Pedagogical moves related to analogy that support a unified understanding of eigentheory concepts in a quantum mechanics class

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It is beneficial for quantum mechanics students to have a unified understanding of eigentheory concepts, so they can recognize the shared structure of mathematized phenomena from the different quantum mechanical systems of spin, energy, or position and recognize those as instantiations of the same overarching concept. Quantum mechanics instructors should, therefore, provide opportunities for their class community to develop a shared unified understanding of eigentheory concepts. One such opportunity can arise by engaging students in analogizing eigentheory concepts in one context with those from another context. We investigate the pedagogical moves related to analogies that can be used by a quantum mechanics course instructor to support a class community in developing a shared unified understanding of eigenequations. We analyze classroom data to characterize an instructor's pedagogical moves as he engaged students in analogical reasoning. Some moves include posing tasks conducive to analogizing; preparing, soliciting, and scaffolding students' participation in analogical reasoning; using deictic gestures and inscriptions; juxtaposing symbols representing the analogized concepts; and explicitly highlighting the sameness of the analogized concepts. We exemplify these pedagogical moves with analytical descriptions of illustrative class episodes. We discuss how these pedagogical moves can support the class community's expansion of their common ground by fostering the development of the class's shared unified understanding of eigentheory concepts.

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I. INTRODUCTION

Quantum mechanics students are expected to reason about eigentheory concepts in various contexts, including spin, energy, and position where eigenequations can have forms such as $\hat{S}_z|+\rangle = \frac{\hbar}{2}|+\rangle$, $\hat{H}|E_n\rangle = E_n|E_n\rangle$, and $\hat{x}|x_i\rangle = x_i|x_i\rangle$. It is nontrivial for students to abstract the shared mathematical structure of these eigenequations and recognize them each as an instantiation of the same overarching concept with analogous physical interpretations. Reasoning about these eigentheory concepts requires students to coordinate the operators, eigenstates, and eigenvalues that comprise eigenequations, as well as the associated physical referents of observables and the results of measuring the observables. There is added complexity when reasoning about the relationships among eigentheory concepts across the varied contexts of spin, energy, and position. Cross-contextual reasoning about a mathematical concept entails having a *unified understanding*, defined as a coherent awareness of the structural relationships, similarities, or shared properties among concepts; a recognition of distinct concepts as instantiations or examples of the same overarching concept; and an ability to reason about those concepts in general without reference to a specific context [1–3]. We posit that it is beneficial for quantum mechanics students to develop a unified understanding of the structure of eigenstates, eigenvalues, and eigenequations so they can recognize varied eigentheory concepts (e.g., $|+\rangle$, $|E_n\rangle$, $|x_i\rangle$) as instantiations of the same overarching concept (e.g., eigenstates). Several mathematics education researchers have characterized individual students' unified understandings or ways of coherently reasoning about a mathematical concept across different contexts [2,4–6]. However, there is little research on how an instructor can support a classroom community in developing unified understandings of mathematical concepts, especially in physics.

We hypothesize that reasoning about eigentheory across the contexts of spin, energy, and position and coming to recognize eigentheory concepts as instantiations of the same overarching concept can be facilitated by engaging in analogical reasoning [7,8]. This assertion stems from

Richland *et al.*'s [9] claim, "Analogical comparison can result in formation of abstract schemas to represent the underlying structure of source and target objects, thereby enhancing reasoners' capacity to transfer learning across context" (p. 38). By making analogies across the contexts of spin, energy, and position, students in a quantum mechanics class community can abstract and generalize the shared structure of the eigenequations that result from the mathematization of those quantum mechanical systems. Analogical reasoning involves mapping concepts from a familiar source domain to a new target domain with the aim of developing new knowledge in that target domain. Making analogies might involve corresponding real-life phenomena with physical phenomena via metaphor, or it might involve drawing comparisons among similar concepts like perimeter and circumference in mathematics. Analogizing has been shown to be a productive pedagogical strategy in physics and mathematics when used appropriately [10–16]. In this study, we focus on the course professor and investigate pedagogical moves of how using analogies during class might contribute to the class community's taken-as-shared [17] unified understanding of eigentheory concepts, such as eigenstates and eigenequations. We address this research question: *What pedagogical moves related to analogies can be used by a quantum mechanics course instructor to support the class community in developing a shared unified understanding of eigentheory concepts?*

II. LITERATURE AND THEORY

For this study on pedagogical moves that support quantum mechanics students in developing a unified understanding of eigenequations, we first include a targeted literature review on student understanding of eigentheory in quantum mechanics. We then introduce three theoretical frameworks that we utilize to shape our study: unified understandings, analogical reasoning, and common ground.

A. Student understanding of eigentheory in quantum mechanics

Grappling with eigenequations may involve reasoning about their constituent objects (i.e., operator, eigenvectors, eigenvalue, equal sign, and operations of matrix-vector multiplication and scalar-vector multiplication) individually or in coordination. For instance, Marshman and Singh [18] reported on various aspects of relating quantum mechanical observables and their corresponding operators in Dirac notation that may be nontrivial for learners. They reported that "after traditional instruction in relevant concepts, students had great difficulty identifying and generating an expression for a generic Hermitian operator $\hat{Q}$ in terms of its eigenvalues and eigenstates" (p. 14). They found that some students concluded incorrectly that the operator $\hat{Q}$ could be expressed as $\hat{Q} = \lambda_j$ (an eigenvalue), $\hat{Q} = \sum \lambda_j$ (a sum of eigenvalues), $\hat{Q} = q_n|q_n\rangle$ (a scalar multiple of an eigenstate), or $\hat{Q} = \sum q_n|q_n\rangle$ (a linear combination of eigenstates). These student difficulties helped guide the creation of a quantum interactive learning tutorial (QuILT) with the aim of helping students grasp the relevant concepts. The study found that students who had worked through the QuILT performed better on the question that guided them through the spectral decomposition of $\hat{Q} = \sum q_n|q_n\rangle\langle q_n|$ for orthonormal eigenstates $\{|q_n\rangle\}$. Overall, it is complex for students to attend to and make sense of the different individual objects that constitute eigenequations.

Physics students must also make sense of the coordinated constituent objects in eigenequations and reason with their different representations in varied contexts. This may involve recognizing the eigenequations, $\hat{S}_z|+\rangle = \frac{\hbar}{2}|+\rangle$, $\hat{H}|E_n\rangle = E_n|E_n\rangle$, and $\hat{x}|x_i\rangle = x_i|x_i\rangle$, as having a shared overarching structure of (operator) $\cdot$ (eigenstate) $=$ (eigenvalue) $\cdot$ (eigenstate), i.e., the symbol template, $\widehat{\square}|\rangle = c|\rangle$ [19,20]. Researchers have recently focused on physics students' cross-contextual reasoning in eigentheory. For instance, in an interview setting, Wawro *et al.* [21] asked students to explain how they think about the eigenequations $Ax = \lambda x$ and $S_x|+\rangle_x = \frac{\hbar}{2}|+\rangle_x$ or $S_z|+\rangle = \frac{\hbar}{2}|+\rangle$ as well as compare them. They found that for the general equation, most students held either a relational view of the equal sign, meaning the resulting object Ax on one side of the equal sign is the same as the resulting object λx on the other side [22], or a functional view of the equal sign, meaning the matrix A acted on or transformed an input x and produced λx. For the spin eigenequation, student interpretations fell into three main categories: that the equation conveyed the act of measurement, the notion of potential measurement, or a correspondence that conveys information about the value of an observable for a given eigenstate. Furthermore, some students conveyed a sense of conceptual incompatibility between the meanings they held for eigenequations in different contexts, such as how an interpretation of an eigenvector is stretched by λ not being sensible in spin where eigenstates are of unit length. In a study investigating students' sensemaking of eigenequations as they transition from spin to position, Pina *et al.* [19] asked students to construct a position eigenvalue equation for the position operator and explain. They found three symbolic forms [23] that typified students' understanding: reproductive transformation, operating as measuring, and potential measurement outcome; these are consistent with Wawro *et al.*'s [21] findings. In particular, the "measurement" meaning from their study [21] and "operating as measuring" symbolic form from Pina *et al.*'s [19] study are consistent with explanations given by Gire and Manogue [24], which aim to illuminate why such an interpretation may seem sensible to students: "In mathematics, operators transform vectors; in physics, measurements change the state of a quantum system; but these are two different uses of the idea 'to change'" (p. 195). Also, in a longitudinal study with written, interview, and classroom data, Karakok [25] investigated students' transfer of learning eigentheory from prerequisite experiences to quantum mechanics. Results of student meanings included findings consistent with the aforementioned relational view [an eigenvalue is "a scalar that accomplishes the same thing when multiplied to a specific eigenvector as is accomplished by the given matrix operation" (p. 7)] and functional view [an eigenvector is "a vector that has an unchanged direction (except in the opposite direction) when operated on by a transformation matrix" (p. 7)]. Overall, reasoning about eigentheory in quantum mechanics can be complex as it involves coordinating objects within eigenequations and making sense of them across varied contexts.

Another complexity about reasoning with eignetheory is that students must mathematize and interpret the varied eigenequations in terms of their physical referents, as shown in Her and Loverude [26], who analyzed written data of students' work on three midterm or final exam tasks about eigentheory concepts. They used the physical-mathematical model by Uhden *et al.* [27] to categorize their tasks according to the skills of mathematization (translating into mathematical terms, mathematical sensemaking) and interpretation ["the ability to read equations and properly identify the physical meaning and importance of symbols involved in a physical system" (p. 212)]. Two of the tasks asked students to write a system of equations (from either a spring system or an LC circuit) as a matrix equation and describe how it fits the profile of an eigenequation, which the authors coded as entailing a high degree of mathematization. The task aspects that involved explaining the physical meaning of the eigenvalue were coded as interpretation. The authors found that both creating the matrix equations and correctly relating the eigenvalue to frequency was nontrivial for some students, with improvement in interpreting over time. The authors suggest that students would benefit from instructors emphasizing these structural skills more. Each of these studies is consistent with the finding that "parsing the conceptual meaning of mathematical expressions and equations can play a key role in mathematical sensemaking" [20] (p. 11).

There is little research on how instructors can support students in cross-contextual reasoning about eigenequations. Karakok [25] demonstrated a physics professor's explicit focus on understanding the various objects within the eigenequation. For example, she wrote the equation $A\vec{v} = \lambda\vec{v}$ on the board and asked, "What kind of a beast is this?" while pointing to each symbol in the equation (p. 9), and she repeated the question frequently during the course. According to Karakok, the professor "explicitly provided her reason for asking this question by stating that before solving an equation, identifying elements in the equation would help to get an insight into 'What's happening on both sides of the equation?'" (p. 10). The study found that the professor's explicit focus on symbolic representations seemed to support students' sensemaking over time. For example, one student, when working to solve the differential equation $i\frac{d}{d\varphi}f(\varphi) - af(\varphi) = 0$ in an end-of-term interview, pointed at $f(\varphi)$ and said, "What kind of a beast is this?" The student also reasoned productively about $f(\varphi)$ as an eigenfunction once asked if the problem could be changed into an eigenvalue problem.

B. Unified understandings of concepts

Having a unified understanding of a concept involves recognizing seemingly disparate notions from different contexts as instances of the same concept with a shared overarching structure [1–3]. Research in undergraduate mathematics education has recently focused on characterizing students' unified understandings of mathematical concepts across contexts in linear algebra and abstract algebra, such as functions, linear transformations, homomorphisms, inverses, identity elements, and factorization [1–6,28,29]. These studies documented how students reason about the structural relationships, similarities, or shared properties among instantiations of a concept in specific contexts (e.g., additive inverses, multiplicative inverses, and inverse functions) and thereby recognize those as instantiations of the same concept (e.g., inverses) with the same overarching structure that holds regardless of context (e.g., inverse can be conceptualized as a way of undoing something, a manipulated element, or something coordinated with a set, binary operation, and identity element). Such understandings can be developed as students "unify their existing understandings of seemingly unrelated mathematical concepts by identifying structural similarities among the concepts, thereby increasing the extent to which the relations among the concepts in their minds become more coherent" [2] (p. 4). It is desirable for quantum mechanics students to have connected, coherent understandings of mathematical concepts and the physical phenomena with which they correspond [30,31]. We posit that quantum mechanics students should have unified understandings of eigenequations, so they can recognize the overarching structure shared by equations, such as $\hat{S}_z|+\rangle = \hbar/2|+\rangle$, $\hat{H}|E_n\rangle = E_n|E_n\rangle$, and $\hat{x}|x_i\rangle = x_i|x_i\rangle$, from varied quantum mechanical contexts. For instance, students can recognize that they all have a shared overarching structure of (operator) $\cdot$ (eigenstate) $=$ (eigenvalue) $\cdot$ (eigenstate), i.e., the symbol template $\widehat{\square}|\rangle = c|\rangle$ [19,20], where the operator represents an observable, the eigenvalue is the result of a measurement of the observable, and the eigenstates correspond to the eigenvalue. This study aims to contribute to what is known about how instructors can support students in developing such unified understandings of quantum mechanics.

C. Analogical reasoning

Developing unified understandings of a concept may involve engaging in *analogical reasoning*, "the ability to perceive and operate on the basis of corresponding structural similarity in objects whose surface features are not necessarily similar" [15] (pp. 37–38). Gentner's [7] structure-mapping theory characterized analogy as a mapping from a base or source domain to a target domain that preserves relations between objects in the source and target. A *source domain* is a collection of knowledge that can be applied in a new domain, the *target*, to develop new knowledge in the target domain. Gentner asserted the objects, attributes, and relations comprising these domains could be mapped between the domains by way of analogy. *Objects* may be whole entities, parts of entities, or a collection of entities in a domain. *Attributes* are characteristics or properties of objects. *Relations* are associations between objects, attributes, or other relations together in a domain. Relations of relations are referred to as *higher-order relations* [32]. When making an analogy, one maps these objects, attributes, and relations from the source domain by aligning them with corresponding objects, attributes, and relations in the target domain and drawing inferences about them [9,33,34]. Such an analogical relationship "is powerful because it comprises an entire set of associative relationships between features of the concepts being compared" [8] (p. 204). In essence, by reasoning with analogies, one can use their knowledge from one domain to develop new knowledge in another domain. Hicks [35] developed a framework for the mathematical activity that one can engage in during analogical reasoning. These activities include exporting aspects from the source domain, importing aspects to the target domain, recalling knowledge about the source domain, distinguishing differences between the source and target domains, and adapting the target to accommodate for those differences. Engaging students in analogical reasoning activities can provide them with an opportunity to develop more connected understandings of mathematical and physical concepts.

Leveraging analogies can be a productive instructional practice in physics and mathematics classrooms to help students conjecture, use higher-order thinking, develop new knowledge based on their prior knowledge, and reduce the abstraction of mathematical or physical concepts [9–12,14,16]. For example, in a study analyzing the analogies used in teaching introductory quantum theory concepts in a Modern Physics course, Didiş [36] identified 48 analogies used by the instructor, such as illustrating the quantization of angular momentum by analogizing the restriction of orbital angular momentum values with Matryoshka dolls. The study's many findings include that analogies were primarily used before conclusions about the target domain were made, as advanced organizers, and as postsynthesizers. From interview data, Didiş [36] also reported that students believed the analogies had a positive effect on their understanding. Podolefsky and Finkelstein [13] investigated the use of analogical scaffolding to foster students' understanding of electromagnetic waves, which is an abstract concept. They developed a tutorial that used analogies to progress students through learning increasingly abstract content, namely waves on a string, sound waves, and electromagnetic waves. They found that "Students taught about electromagnetic waves in a curriculum that builds on the model of analogical scaffolding posted substantially greater gains pre- to post-instruction than students taught using a more traditional (non-analogy-based) tutorial" (p. 1). A study that is particularly relevant for this paper is that of Marshman and Singh [37], which reported on a quantum interactive learning tutorial (QuILT) created to help students develop a procedural and conceptual understanding of Dirac notation. The QuILT uses analogies from familiar three-dimensional real-valued Cartesian coordinate systems in introductory mechanics to Dirac notation in quantum mechanics. In this sense, the analogies are using a familiar notation system for vectors and their operations as the domain and the unfamiliar notation system for kets and their operations as the target, such as mapping $\vec{F} = a\hat{i} + b\hat{j} + c\hat{k}$ to $|F\rangle = a|F\rangle + b|j\rangle + ck\rangle$ or mapping $\hat{i} \cdot \vec{F}$ to $\langle i|F\rangle$. The authors reported that various student difficulties (such as thinking an outer product is a scalar or column vector rather than a matrix) were reduced after engaging with the QuILT.

Instructors' analogies are also productive in mathematics courses to help students recognize parallels between concepts or procedures [9]. Based on a study about analogical reasoning in an upper-division proof-based abstract algebra course, Hicks [12] claimed, "Teachers can introduce students to purposeful reasoning by analogy by allowing students the opportunity to establish new concepts in mathematics through careful attention to similarities and differences between new and old contexts" (p. 14). Instructors may engage students in creating their own analogies, or they may draw analogies among domains in their lectures. To explicate the role of the instructor in making analogies for students, Glynn [8] created the teaching with analogies model comprised of these strategies to: "(i) introduce target concept, (ii) recall analog [source] concept, (iii) identify similar features of concepts, (iv) map similar features, (v) draw conclusions about concepts, and (vi) indicate where analogy breaks down" (p. 208). Although there is a potential for instructors' analogies to support students' development of understandings, some researchers caution that analogies rich in elaborate meanings may be too abstract and inaccessible for students to fully grasp [38,39] or that students may overgeneralize analogies and develop inaccurate understandings [16,40]. Thus, instructors must provide adequate support for students to make sense of analogies and recognize the extent to which the analogy is applicable [9].

D. Common ground

The construct of common ground [41,42] provides a way to characterize a community's development of ideas over time. According to Clark [41], "Common ground based on membership in cultural communities includes facts, beliefs, and assumptions about objects, norms of behavior, conventions, procedures, skills, and even ineffable experiences" (p. 112). Focused on the members of a classroom as a community, common ground includes the, at best, taken-as-shared "knowledge of word meanings and norms for communication, shared knowledge that enables successful communication and coordinated action" [17] (p. 640). It is constituted through communicative displays in which interlocutors tailor forms (e.g., spoken words, written words or symbols, and gestures) to serve certain functions. Shifts in form-function relations over time (what Saxe [43] referred to as ontogenesis) contribute to the evolving common ground of a class community, which expands the class community's taken-as-shared understandings. Furthermore, "common ground gets built up in strata" [41] (p. 120), in that "every new piece of common ground is built on an old piece" (*ibid*, p. 119).

III. METHODS

In this study, we coordinate the frameworks of unified understandings [1–3], analogical reasoning [7], and common ground [42] to characterize how a professor's pedagogical moves facilitate a class community in developing a taken-as-shared unified understanding of eigentheory concepts. We argue that making analogies among concepts is a way for instructors to facilitate expanding the class community's common ground and contribute to the development of a taken-as-shared unified understanding of those concepts. We consider an analogy's source domain to be a subset of a class community's common ground (see Fig. 1). An instructor can make analogies by mapping objects, attributes, and/or relations from that source domain to a target domain. In mapping these objects, attributes, and/or relations to the target domain, new objects, attributes, and/or relations are created and can become part of the common ground. The common ground thereby expands, and the target domain becomes part of a new source domain, from which further analogical mappings can be made. This expansion of the common ground via making analogies among concepts can contribute to the class community's development of taken-as-shared unified understandings of that concept.

Data for this study come from classroom video recordings and associated transcripts collected in a senior-level quantum mechanics course at a medium, public, research-active university in the northeastern United States. The 17 students enrolled in the class were junior and senior Physics majors and a couple of graduate students in a Master of Science in teaching program. The course structure included interactive lectures, when the professor, Dr. Holmes (pseudonym), took responsibility for the delivery of the content but employed various question types that facilitated students giving feedback on their understanding during class. Dr. Holmes also used clicker-question prompts throughout the class sessions, in which students thought and answered on their own, discussed with their neighbors, and then reanswered. There were also class sessions that mostly were spent with students working in small groups, either on tutorials or solving problems, and then discussing as a whole class. All students seemed engaged during small group discussion times, and most students seemed willing to ask or answer questions during interactive lectures and whole class discussions The video recordings in our dataset focused on capturing whole class discussions and lectures, as compared to exchanges during small group work or paired conversations. The class met for 50 min, three days a week. The course used a spins-first textbook [44], in which ket notation is introduced early on. We analyzed data from 22 class days of the first 9 weeks of the semester. The nature of the dataset facilitates analyses of contributions made by the professor or students that live at the classroom community level, rather than on analyses that focus on individual students' understanding. For results about the understanding of individual students from this dataset, see analyses from interview data [21,45–47]. Our analysis in this paper follows naturally from prior studies [48,49], in which we investigated how eigentheory concepts developed over time during whole class discussions and lectures for this classroom community. Those studies, which leveraged ontogenetic analyses of form-function pairs [43] made us aware of rich instances in the data during which the course professor, Dr. Holmes,

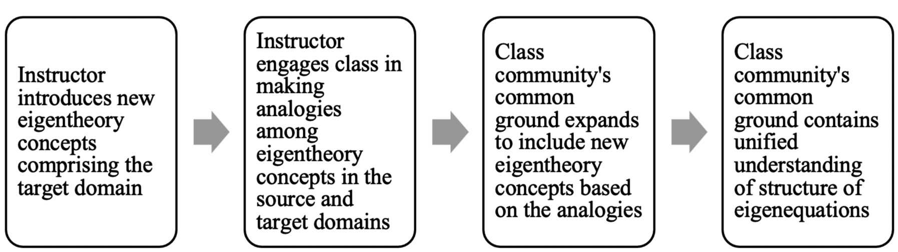


FIG. 1. Theorized connections among a class community's analogical reasoning, common ground, and unified understandings of eigentheory concepts.

TABLE I. Pedagogical moves for supporting analogical activity from the literature.

Pedagogical move	Description
(1) Introducing target concept [8]	Instructor introduces concepts from the target domain to which they will map concepts from the source domain.
(2) Recalling knowledge of the source domain [8,12]	Instructor describes features of concepts in the source domain to help students recall those features that they already know.
(3) Identifying and mapping similar features of concepts [8] via exporting aspects from the source domain and importing aspects to the target domain [12]	Instructor identifies similar objects, attributes, or relations among objects in the source domain and target domain and draws a correspondence from those in the source domain and those in the target.
(4) Distinguishing the differences between the source and target domains [12] to indicate where the analogy breaks down [8]	Instructor identifies how the analogized concepts in the source domain and target domain are different or distinguishable.

enacted pedagogical moves that seemed to support the class in developing and deepening their understanding of eigentheory concepts. We took such instances in the data as indicative of the class community working to expand its common ground for eigentheory concepts, such as by recognizing different forms (e.g., $\frac{\hbar}{2}$ or E_n) as instantiations of the same overarching concept while attending to any shared property, use, or aspect of structure (e.g., both relate to an associated operator as eigenvalues). We then analyzed the broader episodes that contained these instances in terms of Gentner's [7] structure-mapping theory for analogies. In particular, we analyzed the episodes to identify the source domain, target domain, and what objects, attributes, relations, or higher-order relations were being mapped between the two domains. Such analyses helped us answer our research question by indicating the class community's progress toward developing a taken-as-shared unified understanding of the overarching eigentheory concept under consideration, where this understanding is part of the class's common ground.

We then used deductive and inductive coding [50] to identify and characterize the pedagogical moves enacted by the course instructor in each of those class data segments. We conceptualized *pedagogical moves* as "discursive acts an instructor performs during instruction to accomplish a certain instructional goal" [51] (p. 5). We particularly focused our analysis on instances during class discussions or interactive lectures where the instructor gave vocal, written, or gestural

TABLE II. New pedagogical moves for supporting analogical activity.

Pedagogical move	Description
(1) Posing a task conducive to analogizing	Instructor assigns a task or problem that prompts students to engage in analogizing two or more concepts.
(2) Preparing students to look for similar objects, attributes, or relations in source and target	Instructor primes the students to identify features shared by two concepts, perhaps by suggesting there is a similarity that the students should find.
(3) Scaffolding students in identifying and mapping objects, attributes, and relations	Instructor provides support to students as they analogize two concepts, perhaps by asking guiding questions that direct students' attention to the shared features of the concepts.
(4) Juxtaposing the placement of symbols representing objects in the source and target	Instructor writes symbols that represent the concepts being analogized in such a way that the symbols are aligned or are near each other.
(5) Using deictic gestures	Instructor uses their body to make a gesture [53] that points to or emphasizes corresponding aspects in the source and target domains (i.e., objects, attributes, or relations).
(6) Using deictic inscriptions	Instructor writes an inscription (e.g., arrow, label) that points to or emphasizes corresponding aspects in the source and target domains (i.e., objects, attributes, or relations).
(7) Soliciting student participation to map objects, attributes, and relations from the source to the target	Instructor invites students to verbally contribute their ideas in the class discussion as they engage in analogizing aspects in the source and target domains (i.e., objects, attributes, or relations).
(8) Explicitly highlighting the sameness of aspects in the source and target domains	Instructor verbally claims that two aspects in the source and target domains (i.e., objects, attributes, or relations) are equivalent or the same.

communicative displays, in which he described at least two different concepts with similar or analogous features. We formed an *a priori* codebook containing pedagogical moves for supporting analogical activity that have been documented in the literature. These codes are listed and defined in Table I. We then used inductive coding, creating new codes that captured the essence of other pedagogical moves that were not accounted for in the initial codebook. These new pedagogical moves are listed and defined in Table II and are exemplified in Sec. IV. We wrote analytic memos [52] on how those moves seemed to support the class community's development of a unified understanding of eigentheory concepts and thereby contributed to the expansion of the class's common ground.

IV. RESULTS

In addition to the pedagogical moves in the literature (Table I) that support analogical activity, we documented eight other pedagogical moves that Dr. Holmes used. These eight pedagogical moves include: (i) posing a task conducive to analogizing, (ii) preparing students to look for similar objects, attributes, or relations in source and target, (iii) scaffolding students in identifying and mapping objects, attributes, and relations, (iv) juxtaposing the placement of symbols representing objects in the source and target, (v) using deictic gestures, (vi) using deictic inscriptions, (vii) soliciting student participation to map objects, attributes, and relations from the source to the target, and (viii) explicitly stating the sameness of aspects in the source and target domains. We define these pedagogical moves in Table II and exemplify Dr. Holmes's use of them in the following sections. Each section contains an illustrative episode from the class data in which Dr. Holmes engaged the class in analogizing eigentheory concepts, as well as our analysis of Dr. Holmes's pedagogical moves.[1] We chose these episodes because of their mathematical and physical content, which, when considered together, show a coherent thread of content development over time with respect to eigenequations in various contexts. Furthermore, our analysis highlights a frequent and consistent use of the various pedagogical moves for supporting analogical activity (see Tables I and II). We conclude each section by discussing how his pedagogical moves had the potential to support the class in developing a shared unified understanding of the eigentheory concepts addressed in the episode.

A. Analogizing the canonical eigenequation, $Av = \lambda v$, with the spin eigenequations, $\hat{S}_z|\pm\rangle = \pm\frac{\hbar}{2}|\pm\rangle$

On day 5, Dr. Holmes enacted several pedagogical moves that enabled him to guide the class in analogizing the canonical eigenequation $Av = \lambda v$ and the spin

[1]We italicize the instructor's pedagogical moves throughout the Results to direct the reader's attention to them.

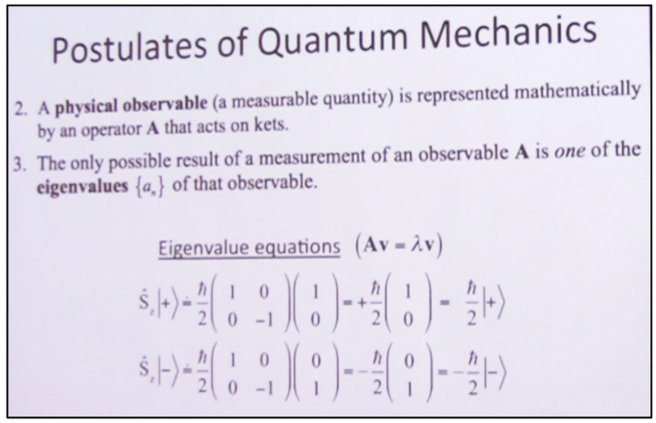


FIG. 2. Slide projected during class on day 5.

eigenequations $\hat{S}_z|\pm\rangle = \pm\frac{\hbar}{2}|\pm\rangle$. In this class episode, we illustrate Dr. Holmes's pedagogical moves for engaging the class in analogical activity. We also explain how the analogical activity seemed to contribute to the development of a shared, unified understanding of eigenequations in the two different contexts of the canonical, generalized eigenequation and the spin-½ eigenequations in terms of the z basis.

Dr. Holmes first introduced the eigenequations related to spin along the z axis by analogizing them with the canonical eigenequation from linear algebra, $Av = \lambda v$. He displayed the slide in Fig. 2 and explained:

Dr. Holmes: When you have a matrix [points to A], you have eigenvectors [points to v on the right side], right. Do people remember what's special about an eigenvector, for a matrix, for a given matrix? What makes it eigen-esque-ish?

Student: When it's operated on by A, it only scales.

Dr. Holmes: That's right… eigenvectors don't rotate at all, they only scale, right? If you operate A on this particular eigenvector [points to Av], all you get is some scaling factor times the original vector [points to λv], alright. That's the magic. Those are magic vectors.

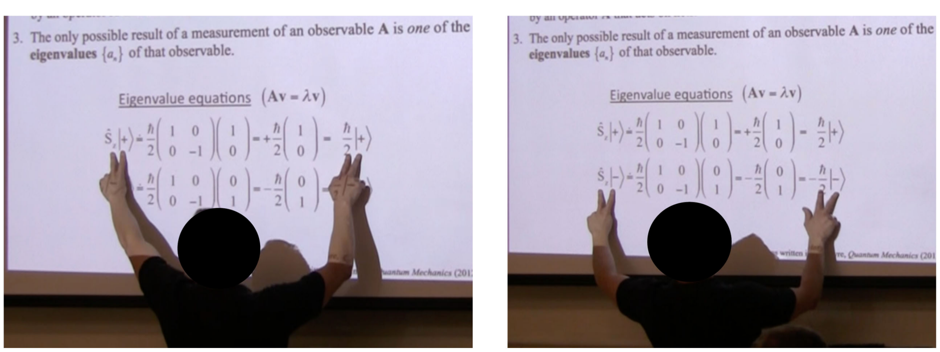
First, Dr. Holmes *recalled aspects of the source domain*, which included the canonical eigenequation from linear algebra, $Av = \lambda v$ and its components: the operator, eigenvectors, and eigenvalue. He *solicited student participation* to *recall* these objects, the relations among them (e.g., the matrix-vector relationships), and their attributes (e.g., Student: "It only scales"). The class community had not discussed the eigenvalues of matrices and operators together in class before day 5 when Dr. Holmes introduced the third postulate (see Fig. 2). The lack of further discussion about "it only scales" implies that eigenvectors of operators are the vectors that are scaled by the operator was a familiar idea to the students, most likely from their linear algebra coursework. Dr. Holmes' comment, "Those are magic vectors," furthers the notion that eigenvectors have special properties.

After this discussion, Dr. Holmes called out the ubiquity of the canonical eigenequation and claimed, "We can write these for our operators." In doing so, he *explicitly highlighted the sameness* of the eigenequations' existence for operators in both the source and target domain, as well as *introduced the target domain,* which included the spin eigenequations and their components. These pedagogical moves of *recalling, soliciting, highlighting,* and *introducing* work in tandem to prepare the class community to expand its common ground by setting the stage to introduce eigenequations for their first quantum mechanical context of spin. Dr. Holmes explained:

Dr. Holmes: "I mean this equation [points to $Av = \lambda v$] is sort of canonical, right? People write these all the time for linear algebra, for matrix-vector relationships. And we can write these for our operators. I have the $\hat{S}_z$ operator… we're gonna have the value, we're gonna have the matrix, and we're gonna have the operators…You operate the $\hat{S}_z$ [points to $\hat{S}_z$ then $|+\rangle$] on the up spin state. Right, we have this matrix [points to matrix] … and you operate it on the $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ state $\left[\text{points to} \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right]$ we know already was just one zero, right. And if you do that you get h bar over 2 times the vector you started with. Right, so that's, this is the eigenvalue part, right [points to two ends of the equation, see Fig. 3(a)]. And the same thing for this, that you get a negative h bar over two on the bottom [points to the two ends of the lower equation, see Fig. 3(b)] for the down state.

By saying, "We're gonna have the value, we're gonna have the matrix, and we're gonna have the operators," he *identified similar objects* and *mapped (exported and imported) them from the source domain to the target.* Dr. Holmes then *mapped relations and attributes* from the source domain to the target domain. For instance, he *mapped the relation* between the operator and the eigenvector by claiming, "You operate the $\hat{S}_z$ on the up spin state," which was parallel to what he previously said about the canonical eigenequation: "You operate A on this particular eigenvector [points to Av]." He also *mapped the relation* between the eigenvalue and eigenvector by saying, "You get h bar over 2 times the vector you started with" which was similar language to what he claimed about the source eigenequation: "all you get is some scaling factor times the original vector [points to λv]."

Beyond the pedagogical moves related to identifying and mapping similar features from the source domain to the target domain, Dr. Holmes used the moves of gesturing and juxtaposing symbols while analogizing. On the projected slide, he *juxtaposed the placement of the symbols representing objects in the source and target*. He vertically aligned the two spin eigenequations underneath the canonical eigenequation (see Fig. 2), which had the potential to support the class in recognizing the shared structure of the different eigenequations as having a product of a matrix (representing an operator) and an eigenvector on the left side of the equation and a product of an eigenvalue and an eigenvector on the right side of the equation. Dr. Holmes accompanied this with the move of *using deictic gestures.* He used two fingers on each hand to simultaneously point to each of the two objects ($\hat{S}_z$ and $|+\rangle$) on the left side of the equation and the two objects ($\frac{\hbar}{2}$ and $|+\rangle$) on the right side of the equation [see Fig. 3(a)]. He then moved the position of his two fingers on each hand to point to the corresponding objects in the other spin eigenequation, $\hat{S}_z|-\rangle = -\frac{\hbar}{2}|-\rangle$ [see Fig. 3(b)]. These deictic gestures directed the class's attention to the corresponding objects in the two spin eigenequations. They also helped *map the relations* between the objects in one eigenequation to the other, so the class could see that those relations were preserved in the latter eigenequation. We posit these moves of using



(a) (b)

FIG. 3. Dr. Holmes's deictic gestures used while juxtaposing the eigenequations containing the eigenstates (a) $|+\rangle$ and (b) $|-\rangle$.

juxtaposition and deictic gestures had the potential to help expand the class community's common ground for eigentheory concepts in quantum mechanics, as well as contribute to the class's development of a unified understanding of eigenequations. In this interaction, the students had an opportunity to recognize different eigenequations as instantiations of the same concept with the shared overarching structure regarding the objects within the eigenequations and the relations among those objects.

B. Analogizing the spin and energy eigenequations $\hat{S}_z|\pm\rangle = \pm\frac{\hbar}{2}|\pm\rangle$ and $\hat{H}|E_n\rangle = E_n|E_n\rangle$ with the position eigenequation $\hat{x}|x_i\rangle = x_i|x_i\rangle$

On day 19, Dr. Holmes used various pedagogical moves to guide the class in making analogies between the spin and energy eigenequations in the source domain, $\hat{S}_z|\pm\rangle = \pm\frac{\hbar}{2}|\pm\rangle$ and $\hat{H}|E_n\rangle = E_n|E_n\rangle$, and the position eigenequation in the target domain, $\hat{x}|x_i\rangle = x_i|x_i\rangle$. In this illustrative class episode, we exemplify Dr. Holmes's pedagogical moves and explain how the analogical activity seemed to contribute to the development of a shared, unified understanding of eigentheory concepts across the contexts of spin, energy, and position.

This class episode involves Dr. Holmes leading a discussion about the students' ideas on this posed task, "Write down an eigenvalue equation for an operator $\hat{x}$ that represents (1D) position." While *posing a task conducive to analogizing*, Dr. Holmes *introduced the target domain,* eigentheory concepts related to measuring position. As he launched the task, Dr. Holmes explained:

Dr. Holmes: We want to be able to measure all these other things. Right, we've measured spin, we've measured all these other things, right? We have such a huge repertoire. We have spins in three directions and energies, right? But let's say we want to measure the position of a particle… So the question is, if it's a thing we're measuring, then it has all the properties of the other things we're measuring so far. So, what do we do about that?

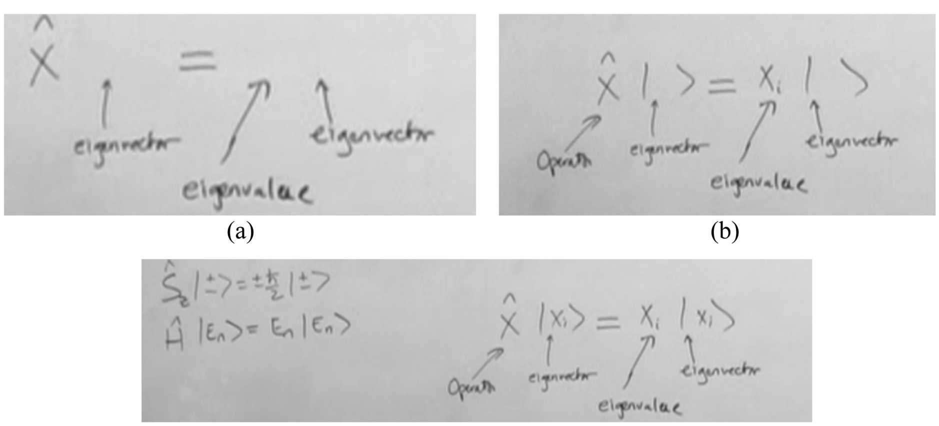
In this explanation, Dr. Holmes *recalled the source domain*, eigentheory concepts related to measuring spin and energy, by alluding to the other observables they have measured thus far. In claiming, "If it's a thing we're measuring, it has all the properties of the other things we're measuring so far," he *prepared students to look for similar attributes* of objects in the source and target domain.

After the students worked on the task in groups, the instructor wrote these symbolic eigenvalue equations on the board: $\hat{S}_z|\pm\rangle = \pm\frac{\hbar}{2}|\pm\rangle$ and $\hat{H}|E_n\rangle = E_n|E_n\rangle$ [see Fig. 4(b)]. The class discussion proceeded:

Dr. Holmes: I want an equation like this right? [points to $\hat{S}_z|\pm\rangle = \pm\frac{\hbar}{2}|\pm\rangle$ and $\hat{H}|E_n\rangle = E_n|E_n\rangle$]… We're talking about positions, by saying operator $\hat{x}$ [writes $\hat{x}$]…If I want to write an eigenvalue equation where that's my operator, what is it going to tell me? What do these eigenvalue equations tell me, in general? [points to $\hat{H}|E_n\rangle = E_n|E_n\rangle$].

Student: If you can operate with that, what your vector gets scaled by.

Here, we analyze Dr. Holmes's interactions as pedagogical moves to begin to map the eigenequation objects in the source ($\hat{S}_z|\pm\rangle = \pm\frac{\hbar}{2}|\pm\rangle$ and $\hat{H}|E_n\rangle = E_n|E_n\rangle$) to the object of the eigenequation containing the operator $\hat{x}$ in the target. He *identified and mapped similar features of the concepts*, mapping the attribute of "what these eigenvalue equations tell me" from the source to the target. This conveyed to the class that the new eigenequation in the target should "tell me" what the other "eigenvalue equations tell me." In doing so, he *exported aspects* (objects and attributes) from the source and *imported aspects* (objects



(a) (b) (c)

FIG. 4. Dr. Holmes's symbol forms written on the board in a progression (a), (b), (c) during the day 19 episode.

and attributes) to the target. This *provided scaffolding for identifying and mapping objects and attributes* of the eigenequations in the source that the students could map to the eigenequation in the target. The discussion continued:

Dr. Holmes: If I operate on an eigenvector [points to $|E_n\rangle$ next to $\hat{H}$] with that operator, what is this then? [points to E_n in $\hat{H}|E_n\rangle = E_n|E_n\rangle$].
Student: A scalar
Dr. Holmes: So I need, what do I need here? [draws an arrow to an empty space to the right of the operator symbol $\hat{x}$ and writes the word, eigenvector]. I need an eigenvector, right, and I need that same eigenvector here, right [writes an equal sign, leaves a space after the equal sign, draws an arrow to a subsequent empty space, and writes the word, eigenvector]. What do I need there? [points to space after the equal sign] [see Fig. 4(a)].
Several Students: A scalar.
Several Students: The position.
Dr. Holmes: The position, right? But yeah, that's the eigenvalue, right? [draws an arrow to the space after the equal sign and writes the word, eigenvalue]. So the, yeah, the general answer is the eigenvalue, but in this case, if this is my operator [draws an arrow to $\hat{x}$ and writes the word, operator], what am I going to put here then? The eigenvalue is a position, so I'm going to call it x_i [writes x_i to the right of the equal sign] because it's a spot, right? It's a point.

In this episode, Dr. Holmes used several pedagogical moves to map objects, attributes, and relations from the source to the target domain. He first *mapped the operator and eigenvector $\hat{H}|E_n\rangle$ objects* from the left side of the source eigenequation, $\hat{H}|E_n\rangle = E_n|E_n\rangle$ to the objects $\hat{x}$_ in the target domain, where the space was labeled with the term, eigenvector. In doing so, he *juxtaposed the placement of the operator and eigenvector symbols in the source and target* eigenequations. This juxtaposition was supported by Dr. Holmes's *use of deictic gestures and inscriptions* (i.e., drawn arrows) to highlight the correspondence between the operator, eigenvector, and eigenvalue objects in the source and target eigenequations. Dr. Holmes thereby mapped the operator and eigenvector objects and the relations between them, namely that the eigenvector is inherently tied to the operator and occupies a specific place in the eigenequation to the right of the operator. Next, he again *used deictic gestures and inscriptions* by saying "I need an eigenvector, right, and I need that same eigenvector here" and drawing arrows, which mapped the relation between the two eigenvectors in the source eigenequations to the relation between the two eigenvectors in the target eigenequation.

Dr. Holmes next asked the students, "What do I need to go there?" and pointed to its empty space in the eigenequation; this directed the class's attention to the eigenvalue object in the source and to its position in the target eigenequation. The students replied "scalar" and "position." Dr. Holmes affirmed their responses and additionally called it the eigenvalue. Here, he *solicited student participation to map objects* and *juxtaposed the placement of eigenvalue symbols in the source and target* eigenequations. Furthermore, he *identified a similar attribute* and *mapped (via importing and exporting) the attribute* of eigenvalues being "what your vector gets scaled by". Then by saying "If this [$\hat{x}$] is my operator…the eigenvalue is a position," he *exported the relation between the operator and eigenvalue from the source domain and imported it to the target domain*, thereby *mapping the relations* that an eigenvalue is a structure inherently tied to a particular operator and that an eigenvalue is associated with observed physical phenomena. Finally, Dr. Holmes claimed the eigenvalue is a position (in the target domain), which served to *distinguish the eigenvalue object in the target from the eigenvalue in the source domain*. These pedagogical moves helped the class map the eigenvalue objects, attributes, and relations between the source and target.

Dr. Holmes went on to say, "Now we're going to be in space space, right, as opposed to energy space or spin space, so you really have to keep these things straight in your head if you can." This further *distinguished the attributes of objects in the source domain and target domain.* He encouraged the students to keep the spin and energy spaces (the context attributes of the eigentheory objects in the source domain) separate in their mind from "space space" (the context attribute of the eigentheory objects in the target domain). This pedagogical move seemed to help the class recognize that even though the spin, energy, and position eigenequations have the same structure, they are distinct instantiations of the same overarching concept.

Dr. Holmes continued the discussion by next focusing on how to symbolize the eigenvectors in the position eigenequation they were creating:

Dr. Holmes: What should I use as how to represent an eigenvector of position? Like given some of the conventions we have for writing stuff [points to $\hat{S}_z|\pm\rangle = \pm\frac{\hbar}{2}|\pm\rangle$ and $\hat{H}|E_n\rangle = E_n|E_n\rangle$], what would be [points to space for the left eigenvector in the target eigenequation]?
Student: x_i
Student: x ket.
Dr. Holmes: Yeah, well first, it better look like that, right [writes kets in equation $\hat{x}|\rangle = x_i|\rangle$, see Fig. 4(b)], and what do I want to put in here?
Student: x_i
Dr. Holmes: x_i right? [writes $\hat{x}|x_i\rangle = x_i|x_i\rangle$, see Fig. 4(c)]. That's fine. Because what the convention was,

> this is the eigenvalue for the eigenvector, right? x_i symbol in the ket on the left side of the equation] So, this expression is not like, in itself, out of the realm of your ability to write, right? Like in the sense that it's $\hat{H}|E_n\rangle = E_n|E_n\rangle$].

In asking students how to represent an eigenvector of position given some conventions they had used for writing spin and energy eigenequations, Dr. Holmes *used deictic gestures* to direct the class's attention to the source eigenequations, *recalled the source analog attributes* of how eigenvectors are conventionally written, and *prepared students to identify similar attributes of objects (eigenvectors) in the source and target*. He also *solicited student participation to map the attributes of eigenvectors* from the source to the target. Considering the students' contributed responses, Dr. Holmes *exported the attributes from the source and imported those attributes to the target* that eigenvectors are symbolized as kets and are labeled by their corresponding eigenvalues. This also served to *map the relation between eigenvectors and eigenvalues from the source to the target domain*.

Overall in this day 19 excerpt, Dr. Holmes used a variety of pedagogical moves to engage the class in analogical reasoning as they created an eigenequation in their new context of position. We posit that this analogizing activity contributed to the class's development of a shared, unified understanding of eigenequations, as the class came to recognize eigenequations from three different quantum mechanical systems as instantiations of the same overarching concept with the same structure. Dr. Holmes used gestures and written arrows to juxtapose the eigenequations $\hat{S}_z|\pm\rangle = \pm\frac{\hbar}{2}|\pm\rangle$ and $\hat{H}|E_n\rangle = E_n|E_n\rangle$ and their composing objects from the source domain with the new eigenequation $\hat{x}|x_i\rangle = x_i|x_i\rangle$ in the target domain. The instructor and students together mapped the objects that compose the eigenequations (i.e., the operator, eigenvectors, and eigenvalue), along with their attributes and relations among the objects from the source domain contexts of spin and energy to the target domain context of position.

In this episode, Dr. Holmes guided the class to see that the position eigenequation had the same arrangement of symbols, operator · eigenvector = eigenvalue · eigenvector, that the spin and energy eigenequations had. He also guided the class to recognize that the relations, respectively, between the operator and eigenvector, the left-hand side eigenvector and the right-hand side eigenvector, and the eigenvector and eigenvalue in the position eigenequation were the same as those in the spin and energy eigenequations. Furthermore, Dr. Holmes scaffolded the class in identifying and mapping several shared attributes of the objects within the source and target domain eigenequations. His pedagogical moves fostered the class's awareness of the structural relationships and shared properties among the three distinct eigenequations, which thereby expanded the class's common ground for eigentheory concepts and contributed to the development of a unified understanding of eigenequations.

C. Analogizing spin and energy eigenstates, operators, and probability amplitudes with the position eigenstates, operators, and probability amplitudes

On day 21, Dr. Holmes juxtaposed the inner product of three different eigenstate kets with a general state $|\psi\rangle$ to analogize their attributes of being used in a computation of inner products for finding probability amplitude. As he introduced wave function, $\psi(x_i) = \langle x_i|\psi\rangle$, as an inner product of a position eigenstate and a state vector, he analogized it with the inner products the students had seen before, $\langle E_n|\psi\rangle$ and $\langle +|\psi\rangle$. He wrote the symbols on the board in Fig. 5 and explained:

Dr. Holmes: We talked about…how to sort of think about the position basis and what do we think about the wave function…which we wrote as that [writes $\psi(x_i) = \langle x_i|\psi\rangle$] …if this is a function of x…this is the eigenstate [points to $\langle x_i|$ in $\langle x_i|\psi\rangle$] of the operator that tells me about the position measurements, and I'm taking the inner product of that with this. So, this is the same thing…to give you the analogy to…$\langle E_n|\psi\rangle$, which we said was c_n, right? So that was the probability amplitude of measuring a particular energy. Or right, when you had $\langle +|\psi\rangle$, which we called a at that point in our generic expression, right, was the probability amplitude of measuring my spin up in the z direction for a generic spin state, right? So, these are all the same idea, so this [points to $\psi(x_i)$] is…the equivalent of c_n. It's the analogous quantity for the probability amplitude.

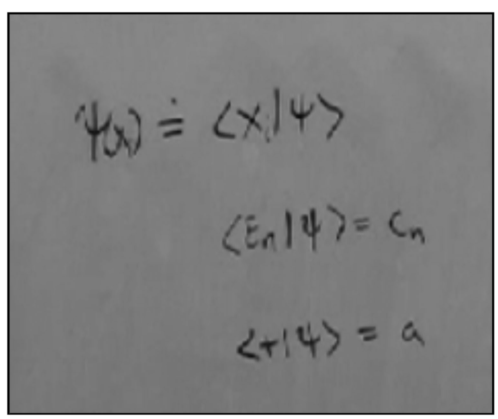


FIG. 5. Dr. Holmes's juxtaposition of probability amplitudes resulting from inner products with eigenstates.

Dr. Holmes drew an analogy among the eigenstates $|x_i\rangle$, $|E_n\rangle$, and $|+\rangle$ to show that they are used in the inner product computations $\langle x_i|\psi\rangle$, $\langle E_n|\psi\rangle$, and $\langle +|\psi\rangle$ to yield the scalar quantities $\psi(x_i)$, c_n, and a, respectively. This involved several pedagogical moves that are typical of analogical activity like identifying similar features and mapping objects, attributes, and relations from a source domain to a target domain via exporting and importing those features. For instance, he *mapped the objects* of eigenstates (i.e., $|E_n\rangle$ and $|+\rangle$), inner products (i.e., $\langle E_n|\psi\rangle$ and $\langle +|\psi\rangle$), and probability amplitudes from the source domain contexts of energy and spin to the target domain context of position. He also *mapped the attribute* of probability amplitudes being quantities from the source to the target domain by stating "It's the analogous quantity for the probability amplitude." Dr. Holmes additionally *mapped the inner product attribute*, i.e., that an inner product of an eigenstate with $|\psi\rangle$ yields a scalar quantity, from the source domain to the target domain.

Dr. Holmes's mapping activity seemed to be supported by his pedagogical moves of *juxtaposing symbols that represent objects in the source and target domains* and *using deictic gestures.* Dr. Holmes wrote the symbols for the three inner products so that they were vertically aligned (see Fig. 5). He then used his fingers to point to the eigenstates within the inner products and to point to the probability amplitudes $\psi(x_i)$ and c_n as he said they were analogous quantities. These moves likely supported the students in recognizing the correspondence or mapping between the eigenstate objects in each line, the inner product objects in each line, and the scalar quantity or probability amplitude in each line. This juxtaposition also aided in mapping the attributes that an eigenstate can be used in an inner product to yield a probability amplitude. Juxtaposing the symbols representing the objects in the source and target domains helped convey the similar structure of the eigenstates, inner products, and probability amplitudes that are consistent across the quantum mechanical contexts of spin, energy, and position.

Finally, Dr. Holmes enacted the pedagogical moves of explicitly highlighting the sameness of aspects in the source and target domains and distinguishing aspects of the source and target domains. As he concluded his explanation, Dr. Holmes explicitly called attention to the three inner products' similarity on the board by claiming "These are all the same idea, so this is…the equivalent of c_n." This pedagogical move of highlighting sameness takes the aforementioned move of identifying similarities in the source and target domains a step further by claiming equivalence or sameness of aspects of the source and target domains. Throughout this analogizing activity, Dr. Holmes also made remarks to distinguish the three inner products in the source and target domains. For example, he differentiated the probability amplitudes by context, stating c_n was "the probability amplitude of measuring a particular energy" and a was "the probability amplitude of measuring my spin up in the z direction for a generic spin state." Differentiating these concepts while also claiming their equivalence or sameness had the potential to support the class's development of a shared unified understanding of eigenstates and the related concepts of inner products and probability amplitudes across varied contexts. Furthermore, the class community's common ground for eigentheory concepts was expanded to include their applicability to probability amplitudes in the position context.

D. Analogizing energy and generic eigenstates and operators with the position eigenstates and operator

In this last illustrative episode from day 21, Dr. Holmes mapped the energy eigenstates $|E_n\rangle$ and generic eigenstates $|a_n\rangle$, as well as their operators, from the source domain to the position eigenstate and operator in the target domain. Here, we present our analysis of Dr. Holmes's pedagogical moves used in this analogy.

Dr. Holmes wrote these eigenstate symbols and their corresponding operators on the whiteboard (see Fig. 6) while explaining:

> $|E_n\rangle$ by itself, this is a ket, right? It's a vector, and this is what it tells me about the eigenstate for whatever energy we're at. In the homework, we had $|a_1\rangle$ and $|a_2\rangle$, and they were eigenstates, eigenvectors of the operator $\hat{A}$, right? So, this goes to $\hat{H}$ [writes $|E_n\rangle \leftrightarrow \hat{H}$], and this goes to $\hat{A}$ [writes $|a_1\rangle \leftrightarrow \hat{A}$], and it gives you the eigenstates of those things, right? So, we're saying $|x_i\rangle$ tells me about the position operator [writes $|x_i\rangle \leftrightarrow \hat{x}$].

Here, we inferred that Dr. Holmes *mapped the eigenstate objects* $|a_n\rangle$ and $|E_n\rangle$ from the source domain to the $|x_i\rangle$ eigenstate object in the target domain. He also *mapped the operators* $\hat{H}$ and $\hat{A}$ from the source domain to the operator

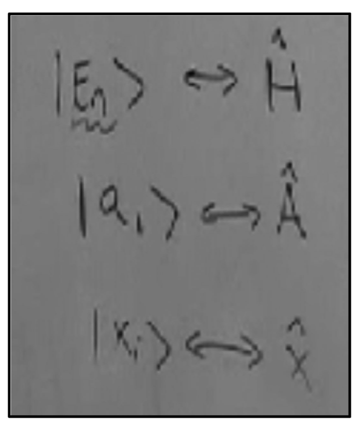


FIG. 6. Dr. Holmes's juxtaposition of eigenstates and operators across three contexts.

$\hat{x}$ in the target. By claiming $|E_n\rangle$ "goes to $\hat{H}$," $|a_1\rangle$ "goes to $\hat{A}$," and $|x_i\rangle$ corresponds to $\hat{x}$, Dr. Holmes *mapped the relations* among eigenstates and operators from the source domain to the target domain.

Dr. Holmes's mapping activity was again supported by his pedagogical moves of *juxtaposing the symbols that represent the objects in the source and target domain*s and *using deictic inscriptions*. Dr. Holmes wrote the three eigenstates and operators in vertical alignment on the board to juxtapose their similar structure and relationship to operators. While doing so, he used deictic inscriptions by drawing bidirectional arrows between each eigenstate and operator. This illustrated the relation between the eigenstate and the operator in each line. Juxtaposing those symbolized relations aided in mapping those relations from the source domain to the target. Overall, these pedagogical moves provided the opportunity to foster the class's shared unified understanding of eigenstates and their relations to operators. His pedagogical moves had the potential to help the class see the overarching structure shared by $|a_n\rangle$, $|E_n\rangle$, and $|x_i\rangle$, so they could recognize these different kets as instantiations of the same eigenstate concept, again solidifying or extending their common ground for eigentheory concepts.

V. DISCUSSION

Given the ubiquity of eigentheory in quantum mechanics, it is essential for students to develop a unified understanding of eigentheory concepts. Such a unified understanding involves an ability to recognize different concepts, like $|+\rangle$, $|a_n\rangle$, $|E_n\rangle$, and $|x_i\rangle$, as instantiations of the same overarching concept (e.g., eigenstate) with shared structural features. Having a unified understanding of eigenstates, eigenvalues, and eigenequations is particularly useful for quantum mechanics students as they can reason about those concepts across different contexts, such as quantum mechanical systems of spin, energy, and position. Quantum mechanics instructors can support students in recognizing the mathematization related to eigenequations for these systems, as well as guide students to understand those different eigenequations as instantiations of the same overarching concept.

We posited that instructors engaging students in analogical reasoning can foster their class's development of unified understandings of the structure underlying various eigentheory concepts in quantum mechanics. Thus, we investigated the specific pedagogical moves that a quantum mechanics instructor used to engage students in analogical reasoning. Through our analysis, we identified and exemplified eight of these pedagogical moves (see Table II) and described how they had the potential to foster the class's development of unified understandings by way of expanding the class's common ground. Our characterization of these pedagogical moves serves as the primary contribution of our study. The pedagogical moves that stand out as most salient overall in the illustrative episodes include juxtaposing the symbols that represent objects in the source and target domain, using deictic gestures, and using deictic inscriptions. These moves served to direct students' attention to the objects and relations that were being analogized in the source and target domains. Doing so fostered the class's awareness of the structural relationships and shared properties among the eigentheory concepts in three different contexts (i.e., spin, energy, and position), which thereby expanded the class's common ground for eigentheory concepts and contributed to the class's development of a unified understanding of eigentheory concepts.

Another contribution of this study lies in our connection of three theoretical constructs of analogical reasoning [7], unified understandings [1–3], and a class community's common ground [42]. Studies have focused on individual students' unified understandings of mathematical concepts [1–6,28,29] but not on a class community's unified understanding. To conceptualize the development of unified understanding at the classroom level, we drew on Saxe *et al.*'s [42] notion of common ground, which refers to a community's at best taken-as-shared understanding that expands during communicative displays, such as those in whole class discussions and lectures. Ideas shared during whole class discussion and lecture become part of the common ground; thus, we posit that a class community can develop a unified understanding of eigentheory concepts and thereby expand their common ground by analogizing different examples of that concept. The class can map objects representing that concept, its attributes, and relations among that concept and others from the source domain to the target domain. Then the target domain can become part of the common ground, and the class can come to recognize those different examples as instantiations of the same overarching eigentheory concept. When a class engages in analogical reasoning, the students can make connections between similar concepts, which is conducive to the development of a unified understanding of mathematical and physical concepts. It is beyond the scope of the data analyzed in this paper to analyze individual students' understanding of the ideas shared at the community level.

A. Limitations and directions for future research

We acknowledge some limitations to our methodology that could impact one's interpretations of our findings. First, we did not consider the instructor's intent in implementing the pedagogical moves we identified. We could infer that the instructor intended to use analogy by his use of certain language, such as "to give you an analogy" or something "is the analogous quantity." However, we cannot claim that the instructor consciously chose to enact the pedagogical moves related to analogizing or fostering students' development of unified understandings. We could only make inferences about the instructor's use of pedagogical moves based on his discursive acts that occurred during whole class discussions or lectures. Furthermore, we cannot explicitly link the instructor's pedagogical moves with the development of each student's unified understanding. Thus, we can only claim that his pedagogical moves related to analogizing were potentially conducive to guiding the class's shared unified understanding. It is outside the scope of this study to analyze individual students' unified understandings of eigentheory concepts. Although the analysis in some studies aligns with investigating students' unified understanding of eigenequations across a couple of contexts [19,21], we suggest this as a direction for future research. We posit that researchers can further characterize students' development of unified understandings of eigentheory concepts and can identify other instructional practices that can support students in developing such unified understandings. Given the productivity of engaging in analogical activities during class, we also suggest that researchers should design research-based instructional materials that are conducive to students engaging in analogical reasoning. Such instructional materials could support the development of students' unified understandings. Another direction for further research involves the investigation of the affordances or drawbacks of students analogizing concepts across quantum mechanical contexts.

B. Implications for instruction

Our findings have implications for instruction regarding how quantum mechanics instructors can support their students in analogizing and developing unified understandings of mathematical concepts. We encourage instructors to engage students in analogical reasoning during class activities and in various content areas. We particularly suggest that instructors can use pedagogical moves related to analogizing to help students connect eigentheory concepts from different contexts in undergraduate physics courses, such as the moment of inertia tensor in torque equations, the Schrödinger equation, the Lorentz transformation matrix in special relativity, linear analysis of fixed points in classical mechanics, resonant frequencies of a cavity, and oscillation frequencies in vibrations of a string or drum. The quantum mechanics instructor in this study employed several pedagogical moves like soliciting students' participation in analogizing and scaffolding their analogical activity by asking leading questions. We recommend that instructors implement these pedagogical moves that are conducive to students' engagement in active learning while analogizing. Other pedagogical moves that seemed productive in supporting the class's development of unified understandings of eigentheory concepts involved directing students' attention to structural similarities by way of juxtaposing analogous written symbols or using deictic gestures (e.g., pointing to corresponding objects) and deictic inscriptions (e.g., drawing arrows). We, therefore, encourage physics instructors to leverage these pedagogical moves to analogize concepts and guide students to develop unified understandings of those concepts.

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