conclusions as Hodges [18]. However, they specified that there is a minimum amount of mistuning necessary to strongly localize the system in the case of finite structural systems such as the two span beam, pair of coupled pendula, and bladed disk assembly. They calculated the localized vibrations of the pendula system by using perturbation methods to derive approximate closed form solutions for the localization. Pierre and Dowell overcame the small coupling assumption in their perturbation analysis by making different approximations for the strong and weak coupling cases.

Hodges [18] studied the propagation of vibration through nearly periodic structures and showed that the localization caused by disorder results in a confinement of energy close to the source of excitation. Furthermore, he showed that the resulting response decays exponentially away from the source. This is different from a tuned system in which the mode shapes extend throughout the structure. In a later work, Hodges and Woodhouse [20] studied the transmission of energy through a localized system again consisting of a stretched string with point masses located at nearly periodic intervals. They showed that a sinusoidal force applied at one end decays exponentially away from the driving point rather than extending throughout the mode as in the tuned case. They made the reasonable assumption of small coupling between the string lengths since the weight of the masses are assumed much greater than the string weight. Furthermore, they modeled the above system with a series of coupled pendula. They found that the attenuation levels were large enough in some cases to be significant in structural vibration problems. Their results were checked experimentally with a nearly periodic stretched string, and they showed reasonable agreement.

Work has been done by Pierre et al. [3] in studying the free vibration of disordered two span beams. The first mistuned mode was calculated using Rayleigh-Ritz and modified perturbation methods. They showed that the degree of localization increases as the system becomes more mistuned and uncoupled. Also, it was shown that localization occurs if the passband of the tuned beam is on the order of, or smaller than the relative spread in the frequencies of the individual spans. These results were verified experimentally.

Later, Pierre [4] studied the free and forced vibration in multi-degree-of-freedom systems using the example of ten coupled pendula and an assembly of disordered coupled beams. He analyzed the occurrence of mode localization in the system. In nearly periodic structures, a small disorder causes drastic changes in the system; therefore, he modified the perturbation method because a classical perturbation analysis can deal only with small perturbations. The modification was to treat the coupling as the perturbation and modify the unperturbed system by mistuning it. The problem of this method is that small amounts of coupling are assumed. As already known, he showed that strong localization occurs for weak coupling between the pendula. Furthermore, Pierre concluded that a system will be localized if the modal coupling is on the order of, or smaller than the modal mistuning. In higher modes the modal coupling decreases; therefore, localization occurs more easily. In addition, Pierre showed that localization is unavoidable in high modes if the amount of mistuning is small and the coupling is large. If the location of coupling stiffness is located at the node of a component mode, the structure is strongly localized.

However, in 1986 Pierre [3] came to the opposite conclusion that if localization does not occur in the lower two modes of a disordered two span beam, then the higher modes are not localized. Also, he hypothesized that if the first two modes are localized then the higher modes will be localized as well. Pierre [4] also analyzed the response of the coupled pendula due to a periodic force applied to a pendula on one end. He found the same result as Hodges which is that the response of nearly periodic structures subject to a sinusoidal force at one end decays exponentially away from the source with a confinement of energy at the source.

Bendiksen [21,22] studied the localization phenomenon in large space structures such as large astronomical telescope reflectors and communication antennas. In Ref. [21] he verified that localization of modes exists and that these structures are especially susceptible because of the high number of weakly coupled substructures. Cornwell and Bendiksen [22] used a multi-degree-of-freedom model with 18 radial rib reflector which they analyzed using finite elements. They concluded that localization becomes more severe with increasing mode number. They also observed strong interactions of the localization peaks as the disorder increased due to a crossing of the natural frequencies and modes of the beam elements

1.3 OBJECTIVES

As it was mentioned in Section 1.2.2, there is a lack of complete understanding of the localization phenomenon in mechanical systems. Furthermore, it is widely accepted that a complete approach to the problem of mistuning should be probabilistic [16]. The scope of such an approach would be to derive the statistics of the vibratory stress and displacements from the tolerances in the dynamic parameters of the individual subsystems.

The scope of this work is to develop an approach for estimating the statistics of the response of some simple mistuned structural systems such as multispanned beams and coupled pendula. Furthermore, attention is given to a loading case that has been ignored by previous studies. This is the case of two or more harmonic loads acting simultaneously in phase or out of phase at different points of the structure. This is an important loading case for two reasons. First, it simulates the loading applied to some real life engineering structures such as bladed disk assemblies. Second, in this particular loading case, the effects of mistuning on the response level are different than that in the loading cases considered in previous studies. More specifically, it is shown that the forced response level may not be sensitive or may even decrease when the degree of mistuning is increasing. The consideration of the above loading cases enables an explanation of the conflict in the conclusions of previous studies concerning the effect of mistuning on the dynamic response.

The objectives of this thesis are:

- To study the statistical characteristics of the localized modes of two nearly periodic structures (the two span beam and the pair of coupled pendula) for random configurations of mistuning and coupling in the systems.
- To obtain the dynamic response in the case of two in and out of phase loads acting simultaneously at different parts of the structure.
- To calculate the probability of failure of the disordered two span beam for a range of failure limits given that the system is randomly configured in its degree of mistuning and coupling of the substructures.

- To study the sensitivity of the probability of failure of the mistuned two span beam for various mean values of the statistics of the system parameters.

1.4 SUMMARY OF PRESENT RESEARCH

The present study uses Pierre, Tang, and Dowell's [3] formulation of the two span beam system and expands the scope of the research to the dynamic response. The response is calculated due to both in and out of phase forces acting simultaneously. A study of this loading case is particularly important as an aid in understanding the dynamic behavior of bladed disk assemblies. This simulates the limiting case of a turbine with only two blades situated opposite one another where two periodic forces of equal magnitude and a phase lag of zero or 180 degrees are applied to the blades. The phase lag is a result of the blades' rotation through the flow of gas or steam.

In the free vibration study, the first and second mistuned modes of the nearly periodic beam are calculated using the Rayleigh-Ritz method. After calculating the free response for random configurations of the system, a probabilistic analysis is made. The objective is to find the probability of localization of the first mistuned mode of the two span beam. Here the localization of a mode is defined as the event that the maximum vibratory amplitude in the span is lower than a prescribed percentage of the maximum amplitude in the other span. This uses amplitude ratios to express the fact that the vibratory energy is concentrated in one span of the beam. The probability of localization is found using a second moment method [23], and the stiffness and mistuning of the beam are assumed to be normally distributed random variables. As expected, the probability of localization increases with an increase in the degree of mistuning or a decrease in the coupling of the subsystems.

The forced vibration response is calculated using the Rayleigh-Ritz and mode superposition methods for the beam and the impedance method for the coupled pendula. Both in phase and out of phase forces are considered which excite the first or second mistuned mode shape. The probability of failure is also calculated by the second moment method for random amounts of coupling and mistuning in the systems. Failure is defined as the event that the maximum response is greater than a critical limit. It is shown that the localization of a mode shape does not necessarily mean an increased maximum amplitude of the forced response. Furthermore, the analysis shows that the localization of the mode shape may or may not cause the forced response to increase depending on the mode which is excited and the phases of the applied forces. Therefore, it is not necessarily true that the probability of failure increases with the degree of localization. The in phase excitation of an antisymmetric mistuned mode shape is sensitive to the degree of localization of the system. Similarly, the out of phase excitation of a symmetric mode results in a large response increase. However, the in phase excitation of a symmetric mode and the out of phase excitation of an antisymmetric mode are relatively insensitive to the degree of localization, and they may even show a decrease in the forced response.

Chapter 2. Localization of Modes in Nearly Periodic Structures

The first mention of localization was by Anderson [23] in the field of solid state physics. He studied its effects in metals to predict electron transport and spin diffusion. Then, Mott [24] used Anderson's ideas to analyze the localization of electron eigenstates of disordered or amorphous solids. The localization of the electrons causes a metallic conductor to behave like a semiconductor. Mode localization in disordered systems is known as Anderson localization in the area of condensed matter physics. In fact, the combined work of Mott and Anderson [25] contributed to a Nobel Prize which they earned in 1977.

Localization can occur in any of the modes of a nearly periodic structure which is weakly coupled. In the previous chapter it was shown that these structures range from bladed disk assemblies to large space reflectors. Localization has been demonstrated easily in these examples. However, it is much more difficult to predict the configuration which just causes the modes to localize. Hodges [18] discussed this difficulty of finding a minimum disorder strength required to localize all of a system's modes. This difficulty is a result of the dependence of the phenomenon of localization on the dimensionality of the system. Hodges in Ref. [17] gave the following account based on current knowledge in solid state physics. In a one-dimensional system any amount of disorder is sufficient to localize all modes of the system provided the scale is large enough. In two dimensions it is also thought that small amounts of mistuning will localize the whole system. However, in three dimensions the problem becomes more difficult: it is believed that there is a minimum or threshold disorder necessary to localize a three-dimensional system.

The more difficult, three-dimensional case is usually studied in the field of solid state physics because of its application to a solid lattice and electron motion. However, in structural dynamics the one-dimensional case is the most common such as the vibrating string or bending and axial motion of a beam. The two span beam and coupled pendula are examples of one-dimensional systems which greatly simplifies the analysis of these structures. Pierre and Dowell [19] modified Hodges' conclusion concerning the localization of a one-dimensional system for any disorder. They concluded that there does exist a minimum amount of mistuning needed to substantially localize the modes of a one-dimensional system if the system is finite.

2.1 ANALYSIS OF TWO SPAN BEAM

The problem of localization of modes in a simply supported two span beam is useful for understanding this phenomenon in more complicated structures (Figure 1). Pierre et al. [3] studied the free vibration of a two span beam and experimentally verified the existence of localized vibration in the beam. Applications of this research include the localization in bladed disk assemblies, large space antennas, and other nearly periodic structures. As discussed in Section 1.1 the mistuning and coupling parameters in a two span beam model the same properties as those in a bladed disk assembly. The degree of mistuning, or disorder, is determined by the distance of the center support from the middle of the beam. This distance, ΔL , is only a small fraction of the beam length. Furthermore, the degree of coupling is controlled by the stiffness of a torsional spring. For a large stiffness value, the two spans are essentially independent and the system is uncoupled. The mistuning in the beam is represented by its nondimensional value which is divided by the length of the beam, L, such that $\overline{\Delta L} = \Delta L/L$.

The localization of modes in a two span beam is studied by prescribing the degree of mistuning and coupling in the system and then computing the modes of free vibration. This is done in this thesis by the Rayleigh-Ritz method discussed by Pierre et al. [3]. To find the localized modes, the known modes of a single span beam are used. The single span beam has the same properties as the two span beam with a length, L, but it has no middle support. Pierre [3] showed that for a two span beam, the degree of localization decreases as the coupling of the system increases for a given amount of mistuning. Furthermore, as the degree of mistuning increases for a given amount of coupling, the localization increases.

One of the objectives of the present study is to evaluate the probability of localization of the free vibrational modes of the two span beam using the degree of mistuning and coupling as parameters. Therefore, the degree of localization must be calculated for varying amounts of disorder and coupling in the structure. In order to illustrate the localization of mode shapes, consider the first and second modes of the tuned two span beam (both spans of equal length) in Figure 4. The stiffness in the problem is nondimensionalized as in Pierre's [3] work such that $\bar{c} = 2cL/EI$. In Figure 4 the nondimensional stiffness is $\bar{c} = 1000$. The energy in both modes is evenly balanced along the beam length. The first mode is antisymmetric while the second mode is symmetric. Both modes become localized when the system is mistuned and weakly coupled. For example, consider the case when the center support is moved two percent of the full beam length to the left or right such that disorder in the system is $\Delta L = 0.01L$. Without a torsional spring, the two spans are completely coupled. When stiffness is added to the system, the coupling of the two spans is weakened, and the susceptibility of the system to localization increases. For a stiffness, $\bar{c} = 400$ and $\overline{\Delta L} = 0.04$, the modes are localized with a larger mode amplitude in one of the spans (Figure 5).

As in Ref [20], the localization of the system is characterized by the ratio of the amplitudes of the first vibrational mode of the mistuned system, $A = \frac{a^*}{a}$, where a^* is the smaller amplitude and a is the larger. Therefore, $A = 1$ is the perfectly tuned case, and $A = 0$ is completely localized. In Figure 5 the localization factor for the first mode is $A = 0.0577$; therefore, the smaller amplitude is 0.0577 times that of the larger amplitude. For the second mode the localization factor is $A = 0.0607$. Localization of the system can be arbitrarily defined such that the ratio of amplitudes of the first or second natural mode is less than a given fraction. Then, the probability of localization is the probability that the mode amplitude ratio is less than this value. If the mode shape is not drastically altered from the tuned case, then the system is not strongly localized. In this case the dynamics of the system will not be significantly different from that in the tuned case, and the system is not considered localized. For example, if $A = 0.1$ is defined as the limiting value for a localized first or second mode, then localization occurs when the smaller amplitude of the mode shape is less than one-tenth of the larger amplitude.

In Chapter 4 it will be shown by probabilistic methods that as the definition of localization is broadened to include larger fractions of the amplitude ratio as being localized, then the probability of the system being localized increases. Furthermore, when the definition of localization is limited to include only small amplitude ratios, the probability of localization approaches zero.

2.1.1 Calculation of the Localized Modes

In order to calculate the probability of localization for the two span beam, the localized modes must be calculated for a given degree of mistuning (ΔL) and coupling (c). The two span beam is a one-dimensional problem with its motion in bending. Since it is a finite system, a minimum disorder in the structure will cause all modes to localize to some degree. Furthermore, higher modes are more susceptible to localization than the lower ones. Cornwell and Bendiksen [22] have shown that localization becomes more severe with increasing mode number. Therefore, if the lower modes of a one-dimensional, mistuned structure are localized, then the higher modes will be localized. Therefore, in this thesis it is deemed sufficient to calculate only the first and second modes and use this as a measure of localization of the system.

Figure 1 shows the two span beam system composed of a simply supported beam of length L with a Young's modulus, E, mass per unit length, m, and moment of inertia, I. In addition, there is a center support which can be moved a distance ΔL from the center thus leaving the support a distance x_c from the left. The coupling of the system is represented by a torsional spring at x_c. In the extreme case of an infinitely stiff spring, the two spans act independently, and the system is uncoupled. Changing the amount of mistuning by moving the center support and changing the coupling of the individual single span beams affects the mode shapes of the system.

The first and second modes are calculated as in Pierre et al.'s [3] study by a Rayleigh-Ritz technique. This method defines the deflection of the two span beam as a weighted sum of the known modes of the single span beam. The transverse deflection of the beam is then given by,

$$w(x, t) = \sum_{i=1}^{NM} a_i(t)\phi_i(x) \tag{1}$$

where the $a_i(t)$ terms are the generalized coordinates, the $\phi_i(x)$ are the normalized modes of the single span beam, and NM is the number of modes used in the summation. The normalized modes of the single span beam are taken as generalized coordinates and are given by,

$$\phi_i(x) = \sqrt{\frac{2}{mL}}\,\sin\left(\frac{i\pi x}{L}\right) \tag{2}$$

with corresponding natural frequencies,

$$\omega_i = (i\pi)^2\sqrt{EI/mL^4} \tag{3}$$

In order to calculate the mode shape, the generalized coordinates, $a_i(t)$, must first be computed. This is done using Hamilton's principle to find the Langrange equation of motion and solve for the generalized coordinates. The strain energy of the two span beam is

$$U = \frac{1}{2}\sum_{i=1}^{NM}\omega_i^2 a_i^2 + \frac{1}{2}\,c[w'(x_c)]^2 \tag{4}$$

and the kinetic energy is

$$T = \frac{1}{2}\sum_{i=1}^{NM} \dot{a}_i^2 \qquad (5)$$

where w' is the derivative with respect to x, and $\dot{a}$ is a derivative with respect to time.

There are two constraints on the system. The first specifies that the displacement of the beam is zero at the point x_c where the beam is supported. The second requires that the slope of the beam at x_c is continuous. These constraints f_1 and f_2 can be written as

$$f_1 = \sum_{i=1}^{NM} a_i(t)\phi_i(x_c) = 0 \qquad (6)$$

$$f_2 = w'(x_c) - \sum_{i=1}^{NM} a_i(t)\phi'_i(x_c) = 0 \qquad (7)$$

Hamilton's principle is then applied to find the Lagrangian equations of motion of the system with the constraints added using Lagrange multipliers β_1 and β_2. The Lagrangian is then

$$L = U - T + \beta_1 f_1 + \beta_2 f_2 \qquad (8)$$

Then, the Lagrangian is minimized with respect to $\bar{a}_i$, β_1 and β_2. Simple harmonic motion of a_i and β_i is assumed with frequency Ω where $a_i = \bar{a}_i\, e^{j\Omega t}$ and $\beta_k = \bar{\beta}_k e^{j\Omega t}$ for $i = 1,\ldots,NM$, $k = 1,2$ and $j = \sqrt{-1}$. The minimization results in the following equations of motion,

$$\ddot{a}_i + \omega_i^2 a_i^2 - \beta_1\phi_i(x_1) + \beta_2\phi_i'(x_1) = 0 \qquad i = 1,\ldots,NM$$

$$\beta_2 = cw'(x_1)$$

$$f_1 = 0 \qquad f_2 = 0$$

The solution of these equations results in the following eigenvalue problem where the harmonic terms have been cancelled,

$$\overline{\beta}_1 \left[\sum_{i=1}^{NM} \frac{\phi_i^2(x_c)}{\omega_i^2 - \Omega^2} \right] - \overline{\beta}_2 \left[\sum_{i=1}^{NM} \frac{\phi_i(x_c)\phi_i'(x_c)}{\omega_i^2 - \Omega^2} \right] = 0 \tag{10}$$

$$\overline{\beta}_1 \left[\sum_{i=1}^{NM} \frac{\phi_i(x_c)\phi'_i(x_c)}{\omega_i^2 - \Omega^2} \right] - \overline{\beta}_2 \left[\frac{1}{c} + \sum_{i=1}^{NM} \frac{\phi'^2_i(x_c)}{\omega_i^2 - \Omega^2} \right] = 0 \tag{11}$$

The unknowns are the coefficients, a_i, $\overline{\beta}_1$, $\overline{\beta}_2$, and the frequency, Ω

A non-zero solution for $\overline{\beta}_1$ and $\overline{\beta}_2$ results when the determinant of the above system of equations is zero. In order to calculate the frequency Ω, a bracketing method is used to find the root of the eigenvalue problem defined by (10) and (11). The first nondimensional frequency lies in the range of $\overline{\Omega} = 40$ to 60 while the second is from 60 to 80. Depending on which modal frequency is found, either the first or second mode is then calculated. Once $\overline{\Omega}$ is calculated, β_1, β_2, and $\overline{a}_i$ are computed from the equations of motion and constraints. Finally, the displacement of the beam w(x,t) can be computed from equation (1) or the mode shape, $\overline{\Psi}$, can be calculated over the nondimensional length of the beam $\overline{x} = x / L$,

$$\overline{\Psi}(\overline{x}) = \sum_{i=1}^{NM} \overline{a}_i \sin(i\pi\overline{x}) \tag{12}$$

This is the mistuned mode shape, and once it is calculated for a specified coupling and mistuning of the system, the ratio A of the minimum to maximum amplitudes can be computed. To calculate the mistuned mode shape, a summation of tuned modes was used in equation (12). A sufficient degree of accuracy was obtained when 2000 tuned modes were used to calculate the first and second mistuned modes. Pierre et al. [3] have calculated the localization factors for the first mode of the two span beam for various various configurations of mistuning and coupling. These results have been recreated in Figure 6. In the present study, this analysis is extended to the localization of the second mode. The localization factors of the second mode for varying amounts of mistuning and coupling appear in Figure 7. They show the same trends of localization in terms of the disorder and coupling in the system as the localization of the first mode. These trends are discussed in the next section.

2.1.2 Sensitivity of Localization to the System Parameters

For the disordered two span beam, mistuning and coupling affect the degree of localization as follows. A system which is weakly coupled requires relatively small amounts of mistuning to become localized. In the case of the system in Figure 6, for example, if localization is defined as an amplitude ratio of the first mode with $A = 0.1$, the first mode will be localized with a nondimensionalized stiffness of $\overline{c} = 800$ when $\overline{\Delta L} \geq 0.012$. Any amount of mistuning below 0.012 causes some degree of energy concentration; however, it is less than the ten percent specified in this example as the limit for localization. If the coupling is made stronger by decreasing the stiffness of the spring, then more mistuning is needed to localize the beam. For example, if the stiffness is reduced to $\bar{c} = 400$, then the central support must be moved 2.2 percent of the beam length ($\overline{\Delta L} = 0.022$) in order to achieve a localization factor of $A = 0.1$. Finally, if the coupling is reduced to $\bar{c} = 100$, the mistuning would have to be raised above $\overline{\Delta L} = 0.07$ in order to localize the mode. This amount of disorder is too large to constitute a system which is considered nearly periodic. Therefore, for strongly coupled, mistuned systems, there is a lower limit to the susceptibility of localization.

The localization factors of the first and second modes are close to the same value for a given amount of mistuning and coupling. In some cases they are the same, while in others they differ by a small fraction. For example, for $\bar{c} = 1000$ and $\overline{\Delta L} = 0.094$, the localization factor of both the first and the second mode is 0.094. Therefore, the concentration of energy is the same for both modes.

2.2 ANALYSIS OF COUPLED PENDULA

A system of nearly periodic coupled pendula can also be used as a model for the vibration of more complicated structures. For example, the study of wave motion in a homogeneously disorderd medium is commonly analyzed using finite differences across a fine grid [18]. This discretization of a continuous problem produces equations of motion comparable to those of coupled pendula. An analysis of these two examples indicates that localization occurs for wave propagation in a discrete medium as well as a continuous one. It is not well understood how these waves interact with the irregularities in continuous systems to cause the localization. Hodges [18] gives an intuitive explanation of this localization phenomenon as the backscattering of waves by the inhomogeneities in the structure. In general, the pendula system can model any nearly periodic structure which is composed of a number of subsystems.

Hodges showed in Ref. [17] that localization of a series of coupled pendula is likely to occur if the spread in natural frequencies is small compared to the width of the tuned passband. This passband is obtained from the tuned case in which each pendula has the same frequency. Pierre and Dowell [19] concluded from a study of ten mistuned, coupled pendula that the disordered system has strongly localized modes if the coupling frequency between the subsystems is on the order of, or smaller than the spread in natural frequencies of the component subsystems. That is equvalent to say that localization occurs if the modal coupling is less than the modal mistuning [4].

Localization in a series of coupled pendula is achieved by mistuning the system so that it is nearly periodic. If the system is altered so that each pendulum has a slightly different natural frequency, then the mode shapes are drastically altered much like those of the two span beam. In this section an analysis is made of two pendula which are connected by a linear spring (Figure 2). The disorder is the small length differential ΔL of the supports of the pendula. Similar to the two span beam, the disorder is approximately one to seven percent of the supporting length. Contrary to the previous beam example, a large spring stiffness produces a strongly coupled system. It will again be shown that strong localization occurs for the case of weakly coupled nearly periodic structures.

2.2.1 Calculation of the Localized Modes

The mode shapes of the disordered pair of coupled pendula is obtained from a straightforward solution of the equations of motion of the system. The system shown in Figure 2 is disordered since the length of the left pendulum differs from that of the right by a small fraction of the support, ΔL. There are two degrees of freedom, x_1 and x_2 which are the displacements of the first and second pendula. The equation of motion is,

$$M\ddot{X} + C\dot{X} + KX = 0 \tag{13}$$

where M, C, and K are two by two matrices of mass, damping, and stiffness respectively. The displacements x_1 and x_2 are written as the vector, X. For the free vibration study, the system is considered undamped; therefore, the C matrix is set equal to zero. In matrix form this equation is,

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} \frac{m_1 g}{L_1} + k & -k \\ -k & \frac{m_2 g}{L_2} + k \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{14}$$

In this example the masses are $m_1 = m_2 = m$, and the supporting lengths are L_1 and L_2. The spring stiffness is k, and the gravitational acceleration, g. In order to solve for the two natural frequencies, the characteristic equation is written

$$\det([K] - \omega^2[M]) = 0 \tag{15}$$

where ω is the first or second natural frequency of the system. The solution of this eigenvalue problem gives the eigenvalues and eigenvectors. The eigenvalues, $\mu_{1,2}$, are the square of natural frequencies and the eigenvectors are their associated modes. The solution of the characteristic equation gives the following expression for the eigenvalues,

$$\mu_{1,2} = \frac{g}{2}\left(\frac{1}{L_1} + \frac{1}{L_2}\right) + \frac{k}{m} \mp \sqrt{\frac{g^2}{4}\left(\frac{1}{L_1} + \frac{1}{L_2}\right)^2 + \frac{k^2}{m^2}} \tag{16}$$

where $\omega_1 = \sqrt{\mu_1}$ and $\omega_2 = \sqrt{\mu_2}$. The mode shapes are found by substituting the eigenvalues into the determinant and normalizing the modes such that

$$\Psi_1 = \begin{Bmatrix} 1 \\ \phi_1 \end{Bmatrix} \quad \text{and} \quad \Psi_2 = \begin{Bmatrix} 1 \\ \phi_2 \end{Bmatrix} \tag{17}$$

where ϕ_1 and ϕ_2 are

$$\phi_{1,2} = \frac{m\,g}{2k}\left(\frac{1}{L_1} + \frac{1}{L_2}\right) \pm \sqrt{\left(\frac{m\,g}{2k}\right)^2\left(\frac{1}{L_1} + \frac{1}{L_2}\right)^2 + 1} \tag{18}$$

In the limiting case where the supporting lengths are equal ($L_1 = L_2 = L$), the system is tuned and the frequencies simplify to

$$\omega_1 = \sqrt{\frac{g}{L}} \quad \text{and} \quad \omega_2 = \sqrt{\frac{g}{L} + \frac{2k}{m}} \tag{19}$$

with associated mode shapes,

$$\Psi_1 = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \quad \text{and} \quad \Psi_2 = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} \tag{20}$$

2.2.2 Analysis of Localization

The modes of coupled pendula are discrete. This is different from the two span beam which has continuous mode shapes. The free vibrational response of the pair of tuned pendula is given by the modes in equation (20) and shown in Figure 8. The system is localized when it is weakly coupled and disordered. In the present analysis, the following values are used. Each mass is 100 slugs, and the length of each support is initially nine inches. The localized modes are also normalized such that the larger amplitude of motion is scaled to one as in equation (17). Figure 9 shows an example of mode localization in a pair of weakly coupled pendula with a spring stiffness $k = 200$ and a mistuning of 4.4 percent disorder in the support ($\Delta L/L_2 = 0.044$). As in the two span beam analysis, the first and second modes both localized to approximately the same degree. The localization factor is defined as the free vibratory amplitude of the pendulum which decreased from the amplitude of normalized tuned modes. Therefore, the localization factors are the values of ϕ_1 or ϕ_2 in equation (17). In this example, the localization factor for the first mode is 0.634 and that for the second is 0.632.

As for the analysis of the two span beam, the localization factor is calculated for varying configurations of the system in terms of spring stiffness and support length (Figure 10). The disorder is represented in terms of the nondimensional stiffness, and the coupling is given in terms of a parameter equal to 1000 lbf/in - k. The stiffness, k, varies from one to 1000 lbf/in. It is plotted in this way in order to compare with Figures 6 and 7 of the beam which are weakly coupled for higher values along the horizontal axis. Figure 10 shows that localization in the system increases for larger amounts of disorder and weaker coupling with the highest amounts of localization occurring in the upper right corner of the graph. Therefore, both the two span beam and pair of coupled pendula behave in the same manner in terms of their free vibrational response. Both systems are susceptible to localization when a dynamic property of the system is slightly altered and the subsystems are weakly coupled.

Chapter 3. Dynamic Response of Nearly Periodic Structures

As mentioned earlier, understanding the forced response of nearly periodic structures is far from being complete. Nevertheless, predicting the forced response is crucial to the analysis of such structures. For example, a large space structure is subject to forces which are applied to control its movement and vibration. Also, bladed disk assemblies have the applied forces induced by the flow of gas over the blades. The forces in the present study are assumed to be deterministic. Randomness is introduced into the problem only through the statistical deviations of coupling and mistuning in the system configuration. The responses due to both in and out of phase forces are analyzed. In the case of a bladed disk assembly, the out of phase forced response is particularly important because the forces applied to the blades have equal magnitude, and their phases are equal to the angle of blade separation around the disk or an integer multiple of this quantity.

A review of previous research shows that the forced response of nearly periodic structures has been studied to only a limited degree. The studies generally involve the periodic excitation of one substructure and the analysis of the propagation of vibration through the system. For example, both Hodges [18] and Pierre [4] have studied the effects of a sinusoidal force applied at one end of a nearly periodically weighted string or row of coupled pendula. Hodges in Ref. [18] discussed the confinement of energy close to the source of excitation and its exponential decay away from the driving point. He concluded that the rate of power input for a broad band source with a continuous frequency spectrum is insensitive to the amount of disorder in the system. Furthermore, the rate of power input is roughly the same as that of the comparable problem using a perfectly tuned structure. The localization of the response close to the source causes a concentration of energy at the source until the total rate of dissipation is equal to the rate of power input. Therefore, the response close to the source is greater than that in a periodic structure. Hodges has pointed out that since the power input is dissipated entirely within the system and since it is insensitive to the degree of mistuning, the total vibrational energy within the system is also insensitive to mistuning. Other research has analyzed the affects of resonant forces on the structure [19].

As discussed in Chapter 1, most of these past studies indicate that the localization of modes in a structure causes the forced response to be greater than that of the tuned system [1,16,27,28]. However, there have also been reports that the opposite is true [8,11,17,29]. No theoretical explanation of the above conclusions based on sound theoretical arguments have been provided. In this thesis it is shown that the forced response of a localized structure may increase or decrease depending on the phase of the forces and the mode of the structure which is being excited. Contingent upon these conditions, the change in the response may or may not be sensitive to changes in the degree of localization.

3.1 ANALYSIS OF TWO SPAN BEAM

The forced response of the two span beam is calculated by two different methods. The first is the Rayleigh-Ritz method. This method requires only slight modifications of the free vibrational analysis presented in section 2.1.1. Then, these results are verified using the mode superposition method. For each method the response was calculated due to two in phase or out of phase forces located at one-quarter and three-quarters of the beam length (Figure 11). The phase difference in the out of phase case was 180 degrees. This phase angle was chosen to simulate the forces applied to two blades rotating on a central disk such as that of a bladed disk assembly.

3.1.1 Rayleigh-Ritz Method

The applied forces are

$$F_1 = F\cos(\Omega t - \alpha) \quad \text{and} \quad F_2 = F\cos(\Omega t)$$

where the phase difference, α, is either 0° or 180°, and the amplitudes are equal. The frequency of the applied forces is set close to the frequency of the mode to be excited, Ω_1 or Ω_2. It is not set at exactly this frequency in order to prevent the responses from having excessively large resonant amplitudes. Therefore, the nondimensional frequency of the forces for the in and out of phase cases is

$$\overline{\Omega} = \overline{\Omega}_1 - 0.0001 \ . \tag{21}$$

The additional energy of the applied forces is taken into account in the Lagrangian equation of motion as nonconservative work, W_{nc}. The Lagrangian of equation (8) becomes

$$L = U - W_{nc} - T + \beta_1 f_1 + \beta_2 f_2 \tag{22}$$

The virtual work of the nonconservative forces is ∂W_{nc} is given by

$$\partial W_{nc} = \sum_{i=1}^{N} Q_i \partial a_i$$

where the Q_i are the N generalized nonconservative forces and the a_i are the generalized coordinates. In the two span beam analysis, there are two applied forces; therefore, the virtual work is

$$\partial W_{nc} = F_1 \partial w(L/4, t) + F_2 \partial w(3L/4, t) \tag{23}$$

The values $w(L/4, t)$ and $w(3L/4, t)$ are the displacements of the beam at the position of the applied forces. The displacement can be written as a general expression of position and time by a summation of the tuned modes given in equation (1) as,

$$w(x, t) = \sum_{i=1}^{NM} a_i(t)\phi_i(x) \tag{24}$$

The terms $a_i(t)$ are the generalized coordinates of the free vibration problem, and $\phi_i(x)$ are the tuned modes of the single span beam. These are given as in Chapter 2 as

$$\phi_i(x) = \sqrt{\frac{2}{ml}} \sin\left(\frac{i\pi x}{L}\right)$$

with corresponding natural frequencies,

$$\omega_i = (i\pi)^2 \sqrt{EI/mL^4}$$

The terms $a_i(t)$ and $\phi_i(x)$ are summed over the number of tuned modes, NM, in order to calculate the displacement. The terms U and T are the strain energy and kinetic energy respectively, and they can be given by,

$$U = \frac{1}{2}\sum_{i=1}^{NM} \omega_i^2 a_i^2 + \frac{1}{2} c[w'(x_c)]^2 \tag{25}$$

and

$$T = \frac{1}{2}\sum_{i=1}^{NM} \dot{a}_i^2 \tag{26}$$

The value f_1 in the Lagrangian is the constraint which specifies that the displacement of the beam at the position of the central support, x_c, is zero:

$$f_1 = \sum_{i=1}^{NM} a_i(t)\phi_i(x_c) = 0 \tag{27}$$

Furthermore, the constraint f_2 requires that the slope of the beam is continuous at the central support which can be written in terms of equation (24) as

$$f_2 = w'(x_c) - \sum_{i=1}^{NM} a_i(t)\phi_i'(x_c) = 0 \tag{28}$$

These constraints are represented in the Lagrangian (eqn. 2) with β_1 and β_2 which are the Lagrange multipliers.

Then Hamilton's principle applied to equation (22) gives the following equations of motion,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{a}_i}\right) - \frac{\partial T}{\partial a_i} + \frac{\partial U}{\partial a_i} - \beta_1\frac{\partial f_1}{\partial a_i} - \beta_2\frac{\partial f_2}{\partial a_i} = Q_1 + Q_2 \qquad i = 1,..,NM \tag{29}$$

Evaluation of each of the terms in eqn. (29) results in four equations of motion which are given in terms of the unknowns, a_i, β_1, β_2, and, $w'(x_c)$ which are respectively the generalized coordinates, Lagrange multipliers, and slope of the beam at the central support. These are

$$\ddot{a}_i + \omega_i^2 a_i^2 - \beta_1\phi_i(x_c) + \beta_2\phi_i'(x_c) - F_1\phi_i(L/4) - F_2\phi_i(3L/4) = 0 \tag{30}$$

$$\beta_2 = cw'(x_c) \tag{31}$$

$$f_1 = 0 \qquad\qquad f_2 = 0 \tag{32}$$

where i = 1,...,NM. As in the free vibration solution, simple harmonic motion is assumed such that the generalized coordinate is written as a harmonic function of time with frequency of oscillation, Ω,

$$a_i = \bar{a}_i\, e^{j\Omega t} \qquad i = 1,...,NM \tag{33}$$

where $j = \sqrt{-1}$. Furthermore, the Lagrange multipliers are assumed to be harmonic and are given by

$$\beta_k = \bar{\beta}_k e^{j\Omega t} \qquad k = 1,2 \tag{34}$$

If the forces are written in exponential form with

$$F_1 = F\, e^{(j\Omega t - \alpha)} \qquad \text{and } F_2 = F\, e^{j\Omega t}$$

then the solution of equations (30-32) gives the following expression for the amplitude of the generalized coordinate, $\bar{a}_i$,

$$\bar{a}_i = \frac{1}{\omega_i^2 - \Omega^2}\left[\bar{\beta}_1\, \phi_i(x_c) - \bar{\beta}_2\, \phi'_i(x_c) + Fe^{-\alpha}\, \phi_i(L/4) + F\phi_i(3L/4)\right] \tag{35}$$

This equation can then be substituted into the two constraints, f_1 and f_2 of equations (32) to give the resulting system of equations in terms of $\bar{\beta}_1$ and $\bar{\beta}_2$:

$$\bar{\beta}_1\left[\sum_{i=1}^{NM}\frac{\phi_i^2(x_c)}{\omega_i^2 - \Omega^2}\right] - \bar{\beta}_2\left[\sum_{i=1}^{NM}\frac{\phi_i(x_c)\phi_i'(x_c)}{\omega_i^2 - \Omega^2}\right]$$

$$= -F\left[\sum_{i=1}^{NM}\frac{(e^{-\alpha}\,\phi_i(L/4) + \phi_i(3L/4))\phi_i(x_c)}{\omega_i^2 - \Omega^2}\right] \tag{36}$$

$$\bar{\beta}_1\left[\sum_{i=1}^{NM}\frac{\phi_i(x_c)\phi_i'(x_c)}{\omega_i^2 - \Omega^2}\right] - \bar{\beta}_2\left[\frac{1}{c} + \sum_{i=1}^{NM}\frac{\phi_i'^2(x_c)}{\omega_i^2 - \Omega^2}\right]$$

$$= -F\left[\sum_{i=1}^{NM}\frac{(e^{-\alpha}\,\phi_i(L/4) + \phi_i(3L/4))\phi_i'(x_c)}{\omega_i^2 - \Omega^2}\right] \tag{37}$$

The values of $\bar{\beta}_1$ and $\bar{\beta}_2$ can in turn be used to evaluate the generalized coordinates from which the beam displacement can finally be calculated from equation (24). The substitution of the expressions for the generalized coordinates and the tuned mode shapes of equation (2) results in a final equation for the displacement of the two span beam as a function of the dimensionless length along the beam, $\overline{x}$, and time and is written as follows,

$$w(\overline{x}, t) = \frac{2L^3}{EI} \sum_{i=1}^{NM} \frac{1}{\overline{\omega}_i^2 - \overline{\Omega}^2}$$

$$x \left[\overline{\beta}_1 \sin(i\pi\overline{x}_c) - \overline{\beta}_2 \left[\frac{i\pi}{L} \right] \cos(i\pi\overline{x}_c) + Fe^{-\alpha} \sin(i\pi/4) + F \sin(3i\pi/4) \right] e^{j\Omega t} \sin(i\pi\overline{x}) \qquad (38a)$$

where the displacement, w, has units of inch. In this equation $\overline{x} = x/L$ is the nondimensional length of the beam and $j = \sqrt{-1}$. The frequencies have been nondimensionalized so that

$$\overline{\omega}_i = \omega_i \sqrt{mL^4/EI} = (i\pi^2) \qquad \text{and} \qquad \overline{\Omega} = \Omega \sqrt{mL^4/EI}$$

The vibratory amplitude occuring at $t = 0$ is important in our study because it may cause structural failure. This is found by evaluating the expression in (38a) for $t = 0$. The displacement of the beam can be made unitless by dividing by the length of the beam, L. The resulting nondimensional displacement is then,

$$\overline{w}(\overline{x}, 0) = \sum_{i=1}^{NM} \frac{1}{\overline{\omega}_i^2 - \overline{\Omega}^2}$$

$$\left[\overline{\beta}_1 / F \sin(i\pi\overline{x}_c) - \overline{\beta}_2 / F \left[\frac{i\pi}{L} \right] \cos(i\pi\overline{x}_c) + e^{-\alpha} \sin(i\pi/4) + \sin(3i\pi/4) \right] \sin(i\pi\overline{x}) \qquad (38b)$$

This nondimensional displacement is given as follows,

$$\overline{w}(\overline{x}, 0) = \frac{w(\overline{x}, 0)}{2L^2F/E\,I} \tag{38c}$$

3.1.2 Mode Superposition Method

The mode superposition method was used to check the results of the forced response calculations of the Rayleigh-Ritz method. The mode superposition method finds the beam displacement

$$w(x, t) = \sum_{i=1}^{N} \Psi_i(x) q_i(t) \tag{39}$$

where the Ψ_i are the mistuned modes of the beam, q_i are generalized coordinates as a function of time, and N is the number of localized modes used in the summation. The frequency of the applied force is given as in eqn. (20) as $\overline{\Omega} = \overline{\Omega}_i - 0.0001$, $i = 1,2$ which excites the first or second mode. In this case the frequency of the applied forces is very close to one of the system's natural frequencies, and therefore, only the associated mode is excited. As a result, the response is dominated by the term in the sum contributed by that mode. Since the system is undamped, this approximation yields accurate results even though the first and second modes are close in frequency. For example, the the modal frequencies of the mistuned two span beam with $\overline{c} = 400$ and $\overline{\Delta L} = 0.04$ are $\overline{\Omega}_1 = 51.94$ and $\overline{\Omega}_2 = 71.51$. The forced excitation of the first mode is far enough from $\overline{\Omega}_2$ so that the second mode is not excited. In this case that the first mode is excited, the equation for the displacement becomes:

$$w(x,t) = \Psi_1 q_1(t) \qquad \text{for} \quad \Omega \sim \Omega_1 \tag{40}$$

where Ω is the frequency of the applied force, and Ω_1 is the associated frequency of the first localized mode. A similar expression can be written for the excitation of the second mode.

In order to solve equation (39) the generalized coordinate is calculated as the solution of the following equation of motion,

$$m\ddot{q} + kq = P = P_o \cos(\Omega t) \tag{41}$$

where m, k, and P are the generalized mass, stiffness, and force respectively corresponding to the first mode. These values are derived by the principle of virtual displacements to produce the following expressions,

$$m = \int_0^L \rho A \Psi_1^2(x) dx, \tag{42}$$

$$k = \int_0^L EI[\Psi''_1(x)]^2 dx \; + \; c[\Psi'_1(x_c)]^2, \text{ and} \tag{43}$$

$$P = \int_0^L F(x,t)\Psi_1(x) dx. \tag{44}$$

In order to solve equation (41) for q(t), the solution is assumed to be $q = q_o\cos(\Omega t)$. Substituting this into the equation of motion and solving for q_o gives

$$q_o = \frac{P_o}{k - \Omega^2 m} \tag{45}$$

Since the steady-state solution is desired, the initial conditions are not necessary.

The localized modes which were calculated in Section 2.1.1 in equation (12) are normalized since q_o and w(x,t) both have units of length in equation (40). Therefore, the mode shape is scaled in the following manner,

$$\overline{\Psi}(\overline{x}) = \frac{1}{\sqrt{m\ L^3}} \sum_{i=1}^{NM} \overline{a}_i \sin(i\pi\overline{x}) \tag{46}$$

where the length along the beam is again nondimensionalized such that $\overline{x} = x/L$ and the differential, dx, becomes $d\overline{x} = (1/L)dx$. In addition, the stiffness of the torsional spring in equation (43) is nondimensionalized so that $\overline{c} = \frac{2L}{EI}c$, and the frequencies are also dimensionless with $\overline{\Omega} = \sqrt{\frac{m\ L^4}{E\ I}}\ \Omega$. The substitution of the normalized mode shape (eqn. 46) into equations (42-44) and the evaluation of these integrals gives expressions for m, k, and P. These values are then given in terms of summations of the generalized coordinates, $a_i(t)$ over the number of tuned modes, NM used in the calculation of the localized modes. These reduced relations are

$$m = \frac{1}{2L^2} \sum_{i=1}^{NM} \overline{a}_i^2 \tag{47}$$

$$k = \frac{EI}{2m\ L^6} \left[\sum_{i=1}^{NM} \overline{a}_i^2 (i\pi)^4 \ + \ \overline{c} \left[\sum_{i=1}^{NM} \overline{a}_i (i\pi) \cos(i\pi\overline{x}_c) \right]^2 \right] \tag{48}$$

$$P(t) = F\cos(\Omega t)\,\overline{\Psi}(x = L/4) + F\cos(\Omega t)\,\overline{\Psi}(x = 3L/4)$$

$$P_o = F\,\overline{\Psi}(x = L/4) + F\,\overline{\Psi}(x = 3L/4) \tag{49}$$

where m has units of mass, k has dimensions of force per length, and $P(t) = P_o \cos\Omega t$ and P_o have units of force. Substituting these expressions for m, k, and P_o into equation (45) gives the following expression for q_o ,

$$q_o = (2L^3/EI)\frac{\left[F\sqrt{mL^3}\sum_{i=1}^{NM}\bar{a}_i(\sin(i\pi/4) + \sin(3i\pi/4))\right]}{\sum_{i=1}^{NM}\bar{a}_i^2\left[(i\pi)^4 - \bar{\Omega}^2\right] + \bar{c}\left[\sum_{i=1}^{NM}\bar{a}_i(i\pi)\cos(i\pi\bar{x}_c)\right]^2}$$

Once q_o is found, the response is easily calculated using equation (39) which can be written as a function of the nondimensional length along the beam,

$$w(\bar{x}, t) = \frac{q_o \cos\Omega t}{\sqrt{m\,L^3}}\sum_{i=1}^{NM}\bar{a}_i \sin(i\pi\bar{x})$$

which has units of inch. The diplacement is nondimensionalized as in the Rayleigh-Ritz method by dividing by the length of the beam, L. Furthermore, this nondimensional displacement at $t = 0$ is given by,

$$\bar{w}(\bar{x}, t) = \frac{q_o}{\sqrt{m\,L^5}}\sum_{i=1}^{NM}\bar{a}_i \sin(i\pi\bar{x}) \qquad (50)$$

3.1.3 Results

The equations for the forced response as calculated by the Rayleigh-Ritz method and the mode superposition method can be compared because both are nondimensional values. Both methods first require the solution of the free vibrational problem for the mode shape and frequency of the localized system which is excited in order to obtain values of $a_i(t)$ and Ω to be used in the expressions for $w(\bar{x}, t)$. Therefore, the method first involves the relatively costly solution of the eigenvalue problem (eqns. 10,11) before starting either method. Once $a_i(t)$ and Ω are calculated, however, the mode superposition method requires fewer calculations than the Rayleigh-Ritz. This lower cost is a result of equations which have fewer sets of summations over the number of tuned modes. Most of the effort in these calculations is in the summing over the NM number of tuned modes used. In the case of the excitation of the first mode, 2000 tuned modes of the single span beam were used in both methods to calculate the beam displacement. However, the calculation of the response due to the excitation of the second mode required 3500 modes in order to obtain sufficiently accurate results. When fewer modes were used, there were irregularities in the monotonicity of the results. Table 1 shows the sensitivity of the of the free and forced excitation of the second mode to the number of single span modes used in the Rayleigh-Ritz solution for the case when $\bar{c} = 800$ and $\overline{\Delta L} = 0.02$.

A comparison of the Rayleigh-Ritz and mode superposition methods shows that they are in agreement. For example, for a beam with mistuning 0.04 and coupling 400, the responses due to the out of phase excitation of the first mode are compared at the point of maximum amplitude which is at $x = 0.77L$. At this point, the Rayleigh-Ritz gives a dimensionless displacement of $\bar{w} = 210.00$ using one mistuned mode which in turn was calculated using 2000 modes of the single span beam in the calculations. The mode superposition method gives an amplitude of $\bar{w} = 210.05$ at the same point on the beam and also using 2000 tuned modes. A similar agreement was found for other values of coupling and mistuning and also for the in phase excitation of the first mode. Because the results are well correlated and the mode superposition method calculates the response more efficiently than the Rayleigh-Ritz, the former method is used for subsequent calculations.

An example of the forced response follows for the disordered two span beam with a torsional stiffness of $\bar{c} = 400$ and mistuning $\overline{\Delta L} = 0.04$. First, the in and out of phase excitation of the first mode will be analyzed. Second, the case will be studied when in and out of phase forces excite the second localized mode of the beam.

The first localized mode shape of the system with $\bar{c} = 400$ and $\overline{\Delta L} = 0.04$ is shown in Figure 6. It has a localization factor of 0.059 and a nondimensional natural frequency, $\overline{\Omega}_1 = 51.93778$. From the figure and the low localization factor, it is seen to be strongly localized as expected since the coupling is weak and the there is a large disorder in the beam span length. The forcing frequency for both the in and out of phase cases is, therefore, $\overline{\Omega} = 51.93768$ as calculated by eqn. 21. Equation (49) for the displacement by the mode superposition method indicates that the response is a constant, q_o, times the first mode shape. The results of the in phase forced response shows that this is indeed true. The generalized coordinate is $q_o = 25203.49$ and the resulting response has the same shape as the localized first mode in Figure 5. The maximum amplitude of the response is 186.94 at $x = 0.77L$ which is sligthly to the right of the center of the longer span. The constraints, f_1 and f_2 are satisfied in Figure 12 since the displacement at the central support, $\bar{x}_c = 0.46$, is zero and the slope is continuous there.

Next, the response of the disordered beam is analyzed when the forces excite the first mode but are 180° out of phase. The system is again configured with $\bar{c} = 400$ and $\overline{\Delta L} = 0.04$. The following results are for a phase lag on the force, F_1, which is applied at one quarter of the beam length. As with the in phase analysis, the displacement of the beam is a constant times the first mode shape. The generalized coordinate is $q_o = 28320.04$, and therefore, the response of the out of phase case is larger than that of the in phase forced response. The maximum response of the beam is at $\bar{x} = 0.77$ with a nondimensional displacement of $\bar{w} = 210.054$, which is larger than the in phase response of 186.94. This maximum occurs in the same position on the beam as the in phase case which is the the position of maximum concentration of energy in the localized mode. Alternatively, if the phase lag is placed on the force in the right span, F_2, the response is the negative of the previous value, or $\bar{w} = -210.054$.

An analysis of the in and out of phase excitation of the second mode provides a further insight into the forced response of the disordered two span beam. The second mode is shown in Figure 5 for the same coupling and mistuning, $\bar{c} = 400$, $\overline{\Delta L} = 0.04$. The calculations for the second mode are done using 3500 tuned modes in order to achieve results with good accuracy. The localization factor is $A = 0.060$ which is approximately the same degree of localization as in the first mode. The natural frequency is $\bar{\Omega}_2 = 71.50416$, and the frequency of the applied forces is calculated as in equation (21) to be $\bar{\Omega} = 71.50406$. The in phase excitation of this mode results in a response which is 27341.0 times the mode and a shape like the localized second mode in Figure 5. The maximum response is at $\bar{x} = 0.19$ with an amplitude of $\bar{w} = 164.55$. This maximum occurs at the position along the beam where there is a maximum concentration of energy in the free vibration problem (Figure 7). The out of phase excitation of the second mode results in lower response amplitudes than for the in phase forces. The generalized coordinate q_o is 24255.0, and the maximum response due to the out of phase excitation is $\bar{w} = 145.97$ at $\bar{x} = 0.19$.

Therefore, for the excitation of the first mode, the out of phase forces cause a response larger than the in phase forces. However, the opposite is true for the excitation of the second mode: the in phase forced response is larger than the out of phase forced response.

Further analysis was done to study the changes in the response of the disordered two span beam when the degree of mistuning and coupling is changed in the system. As in the analysis done in Chapter 2, the stiffness has a range of nondimensional values from 200 to 1000, and the mistuning varies from a disorder of 1 to 7 percent of the beam length. As before, a large spring stiffness causes the two spans to uncouple and act independently, and a large value of ΔL results in a highly mistuned structure. Furthermore, the beam has strongly localized modes when it is weakly coupled and has a large mistuning. In order to study the sensitivity of the forced response to the amount of mode localization, the response of the beam is calculated for various amounts of mistuning and coupling in the structure for given applied forces. Then, the trends can be studied in terms of the increase or decrease of the response when localization is increased as well as the sensitivity of that change.

First an analysis is made of the response of the disordered two span beam due to in phase forces exciting the first mode. Figure 12 shows a graph of these results. The values of the maximum amplitude of the forced in phase response are plotted on the graph for the amount of coupling and mistuning in the structure. The important result of this graph is that the maximum response increases monotonically with an increase in the spring stiffness implying a decrease in the coupling between the two subsystems, and an increase in disorder, $\overline{\Delta L}$. Note that the first mode of the beam becomes more localized for large $\bar{c}$ and large $\overline{\Delta L}$; thus, localization is high in the upper right corner of the graph and weak in the lower left region. The conclusion is that the in phase forced excitation of the first mode results in a response which increases with an increase in the degree of localization in that mode. For example, the in phase forced response of a strongly coupled, weakly mistuned beam with values of $\bar{c} = 200$ and $\overline{\Delta L} = 0.01$ results in a maximum deflection of $\overline{w} = 105.22$. A doubling of the stiffness to 400 results in a new response of 137.90 which is a 31.1 percent increase from the lower response.

Furthermore, a doubling of the amount of mistuning results in a response of $\overline{w} = 146.40$. This is a 39.1 percent increase in the maximum beam displacement. Curves have been drawn on the graph to show the values of stiffness and mistuning which produce the same maximum response. These will be useful in the next chapter in the probabilistic analysis of beam failure.

Next, an analysis will be made of the maximum response of the beam due to applied out of phase forces where the two forces have a 180 degree phase lag. These results are shown in Figure 13. In order for later comparison with the in phase forced response, the same configurations of stiffness and mistuning are considered. Therefore, a weakly localized system with $\overline{c} = 200$ and $\overline{\Delta L} = 0.01$ is studied and, the maximum response is found to be $\overline{w} = 229.75$ at $\overline{x} = 0.78$. Then the first mode is made more localized by weakening the coupling. This is done by doubling the stiffness to $\overline{c} = 400$ to get a resultant response of $\overline{w} = 215.25$ at the same location, $\overline{x} = 0.78$. This is a 6.31 percent decrease in the amplitude of the beam deflection. Similarly, an increase in the localization of the first mode by doubling the amount of mistuning to 0.02 results in a decrease in the response by 2.96 percent to $\overline{w} = 222.95$. Therefore, an increase in the localization of the first mode of the beam results in a decrease in the maximum response due to applied out of phase forces exciting that mode. This is opposite to the case of in phase forces discussed above. In addition, the response due to out of phase forces is much less sensitive to changes in the degree of localization than the response caused by in phase forces. From the examples above, the in phase forced response is on the average seven times more sensitive than that in the out of phase case.

Next, a similar comparison of responses due to in and out of phase forces will be made for the excitation of the second mode of the two span beam. Again, for the case of a weakly localized system with $\overline{c} = 200$ and $\overline{\Delta L} = 0.01$, the maximum response due to two in phase forces is $\overline{w} = 210.05$ at $\overline{x} = 0.19$ (Figure 14). When the localization is increased by doubling the coupling, the response decreases by 4.05 percent to 201.55. The response also decreases as the localization in the second mode is increased by doubling the amount of mistuning. The result is that the maximum beam displacement decreases by 6.62 percent to $\overline{w} = 196.14$. These responses are relatively insensitive to changes in the degree of localization.

On the other hand, the maximum response of the disordered two span beam due to two out of phase forces exciting the second mode increases with an increase in localization and is sensitive to these changes (Figure 15). For example, here a doubling in the coupling from 200 to 400 at a mistuning of 0.01 raised the response by 34.08 percent. Furthermore, a doubling of the mistuning from 0.01 to 0.02 at a stiffness of $\overline{c} = 200$ results in a 32.82 percent increase in the beam displacement. Therefore, for the excitation of the second mode, the out of phase force response increases and is sensitive to increases in localization. However, the in phase force response decreases and is less sensitive to an increase in localization. These conclusions are opposite to those reached for the excitation of the first mode.

It should also be noted in Figures 14 and 16, that the maximum response of the beam due to in phase forces consistently increases or decreases depending on the mode which is being excited. However, this is not true for the out of phase forcing of the two span beam (Figures 13, 15). These figures show that the trend of increasing or decreasing response is basically consistent with changes in stiffness; however, changes in the mistuning cause first one trend and then another. For example, in Figure 13 the response at first decreases slightly with an increase in $\overline{\Delta L}$, and then it increases again as the mistuning is increased further. In the case of out of phase excitation of the second mode (Figure 15), there is a similar trend except that the response increases and then decreases with $\overline{\Delta L}$.

Following a check of these results with the example of the forced response of a nearly periodic pair of coupled pendula, an explanation of both results will be given in terms of the transfer admittance from one substructure to another.

3.2 ANALYSIS OF COUPLED PENDULA

An analysis of the coupled pendula is important because the propagation of the response in this system can be intuitively understood. For the tuned case the mode shapes extend throughout the structure. Hodges [18] showed that this is true because there is a degeneracy of frequencies. For the case of N uncoupled pendula each with the same individual properties, each pendulum vibrates at the same natural frequency. However, the introduction of a small amount of coupling causes the degeneracy of frequencies to split and produce a cluster of frequencies close to the resonant frequency of the uncoupled pendula [20]. This clustering of frequencies is the passband of the system.

When disorder is introduced into the system of N weakly coupled pendula, the energy in the modes is localized close to one individual pendulum. The normal mode frequencies are approximately equal to those of that pendulum. Therefore, for a particular mode, one pendulum oscillates close to its natural frequency with a large resonance. Since the system is disordered, the neighboring pendula have different frequencies and are driven off resonance. Furthermore, since the system is weakly coupled, the neighboring pendula have small response amplitudes. The localization then drives each pendulum further from resonance with smaller response amplitudes. Hodges [18] showed in a series of weakly coupled pendula that the vibration level decays exponentially from the source when it is considered a function of the distance. He went on to analyze the case when one pendulum far from the source happens to have a frequency equal to that of the normal mode frequency. He showed that there will be an increased response in that pendulum, but since the force is weakened away from the source, the response is small and local to that pendulum.

3.2.1 Calculation of the Forced Response

The two coupled pendula are excited by applying a pair of in phase or out of phase forces to the masses (Figure 16). Similar to the the analysis of the two span beam, the forces are harmonic functions with frequency, Ω. The calculation of the forced response of a pair of coupled pendula is done by the impedance method. This involves the solution of the following two-degree-of-freedom equation of motion,

$$\mathrm{M}\ddot{X} + \mathrm{C}\dot{X} + \mathrm{K}X = F \tag{51}$$

In this equation M, C, and K are the matrices given in equation (14), and X is the vector of displacements, x_1 and x_2. In order to solve this equation, simple harmonic motion is assumed with $F = \mathrm{F}e^{j\Omega t}$ and $X = \mathrm{X}e^{j\Omega t}$ where $\mathrm{j} = \sqrt{-1}$. Substituting these into the equation of motion gives the following expression,

$$\left([\mathrm{K}] + \mathrm{j}\Omega[\mathrm{C}] - \Omega^2[\mathrm{M}]\right)\begin{Bmatrix} \mathrm{x}_1 \\ \mathrm{x}_2 \end{Bmatrix} = \begin{Bmatrix} \mathrm{F}_1 \\ \mathrm{F}_2 \end{Bmatrix} \tag{52}$$

In this problem all values are known except for x_1 and x_2. Therefore, the equation is rearranged to solve for the response of the two pendula,

$$\begin{Bmatrix} \mathrm{x}_1 \\ \mathrm{x}_2 \end{Bmatrix} = \left([\mathrm{K}] + \mathrm{i}\Omega[\mathrm{C}] - \Omega^2[\mathrm{M}]\right)^{-1}\begin{Bmatrix} \mathrm{F}_1 \\ \mathrm{F}_2 \end{Bmatrix} \tag{53}$$

As in the free vibrational analysis, the masses are $m_1 = m_2 = m = 100$ lbf-s²/in and the spring stiffness varies from 200 to 1000 lbf/in. In the forced vibration solution the damping is not neglected, and the components of the damping matrix are $c_{11} = c_{22} = 1$ lbf-s²/in and $c_{12} = c_{21} = 0$ lbf-s²/in. Equation (53) can be simplified and given in terms of the inverse dynamic matrix as

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{bmatrix} \frac{m\,g}{L_1} + k - \Omega^2 m + jc_{11}\,\Omega & -k \\ -k & \frac{m\,g}{L_2} + k - \Omega^2 m + jc_{22}\Omega \end{bmatrix}^{-1} \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} \tag{54}$$

with $g = 32$ ft/s². The inverse matrix above is the inverse dynamic matrix, $[D]^{-1}$, and it is a complex matrix. As a result, this problem is solved using a FORTRAN computer program which inverts a complex matrix. The amplitudes of the sinusoidal forces have unit values. For the out of phase forced response where the phase difference is 180°, the amplitudes of the applied forces become $F_1 = 1$ and $F_2 = -1$. The value of L_2 remains constant at nine inches. The length of the first support varies from 9 to 10 inches to introduce disorder into the system. Mistuning can then be represented in this problem as $\Delta L = L_1 - L_2$ which can be nondimensionalized such that $\overline{\Delta L} = \Delta L / L_2$. The frequency Ω is the frequency of the applied force which is set exactly equal to the frequency of the excited mode. This does not cause a large resonance because the system is lightly damped.

3.2.2 Results

In the previous example of the two span beam, the forced response was measured in terms of the maximum amplitude of the displacement along the beam length. However, the response of the pendula is not continuous, but is expressed in terms of

However, the response of the pendula is not continuous, but is expressed in terms of motion of the two pendula, x_1 and x_2 . As in the beam analysis, the response of the system is either sensitive or not sensitive to the degree of localization of the excited mode depending on the the mode which is excited and the phase difference of the applied forces. This phenomenon is analyzed by applying in and out of phase forces which excite the first and second modes of the disordered, coupled pendula. Then, the response of the pendulum which has the larger amplitude is measured. In the tuned case where $L_1 = L_2$, each pendulum has the same response. In the first mode, the pendula move in tandem, while in the second mode, they move opposite directions. The forced response of the ordered system is the same for any value of coupling and has nondimensional displacements of $\bar{x}_1 = \bar{x}_2 = 1.526$ inches for the case of the first mode. The system is made localized by weakening the stiffness of the spring while increasing the amount of mistuning.

First, the in phase excitation of the first mode is analyzed (Figure 17). The horizontal axis is in terms of a coupling indicator 1000-k. Small values along the axis represent a strongly coupled system, and large values are for weakly coupled pendula. This is for better comparison with the beam analysis where the coupling decreases at higher values of $\bar{c}$ along the axis. The mistuning is also scaled so as to compare with the beam example, from $\Delta L = 0.01$ to 0.07. As was discussed in Chapter 2, localization increases for a weakly coupled, strongly mistuned structure. Therefore, mode localization is strongest in the upper right corner of the graph. When two in phase forces are applied to a localized configuration, the result is that one pendulum has a response greater than the tuned response, and the response of the other pendulum is smaller. For example, a coupling of 400 lbf/in and a disorder of 4.4 percent causes the first pendula to move a distance of $x_1 = 1.697$ inches and the second pendulum to move only 1.353 inches. However, for less mistuning and stronger coupling, the first mode is less localized, and the responses are closer to those of the tuned case. An example is a coupling of k= 1000 lbf/in and $\overline{\Delta L} = 0.022$. In this case $x_1 = 1.570''$ and $x_2 = 1.498''$ both of which are closer to the tuned response of 1.526 inches. Figure 17 shows the response of the first pendula which has an increasing amplitude. The largest response of the structure is shown in the figure because it increased amplitude of the forced response could cause failure of the structure.

In general, an increase in the localization of the excited first mode causes the forced in phase response of the first mass to increase and that of the second mass to decrease. However, these changes are relatively insensitive to the degree of localization in that mode. For example, the responses of the first and second masess are 1.489 and 1.578 inches respectively for a configuration which has a weakly localized first mode such as the case when $\bar{k} = 800$lbf/in and $\overline{\Delta L} = 0.022$. The response of the first mass is then 1.489", and the displacement of the second is 1.578". These responses can then be compared to a configuration which has a more localized first mode. For example, doubling the amount of mistuning to $\overline{\Delta L} = 0.044$ results in only a 2.55 percent decrease in x_1 and a 3.04 percent increase in x_2. A reduction in coupling, and thus, an increase in localization of the first mode likewise has little effect on the in phase forced response of the pendula. When the coupling is cut in half from k = 800 to k = 400 lbf/in, the response of m_1 decreases by 3.22 percent and m_2 increases by only 2.60 percent.

The effect of mistuning on the response of the nearly periodic, coupled pendula due to out of phase forces exciting the first mode is opposite to the above in phase case. The force applied to m_1 has a phase lag of 180° with respect to the force on m_2. These out of phase forces cause the displacements of both pendula to increase with an increase in the degree of localization. The vibration amplitude of the first pendulum, x_1, is always larger than that of the second pendulum. Therefore, only the response of the first mass is presented in Figure 18 since x_1 is the maximum response of the system. As in the above case of in phase forces, the responses of the pendula are equal for the tuned case. However, the out of phase forced response of the tuned system increases with a weakening of the coupling between the pendula whereas they always remained the same under the application of in phase forces. The response of the tuned system due to the out of phase excitation of the first mode is $x_1 = x_2 = 0.00062''$ for a strong coupling of $k = 800$ lbf/in. When the coupling is decreased by reducing the stiffness to 400, the displacements are $x_1 = x_2 = 0.00125''$.

A study of the sensitivity of the out of phase response of the nearly periodic system is made by changing the system parameters as done in the previous in phase case. When the system is again configured to be weakly localized with $k = 800$ lbf/in and $\overline{\Delta L} = 0.022$, the displacement of the first mass is $x_1 = 0.046$. However, if the mistuning is doubled to $\overline{\Delta L} = 0.044$ such that the disorder is raised from 2.2 to 4.4 percent, then the response of m_1 is doubled to $x_1 = 0.092''$ which is a 100.0 percent increase. Similarly, a halving of the spring stiffness from 800 to 400 lbf/in weakens the coupling of the system and increases the degree of localization. An analysis of the forced response shows that it is also sensitive to such changes in coupling: the displacement of the second mass increases by 104.3 percent. Therefore, for the case of the forced excitation of the first mode of the disordered, coupled pendula the sensitivity of the responses to the degree of localization depends on the phase lag between the forces applied to each pendulum. If the phases are 180° out of phase, the response of the structure is extremely sensitive to the amount localization of the excited modes with increases in amplitudes of over one hundred percent. On the contrary, the response due to in phase forces is quite insensitive to changes in the degree of localization in the system with only a two to three percent change in response.

A similar analysis now follows for the in and out of phase excitation of the second mode of a nearly periodic pair of coupled pendula. First, the in phase excitation is considered. Figure 19 shows the trends for the response of the second mass. As the coupling between the pendula weakens and the amount of disorder increases, the responses of both pendula increase. Furthermore, the displacement of m_2 is always larger than m_1, and as a result, an analysis is only presented for the second mass. It is interesting to note that the forced response of the tuned system is the same as that of the out of phase excitation of the first mode. This would seem to be because the forces are in opposition to the motion of the pendula in both cases, and therefore, the vibration is damped out to the same extent.

The response of the second mass is shown in the Figure 19 to be sensitive to changes in the degree of localization in the system's modes. For example, when as in the above examples the mistuning is doubled from 0.022 to 0.044, the displacement x_1 increases by 102.6 percent. In addition, a reduction in coupling by one half causes the response to increase from $x_1 = 0.039$ to $x_1 = 0.086$ inches. This is an increase to 2.2 times the first response or 120.5 percent of the tuned case.

The out of phase forced excitation of the second mode gives different results. As in the case of the in phase excitation of the first mode, an increase in the localization of the excited mode causes one mass to increase while the other to decrease. In the present example, x_1 decreases and x_2 increases with added localization of modes. Figure 20 shows the displacement of the second mass for varying values of coupling and disorder in the structure. In can be seen that the response is relatively insensitive to changes in the degree of localization compared to the in phase case given in the preceding paragraph. For example, the response increases only 2.75 percent due to a doubling in the amount of disorder. Furthermore, the x_1 increases 10.33 percent as a result of a lessening in spring stiffness from 800 to 400 lbf/in. Therefore, in the case where the structure has applied forces which excite the second mode, the sensitivity of the response depends on the phase difference of the forces. If the forces are in phase, the displacements of the pendula change drastically with the degree of localization. However, out of phase forces cause the responses of the pendula to be relatively insensitive to any such change.

3.3 DISCUSSION OF RESULTS

The results presented in this chapter are used in this section to explain the contrasting results of previous work for the forced response of nearly periodic structures. Then, these trends of the effect of localization on the forced response are explained using the theory of transfer admittance.

3.3.1 Comparison With Previous Work

The results in this chapter might offer an explanation of the inconsistent conclusions of past researchers as to the effects of mode localization on the forced response of nearly periodic structures. As already discussed in section 1.2.2, most previous work shows that the forced response increases drastically when a disordered structure is subject to excitation of localized modes. However, there have also been reports to the contrary. These latter studies conclude that the response of a mistuned structure subject to applied loads decreases and is relatively insensitive to larger amounts of disorder in the structure. This discrepancy can be explained by the results of sections 3.1.3 and 3.2.2.

In brief, the analyses of the two span beam and pair of coupled pendula indicate that the in phase excitation of an antisymmetric mode and the out of phase excitation of a symmetric mode result in large a response for strong localization. On the other hand, the in phase excitation of a symmetric mode and the out of phase excitation of an antisymmetric mode result in small changes in the response due to this combination of phases and excited modes. It is hypothesized that the past research which shows an increase in forced response for a more localized system used a model which consisted of a combination of forcing phases and excited modes to result in this sensitivity. For example, the studies may have been done on in phase forces applied to the first mode of a row of coupled pendula or length of weighted string which is symmetric. Furthermore, it is believed that the research which shows that the forced response is relatively insensitive to localization was based on a study such as the in phase excitation of an antisymmetric mode such as the first mode of the two span beam.

In order to verify this hypothesis, a further study is necessary to substantiate this claim in terms of symmetric and antisymmetric mistuned structures. Therefore, a larger number of symmetric and antisymmetric modes are necessary to verify these conclusions. However, the results of the two span beam are compatible with those of the pair of coupled pendula. Therefore, the conclusions are not limited to a single disordered system, but can be tentatively generalized to the forced response of nearly periodic structures.

3.3.2 Transfer Admittance

It was discussed in Chapter 2 that the modes of an ordered structure extend throughout the system [18,20]. When the structure is weakly coupled, the individual natural frequencies of the substructures form a passband which is a clustering of individual frequencies of neighboring pendula. Because the modes are extended throughout the structure, there is an unattenuated transmission of vibration in the passband [4,18,20]. Hodges and Woodhouse [20] explain this phenomenon in terms of the transfer admittance from one point to another within the structure. Transfer admittance is defined as the reciprocal of impedance. It indicates the ease or lack of resistance with which energy passes within a structure. More specifically, the transfer admittance, denoted by $Y(x, x'; \omega)$, represents the complex vibratory amplitude at x due to a unit amplitude harmonic force of frequency ω applied to x'. In the example of the beam or pendula systems, the admittance represents the amount of vibrational energy which passes between the nearly periodic substructures. The node of a localized mode acts as a point of high impedance which prevents the transfer of energy from one part of the mode to the other. Therefore, the transfer admittance of a mistuned structure is useful in the analysis of the forced response and the propagation of energy due to applied forces. The following analysis is based on the nearly periodic pair of coupled pendula.

The transfer admittance $Y(x, x'; \omega)$, is measured across the two points, x and x', and is given by

$$Y(x, x'; \omega) = \sum_{r=1}^{NM} \frac{j\omega \Psi_r(x) \Psi_r(x')}{\Omega_r^2 + 2j\omega\Delta_r - \omega^2} \tag{55}$$

where $\Psi_r(x)$ is the rth mode with an associated natural frequency of Ω_r and $j = \sqrt{-1}$ [20]. The term ω is frequency of excitation, Δ_r is the damping factor, and NM is the number of modes used in the summation. Since the transmittance depends on the mode amplitudes at x and x', a strong transmission of energy through the structure depends on a full extension of the modes throughout the system [20]. For the case of two coupled pendula, the transfer admittance becomes,

$$Y(x_1, x_2; \omega) = \sum_{r=1,2} \frac{j\omega \Psi_r(x_1)\Psi_r(x_2)}{\Omega_r^2 + 2j\omega\Delta_r - \omega^2} \tag{56}$$

with x_1 and x_2 as the displacements of the first and second masses respectively.

When disorder is introduced by lengthening the first support, the first and second modes will localize with a concentration of energy in one of the masses, and the pendula frequencies are no longer degenerate. In addition, if the coupling is weak between the two masses, the coupling is no longer strong enough to produce extended modes in which both masses have approximately the same free vibrational amplitude [20]. The result is that localization decreases the transmission of energy from m_1 to m_2 . In a localized mode the amplitude contributed by one mass is very small. Therefore, the product of a mode amplitude at x_1 times the amplitude at x_2 is always small if the mode is strongly localized.

The results of the forced response of the coupled pendula given in Section 3.2.2 can be analyzed in terms of transfer admittance. For two in phase forces, F_1 and F_2, the displacements of the first and second masses can be expressed as

$$x_1 = Y_{1,1}F_1 + Y_{1,2}F_2 \tag{57}$$

$$x_2 = Y_{2,1}F_1 + Y_{2,2}F_2 \tag{58}$$

where $Y_{i,j}$, $i,j = 1,2$ is an abbreviation of the transfer admittance between x_i and x_j. If the frequency of the applied forces is Ω_1 then these forces excite the first mode. Provided that the system is lightly damped, the term in the sum of eqn. (56)

corresponding to the first mode dominates. Therefore, only the contribution of one term of the sum is shown in the following admittance terms. For example, the term $Y_{1,2}$ is

$$Y_{1,2} = Y(x_1, x_2; \omega) = \frac{j\omega\Psi_1(x_1)\Psi_1(x_2)}{\Omega_1^2 + 2j\omega\Delta_1 - \omega^2} \tag{59}$$

As mentioned above, for a strongly localized system, the contribution to the mode amplitude from one pendulum is small, and the product of the mode amplitude at x_1 and at x_2 is almost zero. As a result, the terms $Y_{1,2}$, and $Y_{2,1}$ become small. Hence, the contribution of the terms corresponding to $Y_{1,2}$ and $Y_{2,1}$ to the transmittance of energy between m_1 and m_2 approaches zero. However, the transfer admittance terms $Y_{1,1}$ and $Y_{2,2}$ do not show this large decrease. For example, $Y_{1,1}$ and $Y_{2,2}$ are given by,

$$Y_{1,1} = Y(x_1, x_1; \omega) = \frac{j\omega\Psi_1(x_1)\Psi_1(x_1)}{\Omega_1^2 + 2j\omega\Delta_1 - \omega^2} \tag{60}$$

$$Y_{2,2} = Y(x_2, x_2; \omega) = \frac{j\omega\Psi_1(x_2)\Psi_1(x_2)}{\Omega_1^2 + 2j\omega\Delta_1 - \omega^2} \tag{61}$$

First, consider the case that the exciting forces are applied in phase, and the frequency of the exciting forces is equal to the first natural frequency, Ω_1. In addition, the system is assumed to be lightly damped. Without loss of generality, it is assumed that the forces have unit amplitudes. Since the masses in the first mode move in tandem, $\Psi_1(x_1)$ and $\Psi_1(x_2)$ in eqn. (59) have the same sign. The forces F_1 and F_2 are acting in phase, and therefore, the contribution of the terms $Y_{1,2}F_2$ in eqn. (57), and $Y_{2,1}F_1$ in eqn. (58) is towards increasing the displacement. The results of the forced vibration solution verified this analysis (Figure 17). The response of the first mass due to in phase forces exciting the first mode increases but with a small degree of sensitivity to increases in localization. The second mass is also relatively insensitive but decreases slightly with increases in localization.

For the case of forces exciting the first mode which are 180° out of phase, the terms $Y_{1,1}F_1$ and $Y_{1,2}F_2$ in eqn. (57) as well as $Y_{2,1}F_1$ and $Y_{2,2}F_2$ in eqn. (58) have opposite signs. Therefore, the contribution of the two forces in the displacements are opposing each other. As the degree of mistuning increases and the first mode becomes localized, some of the cancellation between the terms in the right hand sides of eqns. (57) and (58) is eliminated. This is the reason that the vibratory amplitude increases when the first mode becomes strongly localized as illustrated in Figure 18.

The in phase excitation of the second mode also results in a sensitivity of the response to the degree of mode localization (Figure 19). The mode is antisymmetric with amplitudes of opposite signs, but since the forces are in phase, the amplitudes of the forces have unit amplitudes of the same sign. Therefore, the analysis is the same as in the previous case with the result that the terms $Y_{1,2}$ and $Y_{2,1}$ diminish as localization increases. Therefore, as before, the terms in eqns. (56) and (57) containing these values contribute less to the transfer admittance. In the tuned case, applied forces which are in phase cancel the energy in this antisymmetric mode. Yet, the introduction of mistuning isolates the cancellation of energy between the two masses because the transfer admittance is low. Therefore, localization causes the cancellation of energy to decrease so that the forced response is significantly higher than that of a tuned system.

For the case of out of phase forces which excite the second mode, there is a relative insensitivity of the response to changes in mistuning and coupling in the system (Figure 20). The second mode is antisymmetric with pendula displacements having opposite signs. In addition, the applied forces have unit amplitudes and opposite signs since they are 180° out of phase. For the tuned case, out of phase forces add energy to each mass. However, the localization of a mode shape concentrates the energy in one mass and isolates it from the other. Therefore, when forces are applied, energy is added to a pendulum by the force applied to it; however, the high impedance at the node prevents energy additions from the force applied to the other mass. As a result, the amplitude of the forced response is relatively insensitive to the degree of mistuning in the system.

Chapter 4. Probabilistic Analysis of Nearly Periodic Structures

It is important to calculate the statistics of the forced response because the structural parameters of nearly periodic systems are random. In general, the optimum values of the stiffness and mistuning can be calculated and then incorporated in the design of a structure; however, inaccuracies in these calculations, imperfections in the manufacturing process, and aging contribute to uncertainties in the design. As a result, the deviations in the properties of the blades are statistically distributed. Therefore, there is a large number of possible configurations of the system, depending on the number of random variables.

It is important to analyze the statistics of the free and forced response because the structure will have a different degree of localization and maximum response amplitude for each configuration. Also, the vibration of nearly periodic structures is very sensitive to the amount of mistuning, so that the consequences of statistical deviations in the properties will have a drastic effect on the structural dynamics of the system. The objective of this chapter is to evaluate the probability of mode localization and probability of failure in the forced vibration case for the two span beam and coupled pendula systems. Before proceeding to the probabilistic analysis, a brief review follows of some of the basic elements of reliability theory is presented.

4.1 RELIABILITY

In order to explain the basic ideas of reliability, a simple case is considered where there is a supply, X, and demand, Y. Success is defined when supply is greater than deman, or $X > Y$. Reliability is the probability of success denoted by $P(X > Y)$. The corresponding measure of unreliability is $P(X < Y)$. The probability of failure, P_F, can be represented as the following integral,

$$P_F = \int_0^{\infty} F_X(y) f_Y(y) dy \tag{62}$$

for the case of uncorrelated or statistically independent random variables. The function $F_X(y)$ is the cumulative distribution function which is the probability of observing a value of of x less than or equal to Y where $Y = y$. The function $f_Y(y)$ is the probability density function. The term $f_Y(y)dy$ is the probability of observing a value y equal to Y for values of y in the range $[y, y + dy]$. The corresponding probability of sucess, P_S, is

$$P_S = 1 - P_F \tag{63}$$

This can be graphically represented as the shaded area $F_X(y)$ in Figure 21.

The problem can be reformulated in terms of a margin of safety, M, defined as the difference between supply and demand. Therefore, failure is now given by $M < 0$ and

$$P_F = \int_{-\infty}^{0} f_M(m)dm = F_M(0) \quad (64)$$

which is the area under the density function $f_M(m)$ for values of m less than zero. This function $f_M(m)$ is the distribution of values of $M = m$ where m ranges from minus infinity to zero. The case of supply and demand can be generalized to a problem where failure is defined by a more complicated function of the design variables and where the number of design variables is large. The equation $M = X - Y$ is linear but the failure of a system may also be governed by a nonlinear equation. Such a nonlinear or linear expression defining the failure region is called the performance function, $g(X) = g(x_1, x_2, ...x_n)$ where X is the vector of n design variables, $X = x_1, x_2, ...x_n$. In this generalized form,

$$g(X) > 0 \quad \text{defines a safe system,}$$

$$g(X) < 0 \quad \text{defines an unsafe system,} \quad (65)$$

$$\text{and} \quad g(X) = 0 \quad \text{is the limit state equation.}$$

This last equation describes the failure surface which divides the safe and unsafe regions of the design space. The probability of failure can be expressed in terms of the performance function as

$$P_F = \int_{g(X)<0} F_X(X)dx \quad (66)$$

where $f_X(X)$ is the joint probability density function of the random variables $X = x_1, x_2, ...x_n$.

The above method requires the knowledge of the probability density function of each of the random variables. However, the analytical representation of this function is not generally known in practice. A numerical distribution of the design variables may be known, but then the calculation of the probability of failure involves numerical integration over the space of the design variables. On the contrary, in many cases there is a knowledge of the first and second moments of the system. These moments are mean values and variances of the random variables. When these are known, the reliability can be measured as a function of the first and second moments. This method of calculating the probability of failure is the second moment method which was formulated by Cornell [31] in 1969, and later in 1974 by Ang and Cornell [32], and Hasofer and Lind [33].

4.2 SECOND MOMENT METHOD

In the following sections, the results of the free and forced vibration analysis are used to study the statistics of the free and forced response of simple nearly periodic structures. The probability that the mode shapes are localized indicates that the response of the system is also large since the displacement of the beam is a weighted sum of the free vibrational modes. However, it was shown that this is not always true: there are cases where the forced response of nearly periodic systems is relatively insensitive to the amount of localization in the modes (Chapter 3). If there is a large response, it may result in increased stresses in the structure which may cause structural failure. For the purposes of the analysis of the probability of localization and the probability of failure, the advanced second moment method will be used.

The second moment method finds the reliability of a system as a function of the first and second moments of the design variables [23]. In this method the performance function, g(X) , defines the regions of failure and survival as in equation (65). In the examples of the two span beam and the pair of coupled pendula, the vector of design variables, X, is composed of the spring stiffness, $\bar{c}$ or k, and the mistuning, $\overline{\Delta L}$. Hence, the performance function can be written as $g(\bar{c}, \overline{\Delta L})$. In this chapter the second moment method will be used to study two different probabilistic analyses of the two span beam. First, the probability that the first or second mode of the two span beam are localized is calculated for random values of the system parameters. Also, the second moment method is used to evaluate the probability that the forced response of the two span beam exceeds a prescribed level. These two cases require different performance functions although both functions are dependent on the same design variables, $\bar{c}$ and $\overline{\Delta L}$.

The performance function can also be expressed in terms of the nondimensional reduced design variables, $\bar{c}'$ and $\overline{\Delta L}'$ which are the stiffness and mistuning in the reduced space respectively. Reduced space variables are defined such that they have a mean value of zero and a standard deviation of one which makes them useful in probability studies. The reduced design variables are

$$\bar{c}' = \frac{(\bar{c} - \mu_{\bar{c}})}{\sigma_{\bar{c}}} \tag{67}$$

and

$$\overline{\Delta L}' = \frac{(\overline{\Delta L} - \mu_{\overline{\Delta L}})}{\sigma_{\overline{\Delta L}}} \tag{68}$$

where $\mu_{\bar{c}}$ and $\mu_{\overline{\Delta L}}$ are the mean values of $\bar{c}$ and $\overline{\Delta L}$, and $\sigma_{\bar{c}}$ and, $\sigma_{\overline{\Delta L}}$ are their standard deviations.

In terms of these variables, the limit state equation becomes

$$g(\sigma_{\bar{c}}\,\bar{c}' + \mu_{\bar{c}} \;\; , \;\; \sigma_{\overline{\Delta L}}\,\overline{\Delta L} + \mu_{\overline{\Delta L}}) \;=\; 0 \tag{69}$$

This is the limiting surface in the space of the reduced design variables as shown in Figure 22. The system is considered safe for values $\bar{c}'$ and $\overline{\Delta L}'$ giving a positive value of g and unsafe for negative values of g. Note that for a limit state equation which is close to the origin of the failure region becomes larger, there are fewer values of $\bar{c}'$ and $\overline{\Delta L}'$ which produce a safe state.

The most probable point of failure is the configuration of the system which will most likely give a value of the performance function in the failure region. Since the design variables have mean values within the safe region, it is most likely that the deviation from this mean produce failure points not far from the safe region. Shinozuka [33] has shown that the most probable point of failure lies on the failure surface. This point is found where a line of minimum length is drawn from the origin of the reduced space (Figure 22). This minimum distance is called the safety index which is represented by the symbol β. The distance from the origin to any point $(\bar{c}', \overline{\Delta L}')$ on the limit state can be simply written as

$$D = \sqrt{\bar{c}'^2 + \overline{\Delta L}'^2}$$

The point on the limit state giving the minimum distance can be found by minimizing D subject to the constraint $g(\bar{c}, \overline{\Delta L}) = 0$. This optimization problem is solved using the method of the Lagrangian multiplier as done by Ang and Tang [3]. The problem is thus to

$$\text{minimize} \quad D = \sqrt{\bar{c}'^2 + \overline{\Delta L}'^2}$$

$$\text{subject to} \quad g(\bar{c}, \overline{\Delta L}) = 0 \tag{70}$$

Using the Lagrange multiplier, λ, the Lagrangian becomes, $L = D + \lambda\, g(\bar{c}, \overline{\Delta L})$ where $\bar{c}'$ and $\overline{\Delta L}'$ are given by equations (67) and (68). The solution of this optimization problem is obtained by minimizing the Lagrangian with respect to $\bar{c}'$, $\overline{\Delta L}'$, and λ to give the most probable failure point, $(\bar{c}'^*, \overline{\Delta L}'^*)$. Once this is known, the distance D is calulated which is the minimum distance from the origin to the failure surface. This minimum value is the safety index β which is given in Ref. [22] as

$$\beta = -\frac{\left[\bar{c}'^* \left(\frac{\partial g}{\partial c'}\right)_* + \overline{\Delta L}'^* \left(\frac{\partial g}{\partial \overline{\Delta L}'}\right)_*\right]}{\sqrt{\left(\frac{\partial g}{\partial c'}\right)_*^2 + \left(\frac{\partial g}{\partial \overline{\Delta L}'}\right)_*^2}}$$

where the derivatives are calculated at the most probable failure point $(\bar{c}'^*, \overline{\Delta L}'^*)$. This point can be given in terms of the safety index as

$$\bar{c}'^* = -\alpha_{\bar{c}}^* \beta \tag{71}$$

and

$$\overline{\Delta L}'^* = -\alpha_{\overline{\Delta L}}^* \beta \tag{72}$$

The values $\alpha_{\bar{c}}^*$ and $\alpha_{\overline{\Delta L}}^*$ are the direction cosines along the $\bar{c}'$ and $\overline{\Delta L}'$ axes and are given by the following expressions,

$$\alpha_{\bar{c}}^* = \frac{\left(\frac{\partial g}{\partial \bar{c}'}\right)_*}{\sqrt{\left(\frac{\partial g}{\partial \bar{c}'}\right)_*^2 + \left(\frac{\partial g}{\partial \overline{\Delta L}'}\right)_*^2}} \tag{73}$$

and

$$\alpha_c^* = \frac{\left(\frac{\partial g}{\partial \overline{\Delta L}'}\right)_*}{\sqrt{\left(\frac{\partial g}{\partial \bar{c}'}\right)_*^2 + \left(\frac{\partial g}{\partial \overline{\Delta L}'}\right)_*^2}} \tag{74}$$

For the case of a nonlinear performance function, the safety index is the shortest distance from the origin of the reduced space to the tangent surface of the limit state $g(\bar{c}, \overline{\Delta L}) = 0$. If the limit state is convex with respect to the reduced origin, then the calculated probability of failure is a conservative approximation since the unsafe region is estimated as the larger area above the straight line approximation (Figure 23). On the other hand, a performance function which is concave with respect to the reduced origin is approximated by a failure surface which creates a smaller failure region. In the examples of both of two span beam and the coupled pendula, the performance functions defining localization are nonlinear (Figures 7,10). Furthermore, they are both convex with respect to the origin, and therefore, the results obtained by the second moment method are on the conservative side. For the dynamic response, the curves are convex with respect to the origin in Figures 12-15 and 17-20. However, Figures 13 and 15 also have a region which is concave.

The following numerical algorithm used to calculate the safety index was taken from Rackwitz [34]:

1) Assume initial values of $\bar{c}^*$ and $\overline{\Delta L}^*$ and obtain

$$\bar{c}'^* = \bar{c}^* - \frac{\mu_{\bar{c}}}{\sigma_{\bar{c}}} \qquad \text{and} \qquad \overline{\Delta L}'^* = \overline{\Delta L}^* - \frac{\mu_{\overline{\Delta L}}}{\sigma_{\overline{\Delta L}}}$$

2) Evaluate $(\partial g/\partial \bar{c}')_*$, $(\partial g/\partial \overline{\Delta L}')_*$, α_c^* , and $\alpha_{\Delta L}^*$ at $(\bar{c}^*, \overline{\Delta L}^*)$.

3) Form $\bar{c}^* = \mu_{\bar{c}} - \alpha_{\bar{c}}^* \sigma_{\bar{c}} \beta$ and $\overline{\Delta L}^* = \mu_{\overline{\Delta L}} - \alpha_{\overline{\Delta L}}^* \sigma_{\overline{\Delta L}} \beta$.

4) Substitute above $\bar{c}^*$ and $\overline{\Delta L}^*$ in $g(\bar{c}^*, \overline{\Delta L}^*) = 0$ and solve for β .

5) Using the β obtained in Step 4, reevaluate $\bar{c}'^* = -\alpha_{\bar{c}}\,\beta$ and $\overline{\Delta L}'^* = -\alpha_{\overline{\Delta L}}\,\beta$.

6) Repeat Steps 2 through 5 until convergence is obtained.

Since the point of tangency is not known for the minimum distance $D = \beta$, the mean values of $\bar{c}$ and $\overline{\Delta L}$ are chosen as the initial point of iteration in Step 1. Then successive iterations of Steps 2-5 are performed until β converges to the minimum value. Once β has been calculated, the probability of localization, P_L or the probability of failure P_F, is calculated using the expressions,

$$P_L = 1 - \Phi(\beta) \tag{75}$$

$$P_F = 1 - \Phi(\beta) \tag{76}$$

where Φ is the probability distribution function of a canonically normal random variable [22]. The function $\Phi(\beta)$ is found from a table of normal distribution functions of random normal variables.

4.3 PROBABILITY OF LOCALIZATION

The second moment method can be formulated in terms of the localization of modes where localization is arbitrarily chosen such that the smaller amplitude is below a given percentage of the larger amplitude. Therefore, if localization is defined when $A = a^*/a = 0.1$, meaning an amplitude ratio of one tenth that of the tuned case, then the event that the system is localized is defined by the limit state equation,

$$g(\bar{c}, \overline{\Delta L}) = A - 0.1 = 0 \tag{77}$$

The surface given by this equation cannot be called a failure surface because localization of a system's mode may cause the response to be large, but this does not necessarily imply failure. The system is localized if $g \leq 0$ because then the ratio is smaller than the limiting value of 0.1, and it is within the definition of localization. However, if $g > 0$, then the the two span beam is not localized because the amplitude ratio A is greater than the limiting value of 0.1. In this case the mode shape is not sufficiently altered to be considered localized. This development can be graphically represented in Figure 6 where $A = 0.1$ is the nonlinear curve which passes through all of the values of A for which $A = 0.1$. Above this curve g is negative, and the system is localized, while below the curve g is positive and the system is not localized. The condition for localization is minimally satisfied by the limit state $g = 0$. This means that the system will no longer be localized if either the spring stiffness or amount of disorder is reduced. Furthermore, for a system with mean values of $\bar{c}$ and $\overline{\Delta L}$ causing the system to be not localized, the most probable point of localization is on the surface $A = 0.1$. It is plausible that the most probable point of localization lies on this curve because it defines the minimum conditions for a localized mode. This most probable point is similar to the most probable point of failure discussed in the previous section.

The probability of localization, P_L, which is the probability that $g(\bar{c}, \overline{\Delta L}) < 0$, is is given by equation (75) of the previous section,

$$P_L = 1 - \Phi(\beta)$$

where β is the shortest distance from the origin of the reduced space to the surface, $A = 0.1$. In the example of the two span beam the mistuning, $\overline{\Delta L}$, has mean value of zero because the middle support of the beam can be moved either to the left or right of center. The localization factors at negative values of $\overline{\Delta L}$ are the same as that with positive $\overline{\Delta L}$. Therefore, the localization factors of Figures 6 and 7 have have the same values in the plane of positive $\bar{c}$ and negative $\overline{\Delta L}$ below it. As a result, there are two surfaces which define regions of localization which are mirror images of each other. The second moment method calculates the probability of localization by finding the distance to just one of these failure surfaces, depending on the region from which the initial values of $\bar{c}$ and $\overline{\Delta L}$ are chosen.

The probability of localization is then doubled in order to take into account both regions of localization. It the case of the beam, the probability of localization is modified to give the following expression,

$$P_L = 2[1 - \Phi(\beta)] \tag{78}$$

The corresponding probability of not being loclalized, P_{NL} , is given by

$$P_{NL} = 1 - P_L$$

The statistical distributions of the random variables, $\bar{c}$ and $\overline{\Delta L}$ are normal with a Gaussian distribution about the mean. A Gaussian distribution is assumed because this is the simplest case for which the second moment method can be applied. It is most probable that the stiffness and mistuning will be close to their mean values. The means are chosen so that the resultant value of $g(\bar{c}, \overline{\Delta L})$ is less than zero in the safe or unlocalized region. However, there is a probability that the system will occupy states away from the mean and in the region of localization. This probability that the structure will randomly assume a configuration of localized modes is the probability of mode localization.

The second moment method was used to calculate the probability of localization of the first and second modes of the two span beam. The beam, (Figure 1), is assumed to have random amounts of coupling and mistuning in the system. The mean value of stiffness is $\mu_{\bar{c}} = 400$ with a standard deviation of $\sigma_{\bar{c}} = 40$. For the mistuning parameter, the mean value is $\mu_{\overline{\Delta L}} = 0$ with a standard deviation of $\sigma_{\overline{\Delta L}} = 0.015$. At this mean configuration, the degree of localization of the first and second mode is $A = 1.0$.

For example, if localization is defined such that the smaller amplitude of the first mode is one tenth that of the larger, then the limit state equation is given by equation (77). The initial value of $\bar{c}$ and $\overline{\Delta L}$ are chosen as the mean values, $\bar{c} = 400$ and $\overline{\Delta L} = 0$. The numerical algorithm given at the end of the previous section was used to calculate the safety index, β . After three iterations, the most probable point of localization is calculated to be $\bar{c} = 408.5$ and $\overline{\Delta L} = 0.022$ which lies on the surface defining localization, $A = 0.1$. The distance from the origin, $\bar{c}' = \overline{\Delta L}' = 0$, to the above point on the limit state equation is the safety index, $\beta = 1.502$. Then $\Phi(\beta)$ is found in a table of cumulative distribution functions of the standard normal distribution. The probability of localization is calculated from equation (78) and found to be 0.1336 with a corresponding probability of no localization of 0.8664. Therefore, there is a 13.36 percent probability that the mistuning and coupling will be such that the first mode will be localized in the sense that A will be less than 0.1.

The result of further analysis using the second moment method shows that the probability of localization increases for a less strict criterion of localization. For example, in the case of localization defined by a value of A less than 0.2, the probability of localization of the first mode of the disordered two span beam is 0.4658. However, when localization is set for $A \leq 0.05$, the probability of localization is 0.0016 or 0.16 percent. The results for a range of localization factors from $A = 0$ to 0.3 is given in Figure 23, and Table 2 gives the localization factors defining the limit state, the corresponding most probable points of localization, the safety indices, and the probabilities of localization. It can be seen in Fig. 23 that the probability of localization goes to zero in the limiting case of complete concentration of energy in one half of the mode shape or $|A| = 0.0$. The usefulness of this diagram is the ability to predict the probability that the modes of a system will be localized to a critical degree.

4.4 MONTE CARLO SIMULATION

The results of the probability of localization of the disordered two span beam calculated by the second moment method are checked using a Monte Carlo simulation. This method is an expensive yet straightforward and reliable means of calculating the probability of localization. In general, the Monte Carlo method repeats a simulation using a set of random values in each simulation which are generated in accordance with the probability distributions of the variables. A repetition of the process, therefore, generates a sample of solutions each corresponding to a different set of randomly generated variables. It is important that the randomly generated numbers have a prescribed statistical distribution because then the simulation process is deterministic [23]. Monte Carlo simulation is not appropriate for large, complex systems because the method simplifies the dependence of the simulation on the random parameters. As a result, the gross simplification of many problems as well as large amounts of computer time, cause this method to be used primarily as a means of verifying solutions of approximate analytical solution methods.

In terms of the calculation of the probability of localization of the two span beam, the Monte Carlo method does not simplify the problem further from the second moment method, and therefore, it is a good check of the results of the second moment method. The simulation is done by first randomly generating a large number of random values of mistuning, $\overline{\Delta L}$ and torsional stiffness, $\bar{c}$. In order to compare these results to those of the second moment method, the same mean and standard deviations of these values are used: $\mu_{\bar{c}} = 400.0$, $\mu_{\overline{\Delta L}} = 0.0$, $\sigma_{\bar{c}} = 40.0$, and $\sigma_{\overline{\Delta L}} = 0.015$. Each pair of randomly generated $\bar{c}$ and $\overline{\Delta L}$ is then used to calculate an associated localization factor, A. The localization event is prescribed to occur when the localization factor falls below a set value. Then, the number of values of A which is less than this limiting value of the amplitude ratio gives the number of localized states in the population. This sum divided by the total number of randomly generated points is the fraction of localized states or the probability of localization, P_L, given by

$$P_L = \frac{\text{No. of localized points}}{\text{Total no. of points}}$$

It was observed that the results of the Monte Carlo simulation compare well with those of the second moment method. As the number of randomly generated points increases, the calculated probability of localization approaches the analytical solution. For example, a comparison follows of the probability of localization of the first mode of the two span beam with a statistical distribution of mistuning and coupling as given above. Localization is defined as $A \leq 0.1$. The solution by the second moment is $P_L = 0.1336$. Using 100 randomly generated points, 11 had localization factors above 0.1, and therefore, the probability of localization is 0.1100. As the number of simulations increases to 700, the P_L increases to 0.1300. Larger numbers of generated states should continue to converge to 0.1336. A comparison of the computer cost illustrates the limitation of this method to only a rough check of the analytical result. The second moment costs required approximately 10 C.P.U. minutes on the IBM 3090 mainframe computer to calculate the probability of localization for the above mean and standard deviation of $\bar{c}$ and $\overline{\Delta L}$. However, the Monte Carlo simulation used approximately 70 C.P.U. minutes using 700 simulations.

4.5 PROBABILITY OF FAILURE

An analysis of the probability of failure is more important than the the probability of localization because it is a consideration of possible structural failure that places limits on the design variables. The localization of a nearly periodic structure was shown in Chapter 3 to result in large response amplitudes in only certain cases. In the case of the in phase excitation of the first mode of the two span beam, the response levels showed a drastic increase with an increase in localization. However, the out of phase excitation of this mode resulted in an insensitivity to the degree of localization and a slight decrease in the response. For the latter case of insensitivity, the probability of failure is not meaningful because the system is improved since there is a lower chance of failure as a result of localization. Yet, in the former case of large increases, a calculation of the probability of failure is important because it provides a measure of the chance of failure due to excessive levels of vibration. The probability of failure for this case is discussed in the remainder of this section

The performance function for the calculation of the probability of failure is nonlinear and convex with respect to the origin for the in phase excitation of the first mode (Figures 12). This function defines failure where the maximum amplitude exceeds a given displacement value. For example, if failure is arbitrarily defined to occur when the maximum nondimensional displacement of the two span beam exceeds the value of $\overline{w} = 150$, then the performance function is $g = 150 - w$. As before, the middle support of the two span beam moves to the left or right of the center to cause a disorder in the system. Therefore, there are again two failure surfaces: one for positive $\overline{\Delta L}$ and another for negative $\overline{\Delta L}$. As a result, the probability of failure is doubled as it was done in eqn. (78).

$$P_F = 2[1 - \Phi(\beta)] \quad (79)$$

The corresponding probability of survival is

$$P_S = 1 - P_F \quad (80)$$

The safety index, β, is calculated by the second moment method described in Section 4.2.

In this analysis the mean and standard deviations of the $\bar{c}$ and $\overline{\Delta L}$ are the same as those used in the probability of localization study with $\mu_{\bar{c}} = 400.0$, $\mu_{\overline{\Delta L}} = 0.0$, $\sigma_{\bar{c}} = 40.0$, and $\sigma_{\overline{\Delta L}} = 0.015$. Table 3 shows the probabilities of failure and survival for a range of failure limits. For the case of two in phase forces which excite the first mode of the two span beam, the failure surfaces are isoquants of a given failure limit (Figure 12). An example of this is a failure defined as a maximum allowable response amplitude of $\overline{w} = 170$. The second moment method gives the following results in the calculation of the probability of survival. The initial values of iteration are the mean values, $\bar{c}^* = 400.0$ and $\overline{\Delta L}^* = 0.0$. Rackwitz's numerical algorithm given on page 67 is used to find the the safety index, β, and after two iterations the method converges to a value of $\beta = 1.5017$. Using equations (79) and (80), the probability of failure, P_F, is 0.1332 and the probability of survival is 0.8668. Therefore, there is an 86.68 percent chance that the in phase forced response has a maximum response amplitude less than $\overline{w} = 170$.

In real life, high probabilities of survival (ie. greater than 99 percent) are required for the design of a reliable structure. A high survival rate such as this occurs when failure is set at larger response levels. For example, a higher limit of failure such as $\overline{w} = 180$ has a probability of the beam having a maximum amplitude less than this limit of $P_S = 0.9642$. Furthermore, the probability of survival is 0.9994 for failure occurring at maximum amplitudes above $\overline{w} = 195$. Table 3 gives the failure points and the probabilities of failure and survival for varying failure limits. The analysis shows that the probability of survival decreases for lower limits on the maximum allowable response amplitude. Figure 24 shows this trend for maximum amplitudes set from zero to two hundred. A study of the limiting cases shows that the probability of survival is equal to 1 (100 percent) when arbitrarily large response amplitudes are considered safe. However, when an arbitrarily large response constitutes structural failure, the probability of survival is zero. These results would be helpful in the design of a periodic structure which has statistically distributed dynamic properties causing the system to occupy disordered states. Then, if the maximum tolerable response amplitudes are known, the consequent probability of remaining below this limit can be measured.

A calculation of the probability of survival for the out of phase forced excitation of the first mode is not meaningful. As shown in Figure 13 the maximum amplitude of the beam decreases for a more localized mode. Therefore, deviations from the mean values of mistuning and coupling which cause localization result in a system which is less likely to fail.

A further study was done to analyze the sensitivity of the probability of failure to changes in the mean values of the system parameters. Table 4 shows the probabilities of failure and survival and the corresponding failure point and safety index for varying mean values of $\bar{c}$. The failure criterion is kept constant at $\overline{w} = 185$ in order to compare the probabilities of survival. In addition, the standard deviation of $\bar{c}$ remains $\sigma_{\bar{c}} = 40.0$ and the mean and standard deviation of $\overline{\Delta L}$ are $\mu_{\overline{\Delta L}} = 0.0$ and $\sigma_{\overline{\Delta L}} = 0.015$. The results show that the probability of failure increases for higher mean values of coupling. For example, a mean stiffness of 400.0 results in a probability of failure of $P_F = 0.0128$. However, if the mean is raised to 500.0, the probability of failure increases to 0.0214. Figure 25 shows this trend for a range of mean stiffness levels. The conclusion is that a system which is more weakly coupled in the mean is more likely to fail. This is true because the in phase excitation of the localized first mode results in a sensitivity and large response increase with increases in stiffness and mistuning.

Similarly, changes in the standard deviation of the mistuning, $\sigma_{\overline{\Delta L}}$, results in changes in the probability of failure of the two span beam. The same failure criterion is used such that failure occurs when the maximum response amplitude due to two simultaneous in phase forces exciting the first mode exceed $\overline{w} = 185$. The case of $\sigma_{\overline{\Delta L}} = 0.015$ was calculated previously in Table 3 to have a probability of failure of 0.0128. When the standard deviation is decreased to 0.010, the probability of failure decreases to 0.0002. As $\sigma_{\overline{\Delta L}}$ approaches zero, P_F likewise goes to zero. Furthermore, the probability of failure approaches one in the limit of large variance in the mistuning from its mean. Such a diagram is useful in the design of a structure where the failure limit is known and tolerances expressed in terms of the standard deviations of the system parameters must be considered.

Chapter 5. Conclusions and Recommendations for Future Research

In this chapter, conclusions will be made concerning the localization of modes of nearly periodic structures. Then, recommendations for future work in this area are given.

5.1 CONCLUSIONS

Localization of the modes of a nearly periodic structure is caused by the disorder and weak coupling between its nearly periodic substructures. For example, the first and second modes of a two span beam are localized when there is a difference in the length of the two spans and when the torsional spring at the middle support is stiff to provide weak coupling. For the system of two pendula coupled by a linear spring, the first and second modes are localized when the supporting lengths differ and when the spring stiffness is small so that the masses are weakly coupled.

The localization of the modes of a nearly periodic structure affects the forced response. In this study the case that the applied forces are simultaneously applied to the structure to excite the first or second mode is considered. Both of the systems of the two span beam and the pair of coupled pendula consist of two almost identical substructures. Two forces acting in phase or out of phase are applied to each substructure simultaneously. From the study of the two span beam, it is concluded that the sensitivity of the forced response to the degree of localization depends on the particular combination of mode symmetry and force phases in the problem. The first mode of the two span beam is antisymmetric with respect to the beam center support (Fig. 4). The second mode, however, is symmetric about the center of the beam.

Therefore, the conclusion is that the in phase excitation of an antisymmetric (ie. first) mode causes the maximum response amplitude of the beam to increase dramatically when localization increases. The out of phase excitation of a symmetric (ie. second) mode of the beam also increases with localization increasing. The opposite conclusion is made for the out of phase excitation of an antisymetric (first) mode and the in phase excitation of a symmetric (second) mode of the beam. In both of these cases, the maximum amplitude of the forced response is insensitive to the degree of localization in the modes. In fact, the response decreases slightly as the localization increases.

These conclusions are supported by the results of the forced response of the pair of nearly periodic, coupled pendula. The symmetry of the modes of the pendula is opposite to those of the two span beam. The first mode of the pendula system is symmetric, while the second is antisymmetric. Therefore, the results of the beam and pendula are compared in terms of forced excitation of symmetric and antisymmetric modes.

As in the beam analysis, the in phase excitation of an antisymmetric (second) mode causes the maximum response of the two pendula to drastically increase with an increase in the degree of localization of that mode. Also, the out of phase excitation of a symmetric mode increases and is sensitive to the degree of localization. Other combinations of loading and mode symmetry result in a relative insensitivity of the forced response to localization. It was found that the in phase excitation of a symmetric (first) mode and the out of phase excitation of an antisymmetric (second) mode are both relatively insensitive to the degree of localization. Furthermore, in both cases of insensitivity, the maximum response of the system increases and then decreases as localization increases.

From the above results we may conclude that an increase in the localization of a mode does not necessarily mean that the forced response due to the excitation of that mode will also increase. In the last two cases, the dynamic response can be evaluated by assuming a perfectly periodic structure. These previous conclusions are summariized as,

1) The response due to the in phase excitation of an antisymmetric mode and the out of phase excitation of a symmetric mode increases drastically with an increase in localization of the mode.

2) The in phase excitation of a symmetric mode and the out of phase excitation of an antisymmetric mode result in a forced response which is relatively insensitive to the localization of the mode.

3) For the two cases given in (2), the forced response may even decrease with the degree of localization increasing.

It is hypothesized that the above conclusions can be generalized for the forced response of other, more complex, nearly periodic structures such as bladed disk assemblies.

A probabilistic analysis of the localization of modes of nearly periodic structures gives a quantitative measure of the chance of failure of mistuned structures due to excessive vibratory levels. A study of the localization of the first mode of the two span beam shows the apparent result that the probability of localization increases if the mode is considered localized for less concentration of energy in one part of the mode. Although this result is intuitively understood, Figure 23 allows the designer who knows the approximate amount of disorder in a structure, to predict the probability that the modes will be highly localized.

Furthermore, an analysis of the probability of failure of a nearly periodic structure gives an indication of the sensitivity of the forced response to the degree of localization. Failure is defined when the maximum response is larger than a predefined limit. For the case of the forced response of the two span beam due to two in phase forces exciting the first mode, it was shown that this response is sensitive to the degree of localization of the modes. When the spans of this case are more weakly coupled in the mean, then the probability of failure increases. Therefore, the probability that the forced response exceeds a failure limit is high for a system that is likely to be localized because of the disorder in its geometry and the weak coupling between its substructures.

To further analyze the effects of localization on the forced response, a study of the probability of survival for varying failure criteria indicates that survival is more likely when the failure limit is set at higher allowable maximum amplitudes. This general result is apparent; however, a graph showing the exact correlation between failure limits and survival probabilities is important as an aid to the designer of nearly periodic structures.

In conclusion, localization occurs in weakly coupled, nearly periodic systems. Localization causes changes in the forced response of such structures. Depending on the phase of the forces and the mode shape which is excited, the response may or may not be sensitive to increases in localization. If the response is sensitive to localization, then the result is a drastic increase in the maximum response amplitude which may be over a hundred percent when the disorder is doubled. In this case it is important to know the probability of survival of the structure given statistics of the system parameters as well as the maximum amplitude of the forced response which can be tolerated.

5.2 RECOMMENDATIONS FOR FUTURE WORK

The limitations of the present research is that only the first two modes of the two span beam are analyzed. (The pendula pair has only two mode shapes.) One of the first two modes of each system is symmetric and the other is antisymmetric. From a study of these two modes, hypotheses are made concerning the dependance of the forced response of nearly periodic structures on the phases of the applied forces in conjunction with the symmetry of the mode excited by the forces. Therefore, an area of additional work could be the investigation of the forced response of a disordered structure due to in phase and out of phase forces exciting higher symmetric and antisymmetric modes. The results of such work could be used to verify the generalizations made in this thesis.

In addition, an extension of the probabilistic analyses to a greater number of cases of the forced response of disordered systems is recommended. For example, the analysis of other loading situations causing a sensitivity of the forced response to localization could be compared with the results of this thesis in the study of the in phase excitation of the two span beam. Furthermore, the addition of the effects of damping between the substructures of the two span beam would make a model which is more true to life although this would increase the difficulty of the solution.

It would also be interesting to continue this study using more than two substructures so that the systems would more closely model real life engineering structures such as a bladed disk assembly or a large space structure. Studies previously conducted in this area have not considered the effects of localization on the response due to forces which are simultaneously applied to each substructure with a difference in phase. This other research is concentrated in the area of a single excitation at one end of a row of nearly periodic substructures and the subsequent decay of the vibratory energy. However, a bladed disk assembly with N blades, has N applied forces due to its rotation which have an associated phase difference as a result of circumferential placement of the blades about the disk. The localization of modes has been shown in this paper to sometimes drastically affect the forced response of nearly periodic structures; therefore, a complete understanding of this phenomenon is necessary before implementing these results in structural design.

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Tables

Table 1: Sensitivity of 2nd Mode of Two Span Beam to Number of Tuned Modes, NM, Used in Solution

$\bar{c} = 600$ and $\overline{\Delta L} = 0.02$

No. of Tuned Modes, NM	Frequency of Second Mode	Localization Factor	In Phase Response	Out of Phase Response
2000	66.123	0.0784	179.09	153.11
3000	66.119	0.0788	179.15	153.05
3500	66.119	0.0789	179.16	153.03
4000	66.118	0.0790	179.18	153.01
5000	66.117	0.0791	179.20	152.99

Table 2: Probability of Localization of 1st Mode of Two Span Beam for Varying Mode Amplitude Ratios Defining the Localization Factors

Localization Factor $A = a^*/a$	Most Probable Localization Point $(\overline{c}'^*, \overline{\Delta L}'^*)$	Safety index β	Probability of Localization, P_L	Probability of No Localization
0.050	(395.56,0.0474)	3.1591	0.0016	0.9984
0.075	(398.16,0.0308)	2.0563	0.0398	0.9602
0.090	(398.74,0.0255)	1.6985	0.0888	0.9112
0.100	(408.48,0.0223)	1.5018	0.1336	0.8664
0.150	(399.55,0.0149)	0.9945	0.3222	0.6778
0.200	(399.74,0.0110)	0.7307	0.4658	0.5342
0.250	(399.89,0.0085)	0.5700	0.5686	0.4314

Table 3: Probability of Survival of Two Span Beam With Failure Defined by the Maximum Response Due to In Phase Forces Exciting the First Mode

Failure limit $\overline{w}$	Most Probable Failure Point $(\bar{c}'^*, \overline{\Delta L}'^*)$	Safety index β	Probability of Failure, P_F	Probability of Survival, P_S
50.0	(400.23,0.0023)	0.1522	0.8792	0.1208
75.0	(400.28,0.0036)	0.2375	0.8123	0.1877
100.0	(400.32,0.0052)	0.3496	0.7267	0.2733
150.0	(402.41,0.0129)	0.8654	0.3869	0.6131
170.0	(405.25,0.0224)	1.5017	0.1332	0.8668
180.0	(407.57,0.0314)	2.0996	0.0358	0.9642
185.0	(408.49,0.0372)	2.4881	0.0128	0.9872
195.0	(411.09,0.0520)	3.4790	0.0006	0.9994

Table 4: Probability of Failure of Two Span Beam for Varying Mean Coupling Values Due to In Phase Forces Exciting the First Mode

Failure limit $\overline{w}$	Most Probable Failure Point $(\overline{c}'^{*}, \overline{\Delta L}'^{*})$	Safety index β	Probability of Failure, P_F	Probability of Survival, P_S
300.0	(313.86,0.0409)	2.7485	0.0060	0.9940
350.0	(360.79,0.0389)	2.6056	0.0092	0.9908
400.0	(408.49,0.0370)	2.4881	0.0128	0.9872
450.0	(456.92,0.0357)	2.3896	0.0168	0.9832
500.0	(505.80,0.0340)	2.3059	0.0214	0.9786
550.0	(554.93,0.0334)	2.2337	0.0255	0.9745
600.0	(604.28,0.0320)	2.1673	0.0294	0.9706

Figures

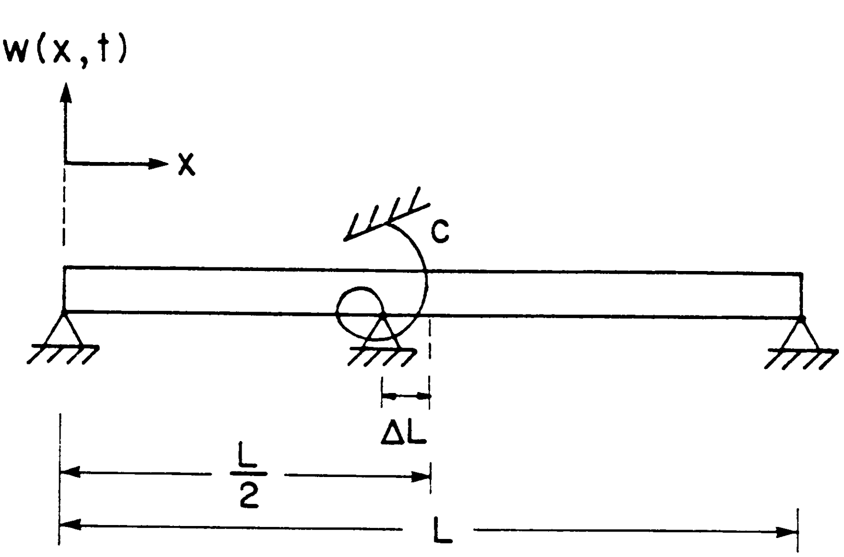

Figure 1 Disordered Two Span Beam

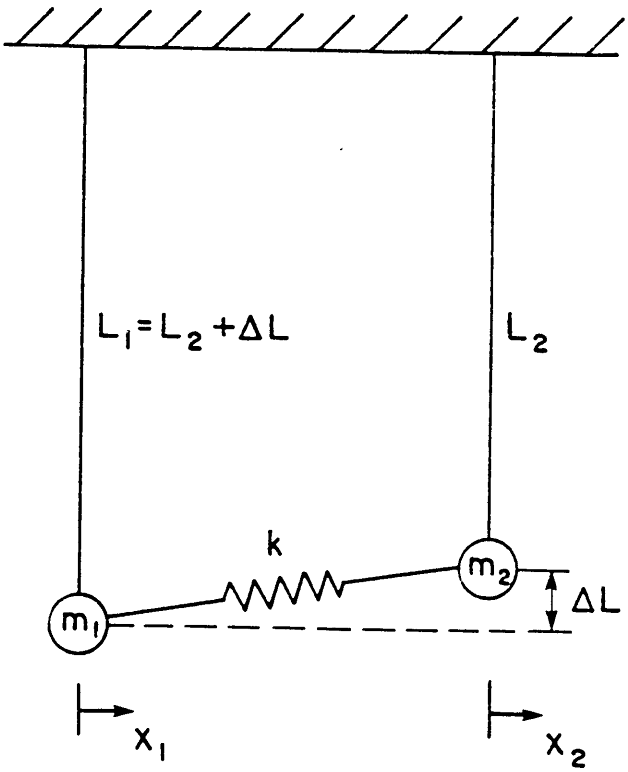

Figure 2 Nearly Periodic Pair of Coupled Pendula

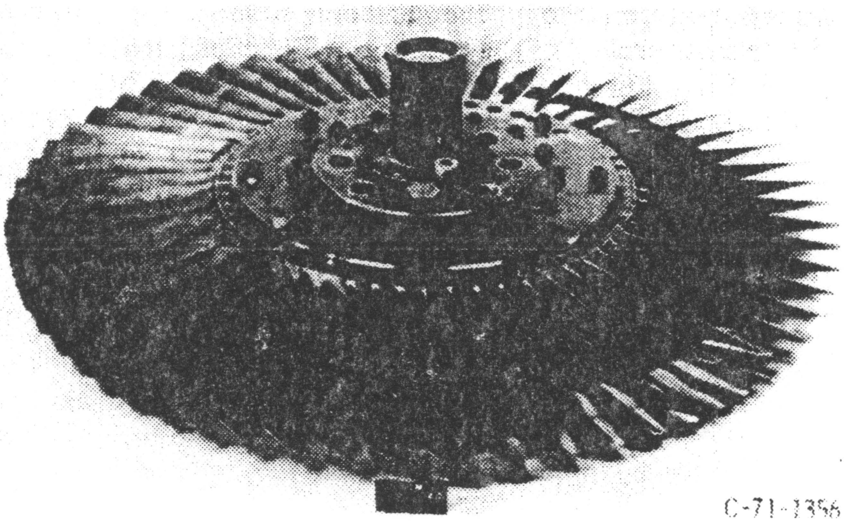

Figure 3 Bladed Disk Assembly

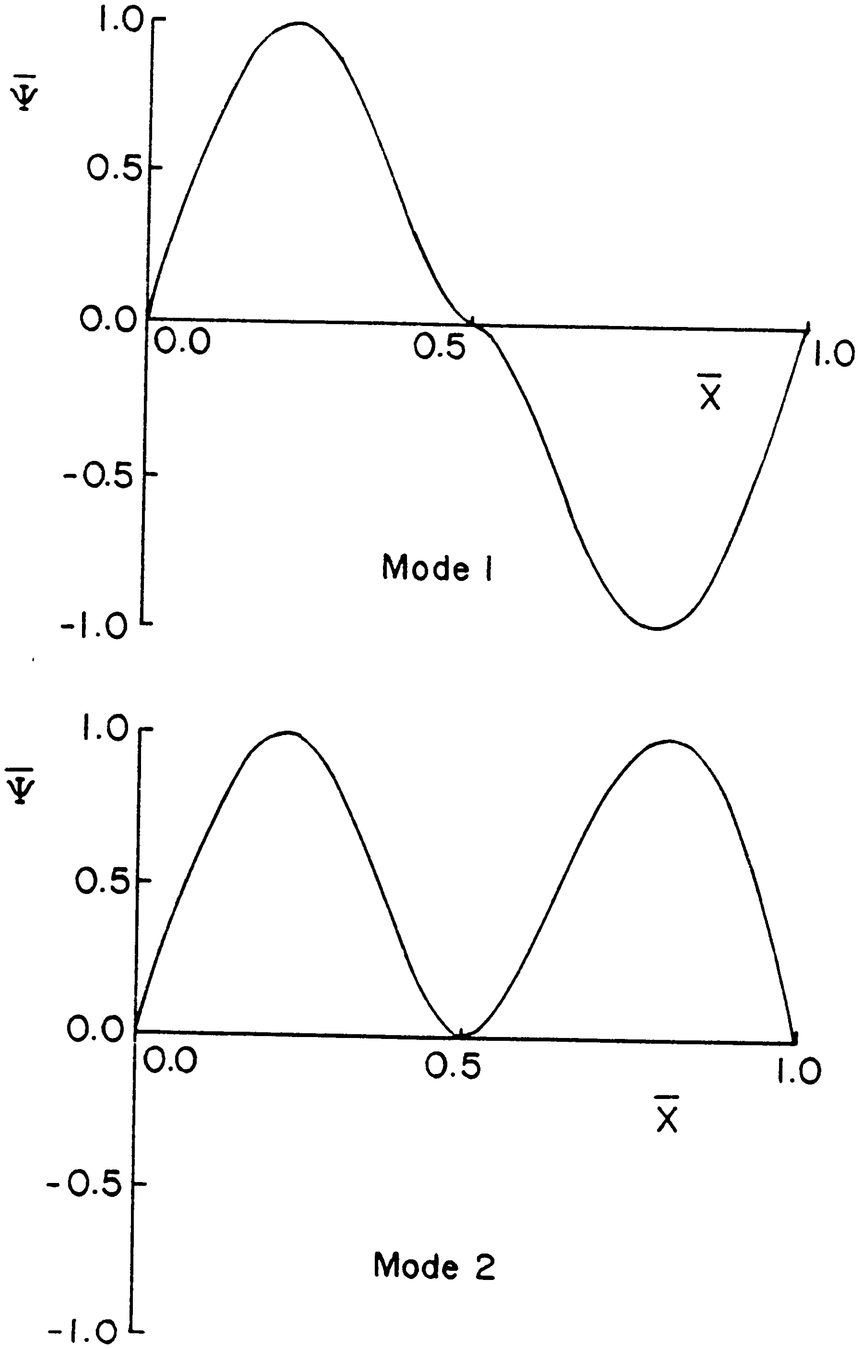

Figure 4 1st and 2nd Tuned Modes of the Two Span Beam

$\bar{c} = 1000, \overline{\Delta L} = 0$

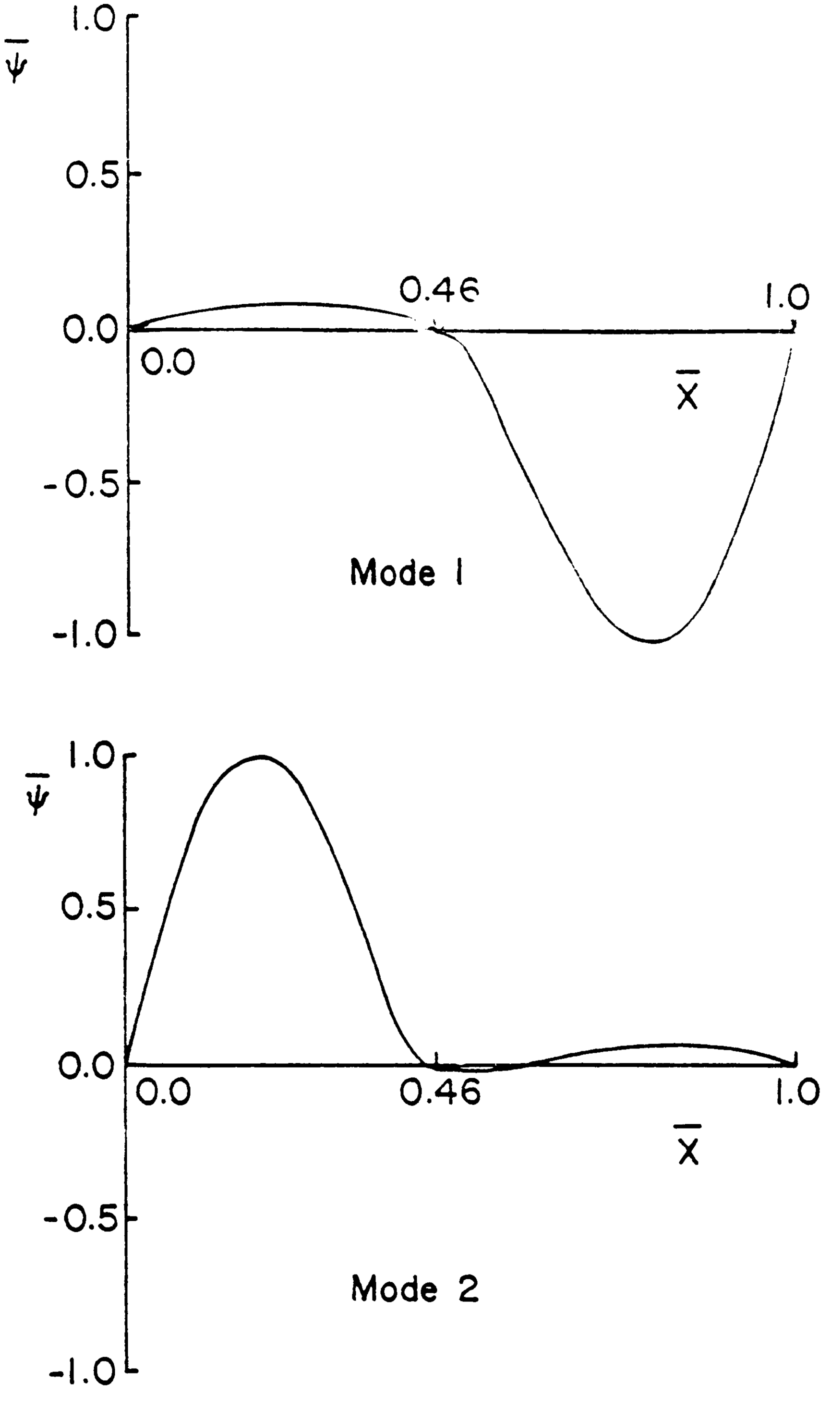

Figure 5 1st and 2nd Localized Modes of Mistuned Two Span Beam

$\bar{c} = 400, \overline{\Delta L} = 40$

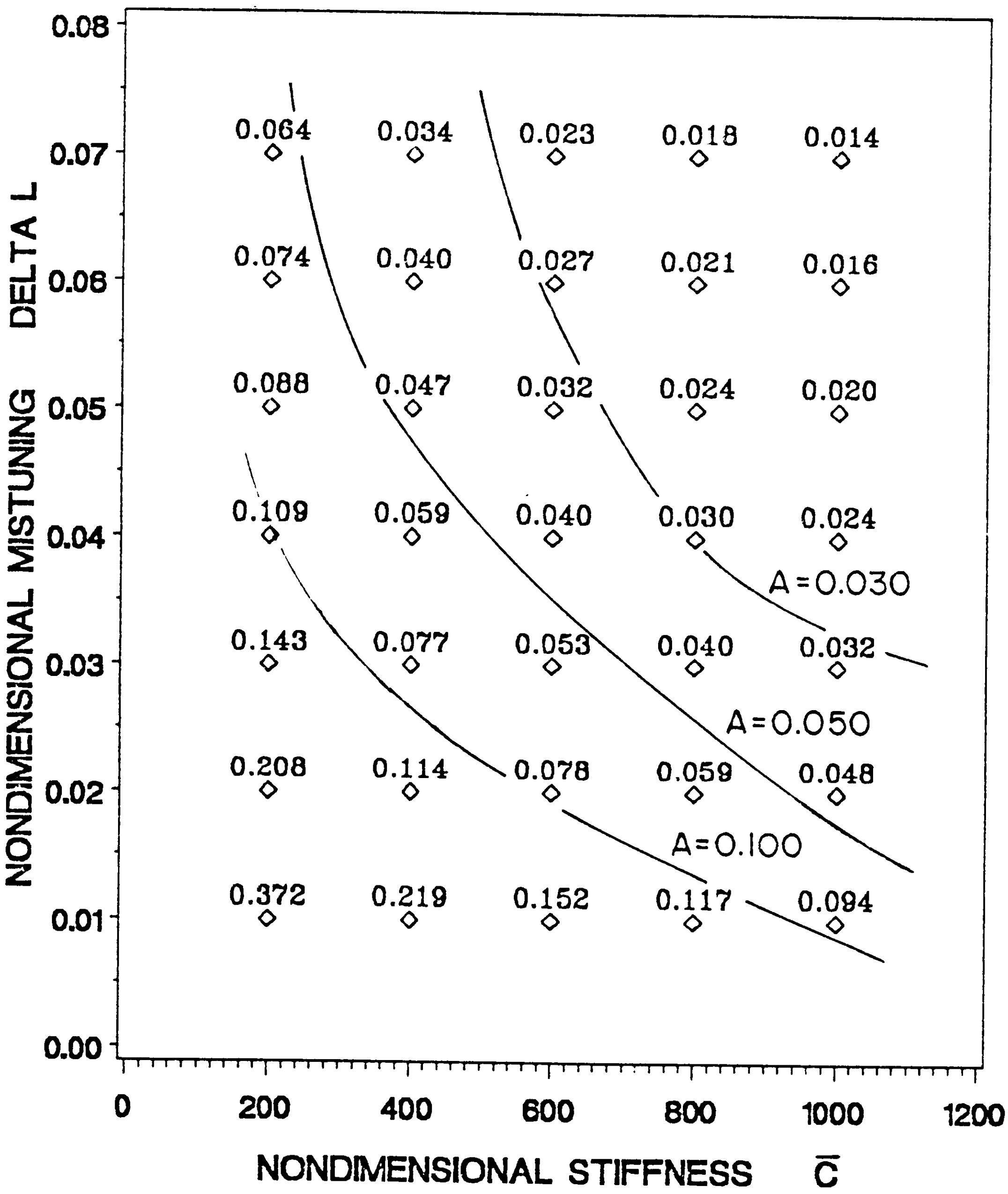

Figure 6 Localization Factors of 1st Mode of the Two Span Beam

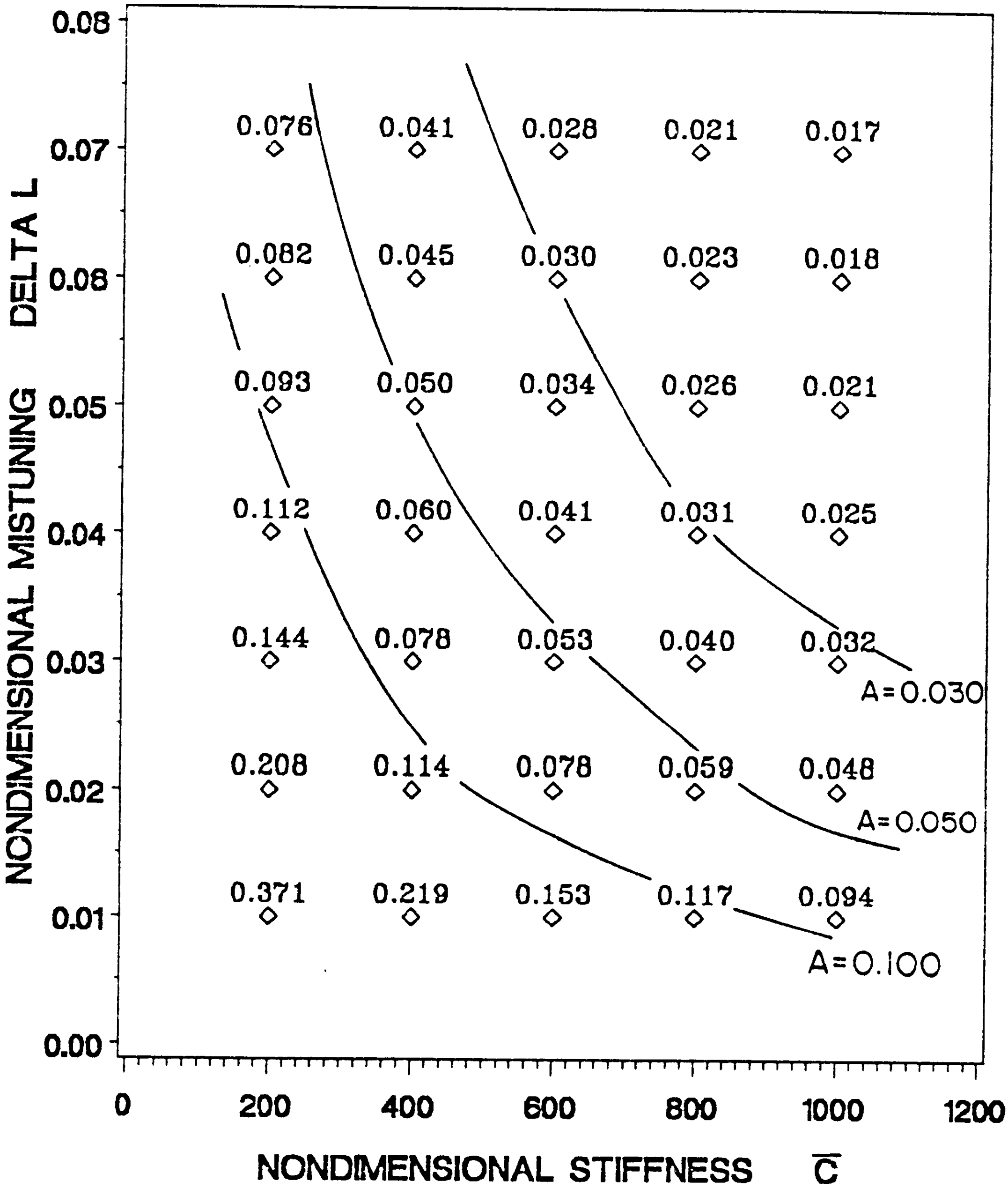

Figure 7 Localization Factors of 2nd Mode of the Two Span Beam

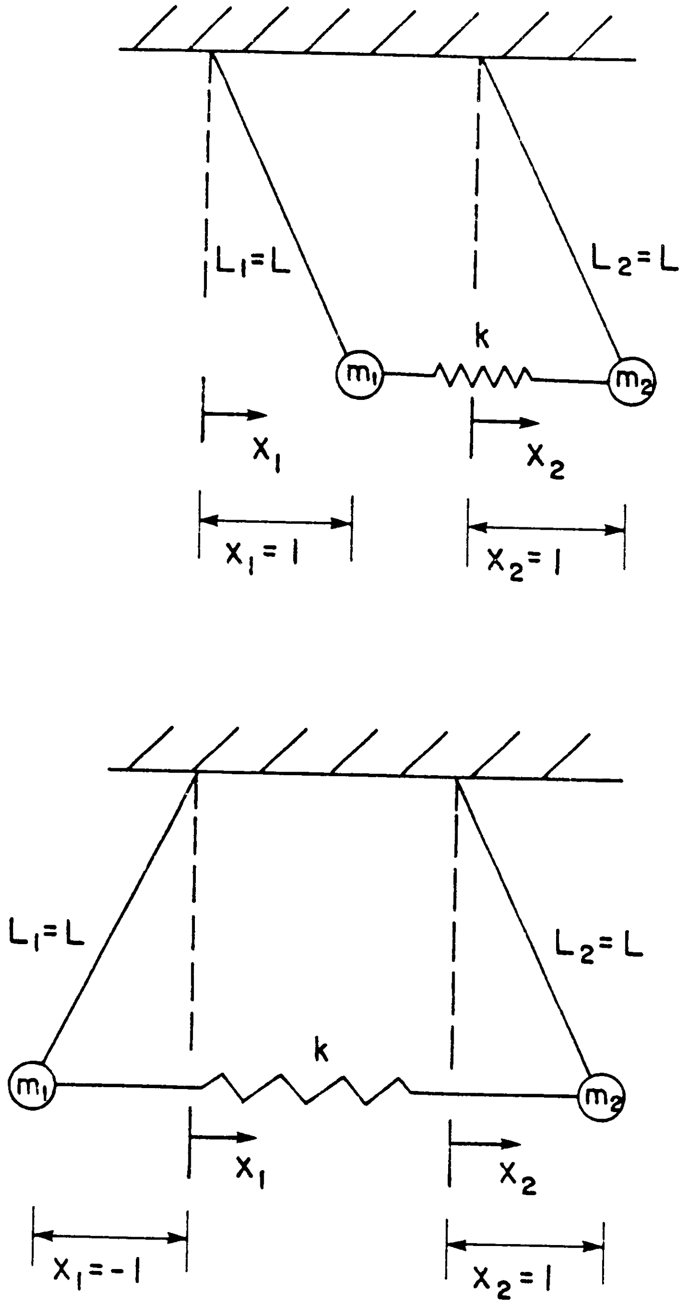

Figure 8 1st and 2nd Modes of Ordered, Coupled Pendula,

$m_1 = m_2 = m$ and $k = 200$, $\overline{\Delta L} = 0$

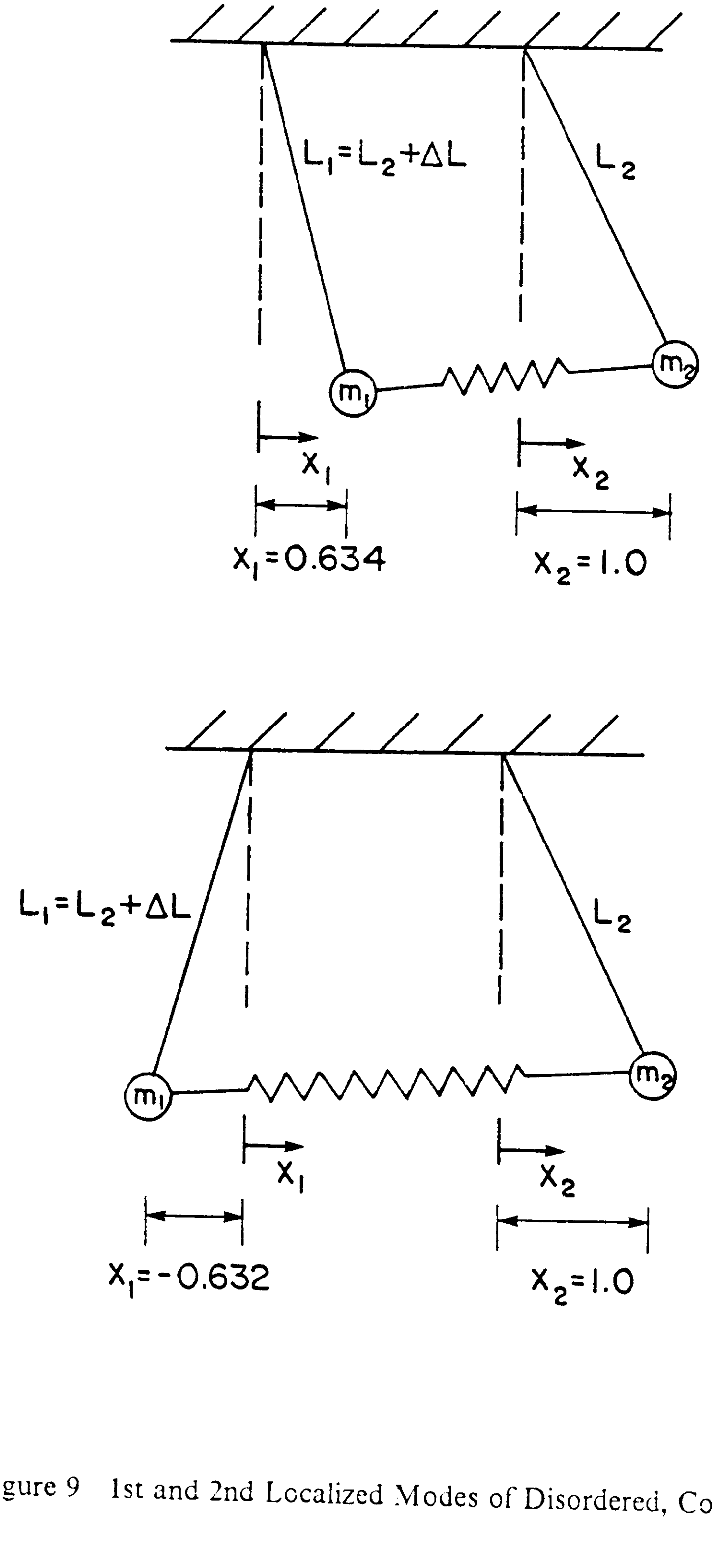

Figure 9 1st and 2nd Localized Modes of Disordered, Coupled Pendula

$k = 200, \overline{\Delta L} = .044$

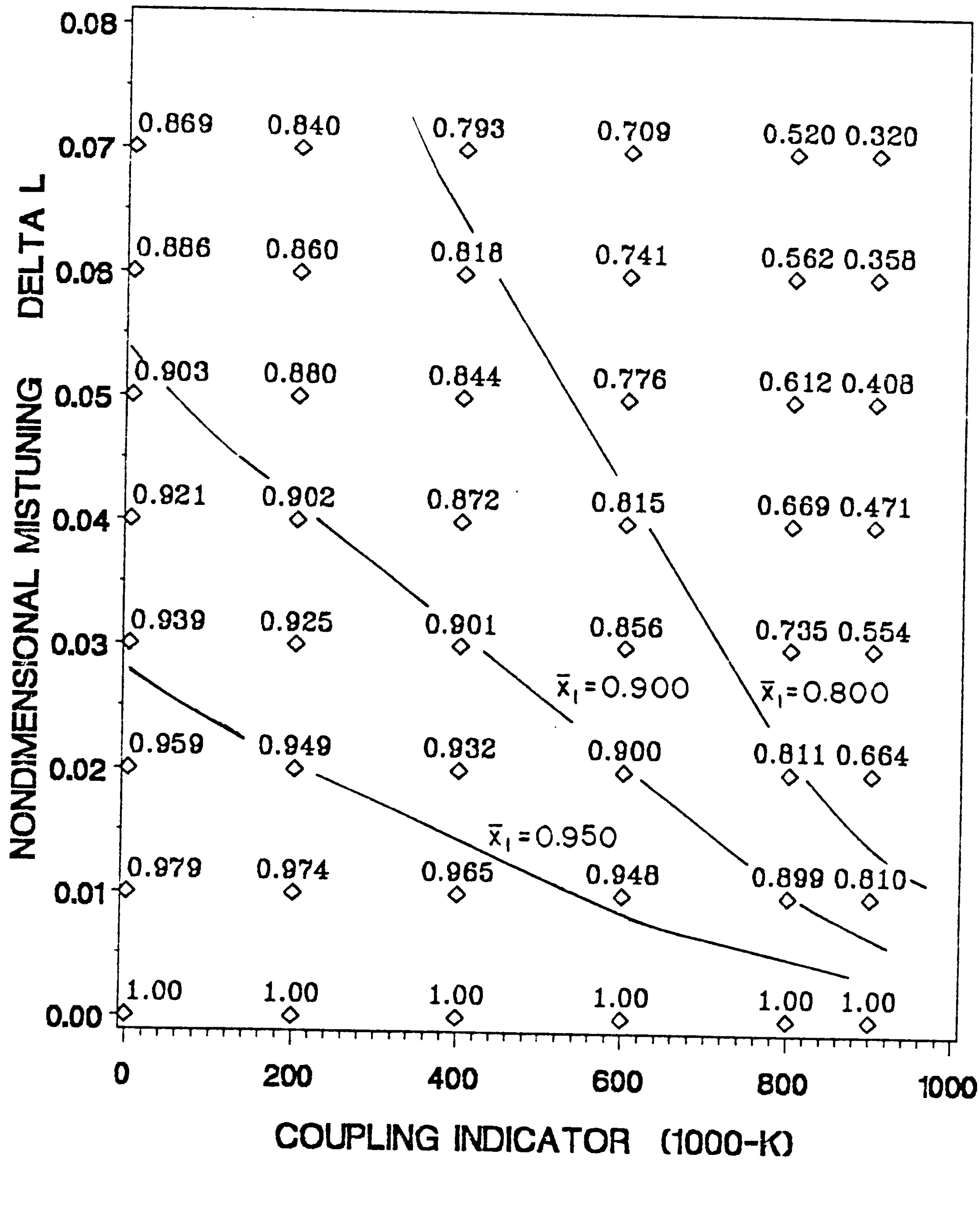

Figure 10 Localization Factors of 1st and 2nd Modes of Disordered Pendula

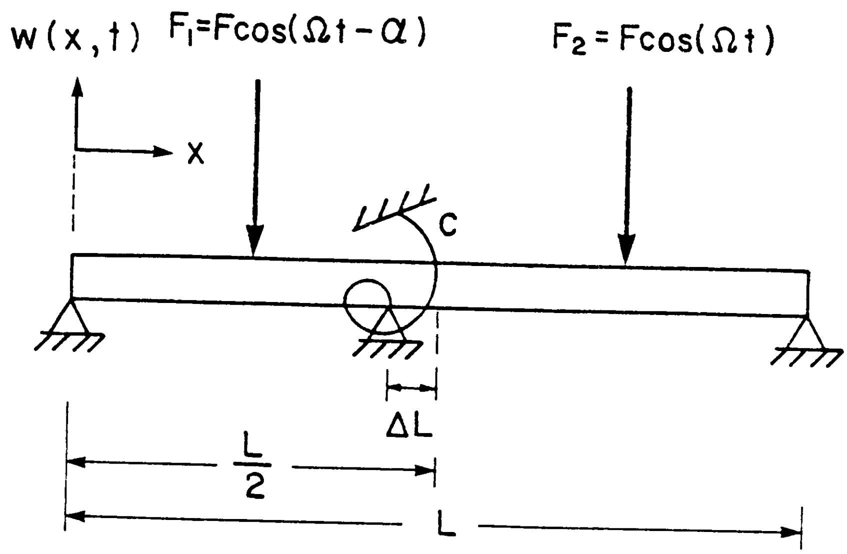

Figure 11 Forced Excitation of the Two Span Beam

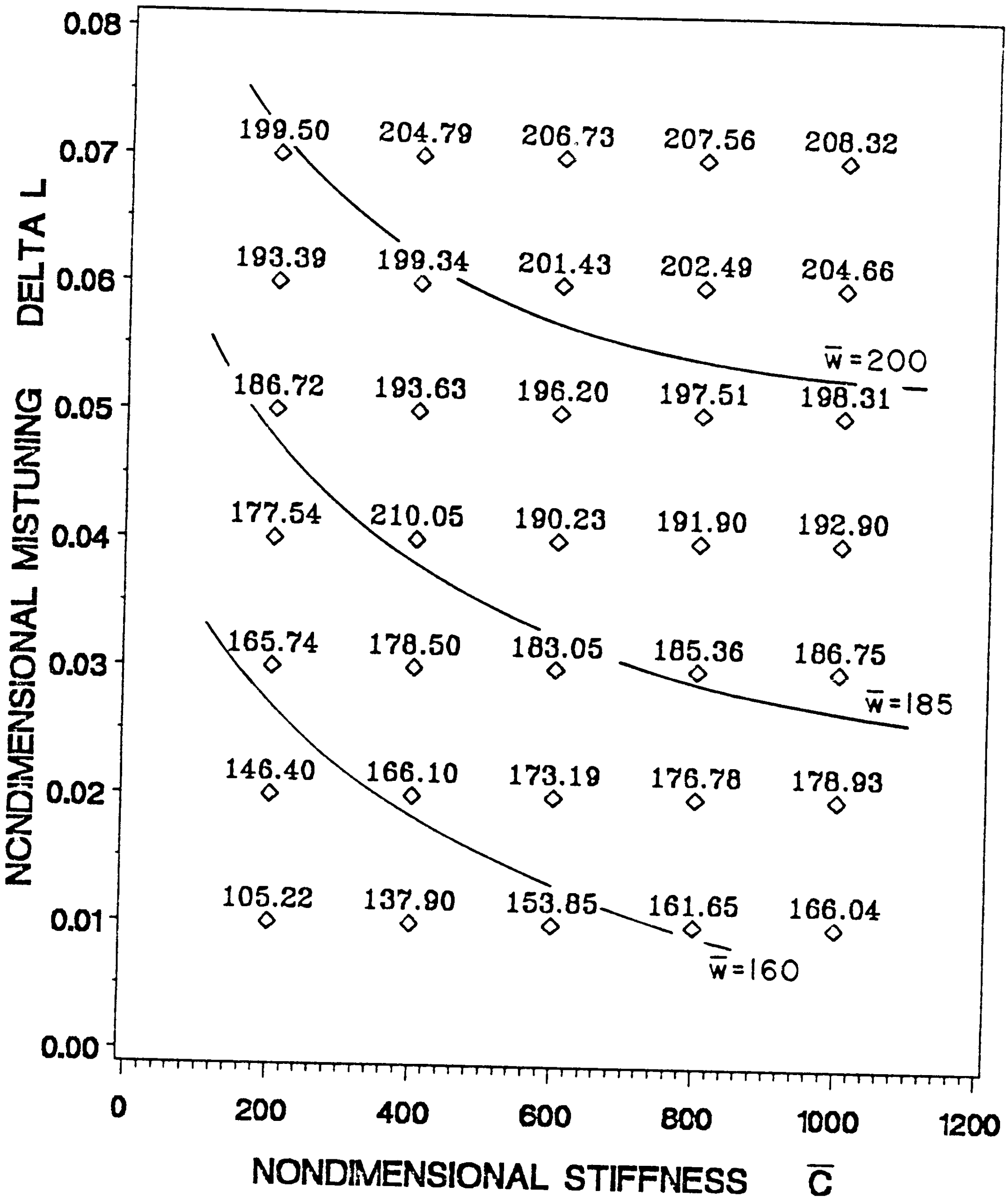

Figure 12 In Phase Forced Response of the Beam Exciting the 1st Mode

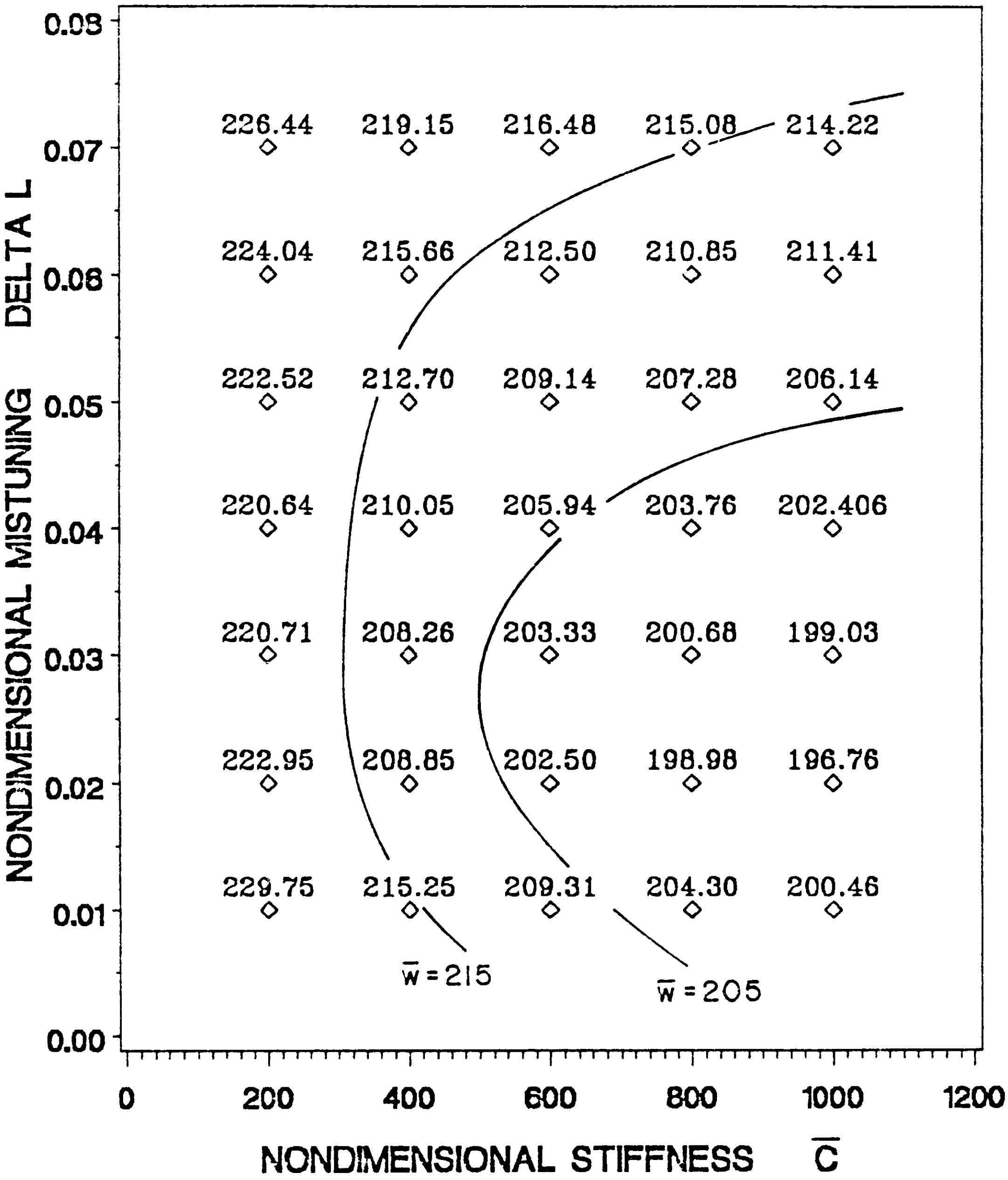

Figure 13 Out of Phase Forced Response of the Beam Exciting the 1st Mode

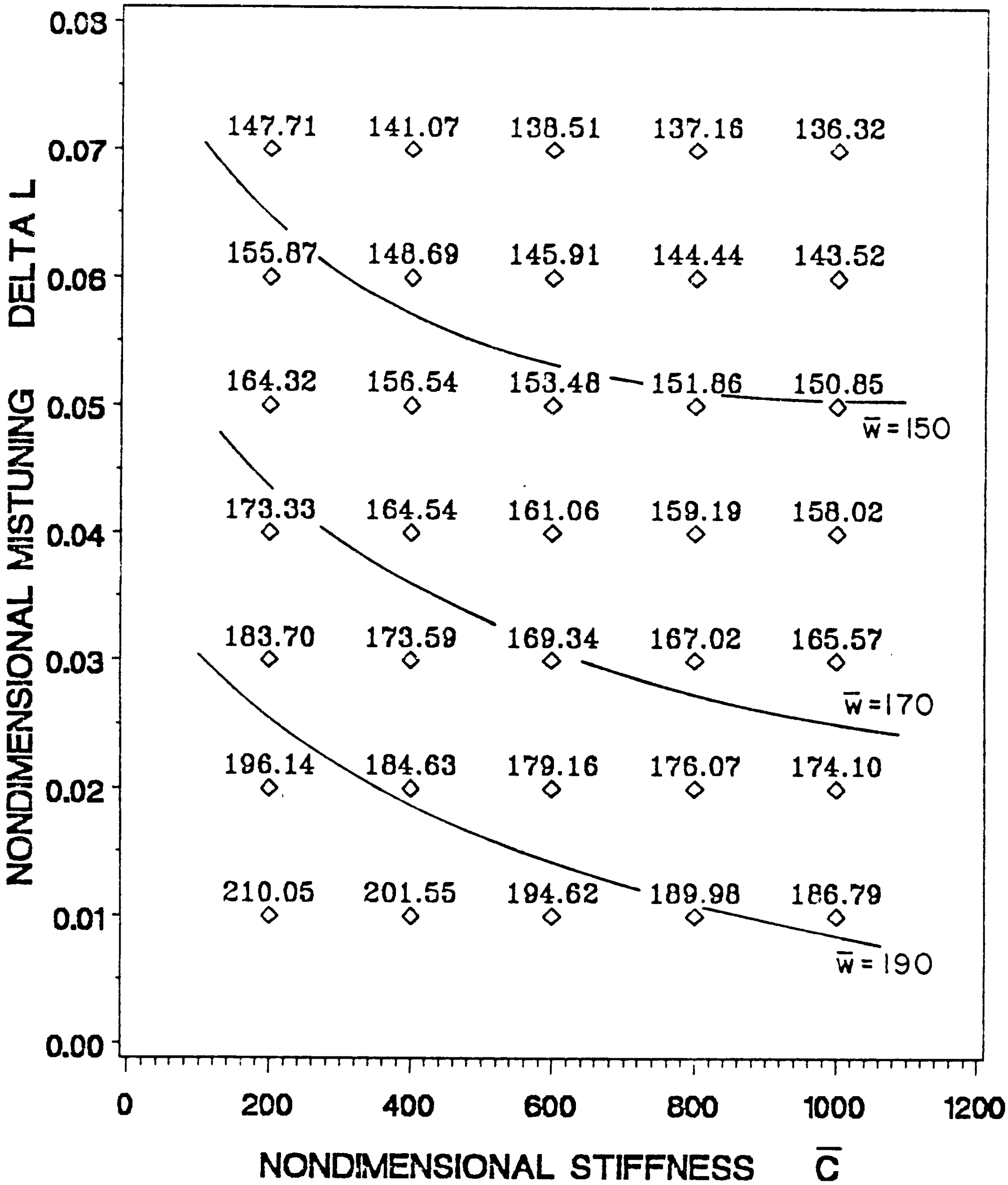

Figure 14 In Phase Forced Response of the Beam Exciting the 2nd Mode

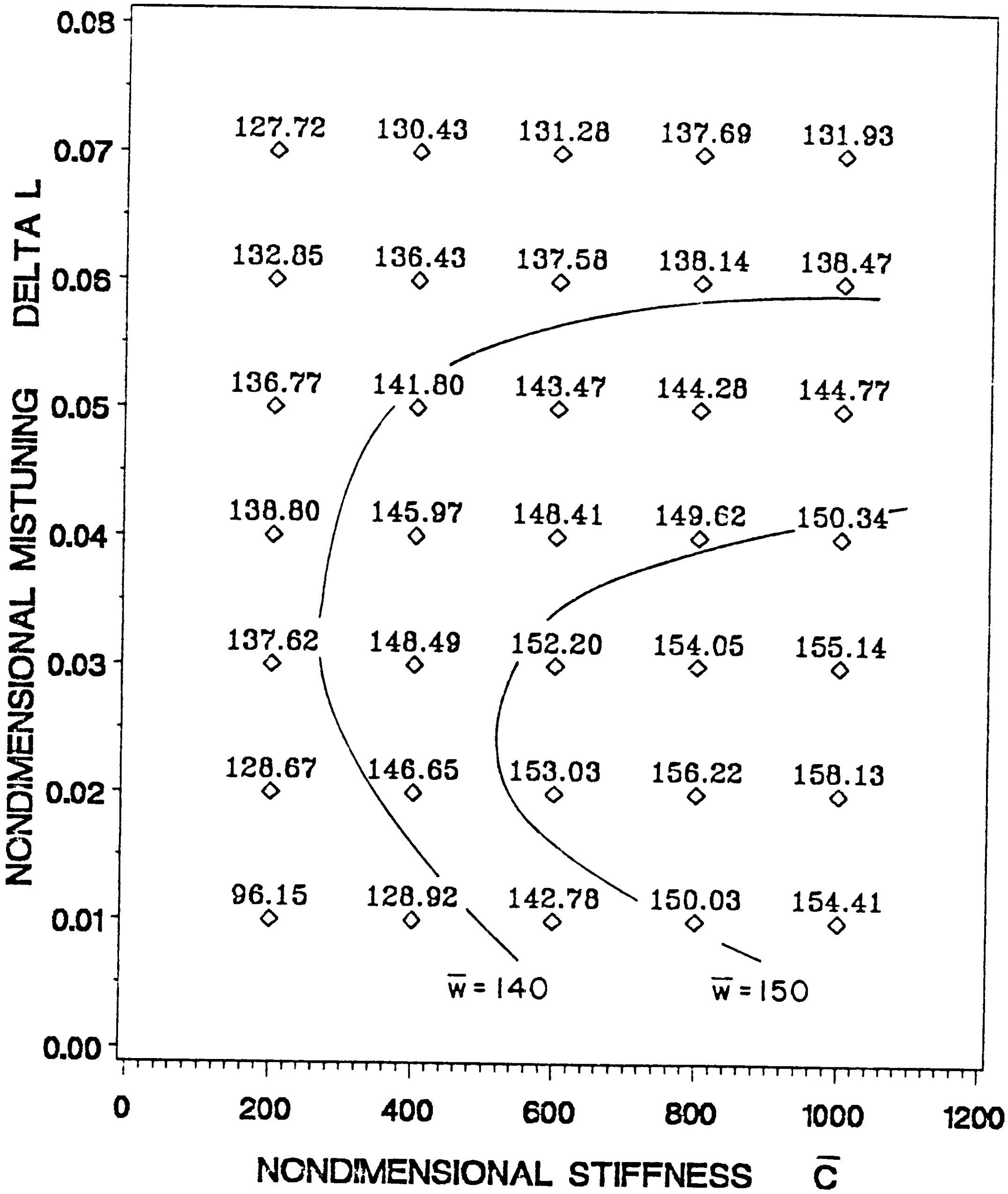

Figure 15 Out of Phase Forced Response of the Beam Exciting the 2nd Mode

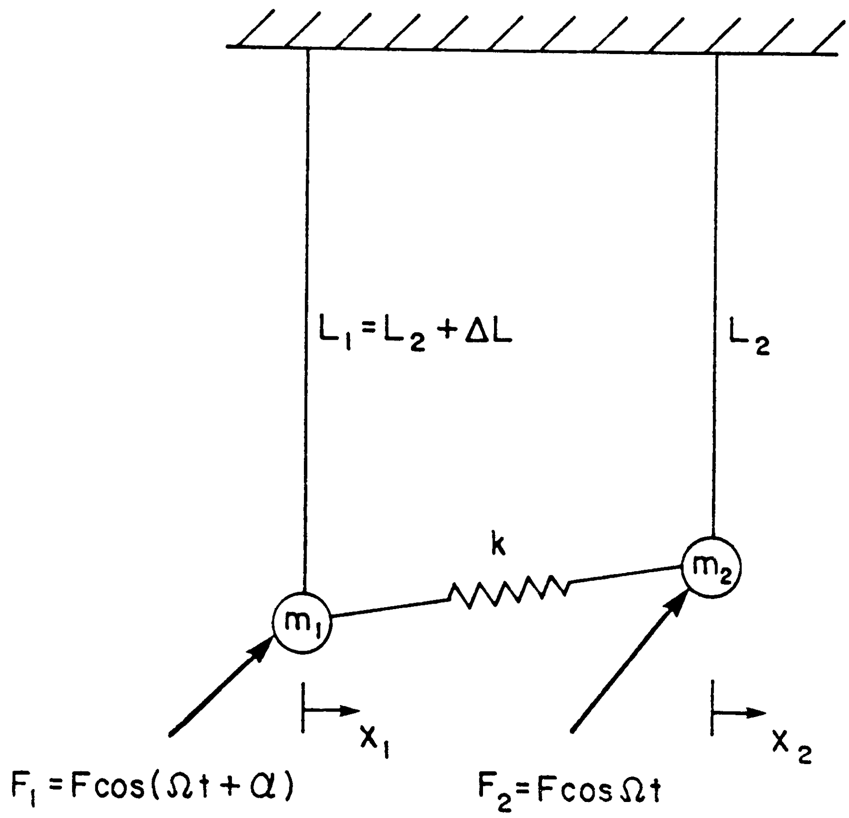

Figure 16 Forced Excitation of Disordered, Coupled Pendula

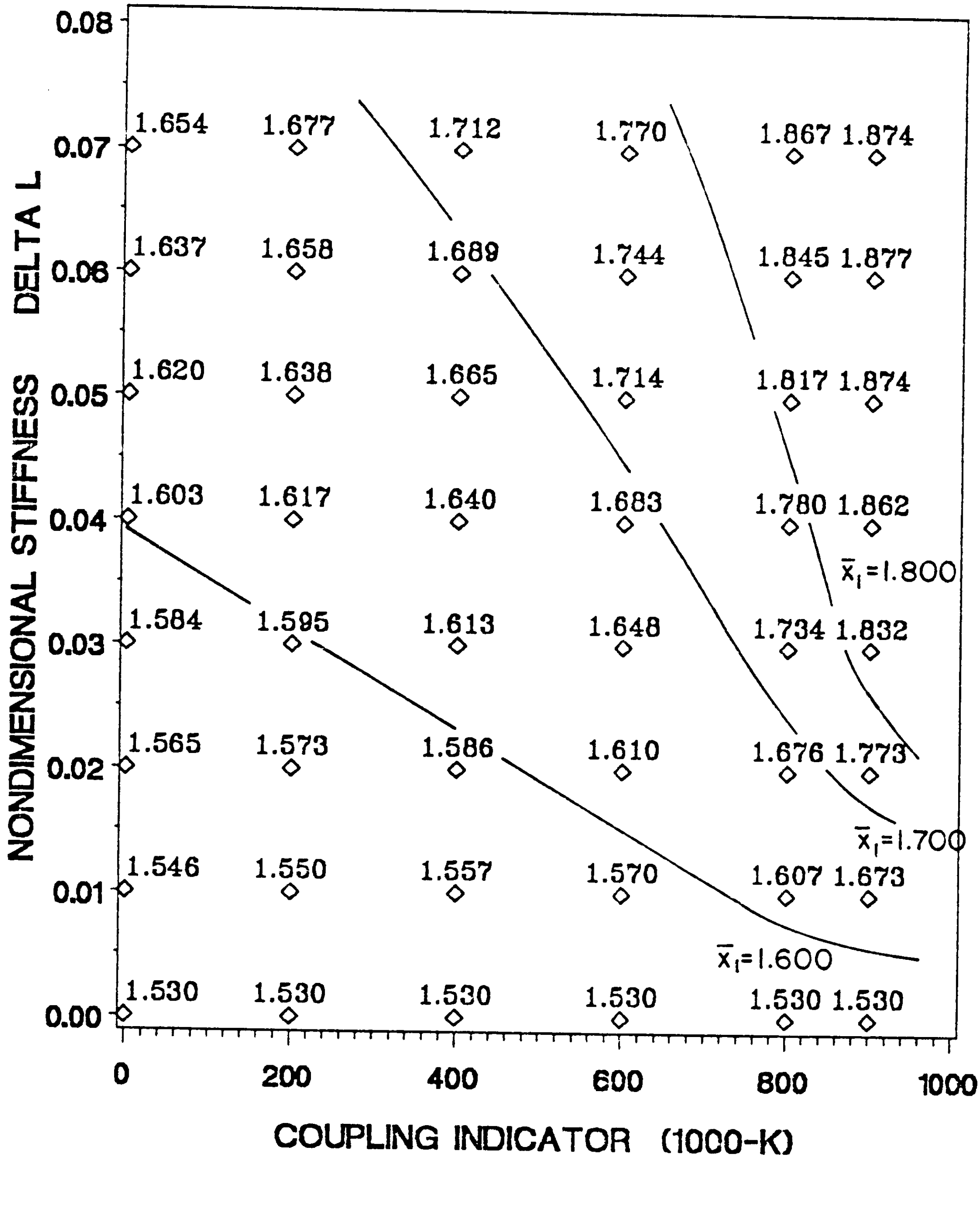

Figure 17 In Phase Forced Response Exciting 1st Mode of the Pendula

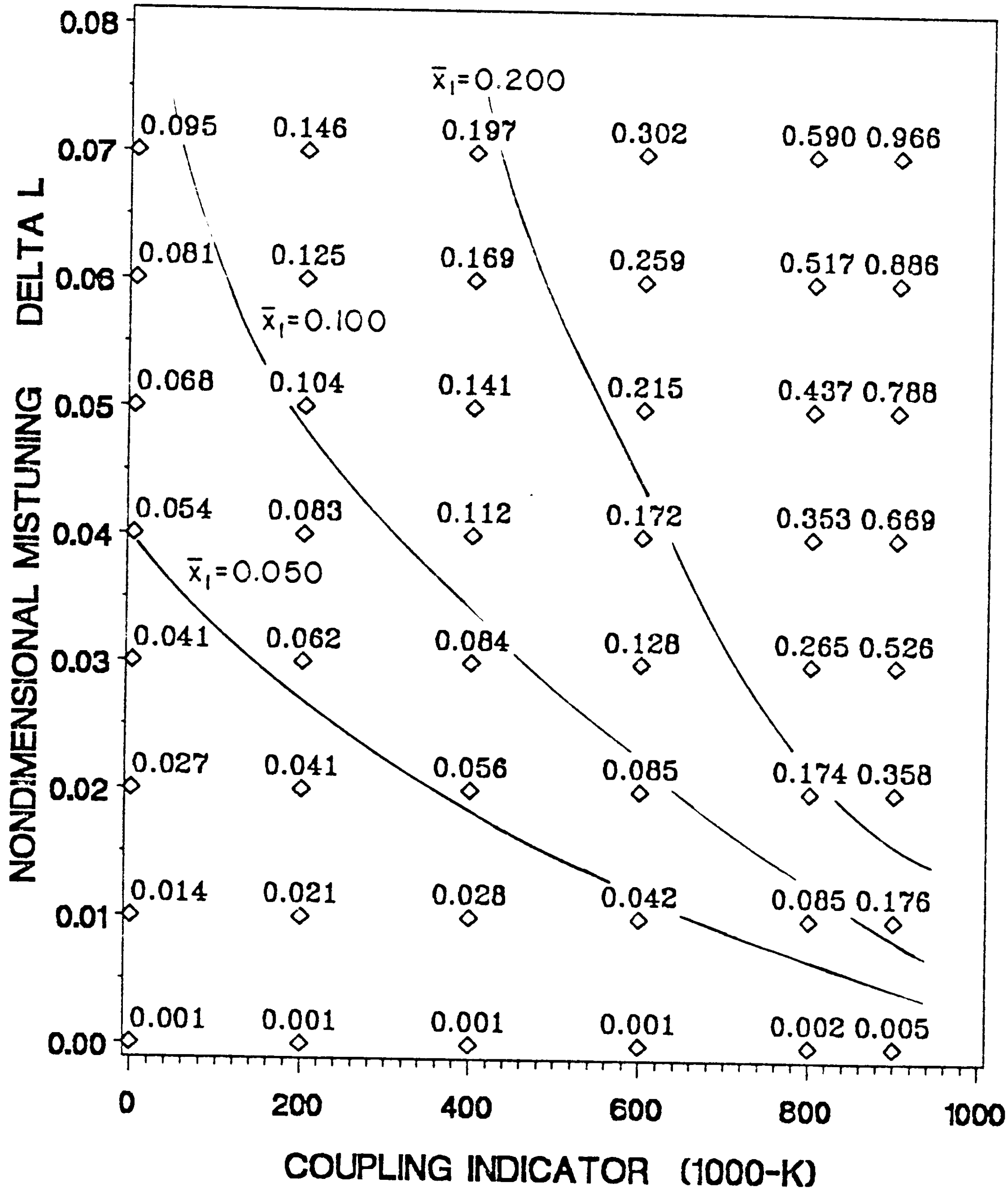

Figure 18 Out of Phase Forced Response Exciting 1st Mode of the Pendula

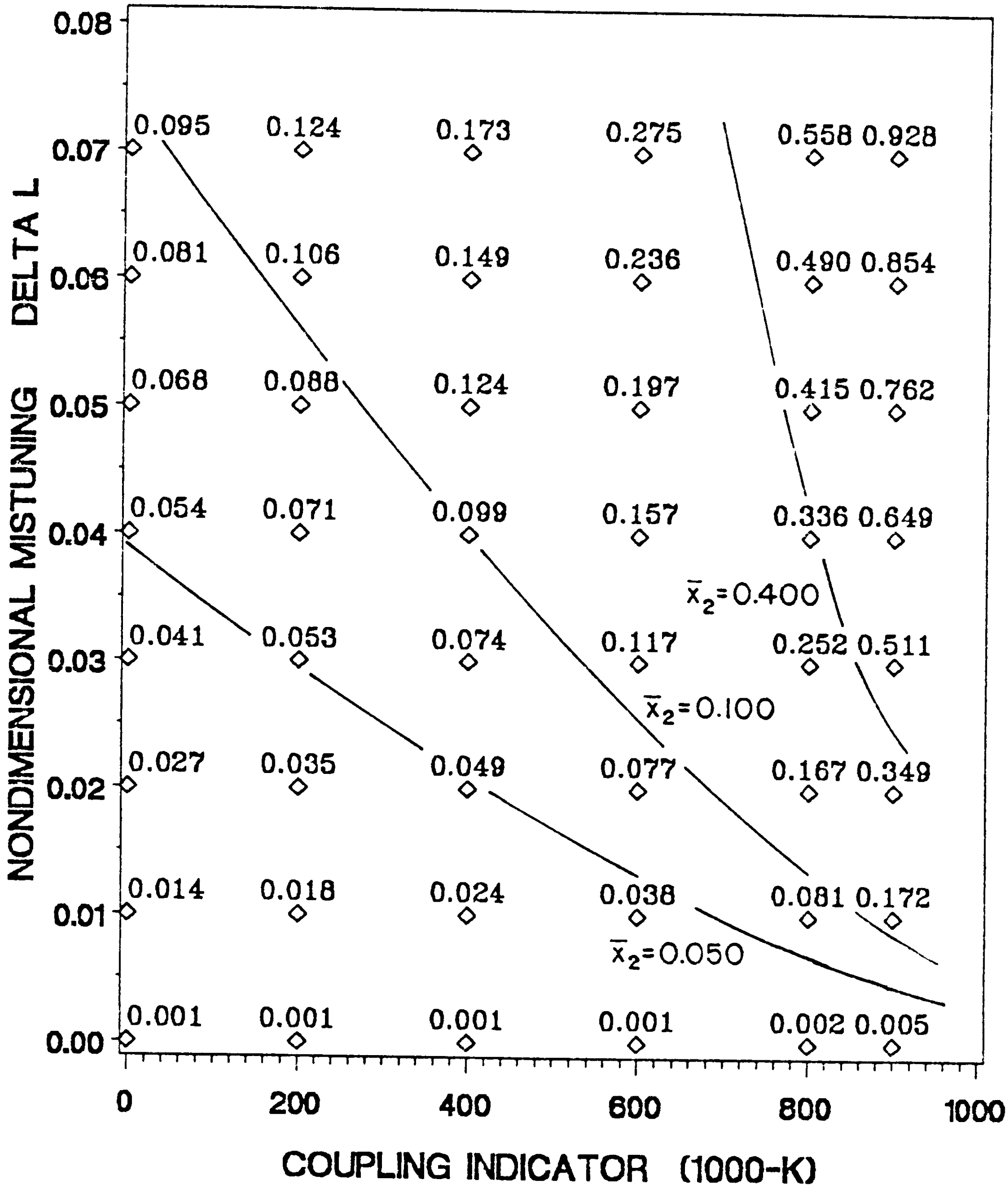

Figure 19 In Phase Forced Response Exciting 2nd Mode of the Pendula

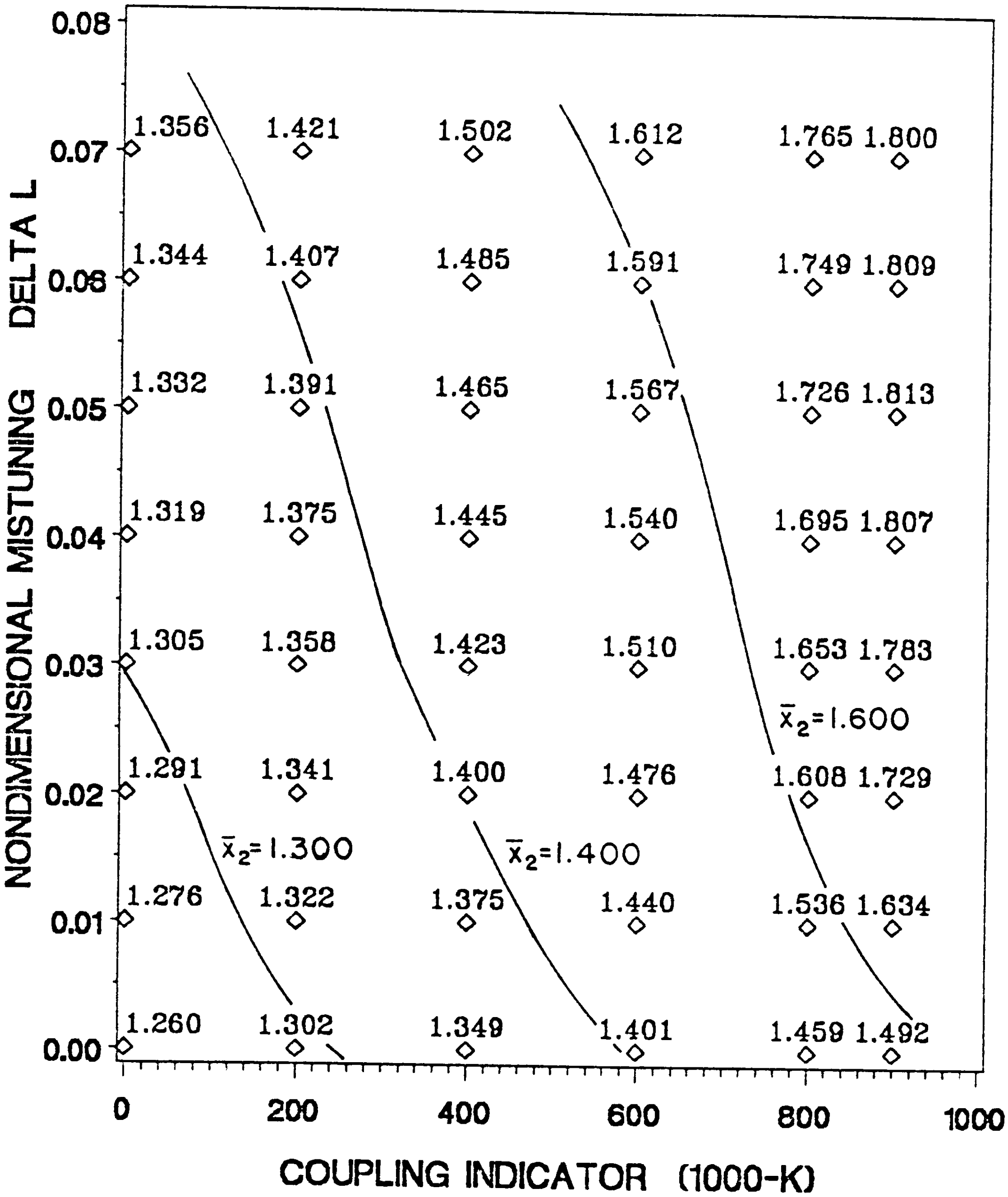

Figure 20 Out of Phase Forced Response Exciting 2nd Mode of the Pendula

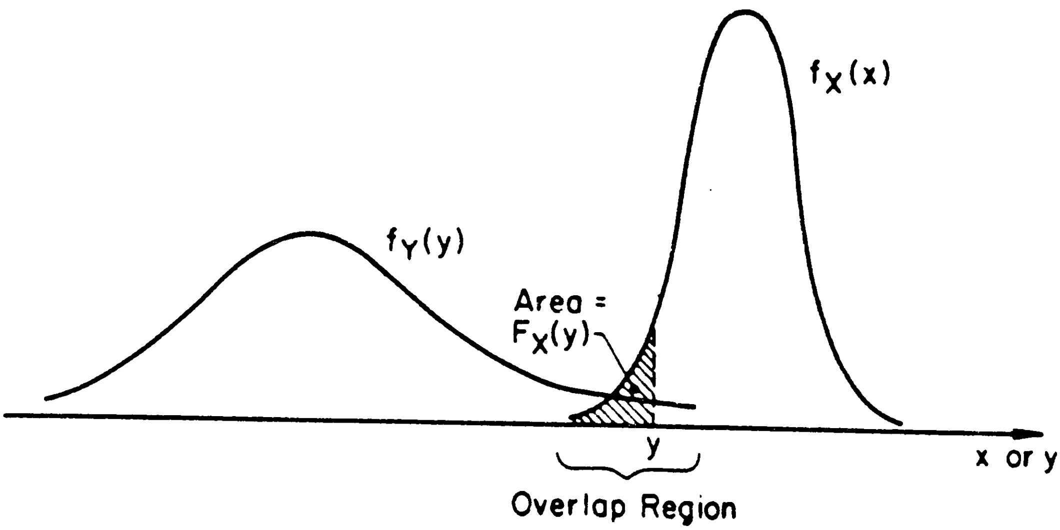

Figure 21 Probability Distribution Function

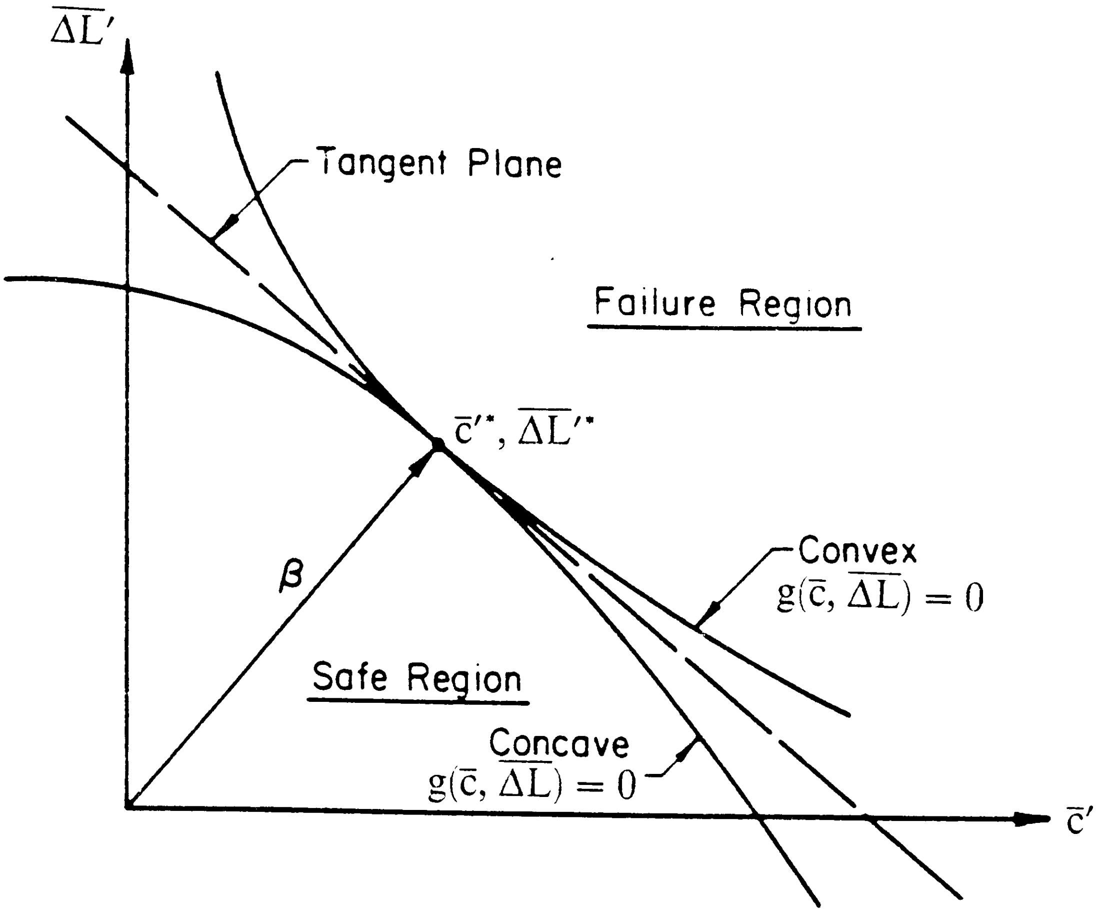

Figure 22 Linear Approximation of Failure Surface in Reduced Space

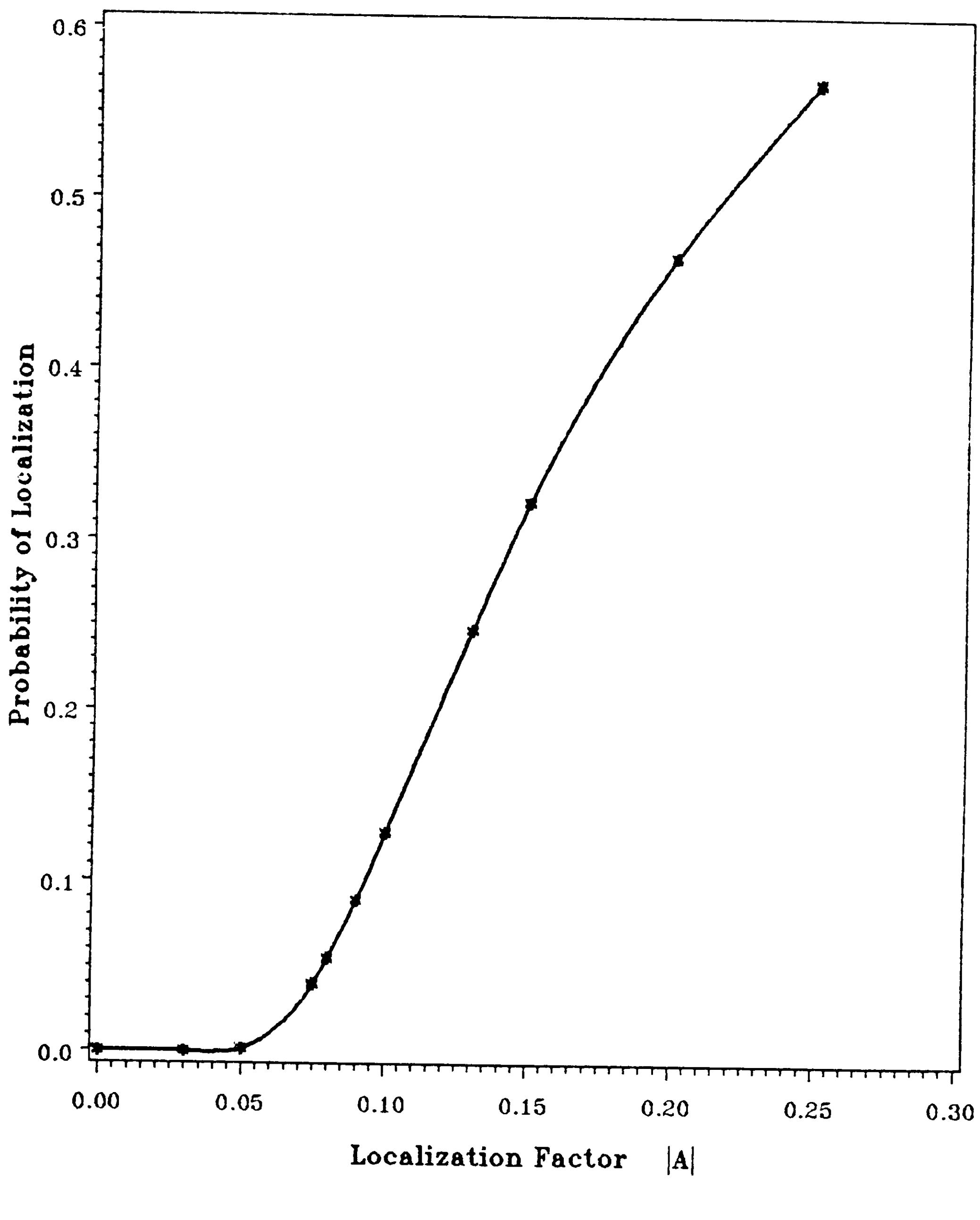

Figure 23 Probability of Localization of 1st Two Span Beam Mode

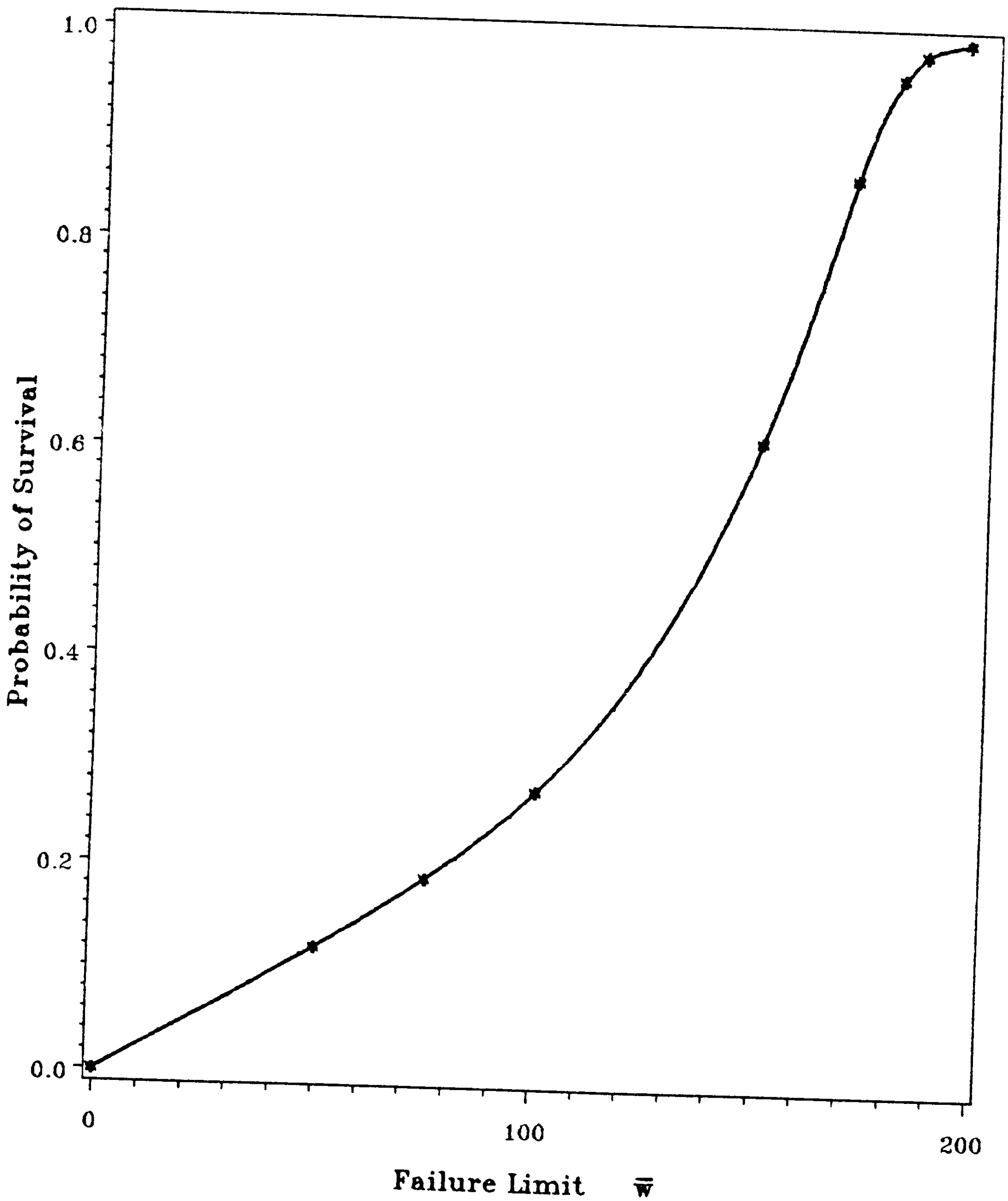

Figure 24 Probability of Survival for Varying Mean Stiffness, $\mu_{\bar{c}}$

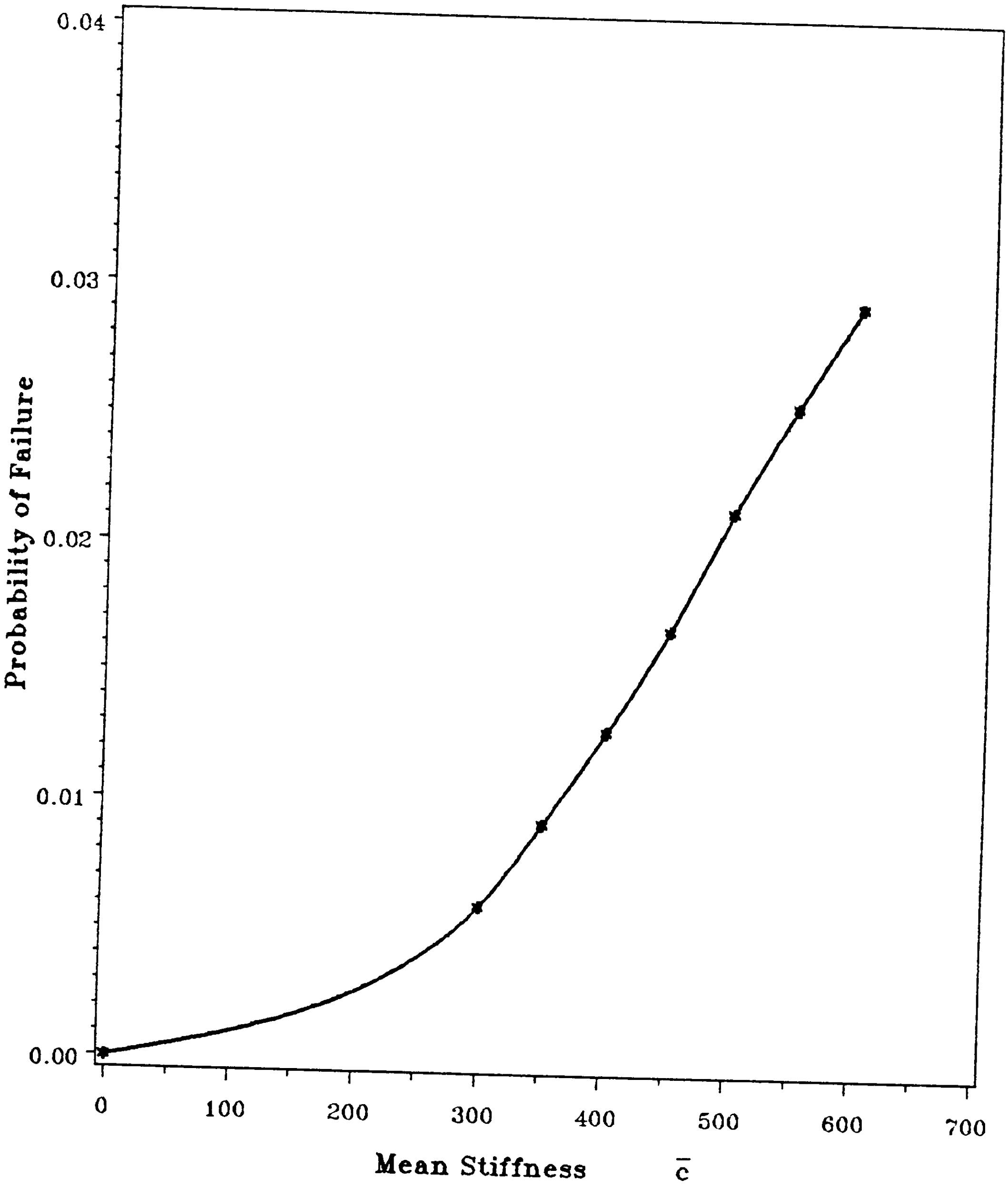

Figure 25 Probability of Failure for Varying Coupling Means

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