COVERING PROPERTIES AND QUASI-UNIFORMITIES

OF TOPOLOGICAL SPACES

by

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INTRODUCTION

The concept of compactness is fundamental in the theory of general

topology and in the applications of this theory to other branches of

mathematics; in general, covering properties are among the central

factors determining the structure of a topological space. Compact spaces

and Lindelöf-spaces were introduced early in the development of general

topology. In 1940, J. Tukey characterized normality in terms of

refinements of finite open covers and he introduced a covering property,

full normality, modeled on these characterizations. Tukey showed that

besides compact Hausdorff spaces, all metric spaces are fully normal;

hence full normality is very different in nature from the earlier

covering properties. In 1944, J. Dieudonné defined paracompact spaces.

The notion of paracompactness has become the cornerstone of what is

sometimes called "modern general topology". In 1948, K. Morita showed

that every regular Lindelöf-space is paracompact and A. H. Stone

proved his famous "coincidence theorem". According to Stone's theorem,

full normality and paracompactness are equivalent properties in the class

of Hausdorff spaces. As a consequence of the coincidence theorem and

Tukey's result, every metrizable space is paracompact. Stone's result

soon led to a solution of the metrization problem, as given by Bing,

Nagata and Smirnov, and in this way it eventually led to the introduction

of the various "generalized metric spaces", whose study is nowadays one

of the most active subfields of general topology. Paracompactness and

full normality also led to the introduction of many new covering proper-

ties. Strong paracompactness and metacompactness were defined around

1950, and in 1957, J. Mansfield defined and studied several generalizations

of full normality.

Since the properties of real numbers play a great role in the study of metric spaces, this study does not quite fit the framework of general topology. In 1938, A. Weil introduced uniform spaces as a general setting in which properties like uniform continuity and uniform convergence can be studied. Every uniformity on a set is "compatible" with a topology on the set and conversely, given any completely regular topological space, there is a uniformity on the underlying set of the space which is compatible with the topology of the space. Hence uniform spaces allow a study of metric-like properties in a setting which is almost as general as that of topological spaces. The concept of a uniformity has also been useful in the study of topological properties. In 1940, Tukey characterized compatible uniformities of a topological space in terms of certain collections of open covers of the space, and he showed that a completely regular space is fully normal if, and only if, the fine uniformity of the space has the Lebesgue property.

In 1950, L. Nachbin generalized the concept of a uniform space by introducing quasi-uniform spaces. Nachbin showed that every quasi-uniformity on a set is compatible with a topology on the set, and around 1960, A. Császár and W. Pervin obtained a converse to this result by showing, independently, that every topological space has a compatible quasi-uniformity. Hence the study of quasi-uniform spaces provides an alternative approach to the study of topological spaces. In 1971, P. Fletcher introduced the concept of a covering quasi-uniformity, and he used this concept to show that a topological space usually has several distinct compatible quasi-uniformities. In their

work on covering quasi-uniformities, P. Fletcher and W. Lindgren have studied the relationships between covering properties and compatible quasi-uniformities. In this dissertation, we pursue the study of these relationships. In particular, we try to illustrate the role that compatible quasi-uniformities have in connection with the covering properties of paracompactness, metacompactness and orthocompactness.

In Chapter I of this work we show that paracompactness and metacompactness can be characterized in terms of the existence of "sufficiently large" compatible quasi-uniformities with certain weak symmetry properties. We derive these characterizations by extending a result of J. M. Worrell Jr. on certain sequences of open covers. In the second part of the chapter we introduce a weak covering property, $\hbar$-double coveredness, and we show that a subspace X of a regular and compact space K is metacompact if, and only if, the product space $X \times K$ is $\hbar$-doubly covered. As a consequence of this result, a Tychonoff space X is metacompact if, and only if, the product space $X \times \beta X$ has a compatible quasi-uniformity with the Lebesgue property.

Chapter II deals with certain generalizations of orthocompactness. We start the chapter by showing that every countably metacompact quasi-metrizable space has a compatible quasi-uniformity with the Lebesgue property. In light of a recent example due to J. Kofner, it follows from this result that a topological space is not necessarily orthocompact, even if the space has a compatible quasi-uniformity with the Lebesgue property; thus the generalizations of orthocompactness studied in this dissertation are indeed genuine generalizations. In Section 2 we introduce a covering property which we call regular $\hbar$-refinability; the definition of this property is obtained by replacing open covers by

neighbornets in the definition of regular refinability (= 2-full normality,
in the terminology of Mansfield). We characterize metacompactness and
regular refinability in terms of regular $\hbar$-refinability. Using these
characterizations and results of H. Corson and S. Peregudov, we show
that uncountable Σ-products of non-trivial T_1-spaces are never regularly
$\hbar$-refinable. This result gives a negative answer to the question of
Mansfield, whether every doubly covered (= almost 2-fully normal) space
is regularly refinable. G. Gruenhage has recently given an example to
show that a continuous image of an orthocompact space under a closed
mapping is not necessarily orthocompact. In Section 3 we give a
modification of Gruenhage's example and we show that besides orthocompact-
ness, regular refinability and regular $\hbar$-refinability are not preserved
under closed, continuous mappings. On the other hand, we show that
every continuous image of an $\hbar$-doubly covered space under a closed
mapping is $\hbar$-doubly covered. We also show that a continuous image
of an orthocompact space under a closed mapping is orthocompact if the
space is submetacompact or if the topology of the space has an orthobase.

In Chapter III we study covering properties and quasi-uniformities
of semi-stratifiable spaces. In Section 1 we show that every $\hbar$-doubly
covered semi-stratifiable space has a compatible quasi-uniformity with
the Lebesgue property. As a consequence of this result, all the
generalizations of orthocompactness studied in this work are equivalent
in the class of semi-stratifiable spaces. To prove this result, we study
unsymmetric neighbornets of semi-stratifiable spaces. We show that in an
$\hbar$-doubly covered semi-stratifiable space, the collection of all
unsymmetric neighbornets of the space forms a base for the fine quasi-
uniformity of the space. It is a major unsolved problem in the theory

of quasi-metrizable spaces, whether every γ-space is quasi-metrizable.
In Section 2 we give a partial solution to this problem by showing that
every developable γ-space is quasi-metrizable. In Section 3 we show
that a topological space is transitive if the space is a countable union
of closed, transitive subspaces. We then use this result to show that
an $\hbar$-doubly covered semi-stratifiable space is orthocompact provided
that the space is either locally orthocompact or the countable union of
closed, orthocompact subspaces.

NOTATION AND TERMINOLOGY

In this section, we define some topological and quasi-uniform concepts that are central to the subject matter of this thesis. We also explain some set-theoretic notation to be used below. For the meaning of concepts and notation used without definition in this work, we refer the reader to [12] and [39].

Sets and relations. The set $\{1,2,\ldots\}$ of the natural numbers is denoted by $\mathbb{N}$. A sequence whose n^{th} term is x_n, for $n \in \mathbb{N}$, is denoted by $\langle x_n \rangle$. For any set A, the cardinality of A is denoted by $|A|$. Ordinal numbers are denoted by small Greek letters. We adhere to the convention that an ordinal is the set of all preceding ordinals. For any ordinal α, the membership relation is a (strict) well-order on the set α. For ordinals β and γ, we often write $\beta < \gamma$ instead of $\beta \in \gamma$. We denote the successor ordinal of α by $\alpha + 1$; note that $\alpha + 1 = \alpha \cup \{\alpha\}$.

By a relation on a set B, we mean a binary relation defined on B. Our notation concerning relations is standard (see e.g. [6]) with the exception that we sometimes write RA instead of R(A) to denote the image of a set $A \subset B$ under a relation R on B; in particular, we always abbreviate R($\{b\}$) to R$\{b\}$. Let S be a relation on B and let $n \in \mathbb{N}$. Then S^n denotes the n-fold composition of S with itself, and $S^{-n} = (S^{-1})^n$. The relations $\underset{n \in \mathbb{N}}{\cup} R^n$ and $\underset{n \in \mathbb{N}}{\cup} R^{-n}$ are denoted by R^∞ and $R^{-\infty}$, respectively. Note that R^∞ and $R^{-\infty}$ are transitive relations and that $R^{-\infty} = (R^\infty)^{-1}$.

In this work, relations serve mainly as abbreviations for certain indexed families of sets. To define a relation R on B, we define a subset R$\{b\}$ of B for every $b \in B$. When Q is a relation on B, we denote by $\mathcal{Q}Q$ the indexed family $\{Q\{b\} \mid b \in B\}$ associated with the relation Q.

Let $\mathcal{L}$ be a family of subsets of B. We define two relations, $S\mathcal{L}$ and $D\mathcal{L}$, on B by setting $S\mathcal{L}\{b\} = \cup\{L \in \mathcal{L}|b \in L\}$ and $D\mathcal{L}\{b\} = \cap\{L \in \mathcal{L}|b \in L\}$ for every $b \in B$. Note that the relation $S\mathcal{L}$ is symmetric and the relation $D\mathcal{L}$ is reflexive and transitive. For each $k \in \mathbb{N}$, we abbreviate $(S\mathcal{L})^k$ to $S^k\mathcal{L}$. Note that in the notation for star-sets (see e.g. [12], p. 376), we have $S^k\mathcal{L}(C) = St^k(C,\mathcal{L})$ for all $k \in \mathbb{N}$ and $C \subset B$. For some results concerning relations of the form $S\mathcal{L}$ and $D\mathcal{L}$, see [22].

Topological concepts. Our terminology and notation relating to topological spaces follows that of [12] with the exception that we adopt "separation-axiom free" definitions for topological properties whenever possible. For example, in our terminology, a paracompact space or a metacompact space is not necessarily a Hausdorff space. Also, a regular space is not necessarily a T_1-space.

To avoid unnecessary repetition of the phrase "a topological space", throughout this thesis we let the symbol X stand for a topological space. For each $x \in X$, we let $\mathcal{N}_x$ denote the neighborhood filter of the point x in the space X.

Let $\mathcal{L}$ be a family of subsets of X. For each $A \subset X$, we let $(\mathcal{L})_A = \{L \in \mathcal{L}|L \cap A \neq \phi\}$; if $A = \{x\}$, then we write $(\mathcal{L})_x$ in room of $(\mathcal{L})_A$. We denote by $\mathcal{L}^F$ the family consisting of all finite unions of sets of $\mathcal{L}$. Note that $\mathcal{L}^F$ is a directed family, that is, whenever $N \in \mathcal{L}^F$ and $M \in \mathcal{L}^F$, there is $K \in \mathcal{L}^F$ such that $N \cup M \subset K$. Let $\mathcal{h}$ be a family of subsets of X. We say that $\mathcal{h}$ is a partial refinement of $\mathcal{L}$ (or that $\mathcal{h}$ partially refines $\mathcal{L}$) if for each $N \in \mathcal{h}$, there is $L \in \mathcal{L}$ such that $N \subset L$. If $\mathcal{h}$ is a partial refinement of $\mathcal{L}$ and $\cup\mathcal{h} = \cup\mathcal{L}$, then $\mathcal{h}$ is a refinement of $\mathcal{L}$. We say that $\mathcal{h}$ is a point-wise (local) W-refinement of $\mathcal{L}$ if

$\cup \mathcal{N} = \cup \mathcal{L}$ and for each $x \in X$, the family $(\mathcal{N})_x$ (for some $U \in \mathcal{N}_x$, the family $(\mathcal{N})_U$) is a partial refinement of some finite subfamily $\mathcal{L}'$ of $\mathcal{L}$; a <u>point</u>-<u>wise</u> (<u>local</u>) <u>star</u>-<u>refinement</u> is defined similarly except that we now require that $\mathcal{L}' = \{L\}$ for some $L \in \mathcal{L}$.

<u>Definition 1</u> ([22]) A family $\mathcal{L}$ of subsets of X is a <u>semi</u>-<u>open</u> <u>cover</u> of X if for each $x \in X$, the set $St(x, \mathcal{L})$ is a neighborhood of x.

<u>Definition 2</u> ([36] and [22]) A family $\mathcal{L}$ of subsets of X is <u>closure</u>-<u>preserving</u> (<u>interior</u>-<u>preserving</u>) if for each $\mathcal{L}' \subset \mathcal{L}$, we have $C\ell(\cup\mathcal{L}') = \cup\{C\ell\ L | L \in \mathcal{L}'\}$ $(Int(\cap\mathcal{L}') = \cap\{Int\ L | L \in \mathcal{L}\})$.

Note that a family $\mathcal{L}$ is closure-preserving if, and only if, the family $\{X \sim L | L \in \mathcal{L}\}$ is interior-preserving. Also note that a family $\mathcal{U}$ is interior-preserving and open if, and only if, $\cap(\mathcal{U})_x$ is a neighborhood of x for each $x \in X$. Interior-preserving open families are called Q-collections in [46] and fundamental open families in [13].

<u>Definition 3</u> A topological space is <u>orthocompact</u> provided that every open cover of the space has an interior-preserving open refinement.

Orthocompact spaces were first studied in [46] and [13]. The term "orthocompact" was introduced in [15].

Our study of covering properties in this work is centered on paracompactness and metacompactness (see [12] and [39]) as well as on orthocompactness and some generalizations of orthocompactness to be introduced later; the two properties defined below play a somewhat lesser role in this work.

<u>Definition 4</u> A topological space is <u>subparacompact</u> provided that every open cover of the space has a σ-discrete closed refinement.

These spaces were studied under the name "F_σ- screenable" in [34]. The term "subparacompact" was introduced in [7], where it was shown that F_σ - screenability is equivalent with the concept of σ-paracompactness defined in [3].

<u>Definition 5</u> A sequence $\langle \mathcal{L}_n \rangle$ of covers of X is a θ-<u>sequence</u> if for each $x \in X$, there is a $n \in \mathbb{N}$ such that the family $(\mathcal{L}_n)_x$ is finite. The space X is <u>submetacompact</u> provided that every open cover of X has a θ-sequence of open refinements.

These spaces were called θ-refinable in [56]. The term "submeta-compact" was introduced in [25].

Through the concept of a covering quasi-uniformity ([14]), open covers provide a link between topological spaces and quasi-uniform spaces; another such link is provided by the following concept.

<u>Definition 6</u> ([22]) A <u>neighbornet</u> of X is a relation V on X such that for every $x \in X$, the set $V\{x\}$ is a neighborhood of x.

Since neighbornets are used throughout the following study, we recall some terminology and notation associated with neighbornets from [22]. Let R be a relation on X. We define a relation $\dot{R}$ on X by setting $\dot{R}\{x\}$ = Int $R\{x\}$ for every $x \in X$. If $\dot{R}$ = R, then we say that R is an <u>open relation</u> on X. For any neighbornet V of X, the relation $\dot{V}$ is an open neighbornet contained in V. Note that if V is an open

neighbornet of X, then the family αV is an open cover of X. A
neighbornet V of X is unsymmetric provided that for all $x \in X$ and
$y \in X$, if $x \in V\{y\}$ and $y \in V\{x\}$, then $V\{x\} = V\{y\}$. A sequence $\langle V_n \rangle$
of neighbornets is a normal sequence provided that $V_{n+1}^2 \subset V_n$ for every
$n \in \mathbb{N}$. A relation V on X is a normal neighbornet of X if V is a member
of some normal sequence of neighbornets of X. Note that if V is a
transitive neighbornet of X, then $V^2 = V$ and hence V is normal and
unsymmetric.

In order to use neighbornets in connection with covering theory,
we employ the following terminology. Let $\mathcal{L}$ be a cover of X. An
$\hbar$-refinement of $\mathcal{L}$ is a neighbornet V of X such that the family αV is
a refinement of $\mathcal{L}$. According to Corollary 3.8 of [22], every open cover
of X has an unsymmetric $\hbar$-refinement. Let $k \in \mathbb{Z} \cup \{\infty, -\infty\}$. An
$\hbar^k$-refinement of $\mathcal{L}$ is a neighbornet V of X such that the family αV^k
is a refinement of $\mathcal{L}$. A cover of X has an $\hbar^\infty$-refinement ($\hbar^{-\infty}$-refinement,
$\hbar^{-1}$-refinement) if, and only if, the cover has an interior-preserving
open (closure-preserving closed, cushioned) refinement (Theorem 2 of
[14], Theorem 3.14 and Lemma 3.4 of [22]). Consequently, X is
orthocompact (fully normal) if, and only if, every open cover of X
has an $\hbar^\infty$-refinement ($\hbar^{-\infty}$-refinement, $\hbar^{-1}$-refinement); see [15,
36 and 37].

To be able to characterize generalized metric spaces in a uniform
way, we adopt the following definition.

Definition 7 ([22]) A sequence $\langle R_n \rangle$ of reflexive relations on X is
a basic sequence provided that for each $x \in X$, if $U \in \eta_x$, then $R_n\{x\} \subset U$
for some $n \in \mathbb{N}$.

<u>Definition 8</u> ([11] and [26]) X is <u>semi-stratifiable</u> if X has a sequence $\langle V_n \rangle$ of neighbornets such that $\langle V_n^{-1} \rangle$ is a basic sequence.

In [26], semi-stratifiable spaces are called pseudo-stratifiable.

<u>Definition 9</u> ([21]) X is a γ-<u>space</u> if X has a sequence $\langle V_n \rangle$ of neighbornets such that $\langle V_n^2 \rangle$ is a basic sequence.

γ-spaces have been defined under different names independently by several authors. The equivalence of the various definitions was established in [31].

We use the condition stated in the following result as our definition of a <u>quasi-metrizable</u> space.

<u>Theorem 10</u> ([43]) X is quasi-metrizable if, and only if, X has a normal basic sequence of neighbornets.

Quasi-metric spaces were introduced in 1931 by A. Wilson ([53]). While there have been numerous characterizations of metrizability in topological terms, the above result is the only known topological characterization of quasi-metrizability.

<u>Quasi-uniform concepts</u>. In this work, we stay within the realm of general topology, and therefore we define quasi-uniformities only in the setting of a topological space. For an exposition of the theory of quasi-uniform spaces, see [38].

<u>Definition 11</u> ([40]) A filter (-base) $\mathfrak{Q}$ of reflexive relations on X is a (<u>base</u> <u>for</u> <u>a</u>) <u>quasi-uniformity</u> on X provided that for each $Q \in \mathfrak{Q}$, there

exists $R \in \mathcal{2}$ such that $R^2 \subset Q$. A quasi-uniformity $\mathcal{2}$ on X is <u>compatible</u> with X provided that for each $x \in X$, the family $\{Q\{x\} | Q \in \mathcal{2}\}$ is a base for $\mathcal{N}_x$.

Let $\mathcal{2}$ be a quasi-uniformity on X. The collection $\{Q^{-1} | Q \in \mathcal{2}\}$ is denoted by $\mathcal{2}^{-1}$. It is easily seen that $\mathcal{2}^{-1}$ is a quasi-uniformity on X.

Note that if $\mathcal{2}$ is a compatible quasi-uniformity on X, then each member of $\mathcal{2}$ is a normal neighbornet of X. The collection of all normal neighbornets of X forms a compatible quasi-uniformity that contains every other compatible quasi-uniformity of X; this quasi-uniformity is called the <u>fine quasi-uniformity</u> of X in [15]. Another compatible quasi-uniformity for X is generated by the collection of all transitive neighbornets of X; this is called the <u>fine transitive quasi-uniformity</u> of X ([15]). In general, a quasi-uniformity $\mathcal{2}$ on X is a <u>transitive quasi-uniformity</u> if $\mathcal{2}$ has a base consisting of transitive relations ([14]).

<u>Definition 12</u> ([14]) Let $\mathcal{G}$ be a collection of interior-preserving open covers of X such that the collection $\{D\mathcal{U} | \mathcal{U} \in \mathcal{G}\}$ is a base for a compatible quasi-uniformity $\mathcal{2}$ of X. Then $\mathcal{2}$ is called a <u>covering quasi-uniformity</u> of X.

Let $\mathcal{R}$ be a compatible quasi-uniformity for X. According to Theorem 2 of [14], $\mathcal{R}$ is a covering quasi-uniformity if, and only if, $\mathcal{R}$ is transitive.

If $\mathcal{G}$ is the collection of all point-finite (locally finite) open covers of X, then the quasi-uniformity with the base $\{D\mathcal{U} | \mathcal{U} \in \mathcal{G}\}$ is called the <u>point-finite</u> (<u>locally finite</u>) <u>covering quasi-uniformity</u> of X ([14]).

Definition 13 ([15]) A quasi-uniformity $\mathfrak{D}$ of X has the Lebesgue property provided that for every open cover $\mathfrak{U}$ of X, there exists $Q \in \mathfrak{D}$ such that the family $\mathfrak{C}Q$ is a refinement of $\mathfrak{U}$.

By Theorem 2.2 of [14], X is orthocompact if, and only if, X has a compatible transitive quasi-uniformity with the Lebesgue property. It follows directly from the definitions that X is metacompact (paracompact) if, and only if, the point-finite (locally finite) covering quasi-uniformity of X has the Lebesgue property.

CHAPTER I

QUASI-UNIFORM CHARACTERIZATIONS OF
METACOMPACTNESS AND PARACOMPACTNESS

In this chapter we show that metacompactness and paracompactness
can be characterized in terms of the existence of compatible quasi-
uniformities with certain properties. Our results follow from two
theorems whose statements do not involve quasi-uniformities. In the
first section of the chapter we show that a cover of a topological
space has a point-finite (locally finite) open refinement if, and only
if, the cover has a point-wise (local) W-sequence of open refinements.
It follows from this result that the existence of point-finite or
locally finite refinements can be characterized in terms of the
existence of certain normal sequences of h-refinements; using this
observation, we arrive at characterizations of metacompactness and
paracompactness in terms of the existence of "sufficiently large"
quasi-uniformities with certain symmetry properties. In the second
section of this chapter, we introduce the concept of an h-doubly
covered space, and we show that a topological space is metacompact
if, and only if, the space is h-doubly covered and every directed open
cover of the space has an h^{-1}-refinement. We then prove that a subspace
X of a regular compact space K is metacompact if, and only if, the
product space $X \times K$ is h-doubly covered. Thus a Tychonoff space X is
metacompact if, and only if, the product space $X \times \beta X$ has a compatible
quasi-uniformity with the Lebesgue-property.

1. Metacompactness, paracompactness and
presymmetric quasi-uniformities

Definition 1.1.1 A sequence $\langle \mathcal{U}_n \rangle$ of covers of X is a point-wise (local)

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W-<u>sequence</u> if for each $n \in \mathbb{N}$, the cover $\mathcal{U}_{n+1}$ is a point-wise (local) W-refinement of the cover $\mathcal{U}_n$.

In [54], J. M. Worrell Jr. showed that a topological space is metacompact if every open cover of the space has an open point-wise W-refinement. In the proof of this result, Worrell showed that if a cover of a topological space has a point-wise W-sequence of open refinements, then the cover has a σ-point-finite open refinement (see also Proposition 2.2 of [23]). We now extend Worrell's result as follows.

<u>Theorem 1.1.2</u> A cover of a topological space has a point-finite (locally finite) open refinement if, and only if, the cover has a point-wise (local) W-sequence of open refinements.

<u>Proof</u>. Necessity follows from the observation that if $\hbar$ is a point-finite (locally finite) cover of X, then the sequence $\langle \hbar_n \rangle$, where $\hbar_n = \hbar$ for each $n \in \mathbb{N}$, is a point-wise (local) W-sequence. Sufficiency. Let $\mathcal{L}$ be a cover of X and let $\langle \mathcal{U}_n \rangle$ be a point-wise (local) W-sequence of open refinements of $\mathcal{L}$. To show that $\mathcal{L}$ has a point-finite (locally finite) open refinement, it suffices to show that $\mathcal{U}_1$ has such a refinement. Represent $\mathcal{U}_1$ in the form $\mathcal{U}_1 = \{U_\alpha | \alpha < \gamma\}$, where γ is an ordinal number. For each $U \in \bigcup_{n \in \mathbb{N}} \mathcal{U}_n$, denote by $\alpha(U)$ the least ordinal $\alpha < \gamma$ such that $U \subset U_\alpha$; note that for any $U' \in \bigcup_{n \in \mathbb{N}} \mathcal{U}_n$, if $U \subset U'$, then $\alpha(U) \leqq \alpha(U')$. For $n > 1$ and $U \in \mathcal{U}_n$, we say that the family $\mathcal{U}_{n-1}$ is "precise" at the set U provided that for any $U' \in \mathcal{U}_{n-1}$, if $U \subset U'$, then $\alpha(U) = \alpha(U')$. For each $n > 1$, we let $\mathbb{w}_n = \{U \in \mathcal{U}_n | \mathcal{U}_{n-1}$ is precise at $U\}$.

Let $\langle K_n \rangle_{n=2}^{\infty}$ be a decreasing sequence of subsets of X such that $(\mathcal{U}_n)_{K_n}$ is a partial refinement of some finite subfamily of $\mathcal{U}_{n-1}$ for every $n \geqq 2$. We show that there exists $m \geqq 2$ such that $(\mathcal{U}_m)_{K_m}$ is a partial refinement of the family $\bigcup_{k=2}^{m} \mathfrak{w}_k$. Let $\mathfrak{R}_n = \{U \in (\mathcal{U}_n)_{K_n} \mid U \not\subset W$ for any $W \in \bigcup_{k=2}^{n} \mathfrak{w}_n\}$ for every $n \geqq 2$. To prove the preceding assertion, it suffices to show that $\mathfrak{R}_m = \emptyset$ for some $m \geqq 2$. Assume on the contrary that $\mathfrak{R}_n \neq \emptyset$ for every $n \geqq 2$. Note that for every $n > 2$, if $R \in \mathfrak{R}_n$ and if $R \subset U \in \mathcal{U}_{n-1}$, then $U \in \mathfrak{R}_{n-1}$; consequently, $\mathfrak{R}_n$ partially refines some finite subfamily of $\mathfrak{R}_{n-1}$. For every $n \geqq 2$, denote by α_n the least upper bound of the set $\{\alpha(R) \mid R \in \mathfrak{R}_n\}$, and note that $\alpha_n < \gamma$ since $\mathfrak{R}_n$ is a partial refinement of some finite subfamily of $\mathcal{U}_{n-1}$. For each $n > 2$, we have $\alpha_n \leqq \alpha_{n-1}$ since $\mathfrak{R}_n$ is a partial refinement of $\mathfrak{R}_{n-1}$. It follows that there exists $k > 2$ such that $\alpha_n = \alpha_k$ for every $n \geqq k-1$. Let $\mathfrak{R}$ be a finite subfamily of $\mathfrak{R}_k$ such that $\mathfrak{R}_{k+1}$ is a partial refinement of $\mathfrak{R}$, and let $R \in \mathfrak{R}$ be such that $\alpha(R') \leqq \alpha(R)$ for every $R' \in \mathfrak{R}$. Since $\mathfrak{R}_{k+1}$ is a partial refinement of $\mathfrak{R}$, we have $\alpha_{k+1} \leqq \alpha(R)$; on the other hand, since $R \in \mathfrak{R}_k$, we have $\alpha(R) \leqq \alpha_k$. Since $\alpha_k = \alpha_{k+1}$, it follows that $\alpha(R) = \alpha_k$. As we also have $\alpha_{k-1} = \alpha_k$, we see that $\mathcal{U}_{k-1}$ is precise at R: if $U \in \mathcal{U}_{k-1}$ is such that $R \subset U$, then $\alpha(R) \leqq \alpha(U)$ and, furthermore, $U \in \mathfrak{R}_{k-1}$ so that $\alpha(U) \leqq \alpha_{k-1} = \alpha(R)$. Consequently, $R \in \mathfrak{w}_k$; this, however, is in contradiction with the definition of the family $\mathfrak{R}_k$. It follows that there exists $m \geq 2$ such that $\mathfrak{R}_m = \emptyset$.

Let $x \in X$. If we let $K_n = \{x\}$ for every $n \in \mathbb{N}$, then it follows from the foregoing that there exists $m \geqq 2$ such that the family $(\mathcal{U}_m)_x$ is a partial refinement of the family $\bigcup_{k=2}^{m} \mathfrak{w}_k$. In particular, it follows that $x \in \cup(\bigcup_{k=2}^{\infty} \mathfrak{w}_k)$. We have shown that the family $\bigcup_{k=2}^{\infty} \mathfrak{w}_k$ covers X. Let

$\mathbb{w}'_2 = \mathbb{w}_2$ and for each $k > 2$, let $\mathbb{w}'_k = \{W \in \mathbb{w}_k | W \not\subset W'$ for any $W' \in \bigcup_{n=2}^{k-1} \mathbb{w}_n\}$.

Since the family $\bigcup_{k=2}^{\infty} \mathbb{w}_k$ is a cover of X, so is the family $\bigcup_{k=2}^{\infty} \mathbb{w}'_k$. For

every $n \geqq 2$, let $V_{\alpha,n} = \bigcup\{W \in \mathbb{w}'_n | \alpha(W) = \alpha\}$ for each $\alpha < \gamma$, and let

$V_n = \{V_{\alpha,n} | \alpha < \gamma\}$. Let $n \geqq 2$ and $K \subseteq X$ be such that there exists a

finite subfamily $\mathcal{U}$ of $\mathcal{U}_{n-1}$ such that the family $(\mathcal{U}_n)_K$ is a partial

refinement of $\mathcal{U}$. We show that the family $(V_n)_K$ is finite. Let

$A = \{\alpha(U) | U \in \mathcal{U}\}$ and let $B = \{\alpha < \gamma | K \cap V_{\alpha,n} \neq \phi\}$. To show that the

family $(V_n)_K$ is finite, it suffices to show that $B \subset A$. Let $\beta \in B$.

Then there exists $W \in \mathbb{w}'_n$ such that $\alpha(W) = \beta$ and $K \cap W \neq \phi$. We have

$W \in (\mathcal{U}_n)_K$ and thus there exists $U \in \mathcal{U}$ such that $W \subset U$. The family $\mathcal{U}_{n-1}$

is precise at W and hence $\alpha(U) = \alpha(W)$. Since $\alpha(W) = \beta$ and $\alpha(U) \in A$, we

have $\beta \in A$. We have shown that $B \subset A$.

Let $V = \bigcup_{n=2}^{\infty} V_n$. Since $\bigcup V_n = \bigcup \mathbb{w}'_n$ for each $n \geq 2$, the family V

covers X. For all $n \geqq 2$ and $\alpha < \gamma$, the set $V_{\alpha,n}$ is open and $V_{\alpha,n} \subset U_\alpha$.

Hence V is an open refinement of $\mathcal{U}_1$. To show that V is point-finite

(locally finite), let $x \in X$. Since $\mathcal{U}_n$ is a point-wise (local)

W-refinement of $\mathcal{U}_{n-1}$ for every $n \geq 2$, we can find a decreasing sequence

$\langle K_n \rangle$ of subsets of X such that for each $n \geqq 2$, we have $x \in K_n (x \in \text{Int } K_n)$

and $(\mathcal{U}_n)_{K_n}$ partially refines some finite subfamily of $\mathcal{U}_{n-1}$. By the

preceding part of the proof, each of the families $(V_n)_{K_n}$, $n \geq 2$, is

finite. As established previously, there exists $m \geqq 2$ such that $(\mathcal{U}_m)_{K_m}$

is a partial refinement of $\bigcup_{k=2}^{m} \mathbb{w}_k$. We show that the family $(V)_{K_m}$ is

finite. For each $j > m$, the family $(\mathcal{U}_j)_{K_m}$ is a partial refinement of

the family $\bigcup_{k=2}^{m} \mathbb{w}_k$ and consequently, $(\mathcal{U}_j)_{K_m} \cap \mathbb{w}'_j = \phi$. It follows that

$(V_j)_{K_m} = \phi$ for every $j > m$. For each $j = 2,\ldots,m$, we have $(V_j)_{K_m} \subset (V_j)_{K_j}$

and so the family $(V_j)_{K_m}$ is finite. We have $(V)_{K_m} = \bigcup_{k=2}^{\infty} (V_k)_{K_m} = \bigcup_{k=2}^{m} (V_k)_{K_m}$

and it follows that the family $(V)_{K_m}$ is finite. Since $x \in K_m$ ($x \in \text{Int } K_m$),

we have shown that the family V is point-finite (locally finite) at

x. □

Corollary 1.1.3 A topological space is metacompact (paracompact) if,

and only if, every open cover of the space has an open point-wise (local)

W-refinement.

As we mentioned above, that part of the above corollary dealing

with metacompactness was proved by Worrell in [54].

Note that in the above proof, the openness of the families

constituting the W-sequence was used only to make the sets in the

resulting refinement open; hence the same proof yields the following

result.

Proposition 1.1.4 A cover of a topological space has a locally finite

refinement if, and only if, the cover has a local W-sequence of

refinements.

To be able to use Theorem 1.1.2 in connection with quasi-uniformities,

we need the following observation.

Lemma 1.1.5 Let R and S be reflexive relations on X such that $S \subset R$.

Assume that for every $x \in X$, there exists a finite subset F of X such

that $S^{-1}\{x\} \subset RF$ ($S^{-1}U \subset RF$ for some $U \in \eta_x$). Then the family $\mathcal{O}S$ is a

point-wise (local) W-refinement of the family $\mathcal{O}R^2$.

Proof. Trivial. □

A quasi-uniformity $\mathfrak{D}$ on X is <u>symmetric</u> provided that for each $Q \in \mathfrak{D}$, there exists $P \in \mathfrak{D}$ such that $P^{-1} \subset Q$. A quasi-uniformity is a uniformity if, and only if, the quasi-uniformity is symmetric. In the following definition, we generalize the concept of a symmetric quasi-uniformity.

<u>Definition 1.1.6</u> A quasi-uniformity $\mathfrak{D}$ on X is (<u>strongly</u>) <u>presymmetric</u> provided that for each $Q \in \mathfrak{D}$, there exists $P \in \mathfrak{D}$ such that for every $x \in X$, there exists a finite set $F \subset X$ such that $P^{-1}\{x\} \subset QF (P^{-1}U \subset QF$ for some $U \in \mathfrak{N}_x)$.

It is obvious that every symmetric quasi-uniformity is presymmetric. Moreover, it is easily seen that every compatible symmetric quasi-uniformity is strongly presymmetric.

It follows from results of J. W. Tukey and A. H. Stone ([52] and [48]) that if $\mathfrak{D}$ is a compatible uniformity for X, then for each $Q \in \mathfrak{D}$, the family $\mathfrak{Q}Q$ has a locally finite open refinement. We now extend this result as follows.

<u>Proposition 1.1.7</u> Let $\mathfrak{D}$ be compatible (strongly) presymmetric quasi-uniformity for X. Then for each $Q \in \mathfrak{D}$, the family $\mathfrak{Q}Q$ has a point-finite (locally finite) open refinement.

<u>Proof</u>. Let $Q \in \mathfrak{D}$. Using induction, we can construct a normal sequence $\langle Q_n \rangle$ of members of $\mathfrak{D}$ such that $Q_1 = Q$ and for all $x \in X$ and $n \in \mathbb{N}$, there is a finite set $F \subset X$ such that $Q_{n+1}^{-1}\{x\} \subset Q_n F$ $(Q_{n+1}^{-1}U \subset Q_n F$ for some $U \in \mathfrak{N}_x)$. From Lemma 1.1.5 we have that for each $n \in \mathbb{N}$, the family $\mathfrak{Q}Q_{2n+1}$ is a point-wise (local) W-refinement of the family $\mathfrak{Q}Q_{2n}^2$; since $Q_{2n}^2 \subset Q_{2n-1}$, the family $\mathfrak{Q}Q_{2n+1}$ is a point-wise (local) W-refinement of

the family $\mathcal{O}Q_{2n-1}$. It follows that the sequence $\langle\mathcal{O}Q_{2n-1}\rangle^{\infty}_{n=1}$, and hence also the sequence $\langle\mathcal{O}\dot{Q}_{2n-1}\rangle^{\infty}_{n=1}$ is a point-wise (local) W-sequence. By Theorem 1.1.2, the family $\mathcal{O}Q$ has a point-finite (locally finite) open refinement. $\square$

The above result has the following partial converse.

Lemma 1.1.8 Let $\mathcal{Q}$ be a quasi-uniformity on X such that $\mathcal{Q}$ contains the point-finite (locally finite) covering quasi-uniformity of X and for each $Q \in \mathcal{Q}$, the family $\mathcal{O}Q$ has a point-finite (locally finite) open refinement. Then $\mathcal{Q}$ is (strongly) presymmetric.

Proof. Let $Q \in \mathcal{Q}$ and let $\mathcal{U}$ be a point-finite (locally finite) open refinement of the family $\mathcal{O}Q$. Let $R = D\mathcal{U}$. Since $\mathcal{Q}$ contains the point-finite (locally finite) covering quasi-uniformity of X, we have $R \in \mathcal{Q}$. Let $x \in X$. Then there exists $K \subset X$ such that $x \in K$ ($x \in$ Int K) and the family $(\mathcal{U})_K$ is finite. For each $U \in (\mathcal{U})_K$, let $\varphi(U) \in X$ be such that $U \subset Q\{\varphi(U)\}$. Let $F = \{\varphi(U) | U \in (\mathcal{U})_K\}$. Then F is a finite set and $R^{-1}K \subset St(K,\mathcal{U}) \subset QF$. It follows from the foregoing that $\mathcal{Q}$ is (strongly) presymmetric. $\square$

A family $\mathcal{L}$ of subsets of X is well-monotone provided that the partial order $\subset$ of set inclusion is a well-order on $\mathcal{L}$. Note that if $\mathcal{L}$ is a well-monotone family of subsets of X, then $\cap\mathcal{L}' \in \mathcal{L}'$ for each $\mathcal{L}' \subset \mathcal{L}$; consequently, the family $\mathcal{L}$ is interior-preserving. It follows that for each well-monotone family $\mathcal{U}$ of open subsets of X, the relation $D\mathcal{U}$ is a neighbornet of X. The quasi-uniformity of X which has as a subbase the collection of all relations $D\mathcal{U}$, where $\mathcal{U}$ is a well-monotone open cover of

X, is called the well-monotone covering quasi-uniformity of X; note that this quasi-uniformity is compatible with the topology of X.

Using the concepts defined above, we can characterize metacompactness and paracompactness in terms of quasi-uniformities.

Theorem 1.1.9 The following conditions are mutually equivalent for a topological space:

(i)   The space is metacompact (paracompact).

(ii) Every compatible quasi-uniformity of the space that contains the point-finite (locally finite) covering quasi-uniformity of the space is (strongly) presymmetric.

(iii) The space has a compatible (strongly) presymmetric quasi-uniformity that contains the well-monotone covering quasi-uniformity of the space.

Proof. (i) $\Rightarrow$ (ii). This follows from Lemma 1.1.8.

(ii) $\Rightarrow$ (iii). Trivial.

(iii) $\Rightarrow$ (i). Assume that X has a compatible (strongly) presymmetric quasi-uniformity $\mathcal{D}$ such that for every well-monotone open cover $\mathcal{U}$ of X, $D\mathcal{U} \in \mathcal{D}$. It follows from results of J. Mack and W. Sconyers (see Theorem 1.1 of [23]) that X is metacompact (paracompact) provided that every well-monotone open cover of X has a point-finite (locally finite) open refinement. Let $\mathcal{U}$ be a well-monotone open cover of X. Then $D\mathcal{U} \in \mathcal{D}$ and it follows from Proposition 1.1.7 that there exists a point-finite (locally finite) open cover $\mathcal{V}$ of X such that $\mathcal{V}$ is a refinement of the family $\mathcal{C}(D\mathcal{U})$. But $\mathcal{C}(D\mathcal{U})$ is a subfamily of $\mathcal{U}$ and hence $\mathcal{V}$ is a refinement of $\mathcal{U}$ □

<u>Corollary 1.1.10</u> The following conditions are mutually equivalent for a topological space:

(i) The space is metacompact (paracompact).

(ii) The fine transitive quasi-uniformity of the space is (strongly) presymmetric.

(iii) The fine quasi-uniformity of the space is (strongly) presymmetric.

<u>Proof</u>. The proof follows directly from Theorem 1.1.9 since the fine transitive quasi-uniformity of a topological space contains all covering quasi-uniformities of the space. □

The remaining results of this section are essentially translations of the result above into the terminology of coverings. We start with a definition.

<u>Definition 1.1.11</u> Let $\mathfrak{D}$ be a quasi-uniformity on X. A cover $\mathcal{L}$ of X is (<u>strongly</u>) $\mathfrak{D}$-<u>normal</u> if there exists $Q \in \mathfrak{D}$ such that for each $x \in X$, there exists $L \in \mathcal{L}$ such that $Q\{x\} \subset L$ ($QU \subset L$ for some $U \in \mathcal{N}_x$).

<u>Remark</u>. J. Tukey introduced the concept of a <u>normal cover</u> of a topological space in [52]. In the above terminology, a cover of a topological space is normal if, and only if, the cover is $\mathfrak{FU}$-normal, where $\mathfrak{FU}$ is the fine uniformity of the space.

Note that if $\mathfrak{D}$ is a compatible quasi-uniformity for X and if $\mathcal{L}$ is a $\mathfrak{D}$-normal cover of X, then $\mathcal{L}$ is strongly $\mathfrak{D}$-normal.

<u>Lemma 1.1.12</u> Let $\mathfrak{D}$ be a compatible quasi-uniformity on X. Then $\mathfrak{D}$ is (strongly) presymmetric if, and only if, every $\mathfrak{D}$-normal directed open

cover of X is (strongly) $\mathfrak{D}^{-1}$-normal.

Proof. Necessity. Assume that $\mathfrak{D}$ is (strongly) presymmetric, and let $\mathcal{U}$ be a $\mathfrak{D}$-normal directed open cover of X. There exists $Q \in \mathfrak{D}$ such that the family $\mathcal{O}Q$ is a refinement of $\mathcal{U}$. Note that since $\mathcal{U}$ is a directed cover, the family $\{QF \,|\, F \subset X$ and F is finite$\}$ is a refinement of $\mathcal{U}$. Since $\mathfrak{D}$ is (strongly) presymmetric, there exists $R \in \mathfrak{D}$ such that for each $x \in X$, there is a finite set $F \subset X$ such that $R^{-1}\{x\} \subset QF (R^{-1}V \subset QF$ for some $V \in \mathfrak{N}_x)$. It follows that for each $x \in X$, there exists $U \in \mathcal{U}$ such that $R^{-1}\{x\} \subset U$ $(R^{-1}V \subset U$ for some $V \in \mathfrak{N}_x)$. We have shown that $\mathcal{U}$ is (strongly) $\mathfrak{D}^{-1}$-normal.

Sufficiency. Assume that every $\mathfrak{D}$-normal directed open cover of X is (strongly) $\mathfrak{D}^{-1}$-normal and let $Q \in \mathfrak{D}$. Note that we have $\dot{Q} \in \mathfrak{D}$ because if $R^2 \subset Q$, then $R \subset \dot{Q}$. Consequently, the directed open cover $(\mathcal{O}\dot{Q})^F$ of X is $\mathfrak{D}$-normal. It follows that there exists $R \in \mathfrak{D}$ such that for each $x \in X$, there exists a finite set $F \subset X$ such that $R^{-1}\{x\} \subset QF$ $(R^{-1}V \subset QF$ for some $V \in \mathfrak{N}_x)$. We have shown that the quasi-uniformity $\mathfrak{D}$ is (strongly) presymmetric. $\qquad\qquad \square$

Let $\mathcal{F}_X$ be the fine quasi-uniformity of a topological space X. It follows from Corollary 1.1.10 and Lemma 1.1.12 that X is metacompact (paracompact) if, and only if, every $\mathcal{F}_X$-normal directed open cover of X is (strongly) $\mathcal{F}_X^{-1}$-normal. In particular, since $\mathcal{F}_X$ is the collection of all normal neighbornets of X, we have the following result.

Proposition 1.1.13 A topological space is metacompact if, and only if, every directed open cover of the space has a normal $\mathfrak{n}^{-1}$-refinement.

Definition 1.1.14 A relation R on X is co-compact provided that for each $x \in X$, the set $R^{-1}\{x\}$ is compact.

By Theorem 4.3 of [23], a topological space is metacompact if the space has a co-compact open neighbornet. It is not known whether this result remains true without requiring the co-compact neighbornet to be open; however, since a co-compact neighbornet is an h^{-1}-refinement of every directed open cover, we have the following consequence of Proposition 1.1.13.

Corollary 1.1.15 If a topological space has a co-compact normal neighbornet, then the space is metacompact.

To close this section, we show that some characterizations of metacompactness and paracompactness given in [23] can easily be derived from the results above.

Theorem 1.1.16 ([23]) A topological space is metacompact (paracompact) if, and only if, every interior-preserving directed open cover of the space has a closure-preserving refinement by closed sets (whose interiors cover the space).

Proof. Let $\mathscr{FT}$ be the fine transitive quasi-uniformity of X. By Corollary 3.15 of [22], for each $U \in \mathscr{FT}$, the family $\{UB \mid B \subset X\}$ is interior-preserving and closed. On the other hand, by Theorem 3.14 of [22], if $\mathscr{U}$ is an interior-preserving and open family of subsets of X and $\mathscr{F}$ is a closure-preserving and closed family of subsets of X, then $D\mathscr{U}$ and $(D\mathscr{F})^{-1}$ are members of $\mathscr{FT}$. It follows that the condition stated

in the theorem is equivalent to the condition that $\mathcal{FT}$ is (strongly)
presymmetric. The conclusion now follows from Corollary 1.1.10.   □

2. On metacompactness and product spaces

Definition 1.2.1 ([25]) A cover $\mathfrak{m}$ of X is a point-star $\dot{F}$-refinement of a
cover $\mathcal{L}$ of X if for each $x \in X$, there exists a finite subfamily $\mathcal{L}'$ of $\mathcal{L}$
such that $x \in \cap\mathcal{L}'$ and $St(x,\mathfrak{m}) \subset \cup\mathcal{L}'$.

By Theorem 3.6 of [25], a topological space is metacompact if, and
only if, every open cover of the space has a semi-open point-star
$\dot{F}$-refinement. We now extend this result by showing that it is not
necessary to find $\dot{F}$-refinements for all open covers of a space.

Definition 1.2.2 Let β be an ordinal number. A family $\{L_\alpha | \alpha < \beta\}$ of
subsets of X is point-convex provided that for each $x \in X$, the set
$\{\alpha < \beta | x \in L_\alpha\}$ is convex in the well-ordering of β. A family $\mathcal{L}$ of
subsets of X is ordinally point-convex provided that for some ordinal
number γ, we can represent $\mathcal{L}$ in the form $\mathcal{L} = \{L_\alpha | \alpha < \gamma\}$ so that the
resulting indexed family is point-convex.

Ordinally point-convex families are closely related to the pairs
of chains of sets considered by H. Tamano in [50].

Note that every well-monotone family of sets is ordinally point-
convex.

Theorem 1.2.3 A topological space is metacompact if, and only if,
every ordinally point-convex open cover of the space has a semi-open
point-star $\dot{F}$-refinement.

Proof. A point-finite open refinement of a cover of a topological

space is a semi-open point-star $\dot{F}$-refinement of the cover. Hence the

condition is necessary. To prove sufficiency, assume that every

ordinally point-convex open cover of X has a semi-open point-star

$\dot{F}$-refinement. Let $\mathcal{U}$ be an open cover of X. To show that $\mathcal{U}$ has a

point-finite open refinement, represent $\mathcal{U}$ in the form $\mathcal{U} = \{U_\alpha | \alpha < \gamma\}$,

where γ is an ordinal number. For each $x \in X$, let $\alpha(x)$ be the least

element of the set $\{\alpha < \gamma | x \in U_\alpha\}$. Let $U_{\alpha,0} = \bigcup_{\beta \leq \alpha} U_\beta$ for each $\alpha < \gamma$,

and let $\mathcal{U}_0 = \{U_{\alpha,0} | \alpha < \gamma\}$. The family $\mathcal{U}_0$ is an open cover of X. For

each $x \in X$, we have $\{\alpha < \gamma | x \in U_{\alpha,0}\} = \{\alpha < \gamma | \alpha \geq \alpha(x)\}$; hence the

family $\{U_{\alpha,0} | \alpha < \gamma\}$ is point-convex. We use induction on n to show

that there exist families $\mathcal{U}_n = \{U_{\alpha,n} | \alpha < \gamma\}$, $\mathcal{V}_n = \{V_{\alpha,n} | \alpha < \gamma\}$ and $\mathcal{L}_n$

of subsets of X such that the following conditions are satisfied for

every $n \in \mathbb{N}$:

1° $\mathcal{L}_n$ is a semi-open point-star $\dot{F}$-refinement of $\mathcal{U}_{n-1}$.

2° $V_{\alpha,n} = U_{\alpha,n} \cap St(X \sim \bigcup_{\beta \neq \alpha} U_{\beta,n-1}, \mathcal{L}_n)^\circ$ for every $\alpha < \gamma$.

3° $U_{\alpha,n} = [\bigcup_{m=1}^{n} (\cup V_m)] \cup [U_{\alpha,0} \cap U_{\alpha+1,n-1} \cap St(X \sim \bigcup_{\beta < \alpha} U_{\beta,n-1}, \mathcal{L}_n)^\circ]$ for
   each $\alpha < \gamma$.

4° $\mathcal{U}_n = \{U_{\alpha,n} | \alpha < \gamma\}$ is a point-convex open cover of X.

5° $x \in U_{\alpha(x),n}$ for every $x \in X$.

To verify that the induction can be carried out, note that we

have already defined $\mathcal{U}_0 = \{U_{\alpha,0} | \alpha < \gamma\}$ so that 4° and 5° hold for $n = 0$.

Now, let $k \in \mathbb{N}$ be such that we have already defined $\mathcal{U}_n = \{U_{\alpha,n} | \alpha < \gamma\}$

for each $n < k$ so that 4° and 5° hold. It follows from the assumption

we have made on X that $\mathcal{U}_{k-1}$ has a semi-open point-star $\dot{F}$-refinement $\mathcal{L}_k$.

Conditions 2° and 3° define the open families $V_k = \{V_{\alpha,k} | \alpha < \gamma\}$ and

$U_k = \{U_{\alpha,k} | \alpha < \gamma\}$. It remains to show that conditions 4° and 5° hold

for $n = k$. To show that 5° holds, let $x \in X$ and let $\alpha = \alpha(x)$. If

$x \in \bigcup_{m=1}^{k} (U V_m)$, then $x \in U_{\alpha,k}$. Assume that $x \notin \bigcup_{m=1}^{k} (U V_m)$. Then $x \notin V_{\alpha,k}$

and it follows, since $x \in U_\alpha$, that $x \notin St(X \sim \bigcup_{\beta \neq \alpha} U_{\beta,k-1}, \mathcal{L}_k)^{\circ}$. As the

cover $\mathcal{L}_k$ is semi-open, we have $x \notin X \sim \bigcup_{\beta \neq \alpha} U_{\beta,k-1}$, that is, $x \in \bigcup_{\beta \neq \alpha} U_{\beta,k-1}$.

Let $\delta < \gamma$ be such that $\delta \neq \alpha$ and $x \in U_{\delta,k-1}$. We have $x \in U_{\delta,k-1} \sim$

$(\bigcup_{m=1}^{k-1} (U V_m)) \subseteq U_{\delta,0}$ and it follows that $\delta \geqq \alpha$; hence $\delta > \alpha$. Since condition

5° holds for $n = k - 1$, $x \in U_{\alpha,k-1}$. We have $\alpha < \delta$ and $x \in U_{\alpha,k-1} \cap U_{\delta,k-1}$;

consequently, as $\{U_{\beta,k-1} | \beta < \gamma\}$ is a point-convex family, we have

$x \in U_{\alpha+1, k-1}$. For each $\beta < \alpha$, $x \notin U_{\beta,0}$ and $U_{\beta,k-1} \sim (\bigcup_{m=1}^{k-1} (U V_m)) \subseteq U_{\beta,0}$;

since $x \notin (\bigcup_{m=1}^{k-1} (U V_m))$, $x \notin \bigcup_{\beta < \alpha} U_{\beta,k-1}$. Hence $x \in X \sim \bigcup_{\beta < \alpha} U_{\beta,k-1}$ and it

follows, since $\mathcal{L}_k$ is a semi-open cover, that $x \in St(X \sim \bigcup_{\beta < \alpha} U_{\beta,k-1}, \mathcal{L}_k)^{\circ}$.

We have $x \in U_{\alpha,0}$ and we have shown that $x \in U_{\alpha+1,k-1}$ and

$x \in St(X \sim \bigcup_{\beta < \alpha} U_{\beta,k-1}, \mathcal{L}_k)^{\circ}$; hence $x \in U_{\alpha,k}$ as is required. We have shown

that condition 5° holds for $n = k$. Consequently, the family U_k covers

X. Since the families U_n and V_n, $n = 0, \ldots, k - 1$, consist of open

sets, we see that the sets $U_{\alpha,k}$, $\alpha < \gamma$, are open. Hence to show that

condition 4° holds for $n = k$, it suffices to show that the family

$\{U_{\alpha,k} | \alpha < \gamma\}$ is point-convex. Let $x \in X$ and let $A = \{\alpha < \gamma | x \in U_{\alpha,k}\}$.

If $x \in \bigcup_{m=1}^{k} (U V_m)$, then $A = \gamma$ and thus A is convex in γ. Assume that

$x \notin \bigcup_{m=1}^{k} (U V_m)$. Then $A = \{\alpha < \gamma | x \in U_{\alpha,0} \cap U_{\alpha+1,k-1} \cap St(X \sim \bigcup_{\beta < \alpha} U_{\beta,k-1}, \mathcal{L}_k)^{\circ}\}$.

We know that the family $\{U_{\alpha,0} | \alpha < \gamma\}$ is point-convex. The family

$\{U_{\alpha,k-1} | \alpha < \gamma\}$ is point-convex and it follows that the family

$\{U_{\alpha+1,k-1}|\alpha < \gamma\}$ has the same property. For all $\alpha < \gamma$ and $\delta < \gamma$, if $\alpha < \delta$, then $St(X \sim \underset{\beta<\delta}{\cup} U_{\beta,k-1},\mathcal{L}_k)^{\circ} \subset St(X \sim \underset{\beta<\alpha}{\cup} U_{\beta,k-1},\mathcal{L}_k)^{\circ}$. It follows that the family $\{St(X \sim \underset{\beta<\alpha}{\cup} U_{\alpha,k-1},\mathcal{L}_k)^{\circ}|\alpha < \gamma\}$ is point-convex. Thus the set A is the intersection of three convex subsets of γ; hence A is a convex subset of γ. We have shown that the family $\{U_{\alpha,k}|\alpha < \gamma\}$ is point-convex. Hence condition 4° holds for $n = k$. This completes the proof of the inductive step.

Let $\mathcal{V} = \underset{n\in\mathbb{N}}{\cup} \mathcal{V}_n$. We show that the family $\mathcal{V}$ covers X. Let $x \in X$. Denote by δ the least ordinal $\alpha \leqq \gamma$ such that $St(x,\mathcal{L}_n) \subset \underset{\beta\leqq\alpha}{\cup} U_{\beta,n-1}$ for some $n \in \mathbb{N}$, and let $h \in \mathbb{N}$ be such that $St(x,\mathcal{L}_h) \subset \underset{\beta\leqq\delta}{\cup} U_{\beta,h-1}$. Note that $\delta < \gamma$, since $\mathcal{L}_n$ is a point-star $\dot{F}$-refinement of $\mathcal{U}_{n-1}$ for every $n \in \mathbb{N}$. We show that $x \in \overset{h+1}{\underset{m=1}{\cup}} (\cup\mathcal{V}_m)$. Assume on the contrary that $x \notin \overset{h+1}{\underset{m=1}{\cup}} (\cup\mathcal{V}_m)$. Note that since $St(x,\mathcal{L}_h) \subset \underset{\beta\leqq\delta}{\cup} U_{\beta,h-1}$, we have $x \notin St(X \sim \underset{\beta'<\beta}{\cup} U_{\beta',k-1},\mathcal{L}_h)$ whenever $\beta > \delta$. Since $x \notin \overset{h+1}{\underset{m=1}{\cup}} (\cup\mathcal{V}_m)$, it follows that $x \notin U_{\beta+1,h}$ whenever $\beta \geqq \delta$; consequently, $x \notin \underset{\beta\leqq\delta}{\cup} U_{\beta,h+1}$. Let B be a finite subset of γ such that $x \in \underset{\beta\in B}{\cap} U_{\beta,h+1}$ and $St(x,\mathcal{L}_{h+2}) \subset \underset{\beta\in B}{\cup} U_{\beta,h+1}$. Let ν be the largest element of the set B. Then $St(x,\mathcal{L}_{h+2}) \subset \underset{\beta\leqq\nu}{\cup} U_{\beta,h+1}$ and it follows that $\delta \leqq \nu$. This, however, is a contradiction since $x \in U_{\nu,h+1}$ and $x \notin \underset{\beta\leqq\delta}{\cup} U_{\beta,h+1}$. Consequently, $x \in \overset{h+1}{\underset{m=1}{\cup}} (\cup\mathcal{V}_m)$. We have shown that the family $\mathcal{V}$ covers X.

Next we show that each of the families $\mathcal{V}_n$, $n \in \mathbb{N}$, is point-finite. Let $n \in \mathbb{N}$ and let $x \in X$. Since $\mathcal{L}_n$ is a point-star $\dot{F}$-refinement of $\mathcal{U}_{n-1}$, there is a finite subset C of γ such that $St(x,\mathcal{L}_n) \subset \underset{\beta\in C}{\cup} U_{\beta,n-1}$. Let $D = \{\alpha < \gamma|x \in V_{\alpha,n}\}$. To show that the family $(\mathcal{V}_n)_x$ is finite, it suffices to show that $D \subset C$. Let $\alpha \in D$. Then $x \in V_{\alpha,n} \subset St(X \sim \underset{\beta\neq\alpha}{\cup} U_{\beta,n-1},\mathcal{L}_n)$.

Let $y \in X \sim \bigcup_{\beta \neq \alpha} U_{\beta,n-1}$ be such that $x \in St(y, \mathcal{L}_n)$. Then $y \in St(x, \mathcal{L}_n) \subset \bigcup_{\beta \in C} U_{\beta,n-1}$. Let $\delta \in C$ be such that $y \in U_{\delta,n-1}$. Since $y \in X \sim \bigcup_{\beta \neq \alpha} U_{\beta,n-1}$, we have $\delta = \alpha$ and hence $\alpha \in C$. We have shown that $D \subset C$; hence the family $\mathcal{V}_n$ is point-finite at x.

Let $V_0 = \phi$ and for each $n \in \mathbb{N}$, let $V_n = \bigcup_{m=1}^{n} (\cup \mathcal{V}_m)$. Since $\mathcal{V}$ is an open cover of X, the family $\{V_n \mid n \in \mathbb{N}\}$ is an open cover of X; moreover, this family is well-monotone and hence ordinally point-convex. Let $\mathcal{L}$ be a semi-open point-star F-refinement of the cover $\{V_n \mid n \in \mathbb{N}\}$. Setting $V'_n = V_n \cap St(X \sim V_{n-1}, \mathcal{L})^o$ for each $n \in \mathbb{N}$, we see that the family $\{V'_n \mid n \in \mathbb{N}\}$ is a point-finite open cover of X. Let $\mathcal{V}'_n = \{v \cap v'_n \mid v \in \bigcup_{m=1}^{n} \mathcal{V}_m\}$ for each $n \in \mathbb{N}$, and let $\mathcal{V}' = \bigcup_{n \in \mathbb{N}} \mathcal{V}'_n$. Then $\mathcal{V}'$ is a point-finite open refinement of the cover $\mathcal{V}$. Since $\mathcal{V}$ is a refinement of the cover $\mathcal{U}$, the cover $\mathcal{V}'$ is a refinement of $\mathcal{U}$. We have shown that X is metacompact. $\square$

Definition 1.2.4 A cover $\mathcal{L}$ of X is an $\hbar$-<u>double</u> <u>cover</u> if there exists a neighbornet U of X such that $U^2 \subset S\mathcal{L}$. The space X is $\hbar$-<u>doubly</u> <u>covered</u> if every open cover of X is an $\hbar$-double cover.

Note that if a cover $\mathcal{L}$ has an $\hbar^2$-refinement, then $\mathcal{L}$ is an $\hbar$-double cover. Consequently, every orthocompact space is $\hbar$-doubly covered. It follows from Proposition 2.2.12 below (see also Example 2.3.2 and Proposition 2.3.3) that an $\hbar$-double cover of a topological space does not necessarily have an $\hbar^2$-refinement; however, the following result obtains.

Lemma 1.2.5 Every open, ordinally point-convex $\hbar$-double cover of a topological space has an $\hbar^2$-refinement.

Proof. Let $\mathcal{U} = \{U_\alpha | \alpha < \gamma\}$ be an open, point-convex $\hbar$-double cover of X. Let V be a neighbornet of X such that $V^2 \subset S\mathcal{U}$. For each $x \in X$, let $\alpha(x)$ be the least element of the set $\{\alpha < \gamma | x \in U_\alpha\}$, and let $W\{x\} = U_{\alpha(x)} \cap V\{x\}$. In this way we define a neighbornet W of X. Let $x \in X$. We show that $W^2\{x\} \subset U_{\alpha(x)}$. Let $y \in W^2\{x\}$. Then there exists $z \in W\{x\}$ such that $y \in W\{z\}$. We have $z \in U_{\alpha(x)}$ and $y \in U_{\alpha(z)}$; consequently $\alpha(y) \leqq \alpha(z) \leqq \alpha(x)$. Since $y \in W^2\{x\} \subset V^2\{x\} \subset St(x,\mathcal{U})$, there exists $\beta < \gamma$ such that $x \in U_\beta$ and $y \in U_\beta$. We have $\alpha(y) \leqq \alpha(x) \leqq \beta$ and it follows, since $y \in U_{\alpha(y)} \cap U_\beta$ and since the set $\{\alpha < \gamma | y \in U_\alpha\}$ is convex, that $y \in U_{\alpha(x)}$. We have shown that $W^2\{x\} \subset U_{\alpha(x)}$. By the foregoing, W is an $\hbar^2$-refinement of $\mathcal{U}$. $\qquad\square$

We now characterize metacompactness and paracompactness in terms of $\hbar$-double coveredness.

Theorem 1.2.6 A topological space is metacompact (paracompact) if, and only if, the space is $\hbar$-doubly covered and every directed open cover of the space has a semi-open point-wise (local) star-refinement.

Proof. Necessity is trivial.

Sufficiency. We prove first the assertion concerning metacompactness. Let X be an $\hbar$-doubly covered space such that every directed open cover of X has a semi-open point-wise star-refinement. We use Theorem 1.2.3 to show that X is metacompact. Let $\mathcal{U}$ be an ordinally point-convex open cover of X. Since X is $\hbar$-doubly covered, it follows from Lemma 1.2.5 that there exists a neighbornet V of X such that V is an $\hbar^2$-refinement of $\mathcal{U}$. Let $\mathcal{L}$ be a semi-open point-wise star-refinement

of the directed open cover $(\dot{\alpha V})^F$ of X, and let $W = S\mathcal{L}$. Then W is

a neighbornet of X. We have $W^{-1} = S\mathcal{L}$ and it follows that for each $x \in X$,

there exists a finite subset A of X such that $W^{-1}\{x\} \subset VA$. By Lemma

1.1.5, the family αW is a point-wise W-refinement of the family αV^2.

Since V is an $\hbar^2$-refinement of the cover $\mathcal{U}$, the family αW is a point-

wise W-refinement of $\mathcal{U}$. It follows that αW is a point-star $\dot{F}$-refinement

of $\mathcal{U}$. Since W is a neighbornet, αW is a semi-open cover of X. We

have shown that X satisfies the condition of Theorem 1.2.3; hence X is

metacompact.

To prove the assertion concerning paracompactness, assume that X

is $\hbar$-doubly covered and that every directed open cover of X has a

semi-open local star-refinement. It follows from the first part of the

proof that X is metacompact. According to Corollary 2.11.1 of [30],

to prove that X is paracompact, it suffices to establish the following

condition: whenever $\mathfrak{m}$ is a locally finite family of subsets of X, there

exists an open family $\{U(M) | M \in \mathfrak{m}\}$ such that for each $M \in \mathfrak{m}$, we have

$M \subset U(M)$ and for each $x \in X$, there exists $V \in \mathfrak{N}_x$ such that the family

$\{M \in \mathfrak{m} | U(M) \cap V \neq \phi\}$ is finite. Assume that $\mathfrak{m}$ is locally finite. For

each $\mathfrak{m}' \subset \mathfrak{m}$, let $V(\mathfrak{m}') = X \sim \cup\{\overline{M} | M \in \mathfrak{m} \sim \mathfrak{m}'\}$. The family $V = \{V(\mathfrak{m}') | \mathfrak{m}' \subset \mathfrak{m}$

and $\mathfrak{m}'$ is finite$\}$ is open and directed. Since $\mathfrak{m}$ is locally finite,

V covers X. Let $\mathcal{L}$ be a semi-open local star-refinement of V. For each

$M \in \mathfrak{m}$, let $U(M) = St(M,\mathcal{L})^o$. We show that the open family $\{U(M) | M \in \mathfrak{m}\}$

has the required properties. Since the cover $\mathcal{L}$ is semi-open, we have

$M \subset U(M)$ for each $M \in \mathfrak{m}$. Let $x \in X$. Since $\mathcal{L}$ is a local star-refinement

of V, there exists a neighborhood W of x and a finite subfamily $\mathfrak{m}'$ of $\mathfrak{m}$

such that $St(W,\mathcal{L}) \subset V(\mathfrak{m}')$. For each $M \in \mathfrak{m} \sim \mathfrak{m}'$, we have $St(W,\mathcal{L}) \cap M = \phi$

and hence $W \cap U(M) = \emptyset$. It follows that $\{M \in \mathfrak{m} | W \cap U(M) \neq \emptyset\} \subset \mathfrak{m}'$. We have shown that the family $\{U(M) | M \in \mathfrak{m}\}$ has the required properties. $\square$

Note that according to Corollary 3.5 of [22] and Lemma 3.9 of [25], an open cover of a topological space has a semi-open point-wise (local) star-refinement if, and only if, the cover has a cushioned (and open) refinement.

We translate a part of Theorem 1.2.6 into the terminology of neighbornets.

<u>Corollary 1.2.7</u> A topological space is metacompact if, and only if, the space is $\hbar$-doubly covered and every directed open cover of the space has an $\hbar^{-1}$-refinement.

<u>Proof</u>. The proof follows directly from Theorem 1.2.6 since an open cover of a topological space has a semi-open point-wise star-refinement if, and only if, the cover has an $\hbar^{-1}$-refinement (see [22], Section 3a). $\square$

We are now ready to characterize metacompactness in terms of $\hbar$-double coveredness of certain product spaces.

<u>Theorem 1.2.8</u> A subspace X of a regular compact space K is metacompact if, and only if, the product space X × K is $\hbar$-doubly covered.

<u>Proof</u>. Since every metacompact space is $\hbar$-doubly covered, necessity of the condition follows from the result that the product of a metacompact space with a compact space is metacompact (see e.g. [12], Example 5.3.H). Sufficiency. Let K be a regular compact space and let X be a subspace of K such that the product space X × K is $\hbar$-doubly covered. Then the

subspace X is $\hbar$-doubly covered. By Corollary 1.2.7, to show that X is metacompact it suffices to show that every directed open cover of X has an $\hbar^{-1}$-refinement. Let Θ be a directed open cover of X. Since X is a subspace of K, there exists a family $\mathcal{U}$ of open subsets of K such that $\Theta = \{U \cap X \mid U \in \mathcal{U}\}$. Let $V = \cup\mathcal{U}$, and let $K' = K \sim V$. The set V is open in K and hence the set K' is closed and compact as a subspace of K. Since K is a regular space, there exists a family $\mathcal{W}$ of open subsets of K such that $\cup\mathcal{W} = V$ and $\overline{W} \subset V$ for each $W \in \mathcal{W}$. It is easily seen that the family $\mathcal{V} = \{X \times V\} \cup \{(X \cap W) \times (K \sim \overline{W}) \mid W \in \mathcal{W}\}$ is an open cover of the product space $X \times K$. Since the space $X \times K$ is $\hbar$-doubly covered, there exists an open neighbornet N of $X \times K$ such that $N^2 \subset S\mathcal{V}$. Denote by p the projection map $X \times K' \to X$. Since K' is a compact space, the map p is closed (see e.g. [12], Theorem 3.1.6). It follows that we can define a neighbornet M of X be setting $M\{x\} = X \sim p(X \times K' \sim N(\{x\} \times K'))$ for each $x \in X$.

We show that M is an $\hbar^{-1}$-refinement of Θ. Let $x \in X$. It is easily seen that the set $L = \{h \in K \mid (x,h) \notin N(\{x\} \times K')\}$ is a closed, and hence compact, subset of K. We have $L \subset V$ and it follows that $L \subset G$ for some $G \in \mathcal{U}^F$. Since $\Theta = \{U \cap X \mid U \in \mathcal{U}\}$ and Θ is a directed family, we have $L \cap X \subset O$ for some $O \in \Theta$. Consequently, to show that $M^{-1}\{x\} \subset O$ for some $O \in \Theta$, it suffices to show that $M^{-1}\{x\} \subset L$. Assume on the contrary that $M^{-1}\{x\} \not\subset L$, and let $y \in M^{-1}\{x\} \sim L$. Since $y \notin L$, there exists $k \in K'$ such that $(x,y) \in N\{(x,k)\}$. Since $y \in M^{-1}\{x\}$, we have $x \in M\{y\}$ and hence $\{x\} \times K' \subset N(\{y\} \times K')$. Consequently, there exists $h \in K'$ such that $(x,k) \in N\{(y,h)\}$. Note that $(x,y) \in N^2\{(y,h)\}$. Since $N^2 \subset S\mathcal{V}$, there exists $Q \in \mathcal{V}$ such that $\{(x,y),(y,h)\} \subset Q$. We have $Q \neq X \times V$,

because h $\not\in$ V, and hence there exists W $\in \mathfrak{w}$ such that Q = (X $\cap$ W) $\times$ (K $\sim \overline{W}$). But then $\{(x,y),(y,h)\} \subset$ W $\times$ (K $\sim \overline{W}$) and this is impossible. It follows that $M^{-1}\{x\} \subset$ L and hence that $M^{-1}\{x\} \subset$ 0 for some 0 $\in \Theta$. We have shown that the neighbornet M is an $\hbar^{-1}$-refinement of Θ. $\square$

Replacing "$\hbar$-doubly covered" by "orthocompact" in the above theorem, we have the result of Theorem 3.1 of [23].

Note that Theorem 1.2.8 makes it possible to exhibit many spaces that are not $\hbar$-doubly covered. For example, since the ordinal space ω_1 is not metacompact (see [12], Theorem 5.3.2 and Example 5.1.21), the product space $\omega_1 \times (\omega_1 + 1)$ is not $\hbar$-doubly covered (in [45], B. M. Scott showed that this product space is not orthocompact).

If X is a Tychonoff space and if we take K = βX in the previous theorem, then we have the following result.

Corollary 1.2.9 A Tychonoff space X is metacompact if, and only if, the product space X $\times$ βX is $\hbar$-doubly covered.

A similar characterization has been obtained by H. Tamano for paracompactness of a Tychonoff space. According to Tamano's result, a Tychonoff space X is paracompact if, and only if, the product space X $\times$ βX is normal ([49]).

The fine quasi-uniformity of a topological space has the Lebesgue property if, and only if, every open cover of the space has a normal $\hbar$-refinement. Consequently, the following result obtains.

Corollary 1.2.10 A Tychonoff space X is metacompact if, and only if, the product space X $\times$ βX has a compatible quasi-uniformity with the Lebesgue property.

CHAPTER II

GENERALIZATIONS OF ORTHOCOMPACTNESS

In this chapter, we study three covering properties that are weaker than the property of orthocompactness. The weakest of these properties, $\hbar$-double coveredness, has already been introduced in Chapter 1. The other two, normal $\hbar$-refinability and regular $\hbar$-refinability, are defined by requiring that every open cover of a topological space has a normal $\hbar$-refinement or a regular $\hbar$-refinement, respectively. That these properties are generalizations of orthocompactness is evident from the observation that a topological space is orthocompact if, and only if, every open cover of the space has an $\hbar^{\infty}$-refinement.

In terms of quasi-uniformities, the relation between normal $\hbar$-refinability and orthocompactness is the following: a topological space is normally $\hbar$-refinable (orthocompact) if, and only if, the fine (transitive) quasi-uniformity of the space has the Lebesgue property. In the first section of this chapter we study normally $\hbar$-refinable spaces and we show that every countably metacompact quasi-metrizable space is normally $\hbar$-refinable. This result and a recent example due to J. Kofner show that normal $\hbar$-refinability is a strictly weaker property than orthocompactness.

In the second section of the chapter we study regularly $\hbar$-refinable spaces. We show that a topological space is metacompact if, and only if, the space is regularly $\hbar$-refinable and semi-metacompact. As is evident from the name we have chosen for regular $\hbar$-refinability, this property is related to the property of regular refinability. We show that a topological space is regularly refinable if, and only if, the space is regularly

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ℏ-refinable and doubly covered. Using regular ℏ-refinability and some
results of H. Corson and S. Peregudov, we answer a question of
J. Mansfield by showing that there exist doubly covered spaces that
are not regularly refinable; we show that one such space is the space
F_0 constructed by A. H. Stone to show that an uncountable product of
infinite discrete spaces is not normal.

In the last section of this chapter we study the behaviour of
orthocompactness and its generalizations under closed, continuous
mappings. It follows from Theorem 1.2.8 that none of the covering
properties considered in this chapter is inversely preserved by closed,
continuous mappings, or even by perfect mappings. G. Gruenhage has
recently constructed an example to show that the continuous image of
an orthocompact space under a closed mapping is not necessarily ortho-
compact; the same example shows that regular refinability, regular
ℏ-refinability and normal ℏ-refinability fail to be preserved by closed,
continuous mappings. We present a simplified version of Gruenhage's
example. We show that a continuous image of an ℏ-doubly covered space
under a closed mapping is ℏ-doubly covered. We also show that a
continuous image of an orthocompact space under a closed mapping is
orthocompact provided that either the space is submetacompact or the
topology of the space has an orthobase.

1. Orthocompactness and normal ℏ-refinability

Definition 2.1.1 A topological space X is normally ℏ-refinable if
every open cover of X has a normal ℏ-refinement. The space X is normally
σℏ-refinable if for every open cover $\mathcal{U}$ of X, there exists a sequence

$\langle V_n \rangle$ of normal neighbornets of X such that for each $x \in X$, we have $V_n\{x\} \subset U$ for some $n \in \mathbb{N}$ and some $U \in \mathcal{U}$.

Since the fine quasi-uniformity $\mathcal{F}_X$ of X consist of all normal neighbornets of X, we see that X is normally $\hbar$-refinable if, and only if, $\mathcal{F}_X$ has the Lebesgue property.

Lemma 2.1.2 A space X is normally $\partial\hbar$-refinable if, and only if, for every open cover $\mathcal{U}$ of X, there exists a normal sequence $\langle W_n \rangle$ of neighbornets of X such that for every $x \in X$, we have $W_n\{x\} \subset U$ for some $n \in \mathbb{N}$ and some $U \in \mathcal{U}$.

Proof. The condition is clearly sufficient. To prove that the condition is necessary, assume that X is normally $\partial\hbar$-refinable. Let $\mathcal{U}$ be an open cover of X. Then there exists a sequence $\langle V_n \rangle$ of normal neighbornets of X such that for each $x \in X$, we have $V_n\{x\} \subset U$ for some $n \in \mathbb{N}$ and some $U \in \mathcal{U}$. For every $n \in \mathbb{N}$, there exists a normal sequence $\langle V_{n,k} \rangle_{k=1}^{\infty}$ of neighbornets of X such that $V_{n,1} = V_n$. Let $W_n = \cap\{V_{m,k} \mid m \leq n \text{ and } k \leq n\}$ for every $n \in \mathbb{N}$. It is easily seen that $\langle W_n \rangle$ is a normal sequence of neighbornets of X. Moreover, we have $W_n \subset V_n$ for every $n \in \mathbb{N}$. It follows that the sequence $\langle W_n \rangle$ has the properties required in the lemma. $\qquad\square$

Proposition 2.1.3 A countably metacompact normally $\partial\hbar$-refinable space is normally $\hbar$-refinable.

This proposition is proved by taking F = X in the following lemma.

Lemma 2.1.4 Let X be a normally $\partial\hbar$-refinable space. If X has a

countably metacompact, closed G_δ-subspace F such that the subspace

$X \sim F$ is a normally $\hbar$-refinable, then X is normally $\hbar$-refinable.

Proof. Assume that F is a subspace of X with the properties stated

in the lemma. To show that X is normally $\hbar$-refinable, let $\mathcal{U}$ be an open

cover of X. By Lemma 2.1.2, there exists a normal sequence $\langle W_n \rangle$ of

neighbornets of X such that for every $x \in X$, we have $W_n\{x\} \subset U$ for some

$n \in \mathbb{N}$ and some $U \in \mathcal{U}$. For each $x \in X$, let $n(x)$ be the least $n \in \mathbb{N}$

such that $W_n\{x\} \subset U$ for some $U \in \mathcal{U}$. For each $n \in \mathbb{N}$, let

$D_n = \dot{W}_{n+1}\{x \in X | n(x) = n\}$. Note that if $x \in D_n$ for some $n \in \mathbb{N}$, then

there exists $z \in X$ and $U \in \mathcal{U}$ such that $x \in W_{n+1}\{z\}$ and $W_n\{z\} \subset U$; since

$\langle W_n \rangle$ is a normal sequence, we have $W_{n+1}\{x\} \subset W_n\{z\}$, and hence $n(x) \leqq n+1$.

Let $\mathcal{Q} = \{D_n | n \in \mathbb{N}\}$. The family $\mathcal{Q}$ is an open cover of X and it

follows, since the subspace F is countably metacompact, that there

exists a point-finite open cover $\{L_n | n \in \mathbb{N}\}$ of the subspace F such

that $L_n \subset D_n$ for every $n \in \mathbb{N}$. Since F is a G_δ-subset of X, there

exists a sequence $\langle G_n \rangle$ of open subsets of X such that $F = \bigcap_{n \in \mathbb{N}} G_n$;

we may assume that $G_1 = X$ and $G_{n+1} \subset G_n$ for every $n \in \mathbb{N}$. Let

$E_n = L_n \cup (G_n \sim F)$ for each $n \in \mathbb{N}$, and let $\mathcal{E} = \{E_n | n \in \mathbb{N}\}$. It is

easily seen that $\mathcal{E}$ is a point-finite open cover of X. For every $x \in X$,

denote by $m(x)$ the largest element of the set $\{m \in \mathbb{N} | x \in E_m\}$. Note

that if $y \in \cap(\mathcal{E})_x$, then $m(y) \geqq m(x)$.

Since the subspace $X \sim F$ is normally $\hbar$-refinable, there exists a

normal sequence $\langle R_n \rangle$ of neighbornets of $X \sim F$ such that for every

$x \in X \sim F$, we have $R_1\{x\} \subset U$ for some $U \in \mathcal{U}$. We extend the neighbornets

R_n of $X \sim F$ to neighbornets V_n of X by setting $V_n\{x\} = R_n\{x\}$ for every

$x \in X \sim F$ and $V_n\{x\} = X$ for every $x \in F$. Note that the sequence $\langle V_n \rangle$ is normal, since $V_n\{x\} = R_n\{x\} \subset X \sim F$ for all $n \in \mathbb{N}$ and $x \in X \sim F$. Let $Q_n = V_n \cap W_n$ for every $n \in \mathbb{N}$. Since the sequences $\langle V_n \rangle$ and $\langle W_n \rangle$ are normal, so is the sequence $\langle Q_n \rangle$.

For every $n \in \mathbb{N}$, define a neighbornet P_n of X by setting $P_n\{x\} = \cap(\mathcal{E})_x \cap Q_{m(x)+n}\{x\}$ for all $n \in \mathbb{N}$ and $x \in X$. We show that $\langle P_n \rangle$ is a normal sequence. Let $x \in X$ and let $n \in \mathbb{N}$. To show that $P_{n+1}^2\{x\} \subset P_n\{x\}$, let $y \in P_{n+1}^2\{x\}$. Then there exists $z \in P_{n+1}\{x\}$ such that $y \in P_{n+1}\{z\}$. We have $z \in \cap(\mathcal{E})_x$ and consequently, $\cap(\mathcal{E})_z \subset \cap(\mathcal{E})_x$ and $m(z) \geqq m(x)$. Also, $z \in Q_{m(x)+n+1}\{x\}$ and hence $Q_{m(z)+n+1}\{z\} \subset Q_{m(x)+n+1}\{z\} \subset Q_{m(x)+n}\{x\}$. It follows that $P_{n+1}\{z\} = \cap(\mathcal{E})_z \cap Q_{m(z)+n+1}\{z\} \subset \cap(\mathcal{E})_x \cap Q_{m(x)+n}\{x\} = P_n\{x\}$, and hence that $y \in P_n\{x\}$. We have shown that $P_{n+1}^2\{x\} \subset P_n\{x\}$. It follows that $\langle P_n \rangle$ is a normal sequence and that P_1 is a normal neighbornet. We show that P_1 is an $\hbar$-refinement of $\mathcal{U}$. Let $x \in X$. If $x \in X \sim F$, then $P_1\{x\} \subset R_1\{x\}$ and hence $P_1\{x\} \subset U$ for some $U \in \mathcal{U}$. Assume that $x \in F$. Let $n \in \mathbb{N}$ be such that $x \in L_n$. We have $x \in L_n \subset D_n$ so that $n(x) \leqq n+1$. Moreover, $x \in L_n \subset E_n$ so that $n \leqq m(x)$. Consequently, $n(x) \leqq m(x) + 1$. We have $P_1\{x\} \subset Q_{m(x)+1}\{x\} \subset W_{m(x)+1}\{x\} \subset W_{n(x)}\{x\}$ and it follows from the definition of the number $n(x)$ that $P_1\{x\} \subset U$ for some $U \in \mathcal{U}$. We have shown that P_1 is an $\hbar$-refinement of $\mathcal{U}$. $\square$

Every quasi-metrizable space has a normal, basic sequence of neighbornets ([43]), and such a space is thus normally $\partial\hbar$-refinable; hence the result of Proposition 2.1.3 yields the following corollary.

Corollary 2.1.5 A countably metacompact quasi-metrizable space is

normally ℏ-refinable.

In [28], J. Kofner has constructed a countably metacompact (in fact, perfect) quasi-metrizable space T^V that is not orthocompact. From Corollary 2.1.5 it follows that T^V is normally ℏ-refinable. Hence we see that normal ℏ-refinability is a strictly weaker property than orthocompactness.

The space T^V mentioned above was obtained by Kofner as a special case of a construction that yields for any quasi-metric space (X,d) another quasi-metric space (X^V, d^V) such that if the space X is not non-archimedeanly quasi-metrizable (i.e. if X does not have a σ-interior-preserving base for its topology; see [17]), then the space X^V is not σ-orthocompact. The space T had been constructed earlier by Kofner ([27]) as an example of a quasi-metrizable space that is not non-archimedeanly quasi-metrizable; applying the v-construction to the space T, Kofner solved a problem in the theory of quasi-metrizable spaces by producing a quasi-metrizable space that is not σ-orthocompact.

We refer the reader to [28] for a description of the v-construction. For our purpose here, it suffices to know that for any quasi-metric space (X,d), the space X^V can be represented as a disjoint union $X^V = F \cup \bigcup_{n=1}^{\infty} U_n$, where each U_n is an open-closed set homeomorphic to X and F is discrete (Theorem 1 of [28]). The set F is clearly a closed G_δ-set, and it is obvious that if X normally ℏ-refinable, then the disjoint sum $\bigcup_{n=1}^{\infty} U_n$ of copies of X is normally ℏ-refinable. These observations and Lemma 2.1.4 yield the following result.

<u>Proposition 2.1.6</u> Let (X,d) be a quasi-metric space. If the space X

is normally $\hbar$-refinable, then the space X^V is normally $\hbar$-refinable.

2. Regularly $\hbar$-refinable spaces

<u>Definition 2.2.1</u> Let $\mathcal{L}$ be a cover of X. A <u>regular refinement</u> of $\mathcal{L}$ is a cover $\hbar$ of X such that for all $N \in \hbar$ and $N' \in \hbar$, if $N \cap N' \neq \phi$, then $N \cup N' \subset L$ for some $L \in \mathcal{L}$. The space X is <u>regularly refinable</u> if every open cover of X has an open regular refinement.

<u>Remark</u>. Regular refinements first appeared in a condition of the Alexandroff-Urysohn metrization theorem ([1]). In [33], regular refinements are called 2-star refinements and regularly refinable spaces are called 2-fully normal. For some results concerning regularly refinable spaces, see [33] and [55].

The above definition has the following analogue in terms of neighbornets.

<u>Definition 2.2.2</u> Let $\mathcal{L}$ be a cover of X. A <u>regular $\hbar$-refinement</u> of $\mathcal{L}$ is a neighbornet V of X such that if $x \in X$ and $y \in V\{x\}$, then $V\{x\} \cup V\{y\} \subset L$ for some $L \in \mathcal{L}$. The space X is <u>regularly $\hbar$-refinable</u> if every open cover of X has a regular $\hbar$-refinement.

If a cover has an open regular refinement, then the cover has a regular $\hbar$-refinement. Indeed, if an open cover $\circledcirc$ of X is a regular refinement of a cover $\mathcal{L}$ of X, then we obtain a regular $\hbar$-refinement V for $\mathcal{L}$ by choosing $V\{x\}$ from $(\circledcirc)_x$ for each $x \in X$. Consequently, every regularly refinable space is regularly $\hbar$-refinable. Also, it is clear that an $\hbar^2$-refinement of a cover is a regular $\hbar$-refinement of the cover;

it follows that every normally h-refinable space is regularly

h-refinable. On the other hand, it is easily seen that if V is a

regular h-refinement of a cover $\mathcal{L}$, then $V^2 \subset S\mathcal{L}$ so that $\mathcal{L}$ is an

h-double cover. Consequently, every regularly h-refinable space is

h-doubly covered.

The following lemmas give some simple, but useful properties of

regular h-refinements and regularly h-refinable spaces.

Lemma 2.2.3 Let V be a regular h-refinement of a cover $\mathcal{L}$ of X and for

each $L \in \mathcal{L}$, let $N(L) = \{x \in X | V\{x\} \subset L\}$. Then the family $\{N(L) | L \in \mathcal{L}\}$

is a semi-open cover of X.

Proof. Let $h = \{N(L) | L \in \mathcal{L}\}$, and let $x \in X$. We show that $V\{x\} \subset St(x,h)$.

Let $y \in V\{x\}$. Since V is a regular h-refinement of $\mathcal{L}$, there exists $L \in \mathcal{L}$

such that $V\{x\} \cup V\{y\} \subset L$. Then $\{x,y\} \subset N(L)$ and hence $y \in St(x,h)$. We

have shown that $U\{x\} \subset St(x,h)$. It follows that h is a semi-open cover

of X.   □

Lemma 2.2.4 Let U be an open cover of a regularly h-refinable space X.

Then there exists a neighbornet W of X such that for all $x \in X$ and $y \in X$,

if $W^{-1}\{x\} \cap W^{-1}\{y\} \neq \phi$, then $W\{x\} \cup W\{y\} \subset U$ for some $U \in U$.

Proof. Let V be a regular h-refinement of U, and let W be a regular

h-refinement of the open cover $\dot{CV}$ of X. We may assume that $W \subset V$. To

show that W has the required property, let $x \in X$ and $y \in X$ be such that

$W^{-1}\{x\} \cap W^{-1}\{y\} \neq \phi$. Let $z \in W^{-1}\{x\} \cap W^{-1}\{y\}$. Then $x \in W\{z\}$ and it

follows, since W is a regular h-refinement of $\dot{CV}$, that there exists

$u \in X$ such that $W\{z\} \cup W\{x\} \subset V\{u\}$. We have $y \in W\{z\} \subset V\{u\}$ and it

follows, since V is a regular h-refinement of U, that there exists $U \in U$

such that $V\{u\} \cup V\{y\} \subset U$. Then $W\{x\} \cup W\{y\} \subset V\{u\} \cup V\{y\} \subset U$. We have shown that the neighbornet W has the property required in the lemma. □

We now characterize metacompactness and regular refinability in terms of regular $\hbar$-refinability.

<u>Definition 2.2.5</u> A family $\mathcal{G}$ of open subsets of X is <u>open-finite</u> if for every non-empty open subset U of X, the family $\{G \in \mathcal{G} | U \subset G\}$ is finite. The space X is <u>semi-metacompact</u> if every open cover of X has an open-finite refinement.

. <u>Remark</u>. These concepts were introduced by S. Peregudov in [42]. Peregudov called semi-metacompact spaces semi-paracompact; the name semi-metacompact has been suggested by P. Fletcher and W. Lindgren.

Clearly, every point-finite family of open sets is open-finite. Peregudov showed in [42] that if an open-finite cover has an interior-preserving open refinement, then the cover has a point-finite open refinement; this result can be extended as follows.

<u>Lemma 2.2.6</u> An open-finite cover of a topological space has a point-finite open (semi-open) refinement if, and only if, the cover has an $\hbar^2$-refinement (a regular $\hbar$-refinement).

<u>Proof</u>. Necessity. Let $\mathcal{U}$ be an open cover of X. If $\mathcal{V}$ is a point-finite open refinement of $\mathcal{U}$, then $D\mathcal{V}$ is an $\hbar^2$-refinement of $\mathcal{U}$. Let $\mathcal{L}$ be a point-finite semi-open refinement of $\mathcal{U}$. For each $L \in \mathcal{L}$, let $U(L) \in \mathcal{U}$ be such that $L \subset U(L)$. Define a neighbornet V of X by setting for each x,

$V\{x\} = St(x,\mathcal{L}) \cap (\cap\{U(L)|L \in (\mathcal{L})_x\})$. We show that V is a regular h-refinement of $\mathcal{U}$. Let $x \in X$ and let $y \in V\{x\}$. Then $y \in St(x,\mathcal{L})$ and hence there exists $L \in \mathcal{L}$ such that $x \in L$ and $y \in L$. Then $L \in (\mathcal{L})_x \cap (\mathcal{L})_y$ and consequently $V\{x\} \cup V\{y\} \subset U(L)$. We have shown that V is a regular h-refinement of $\mathcal{U}$.

Sufficiency. Let V be a regular h-refinement of an open-finite cover $\mathcal{G}$ of X. Let $N(G) = \{x \in X|V\{x\} \subset G\}$ for each $G \in \mathcal{G}$, and let $h = \{N(G)|G \in \mathcal{G}\}$. Since the family $\mathcal{G}$ is open-finite, the family h is point-finite. For each $G \in \mathcal{G}$, we have $N(G) \subset G$. By Lemma 2.2.3, the family h is a semi-open cover of X. Consequently, h is a point-finite semi-open refinement of $\mathcal{G}$. Note that if $V^2\{x\} \subset N(G)$ for some $x \in X$ and $G \in \mathcal{G}$, then $V\{x\} \subset N(G)$. It follows that if V is an h^2-refinement of $\mathcal{G}$, then the point-finite open family $\{Int\ N|N \in h\}$ is a cover and hence a refinement of $\mathcal{G}$. □

We now factor metacompactness into two weaker properties.

Theorem 2.2.7 A topological space is metacompact if, and only if, the space is semi-metacompact and regularly h-refinable.

Proof. Necessity is trivial and sufficiency follows from Lemma 2.2.6 and the result of [24] that a topological space is metacompact if every open cover of the space has a point-finite semi-open refinement. □

Corollary 2.2.8 Every semi-metacompact regularly refinable space is paracompact.

Proof. By a result of [9], every regularly refinable space is collection-wise normal. The conclusion now follows from Theorem 2.2.7 and the result

of Michael and Nagami that every metacompact, collectionwise normal

space is paracompact.   □

Next we characterize regular refinability in terms of regular $\hbar$-

refinability and the following property.

Definition 2.2.9 An open cover Θ of X is <u>doubly</u> <u>open</u> if there exists

an open cover $\mathcal{U}$ of X such that $S^2\mathcal{U} \subset S\Theta$. The space X is <u>doubly</u> <u>covered</u>

if every open cover of X is doubly open.

Remark. Doubly covered spaces were first considered by H. Cohen in [9].

Cohen showed that every doubly covered space is collectionwise normal.

He also showed that a T_1-space X is doubly covered if, and only if, the

collection of all neighborhoods of the diagonal of X X X forms a compatible

uniformity for X. In [33], doubly covered spaces are called almost

2-fully normal.

Note that if a cover $\hbar$ is a regular refinement of a cover $\mathcal{L}$, then

$S^2\hbar \subset S\mathcal{L}$. Consequently, every regularly refinable space is doubly

covered. On the other hand, it is obvious that every doubly covered

space is $\hbar$-doubly covered.

Regular refinability allows the following factorization.

Theorem 2.2.10 A topological space is regularly refinable if, and only

if, the space is doubly covered and regularly $\hbar$-refinable.

Proof. We have already observed that the condition is necessary. To

prove that it is sufficient, let X be a doubly covered regularly

$\hbar$-refinable space. Let $\mathcal{U}$ be an open cover of X. By Lemma 2.2.4, there

exists a neighbornet W of X such that for all $x \in X$ and $y \in X$, if

$W^{-1}\{x\} \cap W^{-1}\{y\} \neq \phi$, then $W\{x\} \cup W\{y\} \subset U$ for some $U \in \mathcal{U}$. Denote by $\mathcal{W}$

the open cover $\dot{\mathcal{O}W}$ of X. Since X is doubly covered, there exists an open

cover $\mathcal{V}$ of X such that $S^2\mathcal{V} \subset S\mathcal{W}$. For each $x \in X$, let V_x be a member of

the family $(\mathcal{V})_x$, and let $0_x = V_x \cap \dot{W}\{x\}$. Let $\mathcal{O} = \{0_x | x \in X\}$ and note that

$\mathcal{O}$ is an open cover of X. We show that $\mathcal{O}$ is a regular refinement of $\mathcal{U}$.

Let $x \in X$ and $y \in X$ be such that $0_x \cap 0_y \neq \phi$. Then $y \in St^2(x,\mathcal{V})$ and

it follows, since $S^2\mathcal{V} \subset S\mathcal{W}$, that there exists $z \in X$ such that $x \in W\{z\}$

and $y \in W\{z\}$. We have $z \in W^{-1}\{x\} \cap W^{-1}\{y\}$ and hence $W^{-1}\{x\} \cap W^{-1}\{y\} \neq \phi$.

Since $0_x \subset W\{x\}$ and $0_y \subset W\{x\}$, we have $0_x \cup 0_y \subset U$ for some $U \in \mathcal{U}$. We

have shown that $\mathcal{O}$ is a regular refinement of $\mathcal{U}$. $\qquad\qquad \square$

Corollary 2.2.11 Every doubly covered orthocompact space is regularly

refinable.

J. Mansfield asked in [33] whether every doubly covered space is

regularly refinable. We now use the results above to show that the

answer to Mansfield's question is in the negative.

Let $\{X_i | i \in I\}$ be a family of topological spaces. In [10],

H. Corson defines the Σ-product of the spaces X_i, $i \in I$, with a base

point $\langle p_i \rangle_I \in \prod_{i \in I} X_i$ to be the subspace of the product space $\prod_{i \in I} X_i$

consisting of all those $\langle x_i \rangle_I \in \prod_{i \in I} X_i$ for which $x_i \neq p_i$ at most for

countably many $i \in I$. Corson showed in [10] that any Σ-product of

separable complete metric spaces is doubly covered; he also noted that

no uncountable Σ-product of non-trivial T_1-spaces is paracompact (see

the remarks following Theorem 4 of [10]; we call a space non-trivial if

it has at least two points). These results can be sharpened as follows.

Proposition 2.2.12   (i) Any Σ-product of separable metric spaces is doubly covered.

(ii)   No uncountable Σ-product of non-trivial T_1-spaces is regularly $\hbar$-refinable.

Proof.   (i).   By the Corollary to Theorem 1 of [29], any Σ-product of separable metric spaces is normal; in light of this result, completeness is not needed in the proof of Corson's result that any Σ-product of separable complete metric spaces is doubly covered (Theorem 3 of [10]).

(ii).   Let $\{X_i \mid i \in I\}$ be an uncountable family of non-trivial T_1-spaces, and let $\langle p_i \rangle_I \in \prod_{i \in I} X_i$.   Denote by Σ the Σ-product of the spaces X_i, $i \in I$, with the base point $\langle p_i \rangle_I$.   To show that the space Σ is not regularly $\hbar$-refinable, assume on the contrary that Σ has this property.   For each $i \in I$, let $x_i \in X_i$ be such that $x_i \neq p_i$, and let Y_i be the subspace $\{p_i, x_i\}$ of X_i.   Denote by Σ' the Σ-product of the spaces Y_i, $i \in I$, with base point $\langle p_i \rangle_I$.   It is known (see e.g. [10]) that any Σ-product of compact T_1-spaces is countably compact; hence Σ' is countably compact. The space Σ' is a closed subspace of Σ; consequently, Σ' is regularly $\hbar$-refinable.   It follows from Assertions E and F of [42] that the topology of Σ' has an open-finite base; hence Σ' is semi-metacompact. It follows from Theorem 2.2.7 that Σ' is metacompact.   Since Σ' is metacompact and countably compact, Σ' is compact, by Theorem 4 of [2]. On the other hand, if we let $U_j = \{\langle y_i \rangle_I \in \Sigma' \mid y_j = p_j\}$ for each $j \in I$, then it is easily seen that the family $\{U_j \mid j \in I\}$ is an open cover of Σ' which has no finite subcover.   This contradiction shows that the space Σ is not regularly $\hbar$-refinable. $\square$

It is not known whether every Σ-product of metric spaces is doubly covered, but M. E. Rudin has recently announced a related result that any such Σ-product is normal ([44]).

If we replace "regularly $\hbar$-refinable" by "orthocompact" in part (ii) of the above theorem, we have a result of B. Scott's ([45], Example 4.3).

According to the above theorem, any uncountable Σ-product of non-trivial separable metric spaces serves as an example of a doubly covered space that is not regularly refinable; moreover, if we choose the factors of the Σ-product to be compact, we have an example of such a space which is also countably compact. We now show that there exists a doubly covered space which is not regularly refinable but which has a compatible complete uniformity.

Example 2.2.13   The space F_0.

Let I be an uncountable set and for each $i \in I$, let Z_i be a copy of the integers equipped with the discrete topology. Let F_0 be the subspace of the product space $\prod\limits_{i \in I} Z_i$ consisting of all those $\langle z_i \rangle_I \in \prod\limits_{i \in I} Z_i$ for which $z_i = n$ at most for one $i \in I$ whenever $n \neq 0$. The space F_0 was constructed by A. H. Stone in [48], in connection with his proof that an uncountable product of copies of the integers is not normal.

Note that F_0 is a closed subspace of $\prod\limits_{i \in I} Z_i$ and that F_0 is contained in the Σ-product of the spaces Z_i, $i \in I$, with base point $\langle 0 \rangle_I$; consequently, by Corson's results, F_0 is a doubly covered space. Corson also observed in [10] that F_0 has a compatible complete uniformity, and he showed that F_0 is not paracompact. P. Fletcher and W. Lindgren have shown that the topology of F_0 has an open-finite base; hence F_0 is

semi-metacompact. These results and Corollary 2.2.8 show that F_0 is

not regularly refinable, and it follows from Theorem 2.2.10 that F_0

is not even regularly $\hbar$-refinable. In particular, as has been shown by

Fletcher and Lindgren, F_0 is not orthocompact.

For the sake of completeness, we include here a proof of the result

of Fletcher and Lindgren that the topology of F_0 has an open-finite base.

For every $L \subset I \times Z$, let $B_L = \{\langle x_i \rangle_I \in F_0 | x_j = n$ for every $(j,n) \in L\}$.

Since F_0 is a subspace of the product space $\prod_{i \in I} Z_i$, the family

$\mathfrak{B} = \{B_L | L \subset I \times Z$ and L is finite$\}$ is a base for the topology of F_0.

To show that $\mathfrak{B}$ is open-finite, let L be a finite subset of $I \times Z$ such that

the set B_L is not empty. We show that for every $T \subset I \times Z$, if $B_L \subset B_T$,

then $T \subset L$. Assume on the contrary that there exists $T \subset I \times Z$ such that

$B_L \subsetneq B_T$ but $T \not\subset L$. Let $(j,m) \in T \sim L$. The set $\{0,m\} \cup \{n \in Z | (i,n) \in L$

for some $i \in I\}$ is finite and hence there is an integer k that is not in

this set. Let $\langle x_i \rangle_I$ be an element of B_L. Let $y_j = k$ and for every

$i \in I \sim \{j\}$, let $y_i = x_i$ if $x_i \neq k$ and let $y_i = 0$ if $x_i = k$. Then it is

easily seen that $\langle y_i \rangle_I \in B_L \sim B_T$. This contradiction shows that we

have $T \subset L$ whenever $B_L \subset B_T$. Since the set L is finite, it follows that

the family $\{B \in \mathfrak{B} | B_L \subset B\}$ is finite. We have shown that $\mathfrak{B}$ is open-

finite. □

3. Closed images of orthocompact spaces

We start this section by giving an example of an orthocompact,

regularly refinable space that admits a closed, continuous mapping onto

a space that is not regularly $\hbar$-refinable. Our example is a modification

of an example recently constructed by G. Gruenhage to show that

orthocompactness is not preserved under closed, continuous mappings.
Gruenhage's example, as well as our modification of this example, is
built using the countable ordinals. To present the example, we need a
result concerning the space of all countable ordinals.

As usual, ω_1 denotes the first uncountable ordinal. Then ω_1 is the
set of all countable ordinals, and this set is well-ordered by the
membership relation. We make ω_1 into a topological space by equipping
it with the order topology (see e.g. [12], Problem 1.7.4). If $\alpha \in \omega_1$,
then the sets $(\alpha \sim \beta) \cup \{\alpha\}$, $\beta \in \alpha$, form a neighborhood basis at the
point α in the space ω_1.

Lemma 2.3.1 The product space $\omega_1 \times \omega_1$ is countably compact, orthocompact
and regularly refinable.

Proof. It is well known that ω_1 is a countably compact first countable
space (see e.g. Examples 3.1.27 and 3.10.16 of [12]). It follows from
Corollary 3.10.15 of [12] that $\omega_1 \times \omega_1$ is countably compact. By
Lemma 3.9 of [45], $\omega_1 \times \omega_1$ is orthocompact. Moreover, from the proofs
of Lemmas 3.8 and 3.9 of [45] it can be seen that every open cover of
$\omega_1 \times \omega_1$ has an open refinement of the form $\underset{s \in S}{\cup} V_s$ where for each $s \in S$,
the family V_s is monotone and $(\cup V_{s'}) \cap (\cup V_s) = \emptyset$ for every $s' \in S \sim \{s\}$;
such a refinement is clearly a regular refinement. It follows that
$\omega_1 \times \omega_1$ is regularly refinable. $\square$

Example 2.3.2 A space X such that X is not regularly $\hbar$-refinable and X
is a continuous image of $\omega_1 \times \omega_1$ under a closed mapping.

Denote by L the set of all limit ordinals in ω_1. Let S be the
subspace $\{\alpha + 1 | \alpha \in \omega_1\}$ of ω_1 consisting of all successor ordinals.

Note that $\omega_1 = \{0\} \cup S \cup L$. The subspace S of ω_1 is discrete. Denote by S^* the one-point compactification of S, where 0 is used as the "point in infinity" (see e.g. [12], Theorem 3.5.11), and denote by X the product space $S^* \times \omega_1$.

We show that X is not regularly $\hbar$-refinable. For each $\alpha \in S$, we shall denote by U_α the set $(S^* \sim \{\alpha\}) \times (\alpha \cup \{\alpha\})$. Then the family $\mathcal{U} = \{U_\alpha | \alpha \in S\}$ is an open cover of X. We show that $\mathcal{U}$ does not have a regular $\hbar$-refinement. Assume on the contrary that there exists a neighbornet V of X such that V is a regular $\hbar$-refinement of $\mathcal{U}$. Let $M_\alpha = \{x \in X | V\{x\} \subset U_\alpha\}$ for each $\alpha \in S$, and let $\mathfrak{m} = \{M_\alpha | \alpha \in S\}$. It follows from Lemma 2.2.3 that the family $\mathfrak{m}$ is a semi-open cover of X. We show that $\mathfrak{m}$ is point-finite at the set $\{0\} \times \omega_1$. Let $\alpha \in \omega_1$. Then there exists a finite subset A of S such that $(S^* \sim A) \times \{\alpha\} \subset V\{(0,\alpha)\}$. If β is a point of S such that $(0,\alpha) \in M_\beta$, then $(S^* \sim A) \times \{\alpha\} \subset U_\beta = (S^* \sim \{\beta\}) \times (\beta \cup \{\beta\})$ and consequently, $S^* \sim A \subset S^* \sim \{\beta\}$, that is, $\beta \in A$. It follows that the family $(\mathfrak{m})_{(0,\alpha)}$ is contained in the finite family $\{M_\beta | \beta \in A\}$. We have shown that the family $\mathfrak{m}$ is point-finite at the set $\{0\} \times \omega_1$. Note that the projection map from $\{0\} \times \omega_1$ onto ω_1 is a homeomorphism. Consequently, if we let $L_\alpha = \{\beta \in \omega_1 | (0,\beta) \in M_\alpha\}$ for each $\alpha \in S$, then the family $\mathcal{L} = \{L_\alpha | \alpha \in S\}$ is a point-finite semi-open cover of ω_1. For each $\alpha \in S$, we have $L_\alpha \subset \alpha \cup \{\alpha\}$ and the subspace $\alpha \cup \{\alpha\}$ of ω_1 is closed and bounded, hence compact. It follows from Corollary 2.3 of [24] that the space ω_1 is metacompact. This, however, is a contradiction since every countably compact metacompact space is compact, according to Theorem 4 of [2], while it is well known that the space ω_1 is not compact (see e.g. [12], Example 3.10.16). It

follows that X is not regularly $\hbar$-refinable.

It remains to show that X is a continuous image of $\omega_1 \times \omega_1$ under a closed mapping. Define a mapping $f : \omega_1 \times \omega_1 \to X$ by setting $f(\alpha,\beta) = (\alpha,\beta)$ if $\alpha \in S$ and $f(\alpha,\beta) = (0,\beta)$ if $\alpha \in L \cup \{0\}$. We show that f is continuous. First observe that $S \times \omega_1$ is an open subspace of $\omega_1 \times \omega_1$ and that the restriction of f to this subspace is a homeomorphism onto a subspace of X. Hence to show that f is continuous, it suffices to show that for every open subset U of X, if $(\alpha,\beta) \in f^{-1}(U)$ and $\alpha \in L \cup \{0\}$, then $f^{-1}(U)$ is a neighborhood of (α,β) in $\omega_1 \times \omega_1$. Let $U \subseteq X$ be open and let $(\alpha,\beta) \in f^{-1}(U)$ be such that $\alpha \in L \cup \{0\}$. Then $f(\alpha,\beta) = (0,\beta)$ and hence $(0,\beta) \in U$. It follows that there exists $\gamma \in \beta$ and a finite set $A \subseteq S$ such that if we let $H = (S^* \sim A) \times [(\beta \sim \gamma) \cup \{\beta\}]$, then $H \subseteq U$. We have $f^{-1}(H) = (\omega_1 \sim A) \times [(\beta \sim \gamma) \cup \{\beta\}]$ and this set is clearly a neighborhood of (α,β) in $\omega_1 \times \omega_1$. We have shown that the mapping f is continuous. Now it is easily seen that f is also closed. First observe that X is a Fréchet-space (that is, whenever $D \subseteq X$ and $x \in \bar{D}$, then $d_n \to x$ for some sequence $\langle d_n \rangle$ of points of D). Clearly, X is a Hausdorff-space. Since $\omega_1 \times \omega_1$ is a countably compact space and the map $f: \omega_1 \times \omega_1 \to X$ is continuous, it follows from Remark (i) of [51] that f is a closed mapping. Obviously, f maps $\omega_1 \times \omega_1$ onto X. We have shown that X is a continuous image of $\omega_1 \times \omega_1$ under a closed mapping. $\square$

Using the Pressing Down Lemma (see e.g. [45], Lemma 3.4), it can be shown that the space X in the above example is doubly covered; hence this space provides us with an example of a locally compact doubly covered space that is not regularly refinable.

Since every orthocompact space is normally ħ-refinable, and every

normally ħ-refinable space is regularly ħ-refinable, the preceding example

shows that neither normal ħ-refinability nor regular ħ-refinability is

preserved under closed, continuous mappings; the following result,

however, does obtain.

Proposition 2.3.3 A continuous image of an ħ-doubly covered space under

a closed mapping is ħ-doubly covered.

Proof. Let X be an ħ-doubly covered space and let f be a closed and

continuous mapping from X onto a space Y. To show that Y is ħ-doubly

covered, let $\mathcal{U}$ be an open cover of Y. Then the family $V = \{f^{-1}(U) | U \in \mathcal{U}\}$

is an open cover of X. Let W be a neighbornet of X such that $W^2 \subset SV$.

Since f is a closed mapping, we can define a neighbornet Q of Y by setting

$Q\{y\} = Y \sim f(X \sim W(f^{-1}\{y\}))$ for each $y \in Y$. We show that $Q^2 \subset S\mathcal{U}$. Let

$y \in Y$, and let $v \in Q^2\{y\}$. Then there exists $u \in Q\{y\}$ such that $v \in Q\{u\}$.

Let x be an element of the set $f^{-1}\{v\}$. Since $v \in Q\{u\}$, we have

$f^{-1}\{v\} \subset W(f^{-1}\{u\})$. Consequently, there exists $z \in f^{-1}\{u\}$ such that

$x \in W\{z\}$. Since $u \in Q\{y\}$, we have $f^{-1}\{u\} \subset W(f^{-1}\{y\})$ and thus there

exists $r \in f^{-1}\{y\}$ such that $z \in W\{r\}$. We have $x \in W^2\{r\}$ and it follows,

since $W^2 \subset SV$, that there exists $U \in \mathcal{U}$ such that $\{x,r\} \subset f^{-1}(U)$.

Consequently, $\{f(x),f(r)\} \subset U$, that is, $\{v,y\} \subset U$. Hence $v \in St(y,\mathcal{U})$.

We have shown that $Q^2 \subset S\mathcal{U}$.   □

In the remainder of this section we indicate some special situations

in which orthocompactness is preserved under closed continuous mappings.

Definition 2.3.4 A topological space X is <u>discretely</u> <u>orthocompact</u> provided

that whenever $\mathcal{F}$ is a discrete family of closed subsets of X and for each $F \in \mathcal{F}$, U_F is an open set containing F, there exists an interior-preserving open family $\{V_F | F \in \mathcal{F}\}$ such that $F \subset V_F \subset U_F$ for each $F \in \mathcal{F}$.

__Lemma 2.3.5__ Every orthocompact space is discretely orthocompact.

__Proof__. Let X be an orthocompact space. To show that X is discretely orthocompact, let $\mathcal{F}$ be a discrete family of closed subsets of X and for each $F \in \mathcal{F}$, let U_F be an open set containing F. Let $U_F' = U_F \sim \cup(\mathcal{F} \sim \{F\})$ for every $F \in \mathcal{F}$, and let $\mathcal{U} = \{X \sim \cup\mathcal{F}\} \cup \{U_F' | F \in \mathcal{F}\}$. Then $\mathcal{U}$ is an open cover of X. Let V be an interior-preserving open refinement of $\mathcal{U}$. For each $F \in \mathcal{F}$, let $V_F = St(F,V)$ and note that $V_F \subset U_F$ since $(\mathcal{U})_F = \{U_F'\}$. Clearly, the family $\{V_F | F \in \mathcal{F}\}$ is open and interior-preserving, and $F \subset V_F$ for every $F \in \mathcal{F}$.   □

__Lemma 2.3.6__ A continuous image of a discretely orthocompact space under a closed mapping is discretely orthocompact.

__Proof__. Let X be a discretely orthocompact space and let f be a closed, continuous mapping from X onto a space Y. To show that Y is discretely orthocompact, let $\mathcal{F}$ be a discrete family of closed subsets of Y and for each $F \in \mathcal{F}$, let U_F be an open subset of Y containing F. Let $K_F = f^{-1}(F)$ and $G_F = f^{-1}(U_F)$ for each $F \in \mathcal{F}$, and let $\mathcal{K} = \{K_F | F \in \mathcal{F}\}$. Then $\mathcal{K}$ is a discrete family of closed subsets of X and for each $F \in \mathcal{F}$, G_F is an open subset of X containing K_F. Hence, there exists an interior-preserving family $\{W_F | F \in \mathcal{F}\}$ of open subsets of X such that $K_F \subset W_F \subset G_F$ for each $F \in \mathcal{F}$. For each $F \in \mathcal{F}$, let $V_F = Y \sim f(X \sim W_F)$ and note that $F \subset V_F \subset U_F$. Since the family $\{W_F | F \in \mathcal{F}\}$ is open and interior-preserving

in X, the family $\{X \sim W_F | F \in \mathcal{F}\}$ is closed and closure-preserving in

X. It follows, since f is a closed mapping, that the family

$\{f(X \sim W_F) | F \in \mathcal{F}\}$ is closed and closure-preserving in Y. Consequently,

the family $\{V_F | F \in \mathcal{F}\}$ is open and interior-preserving in Y. We have

shown that Y is discretely orthocompact. □

Remark. Discrete orthocompactness is of the same nature as the property

of expandability, and the above proof is similar to the proof that almost

expandability is preserved under closed, continuous mappings ([47]).

Expandability, and the various modifications of this property (see e.g.

[47]) can be considered as generalizations of collectionwise normality.

The proof of the next lemma is based on the technique devised by Michael

and Nagami to show that every collectionwise normal metacompact space is

paracompact ([35] and [41]).·

Lemma 2.3.7 Let $\mathcal{U}$ be an open cover of a discretely orthocompact space

X. Then $\mathcal{U}$ has a σ-interior-preserving open partial refinement $\mathcal{V}$ such

that for every $x \in X$, if $\mathcal{U}$ is point-finite at x, then $x \in \cup\mathcal{V}$.

Proof. For each $n \in \mathbb{N}$, let $S_n = \{x \in X | \ |(\mathcal{U})_x| < n\}$, and note that this

is a closed set. We use induction on n to show that for each $n \in \mathbb{N}$,

there exists an interior-preserving open partial refinement $\mathcal{V}_n$ of $\mathcal{U}$

such that $S_n \subset \cup\mathcal{V}_n$. There is nothing to prove for n = 1, since $S_1 = \emptyset$.

Let $k > 1$ be such that $\mathcal{U}$ has an interior-preserving open partial

refinement $\mathcal{V}_{k-1}$ such that $S_{k-1} \subset \cup\mathcal{V}_{k-1}$. To prove the existence of the

family $\mathcal{V}_k$, let $\mathcal{W} = \{\cap\mathcal{U}' | \mathcal{U}' \subset \mathcal{U}$ and $|\mathcal{U}'| = k-1\}$. Note that if $x \in X \sim S_{k-1}$,

then $x \in \cup\mathcal{W}$. Consequently, the open family $\mathcal{W}' = \mathcal{W} \cup \{\cup\mathcal{V}_{k-1}\}$ covers X.

For each $W \in \text{\textbf{\textit{w}}}$, let $F_W = X \sim \cup(\text{\textbf{\textit{w}}}' \sim \{W\})$. Since $\text{\textbf{\textit{w}}}'$ is an open cover, the family $\{F_W | W \in \text{\textbf{\textit{w}}}\}$ is closed and discrete. We have $F_W \subset W$ for each $W \in \text{\textbf{\textit{w}}}$ and it follows, since X is discretely orthocompact, that there exists an interior-preserving open family $\circledcirc = \{0_W | W \in \text{\textbf{\textit{w}}}\}$ such that $F_W \subset 0_W \subset W$ for every $W \in \text{\textbf{\textit{w}}}$. Let $V_k = V_{k-1} \cup \circledcirc$. Then V_k is an interior-preserving family of open subsets of X. Moreover, since V_{k-1} and $\text{\textbf{\textit{w}}}$ are partial refinements of $\mathcal{U}$, the family V_k is a partial refinement of $\mathcal{U}$. It remains to show that $S_k \subset \cup V_k$. Let $x \in S_k$. If $x \in \cup V_{k-1}$, then $x \in \cup V_k$. Assume that $x \notin \cup V_{k-1}$. Then $x \in S_k \sim S_{k-1}$ and hence $|(\mathcal{U})_x| = k-1$. Let $Q = \cap(\mathcal{U})_x$. Then $Q \in \text{\textbf{\textit{w}}}$ and Q is the only set of the family $\text{\textbf{\textit{w}}}$ that contains x. Since $x \notin \cup V_{k-1}$, we have $x \in F_Q$. As $F_Q \subset 0_Q$, we have $x \in 0_Q \subset \cup V_k$. We have shown that $S_k \subset \cup V_k$. This completes the proof of the inductive step.

If we let $V = \cdot \underset{n \in \mathbb{N}}{\cup} V_n$, then the family V has all the properties required in the lemma. $\square$

Proposition 2.3.8 A submetacompact space is orthocompact if, and only if, the space is discretely orthocompact.

Proof. Necessity follows from Lemma 2.3.5. To prove sufficiency, let X be a submetacompact, discretely orthocompact space, and let $\mathcal{U}$ be an open cover of X. Let $\langle V_n \rangle$ be a θ-sequence of open refinements of $\mathcal{U}$. It follows from Lemma 2.3.7 that for each $n \in \mathbb{N}$, there exists an interior-preserving open partial refinement $\text{\textbf{\textit{w}}}_n$ of V_n such that for each $x \in X$, if V_n is point-finite at x, then $x \in \cup \text{\textbf{\textit{w}}}_n$. Since $\langle V_n \rangle$ is a θ-sequence, the family $\text{\textbf{\textit{w}}} = \underset{n \in \mathbb{N}}{\cup} \text{\textbf{\textit{w}}}_n$ covers X. The cover $\text{\textbf{\textit{w}}}$ is a σ-interior-preserving open refinement of $\mathcal{U}$. Since X is submetacompact, X is countably

metacompact (see [18]). The conclusion now follows from the foregoing
and Proposition 3.1 of [17].   □

We are now prepared to show that in the class of submetacompact
spaces, orthocompactness is preserved under closed, continuous mappings.

Proposition 2.3.9 Let X be an orthocompact space and let Y be a contin-
uous image of X under a closed mapping. If either X or Y is submetacompact,
then Y is orthocompact.

Proof. It follows from Lemmas 2.3.5 and 2.3.6 that Y is discretely
orthocompact. If Y is submetacompact, then Y is orthocompact, by
Proposition 2.3.8. If X is submetacompact, then Y is submetacompact,
by Corollary 4.5 of [25], and so Y is again orthocompact.   □

Remark. Certain comments made by D. Burke in [8] concerning a proof of
A. V. Arhangel´skij gave the author the idea to use discrete ortho-
compactness to obtain the above result.

In the end of the section we study a subclass of orthocompact spaces
defined by a base property.

Definition 2.3.10 ([32]) A base $\mathcal{B}$ for the topology of X is an orthobase
provided that for every $\mathcal{B}' \subset \mathcal{B}$, if $x \in \cap\mathcal{B}'$ and $x \notin$ Int $\cap\mathcal{B}'$, then $\mathcal{B}'$ is a
neighborhood base at x.

It follows from Theorem 4.12 of [22] that in an orthocompact semi-
stratifiable space, every unsymmetric neighbornet contains a transitive
neighbornet. We now show that the same result holds in a space that
has an orthobase for its topology.

Lemma 2.3.11 If the topology of a space has an orthobase, then every unsymmetric neighbornet of the space contains a transitive neighbornet.

Proof. Let $\mathcal{B}$ be an orthobase for the topology of X, and let U be an unsymmetric neighbornet of X. Let $\leqq$ be a well-ordering of the family $\mathcal{B}$ and for each $x \in X$, let B_x be the least element, with respect to $\leqq$, of the subfamily $\{B \in (\mathcal{B})_x | B \subset U\{x\}\}$ of $\mathcal{B}$. Let $\mathcal{B}' = \{B_x | x \in X\}$. We show that the family $\mathcal{B}'$ is interior-preserving. Assume on the contrary that there exists $C \subset \mathcal{B}'$ and a point $x \in \cap C$ such that $x \notin \text{Int} \cap C$. Since $C \subset \mathcal{B}$ and $\mathcal{B}$ is an orthobase, the family C is a neighborhood base at x. It follows that there exists $C \in C$ such that $C \subset B_x$ and $C \neq B_x$. We have $x \in C \subset U\{x\}$ and hence $B_x < C$. Let $z \in X$ be such that $C = B_z$. Then $\{x,z\} \subset B_x \cap B_z \subset U\{x\} \cap U\{z\}$ and it follows that $U\{x\} = U\{z\}$. Consequently, $B_x \subset U\{z\}$. Since $z \in B_x$, we have $B_z \leqq B_x$. This, however, is a contradiction, since $B_x < C$. It follows that the family $\mathcal{B}'$ is interior-preserving. Since $\mathcal{B}'$ is an open family, the transitive relation $D\mathcal{B}'$ is a neighbornet. We have $D\mathcal{B}'\{x\} \subset B_x \subset U\{x\}$ for each $x \in X$; hence $D\mathcal{B}' \subset U$. □

Note that it follows from the above lemma that if the topology of X has an orthobase, then X is non-archimedeanly quasi-metrizable (see [17])if, and only if, X has a basic sequence by unsymmetric neighbornets. For some characterizations of non-archimedean quasi-metrizability of a paracompact space whose topology has an orthobase, see [19].

Proposition 2.3.12 If the topology of a space has an orthobase, then every continuous image of the space under a closed mapping is ortho-compact.

Proof. Assume that the topology of X has an orthobase. Let f be a closed and continuous mapping from X onto a space Y. We show that every unsymmetric neighbornet of Y contains a transitive neighbornet. Let U be an unsymmetric neighbornet of Y. Define a relation V on X by setting $V\{x\} = f^{-1}(U\{f(x)\})$ for each $x \in X$. Since the mapping f is continuous, the relation V is a neighbornet of X. It is easily seen that V is unsymmetric. It follows from Lemma 2.3.11 that there is a transitive neighbornet W of X such that $W \subset V$. Define a relation Q on Y by setting $Q\{y\} = Y \sim f(X \sim W(f^{-1}\{y\}))$ for each $y \in Y$. Since the mapping f is closed, the relation Q is a neighbornet of Y. Moreover, it is easily seen that $Q \subset U$ and that Q is transitive. We have shown that every unsymmetric neighbornet of Y contains a transitive neighbornet. By Corollary 3.8 and Theorem 3.14 of [22], Y is orthocompact. $\square$

A related result has been proved in [4] and [28]. According to that result, if the topology of X has a σ-interior-preserving base and if Y is the image of X under a perfect mapping, then the topology of Y has a σ-interior-preserving base.

CHAPTER III

QUASI-UNIFORMITIES OF SEMI-STRATIFIABLE SPACES

In this chapter we study the relationships between covering
properties and quasi-uniformities in a semi-stratifiable space. In
the preceding chapter we saw that an ħ-doubly covered space can fail
to be normally ħ-refinable; in the first section of the present chapter
we show that this cannot happen in the class of semi-stratifiable spaces.
In terms of quasi-uniformities, our result is that the fine quasi-
uniformity of an ħ-doubly covered semi-stratifiable space has the
Lebesgue property. We also exhibit some collections of neighbornets
that serve as bases of the fine quasi-uniformity of an ħ-doubly covered
semi-stratifiable space.

A long-standing unsolved problem concerning quasi-metrizability of
a topological space is whether every γ-space is quasi-metrizable. In
the second section of this chapter, we use the results of the first
section to obtain a partial solution to this problem. We show that
every developable γ-space is quasi-metrizable. We also show that every
quasi-metrizable developable space has a normal basic sequence by
unsymmetric neighbornets.

Every orthocompact space is ħ-doubly covered. As we saw in the
preceding chapter, the converse of this result is not true; however, it
is an open question whether the converse holds in the class of semi-
stratifiable spaces. In the last section of the present chapter, we
obtain some partial answers to that question by studying transitive
spaces. Our results are consequences of the following theorems:
(i) A topological space is transitive provided that the space is the

countable union of transitive closed subspaces; (ii) An $\hbar$-doubly

covered semi-stratifiable space is orthocompact if, and only if, the

space is transitive.

1. $\hbar$-doubly covered semi-stratifiable spaces

We need some auxiliary results to prove that every $\hbar$-doubly

covered semi-stratifiable space is normally $\hbar$-refinable.

Lemma 3.1.1 Let X be an $\hbar$-doubly covered space, let $\mathcal{F}$ be a σ-discrete

closed cover of X and for each $F \in \mathcal{F}$, let 0_F be an open set containing

F. Then for each $n \in \mathbb{N}$, there exists a neighbornet V of X such that

for every $x \in X$, we have $V^n\{x\} \subset 0_F$ for some $F \in (\mathcal{F})_x$.

Proof. Let $\mathcal{K}$ be a discrete subfamily of $\mathcal{F}$. We use induction to show

that for each $k \in \mathbb{N}$, there exists a neighbornet U_k of X such that

$U_k^k(F) \subset 0_F$ for each $F \in \mathcal{K}$. Since $\mathcal{K}$ is closed and discrete, we can define

a neighbornet U_1 of X by setting $U_1\{x\} = X \sim \cup\mathcal{K}$ if $x \notin \cup\mathcal{K}$ and $U_1\{x\} = 0_F$

if $x \in F \in \mathcal{K}$; clearly, we have $U_1(F) = 0_F$ for every non-empty $F \in \mathcal{K}$.

Now let $k > 1$ be such that we have already defined a neighbornet U_{k-1}

so that $U_{k-1}^{k-1}(F) \subset 0_F$ for every $F \in \mathcal{K}$. Let $\mathcal{U} = \{X \sim \cup\mathcal{K}\} \cup \{\dot{U}_{k-1}(F)|F \in \mathcal{K}\}$

and note that $\mathcal{U}$ is an open cover of X. Since X is $\hbar$-doubly covered, there

exists a neighbornet W of X such that $W^2 \subset S\mathcal{U}$. Let $U_k = U_{k-1} \cap W$. To

show that U_k has the required property, let $F \in \mathcal{K}$. We have $St(F,\mathcal{U}) = U_{k-1}(F)$

and it follows, since $U_k \subset W$ and $W^2 \subset S\mathcal{U}$, that $U_k^2(F) \subset U_{k-1}(F)$.

Consequently, $U_k^k(F) = U_k^{k-2}(U_k^2(F)) \subset U_k^{k-2}(U_{k-1}(F)) \subset U_{k-1}^{k-2}(U_{k-1}(F)) = $

$U_{k-1}^{k-1}(F) \subset 0_F$. We have shown that the neighbornet U_k has the required

property. This completes the induction.

To complete the proof of the lemma, let $n \in \mathbb{N}$. Represent $\mathcal{F}$ in the form $\mathcal{F} = \bigcup_{i \in \mathbb{N}} \mathcal{F}_i$, where each of the families $\mathcal{F}_i$ is discrete. By the preceding part of the proof, there exists for each $i \in \mathbb{N}$, a neighbornet V_i of X such that $V_i^n(F) \subset 0_F$ for every $F \in \mathcal{F}_i$. Let $S_0 = \phi$ and for each $i \in \mathbb{N}$, let $S_i = \bigcup \mathcal{F}_i$. For each $x \in X$, let $i(x)$ be the least element of the set $\{i \in \mathbb{N} \mid x \in S_i\}$, and let $V\{x\} = (\bigcap_{i=1}^{i(x)} V_i\{x\}) \sim \bigcup_{i=1}^{i(x)-1} S_i$. Then V is a neighbornet of X. To show that V has the required property, first observe that for all $z \in X$ and $y \in X$, if $y \in V\{z\}$, then $i(y) \geqq i(z)$. Now, let $x \in X$. By the previous observation, we have $i(u) \geqq i(x)$, and hence $V\{u\} \subset V_{i(x)}\{u\}$, for every $u \in V^n\{x\}$. Consequently, $V^n\{x\} \subset V_{i(x)}^n\{x\}$. Let $F \in \mathcal{F}_{i(x)}$ be such that $x \in F$. Then $V^n\{x\} \subset V_{i(x)}^n\{x\} \subset V_{i(x)}^n(F) \subset 0_F$. We have shown that V has the required property. $\square$

Corollary 3.1.2 In an $\hbar$-doubly covered subparacompact space, every open cover has an $\hbar^2$-refinement.

Proof. Let X be an $\hbar$-doubly covered subparacompact space, and let $\mathbb{O}$ be an open cover of X. Let $\mathcal{F}$ be a σ-discrete closed refinement of $\mathbb{O}$ and for each $F \in \mathcal{F}$, let $0_F \in \mathbb{O}$ be such that $F \subset 0_F$. The conclusion that $\mathbb{O}$ has an $\hbar^2$-refinement now follows by applying the result of Lemma 3.1.1 for n = 2. $\square$

Proposition 3.1.3 Let U be an unsymmetric neighbornet of an $\hbar$-doubly covered semi-stratifiable space. Then there exists an unsymmetric neighbornet W of the space such that $W^2 \subset U$.

Proof. Let X be the space in question. By Corollary 4.10 of [22], there

exists a σ-discrete closed cover $\mathcal{F}$ of X such that we have $U\{x\} = U(F)$ whenever $x \in F \in \mathcal{F}$. By Lemma 3.1.1, there exists a neighbornet V of X such that for each $x \in X$, there exists $F_x \in \mathcal{F}$ such that $x \in F_x$ and $V^6\{x\} \subset U(F_x)$. We have $U\{F_x\} = U\{x\}$ for every $x \in X$, and it follows that $V^6 \subset U$. By Theorem 4.7 of [22], there exists an unsymmetric neighbornet W of X such that $W \subset V^3$. This completes the proof since $W^2 \subset (V^3)^2 = V^6 \subset U$. □

Corollary 3.1.4 In an ħ-doubly covered semi-stratifiable space, every unsymmetric neighbornet is normal.

Corollary 3.1.5 Every ħ-doubly covered semi-stratifiable space is normally ħ-refinable.

Proof. This follows from the preceding corollary, since every open cover of a topological space has an unsymmetric ħ-refinement, by Corollary 3.8 of [22]. □

In terms of quasi-uniformities, the above corollary says that in an ħ-doubly covered semi-stratifiable space, the fine quasi-uniformity has the Lebesgue property. In the following results, we exhibit bases for the fine quasi-uniformity of an ħ-doubly covered semi-stratifiable space.

Proposition 3.1.6 In an ħ-doubly covered semi-stratifiable space, the collection of all unsymmetric neighbornets of the space forms a base for the fine quasi-uniformity of the space.

Proof. It follows from Corollary 3.1.4 that every unsymmetric neighbornet

of an $\hbar$-doubly covered semi-stratifiable space belongs to the fine

quasi-uniformity of the space; that the collection of all unsymmetric

neighbornets of such a space forms a base for the fine quasi-uniformity

of the space follows from the result ([22], Theorem 4.7) that if U is

a neighbornet of a semi-stratifiable space, then U^3 contains an

unsymmetric neighbornet. □

For $k \in \mathbb{N}$, denote by $\hbar^k(X)$ the collection consisting of all

relations of the form U^k, where U is a neighbornet of X. We then

have the following result.

<u>Proposition 3.1.7</u> Let X be a semi-stratifiable (developable) $\hbar$-doubly

covered space. Then the collection $\hbar^3(X)$ (the collection $\hbar^2(X)$) forms

a base for the fine quasi-uniformity of X.

<u>Proof</u>. Denote by $\mathscr{F}_X$ the fine quasi-uniformity of X By Theorems 4.7

and 4.4 of [22], every member of $\hbar^3(X)(\hbar^2(X))$ contains an unsymmetric

neighbornet. Consequently, by Corollary 3.1.4, $\hbar^3(X) \subset \mathscr{F}_X$ ($\hbar^2(X) \subset \mathscr{F}_X$).

The rest of the proof follows from the observation that for each $k \in \mathbb{N}$,

every member of $\mathscr{F}_X$ contains a member of $\hbar^k(X)$. □

2. On quasi-metrizability

In this section we show that every developable γ-space is quasi-

metrizable. We state the results of this section in terms of developable

spaces instead of semi-stratifiable spaces, since it is known that every

semi-stratifiable γ-space is developable (see Remark 4.8 of [21]).

To be able to use the results of Section 1 in connection with γ-spaces,

we introduce a covering property shared by all γ-spaces.

Definition 3.2.1 A topological space X is <u>regularly</u> $\sigma\hbar$-<u>refinable</u> if for every open cover $\mathcal{U}$ of X, there exists a sequence $\langle V_n \rangle$ of neighbornets of X such that to each $x \in X$, there corresponds $n \in \mathbb{N}$ such that if $y \in V_n\{x\}$, then $V_n\{x\} \cup V_n\{y\} \subset U$ for some $U \in \mathcal{U}$.

Every regularly $\hbar$-refinable space is regularly $\sigma\hbar$-refinable. If X is a γ-space, then X has a sequence $\langle V_n \rangle$ of neighbornets such that $\langle V_n^2 \rangle$ is a basic sequence; clearly, this sequence satisfies the condition stated in the above definition for any open cover $\mathcal{U}$ of X. Consequently, every γ-space is regularly $\sigma\hbar$-refinable.

We introduced regularly $\sigma\hbar$-refinable spaces in order to show that every semi-stratifiable γ-space is $\hbar$-doubly covered; this result is a consequence of the following lemma.

Lemma 3.2.2 Every countably metacompact regularly $\sigma\hbar$-refinable space is regularly $\hbar$-refinable.

Proof. Let X be countably metacompact and regularly $\sigma\hbar$-refinable, and let $\mathcal{U}$ be an open cover of X. Let $\langle V_n \rangle$ be a sequence of X such that for each $x \in X$, there exists $n(x) \in \mathbb{N}$ such that if $y \in V_{n(x)}\{x\}$, then $V_{n(x)}\{x\} \cup V_{n(x)}\{y\} \subset U$ for some $U \in \mathcal{U}$. For every $n \in \mathbb{N}$, let $Q_n = \dot{V}_n\{x \in X | n(x) = n\}$. Then $\{Q_n | n \in \mathbb{N}\}$ is a countable open cover of X. Since X is countable metacompact, there exists an open cover $\{G_n | n \in \mathbb{N}\}$ of X such that for every $n \in \mathbb{N}$, we have $G_n \subset Q_n$ and for every $x \in X$, the set $N(x) = \{n \in \mathbb{N} | x \in G_n\}$ is finite. For each $x \in X$, let $m(x)$ be a member of the set $N(x)$. For every $x \in X$, we have

$x \in G_{m(x)} \subset Q_{m(x)}$ and hence there exists $\varphi x \in X$ such that $n(\varphi x) = m(x)$ and $x \in \dot{V}_{m(x)}\{\varphi x\}$. Define a neighbornet W of X by setting $W\{x\} = V_{m(x)}\{\varphi x\} \cap (\cap\{G_n \cap V_n\{x\}|n \in N(x)\})$ for every $x \in X$. We show that W is a regular $\hbar$-refinement of $\mathcal{U}$. Let $x \in X$, and let $y \in W\{x\}$. Then $y \in G_{m(x)}$ and consequently, $W\{y\} \subset V_{m(x)}\{y\}$. We have $y \in V_{m(x)}\{\varphi x\}$ and it follows, since $n(\varphi x) = m(x)$, that there exists $U \in \mathcal{U}$ such that $V_{m(x)}\{\varphi x\} \cup V_{m(x)}\{y\} \subset U$ Since $W\{x\} \subset V_{m(x)}\{\varphi x\}$ and $W\{y\} \subset V_{m(x)}\{y\}$, we have $W\{x\} \cup W\{y\} \subset U$. We have shown that W is a regular $\hbar$-refinement of $\mathcal{U}$. $\qquad\qquad\square$

The above result should be compared with Proposition 2.1.3 and with the result of [17] that every countably metacompact σ-orthocompact space is orthocompact. Similarly as in the above proof one can show that in a countably metacompact σ-refinable space (see [31]), every open cover has an $\hbar^2$-refinement.

Theorem 3.2.3 Every developable γ-space has a normal basic sequence by unsymmetric neighbornets.

Proof. Let X be a developable γ-space. Then X is subparacompact ([5]) and hence countably metacompact ([18]). As observed above, X is regularly $\sigma\hbar$-refinable. It follows from Lemma 3.2.2 that X is regularly $\hbar$-refinable. Consequently, X is $\hbar$-doubly covered. It follows from Proposition 3.1.3 that for every unsymmetric neighbornet U of X, there exists an unsymmetric neighbornet W of X such that $W^2 \subset U$. Let $\langle \mathcal{U}_n \rangle$ be a development for X. For each $n \in \mathbb{N}$, let U_n be an unsymmetric $\hbar$-refinement of $\mathcal{U}_n$ (see Corollary 3.8 of [22]). Since the intersection of unsymmetric relations is

unsymmetric, we can use induction to construct a sequence $\langle V_n \rangle$ of unsymmetric neighbornets of X such that for each $n \in \mathbb{N}$, we have $V_n \subset U_n$ and $V_{n+1}^2 \subset V_n$. Then $\langle V_n \rangle$ is a normal sequence. For all $x \in X$ and $n \in \mathbb{N}$, we have $V_n\{x\} \subset U_n\{x\} \subset St(x, \mathcal{U}_n)$; it follows that $\langle V_n \rangle$ is a basic sequence. □

In a private communication, J. Kofner has described a more straight-forward construction of a normal basic sequence by unsymmetric neighbornets for a developable γ-space.

<u>Corollary 3.2.4</u> Every developable γ-space is quasi-metrizable.

3. <u>On transitivity and orthocompactness</u>

<u>Definition 3.3.1</u> ([16]) A topological space is <u>transitive</u> provided that every normal neighbornet of the space contains a transitive neighbornet.

In terms of quasi-uniformities, a space is transitive if, and only if, the fine quasi-uniformity of the space is a transitive quasi-uniformity. Equivalently, a space is transitive if, and only if, the fine quasi-uniformity of the space coincides with the fine transitive quasi-uniformity of the space. Examples of non-transitive quasi-metric spaces are given in [27] and [28]. In [22], it is shown that every orthocompact semi-stratifiable space is transitive.

The following results exhibit some invariance properties of transitivity.

<u>Proposition 3.3.2</u> Let X be a transitive space.

(i)   Every open subspace of X is transitive.

(ii)   Every closed subspace of X is transitive.

(iii)   Every continuous image of X under a closed mapping is transitive.

Proof.   (i). Let G be an open subspace of X, and let U be a normal

neighbornet of G. Define a relation V on X by setting $V\{x\} = U\{x\}$ for

every $x \in G$, and $V\{x\} = X$ for every $x \in X \sim G$. It is easily seen that

V is a normal neighbornet of X. Since X is transitive, there exists a

transitive neighbornet W of X such that $W \subseteq V$. Define a relation Q on

G by setting $Q\{x\} = W\{x\} \cap G$ for every $x \in G$. Then Q is a transitive

neighbornet of the subspace G and $Q \subseteq U$.

(ii). Let S be a closed subspace of X, and let U be a normal neighbornet

of S. Define a relation V on X by setting $V\{x\} = U\{x\} \cup (X \sim S)$ for

every $x \in S$, and $V\{x\} = X \sim S$ for every $x \in X \sim S$. Then V is a normal

neighbornet of X and we can complete the proof as in (i) above.

(iii). Let f be a closed and continuous mapping from X onto a space

Y. To show that Y is transitive, let U be a normal neighbornet of Y.

There exists a normal sequence $\langle U_n \rangle$ of neighbornets of Y such that

$U_1 = U$. For every $n \in \mathbb{N}$, define a neighbornet V_n of X by setting

$V_n\{x\} = f^{-1}(U_n\{f(x)\})$ for every $x \in X$. We show that the sequence $\langle V_n \rangle$

is normal. Let $n \in \mathbb{N}$. To show that $V_{n+1}^2 \subseteq V_n$, let $x \in X$ and let

$z \in V_{n+1}^2\{x\}$. Then there exists $u \in V_{n+1}\{x\}$ such that $z \in V_{n+1}\{u\}$.

We have $f(u) \in U_{n+1}\{f(x)\}$ and $f(z) \in U_{n+1}\{f(u)\}$. Consequently,

$f(z) \in U_{n+1}^2\{f(x)\} \subseteq U_n\{f(x)\}$ and $z \in V_n\{x\}$. It follows that

$V_{n+1}^2\{x\} \subseteq V_n\{x\}$ and $V_{n+1}^2 \subseteq V_n$. We have shown that $\langle V_n \rangle$ is a normal

sequence. Hence V_1 is a normal neighbornet of X. Let W be a transitive

neighbornet of X such that $W \subseteq V_1$. Since f is a closed mapping, we can define a neighbornet Q of Y by setting $Q\{y\} = Y \sim f(X \sim Wf^{-1}\{y\})$ for every $y \in Y$. Since $W \subseteq V_1$, we have $Q \subseteq U_1$. It remains to show that Q is transitive. Let $y \in Y$, and let $v \in Q\{y\}$. Then $f^{-1}\{v\} \subseteq Wf^{-1}\{y\}$. Since W is a transitive relation, it follows that $Wf^{-1}\{v\} \subseteq Wf^{-1}\{y\}$. Consequently, $Q\{v\} \subseteq Q\{y\}$. We have shown that the relation Q is transitive. □

For the proof of our next result, we need the following lemma.

<u>Lemma 3.3.3</u> Let $\langle U_n \rangle$ be a normal sequence of neighbornets of X and let $\langle x_n \rangle$ be a sequence of points of X such that for each $i \in \mathbb{N}$, we have $x_{i+1} \in U_{i+1}\{x_i\}$. Then $\bigcup_{i=1}^{\infty} U_i\{x_i\} \subseteq U_1\{x_1\}$.

<u>Proof.</u> We use induction on m to show that for every $m \in \mathbb{N}$, we have $\bigcup_{i=1}^{m} U_{n+i}\{x_{n+i}\} \subseteq U_n\{x_n\}$ for each $n \in \mathbb{N}$. The assertation holds for m = 1, since $x_{n+1} \in U_{n+1}\{x_n\}$ and $U_{n+1}^2\{x_n\} \subseteq U_n\{x_n\}$ for each $n \in \mathbb{N}$. Assume that the assertion has been proved for m = k. To prove it for m = k+1, let $n \in \mathbb{N}$. We have $\bigcup_{i=1}^{k} U_{n+1+i}\{x_{n+1+i}\} \subseteq U_{n+1}\{x_{n+1}\}$, in other words, $\bigcup_{i=2}^{k+1} U_{n+i}\{x_{n+i}\} \subseteq U_{n+1}\{x_{n+1}\}$. Consequently, $\bigcup_{i=1}^{k+1} U_{n+i}\{x_{n+i}\} \subseteq U_{n+1}\{x_{n+1}\}$. Since $x_{n+1} \in U_{n+1}\{x_n\}$ and $U_{n+1}^2\{x_n\} \subseteq U_n\{x_n\}$, we have $\bigcup_{i=1}^{k+1} U_{n+i}\{x_{n+i}\} \subseteq U_n\{x_n\}$. We have shown that the assertion is true for m = k + 1. It follows by the induction principle that the assertion is true for every $m \in \mathbb{N}$.

For every $m \in \mathbb{N}$, we have $\bigcup_{i=1}^{m} U_{i+1}\{x_{i+1}\} \subseteq U_1\{x_1\}$. Consequently, $\bigcup_{i=1}^{\infty} U_{i+1}\{x_{i+1}\} \subseteq U_1\{x_1\}$ and $\bigcup_{i=1}^{\infty} U_i\{x_i\} \subseteq U_1\{x_1\}$. □

<u>Theorem 3.3.4</u> A topological space is transitive if the space can be

represented as the countable union of closed and transitive subspaces.

<u>Proof</u>. Let $X = \bigcup_{n \in \mathbb{N}} F_n$, where each F_n is a closed and transitive subspace of X. To show that X is transitive, let $\langle U_n \rangle$ be a normal sequence of neighbornets of X. For each $n \in \mathbb{N}$, there exists a transitive neighbornet Q_n of the subspace F_n such that $Q_n\{x\} \subset U_{n+2}\{x\}$ for every $x \in F_n$. For each $x \in X$, denote by $m(x)$ the least element of the set $\{n \in N | x \in F_n\}$ and let $G(x)$ be an open subset of X such that $G(x) \cap F_{m(x)} = Q_{m(x)}\{x\}$. Define a neighbornet of V of X by setting for each $x \in X$,

$$V\{x\} = (G(x) \cap U_{m(x)+2}\{x\}) \sim \cup\{F_n | n < m(x)\}.$$

We show that the transitive neighbornet V^∞ is contained in U_1. Call a sequence $\langle x_m \rangle$ of points of X a V-sequence if $x_{n+1} \in V\{x_n\}$ for each $n \in \mathbb{N}$. To show that $V^\infty \subset U_1$, it suffices to show that for any V-sequence $\langle x_n \rangle$, we have $\bigcup_{n=1}^\infty V\{x_n\} \subset U_1\{x_1\}$. Let $\langle x_n \rangle$ be a V-sequence. For each $n \in \mathbb{N}$, it follows from $x_{n+1} \in V\{x_n\}$ that $m(x_{n+1}) \geqq m(x_n)$. Note that for every $i \in \mathbb{N}$, if $m(x_{i+1}) = m(x_i) = m$, then $x_{i+1} \in V\{x_i\} \cap F_m \subset G(x_i) \cap F_m = Q_m\{x_i\}$ so that $Q_m\{x_{i+1}\} \subset Q_m\{x_i\} \subset U_{m+2}\{x_i\}$, by the transitivity of Q_m. It follows that for all $j \in \mathbb{N}$ and $i \in \mathbb{N}$, if $j > i$ and $m(x_j) = m(x_i) = m$, then $x_j \in Q_m\{x_j\} \subset U_{m+2}\{x_i\}$ and hence $V\{x_j\} \subset U_{m+2}\{x_j\} \subset U_{m+1}\{x_i\}$. Define a subsequence $\langle y_k \rangle$ of the sequence $\langle x_n \rangle$ by dropping out every x_n for which $m(x_n) = m(x_{n-1})$. If $\langle y_k \rangle$ is a finite sequence, let r be the largest $k \in \mathbb{N}$ for which y_{k-1} is defined; if the sequence is infinite, let $r = \infty$. For every $k < r$, let $i(k) = m(y_k)$. Then $\bigcup_{j=1}^\infty V\{x_j\} \subset \bigcup_{k<r} U_{i(k)+1}\{y_k\}$, and $y_{k+1} \in U_{i(k)+1}\{y_k\}$ whenever $k+1 < r$. For every $k < r$, $k > 1$, we have $i(k) > i(k-1)$. Consequently, $i(k) \geqq k$ for every $k < r$. It follows

that $\bigcup\limits_{j=1}^{\infty} v\{x_j\} \subset \bigcup\limits_{k<r} U_{k+1}\{y_k\}$ and that $y_{k+1} \in U_{k+1}\{y_k\}$ for $k + 1 < r$. By

Lemma 3.3.3, we have $\bigcup\limits_{k<r} U_k\{y_k\} \subset U_1\{y_1\}$. Since $y_1 = x_1$, we have

$\bigcup\limits_{j=1}^{\infty} v\{x_j\} \subset \bigcup\limits_{k<r} U_{k+1}\{y_k\} \subset U_1\{x_1\}$. □

Corollary 3.3.5 Every F_σ-subspace of a transitive space is transitive.

Proof. Part (ii) of Proposition 3.3.1, and Theorem 3.3.4. □

Corollary 3.3.6 Every subparacompact locally transitive space is transitive.

Proof. Let X be subparacompact and locally transitive. For each $x \in X$, Let $G_x \in \mathcal{N}_x$ be a transitive subspace of X. Let $\mathcal{F} = \bigcup\limits_{n \in \mathbb{N}} \mathcal{F}_n$ be a closed refinement of the cover $\{G_x | x \in X\}$ such that for every $n \in \mathbb{N}$, the family $\mathcal{F}_n$ is discrete. It follows from part (ii) of Proposition 3.3.1 that for each $F \in \mathcal{F}$, the subspace F of X is transitive. For each $n \in \mathbb{N}$, the subspace $\bigcup \mathcal{F}_n$ of X is closed, and this subspace is transitive since it is the direct sum of its transitive subspaces F, $F \in \mathcal{F}_n$. We have $X = \bigcup\limits_{n \in \mathbb{N}} (\bigcup \mathcal{F}_n)$ and hence X is a transitive space, by Theorem 3.3.4. □

It is easily seen that a topological space is transitive if the space has a point-finite cover by open transitive subspaces; using this observation and Theorem 3.3.4, it can be shown that the result of Corollary 3.3.6 remains true if "subparacompact" is replaced by "submetacompact" in the corollary.

A topological space is orthocompact provided that the space is both transitive and normally $\mathcal{N}$-refinable. Hence the preceding results can be

used to study orthocompact spaces. Note that we cannot expect to find characterizations of orthocompactness in terms of transitivity since there exist non-transitive orthocompact spaces ([28]); however, such characterizations obtain in some special classes of spaces, as is indicated by the following result.

Theorem 3.3.7 An $\hbar$-doubly covered semi-stratifiable space is orthocompact if, and only if, the space is transitive.

Proof. The remarks made after Corollary 4.13 of [22] show that transitivity is a necessary condition for orthocompactness in the class of semi-stratifiable spaces. On the other hand, it follows from Corollary 3.1.5 that every $\hbar$-doubly covered semi-stratifiable space is normally $\hbar$-refinable; consequently, every $\hbar$-doubly covered semi-stratifiable transitive space is orthocompact.   ·   □

It follows from Theorem 3.3.4 that every F_σ-discrete space is transitive (a topological space is F_σ-<u>discrete</u> if the space is the countable union of closed, discrete subspaces); since every F_σ-discrete space is semi-stratifiable, we see that there exist transitive semi-stratifiable spaces that are not orthocompact (e.g. the space described in [20]). The next result shows that orthocompact F_σ-discrete spaces are "super-transitive".

Proposition 3.3.8 In an $\hbar$-doubly covered F_σ-discrete space, every neighbornet contains a transitive neighbornet.

Proof. Let X be an $\hbar$-doubly covered F_σ-discrete space. As was observed above, X is transitive and semi-stratifiable. By Corollary 4.9

of [22], X has an antisymmetric neighbornet, say V. Let U be any neighbornet of X. Then the neighbornet U ∩ V is antisymmetric and hence unsymmetric. By Corollary 3.1.4, U ∩ V is normal. Since X is transitive, U ∩ V contains a transitive neighbornet.

Our last result gives some characterizations of orthocompactness in the class of semi-stratifiable spaces.

Theorem 3.3.9 The following conditions are mutually equivalent for a semi-stratifiable space:

(i)   The space is orthocompact

(ii)   The space is ħ-doubly covered and locally orthocompact

(iii)   The space is ħ-doubly covered and it can be represented as the countable union of closed, orthocompact subspaces.

Proof. It is obvious that (i) ⇒ (ii) and (i) ⇒ (iii). Since every semi-stratifiable space is subparacompact ([11] and [26]), it follows from Theorem 3.3.7 and Corollary 3.3.6 that (ii) ⇒ (i). Since semi-stratifiability is a hereditary property, it follows from Theorems 3.3.7 and 3.3.4 that (iii) ⇒ (i).   □

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Covering properties and quasi-uniformities

of topological spaces

by

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(ABSTRACT)

This thesis deals with the relationships between covering properties
and properties of compatible quasi-uniformities of a topological space.
The covering properties considered in this work are orthocompactness,
metacompactness and paracompactness; some generalizations of orthocompact-
ness are also defined and studied.

In Section 1.1, a result of J. M. Worrell Jr.'s is extended to give
a characterization of the existence of a point-finite or a locally finite
open refinement of a cover of a topological space; this characterization
is then used to obtain characterizations of metacompactness and para-
compactness in terms of quasi-uniformities with generalized symmetry
properties. In Section 1.2, the concept of an ℏ-doubly covered space
is defined, and it is shown that a Tychonoff space X is metacompact if,
and only if, the product space $X \times \beta X$ is ℏ-doubly covered; as a corollary
to this result, X is metacompact if, and only if, the product space
$X \times \beta X$ has a compatible quasi-uniformity with the Lebesgue property.

In Section 2.1, an example due to J. Kofner is used to show that
a topological space may fail to be orthocompact even if the space has a
compatible quasi-uniformity with the Lebesgue property. In Section 2.2,
the concept of a regularly ℏ-refinable space is introduced, and meta-
compactness and regular refinability (=2-full normality) are given

characterizations in terms of this concept. These characterizations

and results of H. H. Corson and S. Peregudov are used to show that

certain Σ-products of metrizable spaces are not regularly refinable

even though they are doubly covered (=almost 2-fully normal); this gives

a negative answer to a question of J. Mansfield. Section 2.3 deals

with continuous images of orthocompact spaces under closed mappings.

G. Gruenhage has recently given an example to show that orthocompactness

is not preserved under closed, continuous mappings. A modification of

Gruenhage's example is presented and it is also shown that regular

refinability and certain generalizations of orthocompactness fail to be

preserved under closed, continuous mappings. It is proved that a

continuous image of an orthocompact space under a closed mapping is

orthocompact provided the space is submetacompact (= θ-refinable) or

the topology of the space has an orthobase.

Section 3.1 deals with quasi-uniformities of semi-stratifiable

spaces. It is shown that in the class of semi-stratifiable spaces, all

the generalizations of orthocompactness considered in this thesis are

equivalent. In Section 3.2, it is shown that every developable γ-space

is quasi-metrizable. In Section 3.3, the concept of a transitive space

is used to study the question whether a semi-stratifiable space is

orthocompact if the space has a compatible quasi-uniformity with the

Lebesgue property. The results in the section follow from the result

that a topological space is a transitive space provided that the space

is the countable union of closed, transitive subspaces.