MONOTONE BOUNDS ON THE PRODUCTIVITY

OF

FIXED-CYCLE PRODUCTION LINES

by

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(ABSTRACT)

This research analyzes a class of fixed-cycle production lines. The main concern is the productivity of the lines. Productivity is defined to be the average number of items produced per unit time in the long run. Closed form solutions are derived for the productivity of a two-machine line with dependent machines. These solutions are used to obtain bounds on the productivity of longer lines. The transition matrix associated with an N-machine line is shown to be stochastically monotone yielding monotone increasing lower bounds and monotone decreasing upper bounds which converge to the productivity of the line. These results are then extended to include lines with Markov machines. With the transition matrix in this case having a conditional monotone property, the monotonicity of the bounds is maintained.

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$A_2^{(i,j)}$	event of two failures at machines i and j	21
$\underset{\sim}{b}$	buffer allocation	42
e_i	i^{th} unit vector	25
$p(\underset{\sim}{b})$	productivity of the line with buffer allocation $\underset{\sim}{b}$ . . .	42
$\mathbb{P}$	transition matrix	15
q_i	reliability of machine i	18
W_n	machine state during cycle n	26
$W_n^{(i)}$	state of machine i	12
X_n	buffer state at the beginning of cycle n	12
$X_n^{(i)}$	state of buffer i	18
γ	distribution for X_0	29
π	invariant distribution for a two-machine line	15

To Mother

with fond memories.

CHAPTER 1

INTRODUCTION

In this dissertation, bounds on the productivity of fixed-cycle transfer lines are analyzed. Chapter 1 consists of five sections. The first section gives the significance and motivation of the research. In the second section, a brief problem description is provided. The third section contains a review of relevant literature. The fourth section outlines and explains the organization of this dissertation. The last section states the contribution of the results.

1.1 Significance and Motivation

The study of production systems has significance from both a scientific and an economical viewpoint. The importance of a good understanding becomes more apparent as automation is increasingly being used within the production world. This is due to the fact that malfunctions in automation usually cause a considerable amount of damage and loss. In addition, production system principles have applications in quite a wide range of areas as cited for example in Gershwin and Schick (1980). Moreover, comprehension of the simplest production models namely transfer lines may lead to an insight of more complex models such as networks of production systems of which very little is known at present.

One major concern in the research of production systems is the productivity of the systems. Productivity as defined here is the average number of items produced per unit time in the long run. Although transfer lines are the simplest production models, evaluating system performance measures such as the productivity of a line with more

than two machines is non-trivial. The model becomes intractable when machines break down coupled with finite storage spaces between machines. This is the motivation for resorting to finding bounds on the productivity which will depend on the storage capacities or on the number of machines in the line or both so that a suitable and optimal allocation of storage spaces can be achieved.

1.2 A Brief Problem Description

A transfer line is a series of machines where workpieces are processed from one machine to another. Between machines, there are storage areas called buffers where unfinished workpieces can be stored temporarily. In the queueing context, the machines are servers, workpieces are customers, and buffers are places where customers may queue up waiting for service. A typical representation of an N-machine transfer line is shown in Figure 1.

It is assumed that there is an unlimited supply of items in front of the first machine and an adequate space for finished products after the last machine in the line. The class of transfer lines to be studied here is limited to fixed-cycle lines. A fixed-cycle transfer line is one in which machine processing times are identical and fixed so that transferring of workpieces from one machine to another occurs simultaneously at the end of each cycle.

1.3 Literature Review

Research on production systems dates as far back to the work by Vladzierskii in 1953. Since then production systems have been studied under several different assumptions. An excellent introduction to literature on transfer lines can be found in Schick and Gershwin (1978)

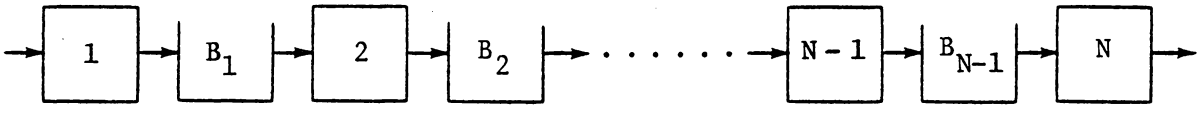


Figure 1: An N-machine Transfer Line.

and Buzacott and Hanifin (1978). One of the earliest models to appear in the literature was by Vladzievskii (1953) who gave results assuming identical machines while Sevasťyanov (1962) extended them to non-identical machines. Apart from these there were works by Hunt (1956) and Koenigsberg (1959).

Several papers by Buzacott (1967), (1968), (1971a), (1971b), (1972) appeared during 1967 to 1972. Analyzed in most of his work were system performances of transfer lines with internal storage areas as oppose to those without storage area. In Buzacott and Hanifin (1978), the authors compared a number of past models on the nature of their assumptions and the validity of results derived when tested with real data. The major problem was the failure of the real data to satisfy the assumptions of the models. Frequently made assumptions include Poisson arrivals of parts at the first machine. This may simplify computation substantially but, as pointed out by Soyster and Toof (1976), it is unrealistic for most of the industrial systems where parts are always available at the first machine. Thus is the assumption made in the present work, i.e., there are always parts available at the first machine.

From 1973 to 1977, Muth (1973), Sheskin (1976), Artamonov (1977), Ignall and Silver (1977), and Okamura and Yamashina (1977) published work on transfer lines under various assumptions using different methods. Recent papers in which extensions to more complicated systems are promising include Schick and Gershwin (1978), Gershwin and Ammar (1979), Gershwin and Schick (1979), (1980), Gershwin and Berman (1981), and Altiok and Stidham (1982).

Although production lines studied in all the research mentioned above vary in their assumptions, the results obtained are mainly for lines with only two or three machines. A closed form solution for the productivity of an N-machine line, $N > 2$, is usually unknown even for independent machines since longer lines are more difficult to analyze owing to machine interference when buffers are full or empty. Much of the work in the analysis of production systems can be boiled down to determining the stationary distribution of a Markov chain. The transition matrix in discrete time processes (or the generator in continuous time processes) normally has a pattern except possibly on the boundaries. Most of the difficulties encountered in determining the stationary distribution are caused by the boundary equations. Because of these, approximation methods have been used by several authors for example, Anderson and Moodie (1969), Masso and Smith (1974), Sheskin (1976), Caseau and Pujolle (1979), and Boxma and Konheim (1981).

Literature on fixed-cycle production systems can be found in Soyster and Toof (1976), Soyster et al. (1979), Dudick (1979), and Foley and Chansaenwilai (1980). In Dudick (1979), the emphasis was on finding repair strategies for the system. In Soyster and Toof (1976), however, the productivity has been computed for a two-machine line in which a machine is up with a fixed probability independent of the state of the other machine and the number of items in the buffer. In contrast, Foley and Chansaenwilai (1980) determined the productivity of a two-machine line where the state of one machine was allowed to depend on the number of items in the buffer and the state of the other machine. The authors showed also that productivity is an increasing function of the

covariance between machines one and two. This is surprising compared to a result by Soyster and Toof (1976) where productivity was shown to be a decreasing function of the covariance between the states of machine two at two consecutive cycles. Moreover, the authors derived bounds on the productivity of an N-machine line in the spirit of Soyster et al. (1979).

1.4 Organization and Outline

There are four chapters in this dissertation. Each chapter is given a number and is organized into several sections identified as a.b where a is the number of the chapter and b is the number of the section. Thus, Section 3.1 refers to the first section of Chapter 3. In each section, definitions, lemmas, theorems, and examples are named a.b.c where a and b are as before while c is the number assigned consecutively within that section. Hence, Theorem 3.1.1 is the first theorem in Section 3.1. Figures are, however, numbered consecutively throughout the dissertation. References are arranged in an alphabetic order with the author's name followed by the year of publication in parentheses.

Chapter 1 gives an overall introduction whereas a detail description of the problem is given in Chapter 2. In Chapter 3, main results are gathered including a closed form for the productivity of a transfer line with two machines, lower and upper bounds for a line with more than two machines, numerical results, and a conjecture for the allocation of storage spaces. Chapter 4 contains the conclusion with some future possible research.

1.5 Contribution

The main results of this dissertation are contained in Chapter 3. Section 3.1 provides closed form solutions for the productivity of a two-machine line with dependent machines. It is also applicable as a special case to lines with independent machines. Furthermore, it can be used in finding bounds on the productivity of longer lines as Theorem 3.3.6 and Theorem 3.3.7 demonstrate. Results in Section 3.2 extend the work in Soyster et al. (1979).

The key contribution is Theorem 3.3.5 where the transition matrix associated with an N-machine line is shown to be stochastically monotone. This together with Theorem 4.2.4 from Stoyan (1977) yield the main results on monotone bounds. These results are then extended to include lines with Markov machines where the transition matrix possesses a conditional monotone property and results on monotone bounds still apply.

CHAPTER 2

PROBLEM DESCRIPTION

There are two sections in this chapter. The first section describes the system under study in detail together with the assumptions made in the model. The second section addresses the question of how realistic the model is.

2.1 Detail Description and Assumptions

The transfer line to be considered here consists of several machines arranged in series. Supply of unprocessed items is always available in front of the first machine. Between each pair of machines, a buffer of finite capacity is provided as a temporary storage for unfinished items. Each item is processed through the machines in the same order starting from the first machine to the second, third, etc., until the last machine. There is always a space to put the finished product from the last machine. Figure 1 shows a transfer line with N machines and N - 1 buffers. A cycle begins when the machines obtain an item to work on from their respective input buffers. Each machine works on its item for one unit of time then transfers the item to its output buffer ending one cycle. As machine processing times are all identical, transferring of an item from each machine occurs simultaneously at the end of each cycle. This synchronous type of production lines is called a fixed-cycle transfer line.

To fully understand the mechanism of the system described above, a fixed-cycle transfer line depicted in Figure 2 with three machines and buffers B_1 and B_2 can be interpreted as in Soyster and Toof (1976) as follows:

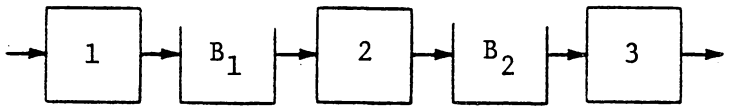


Figure 2: A Three-machine Line.

At the beginning of each cycle, a separate operator at each machine does the following:

First, operator 3 checks

(a) whether machine 3 is up,

(b) whether there is an item available from the input buffer B_2.

If (a) and (b) are satisfied then operator 3 processes an item on machine 3. Otherwise, operator 3 remains idle during this cycle.

Next, operator 2

(a) checks whether machine 2 is up,

(b) if machine 2 is up, checks whether an additional item can be placed in the output buffer B_2,

(c) if the output buffer B_2 can accept another item, checks whether there is an item available from the input buffer B_1.

If (a), (b), and (c) are all satisfied then operator 2 processes an item on machine 2. Otherwise, operator 2 remains idle during this cycle.

Finally, operator 1 checks

(a) whether machine 1 is up,

(b) if machine 1 is up, whether an additional item can be placed in the output buffer B_1.

If (a) and (b) are satisfied then operator 1 processes an item on machine 1. Otherwise, operator 1 remains idle during this cycle.

For an N-machine line where $N > 3$, the same interpretation can be extended with the last machine N in place of machine 3.

2.2 Points to Ponder

Changes in productivity of systems with finite buffers are evidenced when blocking and starving take place. A machine is blocked when its output buffer is full and it is starved when its input buffer is empty. However, there appears to be two different models in the literature for blocking. In one model, the machine will not process any item if its output buffer is full. It will resume work again only when a place in the buffer becomes available. In the other model, blocking does not occur until the buffer is full and the machine finishes an item but has no place to put it. The machine is thus prevented from any further work until a place in the buffer is available again. Altiok and Stidham (1982) discussed via an example that the first model of blocking always gives lower estimates for the efficiencies of the machines. Specifically, the authors comment on the model used in Gershwin and Berman (1981) which is of the first kind with exponential service times and argued that the first model of blocking is inappropriate for many transfer line applications but noted that it is appropriate for models of communication systems. The authors also explained how the model can be modified.

Although blocking of the first kind is assumed in the present model, with a slight amendment namely adding another state to the transition matrix of the system, the model can accommodate the second kind of blocking as well. Furthermore, with infinite supply of items for the first machine and fixed identical processing times at each machine, the model appears to be all the more realistic.

CHAPTER 3

MAIN RESULTS

This chapter consists of five sections, the first of which gives results on a two-machine transfer line to be used in the sequel. Section 3.2 explores some lower bounds on the productivity of an N-machine line when $N > 2$. In the third section, monotone lower and upper bounds are analyzed and derived. The next section on numerical results illustrates, with a three-machine line, the significance of the bounds attained. Finally, the last section presents a rather promising algorithm for the allocation of buffer capacities.

3.1 A Two-machine Line

Consider a two-machine fixed-cycle transfer line with a buffer of capacity b between machines 1 and 2 as shown in Figure 3.

Let X_n be the number of items in the buffer at the beginning of cycle n.

For $i = 1,2$; let $W_n^{(i)}$ be the state of machine i during cycle n where

$$W_n^{(i)} = \begin{cases} 1 \text{ if machine } i \text{ is up during cycle } n \\ 0 \text{ if machine } i \text{ is down during cycle } n \end{cases} .$$

Let X_0 be the initial buffer stock where $0 \leqslant X_0 \leqslant b$ then X_n is recursively defined by

$$X_{n+1} = X_n + \begin{cases} W_n^{(1)} & \text{if } X_n = 0 \\ W_n^{(1)} - W_n^{(2)} & \text{if } 1 \leqslant X_n \leqslant b - 1 \\ W_n^{(2)}(W_n^{(1)} - 1) & \text{if } X_n = b \end{cases}$$

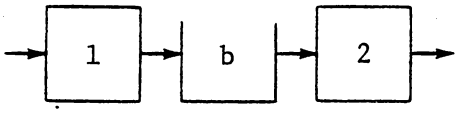


Figure 3: A Two-machine Line with Buffer Capacity b.

Let

$$V_n = \begin{cases} 1 \text{ if an item leaves the line at the end of cycle } n \\ 0 \text{ if no item is produced} \end{cases} .$$

If $\lim_{n\to\infty} \frac{\sum_{i=1}^{n} V_i}{n} = c$ a.s., for some constant c, then the productivity of the line is defined to be c. Otherwise the productivity is undefined.

<u>Hypothesis A</u>. $(W_{n+1}^{(1)},W_{n+1}^{(2)})$ and $(W_n^{(1)},W_n^{(2)},\ldots,W_0^{(1)},W_0^{(2)})$ are conditionally independent given X_{n+1}, i.e.,

$$P(W_{n+1}^{(1)}=i,W_{n+1}^{(2)}=j|X_{n+1}=k,X_n=k_n,\ldots,X_0=k_0,W_n^{(1)}=r_n W_n^{(2)}=s_n,\ldots,$$
$$W_0^{(1)}=r_0,W_0^{(2)}=s_0)$$

$$= P(W_{n+1}^{(1)}=i,W_{n+1}^{(2)}=j|X_{n+1}=k)$$

$$= p_{ij}^{(k)};\ i,j \in \{0,1\},\ 0 \leqslant k \leqslant b.$$

<u>Hypothesis B</u>. $(W_{n+1}^{(1)},W_{n+1}^{(2)})$ and $(X_0,W_n^{(1)},W_n^{(2)},\ldots,W_0^{(1)},W_0^{(2)})$ are independent, i.e.,

$$P(W_{n+1}^{(1)}=i,W_{n+1}^{(2)}=j|X_{n+1}=k,X_n=k_n,\ldots,X_0=k_0,W_n^{(1)}=r_n W_n^{(2)}=s_n,\ldots,$$
$$W_0^{(1)}=r_0,W_0^{(2)}=s_0)$$

$$= P(W_{n+1}^{(1)}=i,W_{n+1}^{(2)}=j)$$

$$= p_{ij};\ i,j \in \{0,1\}.$$

In words, under Hypothesis A the state of one machine is allowed to depend on that of the other machine and the number of items in the

buffer. Under Hypothesis B, however, the state of one machine is allowed to depend on the state of the other machine but not on the number of items in the buffer. A situation where Hypothesis A may hold is when the line is controlled so that the two machines are both in operation only if the buffer is full or empty. For Hypothesis B, the situation may be that there is only one operator available to work on the machines. So if one machine is up, the other machine is down.

Theorem 3.1.1. $\{X_n\}$ forms a time-homogeneous Markov chain with transition matrix $\mathbb{P}$ as shown on the following page.

Proof. The recursive formula for X_n combined with Hypothesis A yield the result. □

Theorem 3.1.2. Under Hypothesis A, if $\{X_n\}$ is irreducible then ν is an invariant distribution for $\{X_n\}$ where $\nu_0 = 1$,

$$\nu_n = \frac{p_{11}^{(0)}+p_{10}^{(0)}}{p_{01}^{(n)}} \prod_{i=1}^{n-1} \frac{p_{10}^{(i)}}{p_{01}^{(i)}}, \quad n = 1,2,\ldots,b$$

and the productivity of the line is $\sum_{k=1}^{b} \pi_k (p_{01}^{(k)}+p_{11}^{(k)})$ where $\pi = \dfrac{\nu}{\sum_{i=0}^{b} \nu_i}$.

Proof. It is straightforward to show that $\nu = \nu\mathbb{P}$.

Now consider the enlarged Markov chain $\{(X_n, W_n^{(1)}, W_n^{(2)})\}$. An invariant probability distribution for this chain is

$P(X_n=k, W_n^{(1)}=i, W_n^{(2)}=j) = \pi_k p_{ij}^{(k)}$. Let

$$f(k,i,j) = \begin{cases} 1 \text{ if } k > 0 \text{ and } j = 1 \\ 0 \text{ otherwise} \end{cases}.$$

$$
\mathbb{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \\ b-1 \\ b \end{array}
\begin{array}{c}
\begin{array}{ccccccc} 0 & 1 & 2 & 3 \cdots & b-2 & b-1 & b \end{array} \\
\begin{bmatrix}
p_{00}^{(0)}+p_{01}^{(0)} & p_{10}^{(0)}+p_{11}^{(0)} & & & & & \\
p_{01}^{(1)} & p_{11}^{(1)}+p_{00}^{(1)} & p_{10}^{(1)} & & & & \\
 & p_{01}^{(2)} & p_{11}^{(2)}+p_{00}^{(2)} & p_{10}^{(2)} & & & \\
 & & & \ddots & & & \\
 & & & & p_{01}^{(b-1)} & p_{11}^{(b-1)}+p_{00}^{(b-1)} & p_{10}^{(b-1)} \\
 & & & & & p_{01}^{(b)} & p_{11}^{(b)}+p_{00}^{(b)}+p_{10}^{(b)}
\end{bmatrix}
\end{array}
$$

Then $V_n = f(X_n, W_n^{(1)}, W_n^{(2)})$ and from Corollary 6.2.22 in Çinlar (1975),

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n} V_i}{n} = \sum_i \sum_j \sum_k f(k,i,j)\pi_k p_{ij}^{(k)}$$

$$= \sum_{k=1}^{b} \pi_k (p_{01}^{(k)} + p_{11}^{(k)})$$

so that the productivity of the line is as claimed. □

<u>Corollary 3.1.3</u>. Under Hypothesis B, if $\{X_n\}$ is irreducible then the productivity is $(1-\pi_0)(p_{01}+p_{11})$ where

$$\pi_0 = \begin{cases} \dfrac{1}{1 + [\frac{p_{11}+p_{10}}{p_{01}}][\frac{1-(p_{10}/p_{01})^b}{1-(p_{10}/p_{01})}]} & \text{if } p_{10} \neq p_{01} \\[2ex] \dfrac{1}{1 + \frac{b(p_{11}+p_{10})}{p_{01}}} & \text{if } p_{10} = p_{01} \end{cases}.$$

<u>Proof</u>. Under Hypothesis B,

$$\sum_{k=1}^{b} \pi_k (p_{01}^{(k)} + p_{11}^{(k)}) = \sum_{k=1}^{b} \pi_k (p_{01}+p_{11})$$

$$= (1-\pi_0)(p_{01}+p_{11}). \ \square$$

3.2 <u>Some Lower Bounds For an N-machine Line</u>

As discussed earlier, a closed form solution for the productivity of an N-machine line with $N > 2$ is usually unknown even for independent machines. In this section, a collection of lower bounds for the productivity of an N-machine fixed-cycle transfer line with $N - 1$ buffers is analyzed where $N > 2$.

Analogous to the two-machine case, under Hypothesis A the machine states are allowed to depend on one another as well as on the number of

items in the buffers while under Hypothesis B the states of the machines are allowed to depend only on one another.

Let $W_n^{(i)}$ be the state of machine i during cycle n for $i = 1,2,\ldots,N$; $X_n^{(j)}$ be the state of buffer j at the beginning of cycle n, $j = 1,2,\ldots,N-1$.

Consider the event U_n where machine N is up during cycle n, machine N - 1 was up during cycle n - 1,..., and machine 1 was up during cycle n - (N - 1), i.e.,

$$U_n = \{W_n^{(N)}=1, W_{n-1}^{(N-1)}=1, \ldots, W_{n-(N-2)}^{(2)}=1, W_{n-(N-1)}^{(1)}=1\}.$$

Note that a part is produced at the end of cycle n if U_n occurs.

Let I_{U_n} be the indicator function of U_n, i.e.,

$$I_{U_n} = \begin{cases} 1 \text{ if } U_n \text{ occurs} \\ 0 \text{ otherwise} \end{cases}.$$

Then, with V_n as defined in Section 3.1, $V_n \geq I_{U_n}$ and $E[V_n] \geq E[I_{U_n}] = P(U_n)$. Thus, a lower bound for the productivity is $\lim_{n\to\infty} \frac{\sum_{i=1}^{n} P(U_i)}{n}$. Hence, $P(U_i)$ must be determined.

<u>Lemma 3.2.1.</u> Under Hypothesis A, $P(U_n) \geq \prod_{i=1}^{N} q_i$ where $q_i = \min_{k_1,k_2,\ldots,k_{N-1}} \{P(W_n^{(i)}=1 \mid X_n^{(1)}=k_1,\ldots,X_n^{(N-1)}=k_{N-1})\}$, the minimum probability of machine i being up during cycle n.

<u>Proof</u>. $P(U_n) = P(W^{(1)}_{n-(N-1)}=1)P(W^{(2)}_{n-(N-2)}=1 \mid W^{(1)}_{n-(N-1)}=1)\cdots$

$$P(W^{(N)}_n=1 \mid W^{(N-1)}_{n-1}=1,\ldots,W^{(1)}_{n-(N-1)}=1)$$

$$\geq \prod_{i=1}^{N} q_i. \ \square$$

Under Hypothesis B, q_i is simply the marginal probability of machine i being up, i.e., $q_i = P(W^{(i)}_n = 1)$, the reliability of machine i.

<u>Corollary 3.2.2</u>. Under Hypothesis B, $P(U_n) = \prod_{i=1}^{N} q_i \quad \forall n \geq N$.

<u>Theorem 3.2.3</u>. A lower bound for the productivity of an N-machine fixed cycle transfer line with N - 1 buffers is $\prod_{i=1}^{N} q_i$ where $N > 2$.

<u>Proof</u>. $\lim_{n\to\infty} \dfrac{\sum_{i=1}^{n} P(U_i)}{n} \geq \lim_{n\to\infty} \dfrac{\sum_{i=1}^{n} (\prod_{i=1}^{N} q_i)}{n} = \prod_{i=1}^{N} q_i. \ \square$

Note that this is also true in the case where all the machines are independent as shown in Soyster et al. (1979). In fact, the approach used here is the same as in Soyster et al. (1979).

Now, instead of considering only the events U_n, better lower bounds can be found by considering the events where one, two or more machines may be down yet a part is produced.

Without loss of generality, assume that a finished product leaves the line at the end of cycle 0. Let $A^{(i)}_1$ be the event where there is a failure at machine i during one cycle disrupting the chain of consecutive up cycles. Specifically, for i = 1,2,...,N,

$$A^{(i)}_1 = \{W^{(N)}_0=1, W^{(N-1)}_{-1}=1, W^{(N-2)}_{-2}=1,\ldots,$$

$$W^{(i)}_{-(N-i)}=1, W^{(i)}_{-(N-i+1)}=0, W^{(i-1)}_{-(N-i+2)}=1,\ldots,W^{(2)}_{-(N-1)}=1, W^{(1)}_{-N}=1\}.$$

Let $C_0 = \{W_0^{(N)}=1, W_{-1}^{(N-1)}=1, \ldots, W_{-(N-1)}^{(1)}=1\}$. Then, by the time-homogeneity of the system, $P(C_0) = P(U_n)$, the lower bound attained before.

Now, with $C_1 = C_0 \cup (\bigcup_{i=1}^{N} A_1^{(i)})$, it is clear that a part is produced if C_1 occurs and $P(C_1) \geqslant P(C_0)$. Hence, a better lower bound is obtained once $P(C_1)$ is known.

<u>Lemma 3.2.4</u>. Under Hypothesis B, with $P(C_0)$ as given by Corollary 3.2.2, $P(C_1) = P(C_0) + (\prod_{j=1}^{N} q_j) \sum_{i=2}^{N} (1-q_i)(1-\prod_{\ell=1}^{i-1} q_\ell)$.

<u>Proof</u>. $P(C_1) = P[C_0 \cup (\bigcup_{i=2}^{N} A_1^{(i)})]$ as $A_1^{(1)} \subset C_0$

$$= P(C_0) + P(\bigcup_{i=2}^{N} A_1^{(i)}) - P[C_0 \cap (\bigcup_{i=2}^{N} A_1^{(i)})]$$

$$= P(C_0) + \sum_{i=2}^{N} P(A_1^{(i)}) - \sum_{i=2}^{N} P(C_0 \cap A_1^{(i)}) \text{ since}$$

$A_1^{(i)}$ and $A_1^{(j)}$ are disjoint for $i \neq j$. Now,

$$P(A_1^{(i)}) = P(W_0^{(N)}=1 \mid W_{-1}^{(N-1)}=1, \ldots, W_{-(N-i)}^{(i)}=1, W_{-(N-i+1)}^{(i)}=0, W_{-(N-i+2)}^{(i-1)}=1, \ldots,$$

$$W_{-N}^{(1)}=1) P(W_{-1}^{(N-1)}=1 \mid W_{-2}^{(N-2)}=1, \ldots, W_{-(N-i)}^{(i)}=1, W_{-(N-i+1)}^{(i)}=0,$$

$$W_{-(N-i+2)}^{(i-1)}=1, \ldots, W_{-N}^{(1)}=1) \cdots P(W_{-(N-i)}^{(i)}=1 \mid W_{-(N-i+1)}^{(i)}=0,$$

$$W_{-(N-i+2)}^{(i-1)}=1, \ldots, W_{-N}^{(1)}=1) P(W_{-(N-i+1)}^{(i)}=0 \mid W_{-(N-i+2)}^{(i-1)}=1, \ldots,$$

$$W_{-N}^{(1)}=1) P(W_{-(N-i+2)}^{(i-1)}=1 \mid W_{-(N-i+3)}^{(i-2)}=1, \ldots, W_{-N}^{(1)}=1) \ldots$$

$$\cdots P(W_{-(N-1)}^{(2)}=1 \mid W_{-N}^{(1)}=1) P(W_{-N}^{(1)}=1)$$

$$= q_N q_{N-1} \cdots q_i (1-q_i) q_{i-1} \cdots q_2 q_1 = (\prod_{j=1}^{N} q_j)(1-q_i).$$

From the definitions of C_0 and $A_1^{(i)}$,

$$C_0 \cap A_1^{(i)} = \{W_0^{(N)}=1, W_{-1}^{(N-1)}=1, \ldots, W_{-(N-i)}^{(i)}=1, W_{-(N-i+1)}^{(i)}=0, W_{-(N-i+2)}^{(i-1)}=1,$$

$$W_{-(N-i+2)}^{(i-1)}=1, \ldots, \ldots, W_{-(N-2)}^{(2)}=1, W_{-(N-1)}^{(2)}=1, W_{-(N-1)}^{(1)}=1, W_{-N}^{(1)}=1\}$$

so that $P(C_0 \cap A_1^{(i)}) = q_N q_{N-1} \cdots q_i (1-q_i) q_{i-1}^2 q_{i-2}^2 \cdots q_2^2 q_1^2$

$$= (\prod_{j=1}^{N} q_j)(1-q_i)(\prod_{\ell=1}^{i-1} q_\ell).$$

Substituting in all these terms, the result follows. □

<u>Remark</u>. Under Hypothesis B, $P(C_1)$ is a better lower bound than $P(C_0)$.

Following the same line of thought, one can consider the events $A_2^{(i,j)}$ where there are two failures at machines i and j with $i = 1,2,\ldots,N$ and $j = 1,2,\ldots,i$, i.e.,

$$A_2^{(i,j)} = \{W_0^{(N)}=1, W_{-1}^{(N-1)}=1, \ldots, W_{-(N-i)}^{(i)}=1, W_{-(N-i+1)}^{(i)}=0, W_{-(N-i+2)}^{(i-1)}=1, \ldots,$$

$$W_{-(N-j+1)}^{(j)}=1, W_{-(N-j+2)}^{(j)}=0, W_{-(N-j+3)}^{(j-1)}=1, \ldots, W_{-N}^{(2)}=1, W_{-(N+1)}^{(1)}=1\}.$$

Then, with $C_2 = C_1 \cup (\bigcup_{(i,j)} A_2^{(i,j)})$, a part is produced if C_2 occurs. Furthermore, $P(C_2) \geq P(C_1)$ implies that an even better lower bound is $P(C_2)$. Indeed, $P(C_2)$ can be determined after some routine manipulations and is given as follows:

$$P(C_2) = P(C_1) + (\sum_{\ell=1}^{N} q_\ell) \sum_{j=2}^{N} \sum_{k=2}^{j} (1-q_j)(1-q_k)(1-\prod_{\ell=1}^{j-1} q_\ell)$$

$$+ (\prod_{\ell=1}^{N} q_\ell) \sum_{i=2}^{N} (1-q_i)^2 (\prod_{\ell=1}^{i-1} q_\ell)(\prod_{\ell=1}^{i-1} q_\ell - 1)$$

$$+ (\prod_{\ell=1}^{N} q_\ell) \sum_{i=3}^{N} \sum_{k=2}^{i-1} (1-q_i)(1-q_k)(\prod_{\ell=1}^{k-1} q_\ell)(\prod_{\ell=1}^{i-1} q_\ell - 1)$$

$$+ (\prod_{\ell=1}^{N} q_\ell) \sum_{i=2}^{N-1} \sum_{j=i+1}^{N} (1-q_i)(1-q_j)^2 (\prod_{\ell=1}^{j-1} q_\ell)(\prod_{\ell=1}^{i-1} q_\ell - 1)$$

$$+ (\prod_{\ell=1}^{N} q_\ell) \sum_{i=2}^{N-2} \sum_{k=i+1}^{N-1} \sum_{j=k+1}^{N} (1-q_i)(1-q_j)(1-q_k)(\prod_{\ell=1}^{j-1} q_\ell)(\prod_{\ell=1}^{i-1} q_\ell - 1)$$

One can attempt to find better and better lower bounds using the foregoing method but, the formulae for the bounds may become more and more cumbersome as can be expected. A more tractable approach is to try to interpret the formulae in the simplest case so as to gain some insights for more general cases.

Take for instance, $q_i = q$ for every i. With $p = 1 - q$, the three lower bounds simplify to

$$P(C_0) = q^N,$$

$$P(C_1) = Npq^N + q^{2N},$$

and

$$P(C_2) = \frac{N}{2}(N+1)p^2q^N + q^{2N} + Npq^{2N} + \frac{N}{2}(N-1)p^2q^{2N}.$$

Obviously, the lower bound q^N is the probability of having all N up cycles. For $P(C_1)$, the first term Npq^N is the probability of any one of the events $A_1^{(i)}$ occurring while the second term q^{2N} is the probability of the event C_0 happening without any occurrence of the events $A_1^{(i)}$. For $P(C_2)$, only the first two terms can be identified satisfactorily as follows:

$\frac{N}{2}(N+1)p^2q^N$ is the probability of any one of the events $A_2^{(i,j)}$ occurring whereas q^{2N} is the probability of the event C_0 occurring with neither the events $A_1^{(i)}$ nor $A_2^{(i,j)}$ happening as well. This can be seen by taking N = 3 for example.

Although the last two terms in $P(C_2)$ ought to be the probability of any of the events $A_1^{(i)}$ happening without the occurrence of any of the events $A_2^{(i,j)}$, it is not clear even with N = 3 as to how this can be justified.

Thus, unless an interpretation leading to a general form for $P(C_k)$ is found, it seems unwise to pursue this method any further.

The following is an example comparing the lower bounds obtained via the above method to the productivity attained numerically.

Example 3.2.6. With N = 3, $q_1 = q_2 = q_3 = q = 0.1$, $p = 1-q = 0.9$, the three lower bounds are $P(C_0) = 0.001$, $P(C_1) = 0.002701$, and $P(C_2) = 0.0048661$ whereas the productivity obtained numerically is 0.0599947 and one can see that they are still quite a distance apart.

3.3 Monotone Bounds

In this section, bounds are derived based on certain stochastic properties of the system. The key to the results lies in Theorem 3.3.5.

Definition 3.3.1. Let $X = (x_1, x_2, \ldots, x_m)$, $Y = (y_1, y_2, \ldots, y_m)$ be two finite dimensional vectors. Then, $X \leqslant Y$ if and only if $x_i \leqslant y_i$ for every $i = 1,2,\ldots,m$ and a function f such that $f(X) \leqslant f(Y)$ for $X \leqslant Y$ is said to be non-decreasing.

<u>Definition 3.3.2.</u> A random vector X is said to be stochastically less than a random vector Y, written as $X \underset{st}{\leq} Y$, if and only if $E[f(X)] \leq E[f(Y)]$ for any non-decreasing function f.

<u>Definition 3.3.3.</u> A transition matrix $\mathbb{P}$ for a Markov chain $\{X_n\}$ is said to be stochastically monotone if and only if $E[f(X_{n+1})|X_n]$ is non-decreasing in X_n for any non-decreasing function f, i.e., if and only if $E[f(X_{k+1})|X_k=\underset{\sim}{i}] \leq E[f(X_{k+1})|X_k=\underset{\sim}{j}]$ for $\underset{\sim}{i} \leq \underset{\sim}{j}$ and any non-decreasing function f.

<u>Theorem 3.3.4.</u> For a time-homogeneous Markov chain $\{X_n\}$ with transition matrix $\mathbb{P}$, if $X_0 \underset{st}{\leq} X_1$ and $\mathbb{P}$ is stochastically monotone then $X_0 \underset{st}{\leq} X_1 \underset{st}{\leq} X_2 \underset{st}{\leq} \ldots\ldots$ [cf. Theorem 4.2.4 in Stoyan (1977)].

<u>Proof.</u> Let $g(X_n) = E[f(X_{n+1})|X_n]$. The proof is by induction as follows:

For n = 1, the result is true since

$$E[f(X_1)] = E[E(f(x_1)|X_0)]$$

$$= E[g(X_0)] \leq E[g(X_1)] \text{ as } X_0 \underset{st}{\leq} X_1 \text{ and g is a non-}$$

decreasing function via $\mathbb{P}$ being stochastically monotone.

Now $E[g(X_1)] = E[E(f(X_2)|X_1)] = E[f(X_2)]$. Hence, $E[f(X_1)] \leq E[f(X_2)]$ and $X_1 \underset{st}{\leq} X_2$ as needed.

Assume the result is true for n = k, i.e., $X_0 \underset{st}{\leq} X_1$ with $\mathbb{P}$ stochastically monotone implies $X_0 \underset{st}{\leq} X_1 \underset{st}{\leq} X_2 \underset{st}{\leq} \ldots \underset{st}{\leq} X_k \underset{st}{\leq} X_{k+1}$.

To show that the result is also true for n = k + 1, i.e., show that $X_0 \underset{st}{\leq} X_1$ with $\mathbb{P}$ stochastically monotone implies $X_0 \underset{st}{\leq} X_1 \underset{st}{\leq} X_2 \underset{st}{\leq} \ldots \underset{st}{\leq} X_{k+1} \underset{st}{\leq} X_{k+2}$.

$$E[f(X_{k+1})] = E[E(f(X_{k+1})|X_k)]$$

$$= E[g(X_k)] \leq E[g(X_{k+1})] \text{ since } g \text{ is a non-decreasing}$$

function by $\mathbb{P}$ being stochastically monotone and $X_k \underset{st}{\leq} X_{k+1}$ by the induction hypothesis.

Now $E[g(X_{k+1})] = E[E(f(X_{k+2})|X_{k+1})] = E[f(X_{k+2})]$. Thus, $E[f(X_{k+1})] \leq E[f(X_{k+2})]$ and $X_{k+1} \underset{st}{\leq} X_{k+2}$ completing the induction steps. □

<u>Theorem 3.3.5</u>. Let $X_n = (X_n^{(1)}, X_n^{(2)}, \ldots, X_n^{(N-1)})$ be the state of the $N - 1$ buffers in an N-machine fixed-cycle transfer line where $X_n^{(i)}$ represents the state of buffer i at the beginning of cycle n for $i = 1,2,\ldots,N-1$. Then, $\{X_n\}$ is a time-homogeneous Markov chain with a stochastically monotone transition matrix $\mathbb{P}$.

<u>Proof</u>. It is routine to see that $\{X_n\}$ does define a time-homogeneous Markov chain.

To show that its transition matrix $\mathbb{P}$ is stochastically monotone, it is sufficient to show that

$$E[f(X_{n+1})|X_n=\underset{\sim}{x}] \leq E[f(X_{n+1})|X_n=\underset{\sim}{x}+e_i]$$

for every $i = 1,2,\ldots,N-1$ where $e_i = (0,0,\ldots,0,1,0,\ldots,0)$ is the $N - 1$ dimensional unit vector with 1 in the ith component and f is a non-decreasing function. Now, since

$$E[f(X_{n+1})|X_n=\underset{\sim}{x}] = \sum_{\underset{\sim}{w}} E[f(X_{n+1})|X_n=\underset{\sim}{x}, W_n=\underset{\sim}{w}]P(W_n=\underset{\sim}{w})$$

and

$$E[f(X_{n+1})|X_n=\underset{\sim}{x}+e_i] = \sum_{\underset{\sim}{w}} E[f(X_{n+1})|X_n=\underset{\sim}{x}+e_i, W_n=\underset{\sim}{w}]P(W_n=\underset{\sim}{w})$$

it suffices to show that $X_{n+1}^{(j)}(\underset{\sim}{x},\underset{\sim}{w}) \leq X_{n+1}^{(j)}(\underset{\sim}{x}+e_i,\underset{\sim}{w})$ $\forall \underset{\sim}{x},\underset{\sim}{w},e_i$, and

$j = 1,2,\ldots,N-1$ where $W_n = (W_n^{(1)}, W_n^{(2)}, \ldots, W_n^{(N)})$ is the state of the machines during cycle n. To do this, let

$$t_i(\underset{\sim}{x},\underset{\sim}{w}) = \begin{cases} 1 \text{ if machine } i \text{ processes an item during cycle } n \\ 0 \text{ otherwise} \end{cases} .$$

for $i = 1,2,\ldots,N$.

Then, for $j = i+2, i+3, \ldots, N-1$,

$$X_{n+1}^{(j)}(\underset{\sim}{x},\underset{\sim}{w}) = X_{n+1}^{(j)}(\underset{\sim}{x}+e_i,\underset{\sim}{w}) \quad \forall \underset{\sim}{x},\underset{\sim}{w}$$

For $j = i + 1$,

$$X_{n+1}^{(i+1)}(\underset{\sim}{x},\underset{\sim}{w}) = x_{i+1} + t_{i+1}(\underset{\sim}{x},\underset{\sim}{w}) - t_{i+2}(\underset{\sim}{x},\underset{\sim}{w})$$

and

$$X_{n+1}^{(i+1)}(\underset{\sim}{x}+e_i,\underset{\sim}{w}) = x_{i+1} + t_{i+1}(\underset{\sim}{x}+e_i,\underset{\sim}{w}) - t_{i+2}(\underset{\sim}{x}+e_i,\underset{\sim}{w}).$$

Now, as $t_{i+2}(\underset{\sim}{x},\underset{\sim}{w}) = t_{i+2}(\underset{\sim}{x}+e_i,\underset{\sim}{w})$ and

$$t_{i+1}(\underset{\sim}{x},\underset{\sim}{w}) \leqslant t_{i+1}(\underset{\sim}{x}+e_i,\underset{\sim}{w}) \text{ since } \underset{\sim}{x} \leqslant \underset{\sim}{x} + e_i,$$

it follows that $X_{n+1}^{(i+1)}(\underset{\sim}{x},\underset{\sim}{w}) \leqslant X_{n+1}^{(i+1)}(\underset{\sim}{x}+e_i,\underset{\sim}{w}) \quad \forall \underset{\sim}{x},\underset{\sim}{w}$.

For $j = i$,

$$X_{n+1}^{(i)}(\underset{\sim}{x},\underset{\sim}{w}) = x_i + t_i(\underset{\sim}{x},\underset{\sim}{w}) - t_{i+1}(\underset{\sim}{x},\underset{\sim}{w})$$

and

$$X_{n+1}^{(i)}(\underset{\sim}{x}+e_i,\underset{\sim}{w}) = x_i + 1 + t_i(\underset{\sim}{x}+e_i,\underset{\sim}{w}) - t_{i+1}(\underset{\sim}{x}+e_i,\underset{\sim}{w}).$$

Now, both $t_i(\underset{\sim}{x},\underset{\sim}{w}) - t_{i+1}(\underset{\sim}{x},\underset{\sim}{w})$ and $t_i(\underset{\sim}{x}+e_i,\underset{\sim}{w}) - t_{i+1}(\underset{\sim}{x}+e_i,\underset{\sim}{w})$ can take the values $-1,0,1$. Every combination from these values for the two quantities results in

$$X_{n+1}^{(i)}(\underset{\sim}{x},\underset{\sim}{w}) \leqslant X_{n+1}^{(i)}(\underset{\sim}{x}+e_i,\underset{\sim}{w})$$

except when

$$t_i(\underset{\sim}{x},\underset{\sim}{w}) - t_{i+1}(\underset{\sim}{x},\underset{\sim}{w}) = 1 \text{ and } t_i(\underset{\sim}{x}+e_i,\underset{\sim}{w}) - t_{i+1}(\underset{\sim}{x}+e_i,\underset{\sim}{w}) = -1$$

However, this can never occur since the same machine state $\underset{\sim}{w}$ is used and $X_n^{(i+1)}(\underset{\sim}{x},\underset{\sim}{w}) = X_n^{(i+1)}(\underset{\sim}{x}+e_i,\underset{\sim}{w})$. Hence, $X_{n+1}^{(i)}(\underset{\sim}{x},\underset{\sim}{w}) \leq X_{n+1}^{(i)}(\underset{\sim}{x}+e_i,\underset{\sim}{w})\ \forall\, \underset{\sim}{x},\underset{\sim}{w}$.

For $j = i - 1$,

$$X_{n+1}^{(i-1)}(\underset{\sim}{x},\underset{\sim}{w}) = x_{i-1} + t_{i-1}(\underset{\sim}{x},\underset{\sim}{w}) - t_i(\underset{\sim}{x},\underset{\sim}{w})$$

and

$$X_{n+1}^{(i-1)}(\underset{\sim}{x}+e_i,\underset{\sim}{w}) = x_{i-1} + t_{i-1}(\underset{\sim}{x}+e_i,\underset{\sim}{w}) - t_i(\underset{\sim}{x}+e_i,\underset{\sim}{w}).$$

Similarly as before, $t_{i-1}(\underset{\sim}{x},\underset{\sim}{w}) - t_i(\underset{\sim}{x},\underset{\sim}{w})$ and $t_{i-1}(\underset{\sim}{x}+e_i,\underset{\sim}{w}) - t_i(\underset{\sim}{x}+e_i,\underset{\sim}{w})$ can have the values $-1,0,1$. Now, $X_{n+1}^{(i-1)}(\underset{\sim}{x},\underset{\sim}{w}) \leq X_{n+1}^{(i-1)}(\underset{\sim}{x}+e_i,\underset{\sim}{w})$ for all combinations except the following three:

(i) $t_{i-1}(\underset{\sim}{x},\underset{\sim}{w}) - t_i(\underset{\sim}{x},\underset{\sim}{w}) = 1,\ t_{i-1}(\underset{\sim}{x}+e_i,\underset{\sim}{w}) - t_i(\underset{\sim}{x}+e_i,\underset{\sim}{w}) = -1$

(ii) $t_{i-1}(\underset{\sim}{x},\underset{\sim}{w}) - t_i(\underset{\sim}{x},\underset{\sim}{w}) = 1,\ t_{i-1}(\underset{\sim}{x}+e_i,\underset{\sim}{w}) - t_i(\underset{\sim}{x}+e_i,\underset{\sim}{w}) = 0$

(iii) $t_{i-1}(\underset{\sim}{x},\underset{\sim}{w}) - t_i(\underset{\sim}{x},\underset{\sim}{w}) = 0,\ t_{i-1}(\underset{\sim}{x}+e_i,\underset{\sim}{w}) - t_i(\underset{\sim}{x}+e_i,\underset{\sim}{w}) = -1$

Again, with the same machine state $\underset{\sim}{w}$ and the fact that $X_n^{(i)}(\underset{\sim}{x},\underset{\sim}{w}) \leq X_n^{(i)}(\underset{\sim}{x}+e_i,\underset{\sim}{w})$ together with $X_n^{(i-1)}(\underset{\sim}{x},\underset{\sim}{w}) = X_n^{(i-1)}(\underset{\sim}{x}+e_i,\underset{\sim}{w})$, the above three combinations can never happen. Thus,

$$X_{n+1}^{(i-1)}(\underset{\sim}{x},\underset{\sim}{w}) \leq X_{n+1}^{(i-1)}(\underset{\sim}{x}+e_i,\underset{\sim}{w})\ \forall\, \underset{\sim}{x},\underset{\sim}{w}.$$

For $j = i-2,i-3,\ldots,3,2,1$, the same argument applies. □

The definition of V_n in Section 3.1 implies that

$$V_n = I_{\{X_n^{(N-1)} > 0\}} W_n^{(N)}$$

Now, with the transition matrix $\mathbb{P}$ being stochastically monotone and a distribution for X_0 such that $X_0 \underset{st}{\leq} X_1$, Theorem 3.3.4 implies that

$$P(X_0^{(N-1)}>0) \leq P(X_1^{(N-1)}>0) \leq P(X_2^{(N-1)}>0) \leq \ldots$$

Hence, by Corollary 6.2.22 of Çinlar (1975), provided the chain is irreducible

$$\lim_{n\to\infty} P(X_n^{(N-1)}>0)P(W_n^{(N)}=1) = \lim_{n\to\infty} \frac{\sum_{i=1}^{n} V_i}{n}$$

which is the productivity. Thus, it follows that

$$P(X_n^{(N-1)}>0)P(W_n^{(N)}=1)$$

is the nth step lower bound which is increasing and tends to the productivity of the line.

As mentioned earlier, Theorem 3.3.5 is the key to the results in this section. With the transition matrix $\mathbb{P}$ being stochastically monotone, the objective is to find a distribution for X_0 such that $X_0 \underset{st}{\leq} X_1$ and the lower bound for the productivity is better than the lower bound $\prod_{i=1}^{N} q_i$ where q_i is the reliability of machine i as before. For then, Theorem 3.3.4 implies that a set of monotone increasing lower bounds can be found. Two approaches can be used to achieve this goal. The first approach employs, as an initial distribution for X_0, an extreme distribution where the probability of having empty buffers is 1. This will yield better lower bounds only after a certain number of transitions. The other approach, a non-extreme distribution for X_0 is used in which a better lower bound is guaranteed from the very beginning.

For a three-machine line, a non-extreme distribution for X_0 can be determined as the following theorems illustrate.

Theorem 3.3.6. In a three-machine line with buffer capacities of 2 each, reliability $q_i = q$, $i = 1,2,3$, and independent machines, a distribution γ for X_0 such that $X_0 \underset{st}{\leqslant} X_1$ is given by $\gamma_{ij} = \pi_i c_j$; $i,j \in \{0,1,2\}$ where c_j is the probability of having j items in the second buffer with $c_2 = 0$, $c_1 = c$, $c_0 = 1 - c$, $0 \leqslant c \leqslant \frac{q}{1+q-q^2}$, and π is the invariant probability distribution for the two-machine line consisting of the first two machines.

Proof. The transition matrix $\mathbb{P}$ for $\{X_n\}$ in this case is shown on the following page.

By Theorem 3.1.2, with independent machines, the probability π_i of having i items in the first buffer amounts to $\pi_0 = \frac{1-q}{3-q}$, $\pi_1 = \pi_2 = \frac{1}{3-q}$. Thus,

$$\gamma_{00} = \frac{(1-q)(1-c)}{3-q},\ \gamma_{01} = \frac{(1-q)c}{3-q},\ \gamma_{02} = 0,$$

$$\gamma_{10} = \frac{1-c}{3-q},\ \gamma_{11} = \frac{c}{3-q},\ \gamma_{12} = 0,$$

$$\gamma_{20} = \frac{1-c}{3-q},\ \gamma_{21} = \frac{c}{3-q},\ \gamma_{22} = 0, \text{ and}$$

$$(\gamma\mathbb{P})_{00} = \frac{(1-q)^2(1-c+cq)}{3-q},\ (\gamma\mathbb{P})_{01} = \frac{(1-q)(c-3cq+2cq^2+q)}{3-q},$$

$$(\gamma\mathbb{P})_{02} = \frac{cq(1-q)^2}{3-q},\ (\gamma\mathbb{P})_{10} = \frac{(1-q)(1-c+cq)}{3-q},\ (\gamma\mathbb{P})_{11} = \frac{c-3cq+2cq^2+q}{3-q},$$

$$(\gamma\mathbb{P})_{12} = \frac{cq(1-q)}{3-q},\ (\gamma\mathbb{P})_{20} = \frac{(1-q)(1-c-cq^2+q)}{3-q},$$

$$(\gamma\mathbb{P})_{21} = \frac{c(1-q)(1-2q^2)+q^2}{3-q},\ (\gamma\mathbb{P})_{22} = \frac{cq^2(1-q)}{3-q}.$$

To show that $X_0 \underset{st}{\leqslant} X_1$ by definition is equivalent to showing that $E[f(X_0)] \leqslant E[f(X_1)]$ for any non-decreasing function f. Now,

$$(3-q)E[f(X_0)] = (1-q)(1-c)f((0,0))+c(1-q)f((0,1))+(1-c)f((1,0))$$
$$+ cf((1,1))+(1-c)f((2,0))+cf((2,1))$$

and

$$(3-q)E[f(X_1)] = (1-q)^2(1-c+cq)f((0,0))+(1-q)(c-3cq+2cq^2+q)f((0,1))$$
$$+ (1-q)(1-c+cq)f((1,0))+(c-3cq+2cq^2+q)f((1,1))$$
$$+ (1-q)(1-c+cq^2+q)f((2,0))+\{c(1-q)(1-2q^2)+q^2\}f((2,1))$$
$$+ cq(1-q)^2f((0,2))+cq(1-q)f((1,2))+cq^2(1-q)f((2,2)).$$

So, the result follows if $(3-q)E[f(X_0)] \leqslant (3-q)E[f(X_1)]$.

Gathering those terms with $f((0,0)),f((0,1)),f((1,0)),f((1,1))$, $f((2,0))$, and $f((2,1))$ onto the left-hand side of the inequality to be shown then, what is left on the right-hand side is

$$cq(1-q)^2f((0,2))+cq(1-q)f((1,2))+cq^2(1-q)f((2,2))$$

and the left-hand side consists of the sum of the followings:

(i) $q(1-q)(1+cq-2c)f((0,0))$

(ii) $q(1-q)(3c-2cq-1)f((0,1))$

(iii) $q(1+cq-2c)f((1,0))$

(iv) $q(3c-2cq-1)f((1,1))$

(v) $\{q^2+cq(q^2-q-1)\}f((2,0))$

(vi) $\{2cq^2(1-q)+cq-q^2\}f((2,1))$

Now, if $3c-2cq-1 \leqslant 0$, i.e., if $c \leqslant \frac{1}{3-2q}$, then

(ii) $\leqslant q(1-q)(3c-2cq-1)f((0,0))$ and

(iv) $\leqslant q(3c-2cq-1)f((1,0))$ as f is non-decreasing.

$$
\mathbb{P} = \begin{array}{c}
\\ (0,0) \\ (0,1) \\ (0,2) \\ (1,0) \\ (1,1) \\ (1,2) \\ (2,0) \\ (2,1) \\ (2,2)
\end{array}
\begin{array}{c}
\begin{array}{ccccccccc}
(0,0) & (0,1) & (0,2) & (1,0) & (1,1) & (1,2) & (2,0) & (2,1) & (2,2)
\end{array} \\
\begin{bmatrix}
1-q & & & q & & & & & \\
(1-q)q & (1-q)^2 & & q^2 & q(1-q) & & & & \\
 & (1-q)q & (1-q)^2 & & q^2 & (1-q)q & & & \\
 & (1-q)q & & (1-q)^2 & q^2 & & (1-q)q & & \\
 & (1-q)q^2 & (1-q)^2q & (1-q)^2q & q^3+(1-q)^3 & (1-q)q^2 & (1-q)q^2 & (1-q)^2q & \\
 & & (1-q)q^2 & & (1-q)^2q & q^3+(1-q)^3+(1-q)q & & (1-q)q^2 & (1-q)^2q \\
 & & & & (1-q)q & & 1-q & q^2 & \\
 & & & & (1-q)q^2 & (1-q)^2q & (1-q)q & q^3+(1-q)^2 & (1-q)q^2 \\
 & & & & & (1-q)q^2 & & (1-q)q & q^3+(1-q)
\end{bmatrix}
\end{array}
$$

Therefore, the sum on the left-hand side is at most

$$cq(1-q)^2 f((0,2))+cq(1-q)f((1,2))+q^2\{f((2,0))-f((2,1))\}$$
$$+\ cq(q^2-q-1)f((2,0))+\{2cq^2(1-q)+cq\}f((2,1)).$$

So, the inequality follows if it can be shown that

$$q^2\{f((2,0))-f((2,1))\}+cq(q^2-q-1)f((2,0))+\{2cq^2(1-q)+cq\}f((2,1))$$
$$-\ cq^2(1-q)f((2,2)) \leqslant 0.$$

Now,

$$q^2\{f((2,0))-f((2,1))\}+cq(q^2-q-1)f((2,0))+\{2cq^2(1-q)+cq\}f((2,1))$$
$$-\ cq^2(1-q)f((2,2)) \leqslant q^2\{f((2,0))-f((2,1))\}+cq(q^2-q-1)f((2,0))$$
$$+\ \{2cq^2(1-q)+cq\}f((2,1))-cq^2(1-q)f((2,1))$$
$$= q^2\{f((2,0))-f((2,1))\}+cq(q^2-q-1)\{f((2,0))-f((2,1))\}$$
$$= \{q^2+cq(q^2-q-1)\}\{f((2,0))-f((2,1))\}$$
$$\leqslant 0 \text{ if } q^2+cq(q^2-q-1) \geqslant 0, \text{ i.e., if } c \leqslant \frac{q}{1+q-q^2}.$$

However, as obtained earlier, it is necessary that $c \leqslant \frac{1}{3-2q}$ as well. Thus, since both $\frac{1}{3-2q}$ and $\frac{q}{1+q-q^2}$ are between 0 and 1 with $\frac{q}{1+q-q^2} \leqslant \frac{1}{3-2q}$, the result $E[f(X_0)] \leqslant E[f(X_1)]$ follows. □

<u>Theorem 3.3.7</u>. With the distribution γ for X_0 as given in Theorem 3.3.6 with $c = \frac{q}{1+q-q^2}$, the lower bound cq obtained is better than the lower bound q^3.

<u>Proof</u>. That cq is a lower bound follows from the fact that the transition matrix $\mathbb{P}$ for $\{X_n\}$ is stochastically monotone with $X_0 \underset{st}{\leqslant} X_1$

and c being the probability of having non-empty content in the second buffer where q is the reliability of the last machine 3.

To show that $cq \geqslant q^3$ is equivalent to showing that $\frac{1}{1+q-q^2} \geqslant q$ or that $q^3-q^2-q+1 \geqslant 0$.

Now, $q^3-q^2-q+1 = (q-1)(q^2-1) = (q-1)^2(q+1) \geqslant 0$. Therefore, the lower bound cq is indeed better than the lower bound q^3. □

To illustrate further, the lower bound cq computed with q = 0.1 is 0.0091743 while the best lower bound obtained in Example 3.2.6 with the same value of q is 0.0048661. This shows that the lower bound cq is superior although it is still far from the true productivity 0.0599947.

In general, an extreme distribution for X_0 can be used where the probability of having empty buffers is 1. Numerical results in the case of three-machine lines show that after a few steps, the lower bounds obtained from the extreme distribution become as good as those from the non-extreme one.

For upper bounds, a similar result holds and is given in Theorem 3.3.8.

<u>Theorem 3.3.8</u>. For a time-homogeneous Markov chain $\{X_n\}$ with transition matrix $\mathbb{P}$, if $X_0 \underset{st}{\geqslant} X_1$ and $\mathbb{P}$ is stochastically monotone then $X_0 \underset{st}{\geqslant} X_1 \underset{st}{\geqslant} X_2 \underset{st}{\geqslant} \ldots\ldots$

<u>Proof</u>. Analogous to Theorem 3.3.4. □

Similarly to Theorem 3.3.4, the above Theorem 3.3.8 implies that with $\mathbb{P}$ stochastically monotone, a set of monotone decreasing upper bounds can be found once a distribution for X_0 such that $X_0 \underset{st}{\geqslant} X_1$ is known. An extreme distribution for X_0 in this case is the one where the

probability of having full buffers is 1. As in the case of lower bounds, a similar argument shows that the upper bounds obtained are decreasing and tend to the productivity.

In summary, the Markov chain $\{X_n\}$ describing the buffer states of an N-machine fixed-cycle transfer line possesses a stochastically monotone transition matrix and by using suitably chosen distributions for X_0, sets of monotone increasing lower bounds and monotone decreasing upper bounds can be determined which converge to the productivity of the line.

The results obtained in this section up to this point are for lines with independent machines. A more realistic model should include some dependence in the assumptions on the machines. To accomplish this, a Markov assumption will be assumed for each machine. Specifically, the state of each machine during each cycle depends on the state during the previous cycle. Hence, the states of the Markov chain for an N-machine line is enlarged to the Markov chain $\{(X_n,W_n)\}$ where X_n is the state of the N - 1 buffers and W_n is the machine states as before. Definition 3.3.3, Theorem 3.3.4, and Theorem 3.3.8 can now be extended.

<u>Definition 3.3.9</u>. A transition matrix $\mathbb{P}$ for a Markov chain $\{(X_n,Y_n)\}$ is said to be conditionally monotone if and only if, for each $\underset{\sim}{i}$, the function $g_{\underset{\sim}{i}}(X_n) = E[f(X_{n+1})|X_n,Y_n = \underset{\sim}{i}]$ is non-decreasing in X_n for any non-decreasing function f.

<u>Theorem 3.3.10</u>. For a time-homogeneous Markov chain $\{(X_n,Y_n)\}$ with transition matrix $\mathbb{P}$, if $X_0 \underset{st}{\leq} X_1$, Y_n is identically distributed, and $\mathbb{P}$ is conditionally monotone then $X_0 \underset{st}{\leq} X_1 \underset{st}{\leq} X_2 \underset{st}{\leq} \cdots\cdots$

Proof. It suffices to show that $X_{k-1} \underset{st}{\leq} X_k$ implies $X_k \underset{st}{\leq} X_{k+1}$. Now, $E[f(X_k)] = E[E(f(X_k)|X_{k-1})]$ and

$$E[f(X_{k+1})] = E[E(f(X_{k+1})|X_k)].$$

Moreover, $E(f(X_k)|X_{k-1}) = \sum_{\underset{\sim}{i}} E(f(X_k)|X_{k-1},Y_{k-1}=\underset{\sim}{i})P(Y_{k-1}=\underset{\sim}{i})$ and $E(f(X_{k+1})|X_k) = \sum_{\underset{\sim}{i}} E(f(X_{k+1})|X_k,Y_k=\underset{\sim}{i})P(Y_k=\underset{\sim}{i})$. However, $E(f(X_k)|X_{k-1},Y_{k-1}=\underset{\sim}{i}) \leq E(f(X_{k+1})|X_k,Y_k=\underset{\sim}{i})$ as $X_{k-1} \underset{st}{\leq} X_k$ and $\mathbb{P}$ is conditionally monotone together with Y_n identically distributed. Hence, $E[f(X_k)] \leq E[f(X_{k+1})]$ so that $X_k \underset{st}{\leq} X_{k+1}$. □

Theorem 3.3.11. For a time-homogeneous Markov chain $\{(X_n,Y_n)\}$ with transition matrix $\mathbb{P}$, if $X_0 \underset{st}{\geq} X_1$, Y_n is identically distributed, and $\mathbb{P}$ is conditionally monotone then $X_0 \underset{st}{\geq} X_1 \underset{st}{\geq} X_2 \underset{st}{\geq} \cdots$.

Proof. Analogous to Theorem 3.3.10. □

Theorem 3.3.12. The time-homogeneous Markov chain $\{(X_n,W_n)\}$ has a transition matrix which is conditionally monotone where X_n is the buffer states and W_n the machine states of an N-machine line with N - 1 buffers.

Proof. From Theorem 3.3.5, one sees that in essence, the property of conditionally monotone of the transition matrix is established in the proof. □

The nth step lower bound for the productivity in this case is

$$P(X_n^{(N-1)}>0|W_n^{(N)}=1)P(W_n^{(N)}=1).$$

To see that these bounds are increasing in n, it suffices to show that

$$E[f(X_k^{(N-1)})|W_k^{(N)}=1] \leq E[f(X_{k+1}^{(N-1)})|W_{k+1}^{(N)}=1]$$

for any k and for any non-decreasing function f since all the W_n are identically distributed.

Now,

$$E[f(X_k^{(N-1)})|W_k^{(N)}=1] = E[E(f(X_k^{(N-1)})|X_{k-1}^{(N-1)},W_k^{(N)}=1)|W_k^{(N)}=1]$$

and

$$E[f(X_{k+1}^{(N-1)})|W_{k+1}^{(N)}=1] = E[E(f(X_{k+1}^{(N-1)})|X_k^{(N-1)},W_{k+1}^{(N)}=1)|W_{k+1}^{(N)}=1].$$

But as $\mathbb{P}$ is conditionally monotone with $X_{k-1} \underset{st}{\leq} X_k$,

$$E[f(X_k^{(N-1)})|X_{k-1}^{(N-1)},W_k^{(N)}=1] \leq E[f(X_{k+1}^{(N-1)})|X_k^{(N-1)},W_{k+1}^{(N)}=1]$$

and the result follows.

A similar argument applies in the case of upper bounds. The fact that these bounds converge to the productivity follows by Corollary 6.2.22 of Çinlar (1975) provided that the Markov chain $\{(X_n,W_n)\}$ is irreducible.

In conclusion, for an N-machine line with Markov machines, monotone increasing lower bounds and monotone decreasing upper bounds which converge to the productivity of the line can be derived according to the last three theorems since the transition matrix is conditionally monotone.

3.4 Numerical Results

The followings are some numerical results on three-machine lines with buffer capacities of 2 each and different values of machine reliabilities. The first columns show lower bounds at each step, obtained by using an extreme distribution for X_0 while the second columns are those obtained via the non-extreme distribution given in Theorem 3.3.7. It can be seen that within 8 steps or less, the values of both columns become very close.

With machine reliability $q_i = 0.1$, $i = 1,2,3$

0.006897	0.014584
0.012762	0.019250
0.017773	0.023290
0.022076	0.026803
0.025793	0.029874
0.029022	0.032571
0.031845	0.034952
0.034328	0.037066
0.036523	0.038950
0.038474	0.040637
0.040216	0.042154
0.041779	0.043523
0.043186	0.044762
0.044458	0.045887
0.045610	0.046911
0.046658	0.047845

With machine reliability $q_i = 0.5$, $i = 1,2,3$

0.200000	0.260000
0.262500	0.292500
0.293750	0.312499
0.313085	0.325741
0.325976	0.334799
0.334869	0.341135
0.341130	0.345646
0.345608	0.348904
0.348853	0.351283
0.351232	0.353039
0.352992	0.354344
0.354304	0.355319
0.355286	0.356053
0.356026	0.356605
0.356584	0.357023
0.357007	0.357340

With machine reliability $q_i = 0.9$, $i = 1,2,3$

0.771428	0.782044
0.782614	0.791064
0.791508	0.798337
0.798666	0.804208
0.804456	0.808943
0.809130	0.812758
0.812900	0.815828
0.815936	0.818296
0.818378	0.820279
0.820341	0.821872
0.821919	0.823149
0.823185	0.824174
0.824202	0.824997
0.825018	0.825656
0.825672	0.826185
0.826197	0.826609

Next, both lower and upper bounds of productivity of a three-machine line are obtained using extreme distributions for X_0. The two buffers may have different capacities. So are the reliabilities of the three machines. In most cases, convergence is achieved within much less than 100 steps.

With $q_1 = 0.1$, $q_2 = 0.2$, $q_3 = 0.3$; and

1st buffer capacity = 2nd buffer capacity = 1

Step	Lower bound	Upper bound
1	0.000000	0.228000
2	0.006000	0.186708
3	0.014436	0.161605
4	0.023296	0.144866
5	0.031547	0.132466
6	0.038745	0.122449
7	0.044778	0.113926
8	0.049704	0.106520
9	0.053658	0.100073
10	0.056795	0.094506
11	0.059267	0.089755
12	0.061207	0.085751
13	0.062728	0.082416
14	0.063918	0.079668
15	0.064852	0.077422
20	0.067258	0.071259
25	0.068004	0.069269
30	0.068240	0.068640
35	0.068315	0.068442
40	0.068339	0.068379
45	0.068346	0.068359
50	0.068349	0.068353
55	0.068349	0.068351
59	0.068350	0.068350
60	0.068350	0.068350

With $q_1 = 0.1$, $q_2 = 0.5$, $q_3 = 0.9$; and

1st buffer capacity = 2nd buffer capacity = 2

Step	Lower bound	Upper bound
1	0.000000	0.900000
2	0.045000	0.717750
3	0.069525	0.617512
4	0.082061	0.422338
5	0.088884	0.287053
6	0.092821	0.210291
7	0.095179	0.166673
8	0.096623	0.140860
9	0.097518	0.125110
10	0.098075	0.115346
15	0.098918	0.100593
20	0.099000	0.099163
25	0.099008	0.099024
30	0.099009	0.099010
31	0.099009	0.099010
32	0.099009	0.099009

With $q_1 = 0.9$, $q_2 = 0.1$, $q_3 = 0.5$; and

1st buffer capacity = 2nd buffer capacity = 2

Step	Lower bound	Upper bound
1	0.000000	0.500000
2	0.045000	0.398750
3	0.069525	0.302562
4	0.082061	0.231687
5	0.088884	0.183832
6	0.092820	0.152723
7	0.095178	0.132859
8	0.096620	0.120287
9	0.097514	0.112367
10	0.098070	0.107388
16	0.098945	0.099511
21	0.098996	0.099050
26	0.099000	0.099006
31	0.099001	0.099001

With $q_1 = 0.1$, $q_2 = 0.2$, $q_3 = 0.3$;

1st buffer capacity = 1, and 2nd buffer capacity = 2

Step	Lower bound	Upper bound
1	0.000000	0.300000
2	0.006000	0.282720
3	0.014436	0.261601
4	0.023316	0.240674
5	0.031630	0.220848
6	0.038945	0.202246
7	0.045148	0.184920
8	0.050284	0.168967
9	0.054471	0.154500
10	0.057850	0.141589
11	0.060557	0.130247
12	0.062716	0.120420
13	0.064434	0.112010
14	0.065798	0.104888
15	0.066880	0.098906
16	0.067738	0.093918
17	0.068419	0.089782
18	0.068961	0.086368
19	0.069391	0.083561
20	0.069734	0.081259
25	0.070653	0.074732
30	0.070960	0.072400
35	0.071065	0.071575
40	0.071102	0.071283
45	0.071115	0.071179
50	0.071120	0.071143
55	0.071122	0.071130
60	0.071122	0.071125
65	0.071122	0.071123
70	0.071122	0.071123
75	0.071122	0.071122

3.5 Allocation of Buffers

A problem frequently addressed in the literature of transfer lines is the allocation of buffer space. For a given set of buffer capacities, attempts have been made to allocate buffer space to achieve maximal productivity. In this section, a buffer allocation scheme is discussed. It includes an algorithm in which more spaces are added to the buffers where they are most needed. The conjecture here is that this algorithm yields the optimal allocation.

Let $\underset{\sim}{b} = (b_1, b_2, \ldots, b_{N-1})$ be a vector representing a buffer allocation of an N-machine transfer line where b_i is the capacity of buffer i, $i = 1,2,\ldots,N-1$.

Let $p(\underset{\sim}{b})$ be the productivity of the line corresponding to buffer allocation $\underset{\sim}{b}$.

For a fixed total buffer capacity, the algorithm proceeds step by step as follows:

(1) Start with $\underset{\sim}{b} = (1,1,\ldots,1)$.

(2) For each $i = 1,2,\ldots,N-1$, determine $p(\underset{\sim}{b}+e_i)$ where e_i is the ith unit vector.

(3) Set $\underset{\sim}{b}^* = \underset{\sim}{b}+e_i$ where

$$p(\underset{\sim}{b}+e_i) = \max_j \{p(\underset{\sim}{b}+e_j)\}, \quad j = 1,2,\ldots,N-1.$$

(4) Repeat steps (1), (2), (3) with $\underset{\sim}{b}$ replaced by $\underset{\sim}{b}^*$.

<u>Conjecture 3.5.1</u>. The algorithm yields the optimal buffer allocation.

By allocating one more space to buffer i where

$$p(\underset{\sim}{b} + e_i) \geq p(\underset{\sim}{b} + e_j) \quad \forall j,$$

other buffer allocations in which no additional space is added to buffer

i are eliminated. Thus, allocations of the form

$$\underset{\sim}{b} + e_j + \sum_{k \neq i} a_k e_k$$

no longer exist where a_k is some positive integer. Hence, it would suffice to show that

$$p(\underset{\sim}{b} + e_i + \sum_{k \neq i} a_k e_k) \geq p(\underset{\sim}{b} + e_j + \sum_{k \neq i} a_k e_k).$$

To do this, it would suffice to show that

$$p(\underset{\sim}{b} + e_i) \geq p(\underset{\sim}{b} + e_j) \quad \forall j \neq i$$

implies

$$p(\underset{\sim}{b} + e_i + e_k) \geq p(\underset{\sim}{b} + e_j + e_k) \quad \forall k \neq i.$$

It would be sufficient since the inequality

$$p(\underset{\sim}{b} + e_i + \sum_{k \neq i} a_k e_k) \geq p(\underset{\sim}{b} + e_j + \sum_{k \neq i} a_k e_k)$$

can be proved inductively by letting $\underset{\sim}{b}' = \underset{\sim}{b} + e_k$ so that

$$p(\underset{\sim}{b}' + e_i) \geq p(\underset{\sim}{b}' + e_j)$$

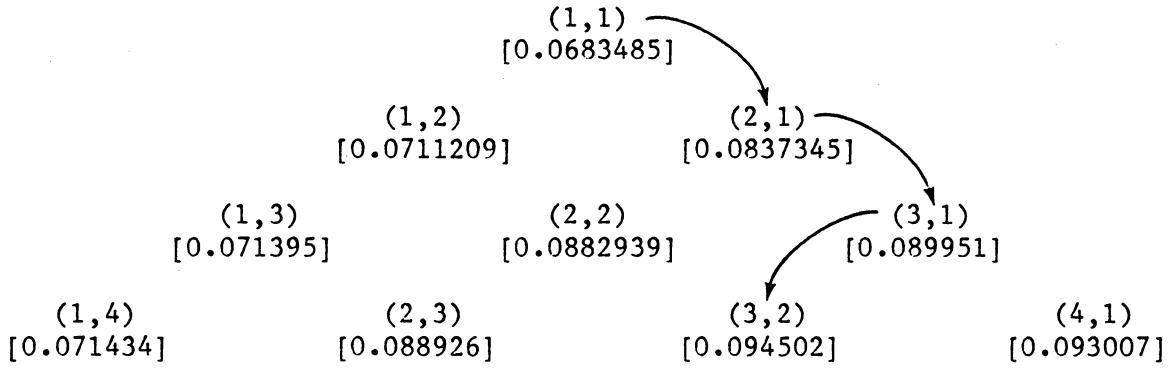
implies

$$p(\underset{\sim}{b}' + e_i + e_\ell) \geq p(\underset{\sim}{b}' + e_j + e_\ell).$$

In words, if buffer i has the greatest need for a unit of buffer space but the unit is assigned to buffer k instead then buffer i still has the greatest need. In other words, once a better productivity is found, the corresponding buffer allocation will lead to an optimum. This can be seen in a three-machine line with the results obtained numerically.

With the machine reliabilities $q_1 = 0.1$, $q_2 = 0.2$, $q_3 = 0.3$, the following diagram shows each possible buffer allocation together with its corresponding productivity enclosed in brackets. The arrows

indicate the path to an optimum with respect to a fixed total buffer capacity.



Instead of having to determine the productivity for each possible buffer allocation, results from Section 3.3 can be applied in comparing any two buffer allocations. Since the upper bounds on the productivity are monotone decreasing and the lower bounds monotone increasing, it suffices to compare the upper bounds obtained according to one buffer allocation with the lower bounds resulting from the other allocation until the lower bound exceeds the upper bound yielding the latter allocation as a better one. This reduces the number of iterations considerably. For example, 215 iterations were needed to obtained convergence for $p((4,2))$ alone whereas only 126 iterations were needed in comparing the lower bounds of $p((4,2))$ with the upper bounds of $p((3,3))$ to conclude that $p((4,2)) \geq p((3,3))$. Using this comparison procedure for the three-machine line mentioned above, the string of buffer allocations leading to the optimum is

$$(1,1) \rightarrow (2,1) \rightarrow (3,1) \rightarrow (3,2) \rightarrow (4,2) \rightarrow \ldots.$$

In conclusion, once Conjecture 3.5.1 is proved, an algorithm for optimal buffer allocations is obtained and knowledge of monotone bounds

from Section 3.3 can be used as an aid in finding the string of buffer allocations leading to the optimum.

CHAPTER 4

EPILOGUE

Included in this chapter at a closing point are discussion and summary of the present work with needed future research.

4.1 Discussion and Summary

An overview of production systems is covered in Chapter 1 with emphases on transfer lines. Chapter 2 describes in detail the class of fixed-cycle transfer lines with some basic assumptions made in the model under study.

In Chapter 3, the main results are presented. It includes sections on closed form solutions for the productivity of a two-machine line with dependent machines and on the extension of some earlier work by Soyster et al. (1979) on lower bounds for the productivity of an N-machine line. Section 3.3 on monotone bounds makes use of the essential monotone property possessed by the transition matrix in connection with an N-machine line. From these results, monotone increasing lower bounds and monotone decreasing upper bounds are derived numerically yielding convergence to the productivity of the line. The last Section 3.5 conjectures an algorithm for optimal allocation of buffer capacities.

It should be pointed out here that results on deriving monotone bounds apply also to the case with Markov machines when the transition matrix is shown to be conditionally monotone. Although work on transfer lines with each machine modeled as a two-state Markov chain has been done in the literature, the approach used here is different. For example, Sheskin (1976) gave approximate solutions by decomposition. Each machine with its input buffer was analyzed separately in terms of

departures to its output buffer and arrivals from the preceding machine. Since each machine except the last in the line can be blocked, departures from and arrivals to each buffer are indeed dependent. However, this dependence was ignored. The algorithm starts with a trial value for the probability of having the last buffer empty and proceeds backwards through successive buffers and machines until a trial value for the first buffer is obtained. This first set of trial values is then used to produce a new trial value and the procedure is repeated through successive iterations until some prespecified convergence rule is satisfied. In the present work, however, the interdependence between machines and buffers is taken into account implicitly through the transition matrix associated with the whole line. Moreover, with the monotone properties of the matrix, only one initial guess is needed in order to find bounds which converge to the productivity.

4.2 Future Research

As transfer lines are the simplest non-trivial production systems, it is natural that their concepts if not their results can be used in future work for more complex models. One of the most needed research is in the area of production system networks of which very little is available in the literature. The earliest work is Jackson (1963) which is an extension of his 1957 classic paper and is motivated by some characteristics of real production systems. More recent work are Buzacott (1976) and Ammar (1980).

Another related topic is the problem of stochastic control on production systems. Work along this line are, for example, Buzacott (1976), Gershwin and Ammar (1979) and Forestier (1980).

Last, but not the least, is the problem of buffer allocations towards which Conjecture 3.5.1 is geared.

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