AN ANALYSIS OF THE BED ALLOCATION PROBLEM

by

John E. Jackson

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APPROVED:

John A. White, Chairman

Marvin H. Agee

A. Eugene Fitzgerald

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Chapter 1

INTRODUCTION

Objectives of Study

Hospital administrators are faced with problems of allocating resources, where there are competing demands for the resources from multiple sources. One such allocation problem is the subject of this research. Namely, the problem of distributing beds among various services within the hospital is an allocation problem confronted by a number of hospital administrators.

The bed allocation problem may be briefly described as follows. Each service within the hospital has a requirement for beds. However, the magnitude of the requirement varies over time in a random fashion. When demand exceeds the number of beds available in a given service, a patient must either be admitted to some other service or be turned away from the hospital. The hospital administrator must weigh the consequence of allocating too few beds to a service, along with information concerning the demand for each service, and make decisions regarding the distribution of beds among services.

It is the objective of this research to bring quantitative analysis to bear on the bed allocation problem. In achieving this objective, models will be developed to assist the hospital administrator in decisions concerning the allocation of beds within the hospital.

Importance of the Topic

The determination of how beds are to be allocated among the services of a general acute hospital is of major importance. The provision of patients' health needs and effective hospital management are both dependent upon the decisions for distributing this resource. Improper distributions can contribute to inadequate care of patients (where patients may be located in hospital areas that are inaccessible for desired treatment and observation), the rising cost of hospital services (due to an overabundance of beds in some service areas), the loss of patients to other hospitals and/or lengthy admission waiting lists. Also, staff sizes and the number and location of treatment facilities are directly proportional to how beds are allocated to meet demands.

The problems indicated have been recognized by Blumberg [1], who states:

> "Beds which are unoccupied in a hospital are insurance against the risk of not having enough beds when the number of patients goes higher. The 'premiums' for this insurance are made up by the cost of having unoccupied beds and include uncompensated depreciation on the facilities and the cost of the staff who are partly idle when beds are unoccupied. The former is almost negligible while the latter is substantial."
>
> "Too few beds result in increased health hazards while too many beds lead to higher dollar costs."

The latter remark implies the dual (but conflicting) objectives of today's hospitals, i.e., maximize service to patients and at the same time minimize health service costs. Likewise, typical reactions of

hospital administrators who have been confronted with the problem have been:

> "My obstetrical facility is either feast or famine. Why is it I'm short of beds one week and have nobody around the next?" [25]

> "If the number of beds that will be available for new admissions on any given day were known in advance, reservations could be made for exactly that number of admissions, much as is done by the airlines, and nearly everyone would be admitted on the date established." [5]

Fortunately, the bed allocation problem can be solved as will be seen in Chapter 4. In addition to contributing to the improvement of health care and hospital management, perhaps this research will also demonstrate the usefulness of applying similar quantitative analyses to other areas of the health field.

Formulation of the Problem

The bed allocation problem is a complex problem involving a number of interacting factors. Therefore, any attempt to capture all facets of the real world problem in a model would not only be a difficult undertaking, but, probably, impossible. One approach which can be taken is to develop a large scale simulation model of the system having a high degree of realism. Unfortunately, such an approach is very expensive. Further, a simulation model is not easily manipulated and cause-effect relationships cannot be easily identified.

As an alternate approach, a very simple, deterministic model can be developed. The resulting model could be solved inexpensively, easily

manipulated, and cause-effect relationships could be easily ascertained. However, the model would not be a realistic representation of the system.

Neither of the above approaches was chosen for this study. It was felt that an analytic model was preferable to the simulation model. Furthermore, many of the complexities of the bed allocation problem are due to the randomness involved. Thus, a deterministic model would fail to capture the essence of the problem.

In order to make the bed allocation problem explicit, let

C = total number of beds available for allocation

n = number of hospital services to which beds are to be allocated, i.e., medical, surgical, obstetrics/gynecology and pediatrics

c_j = number of beds allocated to service j, $j = 1, \ldots, n$

w_j = weighting factor to reflect the penalty for turning a patient away from service j

λ_j = average number of patients arriving per day for service j

$P_j(c_j)$ = proportion of patients turned away per day from service j, given c_j beds available.

The bed allocation problem is formulated in this study as follows:

$$\text{Minimize} \qquad f(C) = \sum_{j=1}^{n} w_j \lambda_j P_j(c_j) \tag{1-1}$$

$$\text{subject to} \qquad \sum_{j=1}^{n} c_j \leq C \tag{1-2}$$

$$c_j \geq 0 \quad j=1, \ldots, n \tag{1-3}$$

Since patients arrive at a daily rate of λ_j at service j and $P_j(c_j)$ percent are turned away, the average number of patients turned away per day at service j equals $\lambda_j P_j(c_j)$. With each patient turned away at service j a penalty of w_j is incurred. Thus, the average daily penalty for turning patients away is given by Equation (1-1). It is desired that beds be allocated among services such that the penalty is minimized, subject to the restrictions given by equations (1-2) and (1-3).

It is pointed out that the bed allocation formulation given by Equation (1-1), (1-2), and (1-3) is a simplification of reality. However, it is felt that the model successfully captures a number of the interesting aspects of the problem. Thus, the model should be of assistance to the hospital administrator in coping with the bed allocation problem.

One aspect of the formulation bears further examination. Namely, the proportion of patients turned away per day from service j must be expressed analytically. The proportion of patients turned away per day can also be interpreted as the probability an arriving patient finds all beds in service j occupied. The latter is a function of the rate of arrival of patients, their length of stay in service j, and the number of beds available. Since these factors are those common to a queueing problem, it was decided to employ a queueing model in obtaining the value of $P_j(c_j)$ in Equation (1-1).

Each service within the hospital can be visualized as a separate queueing system. In the formulation adopted in this research the queueing systems are treated as independent systems. The number of beds allocated to a service is analogous to the number of servers in the

queueing system. Therefore, $P_j(c_j)$ is the probability all "servers" are busy in "queueing system" j. It remains to identify the probability distribution for the arrivals of patients and their corresponding lengths of stay. This determination is treated in Chapter 3.

Scope and Limitations

This research is concerned with the bed allocation formulation given in the previous section. Due to the relative simplicity of the model, a number of factors were not included in this analysis. For example, the following factors were not explicitly accounted for in the formulation: Patient's sex, age, admitting diagnosis, type admission (elective or emergency), and transfers between services once admitted. Physical layout constraints of the hospital have not been considered in relation to the number of beds allocated per service. Further, it was assumed that steady state conditions exist within each service of the hospital. The inclusion of these limitations in the problem solution is discussed in Chapter 6 under the section, Recommendations for Further Study.

Throughout the research the following services were considered: medical, surgical, obstetrics/gynecology, and pediatrics. Data were gathered from one hospital for these services and the bed allocation procedures of four hospitals were analyzed to gain familiarity with the problem. These four services are felt to be representative of the divisions found in most small to medium sized general acute hospitals. Even though this research was limited to a treatment of four services, the techniques employed are equally valid for any number of services.

Order of Discussion

This research was performed in five phases: model formulation, literature search, data analysis, optimization, and analysis of results. In presenting the results of this research the following order of discussion is employed. Chapter 2 presents the results of a search of the literature for research related to the bed allocation problem. Chapter 3 contains the results of an analysis of data gathered from the Radford Community Hospital, Radford, Virginia. Chapter 4 provides a discussion of the procedure for determining the optimal allocation of beds among the four services. Chapter 5 presents both an analysis of results obtained from a study of the sensitivity of the model and solutions to some typical bed allocation problems. Finally, Chapter 6 provides a summary of the research and a listing of recommendations for further study. Included in the Appendices are summaries of the data gathered at the Radford Community Hospital and documentation of the Fortran computer program written in support of this effort.

Chapter 2

SURVEY OF THE LITERATURE

Introduction

Several articles have been reviewed that either address the bed allocation problem directly or use the result of some scheme to assign beds to services when considering the overall admissions scheduling system. However, none of the research has revealed an analytic treatment of the problem of allocating beds to services at a minimum penalty to the hospital. Bed requirements have largely been resolved through forecasting (using regression analysis), analyzing the daily census under assumed probability distributions or through the use of average demands, without a treatment of optimization.

The purpose of this chapter is to present the results of the survey of the literature and to describe the techniques which have been used previously to distribute beds within a hospital. The discussion will be presented in three parts, where articles are grouped that deal with bed allocation techniques, admissions and scheduling and a stochastic model for the behavior of the daily census, respectively.

Bed Allocation Techniques

Blumberg [1] and Weckwerth [25] have assumed a Poisson distribution for the daily census in order to arrive at predictions of bed needs. Blumberg defines distinctive patient facilities, DPF's, as different nursing units in a hospital, where similar care is required

by patients with like illnesses. He has developed a table, based upon Poisson distributed daily censuses, that permits the determination of the number of beds (for a DPF) that will result in a completely occupied facility on an average of one-day in 10, one-day in 100 or one-day in 1,000.

Similarly, Weckwerth formulates the proportion of time that an exact number of beds are full using the Poisson distribution as follows:

$$P(b) = \frac{(ADC)^b \, e^{-ADC}}{b\,!} \qquad (2\text{-}1)$$

where:

P(b) = proportion of the time that exactly b beds are full given an ADC

ADC = average daily census

Then, the following formula is applied:

$$D = T \sum_{b} P(b)$$

where:

D = the number of days for the time interval T (usually 365 days)

$\sum_{b} P(b)$ = overall proportions of time that exactly b or more beds are full.

A table of values, generated from the formula 2-1, is provided to indicate the number of days per year that a hospital might have an insufficient number of beds to satisfy demands. Thus, according to Schonick [21, p. 1483],

> "Blumberg and Weckwerth are examples of advocates of using the Poisson distribution for planning the size of a hospital facility to accommodate a given average daily census with a predetermined probability of overfill."

As will be seen in the section entitled Queueing Model in Chapter 3, a consideration of multiple hospital services implies the use of an M/M/c queueing model. The probability that the system is full is not Poisson distributed for the problem under consideration in this thesis.

Goldman, Knappenberger and Eller [7] have used a computer simulation model to analyze bed allocation policies for:

1) Beds to services, where beds were allocated to restricted services (pediatrics, obstetrics and intensive care) to meet the demand 95%, 85% and 75% of the time, with the remaining beds available to unrestricted services (medical, surgical, orthopedic and gynecology) on the basis of average demand.
2) All beds to private rooms.
3) Beds to private rooms, semi-private rooms and four-bed wards in accordance with average room type demand.
4) Assigning the largest possible number of beds to four-bed wards and any remaining beds to private and semi-private rooms.

The technique used was to test the orthogonal sets of differences between main effect means using the variance estimates obtained through replication. Upon simulating 365 days of operation they observed:

1) No significant effect on average annual total bed utilization was observed for any of the policies tested.

2) The beds to service allocation policies had a significant effect on the average annual total of patient days waited (where patient days waited is assumed to mean waiting for admission).
3) At high levels of bed utilization, satisfaction of demand in the unrestricted services a relatively large per cent of the time resulted in extremely high waiting time; hence, such a policy is unwise under these conditions.
4) At lower levels of bed utilization, the allocation policy had no apparent effect on waiting time.

It was concluded that none of the policies considered independently could be declared generally superior on the basis of the data reported.

A multiple regression model for predicting future bed needs for different types of hospital demands has been developed by Brooks and Beenhakker [2]. They first selected 117 possible independent variables and then determined the relationships that existed with the dependent variable (number of patients) for 19 different case classifications (i.e., obstetrics, newborn, medicine, cardiology, etc.). This resulted in selecting a multiple regression equation for each case classification containing from two to five of the most important independent variables for predicting bed needs.

In another study, Thompson and Fetter [24] considered the extent to which the size of the obstetrical service (as measured by number of patients discharged) affects its average occupancy, for the purpose of predicting maternity service requirements. Using a simulation model

with the distributions of occupancy and service times as input, and varying the numbers of admissions per year, the number of beds required were predicted for the maternity service at 90%, 95% and 99% service levels. Some of their results were:

1) Bed investment costs per 100 patients served level off at 4000 admissions per year, but become increasingly higher below this level.
2) The number of labor, delivery and post partum facilities required as a function of admission rate is greater in investment requirements for the lower admission rates.
3) In considering three hospitals located within approximately seven miles of each other, they found that only one maternity unit was required at the same rate of service. This would not only result in substantial personnel savings, but fewer beds with a higher occupancy level.

Although the article reviewed had little analytical content, the results achieved imply some consideration of optimality.

Admissions and Scheduling

In his article on "Scheduling Elective Admissions", Dunn [5] predicts trends for total bed needs through linear regression, based upon the number of beds required per week, where the number of beds required were computed from the formula:

(the number of general adult deaths and discharges that occur in any day) - (the number of patients in the emergency unit at the beginning of the day) + (the number of empty general adult beds at the beginning of the day).

Fourteen sets of data representing, for example, 105 weeks -- Sunday through Saturday, 105 weeks -- Sunday through Friday, Seasonal, 63 weeks -- Sunday through Friday, 63 weeks -- Sunday, 63 weeks -- Monday, etc., were analyzed and the means were tested for significance using the Duncan Multiple Range Test.

Robinson, Wing and Davis [19] have used simulation to investigate six admission scheduling systems. A fixed number of beds was assumed, classified according to accommodation type (private, semi-private, ward) and with consideration for desired accommodations and personal characteristics (sex, medical isolation needs, etc.).

Kolesar [13] indicated that the purpose of his paper was "to discuss some mathematical approaches to the problem of prescheduling elective admissions and to propose a new Markovian decision model for treating the problem". Subsequently, he presents two queueing models developed by John P. Young of Johns Hopkins University that are applicable to a hospital ward with a finite number of parallel service facilities (beds) with two parallel input streams (one corresponding to elective or scheduled admission and the other for emergency or mandatory admissions).

The first is a rate control model, where the only control exercised on the system is the specification of the input rate for scheduled

admissions. For this model, the service time (occupancy of a bed) is a random variable with an exponential distribution. Scheduled admissions are assumed to be from an L-phase Erlang process and the input stream for emergency admissions is considered to be Poisson. The formulae for the probability of the number in the system and for the mean and variance of the number of occupied beds is then presented. This is very similar to the results achieved from the data analysis in Chapter 3, where combined arrivals are shown to be Poisson and occupancy of a bed is negative exponentially distributed.

The other model, described as adaptive control, assumes that the emergency stream is random while the scheduled input stream is deterministic and depends upon the number of beds occupied. Holding time is again assumed to be exponentially distributed. Emergency arrivals are represented by a Poisson input stream and scheduled arrivals are assumed to constitute a deterministic stream, dependent upon the following:

> As long as B or more of the M available beds are occupied no arrivals are scheduled. Whenever occupancy drops below B, arrivals are scheduled instantaneously, to bring the occupancy level up to B.

Formulae are also presented for the probability of the number in the system, and the expected value and variance of the number of beds occupied for this model.

For the Markovian decision model, Kolesar develops the necessary relationships for the number of beds occupied, non-scheduled arrivals,

discharges, transfers and scheduled arrivals in a given time period.

Optimization methods using linear programming are then presented for the maximization of average occupancy with an overflow constraint and the minimization of overflow with constraints on utilization, where the objective function is to maximize or minimize, respectively, the probability of beds being occupied. This basic model is extended to include scheduling admission reservations over a planning horizon of at least one week and to accommodate simultaneous scheduling of several hospital services in the same ward.

Stochastic Model

A stochastic model for the behavior of the daily census in general acute hospitals is presented by Shonick [21]. The motivation for his work stems from the need for hospitals to optimally utilize their bed resources. He states:

> "One of the important economic characteristics of hospital operation is the relatively large proportion of cost which remains fixed in the face of variability in the percentage of beds occupied. The increment to the cost of maintaining a bed which is occupied is comparatively small compared with the indirect and fixed costs involved in supporting this bed with the entire complement of personnel, equipment and plant which modern medicine requires. Income, on the other hand varies directly with the number of occupied beds; a vacant bed produces no revenue."

Thus, he indicates that the goal of hospital planning might well be to:

> "1) Keep the expected percentage occupancy high.
>
> 2) Keep the expected proportion of time the facility will be full low. Two corollary criteria to this point are:

a) Minimize the expected number of emergency patients who are compelled to seek facilities other than those of their first choice ('lost patients').

b) Minimize the time elective patients who have to wait for admission and the size of the queue or waiting lists.

3) Keep the travel time and distance to the hospital for patients, visitors and attending physicians small."

The minimization of patients turned away, as alluded to in 2a) above, will be considered in this thesis, where the number of "lost patients" will be minimized for both elective and emergency arrivals.

Upon defining the queueing parameters and relationships as a Poisson process Shonick proceeds to formulate the following elements of the stochastic model:

1) Distribution for the number in the system

2) Expected value of the number in the system

3) Distribution for the daily census

4) Mean and variance of the daily census

5) Distribution of the waiting list or queue size

6) Probability that an arriving elective patient has to wait

7) Mean waiting time

8) Expected value of the waiting time for patients who do wait.

Some implications of the model for hospital planning are then examined in detail by considering applicable distributions of the daily census for the cases of infinite and finite numbers of beds. For the latter case, upon considering a Poisson versus a pseudo-Poisson distribution for the census (beds required) he concludes the following:

"If g(c) (the derived distribution of the census) holds then attempting to fit a pseudo-Poisson with γ (the average daily demand generated by elective and emergency arrivals) equal to the observed mean would be expected to give the following results:

1) When P_N (the probability of the number in the system) is close to 0 (say under .1) -- overstate the observed frequencies for c (census) less than s (the number of beds) and understate the observed frequency with which c = s.

2) When P_N is of small to moderate size (about .1 to .5) -- understate the observed frequencies for $c < s$ and overstate the observed frequency with which c = s.

3) When P_N is large (about .6 to 1.0) -- produce erratic results. The 'true' pseudo-Poisson would behave as in (2) above, but the shift to the left due to under estimating γ could produce a sharpened understatement of the middle frequencies and neutralize the understatement of the low frequencies."

Shonick then considers the optimization of the two principal and mutually contradictory criteria of maximizing the expected occupancy ratio and minimizing the expected proportion of the time the facility will be full. Finally, he compares the required bed estimates under:

a) the Poisson method

b) the square root formulae (derived by the Commission on Hospital Care, New York)

c) the distribution of bed census that he derived.

Summary

In summary, this review of applicable studies has shown that methods for determining bed needs have generally been relegated to:

1) Forecasting for groups of beds based upon occupancy related variables such as patients length of stay and the daily census. [1] [25]
2) A multiple regression technique for predicting beds per classification. [2] [5]
3) Analyzing the effects of different bed assignment policies with the technique of Orthogonal Sets of Differences Between Main Effect Means. [7]
4) Prediction of total bed requirements based upon the distributions of arrivals, service times and occupancy. [13] [21] [24]
5) Hospital management decisions. [19]

While significant contributions have been made, none of the research has addressed the problem of optimally allocating beds to the hospital services considering the relationship of patient arrivals, lengths of stay and the minimum penalty for turning patients away from the hospital. These relationships are the prime consideration of this thesis.

Chapter 3

DATA ANALYSIS

Introduction

In Chapter 1 the bed allocation problem was formulated mathematically as the minimization of the expected daily penalty for turning patients away from their desired service within a hospital. It was noted that the values of the terms in the objective function, $P_j(c_j)$, were to be obtained from a queueing model of the service. Before the necessary queueing model can be developed, it is necessary that decisions be reached concerning the probability distributions for the arrivals of patients and their lengths of stay for each of the four services.

This chapter has a two-fold purpose. First, the form of the probability distributions for arrivals and lengths of stay for the four hospital services are to be established based upon 31 days of data collected from the Radford Community Hospital, Radford, Virginia. Second, a technique will be provided for assigning values to the weighting factors. While these factors are not an integral part of the queueing model, the presence of the discussion here simply brings the third of the three required parameters together in the same chapter.

The patient arrival and corresponding length of stay data, as summarized in Appendix A, will be examined by applying Kolmogorov-Smirnov tests to determine if the associated distributions correspond to the Poisson process. The data will also be analyzed to determine

if fluctuations exist by day of the week. Significant variations for any day(s) would imply that different parameters for the mean arrival and/or length of stay rates should be used in the queueing model for the applicable day(s). Thus, the stability of these rates will be tested through analyses of variance.

For convenience, the average number of patients which arrive per day for service j will be denoted as λ_j and the reciprocal of the average length of stay for a patient in service j will be denoted as μ_j. Hereafter, these quantities are referred to as the arrival rate and length of stay rate for service j.

The order of discussion in this chapter will be as follows. First, the arrival data are analyzed using Kolmogorov-Smirnov goodness-of-fit tests. Second, the length of stay data are analyzed. Next, the stability of the arrival data and length of stay data are tested. Then, weighting factor assignments are discussed. Based on the results of the analyses, an appropriate queueing model is then chosen to provide the values of $P_j(c_j)$. Since the queueing analysis is greatly simplified when arrivals are Poisson distributed and lengths of stay are exponentially distributed, the data are compared against these distributions.

Arrival Data

The Kolmogorov-Smirnov one sample goodness-of-fit test was chosen to analyze arrival data, since it is desired to compare the observed data having the characteristic of an interval scale with the Poisson distribution. Also, according to Siegel [22, pg. 51], this test is more powerful than the alternative Chi-Square one sample test.

The null hypothesis, H_0, to be tested is that the arrival data for each of the four services is Poisson distributed. Consequently, the alternative hypothesis, H_1, is that the data is not Poisson distributed. Therefore, a two tailed test is specified. The probability of rejecting H_0 when it is true, α, was selected to be .05, since 5% of the entire space under the curve is assumed to be small enough to adequately register significance if a value falls in this region. The maximum deviation, D, is computed as D = maximum $| F_\chi(x_i) - S(i) |$, where $F_\chi(x_i)$ is the Poisson comulative distribution function and S(i) is the observed cumulative frequency distribution from the sample. Under the Kolmogorov-Smirnov test, the region of rejection consists of all values of D which are so large that the probability associated with their occurrence under H_0 is equal to or less than α, where α = .05 in this test. The number of days observed, N, is equal to 31.

For analysis purposes, the actual data were first grouped into cells representing ascending arrival rates, and applicable frequencies noted for each group. The cumulative distribution for the sample was then calculated as follows:

Let:

n = number of cells

i = cell number = 1, 2, ..., n

f_i = frequency of occurrence of values within the i^{th} cell,

t = total of all frequencies = $\sum_i f_i$

S(i) = observed cumulative frequency distribution value for the i^{th} cell.

$$= \frac{\sum_{j=1}^{i} f_j}{t}$$

The calculated values for the cumulative distribution function for the Poisson distribution were obtained from the expression

$$F_X(x_i) = \sum_{x=0}^{x_i} \lambda_j^x e^{-\lambda_j} / x !$$

where x_i = the upper limit of the i^{th} cell

λ_j = the mean arrival rate in patients per day for service j during the 31 day period

As an example of applying the Kolmogorov-Smirnov test, values for the observed and calculated cumulative frequency distribution functions, S(i) and $F_X(x_i)$ respectively, are shown in Table 3.1. The maximum deviation, D = .0676, occurs for the Cell Assignment for 9 to 11 arriving patients per day. Thus, it is desired to determine whether the probability of occurrence for this value of D falls in the rejection region specified by α. From Table E in Siegel [22, pg. 251], for sample size N = 31, and D = .0676, the probability of occurrence under H_0 is $p > .20$. Therefore, since the probability p is greater than α, we accept H_0 and conclude that the distribution of arrivals for the medical service is Poisson.

Upon applying the same test to arrival data for the remaining three services, results were achieved as shown in Table 3.2. As can be observed, all probabilities of occurrence were greater than .20,

TABLE 3.1

MEDICAL SERVICE ARRIVAL DATA DISTRIBUTION ANALYSIS

$\lambda_1 = 9.2580$

Cell Assignment	Cell Frequency	S(i)	$F_X(x_i)$	$\lvert F_X(x_i) - S(i) \rvert$	D
0 - 2	0	0	.0051	.0051	
3 - 5	5	5/31= .1613	.1009	.0604	
6 - 8	8	13/31= .4194	.4222	.0028	
9 - 11	9	22/31= .7097	.7773	.0676	.0676
12 - 14	7	29/31= .9355	.9496	.0141	
15 - 17	2	31/31=1.0000	.9930	.0070	
Total	31				

TABLE 3.2

SUMMARY OF KOLMOGOROV-SMIRNOV TEST RESULTS
ON ARRIVAL DATA FOR HOSPITAL SERVICES

Service	λ_j	D_j	Probability of Occurrence under H_0	Decision
Medical	9.2580	.0676	$p > .20$	Accept H_0
Surgical	2.9340	.0514	$p > .20$	Accept H_0
Ob/Gyn	4.9030	.1089	$p > .20$	Accept H_0
Pediatrics	2.3550	.0315	$p > .20$	Accept H_0

and, thus the distributions of arrivals for the remaining services are Poisson.

Length of Stay

Data for this analysis were accumulated by the number of days that each arriving patient remained in the hospital. The null hypothesis, H_0, to be tested is that the observed data are negative exponentially distributed. The Kolmogorov-Smirnov one sample test was chosen to compare lengths of stay to the negative exponential distribution, since the data are interval scale. The significance level, α, was selected to be .05. The number of samples per service, N, is shown below:

Service	N
Medical	287
Surgical	91
Ob/Gyn	152
Pediatric	73

The region of rejection consists of all values of D, calculated as in the previous section, which are so large that the probability associated with their occurrence under H_0 is equal to or less than $\alpha = .05$.

For analysis purposes, the lengths of stay data as measured in days for each service were grouped into cells and corresponding frequencies of occurrence were recorded. The observed cumulative frequency distribution value for the i^{th} cell, S(i), was calculated as in the previous section and $F_T(x_i)$, the CDF for the negative exponential

distribution was determined for each cell from the following formulation:

Let

n = number of cells

i = cell number = 1, 2, ..., n

x_i = the upper limit of the i^{th} cell

μ_j = length of stay rate for j type patients; j = 1,2,3,4.

$F_T(x_i)$ = cumulative distribution function for the negative exponential distribution

$$= \int_0^{x_i} \mu_j e^{-\mu_j t} dt$$

$$= 1 - e^{-\mu_j x_i}$$

An example of organizing the data for testing the distribution of lengths of stay is shown in Table 3.3 for the surgical service. The maximum value of D can be observed to occur for the Cell Assignment for 7 - 9 days per patient.

From Table E in Siegel [22, pg. 251], for $N > 35$ and $D = .1079$, the probability of occurrence, p, is greater than .20. Thus, the null hypothesis is accepted and we conclude that the surgical length of stay data is negative exponentially distributed. The results of performing the same tests on length of stay data for the remaining services are shown in Table 3.4.

TABLE 3.3

SURGICAL SERVICE LENGTH OF STAY DATA DISTRIBUTION ANALYSIS

$\mu_2 = .1537 \quad N = 91$

Cell Assignment	Cell Frequency	S(i)	$F_T(x_i)$	$\lvert F_T(x_i) - S(i) \rvert$	D
1 - 3	29	.3187	.3694	.0507	
4 - 6	32	.6703	.6024	.0680	
7 - 9	17	.8571	.7493	.1079	.1079
10 - 12	7	.9341	.8419	.0922	
13 - 15	2	.9560	.7003	.0558	
16 - 18	2	.9780	.9371	.0409	
> 19	2	1.0000	.9604	.0396	
Total	91				

TABLE 3.4

SUMMARY OF KOLMOGOROV-SMIRNOV TEST RESULTS ON LENGTH OF STAY DATA FOR HOSPITAL SERVICES

Service	μ_j	D_j	Probability of Occurrence	Decision
Medical	.1010	.0770	$.05 < p < .10$	Accept H_0
Surgical	.1537	.1079	$p > .20$	Accept H_0
Ob/Gyn	.2249	.0992	$.05 < p < .10$	Accept H_0
Pediatrics	.2500	.0614	$p > .20$	Accept H_0

Stability of Arrival and Length of Stay Data

Stability of a queueing system implies that the mean arrival and service rates are applicable over the entire time period of interest. In the context of this application, we are interested in examining the arrival and length of stay data for each of the four hospital services. If no significant variation of these data exists over a given time period, then the same mean arrival and service rates can be applied over the entire period. A complete examination to determine stability would include an analysis of daily, weekly, monthly and seasonal variations. For purposes of demonstration, this research includes only an evaluation of variations between days of the week, for the 31 days of data collected.

The technique selected for examining the arrival and length of stay data was an analysis of variance for the completely randomized one-factor design. The completely randomized one-factor design is applicable since each mean arrival or length of stay rate per service has an equal probability of occurring on any day of the week. Also, there is only one independent variable, the mean arrival or service rate.

Arrival data (in patients per day) and length of stay data (in days per patient) were grouped by days of the week for each service. Daily average arrival and length of stay rates were then computed. The null hypothesis tested was that the average arrival or length of stay rates for each service were equal for each day of the week. The rejection region, α, was selected as .05. Resulting ANOVA tables for arrival and length of stay data for each of the four services are shown in

Table 3.5 to 3.12. Program BMD01V of the Biomedical Computer Program package [27] was utilized for the analysis of variance calculations.

From Table 4.5 in Myers [17, pg. 384], for α = .05 with 6 and 24 degrees of freedom, values of F greater than 2.51 will result in the rejection of H_0. Thus, in observing Tables 3.5 and 3.7 to 3.12 it can be observed that the computed F ratios are less than the "critical value" of F, 2.51, and thus the null hypothesis can be accepted for these arrival and length of stay data. In Table 3.6, the calculated F ratio is greater than the "critical value", meaning that the average arrival rates for the surgical service are significantly different. However, since the difference between the calculated F ratio and the "critical value" (2.55 - 2.51 = .04) is equivalent to only .023 % of the area under the curve for the ratio of mean squares (F), the difference will be assumed negligible. Therefore, for convenience, the null hypothesis will be accepted for the surgical arrival data also.

In these analyses it has been shown that the Radford Hospital arrivals are Poisson distributed and lengths of stay are negative exponentially distributed over the 31-day period. Thus, it can be concluded that these data were generated from a Poisson process. Also, the stability of the process has been demonstrated. Therefore, λ_j and μ_j are representative of the entire period analyzed. For a longer time span, e.g., a year or greater, it is suspected that significant variations would be detected in arrival and/or length of stay data between months or seasons. This implies that the values of the parameters λ_j or μ_j would have to be adjusted for use in a queueing model to reflect differences during the period(s) of variation.

TABLE 3.5

ANOVA -- MEDICAL ARRIVAL DATA

Source of Variance	Degrees of Freedom	Sum of Squares	Mean Square	F Ratio
Between	6	62.4854	10.4142	0.7824*
Within	24	319.4497	13.3104	
Total	30	381.9351		

*$p > .20$

TABLE 3.6

ANOVA -- SURGICAL ARRIVAL DATA

Source of Variance	Degrees of Freedom	Sum of Squares	Mean Square	F Ratio
Between	6	57.5208	9.5868	2.5466*
Within	24	90.3499	3.7646	
Total	30	147.8708		

*$.025 < p < .05$

TABLE 3.7

ANOVA -- OBSTETRICS/GYNECOLOGY ARRIVAL DATA

Source of Variance	Degrees of Freedom	Sum of Squares	Mean Square	F Ratio
Between	6	9.7597	1.6266	0.2257*
Within	24	172.9500	7.2062	
Total	30	182.7096		

*$p > .20$

TABLE 3.8

ANOVA -- PEDIATRICS ARRIVAL DATA

Source of Variance	Degrees of Freedom	Sum of Squares	Mean Square	F Ratio
Between	6	27.3467	4.5578	1.6146*
Within	24	67.7500	2.8229	
Total	30	95.0967		

*$.20 < p < .10$

TABLE 3.9

ANOVA -- MEDICAL -- LENGTH OF STAY DATA

Source of Variance	Degrees of Freedom	Sum of Squares	Mean Square	F Ratio
Between	6	66.6982	11.1164	1.8902*
Within	24	147.4632	6.1443	
Total	30	214.1614		

*.20 < p < .10

TABLE 3.10

ANOVA -- SURGICAL -- LENGTH OF STAY

Source of Variance	Degrees of Freedom	Sum of Squares	Mean Square	F Ratio
Between	6	320.8159	53.4693	1.3947*
Within	24	920.1165	38.3382	
Total	30	1240.9324		

*$p > .20$

TABLE 3.11

ANOVA -- OB/GYN -- LENGTH OF STAY DATA

Source of Variance	Degrees of Freedom	Sum of Squares	Mean Square	F Ratio
Between	6	10.3804	1.7301	1.0980*
Within	24	37.8169	1.5757	
Total	30	48.1973		

*$p > .20$

TABLE 3.12

ANOVA -- PEDIATRICS -- LENGTH OF STAY DATA

Source of Variance	Degrees of Freedom	Sum of Squares	Mean Square	F Ratio
Between	6	20.9954	3.4992	0.6011*
Within	24	139.7198	5.8217	
Total	30	160.7153		

*$p > .20$

Weighting Factor Assignment

Values for the weighting factor, w_j, must be assigned for each service. This parameter represents a subjective judgment, on the part of a hospital administrator, of the penalty for turning patients away from one of the services. The criteria for selecting the parameter values may be based upon related hospital operating costs, e.g. the cost of an empty bed, staff costs, etc., the severity of illnesses usually associated with the type services, or historical demand. As an example of assigning w_j's, consider a given hospital that has a very high rate of demand, λ_j / μ_j, for the medical service facilities, as compared to the other services. The demand for surgical requirements might be next in order, followed by obstetrics/gynecology and pediatrics. Thus, the weighting factors can be ranked in preferential order w_1, w_2, w_3, w_4, representing medical to pediatrics services, respectively. At this point, the administrator could assign subjective values to each of the factors, say, 1.00, .50, .30 and .20, corresponding to the ranking.

The assignment of initial values to the weighting factors must result from some measurement or scaling criteria. Since the weighting factors are to be ranked in preferential order and, in addition, some judgement will be applied to assign magnitude to the differences between the factors, the assigned values will conform to the interval scale of measurement. Thus, with a knowledge of the magnitude of the differences between the values, an administrator can assign values to the factors starting with any real or integer number, as long as he is

consistent. For example, while 1.00, .50, .30 and .20 were assigned above, the set 200, 100, 60 and 40 could be used since the individual elements comprise the same proportion of the sum of the elements.

Ranking and considering the differences between values only do not provide a complete method for assigning values to the weighting factors. Consistency in assignments and the ability to detect any errors in judgement with respect to the initial values assigned must also be considered. One systematic and organized way of checking these criteria is provided through the objective weighting technique as discussed in Morris [16, pg. 167] and originated by Churchman and Ackoff [3]. Using this technique, an administrator would assign initial values to the factors based upon individual preferential order and required differences between the values. Then, he would examine the ranking and magnitude of the numbers assigned for consistency. For example if, as indicated above, the ranking order for weighting factors is assigned as $w_1 > w_2 > w_3 > w_4$, this implies that w_1 must exceed w_2 by some value, w_2 must exceed w_3 by some value, etc. The values 1.00, .50, .30 and .20 are consistent with the ranking. If any of the values did not correspond to preference, the appropriate values would be adjusted up or down as necessary.

Another method that can be used in objective weighting is to let individual penalties for turning patients away be greater than the total of the penalties for the remaining services. For example, w_1 may be preferred over $(w_2 + w_3 + w_4)$, w_2 may be preferred over $(w_3 + w_4)$, w_3 may be preferred over w_4. In using the above data, the administrator

would compare the value of w_1, 1.00, to the combined values of w_2, w_3, and w_4, i.e., (.50 + .30 + .20) and adjust the value of w_1 upwards if it is less than or equal to the combined sum. Next, the adjusted value of w_2 would be compared to the value of w_3 and w_4 combined, i.e., (.30 + .20) and adjusted as necessary. Finally, w_3 would be compared to w_4. Thus, this process of weighting might result in the assignments:

$$w_{medical} = 1.05$$
$$w_{surgical} = .55$$
$$w_{ob/gyn} = .30$$
$$w_{pediatrics} = .20$$

While the above methods imply that the preferred values must be "greater than" the other values in order of ranking, the process could just as easily be applied for "less than" conditions. In the latter case, the logic for adjustments would, of course, be reversed.

The application of the objective weighting technique is versatile. While only two examples of use have been discussed here for assigning weighting factors, other techniques may be used to order the values according to requirements and/or imagination.

Queueing Model

Based on the results of the data analysis, a queueing model must be developed to provide values of $P_j(c_j)$ for the case of Poisson arrivals and negative exponential lengths of stay. Employing the

classification scheme introduced by Kendall [12] and expanded by Lee [14] and Taha [23], a queueing model of the type: (M/M/c: GD/c/∞) is required where, in order of occurrence, M denotes Markov (Poisson) arrivals, M denotes Markov (negative exponential) lengths of stay, c denotes the number of beds per service, GD indicates patients are served according to any general service discipline, c is the maximum number of patients allowed in the service, and the population size is assumed to be infinitely large.

Fortunately, the probability that an (M/M/c: GD/c/∞) queue is full is a known result [23], and is given as

$$P_j(c_j) = \left[\frac{(c_j\rho_j)^{c_j}}{c_j!}\right]\left[1 + \sum_{n_j=1}^{c_j} (c_j\rho_j)^{n_j} / n_j!\right]^{-1} \qquad (3\text{-}1)$$

where $\rho_j = \lambda_j / c_j\mu_j$. It is noted that Equation (3-1) is often referred to as Erlang's Loss Formula [4, pg. 47].

It has been conveniently assumed that steady state conditions exist. It is felt that this is a reasonable assumption as long as λ_j and μ_j are stationary. In the event that λ_j and μ_j change significantly over time, one might solve the bed allocation problem under steady state conditions for various values of λ_j and μ_j and ascertain the effects on the bed allocations. To explicitly account for non-steady state conditions is an extremely difficulty undertaking, and would necessitate a reformulation of the bed allocation problem.

Summary

In this chapter, typical hospital data have been analyzed to ascertain the underlying probability distributions for parameters that are essential to this problem solution. A Poisson process was verified and the stability of arrival and length of stay data was determined over the 31-day period of interest. Given these results and requirements to consider a Markov process for a multiple server, finite population system, a queueing model of the type (M/M/c: GD/c/∞) will be utilized to provide the probability that the system is full, given c_j beds for service j, where j = 1, 2, 3, 4. It now remains to formulate the optimization model for allocating beds to the medical, surgical, obstetrics/gynecology and pediatrics services. This is accomplished in Chapter 4.

Chapter 4

OPTIMIZATION PROCEDURE

Introduction

The problem addressed by this research is to optimally allocate the bed resources within a hospital. Hospitals of varying sizes are to be considered, each having a given total number of beds, C, to be distributed among the medical, surgical, obstetrics/gynecology and pediatrics patient services.

The mathematical model which expresses a measure of effectiveness for evaluating different allocations was given previously as:

$$\underset{c_j}{\text{minimize}}\ f(C) = \sum_j \lambda_j w_j P_j(c_j)$$

$$\text{subject to}\ \sum_j c_j \leq C, \text{ and } c_j \geq 0,$$

where it is desired to minimize the penalty for turning patients away from each of the services. In Chapter 3, the M/M/c queueing model was justified for providing values of $P_j(c_j)$, the probability that the system is full, given c_j beds for service j. It is now of interest to determine how the measure of effectiveness can be applied to aid in the decision making for optimally assigning beds to services.

The problem can be solved by dynamic programming where four interrelated decisions are to be made, namely, how many beds should be allocated to each of the four services. Therefore, the stages may be

identified as the medical, surgical, obstetrics/gynecology and pediatrics services (in any order). The dynamic programming states may be represented as the number of beds unallocated at each stage. The solution technique will convert this multi-stage decision process into a series of single stage problems with policy decisions required at each stage. An optimal set of allocations will result, based upon the property that regardless of the number of beds allocated to the first service, remaining allocations must be optimal with respect to the outcome which results from the first and successive decisions. This is the dynamic programming principle of optimality [18].

Dynamic Programming Formulation

The following notation is employed for this dynamic programming model:

j = stage variable representing the services of the hospital, $j = 1, 2, 3, 4$.

λ_j = arrival rates of j type patients.

μ_j = length of stay rate for j type patients.

$P_j(c_j)$ = probability that the system is full, given c_j beds for service j.

$$= [(1 / c_j!)(\lambda_j / \mu_j)^{c_j}][1 + \sum_{n=1}^{c_j} (1 / n!)(\lambda_j / \mu_j)^n]^{-1}$$

w_j = weighting factor for turning patients away from service j

x_j = state variable, number of beds unallocated at stage j

$r_j(x_j,c_j)$ = individual stage return, measured as the penalty for turning patients away from service j

$= w_j \lambda_j P_j(c_j)$

$f_j^*(x_j)$ = minimum return through stage j as a function of x_j

$= \min_{c_j} f_j(x_j,c_j)$

$f_1^*(x_1) = r_1(x_1,c_1)$

The dynamic programming formulation for this problem is given by the following recursive relation:

$$f_j(x_j, c_j) = r_j(x_j, c_j) + f_{j-1}^*(x_{j-1})$$

where

$$x_{j-1} = (x_j - c_j).$$

To apply the dynamic programming formulation, a user must first decide upon the total number of beds to be allocated among the four hospital services. Then, values of the mean arrival rate, λ_j (in patients per day), and the mean length of stay rate, μ_j (measured in patients per day), must be determined for each service. The value of the mean arrival rate per service may be determined by summing the numbers of patients that are admitted and dividing by the total number of days that the data represents. The value of the mean length of stay rate may be calculated for each service by dividing the total number of arriving patients by the corresponding total number of days that each

arriving patient remains in the hospital. The weighting factors, w_j may be assigned as shown in Chapter 3.

Example Problem

In this section, an example problem is presented to demonstrate how dynamic programming is used to allocate beds to the services. The total number of beds to be allocated was deliberately chosen to be a small number. The number of services was selected to be three, and λ_j / μ_j was allowed to be the same for all j to facilitate ease of presentation.

Let C, the total number of beds to be allocated, equal 10, with the other pertinent data given in Table 4.1.

Stage 1 Calculations

The probability that the system is full, given c_j beds for service j has been given in Equation 3-1 as

$$P_j(c_j) = [(1 / c_j!)(\lambda_j / \mu_j)^{c_j}][1 + \sum_{n=1}^{c_j} (\lambda_j / \mu_j)^n (1 / n!)]^{-1}.$$

Therefore,

$$P_1(c_j = 0) = \frac{1/0!\ (6/3)^0}{1 + \sum_{n=1}^{0} (6/3)^n (1/n!)} = \frac{1}{1 + 0} = 1$$

TABLE 4.1

PARAMETERS FOR AN EXAMPLE PROBLEM

Service 1 Parameters	Service 2 Parameters	Service 3 Parameters
$\lambda_1 = 6.0$	$\lambda_2 = 4.0$	$\lambda_3 = 6.0$
$\mu_1 = 3.0$	$\mu_2 = 2.0$	$\mu_3 = 3.0$
$w_1 = 0.6$	$w_2 = 0.3$	$w_3 = 0.1$

$$P_1(c_j = 1) = \frac{1/1!\ (6/3)^1}{1 + \sum_{n=1}^{1} (6/3)^n (1/n!)} = \frac{2}{1+2} = \frac{2}{3}$$

.
.
.

$$P_1(c_1 = 10) = .000038$$

The individual stage return formulation is given as

$$r_j(x_j, c_j) = w_j\ \lambda_j\ P(c_j).$$

Therefore,

$$r_1(x_1, 0) = w_1\ \lambda_1\ P(c_1 = 0)$$

$$= (0.6)\ (6.0)\ (1) = 3.6$$

$$r_2(x_1, 1) = (0.6)\ (6.0)\ (2/3) = 2.4$$

.
.
.

$$r_1(x_1, 10) = (0.6)\ (6.0)\ (.000038) = .0001$$

The complete Stage 1 formulation is shown in Table 4.2, with minimum return values indicated under the column $f_1^*(x_1)$ and the number of beds to be allocated under column c_1^*. Note that in this stage, $f_1(x_1)^*$ is equal to $r_1(x_1, c_1 = x_1)$, since $r_1(x_1, c_1)$ monotonically decreases with increasing x_1.

TABLE 4.2

EXAMPLE PROBLEM -- STAGE 1

x_1	$f_1^*(x_1)$	c_1^*
0	3.6000	0
1	2.4000	1
2	1.4400	2
3	0.7579	3
4	0.3429	4
5	0.1321	5
6	0.0435	6
7	0.0124	7
8	0.0031	8
9	0.0007	9
10	0.0001	10

Stage 2 Calculations

Since $\lambda_2/\mu_2 = \lambda_1/\mu_1$, the probabilities that the system is full are also equivalent, i.e., $P_2(c_2) = P_1(c_1)$. Therefore, the individual stage returns, $r_2(x_2, c_2)$ can be shown as follows:

$$r_2(x_2, c_2) = w_2 \lambda_2 P(c_2)$$
$$r_2(1,0) = (1.2)(1) = 1.20$$
$$r_2(1,1) = (1.2)(2/3) = 0.80$$
$$\vdots$$
$$r_2(10, 10) = .00456$$

The expression $f_2(x_2, c_2) = r_2(x_2, c_2) + f_1^*(x_2 - c_2)$ determines the recursive return relationships or tableau elements for the second stage. Thus, using previously calculated values for the return, $r_2(x_2,c_2)$ and the optimal $f_1^*(x_2 - c_2)$ from stage 1, the following values are obtained:

$$f_2(0,0) = 1.20 + 3.60 = 4.80$$
$$f_2(1,0) = 1.20 + 2.40 = 3.60$$
$$f_2(1,1) = 0.80 + 3.60 = 4.40$$
$$\vdots$$
$$f_2(10, 10) = 3.60$$

Results of the stage 2 calculations are shown in Table 4.3. Columnar values of $f_2(x_2, c_2)$ represent the recursive return values, $f_2^*(x_2)$ represents the minimum return for each state x_j and c_2^* indicates the optimum number of beds to be allocated for state x_j. For example, when $x_2 = 9$, a minimum cost of 0.246 is achieved when 4 of the 9 beds are allocated to service 2.

Stage 3 Calculations

Stage 3 calculations result in Table 4.4. As shown under the column c_3^*, the minimum penalty is achieved when 2 beds are allocated to service 3. Due to the relationship, $x_{j-1} = x_j - c_j$, or, in this case,

$$x_2 = x_3 - c_3$$
$$x_2 = 10 - 2 = 8,$$

8 beds remain to be allocated. In observing Table 4.3 for an x_2 value of 8 the minimum cost is reflected when 3 beds are allocated to service 2. Thus, the remaining 5 beds ($x_1 = 8 - 3$) should be assigned to service 1.

Summary

This chapter has presented a procedure for optimally allocating beds to the four services by using a model that has the probability relationships of an (M/M/c: GD/c/∞) queueing model and a dynamic programming formulation as integral components. A computer program has been written for the model to process $C \leq 500$ beds and is documented

TABLE 4.3

EXAMPLE PROBLEM -- STAGE 2

$f_2(x_2,c_2)$

x_2 \ c_2	0	1	2	3	4	5	6	7	8	9	10	$f_2^*(x_2)$	c_2^*
0	4.800											4.800	0
1	3.600	4.400										3.600	0
2	2.640	3.200	4.080									2.640	0
3	1.958	2.240	2.880	3.853								1.958	0
4	1.543	1.558	1.920	2.653	3.714							1.543	0
5	1.332	1.143	1.238	1.693	2.514	3.644						1.143	1
6	1.244	0.932	0.823	1.011	1.554	2.444	3.615					0.823	2
7	1.212	0.844	0.612	0.595	0.872	1.484	2.414	3.604				0.595	3
8	1.203	0.812	0.524	0.385	0.457	0.802	1.455	2.404	3.601			0.385	3
9	1.201	0.803	0.492	0.296	0.246	0.387	0.772	1.444	2.401	3.600		0.246	4
10	1.200	0.801	0.483	0.265	0.158	0.176	0.357	0.762	1.441	2.400	3.600	0.158	4

TABLE 4.4

EXAMPLE PROBLEM -- STAGE 3

$f_3(x_3,c_3)$

x_3 \ c_3	0	1	2	3	4	5	6	7	8	9	10	$f_3^*(x_3)$	c_3^*
10	0.758	0.646	0.625	0.722	0.880	1.165	1.550	1.960	2.641	3.600	4.800	0.625	2

for use in Appendix B. The program may be easily modified to allocate a greater number of beds with the only trade-off being increased core requirements.

Unfortunately, the task is not completed with this development. A determination of how the model reacts to variations of the parameters for arrivals, length of stay and weighting factors must be examined through sensitivity analysis, as shown in Chapter 5.

Chapter 5

ANALYSIS OF RESULTS

Introduction

In this chapter the bed allocation model is analyzed with the objective of gaining additional insights into the bed allocation problem. To achieve this objective the sensitivity of the model to errors in estimating the model parameters is examined. Following the sensitivity analysis, the model is employed to solve bed allocation problems for five small to medium sized hospitals located in Virginia.

Sensitivity Analysis

In the preceding discussion, it has been assumed that the values of the parameters λ_j and μ_j are known exactly and that the w_j's have been adequately applied. This may not always be a realistic assumption, since errors can occur through the sampling procedures used to obtain the mean arrival and service rate values or the values may not be truly representative over the time period of interest. Likewise, the values assigned to the weighting factors may not be exact representations of the penalty for turning patients away simply because they are subjectively assigned on the part of an administrator. The use of inaccurate parameter values, either collectively or individually, will have some effect upon the number of beds allocated to services. The magnitude of this effect is of concern. If the model is very sensitive to errors in estimating parameter values, considerable time

and effort must be devoted to data collection. On the other hand, if the model is insensitive, one can approximate the parameter values.

Two measures of sensitivity will be employed. First, the effect on the optimum bed allocation due to errors in estimating parameter values will be considered. Second, the difference in the true value of f(C) and the minimum value of f(C) will be determined. Since the hospital administrator will be using bed allocation values which are based on the estimated parameter values, the resulting value of the objective function, f(C), will probably be greater than that which could have been achieved if the true parameter values had been known. Let $\hat{f}(C)$ represent the value of the objective function using bed allocation values obtained from the estimated parameter values and let $f^*(C)$ be the minimum value of the objective function based on the true parameter values. Thus,

$$\Phi = \hat{f}(C) \; / \; f^*(C)$$

represents the penalty which results due to errors in estimating parameter values.

In this section it is desired to determine how sensitive the optimization model is to variations in λ_j, μ_j and w_j and to identify which of these parameters have the most (or least) effect upon the model solutions. Thus, sensitivity analyses will be performed to evaluate the results of inaccurate parameter estimates, as well as the effects of parameter values changing over time. The method used will be to determine both the optimum bed allocation and the value of Φ for a range of values for λ_j, μ_j and w_j. Specifically, these values

will be randomly assigned, within a given range, for a large number of model iterations and the resulting effects on bed allocations and Φ will be observed.

Conditions of the Sensitivity Analysis

The true value for the parameters λ_j, μ_j, and w_j are not known. Therefore, the hospital administrator can view these true values as random variables having values distributed in some fashion about their estimated values. The resulting probability distributions for λ_j, μ_j, and w_j can be interpreted as the "estimation error" distributions. Thus, the form of the distribution is largely dependent on the abilities of the person providing the estimates of λ_j, μ_j, and w_j.

For purposes of illustrating the sensitivity of the model, it is assumed that the true value of a parameter is equally likely to be any value from 80 per cent to 120 per cent of the estimated value. The choice of a uniform distribution as a measure of the errors in estimating parameter values is motivated by the Laplace Principle of Choice [16, p. 383]. According to the Laplace principle, if there is insufficient reason to justify an assignment of unequal probabilities to the values of a random variable, then each value of the random variable is assumed to be equally likely to occur.

Additionally, the use of a uniform distribution is motivated by the fact that the sensitivity of the model will tend to be overstated. As an illustration, if the model is found to be insensitive under the assumption of a uniform distribution for parameter values, then the

model will be even more insensitive under assumptions of beta or normally distributed parameter values.

Under the uniform distribution assumption, the sensitivity analysis is conducted by randomly generating values for the desired parameter(s) and solving the bed allocation problem. Thus, through simulation, a probability distribution is developed for the optimum bed allocation and the penalty measure, Φ. Knowing these probability distributions, it is possible to make a probability statement concerning the optimality of the bed allocation using the estimated values of the model parameters.

In the ensuing sub-sections, the results of randomly generating values for λ_j, μ_j, and w_j will be discussed in terms of the variations observed in the numbers of beds allocated by service and the value of Φ. The Radford Hospital data, shown in Table 5.1, will be used as the values for the estimated parameters $\bar{\lambda}_j$, $\bar{\mu}_j$, and $\bar{w}_j$. The total number of beds, C, will be held constant at 150, since for the period that the Radford data was gathered, approximately 150 beds were assigned among the four services. Based on these estimates of the parameters, the bed allocations would be 98, 19, 27, and 6 for the four services, respectively.

Sensitivity of the Model to Arrival Rates

The sensitivity of the model to arrival rates was determined by randomly generating values of λ_j within the interval $.8\ \bar{\lambda}_j \leq \lambda_j \leq 1.2\ \bar{\lambda}_j$ for 1,000 iterations of the model. The observed differences

TABLE 5.1

ARRIVAL AND LENGTH OF STAY RATES FROM THE RADFORD HOSPITAL BY TYPE SERVICE

j	Service	$\bar{\lambda}_j$ [1]	$\bar{\mu}_j$ [1]	$\rho_j = \bar{\lambda}_j/\bar{\mu}_j$	$\bar{w}_j$ [2]
1	Medical	9.2580	0.1010	91.66	.4
2	Surgical	2.9340	0.1537	19.09	.2
3	Ob/Gyn	4.9030	0.2249	21.80	.3
4	Pediatrics	2.3550	0.2500	9.42	.1
				$\sum_j \rho_j = 141.97$	

[1] $\bar{\lambda}_j$ and $\bar{\mu}_j$ represent the "estimated" mean arrival and length of stay rates, respectively.

[2] $\bar{w}_j$ represents the "estimated" penalty factor for turning patients away from service j.

between the number of beds allocated for the estimated values of arrival rates and the randomly generated arrival rates were tabulated according to relative and cumulative frequencies of occurrence. Results are shown in Tables 5.2 to 5.5. In these tables, V represents the absolute difference in the optimal allocation of beds under estimated and randomly generated values of λ_j. Thus, if $c_j(\lambda_j)$ is interpreted as the number of beds allocated to service j given a value of λ_j, then $V = |c_j(\bar{\lambda}_j) - c_j(\lambda_j)|$. $F_j(V)$ can be designated as the cumulative distribution function for service j.

Cumulative distribution curves are shown in Figures 5-1 to 5-4 for the medical, surgical, obstetrics/gynecology and pediatrics services. From the curves, it is possible to observe the variation in the number of beds assigned by first selecting any probability of occurrence ($F_j(V)$) and observing the corresponding absolute difference in the number of beds. For example, if it is desired to determine how sensitive the model is for the medical service at a 90% probability of occurrence, when random arrival rates are assigned, Figure 5-1 reveals that the solution will vary (0 ± 12) beds from the estimated value of 98 beds. Thus, variations of 86 to 110 beds could be expected in allocations to the medical service. The composite results of variations in the medical, surgical, obstetrics/gynecology and pediatrics bed allocations due to randomly assigning λ_j within the interval at a 90% probability of occurrence are shown in the first column of Table 5.6.

The cumulative distribution for Φ was also computed from 1,000 iterations of the model, using the relation $\Phi = \hat{f}(C) / f^*(C)$, where

TABLE 5.2

MEDICAL SERVICE FREQUENCY TABLE FOR VARIATIONS OF λ_1 WITHIN THE INTERVAL $.8\,\bar{\lambda}_1 < \lambda_1 < 1.2\,\bar{\lambda}_1$

$c_1(\bar{\lambda}_1) = 98$ beds

V	Relative Frequency -- p(V)	Cumulative Frequency -- F(V)
0	.029	.029
1	.074	.103
2	.080	.183
3	.083	.266
4	.074	.340
5	.084	.424
6	.087	.511
7	.080	.591
8	.078	.669
9	.084	.753
10	.066	.819
11	.072	.891
12	.052	.943
13	.041	.984
14	.010	.994
15	.004	.998
16	.001	.999
17	.001	1.000

TABLE 5.3

SURGICAL SERVICE FREQUENCY TABLE FOR VARIATIONS OF λ_2 WITHIN THE INTERVAL $.8\,\bar{\lambda}_2 < \lambda_2 < 1.2\,\bar{\lambda}_2$

$c_2(\bar{\lambda}_2) = 19$ beds

V	Relative Frequency -- p(V)	Cumulative Frequency -- F(V)
0	.097	.097
1	.203	.300
2	.207	.507
3	.172	.679
4	.131	.810
5	.092	.902
6	.060	.962
7	.028	.990
8	.010	1.000

TABLE 5.4

OB/GYN SERVICE FREQUENCY TABLE FOR VARIATIONS OF λ_3 WITHIN THE INTERVAL $.8\,\bar{\lambda}_3 \leq \lambda_3 \leq 1.2\,\bar{\lambda}_3$

$c_3(\bar{\lambda}_3) = 27$ beds

V	Relative Frequency -- p(V)	Cumulative Frequency -- F(V)
0	.097	.097
1	.218	.315
2	.220	.535
3	.208	.743
4	.139	.882
5	.095	.977
6	.017	.994
7	.006	1.000

TABLE 5.5

PEDIATRICS SERVICE FREQUENCY TABLE FOR VARIATIONS OF λ_4 WITHIN THE INTERVAL $.8\,\bar{\lambda}_4 \leq \lambda_4 \leq \bar{\lambda}_4$

$c_4(\bar{\lambda}_4) = 6$ beds

V	Relative Frequency -- p(V)	Cumulative Frequency -- F(V)
0	.080	.080
1	.166	.246
2	.158	.404
3	.164	.568
4	.136	.704
5	.118	.822
6	.156	.978
7	.019	.997
8	.003	1.000

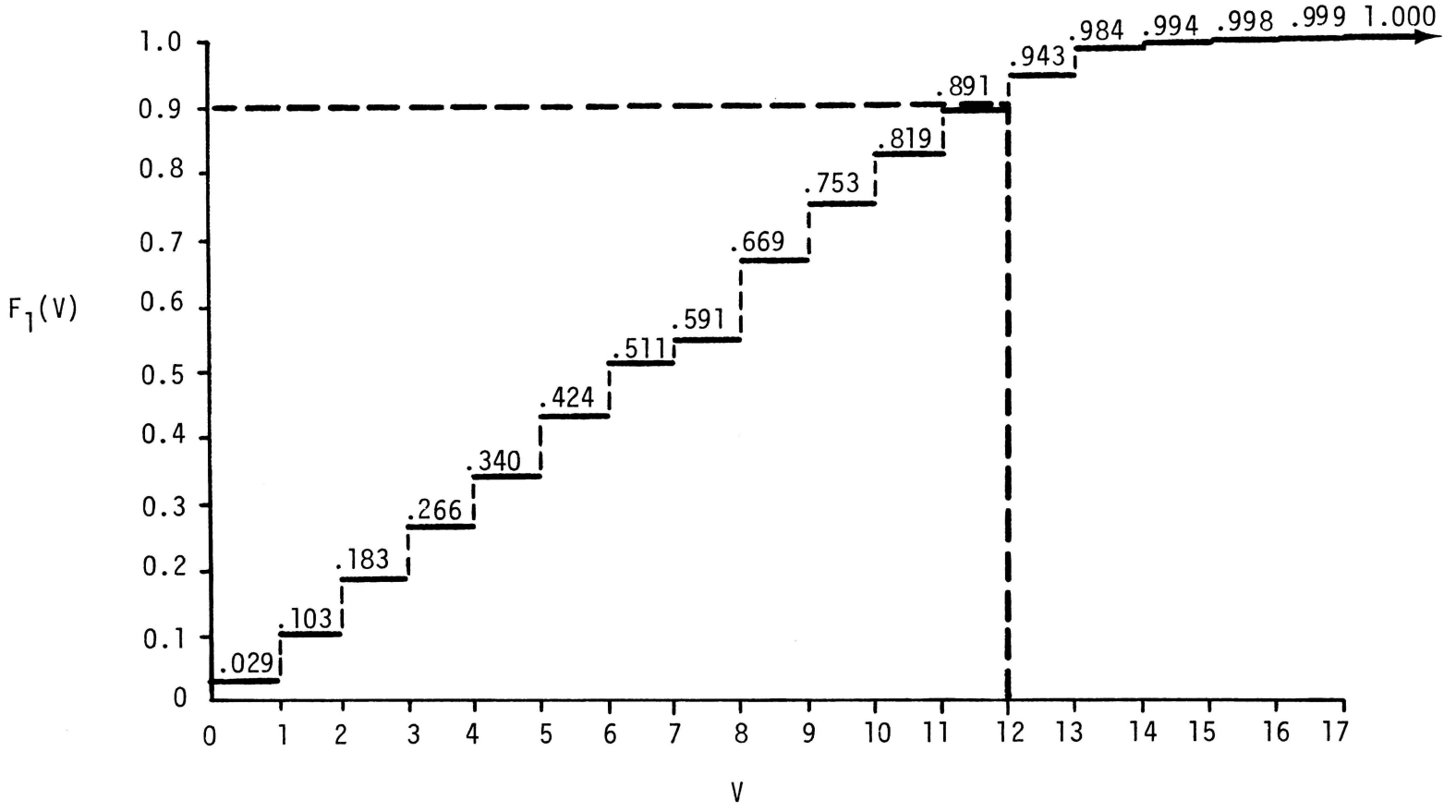


Figure 5-1

Cumulative Frequency Distribution for Variations in Bed Allocations Due to Random Assignments of λ_1 for the Medical Service $c_1(\overline{\lambda}_1)$ = 98 Beds

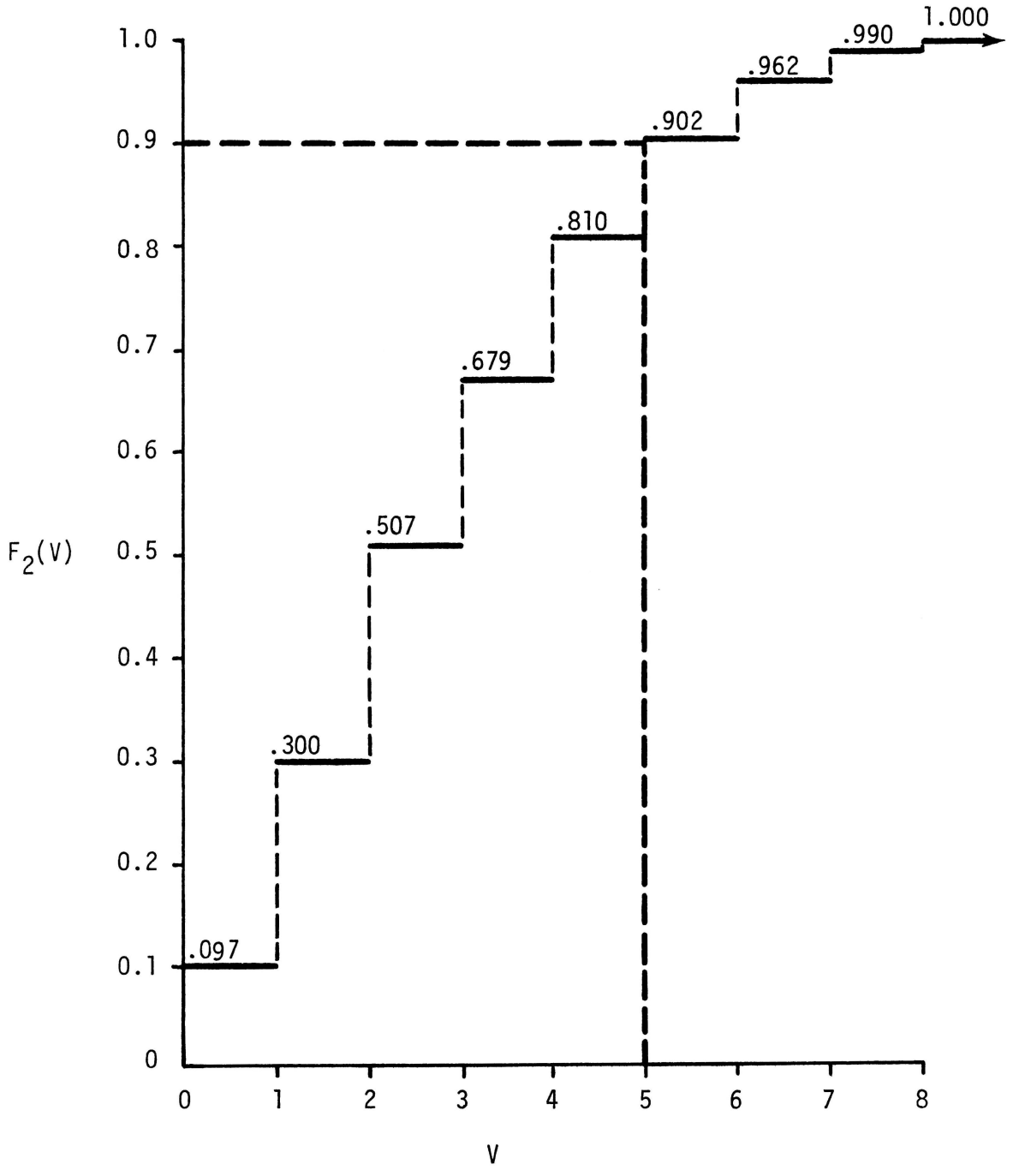


Figure 5-2

Cumulative Frequency Distribution for Variations in Bed Allocations Due to Random Assignments of λ_2 for the Surgical Service

$c_2(\overline{\lambda}_2)$ = 19 beds

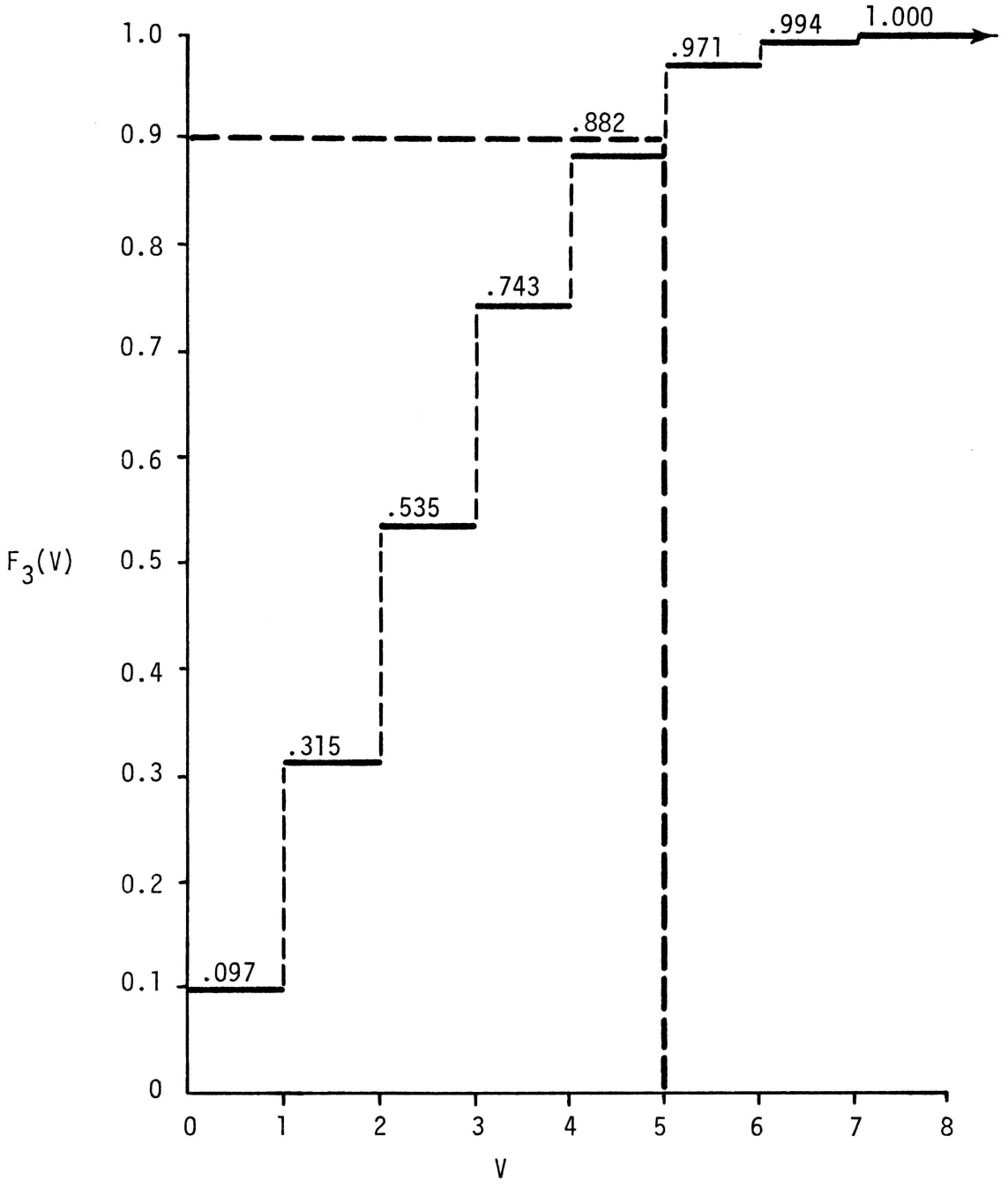


Figure 5-3

Cumulative Frequency Distribution for Variations in Bed Allocations Due to Random Assignments of λ_3 for the Ob/Gyn Service

$c_3(\overline{\lambda}_3)$ = 27 beds

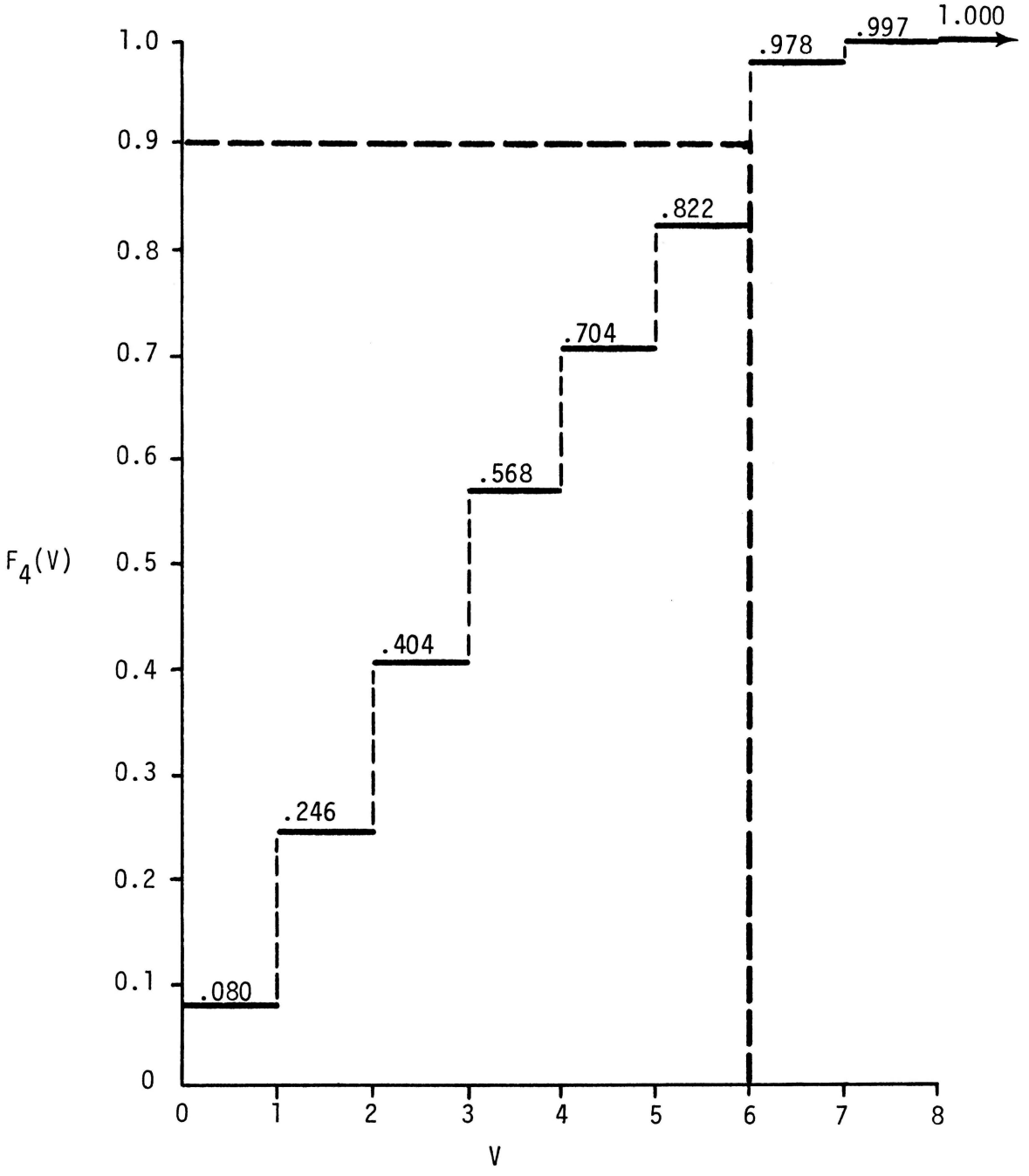


Figure 5-4

Cumulative Frequency Distribution for Variations in Bed Allocations Due to Random Assignments of λ_4 for Pediatrics Services

$c_4(\overline{\lambda}_4)$ = 6 beds

TABLE 5.6

SENSITIVITY RESULTS DUE TO ESTIMATING PARAMETERS AT A 90% PROBABILITY OF OCCURRENCE

Service	Absolute Differences in Bed Assignments			
	Vary λ_j (1)	Vary μ_j (2)	Vary λ_j and μ_j (3)	Vary w_j (4)
Medical	12	9	15	2
Surgical	5	4	8	1
Ob/Gyn	5	4	6	2
Pediatrics	6	5	6	2
$F_{0.90}(\Phi)$	1.80-1.85	1.55-1.60	> 2.00	1.20-1.25

$f^*(C)$ is given by $f_4^*(x_4)$ in the dynamic programming calculation and

$$\hat{f}(C) = \sum_{j=1}^{4} \bar{w}_j \lambda_j \{[(\lambda_j/\bar{\mu}_j)^{\hat{c}_j} / \hat{c}_j!] [\sum_{n=0}^{\hat{c}_j} (\lambda_j/\bar{\mu}_j)^n / n!]^{-1}\}$$

where $\hat{c}_j$ equals 98, 19, 27, and 6 for j = 1, 2, 3, and 4 respectively, λ_j is the true (randomly generated) parameter value, and $\bar{w}_j$ and $\bar{\mu}_j$ are the estimated parameter values. The results of this analysis when values for λ_j were randomly generated are shown in Table 5.7, where $p(\Phi)$ and $F(\Phi)$ represent the relative and cumulative frequencies of occurrence, respectively. In Table 5.6, the value given for $F_{0.90}(\Phi)$ represents the 90% point on the cumulative distribution for Φ. In this case, the administrator can be 90% confident that the true value, $\hat{f}(C)$, will exceed the minimum value, $f^*(C)$ by 80% to 85%. Thus, the model is very sensitive to variations in λ_j.

Sensitivity of the Model to Length of Stay Rates

The sensitivity of the model to errors in estimating values of the length of stay rates was determined by randomly generating values of μ_j within the interval $.8\,\bar{\mu}_j \leq \mu_j \leq 1.2\,\bar{\mu}_j$, while the values of $\bar{\lambda}_j$ and $\bar{w}_j$ were held constant. The absolute differences (V) in the optimal allocations of beds were calculated for each service. The differences observed at a 90% probability of occurrence, i.e., where $F_j(V) = .90$, are shown in column 2 of Table 5.6. The value for $F_{0.90}(\Phi)$ in this column indicates that the true value, $\hat{f}(C)$, will exceed the minimum value, $f^*(C)$, by 55% to 60% with 90% confidence, due to errors in estimating μ_j.

TABLE 5.7

PENALTY DUE TO ESTIMATING ERRORS IN λ_j
$\hat{c}_j$ = 98, 19, 27 and 6 for j = 1, 2, 3, 4, RESPECTIVELY

Φ	$p(\Phi)$	$F(\Phi)$
1.00-1.05	0.0	0.0
1.05-1.10	0.0	0.0
1.10-1.15	0.0030	0.0030
1.15-1.20	0.1980	0.2010
1.20-1.25	0.2500	0.4510
1.25-1.30	0.1550	0.6060
1.30-1.35	0.0620	0.6680
1.35-1.40	0.0380	0.7060
1.40-1.45	0.0360	0.7420
1.45-1.50	0.0280	0.7700
1.50-1.55	0.0300	0.8000
1.55-1.60	0.0290	0.8290
1.60-1.65	0.0180	0.8470
1.65-1.70	0.0140	0.8610
1.70-1.75	0.0120	0.8730
1.75-1.80	0.0170	0.8900
1.80-1.85	0.0120	0.9020
1.85-1.90	0.0160	0.9180
1.90-1.95	0.0120	0.9300
1.95-2.00	0.0070	0.9370
> 2.00	0.0630	1.0000

Sensitivity of the Model to Arrival and Length of Stay Rates

The sensitivity of the model to errors in estimating values of both arrival and length of stay rates was examined next. The mean arrival rate λ_j is applied in the dynamic programming model for calculating both the probability that the system is full and return values. Since the length of stay rate, μ_j, is also used to determine the probability that the system is full, these two parameters, together, have multiple effects upon the model solution. Thus, it is of interest to observe the absolute differences in the optimal allocations of beds when values for λ_j and μ_j are randomly generated within the intervals $.8\ \bar{\lambda}_j \leq \lambda_j \leq 1.2\ \bar{\lambda}_j$ and $.8\ \bar{\mu}_j \leq \mu_j \leq 1.2\ \bar{\mu}_j$, respectively, holding $\bar{w}_j$ constant. The results of this analysis at a 90% probability of occurrence are shown in column 3 of Table 5.6. Also, the value of $F_{0.90}(\Phi)$ in this column indicates that since Φ is greater than 2.00, at a 90% confidence level, $\hat{f}(C)$ will exceed $f^*(C)$ by over 100%.

Sensitivity of the Model to Weighting Factors

The sensitivity of the model to errors in estimating values of the weighting factors, w_j, was observed when values for w_j were randomly generated within the interval $.8\ \bar{w}_j \leq w_j \leq 1.2\ \bar{w}_j$, holding $\bar{\lambda}_j$ and $\bar{\mu}_j$ constant. The absolute differences observed in optimal allocations of beds per service at a 90% probability of occurrence are shown in column 4 of Table 5.6. Also, it can be observed from the value of $F_{0.90}(\Phi)$, $\hat{f}(C)$ will exceed $f^*(C)$ by 20% to 25%, with 90% confidence.

Interpretation of the Sensitivity Analysis

It has been established by analyzing Φ that the model is very sensitive to errors in estimating the values of the three parameters λ_j μ_j and w_j. In Table 5.6 it was shown that the value of the true measure of effectiveness was from 20 to 25 percent greater than the minimum value of the measure of effectiveness, at the 90% confidence level, when w_j was randomly generated, to over 100% when λ_j and μ_j were randomly generated. This magnitude of the penalties due to estimating errors indicates that the values assigned to the parameters must be as precise as possible. Thus, data should be collected over the time period of interest and carefully analyzed to permit factual estimations of the parameter values.

Upon observing the absolute differences in bed assignments in Table 5.6, the relative ranking of parameters that have the greater to the least effect upon bed allocations by service, at the 90% probability of occurrence level is shown as:

$$(\lambda_j,\mu_j) \geq \lambda_j \geq \mu_j \geq w_j,$$

where (λ_j,μ_j) implies errors in estimating values for both λ_j and μ_j. This ranking has some intuitive appeal since, as stated earlier, λ_j is doubly applied throughout the optimization model. Thus, a greater penalty due to errors in estimating these parameter values would be suspected.

Results

In addition to formulating the optimization procedure for allocating beds to hospital services, the intent of this thesis was to provide useable results for hospital administrators. This has been accomplished in two ways. First, for administrators who have access to the computer facilities described in Appendix B, the computer program can be run for specific hospital data. For other hospitals, tables and graphs are provided in this section that approximate bed allocations by service.

Optimal assignments of beds to the medical, surgical, obstetrics/gynecology and pediatrics services are given in Tables 5.8 to 5.12. To ascertain the assignments for the total number of beds to be allocated, C, (i.e., 100 to 500), a user's arrival and length of stay rates and weighting factors must correspond to those shown under "Parameters" for one of the tables.

In Tables 5.8 to 5.12 it can be observed that no beds are assigned to one or more of the services for some allocation policies. This means that, for the given parameters, the optimization model predicts that a minimal number of patients will be turned away from all services when the inventory of beds are allocated as shown. In accepting these policies, the administrator should recognize that it is not necessarily implied that the type patient for which no beds are allocated should be denied admittance. Rather, the hospital would place these patients in other service areas with the general awareness of some loss of allocation efficiency.

TABLE 5.8

BED ALLOCATION TABLE

PARAMETERS :

SERVICE	ARRIVAL RATE	LENGTH OF STAY RATE	WEIGHTING FACTOR
MEDICAL	9.2580	0.1010	1.00
SURGICAL	2.9340	0.1537	0.20
OB / GYN	4.9030	0.2249	0.23
PEDIATRICS	2.3550	0.2500	0.09

ALLOCATIONS:

SERVICE	. TOTAL NO. OF BEDS TO BE ALLOCATED .					
	100	150*	200	300	400	500
MEDICAL	96.	107.	120.	159.	198.	235.
SURGICAL	0.	17.	30.	51.	73.	95.
OB / GYN	4.	25.	34.	57.	79.	102.
PEDIATRICS	0.	1.	16.	33.	50.	68.

* INDICATES THE TOTAL NUMBER OF BEDS TO BE ALLOCATED FOR THE HOSPITAL THAT HAS THE PARAMETERS SHOWN ABOVE

TABLE 5.9

BED ALLOCATION TABLE

PARAMETERS :

SERVICE	ARRIVAL RATE	LENGTH OF STAY RATE	WEIGHTING FACTOR
MEDICAL	6.5000	0.0607	1.00
SURGICAL	2.0000	0.0641	0.29
OB / GYN	2.0000	0.0694	0.27
PEDIATRICS	0.7000	0.1471	0.05

ALLOCATIONS:

SERVICE	TOTAL NO. OF BEDS TO BE ALLOCATED					
	100	104*	200	300	400	500
MEDICAL	100.	104.	128.	164.	202.	241.
SURGICAL	0.	0.	37.	61.	86.	110.
OB / GYN	0.	0.	34.	58.	82.	105.
PEDIATRICS	0.	0.	1.	17.	30.	44.

* INDICATES THE TOTAL NUMBER OF BEDS TO BE ALLOCATED FOR THE HOSPITAL THAT HAS THE PARAMETERS SHOWN ABOVE

TABLE 5.10

BED ALLOCATION TABLE

PARAMETERS :

SERVICE	ARRIVAL RATE	LENGTH OF STAY RATE	WEIGHTING FACTOR
MEDICAL	14.8000	0.0980	1.00
SURGICAL	18.0000	0.1316	0.89
OB / GYN	4.6000	0.2439	0.13
PEDIATRICS	4.5000	0.1923	0.15

ALLOCATIONS:

SERVICE	. TOTAL NO. OF BEDS TO BE ALLOCATED .					
	100	200	300	385*	400	500
MEDICAL	0.	78.	155.	175.	180.	214.
SURGICAL	100.	122.	145.	161.	165.	197.
OB / GYN	0.	0.	0.	22.	25.	41.
PEDIATRICS	0.	0.	0.	27.	30.	48.

* INDICATES THE TOTAL NUMBER OF BEDS TO BE ALLOCATED FOR THE HOSPITAL THAT HAS THE PARAMETERS SHOWN ABOVE

TABLE 5.11

BED ALLOCATION TABLE

PARAMETERS :

SERVICE	ARRIVAL RATE	LENGTH OF STAY RATE	WEIGHTING FACTOR
MEDICAL	9.5000	0.1000	0.86
SURGICAL	13.5000	0.1220	1.00
OB / GYN	9.0000	0.2083	0.40
PEDIATRICS	1.5000	0.2000	0.07

ALLOCATIONS:

SERVICE	. TOTAL NO. OF BEDS TO BE ALLOCATED .					
	100	200	298*	300	400	500
MEDICAL	0.	70.	112.	113.	142.	174.
SURGICAL	100.	109.	132.	132.	162.	195.
OB / GYN	0.	21.	54.	55.	76.	98.
PEDIATRICS	0.	0.	0.	0.	20.	33.

* INDICATES THE TOTAL NUMBER OF BEDS TO BE ALLOCATED FOR THE HOSPITAL THAT HAS THE PARAMETERS SHOWN ABOVE

TABLE 5.12

BED ALLOCATION TABLE

PARAMETERS :

SERVICE	ARRIVAL RATE	LENGTH OF STAY RATE	WEIGHTING FACTOR
MEDICAL	14.5000	0.1075	0.72
SURGICAL	21.2000	0.1136	1.00
OB / GYN	12.0000	0.1818	0.35
PEDIATRICS	5.4000	0.3030	0.09

ALLOCATIONS:

SERVICE	. TOTAL NO. OF BEDS TO BE ALLOCATED .					
	100	200	300	400	450*	500
MEDICAL	0.	75.	128.	164.	187.	198.
SURGICAL	100.	125.	125.	125.	125.	125.
OB / GYN	0.	0.	47.	85.	102.	126.
PEDIATRICS	0.	0.	0.	26.	36.	51.

* INDICATES THE TOTAL NUMBER OF BEDS TO BE ALLOCATED FOR THE HOSPITAL THAT HAS THE PARAMETERS SHOWN ABOVE

The arrival and length of stay parameters used in each table were derived from actual hospital data. The weighting factors are unique to each table and were assigned by using the objective weighting technique, shown in Chapter 3 of this thesis. Only the demand for each service, i.e., (arrival rate/length of stay rate), was used as the criterion for weighting factor assignment, since applicable costs were not available.

The bed allocation tables may be applied for any total number of beds to be allocated between 100 and 500. To verify this, the relationship between assignments and C was investigated to determine if allocations vary linearly by service. A linear regression was applied to the medical, surgical, obstetrics/gynecology and pediatrics allocations in each table, and the results were then compared with the optimal allocations shown. The highest deviation between the regression results and the optimal assignments, 20 beds, occurred for the medical service in Table 5.10 for C = 300. Most of the differences observed, however, were between 0 and 6 beds. As an example of this analysis, Table 5.13 depicts comparisons between the linear regression results (Est.) and optimal results (Opt.) for the allocations shown in Table 5.8. It can be observed that there are minimal deviations between the regression and optimal assignments. Graphically, the optimal results for services shown in Table 5.8 and the linear regression data are given in Figure 5-5. For the linear regression data in Figure 5-5, the curve is discontinuous between C = 100 and C = 150 for the pediatrics service. This was an elective decision to permit a closer "fit" between the regression and optimal curves for $C \geq 150$. Practically, starting at C = 150 means

TABLE 5.13

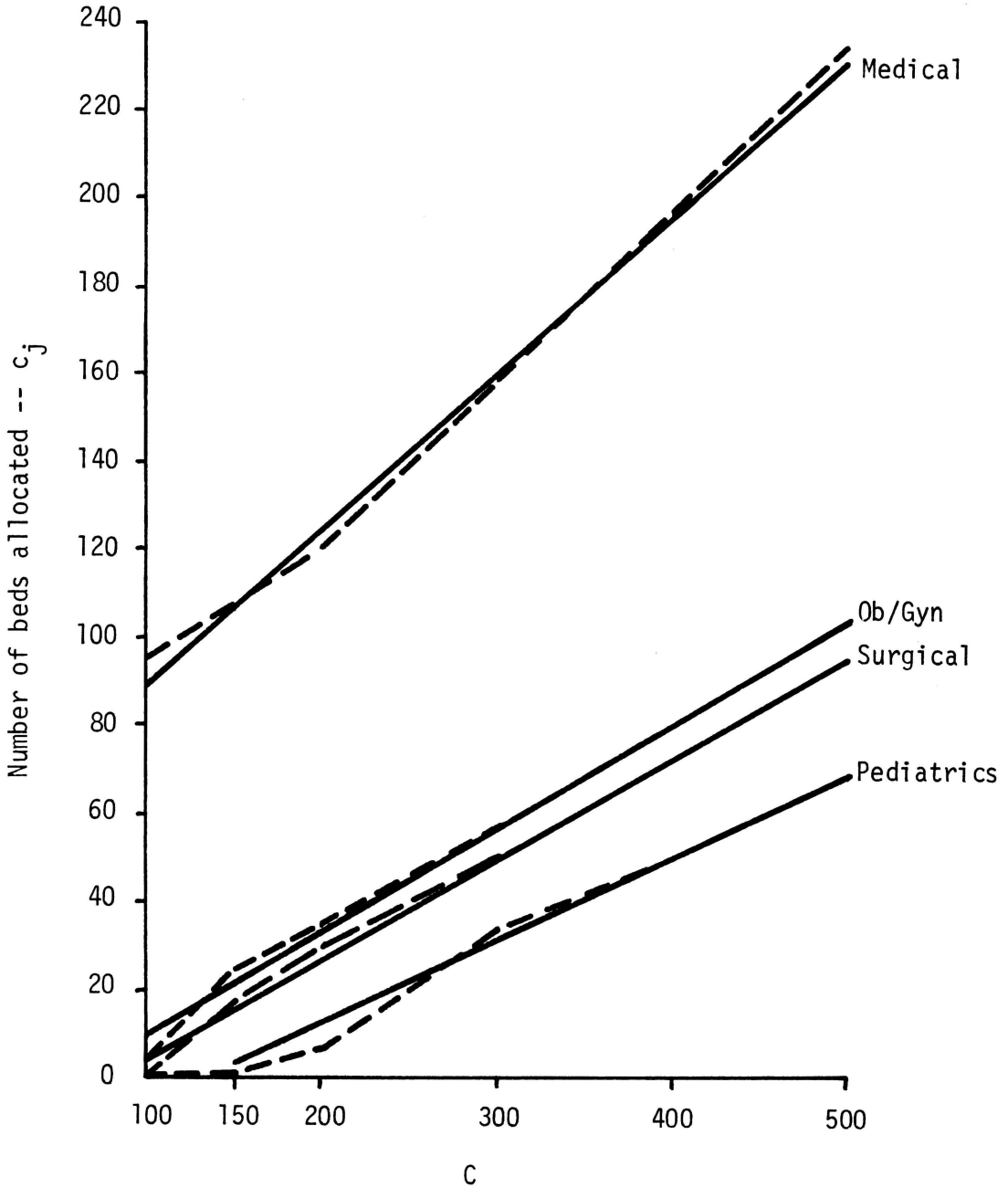
COMPARISON OF LINEAR REGRESSION RESULTS WITH OPTIMAL RESULTS FOR DATA IN TABLE 5.1

	100		200		300		400		500	
Service	Est.	Opt.	Est.	Opt.	Est.	Opt.	Est.	Opt.	Est.	Opt.
Medical	89	96	125	120	161	159	197	198	232	235
Surgical	4	0	27	30	50	51	73	73	96	95
Ob/Gyn	9	4	32	34	56	57	79	79	103	102
Pediatrics	-	0	13	16	31	33	50	50	68	68

that no beds would be allocated to the pediatrics service when C is between 100 and 150, i.e., all beds would be assigned to the medical, surgical and ob/gyn services.

The establishment of a linear relationship is important, for it permits interpolation between the values of C given in the Tables for the desired total number of beds to be allocated to each service. For the user who would rather approximate from graphs the linear regression curves in Figures 5-5 to 5-9 may be used to determine the beds assigned to services. In applying the graphs, the user is cautioned that the sum of the values selected for each service may not equal the value of C for his hospital. This is due to the linear approximation and can be compensated for by reducing or adding the difference obtained to one of the services, with discretion.

In summary, a method has been provided for hospital administrators to use the results of the optimization model without computer processing. Tables and graphs are provided for five sets of typical arrival and length of stay parameters, with unique weighting factors assigned according to the demand relationships per service. Although the table parameters will not exactly represent some hospitals, perhaps at least one set will be close enough to either approximate bed assignments or provide some insight into how beds should be allocated. If, however, none of the tables can be used, the computer program (Appendix B) may be applied to obtain exact bed allocation results.



Legend: Dashed lines (- - -) indicate optimal results shown in Table 5.5.

Solid lines indicate regression results.

Figure 5-5

Optimal and Linear Regression Results for Services Shown in Table 5.8

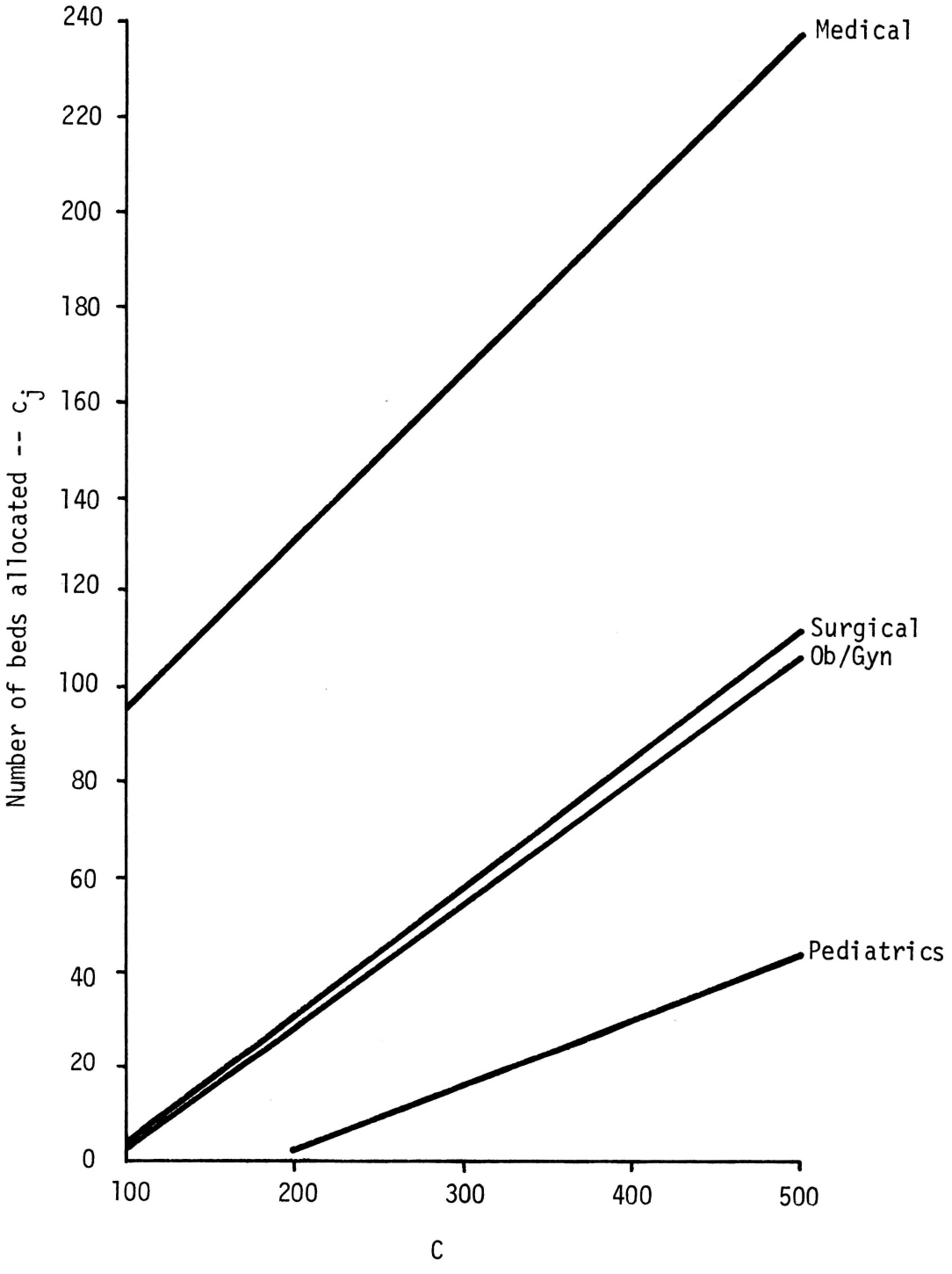


Figure 5-6

Linear Regression Results for Services Shown in Table 5.9

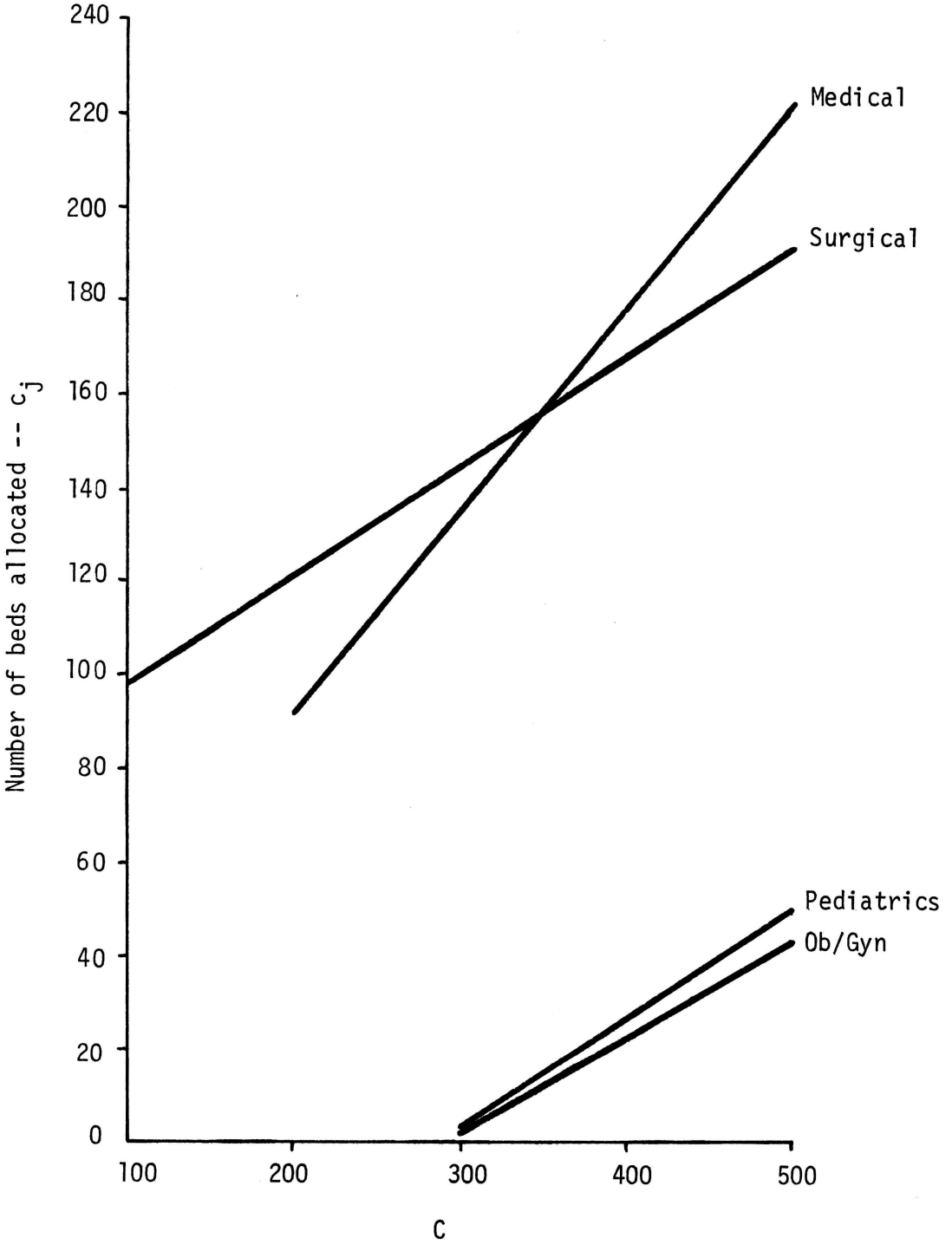


Figure 5-7

Linear Regression Results for Services Shown in Table 5.10

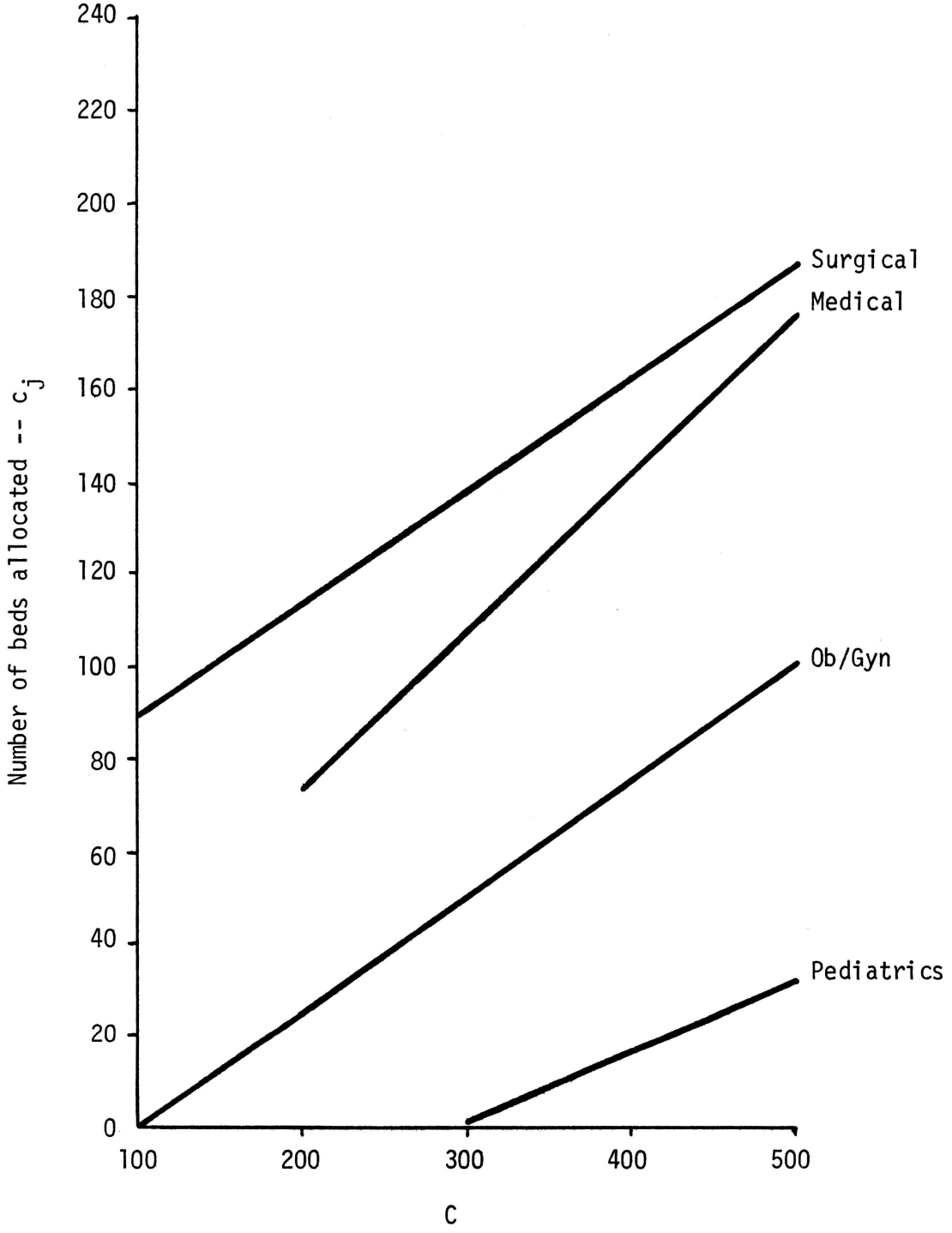


Figure 5-8

Linear Regression Results for Services Shown in Table 5.11

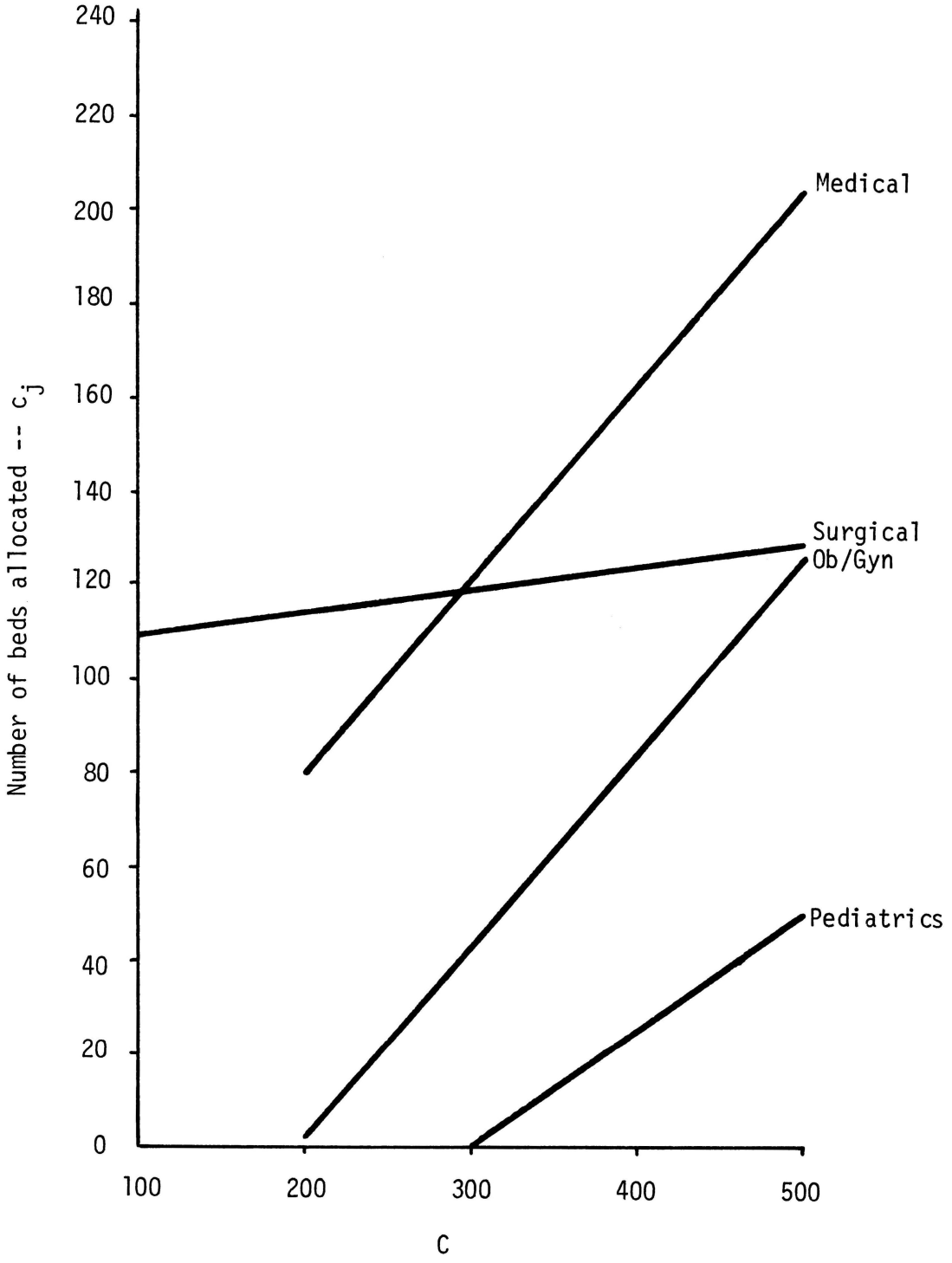


Figure 5-9

Linear Regression Results for Services Shown in Table 5.12

Cautions and Use of the Model

The frequency of applying the bed allocation model will be dependent upon how and when values of the arrival and length of stay rates, the weighting factors and the total number of beds to be allocated vary. It is common assumption that arrival and lengths of stay parameters vary seasonally by service. These parameters are also thought to be predictable as long as the number of beds in the hospital and the administrative policies concerning patient admittances remain stable. Likewise, when the elements that contribute to weighting factor assignments do not change, the weighting factor values will remain constant. The problem arises when unknown changes in the parameter values occur. The bed allocation policy then becomes sub-optimal since, as shown earlier in this chapter, the model is very sensitive to variations in the parameters. Thus, in addition to applying the model initially, considerations should be given to establishing methods for detecting parameter changes and re-running the model as applicable.

One method for detecting significant variations in mean arrival and length of stay rates would be to collect enough data to establish confidence intervals for these parameters. The administrator could then observe rates calculated on a weekly or monthly basis and as trends were observed toward or beyond the upper or lower confidence intervals a decision could be made to revise the parameter values and re-run the bed allocation model. Another technique, which is less

desirable, would be to simply plot the parameter values over time to observe any shifting trends in the values.

The detection of variations in the values of the weighting factors will not be as easily accomplished. Revisions to values of this parameter will result from analyzing the following elements that are used to establish the weighting factors:

1) related hospital administrative costs, e.g., the cost of an empty bed, staff costs, overhead, etc.
2) measurements of the severity of illnesses usually associated with the services
3) demand for the services.

When the administrator does determine the necessity for revising the weighting factors values, then the model should be re-run with the updated factors included.

If the total number of beds, C, changes, the model should be applied to reallocate beds to services. In lieu of running the computer program, graphs or tables, similar to those shown in the previous section may be used to approximate bed allocations to the services if the model parameters do not change.

In this research, the queueing model used to determine the probability that the system is full was dependent upon Poisson distributed arrivals and negative exponentially distributed lengths of stay for each service. Although it is expected that these distributions will be applicable for most hospitals, it is recommended that the distributions be investigated for particular hospitals before the bed

allocation model is applied. In the case where the Poisson process is not representative, the queueing model must be revised and the Fortran program (Subroutine PROB) given in Appendix B must be updated to reflect the different formula for calculating probabilities that the system is full. As indicated in the queueing literature (i.e., [4], [20]), this formula may be quite difficult to derive for some of the other queueing models.

In summary, a model is provided that can assist hospital administrators in allocating beds to services. If appropriately applied, an administrator can be assured of constantly minimizing the penalty for turning patients away.

Chapter 6

SUMMARY AND RECOMMENDATIONS FOR FURTHER RESEARCH

This chapter presents a summary of the accomplishments achieved in this thesis and recommendations for further research.

Summary

The objectives of this research were two-fold. First, the problem of optimally allocating beds between the four hospital services was to be resolved. Second, useable results were to be provided for hospital administrators to assist their decision making regarding how beds were to be distributed for specific hospitals.

In a survey of the literature, most of the articles reviewed either indicated a need for determining how beds should be allocated or employed other than optimization techniques for solving the problem. Typical methods used were forecasting with regression analysis and analyzing the daily census under assumed probability distributions or average demands. Thus, none of the research considered the relationship of random arrivals and lengths of stay, and resolved the problem of minimizing the penalty for turning patients away, which was the prime consideration of this thesis.

In this thesis, the problem has essentially been solved in four phases: problem formulation, data analysis, queueing model recognition and the optimization model formulation. In the first phase, the random nature of bed requirements varying over time led to a consideration of random patient arrivals and lengths of stay. Competing demands

between hospitals and between services within a hospital prompted the introduction of a weighting factor for turning patients away from each service. The application of queueing theory was also necessary due to the random variables in the problem and to provide the probability that the system was full. The consideration of all of these factors led to the problem formulation shown by Equations (1.1), (1.2) and (1.3), where the sum of the penalties for turning a patient away from a desired service was to be minimized.

Arrival and length of stay data from the Radford Community Hospital, Radford, Virginia were analyzed to determine the underlying probability distributions. Upon applying the Kolmogorov-Smirnov one sample goodness-of fit test it was found that the arrivals were Poisson distributed and lengths of stay were exponentially distributed for each service. Analyses of variance justified the use of the mean arrival and service rate parameters over the period of interest, and, thus verified the stability of the process. Based upon these results and the requirements to consider a Markov process for a multiple server, finite population system, a queueing model of the type (M/M/c: GD/c/∞) was utilized to provide the probability that the system is full.

The objective weighting technique, as developed by Churchman and Ackoff, was presented as a method for assigning values to the penalties for turning patients away from each service. It was stated that this technique could be used to test consistency and to detect errors in judgement.

The bed allocation problem was recognized as a multi-stage process that could be solved in single stages to produce optimum results. Thus, for the optimization procedure, a dynamic programming model was formulated to allocate beds to the four hospital services.

Upon developing the dynamic programming model, it was necessary to determine the sensitivity of the model to errors in estimating values of the parameters. By examining the effect on optimum bed allocations, V, and Φ, the penalty which results due to errors in estimating parameter values, it was found that the model is very sensitive to all of the parameters. Further, the sensitivity ranking of the greatest to the least effects upon model solutions was shown as:

$$(\lambda_j, \mu_j) \geq \lambda_j \geq \mu_j \geq w_j.$$

Finally, results were provided for hospital administrators in the forms of a computer program, tables and graphs. The computer program can be used for any specific hospital to allocate up to 500 beds between the services. Modifications to increase this capacity can be accomplished by merely changing the Fortran dimension statements. Tables are presented for five sets of hospital parameters and beds are assigned to the four services for 100 to 500 total beds to be allocated. A linear relationship was established between the number of beds allocated per service, which permits interpolation in the tables. Graphs are also provided which depict the linear allocations.

In conclusion, quantitative, analytical techniques have been used in this research to resolve the bed allocation problem; the use of applicable models has been substantiated by statistical tests; and,

useable results have been provided for those who will benefit from this effort.

Recommendations for Further Study

Several areas for the extension of this research have been recognized and are listed below:

1) The parameter representing the penalty for turning patients away was assumed to be assigned by hospital administrators in this study. It is conceivable that a need will arise to determine the values of this parameter, given the number of beds allocated and the mean arrival and length of stay rates for each service. For example, suppose an administrator is adamant about not changing the bed allocations in his hospital, but does not know what it is "costing" him in terms of service priorities. The ability to impute the w_j values by service would provide the penalty for turning patients away. This insight might be surprisingly revealing, especially when the elements that comprise the cost are examined.
2) Determine the relationship between the mean arrival rate, mean service rate, the penalty for turning patients away and the percentage of beds allocated by service. The knowledge of these relationships, in general, would permit imputing values of some of the factors, given the others. Less data gathering, and less computation would result.

3) Consider the effect upon bed allocation results when additional factors such as a patient's age, sex, admitting diagnosis, transfers between services after admission, and type admission (elective or emergency) are included in the model.

4) The possibility of incorporating this method for allocating beds into an on-line, real time admissions scheduling system is of interest. Complete information regarding arrivals, lengths of stay, the current census including exactly which beds were occupied and how long the beds are forecast to be occupied, coupled with the optimization model would permit a dynamic hospital management information system. It would be possible to not only allocate beds to services for a current period of time, but to project allocations for future considerations.

5) Consider the costs of providing beds for patients in each service. Incorporate a cost model into the optimization procedure to ascertain which service a patient should be transferred to if the desired service is fully occupied.

6) As cited in the thesis, the hospital has dual but conflicting objectives: maximize service and minimize the costs to patients. One consideration of these objectives is bed allocations. As the number of beds is increased, services to patients are increased. However, this increase in service may be provided at the possible expense of unallocated beds. Thus, it would be desirable to construct a cost model for determining the

point on a total cost curve where costs are minimal for some value of the total number of beds to be allocated.

7) Consider the problem of optimally allocating beds to services in the transient state. Steady state conditions were assumed in this research where the probability of the number in the system (i.e., number of beds occupied) is not time dependent. It is expected that while these conditions hold for established, general acute, hospitals, arrivals and/or lengths of stay probably vary significantly over time for "specialty" type hospitals, e.g., Eye, Ear, Nose and Throat, Orthopedic, etc. A reformulation of the bed allocation problem would be necessary to explicitly account for the transient state.

8) This research has been conducted with no consideration for the physical location of beds allocated to the services. Incorporating location or maximum number of beds constraints, by service, into the model would probably make the solution results more beneficial for some hospitals.

9) The sensitivity analysis of the optimization model was conducted in this research by assuming a uniform distribution for the mean values of the model parameters. It would be interesting to observe the results of a similar sensitivity analysis when the actual distributions of these values, as derived from gathering more data, or assumed beta or normal distributions were applied.

10) The implementation of this model in a hospital environment is recommended. The process would be to analyze the arrival

and length of stay data for each service to determine the parameter values and the underlying distributions, establish the weighting factor values, process the computer program (or use the graphs or charts as applicable), and evaluate results after a period of operation.

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APPENDIX A

DATA FROM THE RADFORD COMMUNITY HOSPITAL, RADFORD, VIRGINIA

TABLE I

MEDICAL SERVICE DATA -- AUGUST, 1970

Day	Admissions In Patients Per Day	Length of Stay in Days
1	12	143
2	12	123
3	14	130
4	11	114
5	7	51
6	10	138
7	6	98
8	8	76
9	9	82
10	7	54
11	4	32
12	14	149
13	5	26
14	10	105
15	7	103
16	7	86
17	17	163
18	9	55
19	4	34
20	9	105
21	6	61
22	14	133
23	15	144
24	10	87
25	13	109
26	6	67
27	4	31
28	13	122
29	5	81
30	10	81
31	9	58
Total	287	2841

$\lambda_1 = \frac{287}{31} = 9.2580$ $\mu_1 = \frac{287}{2841} = .1010$

TABLE II

SURGICAL SERVICE DATA -- AUGUST, 1970

Day	Admissions In Patients Per Day	Length of Stay in Days
1	-	--
2	3	14
3	6	35
4	2	12
5	2	10
6	3	22
7	-	--
8	3	11
9	-	--
10	9	39
11	4	15
12	3	12
13	2	12
14	-	--
15	2	8
16	7	38
17	5	26
18	1	2
19	1	18
20	7	42
21	2	7
22	-	--
23	4	18
24	3	40
25	4	24
26	2	7
27	2	68
28	4	38
29	2	17
30	4	31
31	4	26
Total	91	592
	$\lambda_2 = 91/31 = 2.9355$	$\mu_2 = 91/592 = .1537$

TABLE III

OBSTETRICS/GYNECOLOGY DATA -- AUGUST, 1970

Day	Admissions In Patients Per Day	Length of Stay in Days
1	2	9
2	5	14
3	5	22
4	8	39
5	4	19
6	3	9
7	6	20
8	7	26
9	4	33
10	4	20
11	5	20
12	4	12
13	11	48
14	10	38
15	5	16
16	5	29
17	3	11
18	3	15
19	11	54
20	6	30
21	1	1
22	5	24
23	4	20
24	7	28
25	4	17
26	3	12
27	4	18
28	2	10
29	4	25
30	5	25
31	2	12
Totals	152	676

$\lambda_3 = 152/31 = 5.2581$ $\mu_3 = 152/676 = .2249$

TABLE IV

PEDIATRICS DATA -- AUGUST, 1970

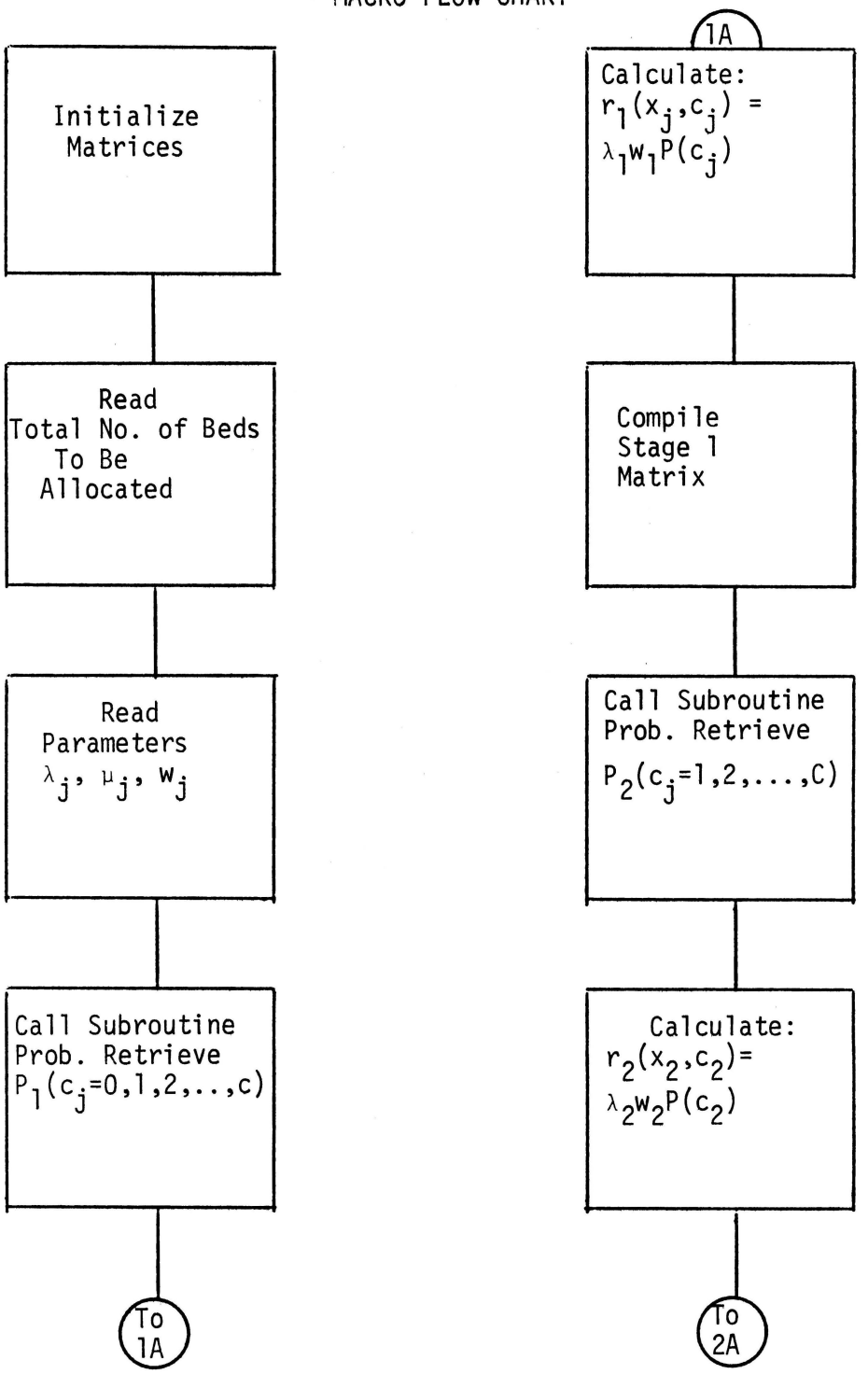
Day	Admissions in Patients Per Day	Length of Stay in Days
1	-	--
2	2	7
3	2	7
4	6	30
5	5	12
6	3	11
7	1	7
8	2	15
9	-	--
10	6	14
11	2	7
12	3	13
13	-	--
14	2	5
15	2	17
16	3	18
17	3	14
18	4	20
19	2	4
20	3	13
21	1	3
22	3	12
23	3	13
24	-	--
25	4	10
26	5	20
27	-	--
28	4	16
29	-	--
30	2	4
31	-	--
Totals	73	292

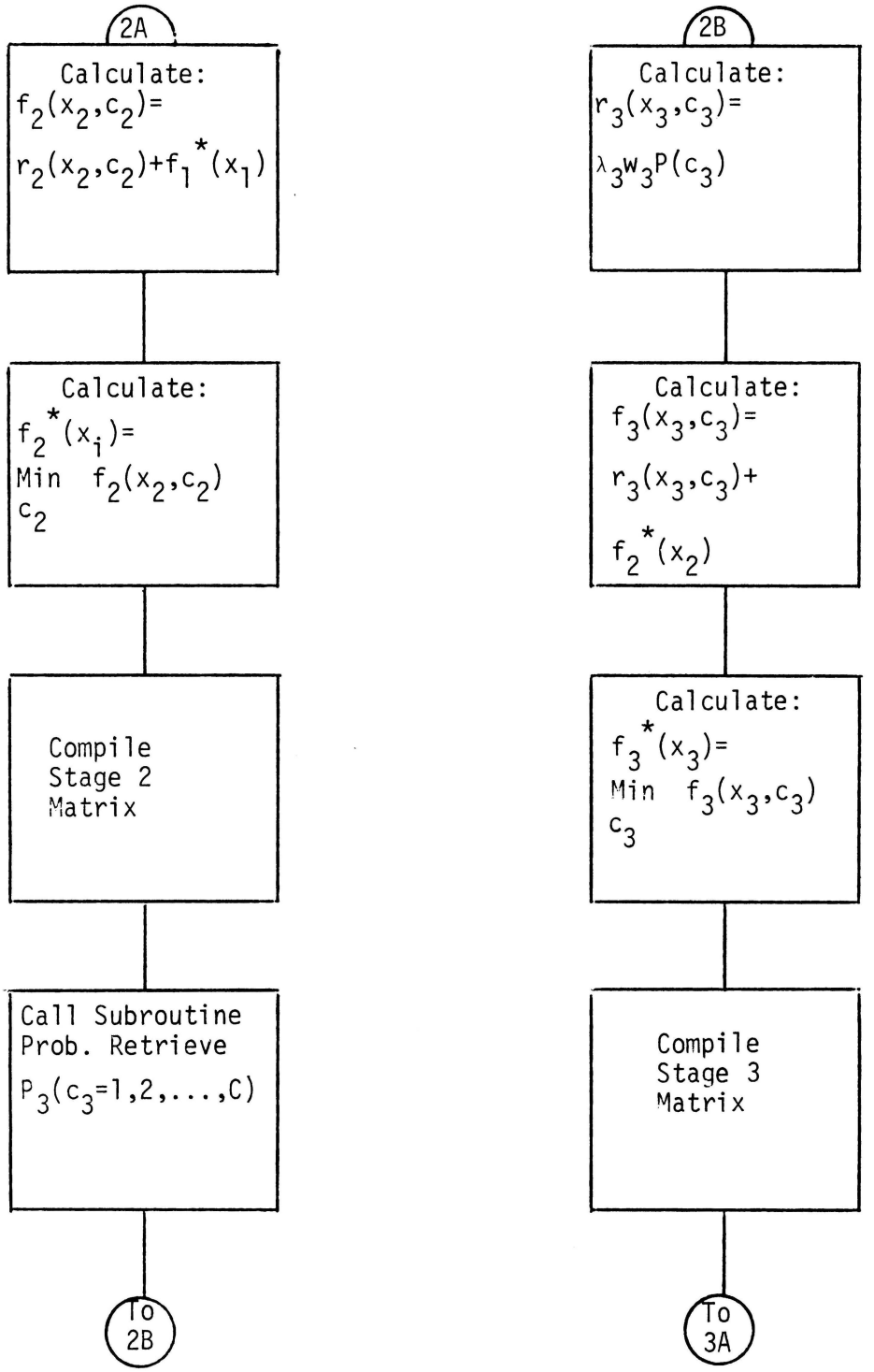
$\lambda_4 = 73/31 = 2.355$ $\mu_4 = 73/292 = .2500$

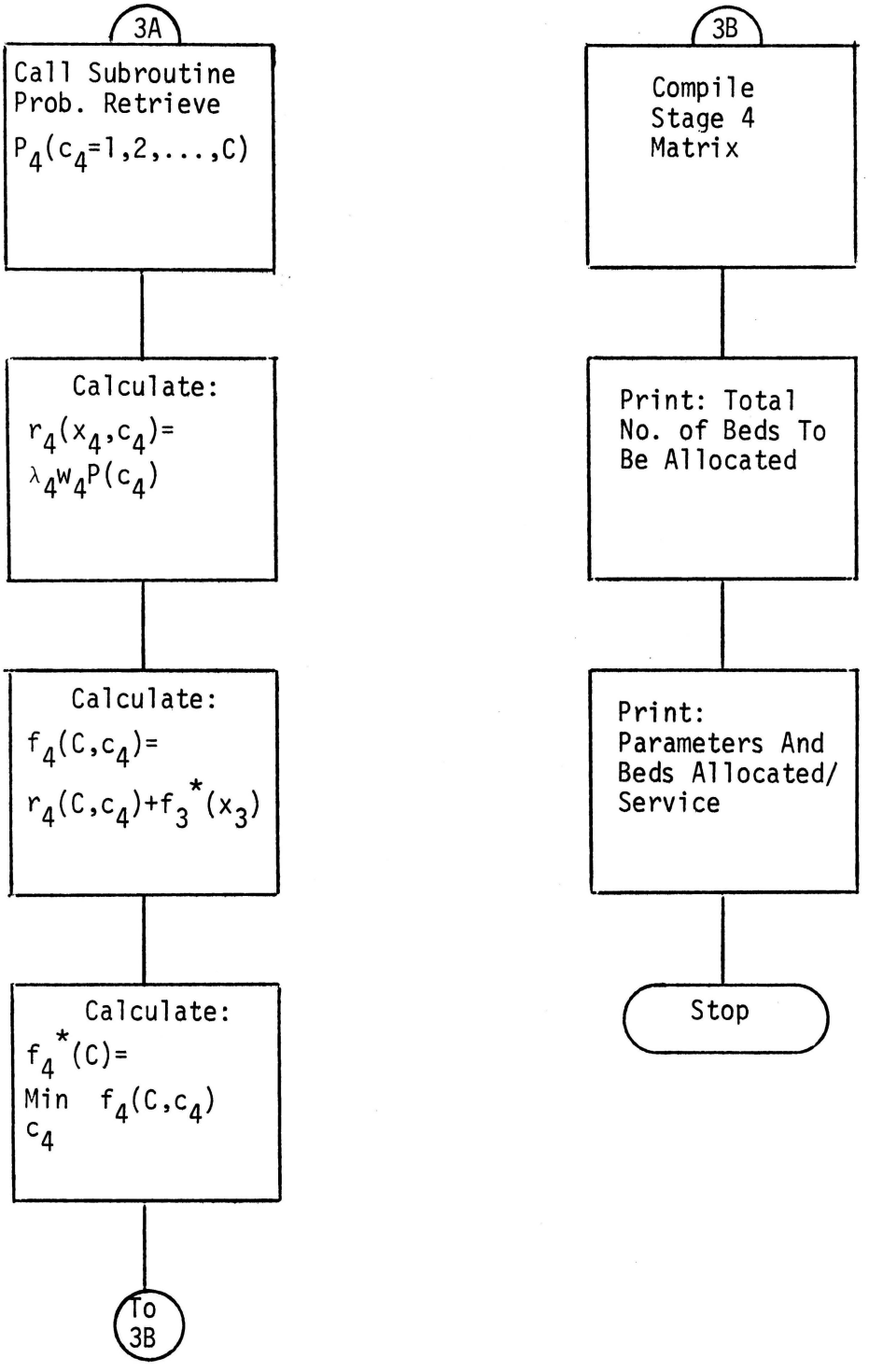
APPENDIX B

MACRO FLOW CHART OF COMPUTER PROGRAM, LISTING AND SAMPLE OUTPUT, AND INSTRUCTIONS FOR RUNNING THE PROGRAM

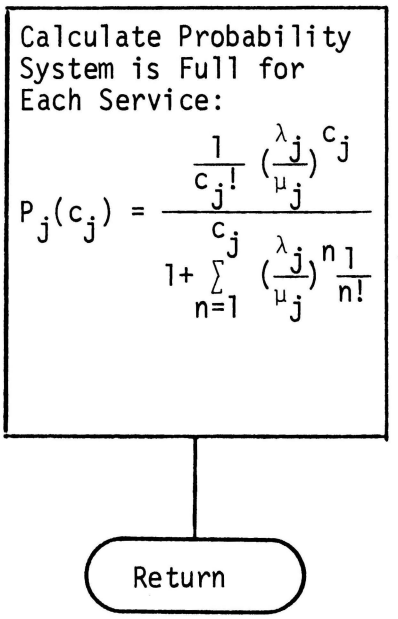
MACRO FLOW CHART




2A
Calculate:
$f_2(x_2,c_2)=$
$r_2(x_2,c_2)+f_1^*(x_1)$
Calculate:
$f_2^*(x_i)=$
Min $f_2(x_2,c_2)$
c_2
Compile
Stage 2
Matrix
Call Subroutine
Prob. Retrieve
$P_3(c_3=1,2,\ldots,C)$
To
2B
2B
Calculate:
$r_3(x_3,c_3)=$
$\lambda_3 w_3 P(c_3)$
Calculate:
$f_3(x_3,c_3)=$
$r_3(x_3,c_3)+$
$f_2^*(x_2)$
Calculate:
$f_3^*(x_3)=$
Min $f_3(x_3,c_3)$
c_3
Compile
Stage 3
Matrix
To
3A


3A
Call Subroutine Prob. Retrieve
$P_4(c_4=1,2,\ldots,C)$
Calculate:
$r_4(x_4,c_4)=$
$\lambda_4 w_4 P(c_4)$
Calculate:
$f_4(C,c_4)=$
$r_4(C,c_4)+f_3^*(x_3)$
Calculate:
$f_4^*(C)=$
Min $f_4(C,c_4)$
c_4
To 3B
3B
Compile Stage 4 Matrix
Print: Total No. of Beds To Be Allocated
Print: Parameters And Beds Allocated/ Service
Stop

SUBROUTINE PROB



```
      DIMENSION STAGE2(1,502),STAGE3(1,502),X3S(502)
      DIMENSION X2S(502),S3LIM(502),S2LIM(502)
      DIMENSION STAGE4(502,2),STAGE1(502)
      DIMENSION G(4),U(4),W(4),C(502,4),IBEDS(4)
      COMMON P(502),NP
C        * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *
C        READ TOTAL NUMBER OF BEDS TO BE ALLOCATED
C        * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *
      READ(5,11)NP
   11 FORMAT(I3)
      DO1 I=1,4
C        * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *
C        READ ARRIVAL RATES-G(J),SERVICE RATES-U(J),AND
C        WEIGHTING FACTORS-W(J)
C        * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *
    1 READ(5,10)G(I),U(I),W(I)
   10 FORMAT(F5.3,2F6.4)
      LIM=NP+1
      DO23 I=1,NP
      X2S(I)=0.
   23 X3S(I)=0.
C        * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *
C        ROUTINES FOR CALCULATING STAGES OF THE DYNAMIC
C        PROGRAMMING MODEL
C        * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *
C        CALL SUBROUTINE TO CALCULATE PERCENTAGES OF
C        PATIENTS TURNED AWAY GIVEN C BEDS FOR
C        SERVICE--STAGE 1
C        * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *
      CALL PROB(G(4),U(4))
C        * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *
C        CALCULATE STAGE RETURNS FOR EACH STATE OF STAGE 1
```

```
C        * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *
      DO74 I=1,NP
   74 C(I,4)=W(4)*G(4)*P(I)
      DO2 I=1,NP
      STAGE4(I,1)=I-1
    2 STAGE4(I,2)=C(I,4)
C        * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *
C        CALL SUBROUTINE TO CALCULATE PERCENTAGES OF
C        PATIENTS TURNED AWAY GIVEN C BEDS FOR
C        SERVICE--STAGE 2
C        * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *
      CALL PROB(G(3),U(3))
      DO75 I=1,NP
C        * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *
C        CALCULATE STAGE RETURNS FOR EACH STATE OF STAGE 2
C        * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *
   75 C(I,3)=W(3)*G(3)*P(I)
C        * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *
C        DETERMINE MINIMUM STAGE RETURNS FOR EACH
C        STATE OF STAGE 2 = X3S(I)
C        * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *
      DO3 I=1,NP
      SMIN3=999999
      IF(I.EQ.1)GO TO 32
      IJ=0
      DO4 J=1,I
      N=I-(J-1)
      STAGE3(1,J)=C(J,3)+STAGE4(N,2)
      IF(STAGE3(1,J).LT.SMIN3)GO TO 20
      GO TO 4
   20 SMIN3=STAGE3(1,J)
      IJ=J-1
```

```
    4 CONTINUE
      S3LIM(I)=SMIN3
      X3S(I)=IJ
      GO TO 3
   32 STAGE3(1,1)=C(1,3)+STAGE4(1,2)
      S3LIM(I)=STAGE3(1,1)
    3 CONTINUE
C        * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *
C        CALL SUBROUTINE TO CALCULATE PERCENTAGES OF
C        PATIENTS TURNED AWAY GIVEN C BEDS FOR
C        SERVICE--STAGE 3
C        * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *
      CALL PROB(G(2),U(2))
      DO76 I=1,NP
C        * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *
C        CALCULATE STAGE RETURNS FOR EACH STATE OF STAGE 3
C        * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *
   76 C(I,2)=W(2)*G(2)*P(I)
C        * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *
C        DETERMINE MINIMUM STAGE RETURNS FOR EACH
C        STATE OF STAGE 3 = X2S(I)
C        * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *
      DO6 I=1,NP
      SMIN2=999999
      IF(I.EQ.1)GO TO 33
      IJ=0
      DO7 J=1,I
      N=I-(J-1)
      STAGE2(1,J)=C(J,2)+S3LIM(N)
      IF(STAGE2(1,J).LT.SMIN2)GO TO 21
      GO TO 7
   21 SMIN2=STAGE2(1,J)
```

```
      IJ=J-1
    7 CONTINUE
      S2LIM(I)=SMIN2
      X2S(I)=IJ
      GO TO 6
   33 STAGE2(1,1)=C(1,2)+S3LIM(1)
      S2LIM(1)=STAGE2(1,1)
    6 CONTINUE
C        * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *
C        CALL SUBROUTINE TO CALCULATE PERCENTAGES OF
C        PATIENTS TURNED AWAY GIVEN C BEDS FOR
C        SERVICE--STAGE 4
C        * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *
      CALL PROB(G(1),U(1))
      DO77 I=1,NP
C        * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *
C        CALCULATE STAGE RETURNS FOR EACH STATE OF STAGE 4
C        * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *
   77 C(I,1)=W(1)*G(1)*P(I)
      DO9 I=1,NP
      IF(I.EQ.1)GO TO 30
      IF(I.EQ.NP)GO TO 31
      N=NP-(I-1)
      STAGE1(I)=C(I,1)+S2LIM(N)
      GO TO 9
   31 STAGE1(I)=C(I,1)+S2LIM(1)
      GO TO9
   30 STAGE1(I)=C(I,1)+S2LIM(NP)
    9 CONTINUE
      SMIN1=999999
C        * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *
C        DETERMINE MINIMUM STAGE RETURNS FOR EACH
```

```
C         STATE OF STAGE 4 = X1S
C         * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *
      DO73 I=1,NP
      IF(STAGE1(I).LT.SMIN1)GO TO 25
      GO TO 73
   25 SMIN1=STAGE1(I)
      X1S=I-1
   73 CONTINUE
      IBEDS(1)=X1S
      SBEDS=NP-X1S
      ISBEDS=SBEDS
      IBEDS(2)=X2S(ISBEDS)
  100 SBEDS1=NP-X1S-IBEDS(2)
      ISBEDS=SBEDS1
      IBEDS(3)=X3S(ISBEDS)
      IBEDS(4)=(NP-1)-X1S-IBEDS(3)-IBEDS(2)
      NP=NP-1
      WRITE(6,19)NP
   19 FORMAT(1H1,15(/),T20,
     1'TOTAL NUMBER OF BEDS TO BE ALLOCATED =',I3,/)
      WRITE(6,12)
   12 FORMAT(       T35,'.',T42,'MODEL  PARAMETERS',T66,
     1'.',//,T35,'ARRIVAL',
     2    T45,'LENGTH OF',T58,'WEIGHTING',T74,'BEDS',/,T21,
     3'SERVICE',T38,
     4 'RATE',T45,'STAY RATE',T60,'FACTOR',T71,
     5'ALLOCATED',/)
      DO 26 I=1,4
      IF(I.EQ.1)GO TO 27
      IF(I.EQ.2)GO TO 28
      IF(I.EQ.3)GO TO 29
      WRITE(6,13)
```

```
 13 FORMAT(T20,'PEDIATRICS')
 40 WRITE(6,14)G(I),U(I),W(I),IBEDS(I)
 14 FORMAT(1H+,T35,F7.4,T47,F7.4,T58,F8.4,T74,I3)
    GO TO 26
 27 WRITE(6,15)
 15 FORMAT(T20,'MEDICAL')
    GO TO 40
 28 WRITE(6,16)
 16 FORMAT(T20,'SURGICAL')
    GO TO 40
 29 WRITE(6,17)
 17 FORMAT(T20,'OB / GYN')
    GO TO 40
 26 CONTINUE
 96 WRITE(6,500)
500 FORMAT(/,1H1)
    STOP
    END
```

```
      SUBROUTINE PROB(G,U)
      DIMENSION FACT(502),N(502),DENOM(502),PROD(502)
      DIMENSION N1(502)
      COMMON P(502),NP
      P(1)=1.
      FACT(1)=1.
      PROD(1)=1.
      N1(1)=0
      N(1)=0
      N2=0
      N3=0
      DENOM(1)=1.
C        * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *
C        CALCULATE :
C          P(C(J)) = (1./C(J)FACTORIAL) * (G(J)/U(J))**C(J)
C        DIVIDED BY 1. + SUM (N=1 TO C(J) OF
C        (G(J)/U(J))**C(J) * 1./N FACTORIAL)
C        * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *
      DO20 I=1,NP
      N(I)=0
   20 N1(I)=0
      DO1 I=2,NP
      J=I-1
      FACT(I)=FACT(J)*J
      IF(FACT(I).GT.9999999999.)GO TO 2
      N(I)=N(I-1)
      GO TO 3
    2 FACT(I)=FACT(I)/10000.
      N2=N2+1
      N(I)=N2
    3 PROD(I)=PROD(I-1)*(G/U)
      IF(PROD(I).GT.9999999999.)GO TO 5
```

```
      N1(I)=N1(I-1)
      GO TO 1
    5 PROD(I)=PROD(I)/10000.
      N3=N3+1
      N1(I)=N3
    1 CONTINUE
      K=0
      DO4 I=2,NP
      IF(K.EQ.1)GO TO 10
      A1=ALOG(PROD(I))
      A2=(N1(I)-N(I))*ALOG(10000.)
      A3=ALOG(FACT(I))
      IF((A1+A2-A3).GT.174.6)GO TO 9
    8 PROBA=EXP(A1+A2-A3)
      IF(DENOM(I-1).GT..1E76)GO TO 9
      DENOM(I)=DENOM(I-1)+PROBA
      IF(P(I-1).LT..1E-75)GO TO 10
      P(I)=EXP(ALOG(PROBA)-ALOG(DENOM(I)))
      GO TO 4
    9 P(I)=.9999999999
      DENOM(I)=DENOM(I-1)
      GO TO 4
   10 K=1
      P(I)=0
    4 CONTINUE
      RETURN
      END
```

SAMPLE PROGRAM DATA

```
151
 92580 1010 1
 2934  1537  2
 4903  2249  23
 2355  2500  09
```

SAMPLE OUTPUT

TOTAL NUMBER OF BEDS TO BE ALLOCATED =150

. MODEL PARAMETERS .

SERVICE	ARRIVAL RATE	LENGTH OF STAY RATE	WEIGHTING FACTOR	BEDS ALLOCATED
MEDICAL	9.2580	0.1010	1.0000	107
SURGICAL	2.9340	0.1537	0.2000	17
OB / GYN	4.9030	0.2249	0.2300	25
PEDIATRICS	2.3550	0.2500	0.0900	1

INSTRUCTIONS FOR USING THE COMPUTER PROGRAM

The computer program shown in this section requires 6,968 bytes of core storage for the object code and 38,220 bytes for the array areas on an S/360, Model 50 computer.

Matrices are dimensioned for a 500 bed hospital. Any number of beds may be allocated up to 500 beds with these statements. For a larger number, the dimension statements must be modified.

The program variable names that are necessary for input and output considerations are given below:

G(I) = arrival rates for service I; I = 1, 2, 3, 4

U(I) = length of stay rates for service I

W(I) = weighting factors for service I

NP = (Total number of beds to be allocated + 1)

IBEDS(1) = optimal number of beds allocated to the 4th stage -- service 1

IBEDS(2) = optimal number of beds allocated to the 3rd stage -- service 2

IBEDS(3) = optimal number of beds allocated to the 2nd stage -- service 3

IBEDS(4) = optimal number of beds allocated to the 1st stage -- service 4.

Two types of data cards are necessary to run the program. The first card must have the total number of beds to be allocated + 1 punched in card columns 1, 2, and 3. For example if 150 beds are to

be allocated among the services, 151 should be punched in the card. The second to the fifth data cards must contain the arrival rates, length of stay rates and weighting factors for the four hospital services. One card is required for each service with the arrival rate punched in card columns 1-5, length of stay rate in card columns 6-11 and weighting factor in card columns 12-17. The format statement in the program implies three decimal places for arrival rates and four decimal places for the length of stay rates and weighting factors.

The program output is of the form shown following the program listing. The format can be easily modified in statements 93 to 114 in the main program.

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AN ANALYSIS OF THE BED ALLOCATION PROBLEM

by

John Elmer Jackson

(ABSTRACT)

The problem of optimally allocating the bed resources of a hospital among the medical, surgical, obstetrical/gynecology and pediatrics patient services was subjected to quantitative analysis in this thesis. The objective of the analytic model, developed to assist hospital administrators with this problem, was taken to be the minimization of the penalty associated with patients not being admitted to the necessary or desired service. Mathematically, the bed allocation problem is given as

$$\underset{c_j}{\text{Minimize}} \quad f(C) = \sum_{j=1}^{4} w_j \lambda_j P_j(c_j)$$

$$\text{Subject to:} \quad \sum_{j=1}^{4} c_j \leq c \text{ and } c_j \geq 0,$$

where

w_j = weighting factor to reflect the penalty for turning a patient away from service j, j = 1, ..., 4,

λ_j = average number of patients arriving per day for service j,

c_j = number of beds allocated to service j,

$P_j(c_j)$ = proportion of arriving patients turned away per day from

service j, given c_j beds available,

C = total number of beds available for allocation.

Data obtained from a representative hospital of the type considered in this study supported the use of the (M/M/c: GD/c/∞) queueing model for determining the values of $P_j(c_j)$. The values of w_j were obtained through the application of an objective weighting scheme.

The bed allocation problem is solved using dynamic programming. Sensitivity analyses are performed on the parameters of the model. Problems are solved for the arrival and length of stay conditions of five representative, small to medium sized, hospitals within Virginia. Results are provided for hospital administrators in the forms of a computer program, tables and graphs. In addition to analyzing and solving the bed allocation problem, this research should be beneficial in the study of similar, but possibly more complex, problems associated with large sized hospitals.