THREE-DIMENSIONAL VIBROMETRY VIA THREE POSITIONS OF A ONE-DIMENSIONAL LASER DOPPLER VELOCIMETER

by

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(ABSTRACT)

A technique to determine the three-dimensional velocity of a point on a vibrating structure is developed. This technique uses a reference-beam type laser-doppler velocimeter in three independent positions to determine the target velocity in a non-orthogonal coordinate system. The transformation from non-orthogonal to orthogonal coordinate systems is analyzed. The sensitivities of the resulting velocity to measurement noise and position errors are also analyzed. Recommendations are made for future testing and applications of this technique.

DEDICATION

This thesis is dedicated to my parents, Kenneth M. and Elena A. Donovan. Their steadfast confidence in my abilities was a constant source of strength especially during the most trying of times. Their genuine pride in my accomplishments provided motivation to achieve what I had previously only dreamed. Without them, this thesis would not exist.

ACKNOWLEDGEMENTS

I would like to acknowledge the efforts of my committee members, Drs. R.L. West and A.L. Wicks. Dr. West gave me many useful pieces of advice about the graduate school experience and human nature. Dr. Wicks was responsible for more of my graduate level instruction than any other single individual. His heuristic teachings were essential for the work contained within this thesis. Dr. Wicks' bits of wisdom were also very practical.

Most of all I would like to acknowledge the efforts of my committee chairman, Dr. L.D. Mitchell. Dr. Mitchell's high level of organization and time management were greatly appreciated. Dr. Mitchell flatly refused to accept any conclusion that could not be communicated effectively, thus requiring a complete understanding on my part. His tireless editing of this document brought out the best in this humble writer.

Finally, I would like to acknowledge the financial support of Zonic Corporation.

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1 INTRODUCTION

1.1 BACKGROUND

The laser doppler velocimeter has become the primary choice for non-contacting velocity measurement in scientific research. The field of fluid dynamics has been greatly furthered by laser-based anemometry technology. Turbulent flows tend to be highly three dimensional. Therefore, three-dimensional laser velocimeters have come into being to provide the fluid dynamicist with such information. In recent years, laser velocimeters have found their way into the laboratories of structural dynamicists as well. All structures to some degree, and some structures to a large degree, vibrate in three-dimensions (if rotation is included, any part of a structure can have six degrees of freedom, as will be discussed later). It is for this reason that three-dimensional laser velocimetry for use on structures is being pursued.

1.1.1 The Need for a Non-contacting Transducer

Conventional structural dynamics measurements are performed with accelerometers. Accelerometers are piezoelectric-seismic transducers which give a charge output proportional to the acceleration of the point on the structure where it is attached.[1*] There are several reasons one would like to avoid the use of accelerometers in a given experiment. First, accelerometers have mass. Even though many manufacturers produce miniature accelerometers, they still can have a significant mass-loading effect on the structure, particularly if the structure has a small effective mass at the point of attachment of the accelerometer. This mass loading is self-defeating since it is the structure's dynamic properties we are trying to measure. Accelerometers also add rotational inertia, local stiffness, and damping to the structure. They cannot be used in high-temperature applications and require electrical leads. Also, accelerometers have finite transverse sensitivities. That is, they are sensitive to motion in directions perpendicular to their designed measurement axis. This transverse sensitivity is less than 5% of its primary sensitivity for a well designed and well constructed transducer. Finally, to measure three-dimensional motion a tri-axial accelerometer is required. A tri-axial accelerometer is simply three single-axis accelerometers mounted in an orthogonal orientation and may be an integral unit. This may be difficult to mount on the structure and certainly worsens the mass-loading problem.

Laser technology offers solutions to many of these problems. A laser beam incident on a structure does not have the effects of mass loading, rotational inertia, local stiffening, or damping. Also, all laser velocimeters are sensitive to velocities along

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[*] Superscript numbers refer to references listed at the end of this thesis.

one axis only and, thus, have no transverse sensitivity by the nature of the phenomenon employed. However, the laser velocimeter does not directly measure any one of the components measured by a conventional orthogonal tri-axial accelerometer unless very special test conditions are arranged. Laser speckle, a problem with some laser systems when there are significant transverse velocities, will be discussed later. The last problem, that of three-dimensional dynamic measurement, is the topic of the remainder of this thesis.

1.1.2 The Need for Three-Dimensional Motion Measurement

Often it is desired that the results of a structural dynamics experiment be comparable in form to the results of a structural dynamics analysis such as that from a finite element model. A common representation of dynamic information from such analyses is with an "outward-normal" orthogonal coordinate system. That is, one axis of the coordinate system is normal to the local tangential plane on the surface of the structure. The other two axes, of course, lie in this plane and are referred to as "in-plane" axes. Obviously, these local coordinate systems change for each point on the structure unless it is a planar structure. Sometimes, the outward-normal motion information from a structural dynamics experiment is sufficient for comparison to the analytical model.

There are two reasons for wanting to measure the full three-dimensional motion of each point on a structure. First, practical considerations of laser systems prohibit direct measurement of the outward-normal motion for structures with complicated geometry. Single accelerometers, because they are mounted on the surface of the structure, easily measure the outward-normal motion of the structure. One-dimensional laser systems, because they measure the line-of-sight motion from a single vantage point, may or may not give the true outward-normal motion. Often, it is simply assumed that the measured motion is close enough to the outward-normal to be acceptable. This assumption is valid for some structures and invalid for others. Mitchell has shown that corrections can be made for non-normal angles of incidence of the laser.[2] Second, structures with complicated dynamics may have significant in-plane motion that must be fully detected to fully document the structures mode shape.

The most desirable information is the full three-dimensional velocity vector of each point on the structure relative to a fixed coordinate system. This information can be easily transformed into any desired coordinate system. If the structure's geometry is known, the information can also be transformed into an outward-normal coordinate system at each point on the structure. Theoretically, the orthogonal motion information need not be measured directly. Any three independent (non-coplanar) measurements can be transformed into any desired orthogonal coordinate system.

1.1.3 Obtaining Six Degrees of Freedom

The motion of any point on a structure can be completely described with six independent quantities called "degrees of freedom". These six degrees of freedom can be described in the context of a rectangular coordinate system. There are three translational degrees of freedom defined along the coordinate axes as x, y, and z. There are also three rotational degrees of freedom defined as rotations about each of the axes: θ_x, θ_y, and θ_z.[3]

Work is being done at Virginia Polytechnic Institute and State University, Department of Mechanical Engineering, under the direction of Dr. L.D. Mitchell to obtain the rotational degrees of freedom from spatially dense three-dimensional translational motion. In general, this work involves mathematical techniques for using information from many points on the structure to arrive at the rotational degrees of freedom.[4,5,6,7] Implicit in the use of these techniques is the need for full-field data.

There are several ways of gathering full-field data. One way is to use a multitude of accelerometers attached to the structure. This presents a host of problems including mass loading, expense, required multiple calibrations, potential for failure of multiple cables and transducers, and long experimentation time. Other techniques gather full-field motion data at once using optics.

A common way of gathering full-field data is by using mirrors to direct a laser beam or beams sequentially to each point on the structure where data is desired. This is called "scanning". Scanning and the problems associated with it are discussed briefly in the Conclusions and Recommendations portion of this thesis. The purpose of this study is to develop the methodology to obtain the three orthogonal translational degrees of freedom of a point on a structure using a laser velocimeter.

1.1.4 Vibrating Structures as Targets

The vast majority of research into laser-based motion measurement instruments has been done in the field of experimental fluid dynamics. The need for non-contacting transduction is most readily apparent there. Also, lasers are used to provide the fluid dynamicist with other information such as the size of particles in a fluid flow.

More recently, structural dynamicists have found the need for such a non-contacting transducer. The advantages of a laser system over conventional accelerometers have already been made clear. However, the use of lasers on vibrating structures, as opposed to fluid flows, raises some new issues.

If the structure's surface is optically reflective (like a mirror) then the majority of the light intensity will be reflected off according to the following law of reflection: *the* *angle of incidence equals the angle of reflection.*[8] Rarely will this reflected light fall on the detection part of the laser system. This will cause problems for laser systems with photodetectors mounted near the source. Making the surface more optically diffuse, by applying a white coating for example, will help reflect light back to the detectors. However, requiring such a coating somewhat reduces the advantage of non-contacting transduction.

Optically diffuse surfaces can also pose problems to a laser system. The laser light incident on the surface is coherent. However, the light reflected from various points within the laser spot are incoherent and may constructively and destructively interfere. Thus, the light returning to the detection system will be "speckled" with light and dark spots. This laser speckle does not pose a problem by itself, but if the structure has significant in-plane motion, these speckles may race across the detector surface, introducing significant random error to the detector output.[9,10] It is important to note that some laser systems are immune to this problem, while others actually make use of laser speckle to measure the in-plane motions. Because of these differences, it would be foolish to assume that a given laser system designed for fluid flow measurement is easily adaptable for use on solid structures.

1.2 LITERATURE REVIEW

Laser systems have been described, up to this point, only in generalities in order to emphasize the needs of the structural dynamics researcher. Laser systems and how they meet these needs will now be described in more detail.

Accelerometers are, obviously, acceleration transducers. In contrast, some laser vibrometers are displacement transducers. J.A. Cafeo, M.W. Trethewey, J.R. Rieker, and H.J. Sommer III have demonstrated the use of one such system for simultaneous measurement of one displacement and two angular rotations of a solid structure.[11] This system uses two two-dimensional photodetectors to locate the reflected light from two lasers. This system has extremely tight dimensional tolerances on the target position and is restricted to small structures.

Other laser systems include holography and laser-speckle based instruments. Holography is an instantaneous full-field technique. That is, motion data is acquired from the entire structure at once. Two different types of holography are photographic holography and TV-holography (sometimes called electronic speckle pattern interferometry or ESPI). Both use sequential images of the structure flooded by laser light.[12,13,14] These methods suffer from stringent stability requirements, difficulty in determining phase, and the need to interpret an image to obtain quantitative information. Laser speckle techniques come in various forms. Most apply some sort of a spatial filter to the speckle pattern produced by a laser beam incident on a diffuse surface. Y. Aizu and T. Asakura have given a complete survey of these techniques along with their basic theory.[15,16,17] J.H. Churnside and H.T. Yura have demonstrated one such technique to measure in-plane velocities on solid structures.[18,19,20] Most laser speckle techniques require relatively large displacements to obtain accurate data, depending on the size of the detector and optics. Also, the optics must be very precise as any distortion of the speckle pattern will be interpreted as motion.

The majority of laser-based motion sensors (or vibrometers) are velocity transducers. Such systems are referred to as laser velocimeters. There are two types of doppler-based velocimetry techniques in common use: differential and reference beam. *The Laser Doppler Technique*, by L.E. Drain is the definitive text on laser doppler velocimetry techniques, especially for one-dimensional measurements.[21] *Principles and Practice of Laser-Doppler Anemometry* by F. Durst, A. Melling, and J.H. Whitelaw is another excellent text with special attention to fluid flow applications.[22] The doppler phenomenon occurs when a laser beam strikes a moving target. Laser light is coherent and of a known frequency (color). The light reflected from the target is shifted slightly in frequency (the doppler phenomenon) proportional to the velocity of the target. Measurement of this light frequency would seem to be the most direct manner in which to measure the target velocity. In fact this has been done with very precise optical spectrometers, with the problem being a slow response time.[23] However, photodetector technology does not currently permit detection at light frequency. Therefore, two clever techniques have been devised to permit the detection of the frequency *shift*, which, after all, is the desired quantity proportional to structural velocity.

In the differential (also called dual-beam) technique two laser beams are focused on a single spot on the structure. Here they constructively and destructively interfere to form a fringe pattern. The light reflected from the spot is bright-to-dark modulated at a frequency proportional the velocity perpendicular to the fringes. This scattered light can be viewed with a photodetector and transduced to give a real-time electric signal proportional to the target velocity. Note that the sensitive axis is perpendicular to the bisector of the angle between the two beams and lies in the plane formed by the two beams. This technique requires very accurate alignment as the two beams must be made to intersect precisely at the surface of the structure. A.D.W. McKie and J.W. Wagner have demonstrated the use of differential velocimetry to measure in-plane vibrations of solid structures.[24]

In the reference beam technique a single beam is split into two beams. One beam is retained for future use as a reference beam while the other, the measurement beam, is allowed to strike the target. The reflected light, having been shifted in frequency by the target structure motion, is then recombined with the reference beam of the original frequency. The result is time variant constructive and destructive interference that can be viewed with a photodetector and transduced to give a real-time electric signal proportional to the target velocity. In this case, the sensitive axis is along the bisector of the angle between the incident and scattered detection beams. In the case when both are collinear, the sensitive axis is, of course, along the line-of-sight of the laser beam.[25]

1.2.1 Three-dimensional Velocimeters

Relatively few laser velocimeters in use are full three-dimensional systems. The vast majority of those are used for investigation of fluid flows. In *Laser Velocimetry: The Elusive Third Component*, J.F. Meyers presents the complete history of three-dimensional laser velocimetry.[26] A. Boutier updates Meyers' survey in *Three-dimensional Laser Velocimetry Systems* and gives a slightly more technical treatment.[27]

All but a few of the three-dimensional velocimetry techniques use a dual-beam arrangement or multiple dual-beam arrangements. Common examples include five-beam and six-beam systems. The five-beam systems may use two dual-beam velocimeters oriented at 90° with respect to each other but aimed coaxially with a single reference-beam down the center.[28] Six-beam systems generally have two dual-beam velocimeters oriented as described above with a third dual-beam velocimeter located off at some angle to the other two.[29,30] In general, these systems are incapable of scanning (they are for point measurement only) although there are exceptions including a product of TSI Incorporated employing a complex arrangement of scanning mirrors.[31,32] One common feature is that, in order to separate the channels, beams of different frequencies (colors) are required.[33]

K.L. Orloff and P.K. Snyder have performed extensive error analyses of a six-beam non-orthogonal velocimeter.[34,35] Portions of the error sensitivity analyses to be found later in this thesis are modelled after their approach. Snyder and Orloff have also made recommendations for reductions of uncertainties related to the use of laser velocimeters with non-orthogonal channels.[36]

Aizu and Asakura suggest the use of spatial filtered laser speckle velocimetry for determination of the two in-plane velocity components.[37] Such a system, in combination with a conventional reference beam velocimeter to measure the out-of-plane velocity, might be a viable three-dimensional velocimeter. Churnside and Yura demonstrated a two-dimensional laser speckle velocimeter and indicate minor modifications needed to produce a full three-dimensional device but never actually did so.[38]

The reference-beam velocimeter lends itself to several possibilities for gathering three-dimensional data. One possibility is the use of three distinct reference-beam systems positioned such that the three velocity directions sensed are independent. For deterministic harmonic vibration this could be performed either simultaneously (all three velocities measured at once) or sequentially (each velocity measured one at a time maintaining some phase reference with the force). The sequential method is also called the three-position (as opposed to three-velocimeter) method. Other possibilities include one integral (laser and detector) velocimeter with two separate detectors or a single laser with three separate detectors. Only one three-dimensional reference beam laser doppler velocimeter is known to have been demonstrated. This system is the topic of the following section.

1.2.2 Three-position Reference-Beam Velocimeters

In 1968, J.D. Fridman, R.M. Huffaker, and R.F. Kinnard unveiled the first three-dimensional laser doppler velocimeter.[39] The work was done at NASA Marshall Space Flight Center in Huntsville, Alabama with Raytheon Company under contract no. NAS8-21293.[40] It was designed for use in fluid flow experiments and used a foreword scatter arrangement. This device used the reference-beam technique with a centrally located laser and three detectors at equal angles from the central axis formed by the laser beam and at 120° increments around it (see Figure 1). The angle that separated the incident beam from the scattered beam was 8.5°.[42] Recall that the axis of sensitivity is along the bisector of this angle. The angle between the central axis and the axis of sensitivity, 4.25° in this case, will be defined as the separation angle α. All three velocities were measured simultaneously. Boutier wrote of Huffaker's apparatus:[43]

> "This system suffered the well-known drawbacks of poor signal to noise ratio of the reference beam configuration, but also the lack of accuracy when measuring a velocity vector in a non-orthogonal coordinate system with small angles between the projection axes. Moreover the apparatus . . . did not include any acousto-optic modulator for velocity sign determination."

Boutier's final statement refers to a problem common to many early laser doppler velocimeters, the inability to determine the sign of the velocity despite knowing its magnitude and the axis along which it is directed. This problem was later solved in various ways.[44] Meyers agrees that Huffaker experienced great difficulties with this apparatus primarily because of stringent alignment requirements and the fact that the small transverse velocities were on the order of the ambient noise level from shot noise at the photodetectors. Shot noise is random noise produced by lasers themselves and is especially troublesome in reference-beam velocimeters. These two sources of error were compounded by the coordinate transformation of a non-orthogonal coordinate system to an orthogonal system.[45]

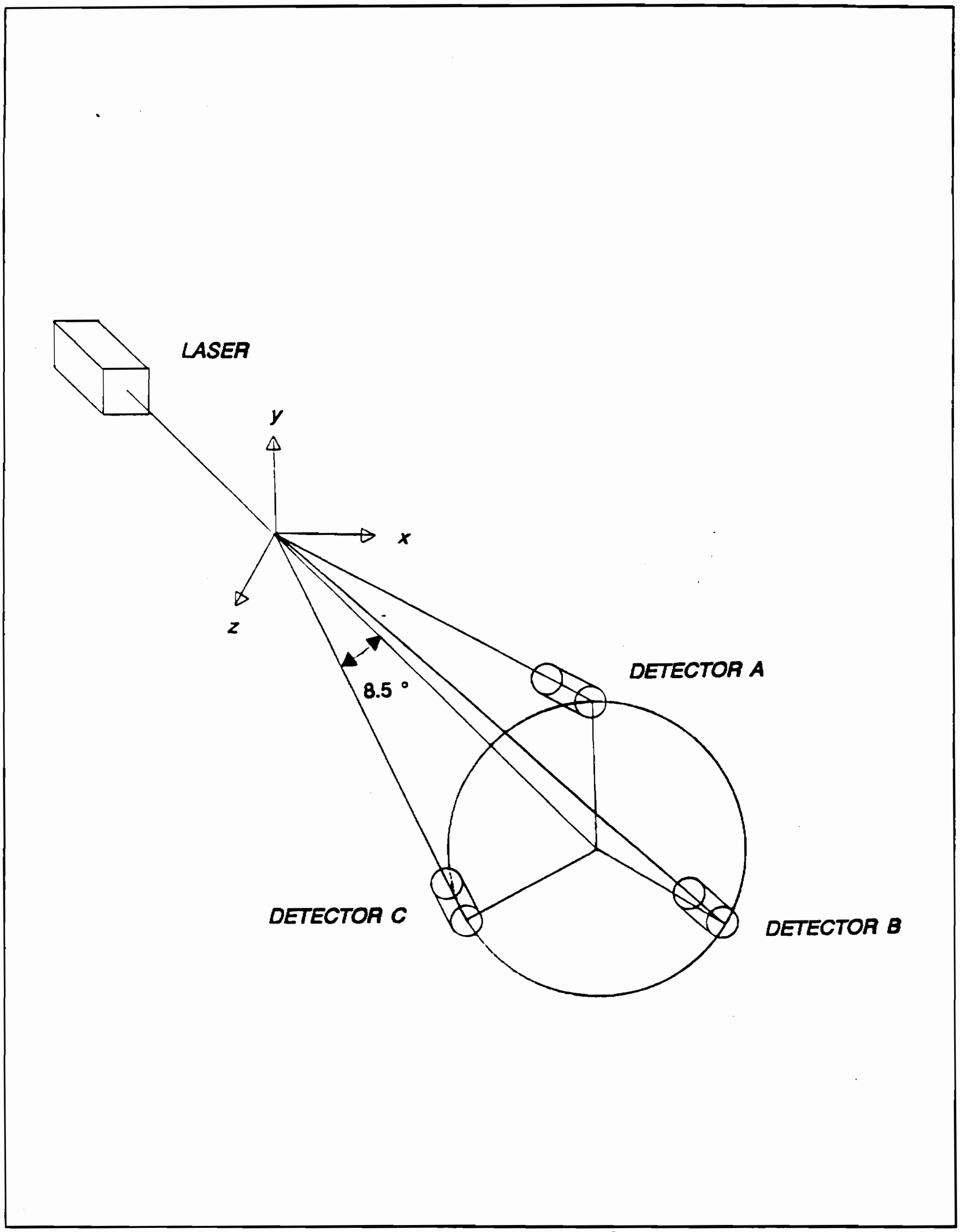

Figure 1 The three-dimensional laser doppler velocimeter of R.M. Huffaker. [40]

The work of Huffaker et al. was not developed further. There has been no significant research into three-dimensional reference-beam laser doppler velocimeters performed since.[46]

1.3 PURPOSE OF THIS RESEARCH

A one-dimensional reference beam laser doppler velocimeter is currently being used in the Modal Analysis Laboratory in the Department of Mechanical Engineering at Virginia Polytechnic Institute and State University, Blacksburg, VA under the direction of Drs. L.D. Mitchell, A.L. Wicks, and R.L. West. The device is a VPI Sensor produced by Ometron, Inc. (London, England).[47] It is capable of scanning in a 25° viewing window in both the horizontal and vertical directions. The scanning mirrors are controlled through analog inputs. It has a range from near zero up to 200m for specially coated surfaces. The Sensor has an analog velocity output with accuracy of about ±3% of full scale for each of three ranges (0-0.01, 0-0.1, 0-1.0 ms^{-1}).[48]

The demands of the modal analysis community are growing. It is envisioned that three-dimensional non-contacting motion information will be essential to the analysis of complicated and/or lightweight structures. The availability and simplicity of reference-beam laser doppler velocimeters along with the obvious lack of research into three-dimensional reference beam systems provide incentive to investigate further their use in three-dimensional motion detection. Also, the choice of a sequential method over a simultaneous one trades off cost (one velocimeter versus three) for data gathering time (more than a factor of three).

The purpose of this research was to investigate the use of a sequential three-dimensional reference beam laser doppler velocimeter system. An analysis of the transformation from non-orthogonal to orthogonal coordinate systems was to be completed. In addition to the general configuration, two specific configurations were to be analyzed. Also, sensitivities of the velocity output to the two cited sources of error (measurement noise and alignment error) were to be analyzed.

1.3.1 Velocities

The most direct way of measuring the three-dimensional velocity would be to set the three laser positions in an orthogonal orientation. However, practical considerations like the interior testing space relative to the structure size require that some other orientation be used. Thus, in general, a transformation from a non-orthogonal coordinate system (which defines the measurement orientation) to an orthogonal coordinate system (which is the desired representation) will be required. This transformation was to be derived. Two specific configurations were to be analyzed as well. One configuration was to be that of Huffaker et al. The other configuration was to be one more likely to be used in the field.

1.3.2 Sensitivities

There are two quantities which are known to be the major sources of error in systems of this type: measurement noise and alignment error. Measurement noise can be caused by laser speckle, electromagnetic interference, or the analog electronic noise from the circuits that convert the frequency-modulated doppler signal into the voltage output. The laser beam alignment error can be the result of operator error, scanning error, or incorrect assumptions about the physical orientation of the velocimeters with respect to the target. The sensitivity of the transformation to these quantities was to be analyzed.

2 ANALYSIS

This chapter describes the theory governing the transformation from non-orthogonal coordinate systems as it applies to a three-dimensional reference-beam velocimeter. Also described are the analyses performed to evaluate the sensitivity of this transformation to two sources of error.

2.1 TRANSFORMATION

The VPI Sensor has a single axis of sensitivity along the laser beam's line of sight.[49] Thus, three velocity measurements of a single point on the structure made from three independent positions in space are sufficient to completely define the three-dimensional velocity vector of that point. The transformation from the general non-orthogonal coordinate system to the desired orthogonal coordinate system will be derived and the concept of position independence will be defined.

Let an orthogonal coordinate system be located at the point on the structure where the velocity measurement is desired, as shown in Figure 2. Let this be a conventional right-handed coordinate system with the axes labeled x, y, and z. Let the points $\boldsymbol{A}$, $\boldsymbol{B}$, and C represent the locations of the origin of the laser beam for each of the three

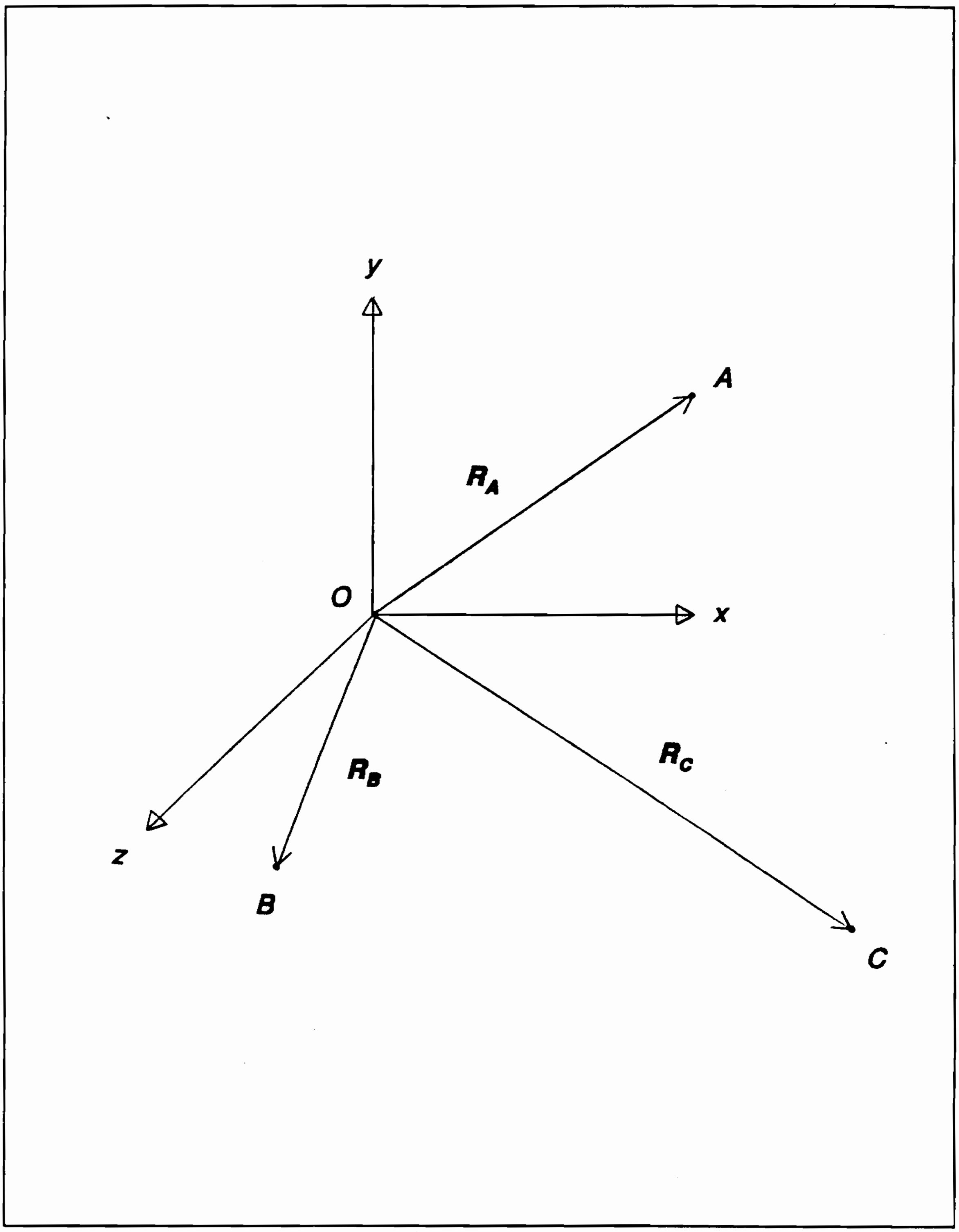

Figure 2 The orthogonal (*x*, *y*, *z*) and non-orthogonal (OA, OB, OC) coordinate systems.

velocimeter positions. It will be shown that these positions must be chosen such that they, along with the origin, do not all lie in a plane. Thus, the vectors $\boldsymbol{R_A}$, $\boldsymbol{R_B}$, and $\boldsymbol{R_C}$ are the three-dimensional position vectors of each of the velocimeter positions relative to the defined coordinate system:

$$\boldsymbol{R_A} = \begin{Bmatrix} R_{Ax} \\ R_{Ay} \\ R_{Az} \end{Bmatrix}, \quad \boldsymbol{R_B} = \begin{Bmatrix} R_{Bx} \\ R_{By} \\ R_{Bz} \end{Bmatrix}, \quad \boldsymbol{R_C} = \begin{Bmatrix} R_{Cx} \\ R_{Cy} \\ R_{Cz} \end{Bmatrix} \tag{1}$$

The magnitudes of each of these vectors are:

$$R_A = \sqrt{R_{Ax}^2+R_{Ay}^2+R_{Az}^2}, \quad R_B = \sqrt{R_{Bx}^2+R_{By}^2+R_{Bz}^2}, \quad R_C = \sqrt{R_{Cx}^2+R_{Cy}^2+R_{Cz}^2} \tag{2}$$

Therefore, the unit vectors in the directions of each of the laser positions are $\boldsymbol{R_A}/R_A$, $\boldsymbol{R_B}/R_B$, and $\boldsymbol{R_C}/R_C$.

The velocity vector of the point on the structure, as shown in Figure 3 is:

$$\boldsymbol{V} = \begin{Bmatrix} V_x \\ V_y \\ V_z \end{Bmatrix} \tag{3}$$

Since the velocimeter is sensitive only along its line of sight, the measured velocities V_A, V_B, and V_C are formed by the scalar product of the velocity vector with the unit

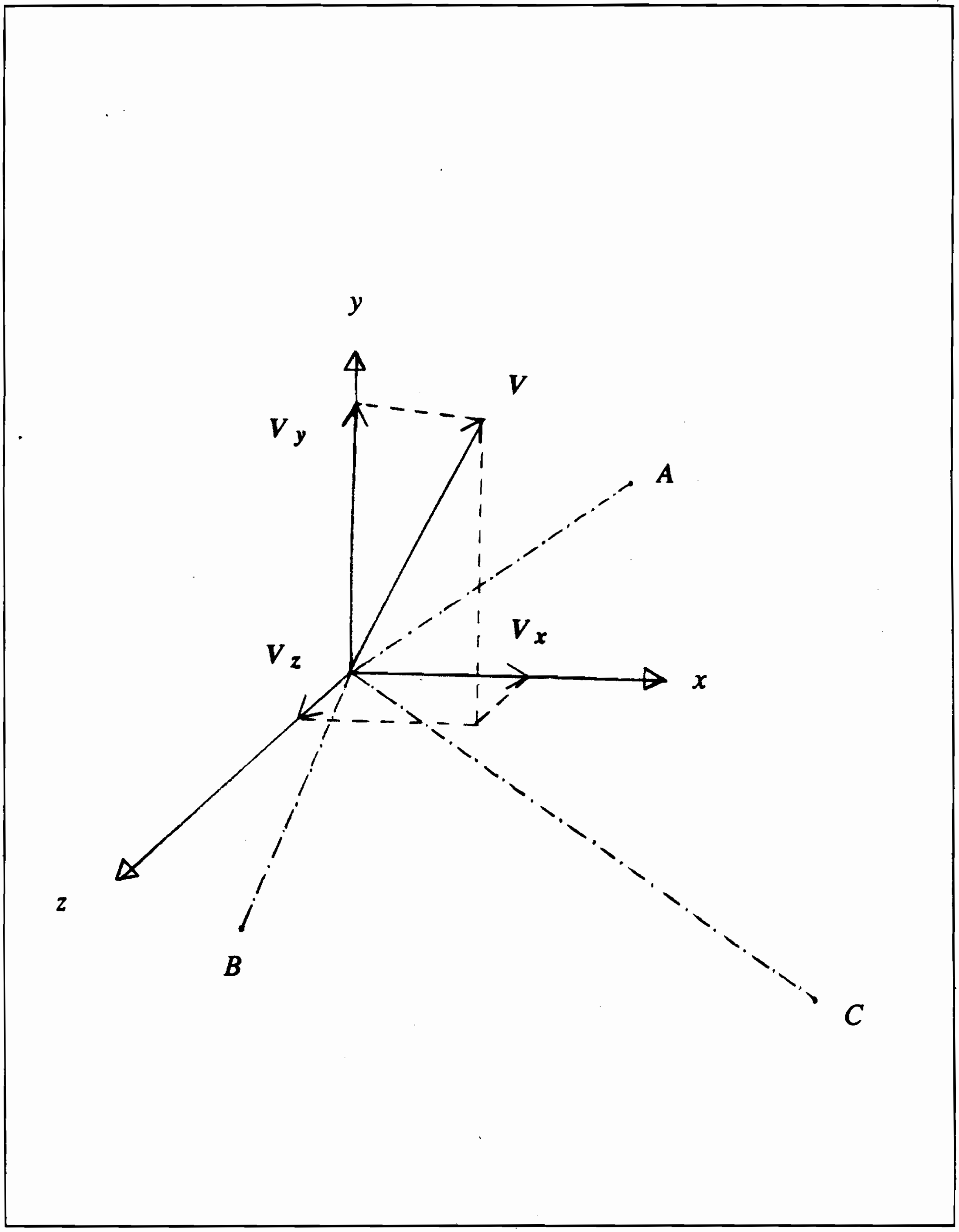

Figure 3 The arbitrary velocity and its components in the orthogonal coordinate system.

vector in the direction of the velocimeter. Figure 4 shows the vector component of **V** in the direction of velocimeter position A. Similar relationships exist for positions B and C. The mathematical relationship is:

$$
\begin{aligned}
V_A &= \mathbf{V} \cdot \frac{\mathbf{R_A}}{R_A} = \begin{Bmatrix} V_x \\ V_y \\ V_z \end{Bmatrix} \cdot \frac{1}{R_A} \begin{Bmatrix} R_{Ax} \\ R_{Ay} \\ R_{Az} \end{Bmatrix} \\
V_B &= \mathbf{V} \cdot \frac{\mathbf{R_B}}{R_B} = \begin{Bmatrix} V_x \\ V_y \\ V_z \end{Bmatrix} \cdot \frac{1}{R_B} \begin{Bmatrix} R_{Bx} \\ R_{By} \\ R_{Bz} \end{Bmatrix} \\
V_C &= \mathbf{V} \cdot \frac{\mathbf{R_C}}{R_C} = \begin{Bmatrix} V_x \\ V_y \\ V_z \end{Bmatrix} \cdot \frac{1}{R_C} \begin{Bmatrix} R_{Cx} \\ R_{Cy} \\ R_{Cz} \end{Bmatrix}
\end{aligned}
\tag{4}
$$

Because the scalar (dot) product is distributive, this can be written as:[50]

$$
\begin{aligned}
V_A &= V_x \frac{R_{Ax}}{R_A} + V_y \frac{R_{Ay}}{R_A} + V_z \frac{R_{Az}}{R_A} \\
V_B &= V_x \frac{R_{Bx}}{R_B} + V_y \frac{R_{By}}{R_B} + V_z \frac{R_{Bz}}{R_B} \\
V_C &= V_x \frac{R_{Cx}}{R_C} + V_z \frac{R_{Cy}}{R_C} + V_z \frac{R_{Cz}}{R_C}
\end{aligned}
\tag{5}
$$

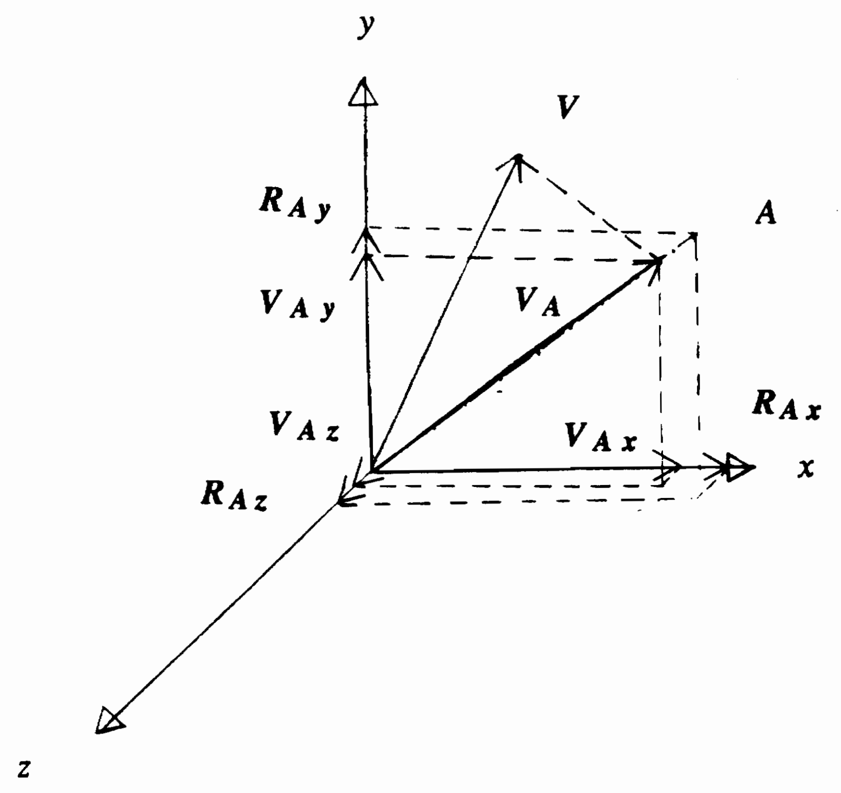

Figure 4 The vector component of **V** in the direction of laser position A.

This set of three equations can be written in matrix form as:

$$\begin{Bmatrix} V_A \\ V_B \\ V_C \end{Bmatrix} = \begin{bmatrix} \frac{R_{Ax}}{R_A} & \frac{R_{Ay}}{R_A} & \frac{R_{Az}}{R_A} \\ \frac{R_{Bx}}{R_B} & \frac{R_{By}}{R_B} & \frac{R_{Bz}}{R_B} \\ \frac{R_{Cx}}{R_C} & \frac{R_{Cy}}{R_C} & \frac{R_{Cz}}{R_C} \end{bmatrix} \begin{Bmatrix} V_x \\ V_y \\ V_z \end{Bmatrix} \tag{6}$$

or simply:

$$\begin{Bmatrix} V_A \\ V_B \\ V_C \end{Bmatrix} = \boldsymbol{T} \begin{Bmatrix} V_x \\ V_y \\ V_z \end{Bmatrix} = \boldsymbol{T}\,\boldsymbol{V} \tag{7}$$

Since we are interested in obtaining $\boldsymbol{V}$ from V_A, V_B, and V_C, we must solve Equation (6). That is:

$$\begin{Bmatrix} V_x \\ V_y \\ V_z \end{Bmatrix} = \begin{bmatrix} \frac{R_{Ax}}{R_A} & \frac{R_{Ay}}{R_A} & \frac{R_{Az}}{R_A} \\ \frac{R_{Bx}}{R_B} & \frac{R_{By}}{R_B} & \frac{R_{Bz}}{R_B} \\ \frac{R_{Cx}}{R_C} & \frac{R_{Cy}}{R_C} & \frac{R_{Cz}}{R_C} \end{bmatrix}^{-1} \begin{Bmatrix} V_A \\ V_B \\ V_C \end{Bmatrix} \tag{8}$$

It is important to note that the inversion exists (the matrix is non-singular) only for $\boldsymbol{R_C}$ not proportional to $\boldsymbol{R_A}$, $\boldsymbol{R_B}$, or a linear sum of $\boldsymbol{R_A}$ and $\boldsymbol{R_B}$. Thus, $\boldsymbol{R_C}$ cannot be collinear with $\boldsymbol{R_A}$ or $\boldsymbol{R_B}$, nor can $\boldsymbol{R_C}$ lie in the plane formed by $\boldsymbol{R_A}$ and $\boldsymbol{R_B}$.[51] This condition is what is referred to as independence of the velocimeter positions.

The solution of Equation (8) is:

$$\begin{Bmatrix} V_x \\ V_y \\ V_z \end{Bmatrix} = \frac{\begin{bmatrix} R_A(R_{By}R_{Cz}-R_{Bz}R_{Cy}) & R_B(R_{Az}R_{Cy}-R_{Ay}R_{Cz}) & R_C(R_{Ay}R_{Bz}-R_{Az}R_{By}) \\ R_A(R_{Bz}R_{Cx}-R_{Bx}R_{Cz}) & R_B(R_{Ax}R_{Cz}-R_{Az}R_{Cx}) & R_C(R_{Az}R_{Bx}-R_{Ax}R_{Bz}) \\ R_A(R_{Bx}R_{Cy}-R_{By}R_{Cx}) & R_B(R_{Ay}R_{Cx}-R_{Ax}R_{Cy}) & R_C(R_{Ax}R_{By}-R_{Ay}R_{Bx}) \end{bmatrix}}{R_{Cx}(R_{Ay}R_{Bz}-R_{Az}R_{By})+R_{Cy}(R_{Az}R_{Bx}-R_{Ax}R_{Bz})+R_{Cz}(R_{Ax}R_{By}-R_{Ay}R_{Bx})} \begin{Bmatrix} V_A \\ V_B \\ V_C \end{Bmatrix} \tag{9}$$

This is the general transformation equation from an arbitrary measurement triad to an orthogonal velocity triad description.

In the section on error sensitivities it will be shown that the velocimeter positions determine how sensitive the calculated velocities are to changes in those positions as well as measurement noise. At this point it will suffice show how the positions determine how near to being singular the matrix $\boldsymbol{T}$ becomes.

Independence of the velocimeter positions is, as discussed previously, essential for ***T*** to be non-singular. ***T*** is singular if the determinant of ***T*** is zero. However, the determinant itself is not a good indicator of how near ***T*** becomes to being singular.[52]

A better measure of near-singularity of ***T*** is the set of singular values of ***T***. The singular values are defined as the square roots of the eigenvalues of ***T'T*** where ***T'*** is the transpose of ***T***.[53]

The condition number of ***T*** is defined as the ratio of the largest singular value of ***T*** to the smallest. If ***T*** is near singular its condition number will be large and ***T*** is said to be "ill-conditioned". If all rows of ***T*** are independent its condition number will be unity and ***T*** is said to be "ideally conditioned". If the condition number is near unity then ***T*** is said to be "well conditioned". There are several algorithms available for finding the condition number of matrices. One such algorithm is employed by the MATLAB numerical computation software package which indicates that the logarithm to the base 10 of the condition number is approximately the number of decimal places that the computer can lose due to roundoff error during the inversion process.[54]

Theoretically, the choice of laser positions is unlimited as long as they are independent. The configuration used by Huffaker et al. is a convenient one to use for analysis and experimentation.[55] This configuration, as shown in Figure 5, defines the

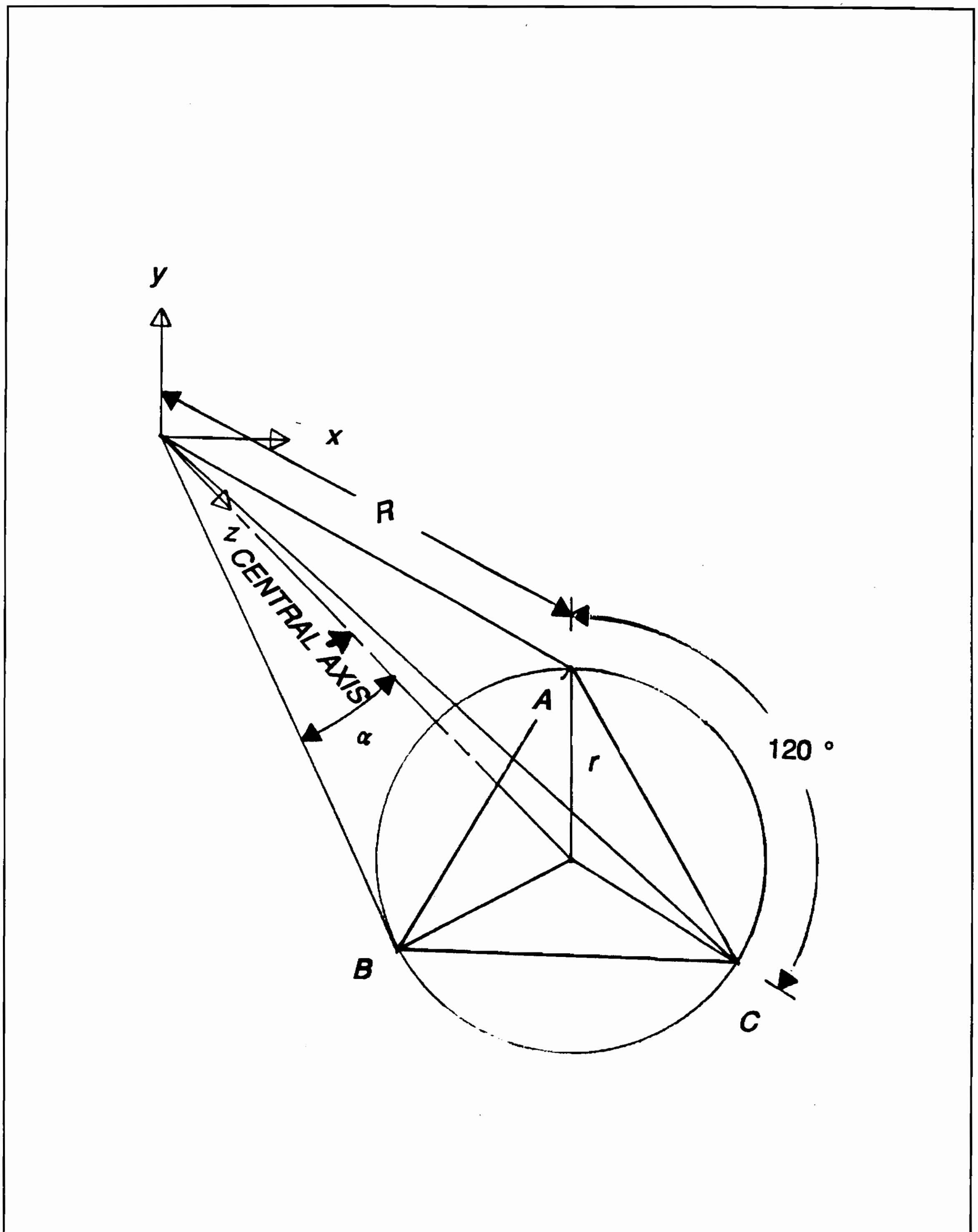

Figure 5 The orthogonal and non-orthogonal coordinate systems for the right pyramid configuration.

laser positions at the base vertices of a right pyramid whose apex lies at the origin, the point common to all three lasers. An alternative way of viewing this configuration results from defining a line through the center of the pyramid as the central axis. If viewed end on, this axis is the center of a circle containing the three velocimeter positions separated by 120° angles. The angle formed by any one of the velocimeter positions and the central axis is the separation angle α. Finally, the orthogonal coordinate system is oriented such that the z-axis lies along the central axis and the y-axis is in the same vector direction as the vector from the central axis to point A. This configuration assumption is represented mathematically by:

$$\begin{aligned} R_{Ax} &= 0,\ R_{Ay} = R\sin\alpha,\ R_{Az} = R\cos\alpha \\ R_{Bx} &= -\tfrac{\sqrt{3}}{2}R\sin\alpha,\ R_{By} = -\tfrac{1}{2}R\sin\alpha,\ R_{Bz} = R\cos\alpha \\ R_{Cx} &= \tfrac{\sqrt{3}}{2}R\sin\alpha,\ R_{Cy} = -\tfrac{1}{2}R\sin\alpha,\ R_{Cz} = R\cos\alpha \end{aligned} \qquad (10)$$

where R is the distance from the origin to any one of the velocimeter positions. Making these simplifications the transformation equation becomes:

$$\begin{Bmatrix} V_x \\ V_y \\ V_z \end{Bmatrix} = \frac{R^3 \begin{bmatrix} -\frac{1}{2}s\alpha c\alpha + \frac{1}{2}s\alpha c\alpha & -\frac{1}{2}s\alpha c\alpha - s\alpha c\alpha & s\alpha c\alpha + \frac{1}{2}s\alpha c\alpha \\ \frac{\sqrt{3}}{2}s\alpha c\alpha + \frac{\sqrt{3}}{2}s\alpha c\alpha & -\frac{\sqrt{3}}{2}s\alpha c\alpha & -\frac{\sqrt{3}}{2}s\alpha c\alpha \\ \frac{\sqrt{3}}{4}s^2\alpha + \frac{\sqrt{3}}{4}s^2\alpha & \frac{\sqrt{3}}{2}s^2\alpha & \frac{\sqrt{3}}{2}s^2\alpha \end{bmatrix}}{R^3\left[\frac{\sqrt{3}}{2}s\alpha\left(s\alpha c\alpha + \frac{1}{2}s\alpha c\alpha\right) - \frac{1}{2}s\alpha\left(-\frac{\sqrt{3}}{2}s\alpha c\alpha\right) + c\alpha\left(\frac{\sqrt{3}}{2}s^2\alpha\right)\right]} \qquad (11)$$

where s means sine and c means cosine. Finally:

$$V = \begin{Bmatrix} V_x \\ V_y \\ V_z \end{Bmatrix} = \frac{1}{3\sin\alpha} \begin{bmatrix} 0 & -\sqrt{3} & \sqrt{3} \\ 2 & -1 & -1 \\ \tan\alpha & \tan\alpha & \tan\alpha \end{bmatrix} \begin{Bmatrix} V_A \\ V_B \\ V_C \end{Bmatrix} \qquad (12)$$

Notice that the transformation, a special case of Equation (9), is dependent only upon the separation angle. This is especially useful for the sensitivity analyses which follow.

The Appendix contains a program written for MATLAB numeric computation software.[56] The program will perform the transformation from non-orthogonal to orthogonal coordinate systems. Given V_A, V_B, and V_C, and some geometry information, the program will output V. The right pyramid configuration is the default. However, with very slight modification (removal of the lines which are commented out) it can handle the general configuration case.

2.2 ERROR SENSITIVITIES

The certainty with which one can state the result of a mathematical operation, like the transformation derived previously, is directly, and solely, related to the degree of certainty one has in the values put into it. In the case of the transformation, there are two categories of input quantities: geometric positioning data and measured velocity data. The next sections analyze the sensitivity of the transformation to errors in these input quantities. Much of the sensitivity analyses which follow are modelled after a similar treatment of a six-beam differential laser doppler velocimeter by K.L. Orloff and P.K. Snyder.[57,58,59]

2.2.1 Position Errors

In order to gain some knowledge about the sensitivity of the transformation to errors in the position information, several assumption are made:

1. The velocimeter positions are in the right pyramid configuration.
2. Whatever form the position error takes, it results in a discrete error in the separation angle of the same magnitude and direction for all velocimeter positions.

The transformation equation for the right pyramid configuration is repeated here as:

$$\begin{Bmatrix} V_x \\ V_y \\ V_z \end{Bmatrix} = \frac{1}{3\sin\alpha} \begin{bmatrix} 0 & -\sqrt{3} & \sqrt{3} \\ 2 & -1 & -1 \\ \tan\alpha & \tan\alpha & \tan\alpha \end{bmatrix} \begin{Bmatrix} V_A \\ V_B \\ V_C \end{Bmatrix} \tag{13}$$

We are interested in the sensitivity of this transform to errors in α. We can either take the partial derivative of each element with respect to α or express the matrix as a power series and take the derivative as defined for power series.[60] We will do the former:

$$\begin{Bmatrix} \frac{\partial V_x}{\partial\alpha} \\ \frac{\partial V_y}{\partial\alpha} \\ \frac{\partial V_z}{\partial\alpha} \end{Bmatrix} = \begin{bmatrix} 0 & \frac{\sqrt{3}\cos\alpha}{3\sin^2\alpha} & \frac{-\sqrt{3}\cos\alpha}{3\sin^2\alpha} \\ \frac{-2\cos\alpha}{3\sin^2\alpha} & \frac{\cos\alpha}{3\sin^2\alpha} & \frac{\cos\alpha}{3\sin^2\alpha} \\ \frac{\sin\alpha}{3\cos^2\alpha} & \frac{\sin\alpha}{3\cos^2\alpha} & \frac{\sin\alpha}{3\cos^2\alpha} \end{bmatrix} \begin{Bmatrix} V_A \\ V_B \\ V_C \end{Bmatrix} \tag{14}$$

We can treat the differentials as finite differences. Also, we divide each equation by the expressions for V_x, V_y, and V_z from Equation (13) respectively and multiply by $\Delta\alpha$. The result is:

$$\begin{Bmatrix} \frac{\Delta V_x}{V_x} \\ \frac{\Delta V_y}{V_y} \\ \frac{\Delta V_z}{V_z} \end{Bmatrix} = \begin{Bmatrix} -\frac{\cos\alpha}{\sin\alpha} \\ -\frac{\cos\alpha}{\sin\alpha} \\ \frac{\sin\alpha}{\cos\alpha} \end{Bmatrix} \Delta\alpha \qquad (15)$$

This equation gives the fraction change in the calculated velocity for a given $\Delta\alpha$ in radians. Plots of these quantities are found in Figures 6 and 7. The vertical axis has been scaled to show the change in the velocity component as a percentage of that velocity component. The horizontal axis shows the nominal separation angle in degrees.

These graphs clearly show that V_x and V_y have high negative sensitivities for small separation angles. Also, V_z has large positive sensitivities for separation angles near 90°. In this case, a positive sensitivity indicates that the calculated value of V_z will be larger than the true value for a positive error in separation angle (where positive is defined for larger separation). A negative sensitivity indicates that the calculated value of V_x and V_y will be smaller than their true values for a positive error in separation angle. A position error of zero results in a sensitivity of zero for all

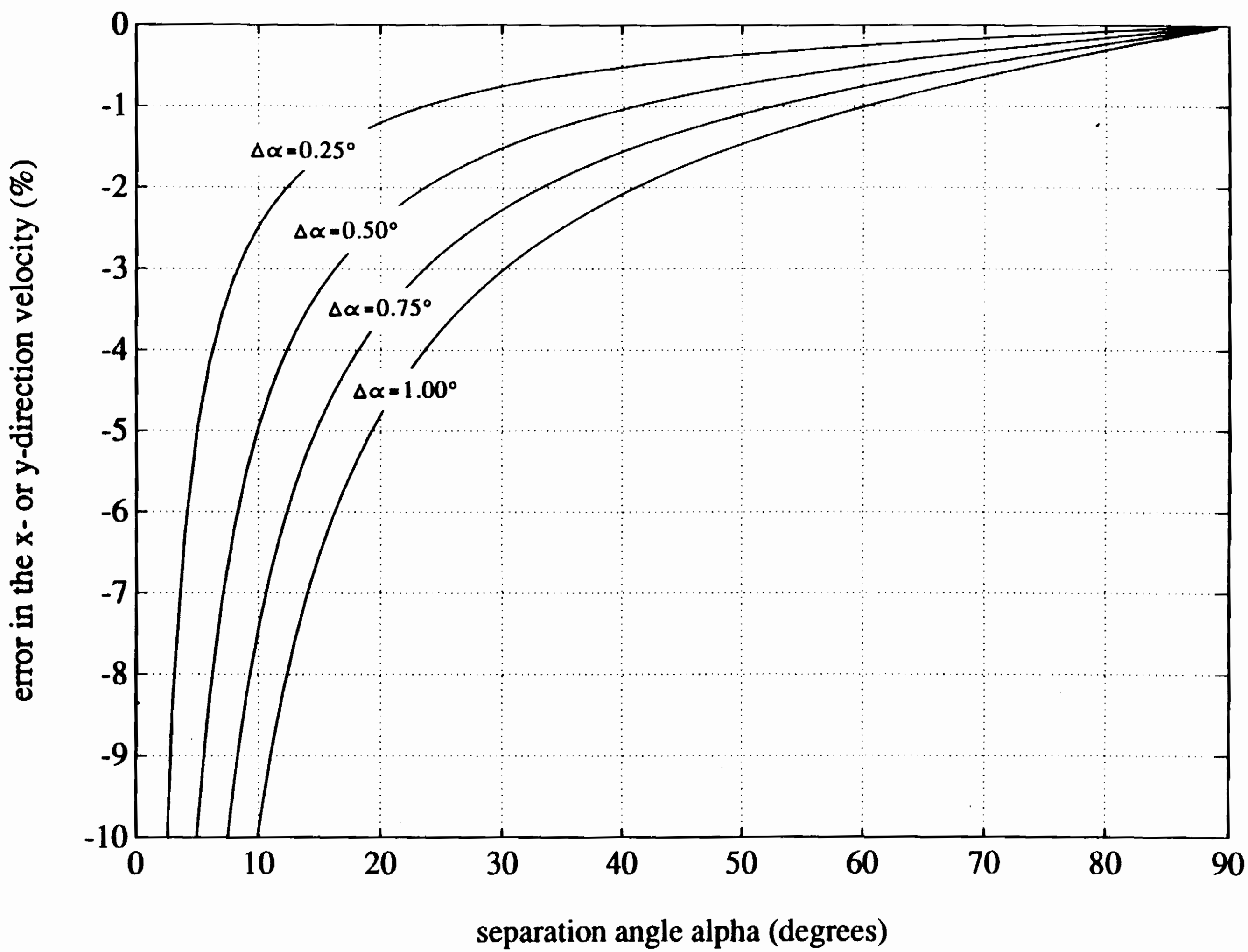

Figure 6 Percentage error in V_x or V_y for error in the separation angle, $\boldsymbol{\Delta\alpha}$, as a function of the laser separation angle.

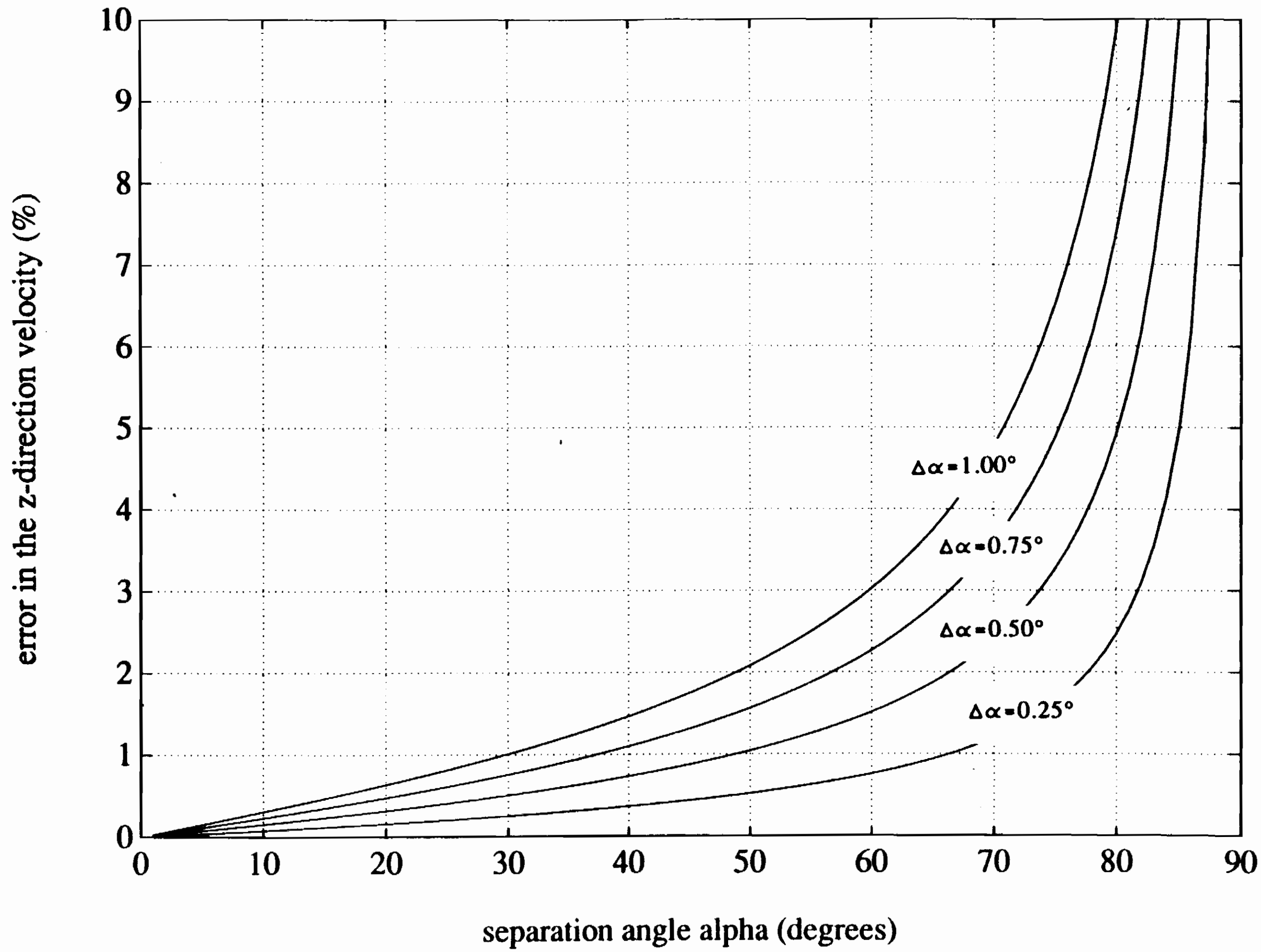

Figure 7 Percentage error in V_z for error in the separation angle, $\boldsymbol{\Delta\alpha}$, as a function of the laser separation angle.

separation angles. These graphs indicates an optimal region of minimum (absolute value) sensitivity for separation angles between 0 and 90°.

To determine a separation angle of minimum sensitivity to position errors the three velocity components must be combined in some way. That is, some assumption must be made about the direction of the velocity *V*. For example, if *V* is assumed to have equal components in all three orthogonal directions, the separation angle of minimum sensitivity can be determined as follows. The sensitivity equation becomes:

$$\frac{\Delta V}{V} = \Delta\alpha\sqrt{\left(-\frac{\cos\alpha}{\sin\alpha}\right)^2+\left(-\frac{\cos\alpha}{\sin\alpha}\right)^2+\left(\frac{\sin\alpha}{\cos\alpha}\right)^2} \tag{16}$$

This sensitivity is shown on Figure 8. Taking the derivative of Equation (16) with respect to α, setting equal to zero, and solving for α gives α=49.9° as the separation angle of minimum sensitivity. With a position error of 1 degree the error introduced is less than 5% for separation angles between about 25° and 85°. These values vary, of course, for different velocity direction assumptions.

A comparison can be made with the work of Huffaker et al. whose three-position reference-beam velocimeter used a separation angle of 4.25°. Their positioning system was claimed to be accurate to ±0.002°.[61] Insufficient evidence exists to dispute this claim. With this small separation angle and relatively precise apparatus, the resulting error in the calculated velocities is less than 0.2%. Recall, however, that this was a

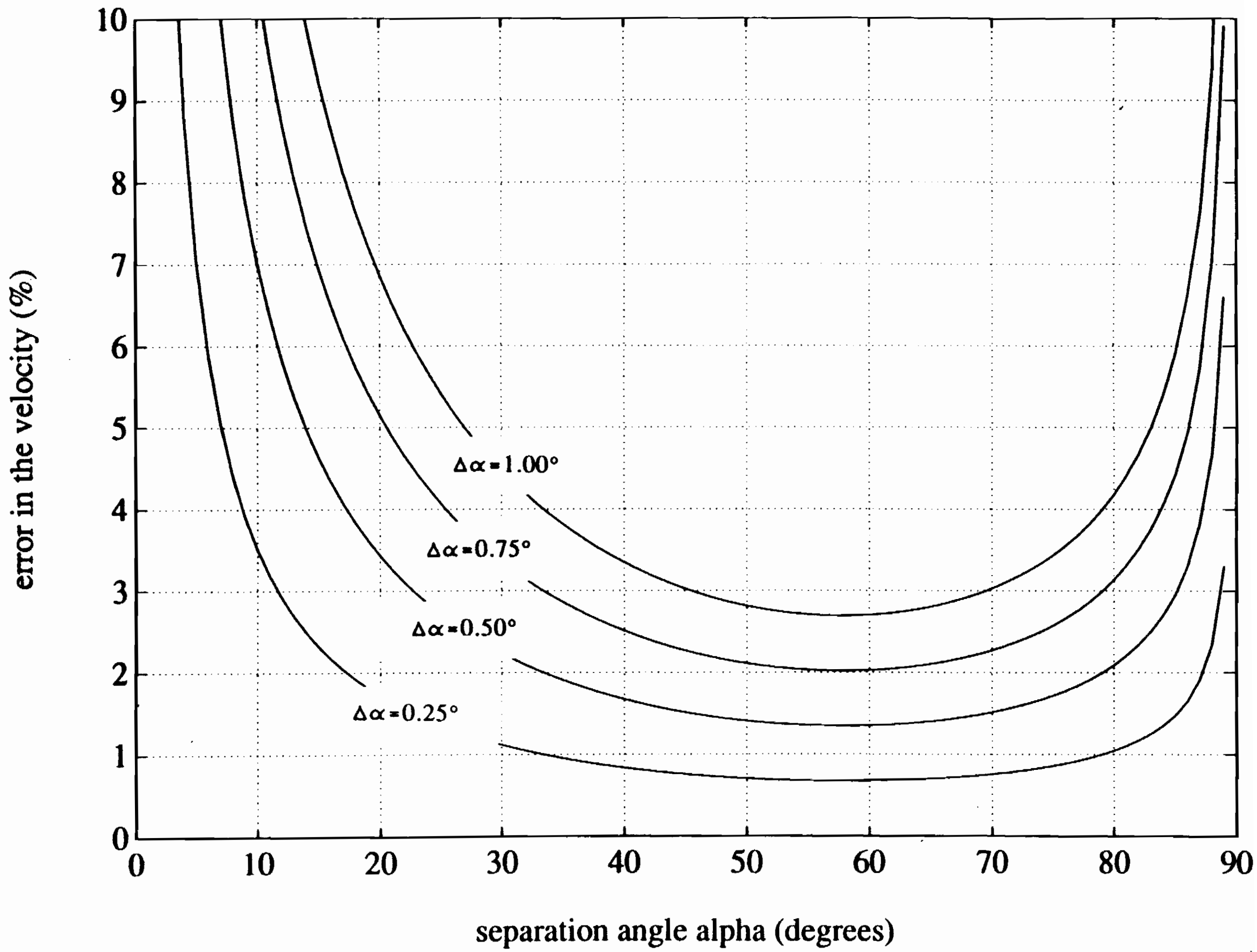

Figure 8 Percentage error in V with equal components in the direction of each laser position for error in the separation angle, $\Delta\alpha$, as a function of the laser separation angle.

stationary non-scanning device. Any velocimeter with scanning capability and/or portability would likely have less accurate positioning capability.

There are two other issues related to positioning errors. First, due to misalignment, each of the three velocity measurements may not be of the exact same spot on the structure. Second, because of elastic deformation and rigid-body motion of the structure, the measurement spot may be constantly in motion over some small region of the surface.[62] These position related error sources were not within the scope of this research.

2.2.2 Measurement Errors

The other possible source of errors in the velocities which result from the transformation are the velocities which go into it. That is, noise in the measured velocity signal will find its way into the result. To analyze the sensitivity of the transformation to noise in the measured velocities, the following assumptions were made:

1. The velocimeter positions are in the right pyramid configuration.
2. The measurement noise is random and has a Gaussian distribution.
3. The measurement noise is isotropic. That is, the noise is of equal magnitude on each of the velocimeter positions.

4. The measurement noise on each position is uncorrelated with that on the other positions.

Once again, the transformation equation for the right pyramid configuration is repeated here for convenience:

$$\begin{Bmatrix} V_x \\ V_y \\ V_z \end{Bmatrix} = \frac{1}{3\sin\alpha} \begin{bmatrix} 0 & -\sqrt{3} & \sqrt{3} \\ 2 & -1 & -1 \\ \tan\alpha & \tan\alpha & \tan\alpha \end{bmatrix} \begin{Bmatrix} V_A \\ V_B \\ V_C \end{Bmatrix} \qquad \textbf{(17)}$$

If the noise is random and Gaussian, then the *best* estimate of the variances (the square of the standard deviation) of the orthogonal velocity components are given by:

$$\begin{aligned} s_{V_x}{}^2 &= \left(\frac{\partial V_x}{\partial V_A}\right)^2 s_{V_A}{}^2 + \left(\frac{\partial V_x}{\partial V_B}\right)^2 s_{V_B}{}^2 + \left(\frac{\partial V_x}{\partial V_C}\right)^2 s_{V_C}{}^2 \\ s_{V_y}{}^2 &= \left(\frac{\partial V_y}{\partial V_A}\right)^2 s_{V_A}{}^2 + \left(\frac{\partial V_y}{\partial V_B}\right)^2 s_{V_B}{}^2 + \left(\frac{\partial V_y}{\partial V_C}\right)^2 s_{V_C}{}^2 \\ s_{V_C}{}^2 &= \left(\frac{\partial V_z}{\partial V_A}\right)^2 s_{V_A}{}^2 + \left(\frac{\partial V_z}{\partial V_B}\right)^2 s_{V_B}{}^2 + \left(\frac{\partial V_z}{\partial V_C}\right)^2 s_{V_C}{}^2 \end{aligned} \qquad \textbf{(18)}$$

where s_{VA}, s_{VB}, and s_{VC} are the standard deviations of the measured velocities.[63] This can be written in matrix form as:

$$\begin{Bmatrix} s_{V_x}^2 \\ s_{V_y}^2 \\ s_{V_z}^2 \end{Bmatrix} = \begin{bmatrix} (\frac{\partial V_x}{\partial V_A})^2 & (\frac{\partial V_x}{\partial V_B})^2 & (\frac{\partial V_x}{\partial V_C})^2 \\ (\frac{\partial V_y}{\partial V_A})^2 & (\frac{\partial V_y}{\partial V_B})^2 & (\frac{\partial V_y}{\partial V_C})^2 \\ (\frac{\partial V_z}{\partial V_A})^2 & (\frac{\partial V_z}{\partial V_B})^2 & (\frac{\partial V_z}{\partial V_C})^2 \end{bmatrix} \begin{Bmatrix} s_{V_A}^2 \\ s_{V_B}^2 \\ s_{V_C}^2 \end{Bmatrix} \tag{19}$$

Note that partial derivative of V_x with respect to V_A is simply the coefficient of V_A in the equation for V_x, Equation (17) for the right pyramid configuration. Therefore, the elements of this matrix are just the squares of the elements of the transform matrix in Equation (17). Evaluating these elements we get:

$$\begin{bmatrix} s_{V_x}^2 \\ s_{V_y}^2 \\ s_{V_z}^2 \end{bmatrix} = \frac{1}{\sin^2\alpha} \begin{bmatrix} 0 & \frac{1}{3} & \frac{1}{3} \\ \frac{4}{9} & \frac{1}{9} & \frac{1}{9} \\ \frac{\tan^2\alpha}{9} & \frac{\tan^2\alpha}{9} & \frac{\tan^2\alpha}{9} \end{bmatrix} \begin{bmatrix} s_{V_A}^2 \\ s_{V_B}^2 \\ s_{V_C}^2 \end{bmatrix} \tag{20}$$

Now, if we assume isotropy, $s_{VA} = s_{VB} = s_{VC} = s$, and we get:

$$\begin{Bmatrix} s_{V_x}^2 \\ s_{V_y}^2 \\ s_{V_z}^2 \end{Bmatrix} = \begin{Bmatrix} \frac{2}{3\sin^2\alpha} \\ \frac{2}{3\sin^2\alpha} \\ \frac{1}{3\cos^2\alpha} \end{Bmatrix} s^2 \tag{21}$$

From the standard deviations of the orthogonal velocity components we can arrive at a confidence interval for the mean value of the velocity component. The velocity components can be stated as:

$$
\begin{aligned}
V_x \pm \Delta V_x &= V_x \pm C\frac{s_{V_x}}{\sqrt{N}} \\
V_y \pm \Delta V_y &= V_y \pm C\frac{s_{V_y}}{\sqrt{N}} \\
V_z \pm \Delta V_z &= V_z \pm C\frac{s_{V_z}}{\sqrt{N}}
\end{aligned}
\tag{22}
$$

where C is a constant which depends upon the confidence which we want to have in the above statements and N is the number of samples taken. Using a Gaussian distribution $C=1.96$ for 95% confidence. However, for small N, which is likely, a Student t distribution is more appropriate. Thus, C=2.23 for 95% confidence.[64] Forming the vector of ΔV_x, ΔV_y, and ΔV_z from Equation (22) and substituting s_{V_x}, s_{V_y}, and s_{V_z} from Equation (21) we get:

$$
\begin{Bmatrix} \Delta V_x \\ \Delta V_y \\ \Delta V_z \end{Bmatrix} = \begin{Bmatrix} \dfrac{C\sqrt{2}}{\sqrt{3}\sqrt{N}\sin\alpha} \\ \dfrac{C\sqrt{2}}{\sqrt{3}\sqrt{N}\sin\alpha} \\ \dfrac{C}{\sqrt{3}\sqrt{N}\cos\alpha} \end{Bmatrix} s
\tag{23}
$$

Plots of these quantities are found in Figures 9 and 10 for $C=2.23$ (95% confidence) and $N=10$.

These graphs are to be interpreted as follows. The vertical axis shows the half-width of the 95% confidence interval on the orthogonal velocities. Actually this graph has a mirror image below the horizontal axis (the lower bound) but only the positive portion is shown for clarity. The horizontal axis is the nominal separation angle in degrees. This graph reveals that V_x and V_y are highly sensitive to measurement noise for small separation angles. Also, V_z is highly sensitive for separation angles near 90°. Measurement noise with a standard deviation of zero (no noise) results in zero sensitivity for all separation angles. This graph indicates an optimal region of minimum sensitivity somewhere in-between 0 and 90°.

As before, to find a separation angle of minimum sensitivity we must make an assumption about the direction of V. For equal components in each of the orthogonal directions the sensitivity equation becomes:

$$\Delta V = \frac{sC}{\sqrt{3}\sqrt{N}}\sqrt{\left(\frac{\sqrt{2}}{\sin\alpha}\right)^2+\left(\frac{\sqrt{2}}{\sin\alpha}\right)^2+\left(\frac{1}{\cos\alpha}\right)^2} \tag{24}$$

This sensitivity is shown on Figure 11. Taking the derivative with respect to α, setting equal to zero, and solving for α gives $\alpha=54.7°$. This value is expected

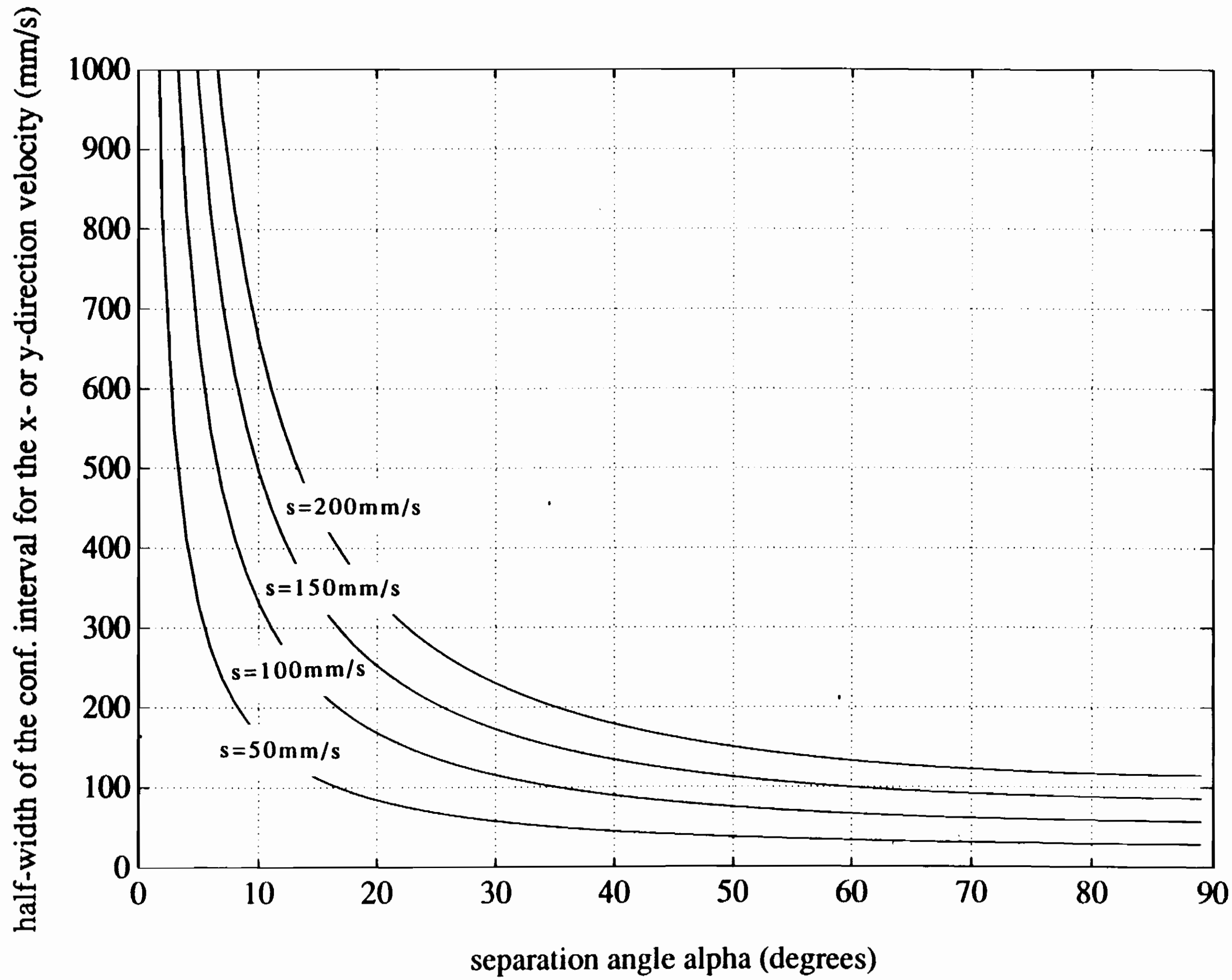

Figure 9 Half-width of the confidence interval using a 95% Student t distribution on V_x or V_y for measurement noise of standard deviation s on each component as a function of the laser separation angle.

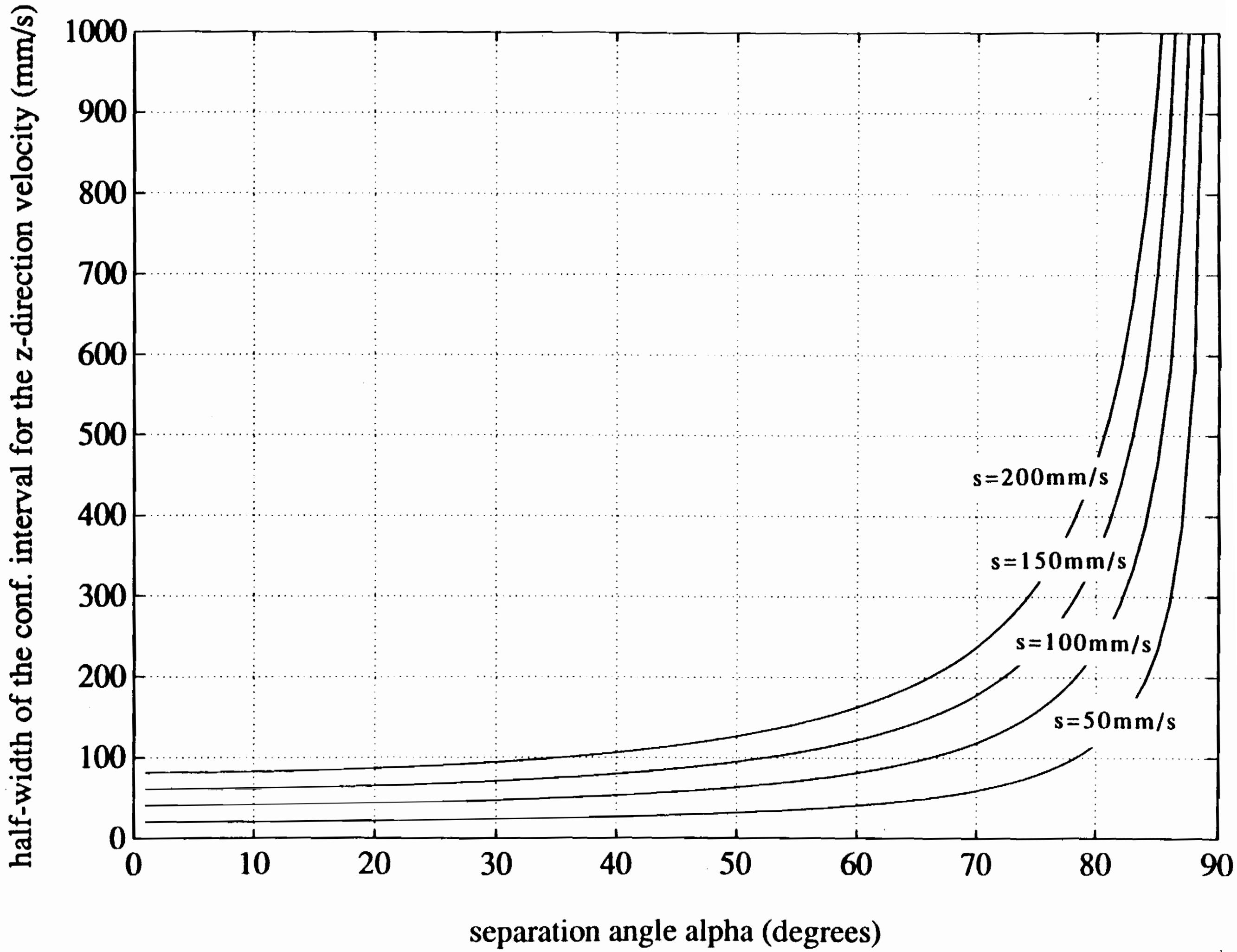

Figure 10 Half-width of the confidence interval using a 95% Student t distribution on V_z for measurement noise of standard deviation s on each component as a function of the laser separation angle.

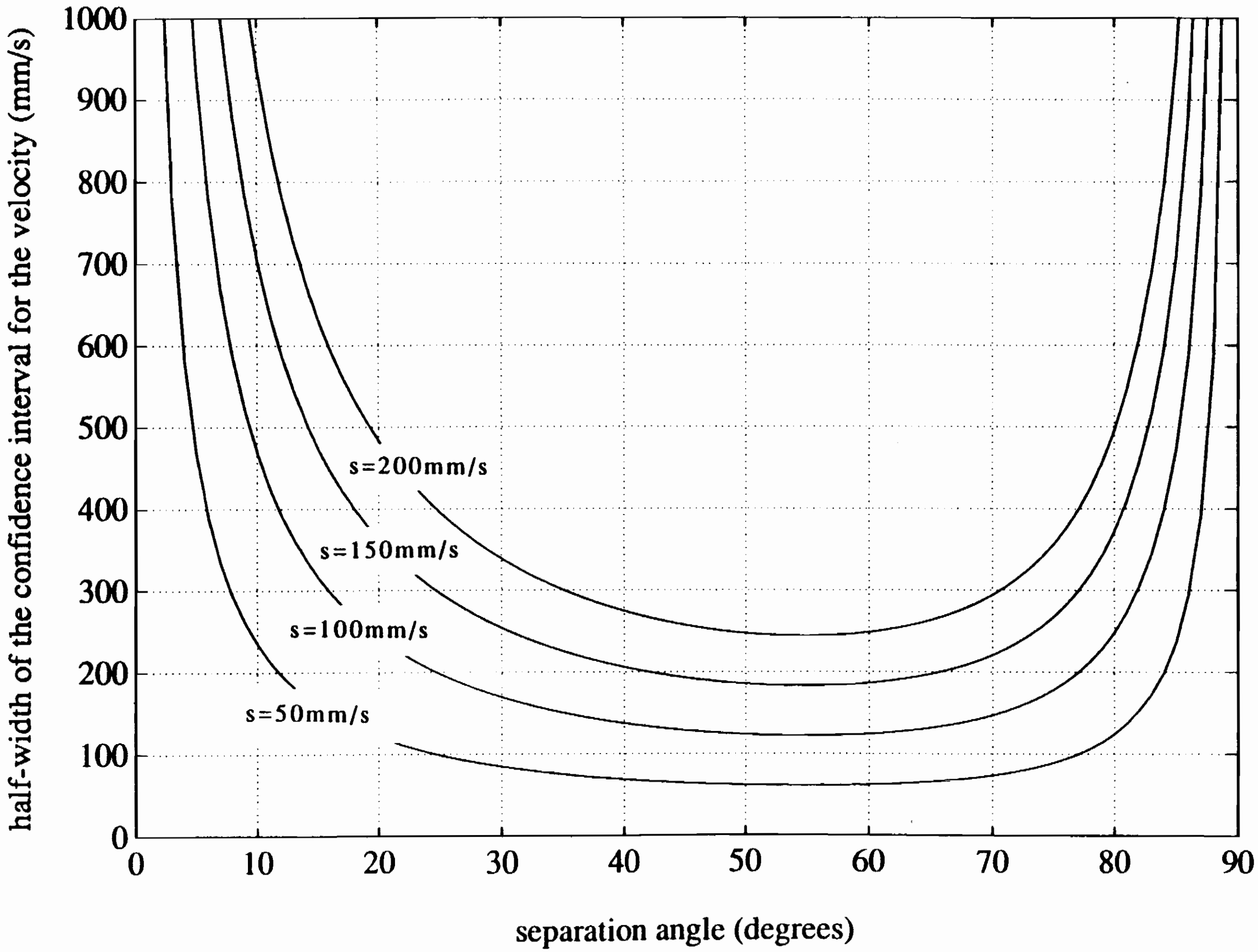

Figure 11 Half-width of the confidence interval using a 95% Student t distribution on V from three equal components for measurement noise as a function of the laser separation angle.

because 54.7° is the separation angle corresponding to an orthogonal laser configuration. For separation angles between about 25° and 75° one can be 95% confident that the coordinate transformation will produce no more than double the noise in the velocity measurement. For separation angles less than 25° and greater than 75° the noise is amplified much more by the transformation.

A comparison with the work of Huffaker et al. is particularly enlightening here.[65] Their small separation angle, $\alpha=4.25°$, would result in amplification of the noise by about ten times. It is no surprise that Huffaker's work was criticized for being highly susceptible to noise which tended to obliterate small in-plane velocity measurements. Incidentally, this had the subsequent effect of turning the majority of researchers toward dual-beam velocimeters for three-dimensional measurement. These, however, are concentrated in the area of fluid flow.

2.2.3 Condition Number

Substitution of the right-pyramid configuration assumptions into Equation (7) yields:

$$\begin{Bmatrix} V_A \\ V_B \\ V_C \end{Bmatrix} = \boldsymbol{T} \begin{Bmatrix} V_x \\ V_y \\ V_z \end{Bmatrix} \tag{25}$$

where:

$$\boldsymbol{T} = \begin{bmatrix} 0 & \sin\alpha & \cos\alpha \\ -\frac{\sqrt{3}}{2}\sin\alpha & -\frac{1}{2}\sin\alpha & \cos\alpha \\ \frac{\sqrt{3}}{2}\sin\alpha & -\frac{1}{2}\sin\alpha & \cos\alpha \end{bmatrix} \tag{26}$$

The condition number of $\boldsymbol{T}$ has been calculated and plotted in Figure 12 as a function of the separation angle. As expected, the condition number is unity (ideally conditioned) for a separation angle of 54.7° corresponding to an orthogonal velocimeter orientation. If the previously discussed bounds of 25° to 75° are to be adhered to, then an equivalent rule would be to accept only transformation matrices $\boldsymbol{T}$ whose condition number is less than about three.

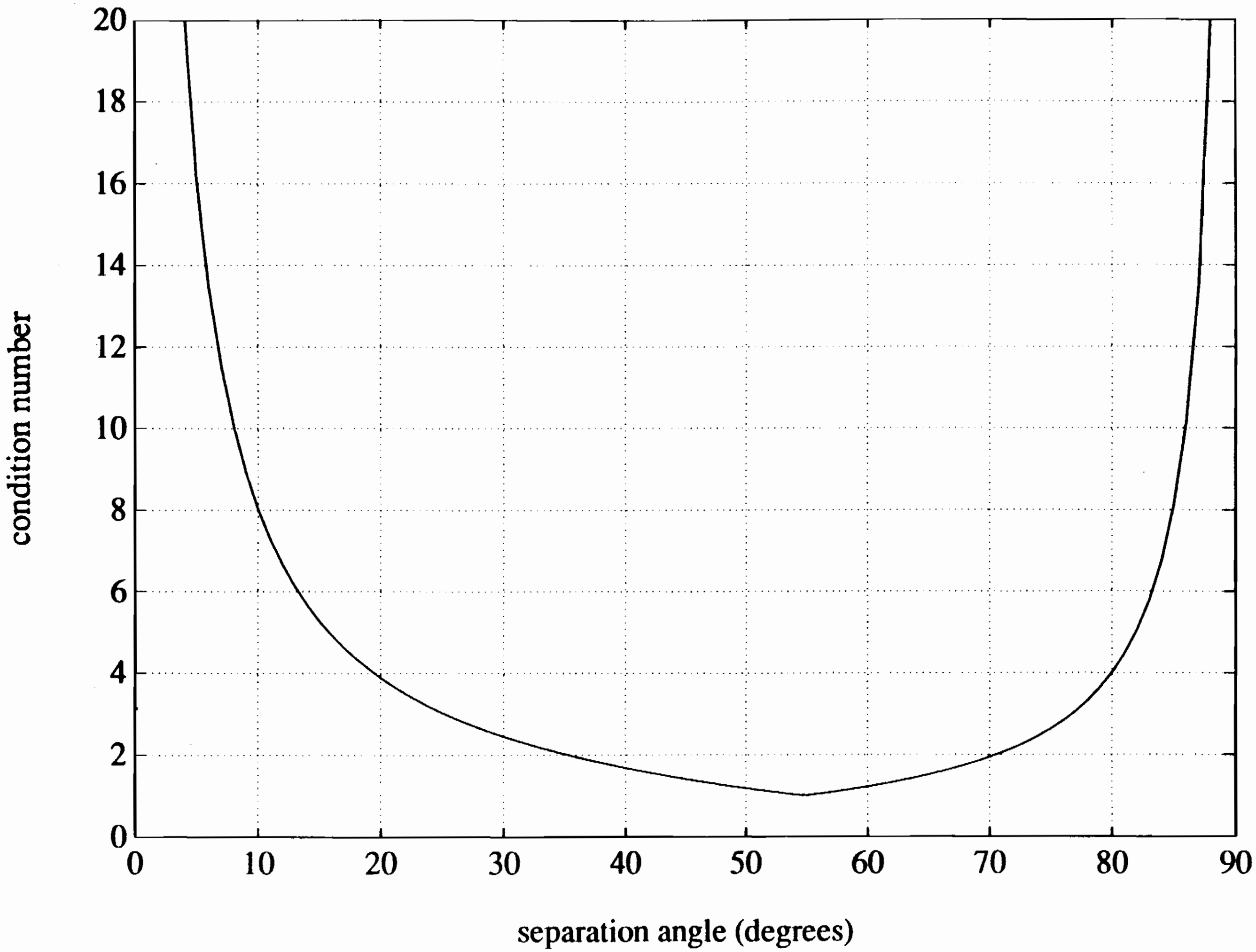

Figure 12 Condition number for the transformation matrix, $\boldsymbol{T}$, in the right-pyramid configuration.

2.3 COMPARISON WITH ACCELEROMETERS

The sensors which a three-position laser velocimeter would likely replace are accelerometers. A typical accelerometer has a sensitivity to motion perpendicular to its axis of primary sensitivity. This sensitivity is called transverse sensitivity.[66]

To facilitate comparison with an accelerometer we must modify the right pyramid configuration. The modified configuration allows the separation angle for position A to vary independent of *B* and *C*. Thus:

$$
\begin{gathered}
R_{Ax}=0,\ R_{Ay}=R\sin(\alpha+\Delta\alpha),\ R_{Az}=R\cos(\alpha+\Delta\alpha),\\
R_{Bx}=-\tfrac{\sqrt{3}}{2}R\sin\alpha,\ R_{By}=-\tfrac{1}{2}R\sin\alpha,\ R_{Bz}=R\cos\alpha\\
R_{Cx}=\tfrac{\sqrt{3}}{2}R\sin\alpha,\ R_{Cy}=-\tfrac{1}{2}R\sin\alpha,\ R_{Cz}=R\cos\alpha
\end{gathered}
\tag{27}
$$

where $\Delta\alpha$ is an error in the separation angle of position *A* only. We desire the transverse sensitivity corresponding to a given $\Delta\alpha$. Substitution of the above assumptions into Equation (9) gives for V_y:

$$
V_y = \frac{V_A\sqrt{3}\,c\alpha\ \ s\alpha-(V_B+V_C)\,\frac{\sqrt{3}}{2}c\alpha+\Delta\alpha\ \ s\alpha}{\frac{\sqrt{3}}{2}s\alpha\,(s\alpha+\Delta\alpha\ \ c\alpha+\frac{1}{2}c\alpha+\Delta\alpha\ \ s\alpha)+\frac{\sqrt{3}}{4}s^2\alpha\ \ c\alpha+\Delta\alpha+\frac{\sqrt{3}}{2}s\alpha\ \ c\alpha\ \ s\alpha+\Delta\alpha}
\tag{28}
$$

where s means sine and c means cosine. If the true velocity is in the z direction only then:

$$V_A = V_B = V_C = V_z \cos\alpha \tag{29}$$

If the calculated velocity in the y direction is written in terms of V_z then:

$$V_y = HV_z \tag{30}$$

where H is the transverse sensitivity. Substitution gives:

$$H = \frac{c\alpha\sqrt{3}\,(c\alpha\ s\alpha - c\alpha+\Delta\alpha\ s\alpha)}{\frac{\sqrt{3}}{2}s\alpha\,(s\alpha+\Delta\alpha\ c\alpha+\frac{1}{2}c\alpha+\Delta\alpha\ s\alpha)+\frac{\sqrt{3}}{4}s^2\alpha\ c\alpha+\Delta\alpha+\frac{\sqrt{3}}{2}s\alpha\ c\alpha\ s\alpha+\Delta\alpha} \tag{31}$$

where s means sine and c means cosine, and which is plotted in Figure 13 for various $\Delta\alpha$. Figure 13 shows the transverse sensitivity in the range of 5% transverse sensitivity, typical for accelerometers. It can be clearly seen that by maintaining one separation angle to within about 4 degrees of the other two, one can expect to have less than 5% equivalent transverse sensitivity.

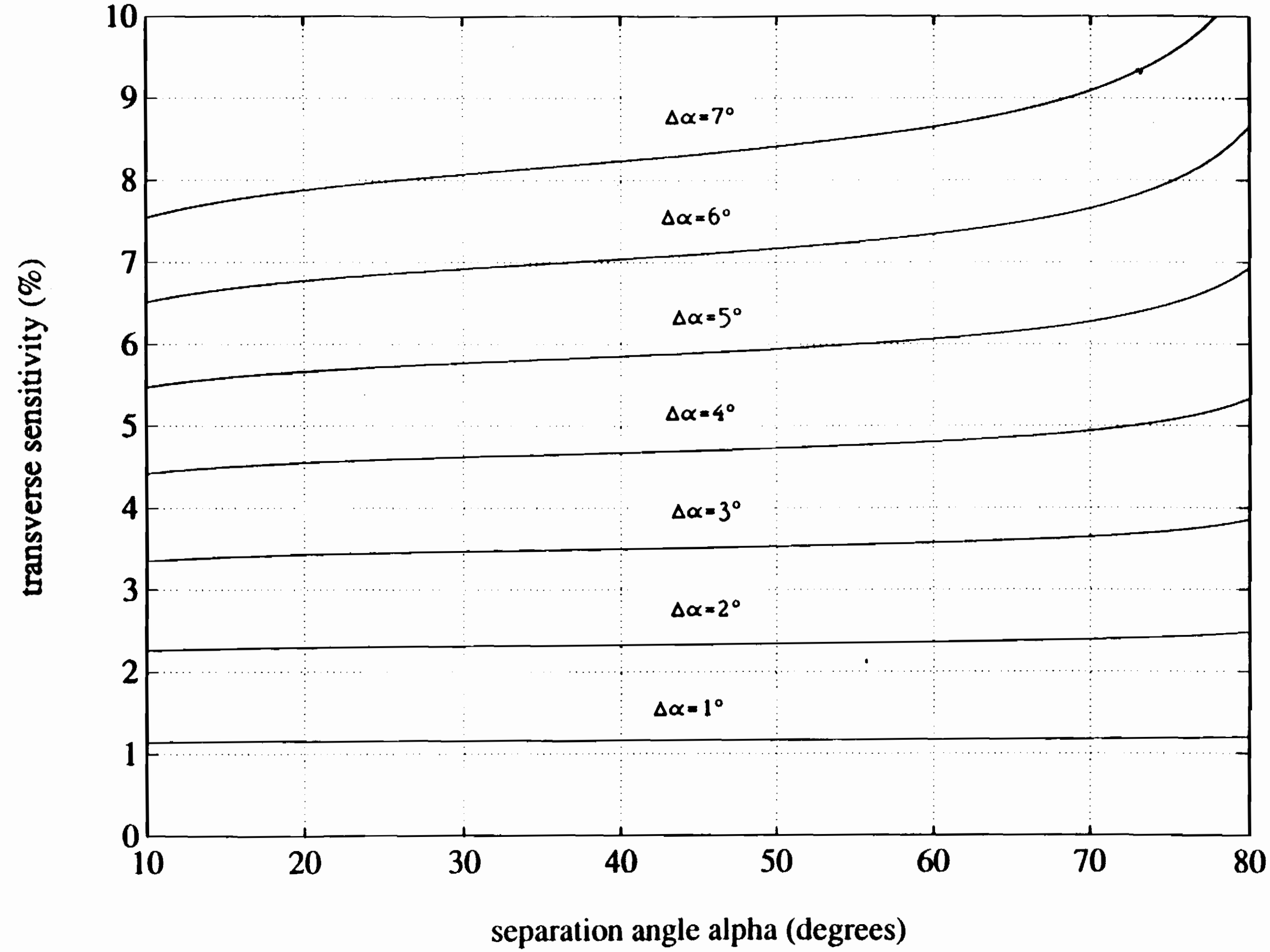

Figure 13 Transverse sensitivity for various separation angle errors, **$\Delta\alpha$**.

2.4 PRACTICAL CONSIDERATIONS

The practical implications of the error sensitivities as developed for the right-pyramid configuration are revealed in the workspace diagram of Figure 14. For the sake of this discussion, let us limit ourselves, based on the sensitivity information, to the use of separation angles between 25° and 70°, or equivalently condition numbers below three. The ordinate of the graph shows the diameter of the circle formed by the three velocimeter positions. The horizontal axis shows the range from the velocimeters to the target. Oftentimes, because of the limitations of the scan window, relatively large ranges are desired. However, as the graph shows, achieving those ranges may require a larger than practical velocimeter circle diameter (generally limited by the ceiling height). A line representing the additional workspace acquired by lowering the minimum separation angle to that of Huffaker et al. is also shown.[67]

As useful as the right-pyramid configuration is for analysis of three-dimensional velocity determination, it is not very practical. A more likely configuration is that shown in Figure 15. This "likely use" configuration places two laser velocimeter positions on the "floor" with the third located equidistant from the other two and raised above the floor. The mathematical relationships describing this configuration are:

$$\begin{aligned} &R_{Ax}=0, \; R_{Ay}=h, \; R_{Az}=k \\ &R_{Bx}=-d, \; R_{By}=0, \; R_{Bz}=k \\ &R_{Cx}=d, \; R_{Cy}=0, \; R_{Cz}=k \end{aligned} \tag{32}$$

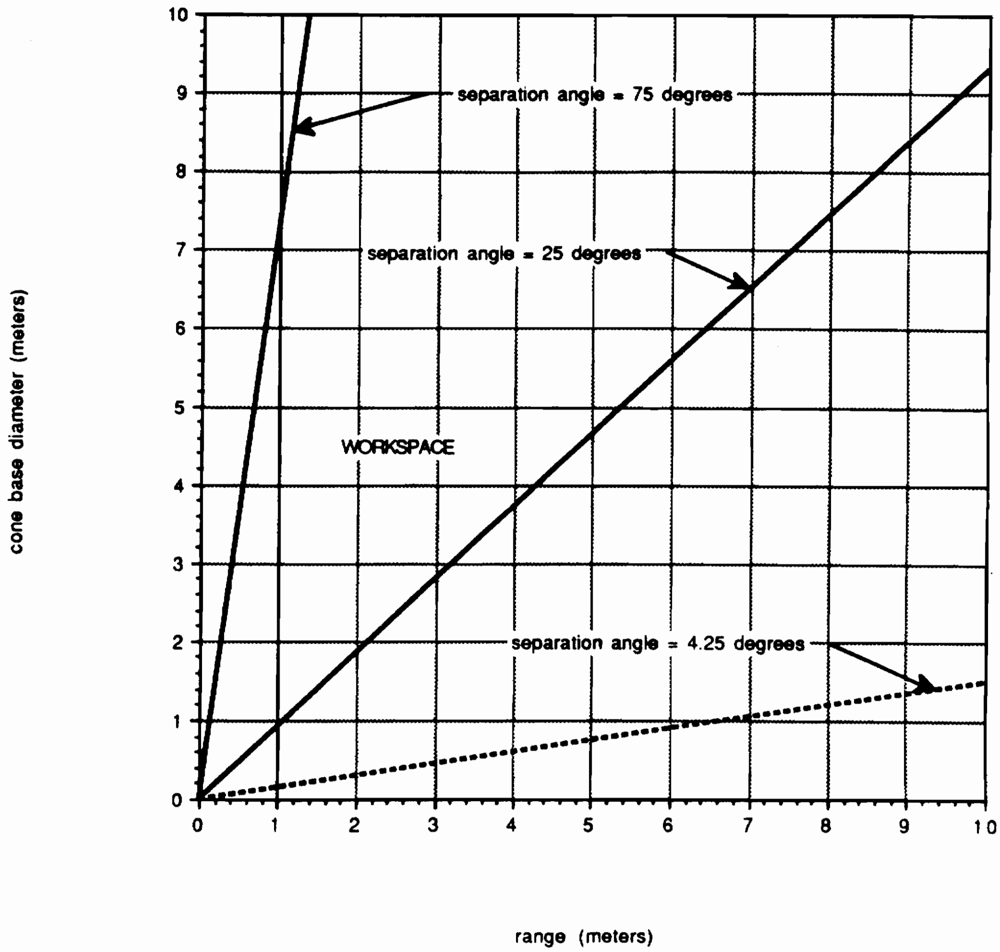

Figure 14 Workspace diagram for the right-pyramid configuration showing the practical limitations resulting from the error sensitivities.

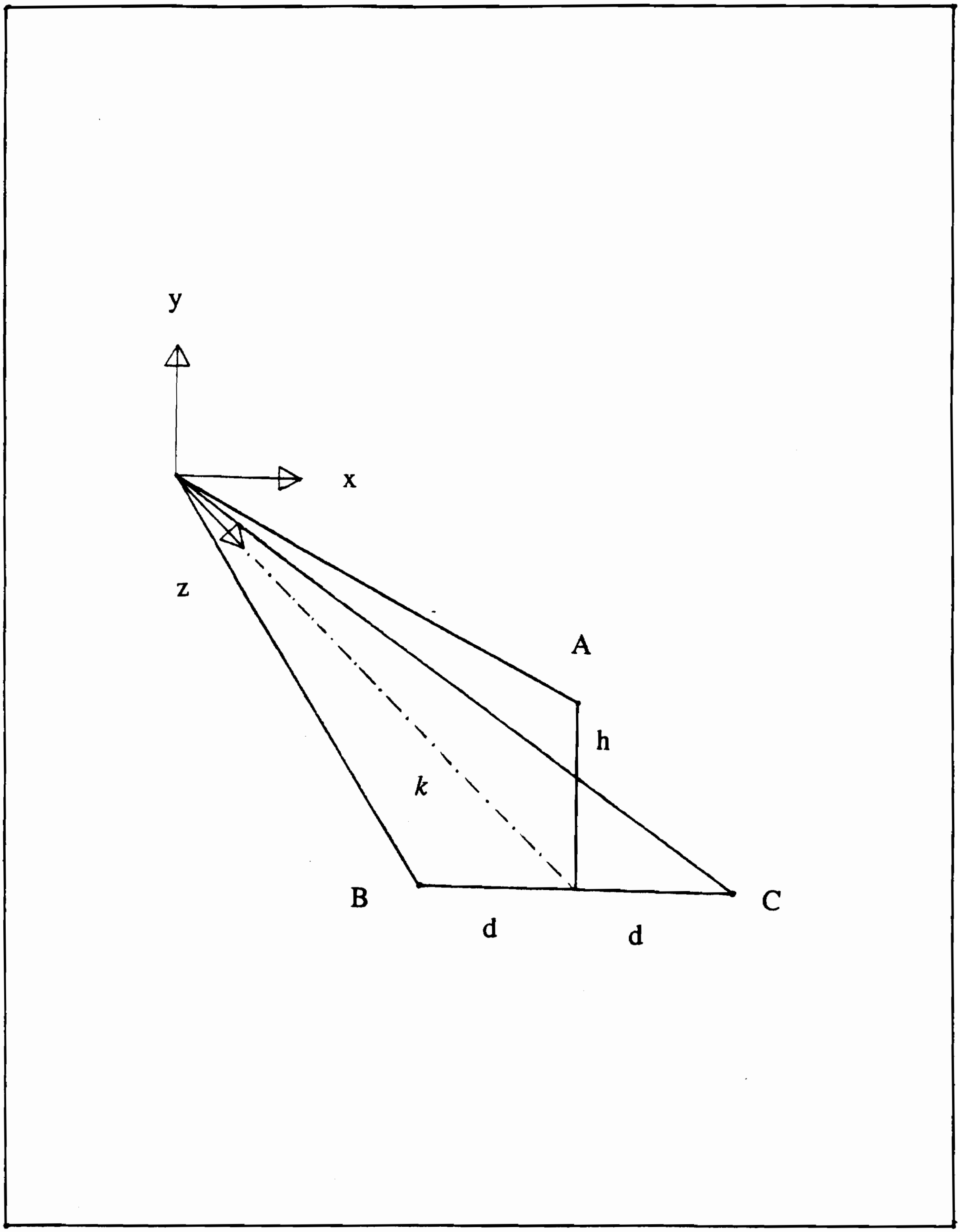

Figure 15 The likely use configuration.

This configuration assumes all three positions lie in a plane normal to the *z*-axis. In practice, the positions need not lie in that plane but need only form the same angles with respect to the orthogonal coordinate system. Thus the distances from each velocimeter position to the target are:

$$R_A=\sqrt{h^2+k^2},\quad R_B=\sqrt{d^2+k^2},\quad R_C=\sqrt{d^2+k^2} \tag{33}$$

Substituting these assumptions into Equation (7) gives:

$$\begin{Bmatrix} V_A \\ V_B \\ V_C \end{Bmatrix} = \boldsymbol{T} \begin{Bmatrix} V_x \\ V_y \\ V_z \end{Bmatrix} \tag{34}$$

where:

$$\boldsymbol{T} = \begin{bmatrix} 0 & \dfrac{(\frac{h}{k})}{\sqrt{(\frac{h}{k})^2+1}} & \dfrac{1}{\sqrt{(\frac{h}{k})^2+1}} \\ \dfrac{-(\frac{d}{k})}{\sqrt{(\frac{d}{k})^2+1}} & 0 & \dfrac{1}{\sqrt{(\frac{d}{k})^2+1}} \\ \dfrac{(\frac{d}{k})}{\sqrt{(\frac{d}{k})^2+1}} & 0 & \dfrac{1}{\sqrt{(\frac{d}{k})^2+1}} \end{bmatrix} \tag{35}$$

where k is the distance of the plane of velocimeters from the target, d is the distance on either side of center at which positions B and C lie, and h is the height of position A from the floor. The condition number of T for various values of (h/k) is plotted versus (d/l) in Figure 16. From our previous analysis we desire condition numbers below three. Thus, the plots indicate the minimum (h/k) to be approximately one. For $(h/k)=1$ the range of acceptable values of (d/k) is from about 0.4 to 1.8, with the upper limit rising to almost 3.0 for $(h/l)=10$.

Substitution of the likely use configuration assumptions into Equation (9) gives:

$$\begin{Bmatrix} V_x \\ V_y \\ V_z \end{Bmatrix} = \frac{\sqrt{d^2+k^2}}{2hdk} \begin{bmatrix} 0 & -hk & hk \\ 2dk\frac{\sqrt{h^2+k^2}}{\sqrt{d^2+k^2}} & -dk & -dk \\ 0 & hd & hd \end{bmatrix} \begin{Bmatrix} V_A \\ V_B \\ V_C \end{Bmatrix} \qquad (36)$$

This is the transformation equation for the likely use configuration.

A workspace diagram can be generated for this configuration as well. If we do not allow the condition number of T to exceed about 3 one arrives at the workspace shown in Figure 17.

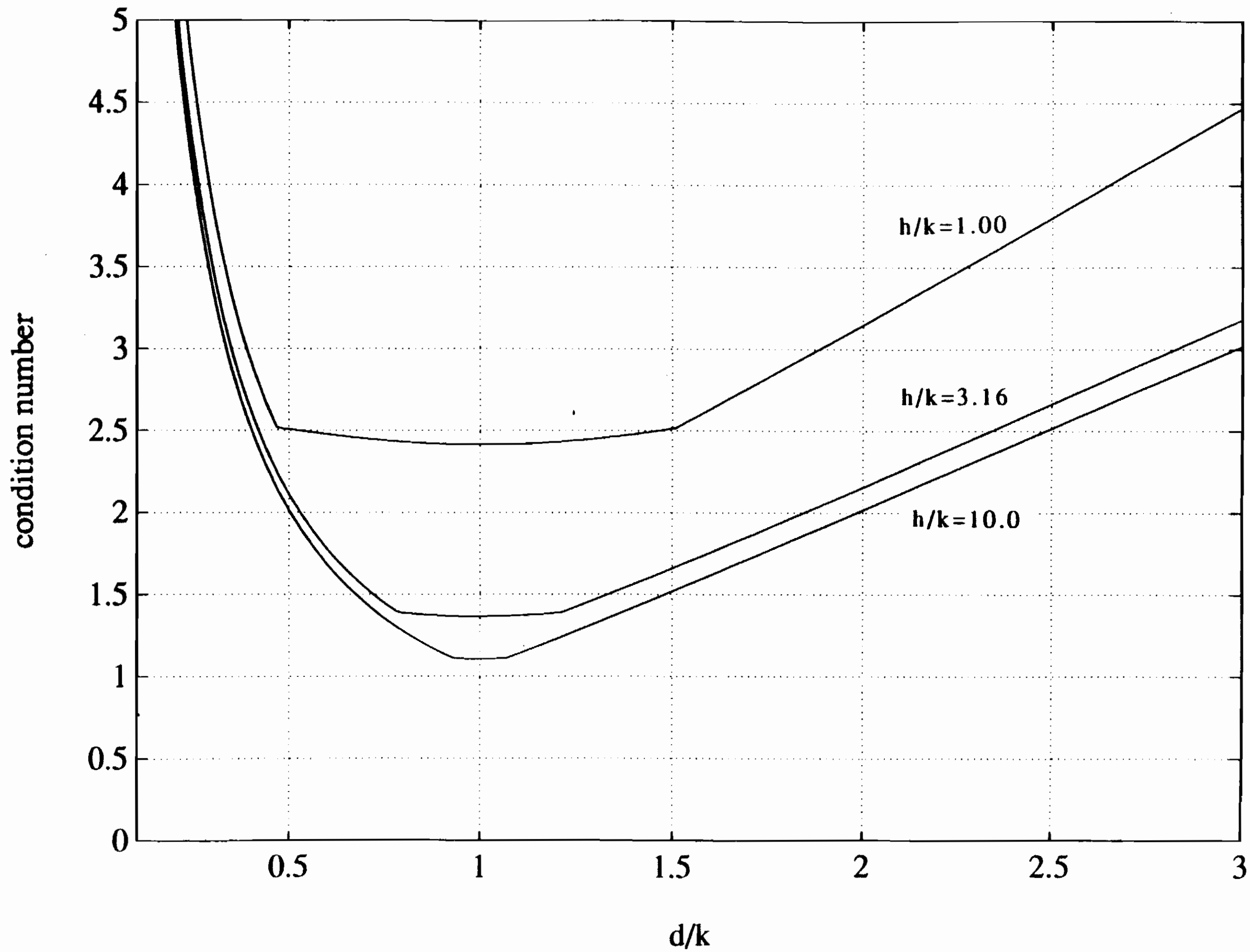

Figure 16 Condition number for the transformation matrix, ***T***, in the likely use configuration.

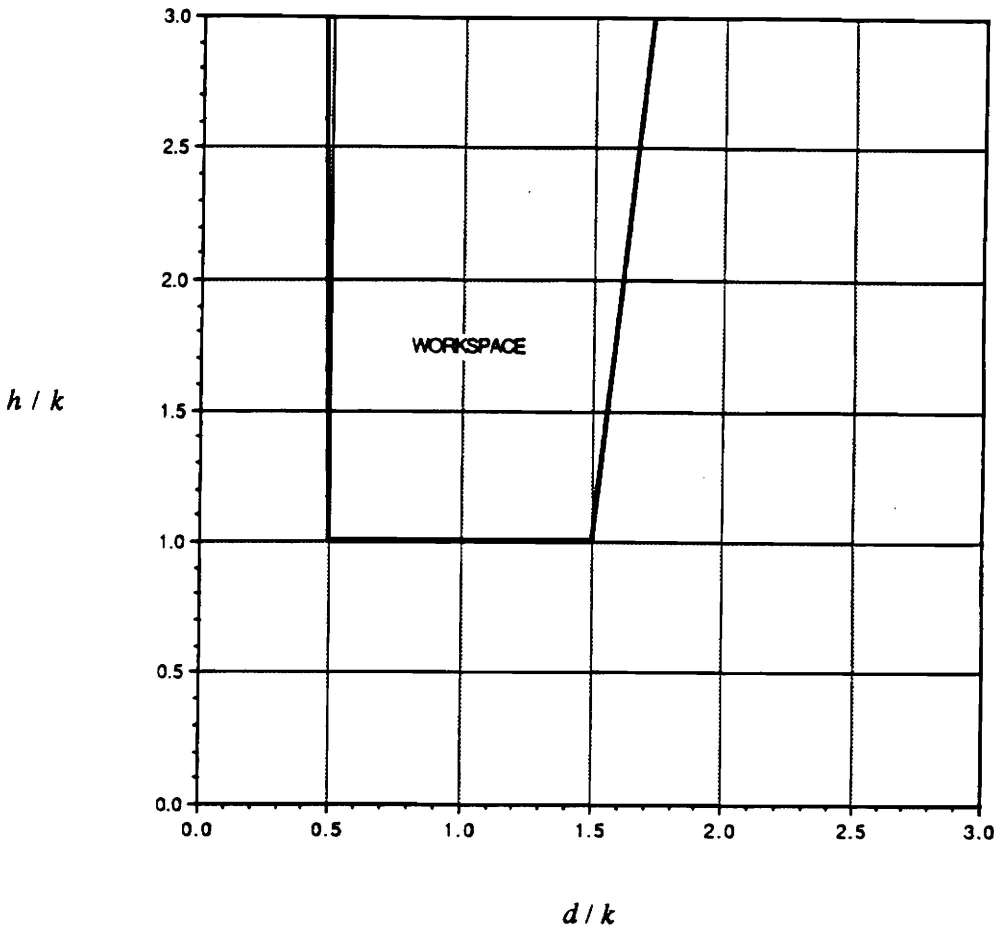

Figure 17 Workspace diagram for the likely use configuration showing the practical limitations resulting from the error sensitivities.

3 CONCLUSIONS AND RECOMMENDATIONS

3.1 CONCLUSIONS

Although much research has been done in the area of three-dimensional velocimetry using lasers, very little has been done using the three-position technique which seems to hold the most promise for vibrating structure applications. In this thesis, the general transformation equations were developed. Two specific configurations were analyzed in depth: the right-pyramid configuration and the likely use configuration. The sensitivities of the transformation to position errors and measurement noise were determined and the condition number was used to give a general criterion by which to judge any arbitrary configuration. Sensitivity of the right-pyramid configuration in particular was compared to accelerometer transverse sensitivity.

1. From the sensitivity analysis, it can be concluded that the three-position technique is capable of performing at least as well as, if not better than,

conventional accelerometers given stringent but perhaps not unreasonable accuracy requirements.

2. Workspace diagrams derived from the analysis indicate fairly severe limitations on acceptable configurations with the practical concern of space available being the other source of limitation.

3. The condition number is a very useful concept when judging various configurations. It should be considered for use in automated testing to provide the computer with a tool for deciding whether data is "good" or "bad" while scanning. The issue of scanning adds a great deal of complexity to the three-position technique. It is possible that some points on the structure are acceptable from a coordinate transformation point of view while others are not. The optimum configuration is that for which the most points on the structure are acceptable. This becomes harder to achieve the smaller the workspace available. It is conceivable that data from different "patches" of the surface could be acquired using different configurations. These patches can be defined as the region of points whose transformation is acceptable. Then the data must be merged in some way.

3.2 RECOMMENDATIONS

It is recommended that an experiment be designed to test the conclusions previously drawn. This experiment should have a velocity target with very consistent motion over the time it takes to move from one position to another. It is unclear exactly what form such a target would take. Calibrated velocity targets for use in fluid dynamics velocimeters are known to exist. The experiment should make use of the reference-beam velocimeter to measure directly the orthogonal velocity components. Then various configurations of non-orthogonal configurations can be used. Noise and position errors can be introduced to verify the sensitivity analyses. It is important that a means for accurate positioning be developed to control position to at least an order of magnitude more precisely than the position errors to be examined. Also, an accurate phase reference must be established because in general the velocities measured will be complex quantities. Such an experiment would verify the conclusions drawn in this thesis and make clear the practical limitations that exist in most laboratory or field settings.

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APPENDIX

MATLAB code for performing the coordinate transformation.

```
%TRANS.M
%Performs the coordinate transformation from three non-orthogonal complex
%velocities to three orthogonal complex velocities. Designed for use in a
%series of three-position experiments run in the summer of 1991.
%All length measures are in m.
%All velocity measures are in m/s.
%All angular measures are in radians except where noted.

%By Joseph B. Donovan
%Last revised 16 July 1991
%______________________________________________________________


clear


disp('')
disp('This program will')
disp('perform the coordinate transformation')
disp('')


alphadeg=input('alpha in degrees : ');
alpha=alphadeg*pi/180;


%Form the transformation matrix for laboratory setup (like R.M. Huffaker)
T=(1/sin(alpha)).*[ 0              -sqrt(3)/3      sqrt(3)/3
                    2/3            -1/3            -1/3
                    tan(alpha)/3   tan(alpha)/3    tan(alpha)/3];


%Form the transformation matrix for the general setup
%Rax=
%Ray=
%Raz=
%Rbx=
```

```
%Rby=
%Rbz=
%Rcx=
%Rcy=
%Rcz=
%Ra=sqrt(Rax^2+Ray^2+Raz^2);
%Rb=sqrt(Rbx^2+Rby^2+Rbz^2);
%Rc=sqrt(Rcx^2+Rcy^2+Rcz^2);
%det=Rcx*(Ray*Rbz-Raz*Rby)+Rcy*(Raz*Rbx-Rax*Rbz)+Rcz*(Rax*Rby-Ray*Rb
x);
%T=(1/det).*[Ra*(Rby*Rcz-Rbz*Rcy) Rb*(Raz*Rcy-Ray*Rcz)
Rc*(Ray*Rbz-Raz*Rby)
%          Ra*(Rbz*Rcx-Rbx*Rcz) Rb*(Rax*Rcz-Raz*Rcx) Rc*(Raz*Rbx-Rax*Rbz)
%          Ra*(Rbx*Rcy-Rby*Rcx) Rb*(Ray*Rcx-Rax*Rcy)
Rc*(Rax*Rby-Ray*Rbx)];


%Input measured complex velocities
disp('Input the measured complex velocities in meters per second')
disp('(complex quantities are entered as a+b*sqrt(-1) )')
A=input('Enter the velocity from position A: ');
B=input('Enter the velocity from position B: ');
C=input('Enter the velocity from position C: ');
disp('')


%Perform the coordinate transformation
V=T*[A;B;C];


%Output orthogonal complex velocities
disp('The orthogonal complex velocities are')
disp(V)
disp('meters per second.')


%End of TRANS.M
```

VITA

Joseph B. Donovan was born July 21, 1968 and raised in the Washington, DC area. He attended Bishop McNamara High School in Forestville, Maryland. He received a Bachelor of Science degree in Mechanical Engineering from Virginia Polytechnic Institute and State University (Virginia Tech) in Blacksburg, Virginia. His professional society involvement included a term as vice president of the Virginia Tech student chapter of the Society of Automotive Engineers and a term as vice president of Pi Tau Sigma, the Mechanical Engineering Honor Fraternity. His work experience included a summer at the General Motors Technical Center working for the Chevrolet-Pontiac-Canada group. He also participated in the Washington Internships for Students in Engineering (WISE) program in Washington, DC. He attended graduate school at Virginia Tech's Department of Mechanical Engineering.