# STOCHASTIC TURNING POINT PROBLEM

by

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## (ABSTRACT)

A one-dimensional refractive, randomly-layered medium is considered in an acoustic context. A time harmonic plane wave emitted by a source is incident upon it and generates totally reflected fields which consist of "signal" and "noise". The statistical properties, i.e., mean and correlation functions, of these fields are to be obtained. The variations of the medium structure are assumed to have two spatial scales; microscopic random fluctuations are superposed upon slowly varying macroscopic variations. With an intermediate scale of the wavelength, the interplay of total internal reflection (geometrical acoustics) and random multiple scattering (localization phenomena) is analyzed for the turning point problem. The problem, in particular, above the turning point is formulated in terms of a transition scale. Two limit theorems for stochastic differential equations with multiple spatial scales, called Theorem 1 and Theorem 2, are derived. They are applied to the stochastic initial value problems for reflection coefficients in the regions above and below the turning point, respectively. Theorem 1 is an extension of a limit theorem on O(1) scaled interval to infinite scale and provides uniformly-valid approximate statistics for random multiple scattering in the region above the turning point (transition as well as outer regions). Theorem 2 deals with stochastic problems with a rapidly varying deterministic component and approximates the reflection process in the region below the turning point which is characterized by the random noise. Finally, the evolution of the reflection coefficient statistics in the whole region is described by combining the two results as a product of a transformation at the turning point and two evolution operators corresponding to the two regions.

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*TO MY PARENTS*
*FOR THEIR INFINITE PATIENCE*

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# Chapter 1

# INTRODUCTION

Waves are expected to undergo some scales of fluctuations when they propagate in a randomly inhomogeneous medium. If the random inhomogeneities are strong, multiple scattering and the interference of waves produce phenomena in which the waves are localized around the region where they are generated. They lose coherence and eventually they are totally converted to random fluctuations; the waves do not propagate at all. Localization phenomena can be found in one or two dimensional settings of layered media even when the random inhomogeneities are weak. Anderson [1] discovered these localization phenomena in connection with electronic waves in semiconductors in 1958. The wave nature of an electron produces allowed energy bands and forbidden gaps for its motion in the solid. Electrical conductivity is hindered by disorder in the crystal and the electronic waves are localized in space for some energies. Recently, an analogous localization of light has also started getting attention from physicists. Light in a certain class [2] of strongly scattering dielectric microstructure exhibits localized modes. It took time, however, before localization of classical waves became fully appreciated. One reason [3] is that localization phenomena are observed usually over the long range of distances unless the random scattering is very strong and the dissipation is rather weak. In the last decade, the localization of various types (cf. [3],[4] for references) of waves in inhomogeneous media has been studied with a great deal of interest.

One typical type of randomly inhomogeneous medium, i.e., a one-dimensional refractive, randomly-layered medium, is considered here in an acoustic context. One model that fits very well our framework is reflection of acoustic waves from ocean sedimentary layers. Over the ages the various physical deposition processes in the ocean have created a heterogeneous material; in the depth direction it possesses a rapid fine-scale layering structure superposed upon a slow-scale increase induced by compactification while variability in the transverse direction is much more gradual to such an extent that one-dimensional modeling is a reasonable approximation. The sediment layer rests upon a semi-infinite rock basement. The sediment acoustic parameters such as density and sound speed become rapidly-varying functions of depth variable, undergoing a slow-scale increase in average. This increase refracts the penetrating acoustic energy back upward towards the water-sediment interface. Within the layer, dissipation can be neglected.

When we have a point source above a medium, one can think of it as launching rays in all directions. We consider an oblique wave incident upon the medium. The problem that we shall consider is schematically shown in Figure 1 in Section 2.1. A plane wave is incident upon a layered slab from an upper homogeneous medium; the slab itself rests upon another homogeneous medium. In the geometrical acoustics picture (high frequency regime), an asymptotic representation of acoustic fields, i.e., pressure and particle velocity, within a layered medium can be obtained explicitly. If the sound speed is assumed to increase in the layer, then there exist depths at which the rays turn. These are called turning points. The waves propagate in the region above the turning point and evanesce in the region below the turning point.

Our problem in this work is to study the effects of fluctuations of acoustic waves in a randomly-layered slab. Average acoustic parameters are assumed to increase with depth so that an obliquely incident wave, undergoing some random fluctuation, turns at some point in the medium, leading to stochastic turning point problem. To see the variations of the large-scale structure of the medium, the wave needs to have short wavelength compared to the length scales of the macrostructure but to acquire the statistical properties of the details of the random inhomogeneities, the wavelength needs to be long compared to the size of the microstructure. This allows us to combine both a geometrical optics-like accomodation of the large-scale macroscopic variations and a Central Limit Theorem or diffusion limit theorem for the random microscopic details. The combined limits can reveal the very rich structures of stochastic problems for suitably defined quantities. Our modeling hypothesis is this separation of scales with random layering and compactification of the medium. The statistics of the reflected signal at the interface is to be obtained based on this hypothesis.

Limit theorems for stochastic differential equations remain our mathematical tool. They say in brief that a large class of random processes defined by the following stochastic differential equation, not of Ito type, converge weakly to a Markov diffusion process which solves a Kolmogorov backward equation:

$$\frac{dx}{d\tau} = \frac{1}{\epsilon} F(\tau, \tau/\epsilon^2, x, \omega), \quad 0 \le \tau \le \tau_0 \sim O(1), \quad \omega \in \Omega, \tag{1.1}$$

$$x(0) = x_0, \tag{1.2}$$

where the random field $F$ has zero-mean and $\Omega$ denotes an underlying probability space. Such a type of theory was called attention to first by Stratonovich [5] in 1963 for problems of nonlinear vibrations in the presence of noise. Then mathematical theory was developed by Khasminskii [6] and extended by Papanicolaou and Kohler [7]. The theory has been applied effectively in a variety of problems [8]–[13], [3] (and references therein). Our problem of interest in the region above the turning point, however, requires an extension of this theory to an asymptotically infinite interval. We present a limit theorem, called Theorem 1, for stochastic equation (1.1)-(1.2) over an extended interval $[0, \tau_0/\epsilon]$, i.e.,

$$\frac{dx}{d\tau} = \frac{1}{\epsilon} F(\tau, \tau/\epsilon^2, x, \omega), \quad 0 \le \tau \le \tau_0/\epsilon \sim O(\epsilon^{-1}), \quad \omega \in \Omega, \tag{1.3}$$

$$x(0) = x_0, \tag{1.4}$$

where the random field $F$ has zero-mean. For the problem below the turning point, we derive another limit theorem, called Theorem 2, which deals with the following stochastic problem with rapidly varying deterministic component:

$$\frac{dx}{d\tau} = \frac{1}{\epsilon} F_0(\tau, x) + \frac{1}{\epsilon} F_1(\tau, \tau/\epsilon^2, x, \omega), \quad 0 \le \tau \le \tau_0, \quad \omega \in \Omega, \tag{1.5}$$

$$x(0) = x_0, \tag{1.6}$$

where $F_0$ is (non-zero) deterministic and the random field $F_1$ has zero-mean.

Our goal in this work is to analyze the interplay of total internal refraction (caused by compactification) and diffusion effects (produced by random scattering). Aimed at this purpose, there has been a series of works by Kohler [13]–[16] using both analysis and simulations; the structure of fields within the medium and interface reflection properties over a wide range of frequencies and incidence angles have been revealed.

We introduce, in Chapter 2, the deterministic problem of acoustic waves in layered media before our stochastic problem is studied. Using the results from [17], explicit results for a uniform slab and the WKB analysis for high-frequency regimes are included.

We consider, in Chapter 3, the problem of time-harmonic acoustic plane waves impinging upon a randomly-layered slab. We first introduce the linearized acoustic equations and quantify the scaling condition for the acoustic parameters. Using the fundamental matrix of an effective system, we formulate a stochastic boundary value problem for the scattering variables. Using the explicit uniformly-valid results of Lynn and Keller [18], we establish the problem above the turning point in terms of a new stretched variable. Finally we recast it as a nonlinear stochastic initial value problem (Riccati differential equation) for the reflection coefficient. We can characterize the quantities of our interest (i.e., mean and correlation functions of reflected pressure) as expectations of functions of the reflection coefficient.

We approach the problem of stochastic characterization of the relection coefficient by using a "two-step decomposition". Using the properties of conditional expectation, we first assume that the region below the turning point is known. Considering the region above the turning point, we solve the Kolmogorov backward equation with final value given at the turning point. This provides us with an asymptotic characterization of the conditional expectation of the reflection coefficient in terms of a Markov diffusion process along the lines of known theory, i.e., limit theorems for stochastic differential equations. But it requires an extension of these ideas in order to get uniformly-valid characteristics throughout the turning point region. In Chapter 4, we derive a limit theorem (Theorem 1) on an extended interval in general terms and subsequently apply this theorem to our particular problem in the region above the turning point. Theorem 1 is proved in Chapter 5.

Now we can characterize uniformly the reflection process from the random slab above the turning point. Then we replace the assumed final value, evaluated at the turning point, of the Markov diffusion process by the small random noise value which arises from our characterizing reflection process from the region below the turning point. The governing statistics of reflected signal from the region below the turning point is estimated by Theorem 2 which is derived in Chapter 6.

Final characterization of the problem in the whole region can be obtained by taking the expectation of the Markov diffusion process containing the random noise. In Chapter 7, from the results of Theorem 1 and Theorem 2, its characterization is expressed by the product of an operator (representing the transformation at the turning point between the two reflection coefficients corresponding to the two regions) and the two evolution operators (representing the Markov diffusion processes in the two regions).

# Chapter 2

# DETERMINISTIC WAVES IN LAYERED MEDIA

## 2.1 The Acoustic Equations and Acoustic Energy

We will focus on the following linearized acoustic equations which describe the space-time evolution of a pressure $P(\mathbf{x}, t)$ and a vector particle velocity $\mathbf{U}(\mathbf{x}, t)$ in the medium:

$$\rho\, \partial_t \mathbf{U} + \nabla P = \mathbf{0}, \tag{2.1}$$

$$\partial_t P + K\, \nabla \cdot \mathbf{U} = 0, \tag{2.2}$$

where $\rho(\mathbf{x})$ and $K(\mathbf{x})$ represent time-independent material density and bulk modulus at a spatial point $\mathbf{x} = (x, y, z)$, respectively, and satisfy the following relation

$$K = \rho\, c^2 \tag{2.3}$$

with sound speed $c(\mathbf{x})$. Equations (2.1)-(2.2) are the fundamental equations in fluid dynamics, i.e., Euler's equation and the equation of continuity coming from conservation laws of momentum and mass, respectively [19]. The linearized equations are usually derived from a consideration of small amplitude acoustic disturbances in a fluid (or gas); the velocity $u$ ($= \sqrt{\mathbf{U} \cdot \mathbf{U}}$) of the fluid particle in the wave is small compared with the sound speed, i.e., $u \ll c$. They are also useful in describing the propagation of acoustic waves through solids if elastic effects can be ignored.

Equations (2.1)-(2.2) can be recast as a pair of second order differential equations, each involving only one of the acoustic variables $P$ or $\mathbf{U}$; by appropriate differentiation of equations (2.1)-(2.2) and by substitution,

$$\partial_{tt}^2 \mathbf{U} - \rho^{-1} \nabla (K\, \nabla \cdot \mathbf{U}) = \mathbf{0}, \tag{2.4}$$

$$\partial_{tt}^2 P - K\nabla \cdot (\rho^{-1}\nabla P) = 0. \tag{2.5}$$

In particular, if $\rho$ and $K$ (and thus $c$) are constants (i.e., a homogeneous medium), equations (2.4)-(2.5) reduce to the following constant coefficient wave equation:

$$\partial_{tt}^2 \phi - c^2 \Delta \phi = 0, \tag{2.6}$$

where $\phi$ is $P$ or $\mathbf{U}$.

Using acoustic equations (2.1)-(2.2) and the time independence of $\rho$ and $K$, one can readily verify the identity

$$\partial_t\left(\frac{1}{2}\rho u^2 + \frac{P^2}{2K}\right) + \nabla \cdot (P\mathbf{U}) = 0. \tag{2.7}$$

Here both $\frac{1}{2}\rho u^2$ and $\frac{P^2}{2K}$ have the dimensions of energy per unit volume; they represent the kinetic and potential energy densities, respectively, of an acoustic wave. On the other hand, the vector quantity $P\mathbf{U}$, which is called Umov vector, has the dimensions of energy per unit area per unit time; it is the energy flux in the acoustic wave. Equation (2.7), therefore, states the conservation of acoustic energy.

## 2.2 Time-Harmonic Plane Waves in Layered Media

We shall consider a scattering problem in which acoustic energy impinges upon a slab, occupying the region $-L \leq z \leq 0$. The slab is assumed to be transversely homogeneous, i.e., layered in $z$-direction; its constitutive parameters, i.e., density and bulk modulus (and thus sound speed), are functions only of the variable $z$. The exterior regions $z > 0$ and $z < -L$ are assumed to be ones in which both density and bulk modulus are constant. For the incident acoustic fields, the question of transients is completely ignored; we assume that the acoustic source emits a monochromatic sinusoidal signal which persists for all subsequent time. In this time-harmonic problem (or continuous wave problem), the steady state solution will be focused on. We refer to [20] as a general text for waves in layered media.

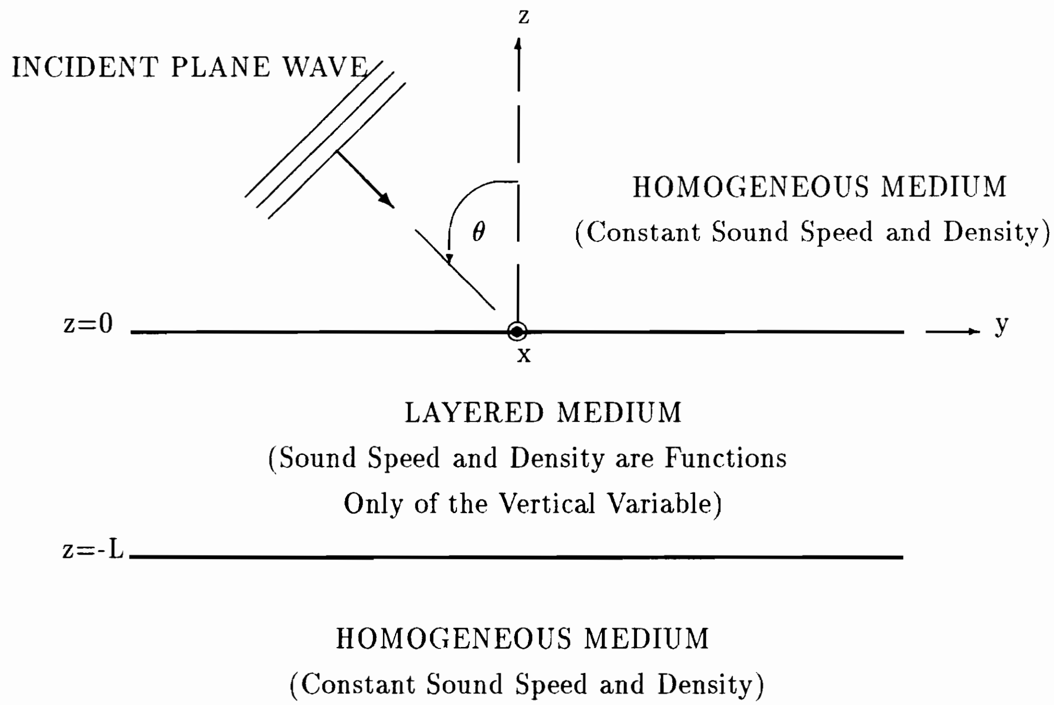


**Figure 1. Problem configuration**

By assuming plane wave excitation, transverse homogeneity considerably simplifies the acoustic propagation problem; the problem can be reduced to the analysis of ordinary differential equations. The plane wave is assumed to impinge upon the slab at an angle $\theta$ from the region $z > 0$. As Figure 1 indicates, this incidence angle is the acute angle that the propagation vector (or displacement velocity vector) makes with the vertical axis; in other words, the incidence angle is the acute angle that the plane wave phase front makes with the horizontal axis.

For the time-harmonic problem, the obliquely incident acoustic fields can be written from (2.6) in the following form:

$$P_{inc}(\mathbf{x},t) = P_0\, e^{-i\omega t} e^{-ik(-y\sin\theta + z\cos\theta)}, \tag{2.8}$$

$$\mathbf{U}_{inc}(\mathbf{x},t) = \frac{P_0}{\rho_0 c_0}\left[\sin\theta \mathbf{y}_0 - \cos\theta \mathbf{z}_0\right] e^{-i\omega t} e^{-ik(-y\sin\theta + z\cos\theta)}, \tag{2.9}$$

where $\rho_0$, $c_0$ and $k$ $(= \omega/c_0)$ are constant density, sound speed and wave number in the region $z > 0$, respectively.

All three regions (i.e., above, below and within the slab) are transversely homogeneous; no physical mechanism exists to create any transverse phase variations other than that impressed upon the problem by the incident fields. Likewise, no physical mechanism exists to create an out-of-plane (or $x$) component of displacement velocity. Since the pressure and normal velocity component must be continuous across $z = 0$ and $z = -L$, the solutions in all three regions must have the following general form:

$$P(\mathbf{x}, t) = P(z)\, e^{-i\omega t} e^{iky\sin\theta}, \tag{2.10}$$

$$\mathbf{U}(\mathbf{x}, t) = [U_2(z)\mathbf{y}_0 + U_3(z)\mathbf{z}_0]\, e^{-i\omega t}\, e^{iky\sin\theta}. \tag{2.11}$$

When (2.10)-(2.11) are substituted into acoustic equations (2.1)-(2.2), one obtains the following ordinary differential equations:

$$\frac{dP}{dz} = i\omega\rho U_3, \tag{2.12}$$

$$\frac{dU_3}{dz} = \frac{i\omega}{K}[1 - c_0^{-2}c^2\sin^2\theta]P, \tag{2.13}$$

$$U_2 = \frac{\sin\theta}{\rho c_0}P. \tag{2.14}$$

In the region $z > 0$, the solution of equations (2.12), (2.13) and (2.14) leads to the following expression for the total fields:

$$P(\mathbf{x}, t) = P_0(e^{-ikz\cos\theta} + re^{ikz\cos\theta})e^{-i\omega t}e^{iky\sin\theta}, \tag{2.15}$$

$$\mathbf{U}(\mathbf{x}, t) = \frac{P_0}{\rho_0 c_0}\{e^{-ikz\cos\theta}[\sin\theta\mathbf{y}_0 - \cos\theta\mathbf{z}_0] + re^{ikz\cos\theta}[\sin\theta\mathbf{y}_0 + \cos\theta\mathbf{z}_0]\}\cdot \cdot e^{-i\omega t}e^{iky\sin\theta}, \quad z > 0. \tag{2.16}$$

The parameter (complex constant) $r$ relates the amplitude of the reflected wave to that of the incident wave and is called the refection coefficient. In our problem formulation, $r$ is the ratio of reflected to incident pressure at the interface $z = 0$ and is called the pressure reflection coefficient at $z = 0$. The above solutions (2.15) and (2.16) simply represent the superposition of the incident wave and a reflected wave propagating in the specular direction.

In the region $z < -L$, one must impose an outgoing wave radiation condition. From the solution of equations (2.12), (2.13) and (2.14) in this region, the total fields become

$$P(\mathbf{x}, t) = P_0 t\, e^{-i\omega t} e^{-ik(-y\sin\theta + (z+L)\cos\theta)}, \tag{2.17}$$

$$\mathbf{U}(\mathbf{x}, t) = \frac{P_0 t}{\rho_0 c_0}\left[\sin\theta \mathbf{y}_0 - \cos\theta \mathbf{z}_0\right] e^{-i\omega t} e^{-ik(-y\sin\theta + (z+L)\cos\theta)}, \quad z < -L, \tag{2.18}$$

where the complex constant $t$ represents the ratio of transmitted pressure at $z = -L$ to incident pressure at $z = 0$ and is called the pressure transmission coefficient of the slab. From equations (2.8)-(2.9) and (2.17)-(2.18), it is clear that the transmitted wave is a wave propagating in the same direction as the incident wave but scaled by the transmission coefficient $t$. One must solve the scattering problem in order to determine these reflection and transmission coefficients.

## 2.3 The Uniform Slab Case

In order to actually determine the pressure reflection and transmission coefficients that were introduced in the previous section, one must determine the fields within the slab region and impose continuity conditions upon the fields at the interfaces $z = 0$ and $z = -L$. In this section, we briefly consider the simplest case, that of the uniform slab, for which one can obtain explicit results.

The slab will now be assumed to have constant density $\rho_1$ and constant bulk modulus $K_1$ (and thus $c_1$ will denote the corresponding constant sound speed). Equations (2.12) and (2.13) can then be explicitly solved; the following representation for the fields within the slab can be obtained:

$$P(z) = e^{i\kappa z} d_1 + e^{-i\kappa z} d_2, \tag{2.19}$$

$$U_3(z) = \frac{\kappa}{\omega\rho_1}(e^{i\kappa z} d_1 - e^{-i\kappa z} d_2), \quad -L \le z \le 0, \tag{2.20}$$

$$(\kappa \equiv \omega\sqrt{c_1^{-2} - c_0^{-2}\sin^2\theta})$$

where $d_1$ and $d_2$ are constants to be determined and radian frequency $\omega$ is assumed to be positive. The solutions of equations (2.12)-(2.13) in the exterior regions $z > 0$ and $z < -L$ are represented by equations (2.15)-(2.16) and (2.17)-(2.18), respectively. It is required that the pressure $P$ and the normal component of the velocity $U_3$ be continuous across the interfaces $z = 0$ and $z = -L$. We have, therefore, a system of four equations for the four unknowns $r$, $t$, $d_1$ and $d_2$ and thus the solution of this system of equations can be determined, being represented in the following manner:

$$r = \frac{\Gamma(1 - e^{i2\kappa L})}{1 - \Gamma^2 e^{i2\kappa L}}, \quad t = \frac{e^{i\kappa L}(1 - \Gamma^2)}{1 - \Gamma^2 e^{i2\kappa L}}, \tag{2.21}$$

$$d_1 = -\Gamma e^{i2\kappa L} d_2, \quad d_2 = \frac{P_0(1 + \Gamma)}{1 - \Gamma^2 e^{i2\kappa L}}, \tag{2.22}$$

where $\Gamma$ is defined to be

$$\Gamma \equiv \frac{\frac{\cos\theta}{\rho_0 c_0} - \frac{\sqrt{1 - c_0^{-2} c_1^2 \sin^2\theta}}{\rho_1 c_1}}{\frac{\cos\theta}{\rho_0 c_0} + \frac{\sqrt{1 - c_0^{-2} c_1^2 \sin^2\theta}}{\rho_1 c_1}}. \tag{2.23}$$

The parameter $\Gamma$ defined by (2.23) is same as the uniform half-space pressure reflection coefficient, i.e., it is the reflection coefficient that would exist at $z = 0$ if the slab thickness $L$ were infinite.

When $c_0 > c_1 \sin\theta$, the vertical wave number parameter $\kappa$ is real. Acoustic waves propagate and undergo multiple reflections within the slab; the total fields are a superposition of upward and downward propagating waves. The half-space reflection coefficient $\Gamma$ is real and bounded in magnitude by unity; the pressure reflection coefficient $r$ and transmission coefficient $t$ are generally complex-valued but also bounded in absolute value by unity. The slab wave coefficients, $d_1$ and $d_2$, respectively, are likewise periodic functions of $\kappa L$. When $c_0 < c_1 \sin\theta$, on the other hand, the acoustic fields within the slab become evanescent, i.e., exponentially decaying. In this case, the interface reflection coefficient $\Gamma$ becomes a complex number and unimodular (since the uniform half-space becomes totally reflecting). The magnitude of the pressure transmission coefficient $t$ now decreases exponentially with increasing slab thickness since $e^{i\kappa L} = e^{-|\kappa| L}$.

## 2.4 From Boundary Value Problem to Initial Value Problem

We have discussed, in Sections 2.2 and 2.3, a linear two-point boundary value scattering problem for the layered slab. The differential equations to be solved in the slab region are equations (2.12)-(2.13). Noting equations (2.15)-(2.16) and (2.17)-(2.18), the corresponding boundary conditions can be expressed in the following manner:

$$P(0) - \frac{\rho_0 c_0}{\cos\theta} U_3(0) = 2P_0, \tag{2.24}$$

$$P(-L) + \frac{\rho_0 c_0}{\cos\theta} U_3(-L) = 0. \tag{2.25}$$

There are two ways by which this two-point boundary value problem can be recast as an initial value problem. The first method is to introduce the fundamental or propagator matrix and to solve the corresponding initial value problem for this matix. Let $\Phi(z)$ be a propagator matrix for the system of equations (2.12)-(2.13); it solves an initial value problem of the form

$$\frac{d}{dz}\Phi = i\omega \begin{bmatrix} 0 & \rho \\ K^{-1}(1 - c_0^{-2}c^2\sin^2\theta) & 0 \end{bmatrix} \Phi, \quad \Phi(\zeta) = \Phi_0, \quad -L \le z \le 0, \tag{2.26}$$

where $\zeta$ is some fixed point within the domain (typically $z = -L$ or $z = 0$) and $\Phi_0$ is any (convenient) nonsingular 2×2 matrix. Then the pressure and velocity can be expressed in terms of this propagator matrix solution as

$$\begin{bmatrix} P \\ U_3 \end{bmatrix} = \Phi(z) \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}, \tag{2.27}$$

where $c_1$ and $c_2$ are constants to be determined by imposing boundary condition (2.24)-(2.25). This propagator matrix approach, therefore, replaces a linear two-point boundary value problem for a vector-valued dependent variable with a linear initial value problem for a matrix-valued dependent variable. In the next section, this method will be demonstrated in the high-frequency regime using the WKB approximation.

The second method involves the formulation of a Riccati differential equation for an appropriately defined reflection coefficient which will be defined using appropriately chosen dependent variables. In this approach, a linear two-point boundary value problem is replaced by a nonlinear (scalar-valued) initial value problem. In order to find appropriate dependent variables, we observe the form of boundary condition (2.24)-(2.25) as well as the structure of the acoustic fields in the regions $z > 0$ and $z < -L$, i.e., equations (2.15)-(2.16) and (2.17)-(2.18). These suggest we consider the following form of dependent variables:

$$A(z) \equiv \frac{1}{2}[(\frac{\cos\theta}{\rho_0 c_0})^{1/2} P(z) + (\frac{\rho_0 c_0}{\cos\theta})^{1/2} U_3(z)], \tag{2.28}$$

$$B(z) \equiv \frac{1}{2}[(\frac{\cos\theta}{\rho_0 c_0})^{1/2} P(z) - (\frac{\rho_0 c_0}{\cos\theta})^{1/2} U_3(z)]. \tag{2.29}$$

In terms of these new variables, basic equations (2.12)-(2.13) together with boundary conditions (2.24)-(2.25) become the boundary value problem

$$\frac{d}{dz}\begin{bmatrix} A \\ B \end{bmatrix} = \frac{i\omega}{2}\begin{bmatrix} a_{11} & a_{12} \\ -a_{12} & -a_{11} \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix}, \quad -L \le z \le 0, \tag{2.30}$$

$$A(-L) = 0, \quad B(0) = (\frac{\cos\theta}{\rho_0 c_0})^{1/2} P_0, \tag{2.31}$$

where the coefficient matrix is given by

$$a_{1i} \equiv (-1)^{i-1}\frac{\cos\theta}{\rho_0 c_0}\rho + \frac{\rho_0 c_0}{\cos\theta} K^{-1}(1 - c_0^{-2} c^2 \sin^2\theta), \quad i = 1, 2.$$

In the region $z > 0$, (2.15)-(2.16) lead to

$$A(z) = (\frac{\cos\theta}{\rho_0 c_0})^{1/2} P_0 r e^{ikz\cos\theta}, \quad B(z) = (\frac{\cos\theta}{\rho_0 c_0})^{1/2} P_0 e^{-ikz\cos\theta}, \quad z > 0. \tag{2.32}$$

In the region $z < -L$, on the other hand, (2.17)-(2.18) lead to

$$A(z) = 0, \quad B(z) = (\frac{\cos\theta}{\rho_0 c_0})^{1/2} P_0 t e^{-ik(z+L)\cos\theta}, \quad z < -L. \tag{2.33}$$

Therefore, $A$ and $B$ are basically "upward"- and "downward"- propagating scattering variables, respectively.

In this scattering formulation, one can verify easily the conservation of energy relation $|r|^2 + |t|^2 = 1$. Basic system (2.30) leads, by direct calculation, to the following relation between amplitudes of the scattering variables:

$$\frac{d}{dz}[|B(z)|^2 - |A(z)|^2] = 0, \quad -L \le z \le 0. \tag{2.34}$$

Thus the quantity $|B|^2 - |A|^2$ is constant throughout the slab; in particular, it follows by (2.32)-(2.33) that

$$\begin{aligned} 1 - |r|^2 &= \frac{\rho_0 c_0}{\cos\theta}|P_0|^{-2}[|B(0)|^2 - |A(0)|^2] \\ &= \frac{\rho_0 c_0}{\cos\theta}|P_0|^{-2}[|B(-L)|^2 - |A(-L)|^2] = |t|^2. \end{aligned} \tag{2.35}$$

In view of these properties, one can identify $|B|^2$ with an incident energy flux and $|A|^2$ with a reflected enegy flux throughout the slab region. Equation (2.35), then, simply asserts that the slab is lossless.

We are now ready to reformulate boundary value problem (2.30)-(2.31) into an initial value problem. Let us define the following reflection coefficient:

$$R(z) \equiv \frac{A(z)}{B(z)}, \quad -L \leq z \leq 0. \tag{2.36}$$

By direct calculation, one can show from (2.30)-(2.31) that the reflection coefficient $R$ satisfies the following Riccati differential equation:

$$\frac{dR}{dz} = i\omega[a_{11}R + \frac{1}{2}a_{12}(1 + R^2)], \quad -L \leq z \leq 0, \tag{2.37}$$

$$R(-L) = 0. \tag{2.38}$$

It is clear from (2.32) that $R(0) = r$. Therefore, the solution of the above initial value problem provides directly information about the pressure reflection coefficient. Since $|t|^2 = 1-|r|^2$, the magnitude of the pressure transmission coefficient is likewise directly determined. In order to obtain full knowledge of the transmission coefficient including the phase, one needs to do the following additional work. Once the pressure reflection coefficient $r$ is known, both $A(0)$ and $B(0)$ are obtained by (2.32). Hence one can now solve differential equation (2.30) as a final value problem to determine $B(-L)$ which is same as $(\frac{\cos\theta}{\rho_0 c_0})^{1/2} P_0 t$ from (2.33). The pressure transmission coefficient $t$ is, therefore, finally determined.

## 2.5 High Frequency Regime - WKB Analysis

In Section 2.3, we have obtained explicitly the acoustic fields for the uniform slab. For nonuniformly layered slabs, i.e., for slabs in which the acoustic parameters actually vary with depth, explicit results are generally not attainable. In this section, we shall consider a high frequency (or geometrical acoustics) regime, in which an asymptotic representation of fields within a nonuniformly layered slab can be obtained. The approximation is referred to as the approximate solution of WKB (Wentzel-Kramers-Brillouin) type. We shall use the ansatz of Ludwig [21] to derive formally these approximate acoustic fields.

Consider again the case of a time-harmonic plane wave obliquely incident upon a layered slab as discussed in Section 2.3. Basic differential equations (2.12)-(2.13) are rewritten as

$$\frac{d}{dz}\begin{bmatrix} P \\ U_3 \end{bmatrix} = i\omega \begin{bmatrix} 0 & \rho(z) \\ K^{-1}(z)(1 - c_0^{-2}c^2(z)\sin^2\theta) & 0 \end{bmatrix} \begin{bmatrix} P \\ U_3 \end{bmatrix}, \tag{2.39}$$

where the $z$-dependence of the coefficient matrix is now explicitly noted. In our regime of interest, the radian frequency $\omega$ is large (and thus acoustic wavelength is small). Compared with this large parameter $\omega$, the constitutive variables $\rho$, $K$ and $c$ are assumed to undergo $O(1)$-length scale variations as functions of $z$; therefore, the slab composition varies slowly with depth relative to acoustic wavelength.

For brevity, let system (2.39) be represented as

$$\frac{d}{dz}\mathbf{v}(z,\omega) = i\omega\mathbf{A}(z)\mathbf{v}(z,\omega). \tag{2.40}$$

One seeks a solution of system (2.40) of the following form:

$$\mathbf{v}(z,\omega) = e^{i\omega\sigma(z)} \sum_{n=0}^{\infty} \omega^{-n}\mathbf{v}_n(z). \tag{2.41}$$

Then the WKB approximation will consist of the leading order term, i.e.,

$$\mathbf{v}_0(z,\omega) = \mathbf{v}_0(z)\, e^{i\omega\sigma(z)}. \tag{2.42}$$

If ansatz (2.41) is substituted into equation (2.40) and coefficients of like powers of $\omega$ are equated, one obtains the following hierarchy of equations:

$$\omega^1 : (\mathbf{A}(z) - \sigma'(z)\mathbf{I})\,\mathbf{v}_0(z) = \mathbf{0}, \quad \sigma' \equiv \frac{d\sigma}{dz}, \tag{2.43}$$

$$\omega^0 : (\mathbf{A}(z) - \sigma'(z)\mathbf{I})\,\mathbf{v}_1(z) = -i\frac{d}{dz}\mathbf{v}_0(z), \tag{2.44}$$

. . . . . . . . . . ..

$$\omega^{-n} : (\mathbf{A}(z) - \sigma'(z)\mathbf{I})\,\mathbf{v}_{n+1}(z) = -i\frac{d}{dz}\mathbf{v}_n(z), \quad n = 1, 2, ..., \tag{2.45}$$

where $\mathbf{I}$ denotes the 2×2 identity matrix. Equation (2.43) requires that $\sigma'$ and $\mathbf{v}_0$ form an eigenpair of the matrix $\mathbf{A}$, while equation (2.44) leads to a solvability constraint upon $\mathbf{v}_0$.

Consider first the eigenvalue problem for the matrix $\mathbf{A}$. The eigenpairs are found to be

$$\sigma'(z) \equiv \lambda_i(z) = (-1)^{i-1}\sqrt{c^{-2}(z) - c_0^{-2}\sin^2\theta}, \tag{2.46}$$

$$\mathbf{X}_i(z) = \begin{bmatrix} \rho^{1/2}(z) \\ (-1)^{i-1}K^{-1/2}(z)(1 - c_0^{-2}c^2(z)\sin^2\theta)^{1/2} \end{bmatrix}, \quad i = 1, 2. \tag{2.47}$$

The factor $c^{-2}(z) - c_0^{-2}\sin^2\theta$ controls the character of the eigenvalues and the integrals of these eigenvalues represent the phases of two WKB solutions which are given by (2.42). Therefore, we now encounter a situation qualitatively similar to that of uniform slab in Section 2.3. If for a given incidence angle $\theta$ and depth $z$ this factor is positive (i.e., $c(z)\sin\theta < c_0$), the eigenvalues $\lambda_i(z)$ will be real; in this case the resulting two WKB solutions will correspond locally to (upward and downward) propagating acoustic waves. On the other hand, if the factor is negative (i.e., $c(z)\sin\theta > c_0$), the eigenvalues become imaginary and the solutions will correpond to exponentially damped or evanescent waves.

We consider the case where $c(z)$ is increasing in the slab such that $c^{-2}(z) - c_0^{-2}\sin^2\theta$ changes sign from positive to negative as $z$ decreases. The depth at which the eigenvalues vanish (i.e., $c(z)\sin\theta = c_0$) are called turning points; these are depths at which, in the geometrical acoustics picture, the rays turn or become tangent to the horizontal. For the present discussion, we shall assume that $c(z)\sin\theta < c_0$ for all $\theta$ and $z$ values of interest; the region above the turning point will be considered in the rest of this section.

In view of the form of (2.42), the two leading order terms corresponding to the eigenpairs are constrainted to have the form

$$\mathbf{v}_0^{(j)}(z,\omega) \sim \mu_j(z)\mathbf{X}_j(z)e^{(-1)^{j-1}i\omega\sigma(z)}, \quad j = 1, 2, \tag{2.48}$$

where $\mu_j$ are arbitrary functions of $z$ to be determined and (for definiteness)

$$\sigma(z) = -\int_z^0 \sqrt{c^{-2}(s) - c_0^{-2}\sin^2\theta}\; ds, \quad -L \le z \le 0. \tag{2.49}$$

The integrand in equation (2.49) has the following ray interpretation. Let $\hat{\theta}(z)$ denote the incidence angle of the acoustic ray at depth $z$ (i.e., let us define $\hat{\theta}(z)$ to be the angle that the tangent to the ray path at depth $z$ makes with the vertical). A consequence of the interface boundary conditions and the fact that the medium is layered is that the transverse (or horizontal) wavenumber must remain constant with depth. We must have, therefore, the following relation:

$$\frac{\sin\theta}{c_0} = \frac{\sin\hat{\theta}(z)}{c(z)}, \quad -L \le z \le 0. \tag{2.50}$$

From equation (2.50), it is clear that the integrand in (2.49) is simply $c^{-1}(s)\cos\hat{\theta}$ and thus the exponent of (2.48), $\omega\sigma(z)$, represents the integrated vertical wavenumber from 0 to $z$. Also $\sigma(z)$ represents the accumulated time from 0 to $z$ and is called travel time.

Since the matrix $\mathbf{A}$ is real-valued, the eigenvalues of the adjoint matrix $\mathbf{A}^*$ are equal to those of $\mathbf{A}$; let the corresponding eigenvectors be denoted by $\mathbf{Y}_i$, $i = 1, 2$. One readily finds that

$$\mathbf{Y}_i(z) = \begin{bmatrix} K^{-1/2}(z)(1 - c_0^{-2}c^2(z)\sin^2\theta)^{1/2} \\ (-1)^{i-1}\rho^{1/2}(z) \end{bmatrix}, \quad i = 1, 2. \tag{2.51}$$

From equation (2.44), the following solvability conditions (from the Fredholm alternative) are imposed:

$$\left(\frac{d}{dz}\mathbf{v}_0^{(i)}(z), \mathbf{Y}_i(z)\right) = 0, \quad i = 1, 2 \tag{2.52}$$

which, in turn, lead to the following first order linear differential equations for $\mu_i$:

$$\frac{d}{dz}\mu_i\,(\mathbf{X}_i, \mathbf{Y}_i) + \mu_i\left(\frac{d}{dz}\mathbf{X}_i, \mathbf{Y}_i\right) = 0, \quad i = 1, 2. \tag{2.53}$$

The solution of equations (2.53) determines the leading order terms $\mathbf{v}_0^{(i)}(z)$ to within a multiplicative constant. In particular, one can obtain the following approximation to the propagator matrix solution of equation (2.39):

$$\Phi(z) \equiv \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} = \begin{bmatrix} (\frac{\rho(z)}{\sigma'(z)})^{1/2} e^{i\omega\sigma(z)} & -(\frac{\rho(z)}{\sigma'(z)})^{1/2} e^{-i\omega\sigma(z)} \\ (\frac{\rho(z)}{\sigma'(z)})^{-1/2} e^{i\omega\sigma(z)} & (\frac{\rho(z)}{\sigma'(z)})^{-1/2} e^{-i\omega\sigma(z)} \end{bmatrix}. \tag{2.54}$$

# Chapter 3

# STOCHASTIC WAVES IN RANDOMLY-LAYERED MEDIA

## 3.1 Stochastic Problem with Turning Points

From now on, we consider a stochastic problem in a randomly-layered medium; the effects of fluctuations of plane waves in a random multilayer are going to be studied. An acoustic plane wave impinges obliquely upon the medium. The average acoustic parameters such as density and sound speed are assumed to increase with depth. This increase refracts the penetrating acoustic energy upward so that an obliquely incident wave, undergoing random scattering, turns at some depth leading to the stochastic turning point problem.

The geophysical nature of ocean sediments [13] exhibits this structure very well. The various physical deposition processes have created a material with a highly-laminated fine structure in the vertical or depth direction. Sediment constitutive parameters vary with depth on two length scales; they possess a fine-scale layering structure superposed upon a gradual slow-scale increase induced by compactification effects. This nominally linear increase tends to refract the penetrating acoustic energy back upward towards the water-sediment interface. In the transverse directions, the sediment acoustic parameters are homogeneous to a degree that makes one-dimensional modeling a reasonable approximation.

The medium considered here is a randomly-layered slab. The schematic picture of our stochastic problem that we shall consider is basically the same as Figure 1 in Chapter 2. An oblique acoustic plane wave impinges on a refracting rapidly-layered random slab occupying the region $-L \leq z \leq 0$ from above. The exterior regions $z > 0$ and $z < -L$ are homogeneous ones; in these regions acoustic parameters are constant. We consider an idealized model in which the random multilayer has a one-dimensional fine structure; it is a homogeneous in the transverse directions. Thus the acoustic parameters are the functions of only one variable $z$.

We formulate the problem of interest as a stochastic boundary value problem for the scattering variables. We shall subsequently recast it as a stochastic initial value problem for the reflection coefficient. We approach the problem by using a "two-step decomposition"; we deal with problems above and below turning points separately using the properties of conditional expectation and then combine the two results using the continuity of the fields at the turning point. The scattering variables are suitably defined in each region.

In this chapter, we consider the problem above the turning point. The problem below the turning point will be considered later in Chapter 6. First a system of linearized acoustic equations with a random coefficient matrix is derived using the continuity condition at the interfaces of the slab. Using the fundamental matrix of an effective system, we derive a stochastic boundary value problem for the scattering variables. Then we use the Lynn and Keller uniformly-valid results [18] to obtain a centered system in the stretched region above the turning point. Finally a stochastic initial value problem, over an interval of infinite scale, for the unimodular reflection coefficient is derived.

## 3.2 Basic Equations and Scaling

In this section, we introduce the basic acoustic equations in the random slab and two kinds of scale dependence of the field parameters. These scales represent a deterministic macroscale and a random microscale.

The basic equations are the linearized acoustic equations coming from linear momentum and mass coservation for a pressure $P(\mathbf{x}, t)$ and a particle velocity $\mathbf{U}(\mathbf{x}, t)$:

$$\rho(z)\partial_t \mathbf{U}(\mathbf{x}, t) + \nabla P(\mathbf{x}, t) = \mathbf{0}, \tag{3.1}$$

$$\partial_t P(\mathbf{x}, t) + K(z)\nabla \cdot \mathbf{U}(\mathbf{x}, t) = 0, \tag{3.2}$$

where $\rho(z)$ and $K(z)$ denote time-independent density and bulk modulus, respectively, satisfying $K = \rho c^2$ with sound speed $c(z)$.

Within the random slab $-L \leq z \leq 0$, the acoustic parameters $\rho$ and $c$ are assumed to vary with depth $z$ in a manner that superposes a randomly-fluctuating component upon a slowly-varying (nominally linear) increasing mean value. We introduce a small positive parameter $\epsilon$ to quantify this two-scale dependence, i.e., randomness on the microscale and smooth deterministic macroscopic variations:

$$\rho(z) = \rho(z, z/\epsilon^2), \quad K(z) = K(z, z/\epsilon^2), \qquad -L \leq z \leq 0, \tag{3.3}$$

where $\rho$ and $K$ are unit correlation length random functions of the second arguments while the first arguments account for deterministic nonstationary modulation; the actual correlation length of the processes is, therefore, $\epsilon^2$. Let the frequency of the incident wave also be scaled by

$$\omega = \overline{\omega}/\epsilon, \tag{3.4}$$

where $\overline{\omega}$ is $O(1)$ (the overbar will subsequently be omitted throughout). Consequently the orders $O(1)$, $O(\epsilon)$ and $O(\epsilon^2)$ correspond to the large-scale structure of the medium, the wavelength of incident wave and the correlation length of the random features of the medium, respectively. This means that the wavelength is "large" enough to average over microscopic random effects and "small" enough to probe macroscopic variation. A limit is, therefore, possible which combines a diffusion limit for fine-scale random fluctuation with a geometric acoustics-like limit for large-scale deterministic variation. Also the high-frequency scale (i.e., $O(\epsilon^{-1})$) will permit the use of WKB analysis.

The monochromatic, time-harmonic, plane wave is incident upon the layer with an incidence angle $\theta$ measured from the normal as shown in Figure 1. We introduce a pressure $P(\mathbf{x}, t)$ and a particle velocity $\mathbf{U}(\mathbf{x}, t)$ incident from the region $z > 0$ of the following form from the constant coefficient wave equation (2.6):

$$P_{inc}(\mathbf{x}, t) = P_0\, e^{-i\frac{k}{\epsilon}(c_0 t - y \sin\theta + z\cos\theta)}, \tag{3.5}$$

$$\mathbf{U}_{inc}(\mathbf{x}, t) = \frac{P_0}{\rho_0 c_0}[\sin\theta \mathbf{y}_0 - \cos\theta \mathbf{z}_0]\, e^{-i\frac{k}{\epsilon}(c_0 t - y \sin\theta + z\cos\theta)}, \; z > 0, \tag{3.6}$$

where $\rho_0$, $c_0$ and $k$ $(= \omega/c_0)$ are the constant density, sound speed and wave number of the upper homogeneous medium, respectively. The quantity $\rho c$ is called the impedance of the medium. As seen above, the incident pressure and particle velocity aree related to each other by the impedance of the obliquely incident wave.

The assumed plane wave excitation and transverse homogeneity imply that the acoustic fields in the random multilayer depend only on depth $z$ and one transverse coordinate $y$; no physical mechanism exists to create any transverse phase variations other than that impressed upon the problem by the incident fields. Since the acoustic pressure and the normal component of the particle velocity must be continuous across the boundaries $z = 0$ and $z = -L$ [20], the acoustic fields in all three regions must have the following general form:

$$P(\mathbf{x}, t) = P(z)e^{-i\frac{k}{\epsilon}(c_0 t - y\sin\theta)}, \tag{3.7}$$

$$\mathbf{U}(\mathbf{x}, t) = [U_2(z)\mathbf{y}_0 + U_3(z)\mathbf{z}_0]e^{-i\frac{k}{\epsilon}(c_0 t - y\sin\theta)}. \tag{3.8}$$

If (3.7)-(3.8) are substituted into acoustic equations (3.1)-(3.2), one obtains the system of ordinary differential equations:

$$\frac{d}{dz}\begin{bmatrix} P \\ U_3 \end{bmatrix} = \frac{i\omega}{\epsilon}\begin{bmatrix} 0 & \rho(z) \\ K^{-1}(z) - \rho^{-1}(z)c_0^{-2}\sin^2\theta & 0 \end{bmatrix}\begin{bmatrix} P \\ U_3 \end{bmatrix}, \tag{3.9}$$

$$U_2 = \frac{\sin\theta}{\rho c_0}P. \tag{3.10}$$

From now on, system (3.9) for the pressure $P$ and the $z$-component of the particle velocity $U_3$ is focused on in the high frequency regime; the $y$-component of the particle velocity will be determined immediately by (3.10) once the pressure is determined. In the random slab, scaling condition (3.3) is assumed to take the form

$$\rho(z, z/\epsilon^2) = \alpha(z)[1 + \eta(z, z/\epsilon^2)], \tag{3.11}$$

$$K^{-1}(z, z/\epsilon^2) - \rho^{-1}(z, z/\epsilon^2)c_0^{-2}\sin^2\theta = \beta(z)[1 + \nu(z, z/\epsilon^2)], \tag{3.12}$$

where $\alpha$ and $\beta$ are defined to be

$$\alpha(z) \equiv E\{\rho(z, z/\epsilon^2)\}, \quad \beta(z) \equiv E\{K^{-1}(z, z/\epsilon^2) - \rho^{-1}(z, z/\epsilon^2)c_0^{-2}\sin^2\theta\}, \tag{3.13}$$

and the fluctuations $\eta$ and $\nu$ are zero-mean, unit correlation length random functions of their second arguments; we define the random functions

$$\alpha_1(z, z/\epsilon^2) \equiv \alpha(z)\, \eta(z, z/\epsilon^2), \quad \beta_1(z, z/\epsilon^2) \equiv \beta(z)\, \nu(z, z/\epsilon^2). \tag{3.14}$$

We assume that $\alpha$, $\beta$, $\alpha_1$ and $\beta_1$ are all bounded in the region $-L \leq z \leq 0$. Using the following notation

$$\mathcal{A}_0 \equiv \begin{bmatrix} 0 & \alpha \\ \beta & 0 \end{bmatrix}, \quad \mathcal{A}_1 \equiv \begin{bmatrix} 0 & \alpha_1 \\ \beta_1 & 0 \end{bmatrix}, \tag{3.15}$$

we decompose (3.9) within the random slab $-L \leq z \leq 0$ into the sum of a deterministic part and a mean-zero random part; (3.9) is rewritten in the simple form

$$\frac{d}{dz}\begin{bmatrix} P \\ U_3 \end{bmatrix} = \frac{i\omega}{\epsilon}(\mathcal{A}_0 + \mathcal{A}_1)\begin{bmatrix} P \\ U_3 \end{bmatrix}, \quad -L \leq z \leq 0. \tag{3.16}$$

In the case that high-frequency system (3.9) is deterministic, as discussed in Section 2.5, there are turning points dividing the behavior of the WKB solutions; they correspond to propagating waves and evanescent waves, respectively, above and below the turning point. These are depths at which the rays turn in the geometrical acoustic picture.

If (3.9) is a random system, in the asymptotic theory of interest, the turning point of the deterministic matrix $\frac{ik}{\epsilon}\mathcal{A}_0$ of the following system will define the turning point for the stochastic problem.

$$\frac{d}{dz}\begin{bmatrix} P \\ U_3 \end{bmatrix} = \frac{i\omega}{\epsilon}\mathcal{A}_0\begin{bmatrix} P \\ U_3 \end{bmatrix}, \quad -L \leq z \leq 0. \tag{3.17}$$

Note that $\mathcal{A}_0$ defines an effective acoustic medium. The eigenvalues of $\mathcal{A}_0$ satisfy

$$\lambda^2 = \alpha\beta, \tag{3.18}$$

where $\alpha$ and $\beta$ are defined by (3.13). Note that $\alpha$ is positive always. We assume that the density and the sound speed increase in such a way that $\beta$ is monotone decreasing and changes sign from positive to negative as $z$ decreases. The eigenvalues of $\mathcal{A}_0$ vanish at some point, say $z_T$, and they are real valued on $z_T < z \leq 0$ and pure imaginary on $-L \leq z < z_T$; in the effective medium, waves propagate in the region $z_T < z \leq 0$ and evanesce in the region $-L \leq z < z_T$. The point $z_T$ will remain the dividing point for the stochastic problem in our scaling and it is called the turning point of our stochastic problem.

## 3.3 Scattering Variables and Boundary Value Problem

In order to obtain quantities exhibiting a limiting stochastic behavior as $\epsilon \downarrow 0$ in our turning point problem, one needs to remove the rapid deterministic phase variations of (3.16) so that there is only a mean-zero term on the right-hand side of (3.16). For this step of centering, we need a fundamental matrix (or propagator matrix) of (3.17). In this section, we use the Lynn and Keller uniformly-valid approximant and derive a system of equations for the scattering variables.

First we consider the case of a uniform slab for which one can obtain explicit results similar to those of Section 2.3; the only difference is that we now have a high-frequency scale. With constant density $\rho_1$ and sound speed $c_1$ of the slab (i.e., with $\mathcal{A}_1$ zero and $\mathcal{A}_0$ a constant matrix), our basic system (3.9) in a uniform slab becomes

$$\begin{bmatrix} P \\ U_3 \end{bmatrix} = \begin{bmatrix} e^{i\frac{\kappa}{\epsilon}z} & e^{-i\frac{\kappa}{\epsilon}z} \\ \frac{\kappa}{\omega\rho_1}e^{i\frac{\kappa}{\epsilon}z} & -\frac{\kappa}{\omega\rho_1}e^{-i\frac{\kappa}{\epsilon}z} \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}, \tag{3.19}$$

where $\kappa$ (vertical wavenumber) is defined to be

$$\kappa = \omega\sqrt{c_1^{-2} - c_0^{-2}\sin^2\theta}. \tag{3.20}$$

In view of the $e^{-i\frac{\omega}{\epsilon}t}$ dependence, a wave of the form $e^{i\frac{\kappa}{\epsilon}z}$ is "up-coming"; a time dependent solution of the form $e^{i\frac{\kappa}{\epsilon}\{z-(c_1^{-2}-c_0^{-2}\sin^2\theta)^{-1/2}t\}}$ travels to increasing values of $z$ as time increases. Similarly, a wave of the form $e^{-i\frac{\kappa}{\epsilon}z}$ is "down-going". System (3.19), therefore, identifies $P$ and $U_3$ as sums of up-coming and down-going scattering variables. Also note that the constants $d_1$ and $d_2$ (to be determined) correspond to the up-coming and down-going wave amplitudes, respectively.

We shall use the constant coefficient case as a guide to analyzing the variable coefficient case of interest. In order to remove rapid deterministic phase variations of random system (3.16), let us introduce a WKB approximant for the fundamental matrix of system (3.17); we denote this approximant by $\Phi$. The new dependent variables $A$ and $B$ are introduced by

$$\begin{bmatrix} P \\ U_3 \end{bmatrix} = \Phi \begin{bmatrix} A \\ B \end{bmatrix}. \tag{3.21}$$

As we will see later when $\Phi$ is expressed explicitly, the dependent variable $A$ will represent a reflected or upward-propagating wave amplitude while $B$ will correspond to an incident downward-propagating one. These amplitudes of scattering variables are constants in the uniform slab case previously discussed. System (3.16) for pressure and velocity transforms into the following system of equations for scattering variables $A$ and $B$:

$$\frac{d}{dz}\begin{bmatrix} A \\ B \end{bmatrix} = \frac{i\omega}{\epsilon}\Phi^{-1}\mathcal{A}_1\Phi \begin{bmatrix} A \\ B \end{bmatrix} + \Phi^{-1}\left(\frac{i\omega}{\epsilon}\mathcal{A}_0\Phi - \frac{d}{dz}\Phi\right)\begin{bmatrix} A \\ B \end{bmatrix}, \tag{3.22}$$

where the second term on the right side of (3.22) would vanish if $\Phi$ were an exact fundamental matrix for (3.17).

To produce a two-point boundary value problem, we consider a unit down-going wave in $z > 0$ impinging on the random slab (i.e., $B = 1$ in $z > 0$), while for simplicity we assume total reflection at $z = -L$. For a physical point of view, the presence of a turning point at $z_T$ makes the nature of the termination at $z = -L < z_T$ irrelevant. Thus the following boundary conditions are imposed:

$$B(0) = 1, \tag{3.23}$$

$$A(-L) = \Gamma_{-L}B(-L), \quad |\Gamma_{-L}| = 1. \tag{3.24}$$

Let

$$\Phi \equiv \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix}. \tag{3.25}$$

Then the right-hand side of system (3.22) becomes

$$\frac{i\omega}{\epsilon}\Phi^{-1}\mathcal{A}_1\Phi = \frac{i\omega}{\epsilon\, det\Phi}\begin{bmatrix} \alpha_1\phi_{21}\phi_{22} - \beta_1\phi_{11}\phi_{12} & \alpha_1\phi_{22}^2 - \beta_1\phi_{12}^2 \\ -\alpha_1\phi_{21}^2 + \beta_1\phi_{11}^2 & -\alpha_1\phi_{21}\phi_{22} + \beta_1\phi_{11}\phi_{12} \end{bmatrix}, \quad (3.26)$$

$$\Phi^{-1}(\frac{i\omega}{\epsilon}\mathcal{A}_0\Phi - \frac{d}{dz}\Phi) = \frac{1}{det\Phi}\Delta, \quad (3.27)$$

where the matrix $\Delta$ is (using a prime to denote derivative)

$$\begin{bmatrix} \frac{i\omega}{\epsilon}(\alpha\phi_{21}\phi_{22} - \beta\phi_{11}\phi_{12}) - (\phi_{11}'\phi_{22} - \phi_{12}\phi_{21}') & \frac{i\omega}{\epsilon}(\alpha\phi_{22}^2 - \beta\phi_{12}^2) - (\phi_{12}'\phi_{22} - \phi_{12}\phi_{22}') \\ \frac{i\omega}{\epsilon}(-\alpha\phi_{21}^2 + \beta\phi_{11}^2) + (\phi_{11}'\phi_{21} - \phi_{11}\phi_{21}') & \frac{i\omega}{\epsilon}(-\alpha\phi_{21}\phi_{22} + \beta\phi_{11}\phi_{12}) + (\phi_{12}'\phi_{21} - \phi_{11}\phi_{22}') \end{bmatrix}$$

If a propagator matrix $\Phi$ can be chosen to have $\Delta$ negligible, then the right-hand side of (3.22) becomes essentially mean-zero. To obtain such a $\Phi$ explicitly, we use results of Lynn and Keller [18] from which it follows that $det\Phi = 2$ and the $\phi_{ij}$'s are

$$\phi_{11} = \alpha^{1/2}B_0\,V_1(\lambda^{2/3}\phi), \quad \lambda \equiv i\frac{\omega}{\epsilon} \quad (3.28)$$

$$\phi_{12} = \alpha^{1/2}B_0\,V_2(\lambda^{2/3}\phi), \quad (3.29)$$

$$\phi_{21} = \lambda^{-1}\alpha^{-1/2}(\alpha' B_0/2\alpha + B_0')\,V_1(\lambda^{2/3}\phi) + \lambda^{-1/3}\alpha^{-1/2}B_0^{-1}\,V_1'(\lambda^{2/3}\phi), \quad (3.30)$$

$$\phi_{22} = \lambda^{-1}\alpha^{-1/2}(\alpha' B_0/2\alpha + B_0')\,V_2(\lambda^{2/3}\phi) + \lambda^{-1/3}\alpha^{-1/2}B_0^{-1}\,V_2'(\lambda^{2/3}\phi), \quad (3.31)$$

where the functions $B_0$ and $\phi$ are defined to be

$$B_0 \equiv (\frac{\phi}{\alpha\beta})^{1/4}, \quad \phi(z) \equiv \{\frac{3}{2}(\tau_T - \tau(z))\}^{2/3}, \quad (3.32)$$

$$\tau(z) \equiv \int_z^0 \sqrt{\alpha\beta}\,ds, \quad \tau_T \equiv \tau(z_T), \quad (3.33)$$

and the functions $V_1$ and $V_2$ are defined as follows in terms of Airy functions ([22] for definition):

$$V_1(\cdot) \equiv \pi^{1/2}\lambda^{1/6}e^{-i\frac{\omega}{\epsilon}\tau_T}\{-iAi(\cdot) + Bi(\cdot)\}, \quad (3.34)$$

$$V_2(\cdot) \equiv -2\pi^{1/2}\lambda^{1/6}e^{i\frac{\omega}{\epsilon}\tau_T}Ai(\cdot). \quad (3.35)$$

As one can notice from (3.13), $(\alpha\beta)^{-1/2}$ roughly represents a mean propagation speed in the random medium (see also Section 2.5). Thus $\tau(z)$ represents an accumulated time from $z$ to 0 and is called the travel time. It is worth noting that $\tau$, $\phi$ and $B_0$ are real positive funtions above the turning point $z_T$ but are complex-valued functions below $z_T$.

If we use properties of Airy functions, the right-hand side of system (3.22) can be simplified and some underlying group structure of (3.22) can be revealed. First we need the following some useful conjugacy relations among the $\phi_{ij}$'s.

**Lemma 3.3.1** *The $\phi_{ij}$, $i,j\in\{1,2\}$, defined by (3.28)-(3.31), satisfy the following relations: $\phi_{12} = -\phi_{11}^*$ and $\phi_{22} = \phi_{21}^*$.*

**Proof**: From (3.34) and the following identity (cf. Abramowitz and Stegun [22])

$$Ai(ze^{-i2\pi/3}) = \frac{1}{2}e^{-i\pi/3}\{Ai(z) + iBi(z)\}, \tag{3.36}$$

one can obtain

$$\begin{aligned} V_1(\lambda^{2/3}\phi) &= \pi^{1/2}|\lambda|^{1/6}e^{-i\frac{\omega}{\epsilon}\tau_T}e^{-i5\pi/12}\{Ai(\lambda^{2/3}\phi) + iBi(\lambda^{2/3}\phi)\} \\ &= 2\pi^{1/2}|\lambda|^{1/6}e^{-i\frac{\omega}{\epsilon}\tau_T}e^{-i\pi/12}Ai(\lambda^{2/3}\phi e^{-i2\pi/3}). \end{aligned} \tag{3.37}$$

From (3.35) and $\lambda = |\lambda|e^{i\pi/2}$, one has

$$V_2(\lambda^{2/3}\phi) = -2\pi^{1/2}|\lambda|^{1/6}e^{i\frac{k}{\epsilon}\tau_T}e^{i\pi/12}Ai(\lambda^{2/3}\phi). \tag{3.38}$$

The identity $\lambda^{2/3}\phi = e^{i\pi/3}|\lambda|^{2/3}\phi$ gives

$$e^{-i2\pi/3}\lambda^{2/3}\phi = e^{-i\pi/3}|\lambda|^{2/3}\phi = (\lambda^{2/3}\phi)^*. \tag{3.39}$$

Upon applying the identity $Ai^*(z)=Ai(z^*)$, (3.37), (3.38) and (3.39) lead to

$$V_2 = -V_1^*. \tag{3.40}$$

It follows that $\phi_{12} = -\phi_{11}^*$. The derivative of (3.36) with respect to z is

$$e^{-i2\pi/3}Ai'(ze^{-i2\pi/3}) = \frac{1}{2}e^{-i\pi/3}\{Ai'(z) + iBi'(z)\}. \tag{3.41}$$

Using (3.41), the derivatives of $V_1$ and $V_2$ with respect to $\lambda^{2/3}\phi$ are given by

$$\begin{aligned} V_1'(\lambda^{2/3}\phi) &= \pi^{1/2}|\lambda|^{1/6}e^{-i\frac{\omega}{\epsilon}\tau_T}e^{-i5\pi/12}\{Ai'(\lambda^{2/3}\phi)+iBi'(\lambda^{2/3}\phi)\} \\ &= 2\pi^{1/2}|\lambda|^{1/6}e^{-i\frac{\omega}{\epsilon}\tau_T}e^{-i3\pi/4}Ai'(\lambda^{2/3}\phi e^{-i2\pi/3}), \qquad (3.42) \\ V_2'(\lambda^{2/3}\phi) &= -2\pi^{1/2}|\lambda|^{1/6}e^{i\frac{\omega}{\epsilon}\tau_T}e^{i\pi/12}Ai'(\lambda^{2/3}\phi), \qquad (3.43) \end{aligned}$$

leading to the identity (from (3.39))

$$V_2' = e^{i\pi/3}(V_1')^*. \qquad (3.44)$$

Since the identity $\lambda^{-1/n} = e^{-i\pi/n}(\lambda^{-1/n})^*$ holds for $n = 1, 2, \ldots$ , (3.40) and (3.44) imply that $\phi_{22} = \phi_{21}^*$. ■

Due to Lemma 3.3.1, one can simplify system (3.22); matrices (3.26) and (3.27) become the following:

$$\frac{i\omega}{\epsilon}\Phi^{-1}\mathcal{A}_1\Phi = \frac{i\omega}{2\epsilon}\begin{bmatrix} \alpha_1|\phi_{21}|^2+\beta_1|\phi_{11}|^2 & \alpha_1(\phi_{21}^*)^2-\beta_1(\phi_{11}^*)^2 \\ -\alpha_1\phi_{21}^2+\beta_1\phi_{11}^2 & -\alpha_1|\phi_{21}|^2-\beta_1|\phi_{11}|^2 \end{bmatrix}, \qquad (3.45)$$

$$\Phi^{-1}(\frac{i\omega}{\epsilon}\mathcal{A}_0\Phi - \frac{d}{dz}\Phi) = \frac{\epsilon}{2\omega}B_0(\alpha' B_0/2\alpha + B_0')\begin{bmatrix} i|V_1|^2 & -i(V_1^*)^2 \\ iV_1^2 & -i|V_1|^2 \end{bmatrix}. \qquad (3.46)$$

Both (3.45) and (3.46) (and thus the coefficient matrix on the right-hand side of (3.22)) have the following common structure that generates fundamental matrices belonging to $SU(1,1)$ [23]:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{12}^* & a_{11}^* \end{bmatrix}, \quad a_{11}^* = -a_{11}. \qquad (3.47)$$

It means that the Lynn and Keller approximant preserves the group structure of the fundamental matrices for the exact problem.

With the help of Lemma 3.3.1 and the above observation, one can prove the invariance of the time average energy flux (see [20] for definition) for a time-harmonic wave.

**Lemma 3.3.2** *The time average energy flux for a time-harmonic wave, defined by* $\bar{\mathbf{I}} = \frac{1}{2}Re(PU_3^*)$, *is same as* $\frac{1}{2}(|A|^2 - |B|^2)$ *and it is invariant.*

**Proof**: From (3.21) and Lemma 3.3.1 and $det\Phi = 2$, one can obtain

$$A = \frac{1}{2}(\phi_{21}^* P + \phi_{11}^* U_3), \quad B = \frac{1}{2}(-\phi_{21} P + \phi_{11} U_3) \tag{3.48}$$

so that

$$AA^* - BB^* = \frac{1}{4}(\phi_{21}^*\phi_{11} + \phi_{21}\phi_{11}^*)(PU_3^* + P^*U_3) = \frac{1}{2}\,det\Phi\; Re(PU_3^*) = Re(PU_3^*). \tag{3.49}$$

Using matrix structure (3.47), system (3.22) can be represented as

$$\frac{d}{dz}A = a_{11}A + a_{12}B, \quad \frac{d}{dz}B = a_{12}^* A + a_{11}^* B \tag{3.50}$$

which lead to the identity $\frac{d}{dz}(AA^* - BB^*) = 0$ by $a_{11}^* = -a_{11}$. ■

This lemma leads to the fact that $|A| = |B|$ everywhere in the random slab since total reflection at $z = -L$ is assumed in our problem of interest.

## 3.4 Transition Region Scale above the Turning Point

Throughout this chapter, only the region above the turning point, i.e., $z_T \le z \le 0$, is considered. The problem in the region below the turning point will be discussed later in Chapter 5. In this section, we stretch the region $z_T \le z \le 0$ by defining a new argument and identify the growth behavior of the coefficient matrix of our system using asymptotic expansions.

Let

$$\eta \equiv (\frac{\omega}{\epsilon})^{2/3}\phi = \{\frac{3\omega}{2\epsilon}(\tau_T - \tau(z))\}^{2/3}. \tag{3.51}$$

Note that $\phi$ is positive above the turning point $z_T$ and therefore so is $\eta$. Then, in terms of this new variable $\eta$, system (3.22) becomes

$$\frac{d}{d\eta}\begin{bmatrix} A \\ B \end{bmatrix} = B_0^2\,\{i(\frac{\omega}{\epsilon})^{1/3}\Phi^{-1}\mathcal{A}_1\Phi + (\frac{\omega}{\epsilon})^{-2/3}\Phi^{-1}(i\frac{\omega}{\epsilon}\mathcal{A}_0\Phi - \frac{d}{dz}\Phi)\}\begin{bmatrix} A \\ B \end{bmatrix}, \tag{3.52}$$

$$0 \le \eta \le \eta_0 \equiv (\frac{3\omega}{2\epsilon}\tau_T)^{2/3} \sim O(\epsilon^{-2/3}), \tag{3.53}$$

$$B(\eta_0) = 1, \qquad A(0) = \Gamma_{z_T} B(0), \tag{3.54}$$

where the reflection coefficient $\Gamma_{z_T}$ at the turning point $z_T$ is assumed for the present discussion to be known. The relation $d\eta = (\frac{\omega}{\epsilon})^{2/3} B_0^{-2} dz$ was used here. The corresponding functions $V_1$, $\phi_{11}$ and $\phi_{21}$ are written as follows in terms of new argument: Using the complex conjugate of (3.36)-(3.37)and $Ai^*(z) = Ai(z^*)$, one obtains

$$V_1 = \pi^{1/2}(\frac{\omega}{\epsilon})^{1/6} e^{-i(\frac{\omega}{\epsilon}\tau_T - \frac{\pi}{4})}\{Ai(-\eta) - iBi(-\eta)\}, \tag{3.55}$$

$$\phi_{11} = \pi^{1/2}(\frac{\omega}{\epsilon})^{1/6} \alpha^{1/2} B_0 e^{-i(\frac{\omega}{\epsilon}\tau_T - \frac{\pi}{4})}\{Ai(-\eta) - iBi(-\eta)\}, \tag{3.56}$$

$$\begin{aligned}\phi_{21} &= \pi^{1/2}(\frac{\omega}{\epsilon})^{-5/6} \alpha^{-1/2}(\alpha' B_0/2\alpha + B_0') e^{-i(\frac{\omega}{\epsilon}\tau_T + \frac{\pi}{4})}\{Ai(-\eta) - iBi(-\eta)\} \\ &\quad -\pi^{1/2}(\frac{\omega}{\epsilon})^{-1/6} \alpha^{-1/2} B_0^{-1} e^{-i(\frac{\omega}{\epsilon}\tau_T + \frac{\pi}{4})}\{Ai'(-\eta) - iBi'(-\eta)\}. \end{aligned} \tag{3.57}$$

Also we note that the correlation length in the $\eta$-scale becomes $\epsilon^{4/3}$ instead of $\epsilon^2$ in the $z$-scale. Let us introduce here the terminology "transition region" and "outer region"; the former is a region in distance of order $O(\epsilon^{2/3})$ from the turning point $z_T$ while the latter is the rest of the transition region. If one uses the following asymptotic expansions of Airy functions and their first derivatives, then one can identify the growth or decay behavior of components of exact transformation (3.52) as they reach the outer region after starting at the turning point $z_T$:

$$Ai(-\eta) = +\pi^{-1/2}\eta^{-1/4}\{\sin(\zeta + \pi/4) - \frac{5}{72}\zeta^{-1}\cos(\zeta + \pi/4) + O(\zeta^{-2})\}, \tag{3.58}$$

$$Bi(-\eta) = +\pi^{-1/2}\eta^{-1/4}\{\cos(\zeta + \pi/4) + \frac{5}{72}\zeta^{-1}\sin(\zeta + \pi/4) + O(\zeta^{-2})\}, \tag{3.59}$$

$$Ai'(-\eta) = -\pi^{-1/2}\eta^{+1/4}\{\cos(\zeta + \pi/4) - \frac{5}{72}\zeta^{-1}\sin(\zeta + \pi/4) + O(\zeta^{-2})\}, \tag{3.60}$$

$$Bi'(-\eta) = +\pi^{-1/2}\eta^{+1/4}\{\sin(\zeta + \pi/4) + \frac{5}{72}\zeta^{-1}\cos(\zeta + \pi/4) + O(\zeta^{-2})\}. \tag{3.61}$$

$$\left(|arg\ \eta| < \frac{2}{3}\pi, \quad \zeta \equiv \frac{2}{3}\eta^{3/2}\right)$$

Now, based on above expansions (3.58)-(3.61), one can arrange the coefficient matrix on the right-hand side of (3.52) into the following four types of terms according to the "size" of growth:

$$\frac{d}{d\eta}\begin{bmatrix} A \\ B \end{bmatrix} = \tag{3.62}$$

$$\Big\{ \frac{i}{2}(\frac{\omega}{\epsilon})^{2/3}\pi\alpha B_0^4\beta_1 \begin{bmatrix} Ai^2(-\eta)+Bi^2(-\eta) & -e^{2i(\frac{\omega}{\epsilon}\tau_T-\frac{\pi}{4})}\{Ai(-\eta)+iBi(-\eta)\}^2 \\ e^{-2i(\frac{\omega}{\epsilon}\tau_T-\frac{\pi}{4})}\{Ai(-\eta)-iBi(-\eta)\}^2 & -Ai^2(-\eta)-Bi^2(-\eta) \end{bmatrix}$$

$$+\frac{i}{2}\pi\alpha^{-1}\alpha_1 \begin{bmatrix} (Ai^{'}(-\eta))^2+(Bi^{'}(-\eta))^2 & e^{2i(\frac{\omega}{\epsilon}\tau_T+\frac{\pi}{4})}\{Ai^{'}(-\eta)+iBi^{'}(-\eta)\}^2 \\ -e^{-2i(\frac{\omega}{\epsilon}\tau_T+\frac{\pi}{4})}\{Ai^{'}(-\eta)-iBi^{'}(-\eta)\}^2 & -(Ai^{'}(-\eta))^2-(Bi^{'}(-\eta))^2 \end{bmatrix}$$

$$+\frac{i}{2}(\frac{\omega}{\epsilon})^{1/3}B_0^2\alpha_1 \begin{bmatrix} \psi_{11} & \psi_{21}^* \\ -\psi_{21} & -\psi_{11} \end{bmatrix}$$

$$+\frac{i}{2}(\frac{\omega}{\epsilon})^{-5/3}B_0^3(\alpha^{'}B_0/2\alpha+B_0^{'}) \begin{bmatrix} |V_1|^2 & -(V_1^*)^2 \\ V_1^2 & -|V_1|^2 \end{bmatrix} \Big\} \begin{bmatrix} A \\ B \end{bmatrix},$$

where $\psi_{11}$ and $\psi_{21}$ are defined to be

$$\psi_{11} = |\phi_\epsilon|^2 - 2\pi(\frac{\omega}{\epsilon})^{-1}(\alpha B_0)^{-1}(\alpha^{'}B_0/2\alpha+B_0^{'})\{Ai(-\eta)Ai^{'}(-\eta)+Bi(-\eta)Bi^{'}(-\eta)\}, \quad (3.63)$$

$$\psi_{21} = \phi_\epsilon^2 - 2\pi(\frac{\omega}{\epsilon})^{-1}(\alpha B_0)^{-1}(\alpha^{'}B_0/2\alpha+B_0^{'})e^{-2i(\frac{\omega}{\epsilon}\tau_T+\frac{\pi}{4})}\{Ai(-\eta)-iBi(-\eta)\}\cdot \\ \cdot\{Ai^{'}(-\eta)-iBi^{'}(-\eta)\}. \quad (3.64)$$

$$\Big(\phi_\epsilon \equiv \pi^{1/2}(\frac{\omega}{\epsilon})^{-5/6}\alpha^{-1/2}(\alpha^{'}B_0/2\alpha+B_0^{'})e^{-i(\frac{\omega}{\epsilon}\tau_T+\frac{\pi}{4})}\{Ai(-\eta)-iBi(-\eta)\}\Big)$$

The major contribution in the diffusion limit comes from the first two terms on the right hand side of (3.62); the first term is a dominant term in both the transition region and the outer region, while the second term will not contribute on a finite $\eta$-interval but it will become as significant in the outer region as the first term. In the next section, we are going to show that the third and fourth terms are "negligible" uniformly, i.e., when one rescales back to $z$. Therefore, we will need to consider only the first two terms in the diffusion limit.

## 3.5 Negligible Terms of the Exact Transformation

In the following lemmas, we will justify that we can neglect the last two terms of exact transformation (3.62) in the diffusion limit. The proofs of the lemmas basically use the asymptotic expansions of the Airy functions and their first derivatives. We note that

$$Ai, Bi, Ai^{'}, Bi^{'} \sim O(1), \quad 0 \le \eta \le \eta_1, \quad (3.65)$$

$$Ai, Bi \sim O(\eta^{-1/4}), \quad Ai', Bi' \sim O(\eta^{1/4}), \quad \eta_1 \leq \eta \leq \eta_0, \tag{3.66}$$

for any finite number $\eta_1$ of order $O(1)$ between 0 and $\eta_0$.

**Lemma 3.5.1** *The third term (i.e., the third coefficient matrix) of exact transformation (3.62) has the form of a zero mean term times an order $O(1)$ term uniformly on $z_T \leq z \leq 0$.*

**Proof**: From asymptotic expansions (3.58) and (3.59) of Airy functions $A_i$ and $B_i$, we have that $\phi_\epsilon \sim \epsilon^{5/6}O(\eta^{-1/4})$. Then, using asymptotic expansions (3.58)-(3.61), the orders of $\psi_{11}$ and $\psi_{21}$ will become

$$\psi_{11},\ \psi_{21} \sim \epsilon^{5/3}O(\eta^{-1/2}) + \epsilon\, O(1) \sim O(\epsilon).$$

Thus the third term on the right-hand side of (3.62) is $\alpha_1$ times an $O(\epsilon^{2/3})$ term, which means that it has order $O(1)$ when one rescales back to $z$. Obviously its mean is zero because of a mean zero term $\alpha_1$ (cf. (3.14)). ■

The fourth term is not mean-zero but, as we will see in the next lemma, it is "small" enough to be ignored in the diffusion limit. Therefore, the second coefficient matix on the right-hand side of (3.22) turns out to be negligible.

**Lemma 3.5.2** *The fourth term of exact transformation (3.62) has an order $O(\epsilon^{2/3})$ uniformly on $z_T \leq z \leq 0$.*

**Proof**: From asymptotic expansions (3.58) and (3.59), $V_1$ has an order $\epsilon^{-1/6}O(\eta^{-1/4})$ and thus the fourth term of the exact transformation has an order $\epsilon^{4/3}O(\eta^{-1/2})$. Even when one rescales back to $z$, this one still becomes $O(\epsilon^{2/3})$ on $z_T \leq z \leq 0$. ■

We next will show that the first two mean-zero terms of the right-hand side of (3.22) are significant and comparable in the outer region. So we will retain these two terms in the diffusion limit.

**Lemma 3.5.3** *The first two terms of exact transformation (3.62) have the same order $O(\epsilon^{-1})$ uniformly in the outer region.*

**Proof**: From asymptotic expansions (3.58) and (3.59), the first term of the exact tranformation has an order $\epsilon^{-2/3}O(\eta^{-1/2})$. From asymptotic expansions (3.60) and (3.61) of the first derivatives of Airy functions, the second term of exact transformation (3.62) has an order $O(\eta^{1/2})$. In particular, in the outer region, both of them are of order $O(\epsilon^{-1/3})$. They become, therefore, of order $O(\epsilon^{-1})$ when one rescales back to $z$. ■

## 3.6 Stochastic Initial Value Problem on an Unbounded Interval

In this section, we recast our stochastic boundary value problem (3.52)-(3.54) without negligible terms as a stochastic initial value problem. Because of the total reflection (i.e., $|A| = |B|$) throughout the region (see Section 3.3), the problem reduces to a consideration of the phase of the unimodular reflection coefficient.

First let us introduce the following moduli and phases for the Airy functions and their first derivatives:

$$Ai(-\eta) + iBi(-\eta) \equiv M(\eta)\, e^{i\{\Omega(\eta)+\pi/4\}}, \tag{3.67}$$

$$Ai'(-\eta) + iBi'(-\eta) \equiv N(\eta)\, e^{i\{\Phi(\eta)+3\pi/4\}}. \tag{3.68}$$

From Abramowitz and Stegun [22], the asymptotic expansions of moduli ($M$ and $N$) and phases ($\Omega$ and $\Phi$) for large $\eta$ are given by

$$M^2(\eta) \sim \frac{1}{\pi}\eta^{-1/2} + O(\eta^{-7/2}), \tag{3.69}$$

$$N^2(\eta) \sim \frac{1}{\pi}\eta^{1/2} + O(\eta^{-5/2}), \tag{3.70}$$

$$\Omega(\eta),\ \Phi(\eta) \sim -\frac{2}{3}\eta^{3/2} + O(\eta^{-3/2}). \tag{3.71}$$

In terms of these moduli and phases for the Airy functions, (3.62) becomes the following system of equations with mean-zero right side after the small terms are neglected:

$$\frac{d}{d\eta}\begin{bmatrix} A \\ B \end{bmatrix} = \frac{i}{2}(\frac{\omega}{\epsilon})^{2/3}\pi\alpha B_0^4 M^2\beta_1 \begin{bmatrix} 1 & -e^{2i\{\frac{\omega}{\epsilon}\tau_T+\Omega\}} \\ e^{-2i\{\frac{\omega}{\epsilon}\tau_T+\Omega\}} & -1 \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix}$$

$$+\frac{i}{2}\pi\alpha^{-1}N^2\alpha_1\begin{bmatrix} 1 & e^{2i\{\frac{\omega}{\epsilon}\tau_T+\Phi\}} \\ -e^{-2i\{\frac{\omega}{\epsilon}\tau_T+\Phi\}} & -1 \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix}. \tag{3.72}$$

The linear two point stochastic boundary value problem for the scattering variables $A$ and $B$ can be recast as a nonlinear stochastic initial value problem for the reflection coefficient. In Section 3.3, we noted that the reflection coefficient is unimodular in the random slab. The problem reduces to a consideration of the phase of the unimodular reflection coefficient. Let

$$\frac{A}{B} \equiv e^{-i\psi}. \tag{3.73}$$

Then system (3.72) for the scattering variables $A$ and $B$ leads to the following Riccati differential equation for the phase $\psi$ of the reflection coefficient as follows:

$$\frac{d\psi}{d\eta} = \frac{1}{\epsilon^{2/3}}\mathcal{F}_1(\eta, \frac{\eta}{\epsilon^{4/3}}, \psi) + \mathcal{F}_2(\eta, \frac{\eta}{\epsilon^{4/3}}, \psi), \quad 0 \le \eta \le \eta_0 \sim O(\epsilon^{-2/3}), \tag{3.74}$$

$$\psi(0) = \psi_0, \tag{3.75}$$

where $e^{-i\psi_0} = \Gamma_{z_T}$, i.e., $\psi_0$ is the phase of the reflection coefficient at the turning point $z_T$ and it is considered known for present discussion and the random fields $\mathcal{F}_1$ and $\mathcal{F}_2$ are given by

$$\mathcal{F}_1(\eta, \frac{\eta}{\epsilon^{4/3}}, \psi) = \omega^{2/3}\pi\alpha B_0^4\, M^2(\eta)\beta_1(\frac{\eta}{\epsilon^{4/3}})\,\{1 - \cos(\psi + 2(\frac{\omega}{\epsilon}\tau_T + \Omega(\eta)))\}, \tag{3.76}$$

$$\mathcal{F}_2(\eta, \frac{\eta}{\epsilon^{4/3}}, \psi) = \pi\alpha^{-1}N^2(\eta)\alpha_1(\frac{\eta}{\epsilon^{4/3}})\,\{1 + \cos(\psi + 2(\frac{\omega}{\epsilon}\tau_T + \Phi(\eta)))\}. \tag{3.77}$$

Here, $\beta_1(\frac{\eta}{\epsilon^{4/3}})$ (for example) is actually

$$\beta_1(\frac{\eta}{\epsilon^{4/3}}) = \beta_1(\phi^{-1}((\frac{\epsilon}{\omega})^{2/3}\eta), \frac{\phi^{-1}((\frac{\epsilon}{\omega})^{2/3}\eta)}{\epsilon^2}) \tag{3.78}$$

corresponding to $\beta_1(z, \frac{z}{\epsilon^2})$, where $\phi^{-1}$ denotes the inverse function of $\phi$.

From the above stochastic initial value problem, a number of facts should be noted. Riccati equation (3.74) is expressed in terms of the stretched variable $\eta$ (transition region scale); $\epsilon^{2/3}\eta$ and $\frac{\eta}{\epsilon^{4/3}}$ correspond to $z$ (or $\phi(z)$) and $\frac{z}{\epsilon^2}$ in the $z$-scale, respectively. The deterministic functions $\alpha$, $\alpha^{-1}$ and $B_0$ depend on only a slow-variable $\epsilon^{2/3}\eta$ and thus they behave like constants in the $\eta$-scale. Both random fields $\mathcal{F}_1$ and $\mathcal{F}_2$ have the form of a mean-zero nonstationary random function, either $\alpha_1$ or $\beta_1$, times deterministic functions. In short, stochastic initial value problem (3.74)-(3.77) has the form of model problem (1.3)-(1.4).

The interval $0 \leq \eta \leq \eta_0$ considered in our case, however, is not a bounded interval; it is order $O(\epsilon^{-2/3})$. Thus the mean-zero random field $\mathcal{F}_2$ must be retained (not like the case of previously known theory). It will contribute in the diffusion limit since it is significant in the outer region (i.e., $N^2(\eta) \sim \eta^{1/2}$ if $\eta$ is large). Hence it requires an extension of limit theorems for stochastic differential equations to understand the diffusion effects caused by random multiple scattering in the region above the turning point. This work will be performed in the following two chapters.

Usual ordinary differential equation existence and uniqueness theory tells us that the solutions of (3.74)-(3.77) are well-defined stochastic processes. Our goal is to characterize quantities of interest (i.e., the mean and correlation function of the reflected pressure) as expectations of functions of solutions of stochastic differential equation (3.74)-(3.77), using the extended limit theorem that will be described in the next chapter.

# Chapter 4

# EXTENDED LIMIT THEOREM AND ITS APPLICATIONS

## 4.1 An Extended Limit Theorem in Banach Space – Theorem 1

In the previous chapter, we have formulated a model of a stochastic initial value problem to which the result of Khasminskii [6] can be used in order to characterize the limiting stochastic behavior as $\epsilon \to 0$. Our problem, however, requires an extention of this idea, because the former result deals with only an $O(1)$ interval, not our $O(\epsilon^{-2/3})$ interval. In this section, we state in general terms a result about the extension of the limit theorem to an unbounded interval. In the next section, we apply the extended limit theorem to the stochastic model established in Chapter 3.

We first establish notation and hypotheses before stating our theorem. Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\mathcal{F}_s^t$, $0 \le s \le t \le \infty$, be a family of $\sigma$-algebra contained in $\mathcal{F}$ such that

$$\mathcal{F}_{s_1}^{t_1} \subset \mathcal{F}_{s_2}^{t_2}, \quad 0 \le s_2 \le s_1 \le t_1 \le t_2 \le \infty. \tag{4.1}$$

We assume the strong mixing condition in the sense that

$$\sup_{s \ge 0} \sup_{A \in \mathcal{F}_{s+t}^{\infty}, B \in \mathcal{F}_0^s} |P(A|B) - P(A)| \equiv \rho(t) \downarrow 0, \quad \text{as t} \uparrow \infty. \tag{4.2}$$

The monotonically decreasing nonnegative function $\rho$ is called the mixing rate which is assumed to satisfy the following rate condition:

$$\int_0^\infty \rho^{1/2}(s)\, ds < \infty. \tag{4.3}$$

For example, ergodic Markov processes on a compact state space are mixing processes with an exponential mixing rate. Note that decreasing monotonicity (4.2) and rate condition (4.3) imply that $\frac{1}{\epsilon}\rho(\frac{1}{\epsilon})$ is uniformly bounded in $\epsilon \in (0,1]$ and $\int_0^\infty \rho(s)ds < \infty$ and $\int_0^\infty s\rho(s)ds < \infty$. The conditional probabilities relative to $\mathcal{F}_0^s$, $0 \le s \le \infty$, are assumed to have a regular version so that we are able to have the following representation almost everywhere:

$$E\{\cdot|\mathcal{F}_0^s\} = \int_\Omega \cdot\, P_s(d\omega|\omega'). \tag{4.4}$$

Let $F(\eta,\nu,x,\omega)$ be a function from $[0,\infty)\times[0,\infty)\times\mathcal{R}\times\Omega$ into $\mathcal{R}$, where $\mathcal{R}$ denotes the set of real numbers. The random field $F$ is assumed to be jointly measurable with respect to its arguments and, for fixed $\eta$, $\nu$ and $x$, $F(\eta,\nu,x,\omega)$ is $\mathcal{F}_\nu^\nu$ measurable as a function of $\omega \in \Omega$.

We introduce the one-point compactification of $\mathcal{R}$, denoted by $\mathcal{R}_c$, and $\mathcal{C}^0$ denotes the space of bounded continuous real valued functions on $\mathcal{R}_c$ with the supremum norm $||\cdot||$. Let $\mathcal{C}^k$ denote the space of real valued functions on $\mathcal{R}_c$ with bounded continuous derivatives up to order $k$ with norm $||\cdot||_k$ the sum of the supremum norm of the function and its derivatives up to order $k$. Then the spaces $\mathcal{C}^k$ are separable Banach spaces and dense subspaces of $\mathcal{C}^0$ such that $\mathcal{C}^k \subset \mathcal{C}^{k-1}$ and $||f||_{k-1} \le ||f||_k$, $f \in \mathcal{C}^k$.

With a positive parameter $\epsilon$, we consider the following stochastic initial value problem on an $O(\epsilon^{-1})$ interval which is a general form of our model problem (3.74)-(3.77):

$$\frac{d}{d\tau}x^\epsilon(\tau,\sigma,x) = \frac{1}{\epsilon}F(\tau,\tau/\epsilon^2,x^\epsilon(\tau,\sigma,x),\omega), \quad 0 \le \sigma \le \tau \le \eta_0 \sim O(\epsilon^{-1}), \tag{4.5}$$

$$x^\epsilon(\sigma,\sigma,x) = x, \tag{4.6}$$

where the solution $x^\epsilon(\tau,\sigma,x)$ is $\mathcal{F}_{\sigma/\epsilon^2}^{\tau/\epsilon^2}$ measurable as a function of $\omega$ for any fixed $x$.

To develop a limit theory for our problem of interest, it is convenient to introduce solution operators $U^\epsilon(\sigma,\tau)$, called random propagators, associated with (4.5)-(4.6); we define $U^\epsilon(\sigma,\tau)$ by

$$(U^\epsilon(\sigma,\tau)f)(x) \equiv f(x^\epsilon(\tau,\sigma,x)), \tag{4.7}$$

for arbitrary $f \in \mathcal{C}^0$. These are contraction operators on $\mathcal{C}^0 \to \mathcal{C}^0$. The function $U^\epsilon(\sigma,\tau)f$, $f \in \mathcal{C}^0$, is strongly $\mathcal{F}_{\sigma/\epsilon^2}^{\tau/\epsilon^2}$ measurable. It is also useful to define the following random differential operator $V(\tau)$:

$$(V(\tau)f)(x) \equiv F(\tau, \tau/\epsilon^2, x)\partial_x f(x), \tag{4.8}$$

for arbitrary $f \in \mathcal{C}^k$, $k \geq 1$. Note that the $\epsilon$ dependence of $V(\tau)$ is not explicitly expressed. For each $f \in \mathcal{C}^k$, $V(\tau)f$ is strongly $\mathcal{F}_{\tau/\epsilon^2}^{\tau/\epsilon^2}$ measurable.

Let the interval $0 \leq \tau \leq \eta_0 \sim O(\epsilon^{-1})$ be covered by $O(1)$ intervals denoted by $[\sigma_{n-1}, \sigma_n]$, $n = 1, 2, ..., m_0 \sim O(\epsilon^{-1})$, i.e.,

$$[0, \eta_0] = \bigcup_{n=1}^{m_0} [\sigma_{n-1}, \sigma_n] \tag{4.9}$$

such that $0 = \sigma_0 < \sigma_1 < \cdots < \sigma_n < \cdots < \sigma_{m_0} = \eta_0$.

We adopt, throughout, the convention that, for each fixed $n$, $\epsilon$ dependent constant $C_{n,\epsilon}$ introduced in the following theorem is not necessarily the same constant; constant multiples of $C_{n,\epsilon}$ will again be denoted by $C_{n,\epsilon}$.

Now we are ready to state one of our main results. The proof of the following theorem will be given in the next chapter.

**Theorem 1** *Let $U^\epsilon(\sigma,\tau)$ and $V(\tau)$, $0 \leq \sigma \leq \tau \leq \eta_0 \equiv \tau_0/\epsilon \sim O(\epsilon^{-1})$, be the operators, respectively, defined by (4.7) and (4.8) corresponding to stochastic initial value problem (4.5)-(4.6). Suppose the strong mixing condition (4.2)-(4.3) and the hypotheses stated above are satisfied. Let us assume the following conditions (i)-(iv) hold:*
*(i) For $f \in \mathcal{C}^1$,*

$$E\{V(\tau)f\} = 0. \tag{4.10}$$

*(ii) There are positive constants $\alpha_k$ and $\beta_k$ independent of $\sigma$, $\tau$ and $\epsilon$ such that for arbitrary $f \in \mathcal{C}^k$, $k = 1, 2$,*

$$\|U^\epsilon(\sigma,\tau)f\|_k \leq \beta_k\{1 + \frac{\tau-\sigma}{\epsilon} + \cdots + (\frac{\tau-\sigma}{\epsilon})^{k-1}\}e^{\alpha_k\frac{\tau-\sigma}{\epsilon}}\|f\|_k \quad a.e.. \tag{4.11}$$

*(iii) For each interval $[\sigma_{n-1}, \sigma_n]$, there is a positive constant $C_{n,\epsilon}$ such that*

$$\sup_{\sigma_{n-1} \leq \tau \leq \sigma_n} \|V(\tau)f\|_{k-1} \leq C_{n,\epsilon}\|f\|_k \quad a.e., \qquad f \in C^k,\ 1 \leq k \leq 4. \tag{4.12}$$

*The above constants $C_{n,\epsilon}$, $n = 1, 2, ..., m_0$, are uniformly bounded in $n$ by a constant independent of $\epsilon$ and satisfy the following decay condition*

$$\sum_{n=1}^{m_0} C_{n,\epsilon} \sim O(\epsilon^{-\gamma}), \tag{4.13}$$

*for some number $\gamma < 1$.*

*(iv) Let $(A(\sigma, \tau)f)(x)$ denote the solution $u(\sigma, \tau, x; f)$ of the backward Kolmogorov equation*

$$\frac{\partial}{\partial \sigma} u(\sigma, \tau, x) + \mathcal{L}_\sigma u(\sigma, \tau, x) = 0, \tag{4.14}$$

$$u(\tau, \tau, x) = f(x), \tag{4.15}$$

*with infinitesimal generator $\mathcal{L}_\sigma$ defined on $C^2 \to C^0$ by the strong limit in $C^0$*

$$\mathcal{L}_\sigma f \equiv \lim_{\epsilon \to 0} \int_0^{1/\epsilon} E\{V(\sigma)V(\sigma + \epsilon^2 t)f\}\, dt, \tag{4.16}$$

*which exists and satisfies the rate of approach*

$$\sup_{\sigma_{n-1} \leq \sigma \leq \sigma_n} \| \int_0^{1/\epsilon} E\{V(\sigma)V(\sigma + \epsilon^2 t)f\}\, dt - \mathcal{L}_\sigma f\|_0 \leq \epsilon C_{n,\epsilon}\|f\|_2, \tag{4.17}$$

*for all sufficiently small $\epsilon > 0$. The (averaged) backward propagators $A(\sigma, \tau)$ satisfy the following inequality for some positive constants $a_k$,*

$$\sup_{0 \leq \sigma \leq \tau \leq \eta_0} \|A(\sigma, \tau)f\|_k \leq a_k\|f\|_k, \qquad f \in C^k, \quad 1 \leq k \leq 4. \tag{4.18}$$

*Then, for arbitrary $f \in C^4$, we have the estimate*

$$\sup_{0 \leq \tau \leq \eta_0} \|E\{\, U^\epsilon(0, \tau)f\} - A(0, \tau)f\,\|_0 \ \leq\ \epsilon^{1-\gamma} C(f; \tau_0), \tag{4.19}$$

*where $C(f; \tau_0)$ is a positive constant depending on $f$ and its derivatives up to order 4 and $\tau_0$ (which comes from $\eta_0 \equiv \tau_0/\epsilon$) but independent of $\epsilon$.*

**Remark.** The known limit theorems so far have dealt with the random field of the form $\frac{1}{\epsilon}F = \frac{1}{\epsilon}F_1 + F_2$ for some mean-zero random fields $F_1$ and $F_2$ on a finite scaled interval. In this case, the random field $F_2$ can be ignored in the diffusion limit. On the interval of infinite scale considered here, however, $F_2$ may grow and become comparable to $\frac{1}{\epsilon}F_1$ as $\eta$ increases while $F_1$ remains controlled. Then $F_2$ can not be ignored anymore in the diffusion limit. This is the case in the stochastic model established in the previous chapter; the random fields $\mathcal{F}_1$ and $\mathcal{F}_2$ there (cf. (3.76) and (3.77)) correspond to $F_1$ and $F_2$, respectively. Unless such a problem is dealt with separately in two different regions, therefore, we need Theorem 1 for a uniform diffusion approximation.

## 4.2 Application to Wave Propagation above the Turning Point

Theorem 1 can be applied to the stochastic model in Chapter 3 for a uniform diffusion limit. In this section, we check the necessary hypotheses described in Theorem 1 for the model problem and obtain the expression of the corresponding infinitesimal generator.

First of all, the phase $\psi$ of the reflection coefficient in the model corresponds to the variable $x$ in Theorem 1. The small parameter $\epsilon^{2/3}$ corresponds to $\epsilon$ and the random fields (3.76) and (3.77) together correspond to the random field $F$;

$$F(\eta, \frac{\eta}{\epsilon^{4/3}}, \psi) = \mathcal{F}_1(\eta, \frac{\eta}{\epsilon^{4/3}}, \psi) + \epsilon^{2/3}\mathcal{F}_2(\eta, \frac{\eta}{\epsilon^{4/3}}, \psi). \tag{4.20}$$

Conditions (4.10)-(4.13) are related with the random field $F$. Since $\mathcal{F}_1$ and $\mathcal{F}_2$ are multiples of the mean-zero functions $\beta_1$ and $\alpha_1$, respectively (see (3.76) and (3.77)), the random field $F$ is centered so that (4.10) holds. The following quantity corresponds to the end point $\eta_0$:

$$\eta_0 = \left(\frac{3\omega}{2\epsilon}\tau_T\right)^{2/3}. \tag{4.21}$$

Let us take a fixed interval $\eta_1 \sim O(1)$ and define

$$\sigma_n \equiv n\eta_1, \quad n = 0, 1, 2, ..., m_0. \quad (\sigma_{m_0} \equiv \eta_0) \tag{4.22}$$

Then $m_0 \sim O(\epsilon^{-2/3})$ and obviously $[0,\eta_0] = \cup_{n=1}^{m_0}[\sigma_{n-1},\sigma_n]$. We assume $\alpha_1$ and $\beta_1$ to be bounded each by some constant; then (3.69)-(3.70) provide the following estimates on each interval $[\sigma_{n-1},\sigma_n]$: For $n = 2,3,...,m_0$, we have

$$|\mathcal{F}_1(\eta,\frac{\eta}{\epsilon^{4/3}},\psi)| \le c_1((n-1)\eta_1)^{-1/2}, \quad |\mathcal{F}_2(\eta,\frac{\eta}{\epsilon^{4/3}},\psi)| \le c_2(n\eta_1)^{1/2}, \tag{4.23}$$

where $c_1$ and $c_2$ are some positive constants. From the way that $\psi$ appears in the random fields $\mathcal{F}_1$ and $\mathcal{F}_2$, therefore, not only $F$ but also $\partial^k_{\psi..\psi}F$, $k \ge 1$, are bounded by

$$C_{n,\epsilon} \equiv c_1((n-1)\eta_1)^{-1/2} + c_2\epsilon^{2/3}(n\eta_1)^{1/2}, \tag{4.24}$$

on each interval $[\sigma_{n-1},\sigma_n]$ for $n = 2,3,...,m_0$. Since $\eta_1$ is finite, $C_{1,\epsilon}$ can be defined as a fixed positive number. In view of definition (4.8), therefore, (4.12) holds. Obviously $\{C_{n,\epsilon} : n = 1,2,...,m_0\}$ is bounded uniformly in $n$ by a constant since $m_0 \sim O(\epsilon^{-2/3})$. From stochastic initial value problem (4.5)-(4.6) (using $\psi$ instead of $x$),

$$\frac{\partial\psi^\epsilon}{\partial\psi}(\tau,\sigma,\psi) = e^{\frac{1}{\epsilon}\int_\sigma^\tau F'ds}, \tag{4.25}$$

$$\frac{\partial^2\psi^\epsilon}{\partial\psi^2}(\tau,\sigma,\psi) = e^{\frac{1}{\epsilon}\int_\sigma^\tau F'ds}\cdot\frac{1}{\epsilon}\int_\sigma^\tau F''\frac{\partial\psi^\epsilon}{\partial\psi}ds. \tag{4.26}$$

Since $\partial^n_{\psi..\psi}F(s,\frac{s}{\epsilon^{4/3}},\psi)$, $n \ge 0$, are each bounded by a constant independent of $s$, $\psi$ and $\epsilon$, there exist positive constants $\hat{\alpha}_n$ and $\hat{\beta}_n$ independent of $\tau$, $\sigma$, $\psi$ and $\epsilon$ such that from (4.25)-(4.26)

$$\left|\frac{\partial^n\psi^\epsilon}{\partial\psi^n}\right| \le \hat{\beta}_n\{\frac{\tau-\sigma}{\epsilon} + \cdots + (\frac{\tau-\sigma}{\epsilon})^{n-1}\}e^{\hat{\alpha}_n\frac{\tau-\sigma}{\epsilon}} \quad a.e., \qquad n = 1,2. \tag{4.27}$$

Using the chain rule for the derivatives of $U^\epsilon(\sigma,\tau)f$, one can obtain

$$\|\partial^n_{\psi..\psi}U^\epsilon(\sigma,\tau)f\|_0 \le \check{\beta}_n \sup_\psi \sum_{i_1+2i_2+\cdots+ni_n=n}\left|\frac{\partial\psi^\epsilon}{\partial\psi}\right|^{i_1}\left|\frac{\partial^2\psi^\epsilon}{\partial\psi^2}\right|^{i_2}\cdots\left|\frac{\partial^n\psi^\epsilon}{\partial\psi^n}\right|^{i_n}\|f\|_n$$
$$a.e., \quad n = 1,2, \quad (\psi^\epsilon \equiv \psi^\epsilon(\tau,\sigma,\psi)) \tag{4.28}$$

for some constant $\check{\beta}_n$ independent of $\tau$, $\sigma$ and $\epsilon$. We apply inequalities (4.27) to (4.28) and rearrange the result to obtain the right side of (4.28) as a form of the right side of

(4.11). Since $\|U^\epsilon(\sigma,\tau)f\|_k$ is simply the sum of $\|\partial^n_{\psi..\psi}U^\epsilon(\sigma,\tau)f\|_0$, $0 \le n \le k$, (4.11) can be obtained for some suitably chosen constants $\alpha_k$ and $\beta_k$.

Let us check now whether the $C_{n,\epsilon}$ satisfy the decay condition (4.13) required as one of the hypotheses of Theorem 1: From (4.24),

$$\begin{aligned}\sum_{n=2}^{m_0} C_{n,\epsilon} &= c_1\,\eta_1^{-1/2}\sum_{n=2}^{m_0}(n-1)^{-1/2} + c_2\,\epsilon^{2/3}\eta_1^{1/2}\sum_{n=2}^{m_0} n^{1/2}\\ &\le c_1\,\eta_1^{-1/2}(1+\int_2^{m_0}(s-1)^{-1/2}ds) + c_2\,\epsilon^{2/3}\eta_1^{1/2}\int_0^{m_0+1}s^{1/2}ds\\ &\le 2c_1\,\eta_1^{-1/2}(m_0-1)^{1/2} + \frac{2}{3}c_2\,\epsilon^{2/3}\eta_1^{1/2}(m_0+1)^{3/2}. \end{aligned}\tag{4.29}$$

Since $m_0 \sim O(\epsilon^{-2/3})$, therefore, the following estimate for the decay condition (4.13) corresponding to $\gamma = 1/2$ is obtained:

$$\sum_{n=1}^{m_0} C_{n,\epsilon} \sim O(\epsilon^{-1/3}). \tag{4.30}$$

As is shown in (4.31), the diffusion and drift coefficients of our generator $\mathcal{L}_\eta$ behave like an $O(1)$ term times $\eta^{-1}$ when $\eta$ is large (cf. (3.69)-(3.70)); thus the corresponding $A(\sigma,\tau)f$ satisfies condition (4.18) on our $O(\epsilon^{-2/3})$ interval as the solution of parabolic differential equation (4.14)-(4.15) (cf. Lemma 4 in [7] or [24]-[25]).

We assume that the random functions $\alpha_1$ and $\beta_1$ satisfy the strong mixing condition (4.2) with the appropriate rate (4.3). Also these random functions are assumed to satisfy the existence of limits (4.32)-(4.35) and the rate of approach (4.17).

Therefore, every hypothesis of Theorem 1 is fulfilled; the limiting behavior of stochastic initial value problem (3.74)-(3.77) can be characterized by Theorem 1. According to Theorem 1, expectations of functions of solutions of stochastic initial value problem (3.74)-(3.77) can be approximated by the solution of backward Kolmogorov equation (4.14)-(4.15) with generator given by (4.16).

We present here explicitly the single-frequency generator of the corresponding diffusion process for the stochastic model in Chapter 3. If one substitutes the operator $V(\tau)$ with the random field given by (4.20) into (4.16), a direct calculation gives the following uniform expression for the infinitesimal generator:

$$\mathcal{L}_\eta f = \Big[ \rho_{\beta\beta}\omega^{4/3}\pi^2\alpha^2 B_0^8 M^4(\eta)\left\{1 - \cos(\psi + 2(\frac{\omega}{\epsilon}\tau_T + \Omega(\eta)))\right\}^2 \tag{4.31}$$

$$+\epsilon^{2/3}\rho_{\beta\alpha}\omega^{2/3}\pi^2 B_0^4 M^2(\eta)N^2(\eta)\left\{1-\cos(\psi+2(\frac{\omega}{\epsilon}\tau_T+\Omega(\eta)))\right\}\left\{1+\cos(\psi+2(\frac{\omega}{\epsilon}\tau_T+\Phi(\eta)))\right\}$$

$$+\epsilon^{2/3}\rho_{\alpha\beta}\omega^{2/3}\pi^2 B_0^4 M^2(\eta)N^2(\eta)\left\{1-\cos(\psi+2(\frac{\omega}{\epsilon}\tau_T+\Omega(\eta)))\right\}\left\{1+\cos(\psi+2(\frac{\omega}{\epsilon}\tau_T+\Phi(\eta)))\right\}$$

$$+\ \epsilon^{4/3}\rho_{\alpha\alpha}\pi^2\alpha^{-2}N^4(\eta)\left\{1 + \cos(\psi + 2(\frac{\omega}{\epsilon}\tau_T + \Phi(\eta)))\right\}^2\Big]\ \frac{\partial^2 f}{\partial\psi^2}$$

$$+\Big[\rho_{\beta\beta}\omega^{4/3}\pi^2\alpha^2 B_0^8 M^4(\eta)\left\{1 - \cos(\psi + 2(\frac{\omega}{\epsilon}\tau_T + \Omega(\eta)))\right\}\ \sin(\psi + 2(\frac{\omega}{\epsilon}\tau_T + \Omega(\eta)))$$

$$-\epsilon^{2/3}\rho_{\beta\alpha}\omega^{2/3}\pi^2 B_0^4 M^2(\eta)N^2(\eta)\left\{1 - \cos(\psi + 2(\frac{\omega}{\epsilon}\tau_T + \Omega(\eta)))\right\}\ \sin(\psi + 2(\frac{\omega}{\epsilon}\tau_T + \Phi(\eta)))$$

$$+\epsilon^{2/3}\rho_{\alpha\beta}\omega^{2/3}\pi^2 B_0^4 M^2(\eta)N^2(\eta)\ \sin(\psi + 2(\frac{\omega}{\epsilon}\tau_T + \Omega(\eta)))\left\{1 + \cos(\psi + 2(\frac{\omega}{\epsilon}\tau_T + \Phi(\eta)))\right\}$$

$$-\ \epsilon^{4/3}\rho_{\alpha\alpha}\pi^2\alpha^{-2}N^4(\eta)\ \sin(\psi + 2(\frac{\omega}{\epsilon}\tau_T + \Phi(\eta)))\left\{1 + \cos(\psi + 2(\frac{\omega}{\epsilon}\tau_T + \Phi(\eta)))\right\}\Big]\ \frac{\partial f}{\partial\psi},$$

where $\rho_{\alpha\alpha}$, $\rho_{\alpha\beta}$, $\rho_{\beta\alpha}$ and $\rho_{\beta\beta}$ are given by

$$\rho_{\alpha\alpha} = \lim_{\epsilon\to 0}\int_0^{\epsilon^{-2/3}} E\{\alpha_1(\frac{\eta}{\epsilon^{4/3}})\alpha_1(\frac{\eta}{\epsilon^{4/3}} + t)\}\, dt, \tag{4.32}$$

$$\rho_{\alpha\beta} = \lim_{\epsilon\to 0}\int_0^{\epsilon^{-2/3}} E\{\alpha_1(\frac{\eta}{\epsilon^{4/3}})\beta_1(\frac{\eta}{\epsilon^{4/3}} + t)\}\, dt, \tag{4.33}$$

$$\rho_{\beta\alpha} = \lim_{\epsilon\to 0}\int_0^{\epsilon^{-2/3}} E\{\beta_1(\frac{\eta}{\epsilon^{4/3}})\alpha_1(\frac{\eta}{\epsilon^{4/3}} + t)\}\, dt, \tag{4.34}$$

$$\rho_{\beta\beta} = \lim_{\epsilon\to 0}\int_0^{\epsilon^{-2/3}} E\{\beta_1(\frac{\eta}{\epsilon^{4/3}})\beta_1(\frac{\eta}{\epsilon^{4/3}} + t)\}\, dt. \tag{4.35}$$

The above generator (4.31) is uniformly valid in the region above the turning point (transition as well as outer regions). Note that there is a competition among diffusion (and drift) coefficients due to the asymptotic behavior of $M(\eta)$ and $N(\eta)$ (cf. (3.69)-(3.70)); the terms involving $\rho_{\beta\beta}$, $\rho_{\beta\alpha}$, $\rho_{\alpha\beta}$ and $\rho_{\alpha\alpha}$ are of orders $O(1)$, $O(\epsilon^{2/3})$, $O(\epsilon^{2/3})$ and $O(\epsilon^{4/3})$, respectively, in the transition region whereas all of these are of comparable $O(\epsilon^{2/3})$ order in the outer region. When one rescales back to $z$ by the change of variable $\eta = (\frac{\omega}{\epsilon})^{2/3}\phi(z)$, the orders (in the $z$-scale) of the diffusion and drift coefficients of the generator $\mathcal{L}_\eta$ are determined by dividing the orders in the $\eta$-scale by $\epsilon^{2/3}$; then one gets $O(1)$ diffusion equation in the outer region.

In (4.32)-(4.35), the limits are finite in view of the mixing rate condition (4.3). These quantities represent the "noise intensity"; $\epsilon^{2/3}\eta$ $(= \omega^{2/3}\phi(z))$ represents the (nonstationary) slow-scale ($z$) dependence even though it is not expressed explicitly (cf. (3.78)) while $\frac{\eta}{\epsilon^{4/3}}$ $(= \frac{\omega^{2/3}\phi(z)}{\epsilon^2})$ represents the fast-scale ($\frac{z}{\epsilon^2}$) dependence.

## 4.3 Application to Random Scattering at Low Grazing Angles

If the incidence angle $\theta$ is near grazing, i.e., if it is sufficiently close to a right angle, then the region above the turning point $z_T$ will become only transition region. Let us consider this particular case in this section. The Riccati equation and the corresponding generator for the stochastic initial value problem will have simpler forms.

The turning points of our stochastic problem are given by the zero eigenvalues of the matrix $\mathcal{A}_0$ that defines the effective medium (cf. (3.17)). From (3.18), these are solutions $z_T$ (as functions of $\theta$) of the following equation:

$$E\{\rho(z_T)\}E\{K^{-1}(z_T) - \rho^{-1}(z_T)c_0^{-2}\sin^2\theta\} = 0. \tag{4.36}$$

Knowing that the density is positive always in the medium, then, $\sin\theta$ is given by

$$\sin\theta = \left[\frac{E\{\rho^{-1}(z_T)\frac{c_0^2}{c^2(z_T)}\}}{E\{\rho^{-1}(z_T)\}}\right]^{1/2}. \tag{4.37}$$

Here $c_0 \leq c(z_T)$ always. Note that $\sin\theta \uparrow 1$ as $z_T \uparrow 0$. If $\theta$ is so close to a right angle that equation (4.37) holds for $z_T$ satisfing

$$z_T \sim O(\epsilon^{2/3}) \tag{4.38}$$

in the $z$-scale, then the region above the turning point $z_T$ will be only a transition region and the interval $\eta_0$ $(\equiv \tau_0/\epsilon^{2/3})$ is of order $O(1)$. In this case, the second term of the right side of system (3.72) will not contribute in the diffusion limit because it is significant only in the outer region. The relevant system for the scattering variables in the case of grazing angle incidence, therefore, is given by

$$\frac{d}{d\eta}\begin{bmatrix} A \\ B \end{bmatrix} = \frac{i}{2}(\frac{\omega}{\epsilon})^{2/3}\pi\alpha B_0^4 M^2(\eta)\beta_1 \begin{bmatrix} 1 & -e^{2i\{\frac{\omega}{\epsilon}\tau_T+\Omega(\eta)\}} \\ e^{-2i\{\frac{\omega}{\epsilon}\tau_T+\Omega(\eta)\}} & -1 \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix}, \quad (4.39)$$

and the corresponding Riccati differential equation for the phase of the refection coefficient is given by

$$\frac{d}{d\eta}\psi = \frac{1}{\epsilon^{2/3}}\mathcal{F}_1(\eta, \frac{\eta}{\epsilon^{4/3}}, \psi), \quad 0 \le \eta \le \eta_0 \sim O(1), \quad (4.40)$$

$$\psi(0) = \psi_0, \quad (4.41)$$

where the random field $\mathcal{F}_1$ is defined to be (3.76). The single-frequency generator also has the simpler form since there is no contribution from the terms involving $\rho_{\alpha\alpha}, \rho_{\alpha\beta}$ and $\rho_{\beta\alpha}$. In this case, we have

$$\mathcal{L}_\eta f = \rho_{\beta\beta}\omega^{4/3}\pi^2\alpha^2 B_0^8 M^4(\eta)\left\{1 - \cos(\psi + 2(\frac{\omega}{\epsilon}\tau_T + \Omega(\eta)))\right\}^2 \frac{\partial^2 f}{\partial\psi^2} \quad (4.42)$$

$$+ \rho_{\beta\beta}\omega^{4/3}\pi^2\alpha^2 B_0^8 M^4(\eta)\left\{1 - \cos(\psi + 2(\frac{\omega}{\epsilon}\tau_T + \Omega(\eta)))\right\} \sin(\psi + 2(\frac{\omega}{\epsilon}\tau_T + \Omega(\eta)))\frac{\partial f}{\partial\psi},$$

where the noise intensity $\rho_{\beta\beta}$ is defined by (4.35). Note that the diffusion and drift coefficients are both $O(1)$.

# Chapter 5

# PROOF OF THE EXTENDED LIMIT THEOREM

## 5.1 Propagator Formulas

The proof of Theorem 1 is based on a "bootstrapping" idea; we decompose the quantity to be estimated, i.e., the left side of (4.19), into many parts such that each part corresponds to the quantity on the $O(1)$ interval $[\sigma_{n-1}, \sigma_n]$. These estimates on each interval are then added up to obtain the desired estimate for the whole $O(\epsilon^{-1})$ interval. For the estimate of each part, a mixing lemma is going to be applied or a limit theorem on each interval $[\sigma_{n-1}, \sigma_n]$ in our framework is going to be derived and used.

In this section, we first derive some useful properties about the random propagators $U^\epsilon(\sigma, \tau)$ and the backward propagators $A(\sigma, \tau)$ defined in Section 4.1. These properties will be used for the appropriate decomposition of the left side of (4.19).

**Lemma 5.1.1** *The random propagators $U^\epsilon(\sigma, \tau)$ and the operator $V(\tau)$, defined by (4.7) and (4.8), respectively, satisfy the following properties:*
*(i) finite propagator property:*

$$U^\epsilon(\sigma, \eta)\, U^\epsilon(\eta, \tau) = U^\epsilon(\sigma, \tau), \tag{5.1}$$

$$U^\epsilon(\tau, \tau) = I. \tag{5.2}$$

*(ii) infinitesimal forward and backward propagator properties:*

$$U^\epsilon(\sigma, \tau) = I + \frac{1}{\epsilon} \int_\sigma^\tau U^\epsilon(\sigma, \eta) V(\eta)\, d\eta, \tag{5.3}$$

$$U^\epsilon(\sigma, \tau) = I + \frac{1}{\epsilon} \int_\sigma^\tau V(\eta) U^\epsilon(\eta, \tau)\, d\eta. \tag{5.4}$$

**Proof**: Let $x^\epsilon(\tau,\sigma,x)$ be the solution of stochastic initial value problem (4.5)-(4.6) and $f \in C^4$. Then from definition (4.7) of the propagators $U^\epsilon(\sigma,\tau)$

$$\begin{aligned}(U^\epsilon(\sigma,\eta)U^\epsilon(\eta,\tau)f)(x) &= f(x^\epsilon(\tau,\eta,x^\epsilon(\eta,\sigma,x)))\\ &= f(x^\epsilon(\tau,\sigma,x)) = (U^\epsilon(\sigma,\tau)f)(x),\end{aligned} \tag{5.5}$$

where the second equality holds because of the uniqueness of the solution of the differential equation. Thus (5.1) is proved. From (4.7),

$$(U^\epsilon(\tau,\tau)f)(x) = f(x^\epsilon(\tau,\tau,x)) = f(x). \tag{5.6}$$

Thus (5.2) is proved. For the proof of $(ii)$, if one differentiates the identity $(U^\epsilon(\sigma,\tau)f)(x)$ $= f(x^\epsilon(\tau,\sigma,x))$ with respect to $\tau$, then by using (4.7), (4.5), (4.8) and (4.7) consecutively one obtains

$$\begin{aligned}\partial_\tau(U^\epsilon(\sigma,\tau)f)(x) &= f'(x^\epsilon(\tau,\sigma,x))\,\epsilon^{-1}F(\tau,\frac{\tau}{\epsilon^2},x^\epsilon(\tau,\sigma,x))\\ &= \epsilon^{-1}(V(\tau)f)(x^\epsilon(\tau,\sigma,x)) = \epsilon^{-1}(U^\epsilon(\sigma,\tau)V(\tau)f)(x),\end{aligned} \tag{5.7}$$

which is equivalent to the integral form (5.3). For the proof of (5.4), let us differentiate the identity $U^\epsilon(\sigma,\tau) = U^\epsilon(\sigma,\eta)U^\epsilon(\eta,\tau)$ with respect to $\eta$:

$$\begin{aligned}0 &= (\partial_\eta U^\epsilon(\sigma,\eta))U^\epsilon(\eta,\tau) + U^\epsilon(\sigma,\eta)(\partial_\eta U^\epsilon(\eta,\tau))\\ &= \epsilon^{-1}U^\epsilon(\sigma,\eta)V(\eta)U^\epsilon(\eta,\tau) + U^\epsilon(\sigma,\eta)(\partial_\eta U^\epsilon(\eta,\tau)),\end{aligned} \tag{5.8}$$

where the product rule and (5.7) were used. Let $\eta \downarrow \sigma$. Then one obtains

$$\partial_\sigma U^\epsilon(\sigma,\tau) = -\epsilon^{-1}V(\sigma)U^\epsilon(\sigma,\tau), \tag{5.9}$$

which is equivalent to the integral form (5.4). ∎

**Lemma 5.1.2** *The backward propagators $A(\sigma,\tau)$ defined in Theorem 1 satisfy the following properties:*

$$\partial_\sigma A(\sigma,\tau) + \mathcal{L}_\sigma A(\sigma,\tau) = 0, \tag{5.10}$$

$$A(\sigma,\eta)A(\eta,\tau) = A(\sigma,\tau), \quad A(\sigma,\sigma) = I, \tag{5.11}$$

$$A(\sigma,\tau) = I + \int_\sigma^\tau \mathcal{L}_\eta A(\eta,\tau)d\eta. \tag{5.12}$$

**Proof**: Equation (5.10) comes from equation (4.14) directly. The uniqueness of the solution of final value problem (4.14)-(4.15) leads to (5.11). Equation (5.12) is just equivalent to the integral form of (5.10). ■

For the proof of Theorem 1, using Lemmas 5.1.1–5.1.2, we now decompose the left side of (4.19) into many terms on $O(1)$ intervals $[\sigma_{n-1}, \sigma_n]$ that can be estimated by a limit theorem on these intervals. Let us first decompose the interval $[0, \tau]$ into intervals $[\sigma_{n-1}, \sigma_n]$, $n = 1, \cdots, m$, (such that $0 = \sigma_0 < \sigma_1 < \cdots < \sigma_m = \tau$) which, without loss of generality, are the same as the intervals introduced in Section 4.1. Using a "telescope" argument and Lemmas 5.1.1–5.1.2, one obtains for arbitrary $f \in \mathcal{C}^4$

$$
\begin{aligned}
& E\{U^\epsilon(0,\tau)f\} - A(\sigma,\tau)f \\
&\quad = \sum_{n=1}^{m} E\{U^\epsilon(0,\sigma_n)A(\sigma_n,\tau)f - U^\epsilon(0,\sigma_{n-1})A(\sigma_{n-1},\tau)f\} \\
&\quad = \sum_{n=1}^{m} E\{U^\epsilon(0,\sigma_{n-1})[U^\epsilon(\sigma_{n-1},\sigma_n) - A(\sigma_{n-1},\sigma_n)]A(\sigma_n,\tau)f\} \\
&= \sum_{n=1}^{m} E\{U^\epsilon(0,\sigma_{n-1})[E\{U^\epsilon(\sigma_{n-1},\sigma_n)g_n\} - A(\sigma_{n-1},\sigma_n)g_n]\} \\
&\quad + \sum_{n=1}^{m} E\{U^\epsilon(0,\sigma_{n-1})[U^\epsilon(\sigma_{n-1},\sigma_n)g_n - E\{U^\epsilon(\sigma_{n-1},\sigma_n)g_n\}]\}, \qquad (5.13)
\end{aligned}
$$

where $g_n \equiv A(\sigma_n, \tau)f$ is a deterministic function which is in $\mathcal{C}^4$ and $\|g_n\|_4 \le c_4\|f\|_4$ because of condition (4.18). Since the propagators $U^\epsilon(0, \sigma_{n-1})$ are contraction operators on $\mathcal{C}^0 \to \mathcal{C}^0$, the $\|\cdot\|_0$ norm of the above (5.13) satisfies the following inequality:

$$
\begin{aligned}
& \|E\{U^\epsilon(0,\tau)f\} - A(0,\tau)f\|_0 \\
&\quad \le \sum_{n=1}^{m} \|E\{U^\epsilon(\sigma_{n-1},\sigma_n)g_n\} - A(\sigma_{n-1},\sigma_n)g_n\|_0 \\
&\quad + \sum_{n=1}^{m} \|E\{U^\epsilon(0,\sigma_{n-1})U^\epsilon(\sigma_{n-1},\sigma_n)g_n\} - E\{U^\epsilon(0,\sigma_{n-1})E\{U^\epsilon(\sigma_{n-1},\sigma_n)g_n\}\}\|_0 \\
&\quad \equiv \sum_{n=1}^{m} \left(I_n^1(f) + I_n^2(f)\right). \qquad (5.14)
\end{aligned}
$$

From now on, therefore, we are left to obtain the desired estimate for each term of the right side of (5.14). In the following two sections, we will show that

$$I_n^1(f) \le \epsilon C_{n,\epsilon} \|f\|_4, \tag{5.15}$$

$$I_n^2(f) \le \epsilon C_{n,\epsilon} \|f\|_1. \tag{5.16}$$

Once (5.15)-(5.16) are proved, then the estimate (4.19) of Theorem 1 follows from the decay condition (4.13) for $C_{n,\epsilon}$;

$$\begin{aligned} &\|\{U^\epsilon(0,\tau)f\} - A(0,\tau)f\|_0 \\ &\le \sum_{n=1}^{m} \epsilon C_{n,\epsilon}(\|f\|_4 + \|f\|_1) \le 2\epsilon\|f\|_4 \sum_{n=1}^{m} C_{n,\epsilon} \equiv \epsilon^{1-\gamma} C(f;\tau_0), \end{aligned} \tag{5.17}$$

where $C(f;\tau_0)$ is a positive constant depending on $f$ and its derivatives up to order 4 and $\tau_0$ but independent of $\epsilon$.

## 5.2 $\mathbf{I}_n^1$–estimation

In this section, we derive a lemma (Lemma 5.2.1) which corresponds to a limit theorem on each $O(1)$ interval $[\sigma_{n-1}, \sigma_n]$ in our context. Lemma 5.2.1 claims the desired estimate for $I_n^1(f)$, i.e., (5.15).

**Lemma 5.2.1** *Let $U^\epsilon(\sigma_{n-1}, \sigma_n)$ and $A(\sigma_{n-1}, \sigma_n)$, be the propagators defined in Section 4.1. Then, under the same hypotheses as described in Theorem 1, we have the estimate*

$$I_n^1(f) \equiv \|E\{U^\epsilon(\sigma_{n-1}, \sigma_n)g_n\} - A(\sigma_{n-1}, \sigma_n)g_n\|_0 \le \epsilon C_{n,\epsilon}\|f\|_4, \tag{5.18}$$

*where $g_n = A(\sigma_n, \tau)f \in C^4$.*

**Proof**: Let us first define the notation

$$\tau_k \equiv \sigma_{n-1} + k\epsilon, \quad k = 0, 1, ..., \tilde{m}. \quad (\tau_{\tilde{m}} \equiv \sigma_n) \tag{5.19}$$

Note that $\tau_k - \tau_{k-1} = \epsilon$ and $\tilde{m} \sim O(\epsilon^{-1})$. The $n$ (or $\sigma_{n-1}$) dependence of $\tau_k$ is not explicitly expressed. Again using a telescope argument, the finite propagator properties for $U^\epsilon(\sigma, \tau)$

and $A(\sigma,\tau)$, and the contraction property for $U^\epsilon(\sigma,\tau)$ on $\mathcal{C}^0$ as we did in Section 5.1, one obtains the inequality

$$\begin{aligned}
I_n^1(f) &\equiv \|E\{U^\epsilon(\sigma_{n-1},\sigma_n)g_n\} - A(\sigma_{n-1},\sigma_n)g_n\|_0 \\
&\leq \sum_{k=1}^{\tilde{m}} \|E\{U^\epsilon(\tau_{k-1},\tau_k)h_k\} - A(\tau_{k-1},\tau_k)h_k\|_0 \\
&+ \sum_{k=1}^{\tilde{m}} \|E\{U^\epsilon(\tau_0,\tau_{k-1})U^\epsilon(\tau_{k-1},\tau_k)h_k\} - E\{U^\epsilon(\tau_0,\tau_{k-1})E\{U^\epsilon(\tau_{k-1},\tau_k)h_k\}\}\|_0 \\
&\equiv \sum_{k=1}^{\tilde{m}} \left(I_{n,k}^{11}(f) + I_{n,k}^{12}(f)\right) \qquad (5.20)
\end{aligned}$$

where $h_k \equiv A(\tau_k,\sigma_n)g_n = A(\tau_k,\sigma_n)A(\sigma_n,\tau)f = A(\tau_k,\tau)f$ which is a deterministic function in $\mathcal{C}^4$ from condition (4.18).

Let us estimate $I_{n,k}^{11}(f)$ first. We use the backward and forward infinitesimal propagator properties (5.3)-(5.4) iterated four times to obtain the expansion

$$\begin{aligned}
&U^\epsilon(\tau_{k-1},\tau_k)h_k \\
&= h_k + \frac{1}{\epsilon}\int_{\tau_{k-1}}^{\tau_k} V(\sigma)h_k\, d\sigma \\
&+\frac{1}{\epsilon^2}\int_{\tau_{k-1}}^{\tau_k}\int_{\sigma}^{\tau_k} V(\sigma)V(\lambda)h_k\, d\lambda d\sigma \\
&+\frac{1}{\epsilon^3}\int_{\tau_{k-1}}^{\tau_k}\int_{\sigma}^{\tau_k}\int_{\lambda}^{\tau_k} V(\sigma)V(\lambda)V(\nu)h_k\, d\nu d\lambda d\sigma \\
&+\frac{1}{\epsilon^4}\int_{\tau_{k-1}}^{\tau_k}\int_{\sigma}^{\tau_k}\int_{\lambda}^{\tau_k}\int_{\lambda}^{\nu} V(\sigma)V(\lambda)U^\epsilon(\lambda,\mu)V(\mu)V(\nu)h_k\, d\mu d\nu d\lambda d\sigma. \qquad (5.21)
\end{aligned}$$

We iterate (5.12) two times to obtain the expansion

$$\begin{aligned}
A(\tau_{k-1},\tau_k)h_k = h_k &+ \int_{\tau_{k-1}}^{\tau_k} \mathcal{L}_\sigma h_k\, d\sigma \\
&+ \int_{\tau_{k-1}}^{\tau_k}\int_{\sigma}^{\tau_k} \mathcal{L}_\sigma\mathcal{L}_\lambda A(\lambda,\tau_k)h_k\, d\lambda d\sigma. \qquad (5.22)
\end{aligned}$$

Now we subtract (5.22) from the expected value of (5.21), use (4.10), use the change of variables and take the $\|\cdot\|_0$ norms to obtain the inequality

$$I_{n,k}^{11}(f) \equiv \|E\{U^\epsilon(\tau_{k-1},\tau_k)h_k\} - A(\tau_{k-1},\tau_k)h_k\|_0$$

$$\leq \|\frac{1}{\epsilon^2}\int_{\tau_{k-1}}^{\tau_k}\int_{\sigma}^{\tau_k} E\{V(\sigma)V(\lambda)h_k\}\, d\lambda d\sigma - \int_{\tau_{k-1}}^{\tau_k} \mathcal{L}_\sigma h_k d\sigma\|_0$$
$$+\|\int_{\tau_{k-1}}^{\tau_k}\int_{\sigma}^{\tau_k} \mathcal{L}_\sigma\mathcal{L}_\lambda A(\lambda,\tau_k)h_k\, d\lambda d\sigma\|_0$$
$$+\epsilon^3\|\int_{\tau_{k-1}/\epsilon^2}^{\tau_k/\epsilon^2}\int_{\sigma}^{\tau_k/\epsilon^2}\int_{\lambda}^{\tau_k/\epsilon^2} E\{V(\epsilon^2\sigma)V(\epsilon^2\lambda)V(\epsilon^2\nu)h_k\}\, d\nu d\lambda d\sigma\|_0$$
$$+\epsilon^4\|\int_{\tau_{k-1}/\epsilon^2}^{\tau_k/\epsilon^2}\int_{\sigma}^{\tau_k/\epsilon^2}\int_{\lambda}^{\tau_k/\epsilon^2}\int_{\lambda}^{\nu} E\{V(\epsilon^2\sigma)V(\epsilon^2\lambda)U^\epsilon(\epsilon^2\lambda,\epsilon^2\mu)\cdot$$
$$\cdot V(\epsilon^2\mu)V(\epsilon^2\nu)h_k\}\, d\mu d\nu d\lambda d\sigma\|_0. \tag{5.23}$$

In the following lemmas, we proceed with the term-by-term estimation of (5.23). We first cite the Banach space version of mixing lemma which is proven in [26].

**Lemma 5.2.2** *Let $\mathcal{L}$, $\mathcal{M}$ and $\mathcal{N}$ be Banach spaces and let $W(\omega)$ and $U(\omega)$, $\omega \in \Omega$, be bounded operators from $\mathcal{N} \to \mathcal{M}$ and $\mathcal{M} \to \mathcal{L}$, respectively. Let $U$ be strongly $\mathcal{F}_0^s$ measurable and $W$ strongly $\mathcal{F}_{s+t}^\infty$ measurable. Then, for arbitrary $f \in \mathcal{N}$,*

$$\|E\{UWf\} - E\{UE\{Wf\}\}\|_{\mathcal{L}} \leq 2\rho(t)\sup_{\omega,\omega'}\|U(\omega)W(\omega')f\|_{\mathcal{L}}. \tag{5.24}$$

In the following four lemmas, we repeatedly use the above Lemma 5.2.2. The following inequality from condition (4.11) is necessary for the estimate of $I_{n,k}^{12}(f)$ and the last term of (5.23): If $0 \leq \tau - \sigma \leq \epsilon$, then

$$\|U^\epsilon(\sigma,\tau)f\|_k \leq \beta_k k e^{\alpha_k}\|f\|_k \quad a.e., \qquad f \in \mathcal{C}^k, \quad k = 1, 2, \tag{5.25}$$

where $\alpha_k$ and $\beta_k$ are independent of $\sigma$, $\tau$ and $\epsilon$.

**Lemma 5.2.3** *For arbitrary $f \in \mathcal{C}^2$ and $k = 1, 2, ..., \tilde{m}$,*

$$\|\frac{1}{\epsilon^2}\int_{\tau_{k-1}}^{\tau_k}\int_{\sigma}^{\tau_k} E\{V(\sigma)V(\lambda)f\}\, d\lambda d\sigma - \int_{\tau_{k-1}}^{\tau_k} \mathcal{L}_\sigma f\, d\sigma\|_0 \leq \epsilon^2 C_{n,\epsilon}\|f\|_2. \tag{5.26}$$

**Proof**: Using the change of variables $\lambda = \sigma + \epsilon^2 t$ for (4.16), we have

$$\int_0^{1/\epsilon} E\{V(\sigma)V(\sigma+\epsilon^2 t)f\}\, dt = \frac{1}{\epsilon^2}\int_\sigma^{\sigma+\epsilon} E\{V(\sigma)V(\lambda)f\}\, d\lambda \equiv \mathcal{L}_\sigma^\epsilon f, \quad f \in \mathcal{C}^2. \tag{5.27}$$

Then from (4.16) $\mathcal{L}_\sigma f$ is the limit of $\mathcal{L}_\sigma^\epsilon f$ as $\epsilon \to 0$ and condition (4.17) implies that

$$\sup_{\sigma_{n-1}\leq\sigma\leq\sigma_n} \|\mathcal{L}_\sigma^\epsilon f - \mathcal{L}_\sigma f\|_0 \leq \epsilon C_{n,\epsilon}\|f\|_2, \quad f \in \mathcal{C}^2. \tag{5.28}$$

By subtracting and adding the integral (from $\tau_{k-1}$ to $\tau_k$) of $\mathcal{L}_\sigma^\epsilon f$ and using the triangle inequality, the left side ($\equiv J$) of (5.26) becomes

$$\begin{aligned} J &\leq \|\frac{1}{\epsilon^2}\int_{\tau_{k-1}}^{\tau_k}\int_\sigma^{\tau_k} E\{V(\sigma)V(\lambda)f\}\, d\lambda d\sigma - \int_{\tau_{k-1}}^{\tau_k} \mathcal{L}_\sigma^\epsilon f\, d\sigma\|_0 \\ &\quad +\|\int_{\tau_{k-1}}^{\tau_k} \mathcal{L}_\sigma^\epsilon f\, d\sigma - \int_{\tau_{k-1}}^{\tau_k} \mathcal{L}_\sigma f\, d\sigma\|_0 \\ &\equiv J_1 + J_2. \end{aligned} \tag{5.29}$$

For $J_1$-estimate, we use equality (5.27) and Lemma 5.2.2 to obtain the inequalities

$$\begin{aligned} J_1 &= \| -\frac{1}{\epsilon^2}\int_{\tau_{k-1}}^{\tau_k}\int_{\tau_k}^{\sigma+\epsilon} E\{V(\sigma)V(\lambda)f\}\, d\lambda d\sigma\|_0 \\ &\leq \frac{1}{\epsilon^2}\int_{\tau_{k-1}}^{\tau_k}\int_{\tau_k}^{\sigma+\epsilon} 2\rho(\frac{\lambda-\sigma}{\epsilon^2})\, d\lambda d\sigma \sup_{\tau_{k-1}\leq\sigma\leq\tau_k\leq\lambda\leq\tau_k+\epsilon}\ \sup_{\omega,\omega'} \|V(\sigma)V(\lambda)f\|_0 \\ &\leq \frac{1}{\epsilon^2}\int_0^\epsilon\int_{\tau_k-u}^{\tau_k} 2\rho(\frac{u}{\epsilon^2})\, dv du \cdot C_{n,\epsilon}^2\|f\|_2 \\ &\leq 2\epsilon^2\int_0^{1/\epsilon} \tilde{u}\rho(\tilde{u})\, d\tilde{u} \cdot C_{n,\epsilon}\|f\|_2 \\ &\leq \epsilon^2 C_{n,\epsilon}\|f\|_2, \end{aligned} \tag{5.30}$$

where condition (4.12), the uniform boundedness of $\{C_{n,\epsilon} : n = 1, 2, ..., m_0\}$, the notational convention about $C_{n,\epsilon}$ and $\int_0^\infty s\rho(s)\, ds < \infty$ were used. For $J_2$-estimate, we use inequality (5.28) and $\tau_k - \tau_{k-1} = \epsilon$ to obtain

$$J_2 \leq \int_{\tau_{k-1}}^{\tau_k} \|\mathcal{L}_\sigma^\epsilon f - \mathcal{L}_\sigma f\|_0\, d\sigma \leq \epsilon^2 C_{n,\epsilon}\|f\|_2. \tag{5.31}$$

Now we combine the above (5.30) and (5.31); from inequality (5.29)

$$J \leq J_1 + J_2 \leq \epsilon^2 C_{n,\epsilon}\|f\|_2, \quad f \in \mathcal{C}^2. \tag{5.32}$$

Therefore, the desired estimate (5.26) of Lemma 5.2.3 is obtained. ■

**Lemma 5.2.4** *For arbitrary $f \in C^4$ and $k = 1, 2, ..., \tilde{m}$,*

$$\| \int_{\tau_{k-1}}^{\tau_k} \int_{\sigma}^{\tau_k} \mathcal{L}_\sigma \mathcal{L}_\lambda A(\lambda, \tau_k) f \, d\lambda d\sigma \|_0 \leq \epsilon^2 C_{n,\epsilon} \|f\|_4. \tag{5.33}$$

**Proof**: Let $g_\lambda^k \equiv A(\lambda, \tau_k) f$ which is in $C^4$ from condition (4.18). For arbitrary $f \in C^k, k \geq 2$, using the notation $\mathcal{L}_\sigma^\epsilon f$ defined by (5.27), Lemma 5.2.2, and condition (4.12), one obtains

$$\begin{aligned} \|\mathcal{L}_\sigma^\epsilon f\|_{k-2} &= \|\frac{1}{\epsilon^2} \int_\sigma^{\sigma+\epsilon} E\{V(\sigma)V(\lambda)f\} \, d\lambda\|_{k-2} \qquad k \geq 2 \\ &\leq \frac{1}{\epsilon^2} \int_\sigma^{\sigma+\epsilon} 2\rho(\frac{\lambda - \sigma}{\epsilon^2}) \, d\lambda \cdot C_{n,\epsilon}'^2 \|f\|_k \leq 2 \int_0^\infty \rho(\tilde{\lambda}) \, d\tilde{\lambda} \cdot C_{n,\epsilon}'^2 \|f\|_k \equiv C_{n,\epsilon} \|f\|_k, \end{aligned} \tag{5.34}$$

where the condition that $\{C_{n,\epsilon} : n = 1, 2, ..., m_0\}$ is uniformly bounded in $n$ by a constant and the notational convention about $C_{n,\epsilon}$ were used. Since this uniform boundedness implies that, for each fixed $n$, $C_{n,\epsilon}$ is monotone decreasing as $\epsilon \to 0$, therefore, (5.34) leads to the following inequality for (4.16):

$$\|\mathcal{L}_\sigma f\|_{k-2} = \lim_{\epsilon \to 0} \|\mathcal{L}_\sigma^\epsilon f\|_{k-2} \leq \lim_{\epsilon \to 0} C_{n,\epsilon} \|f\|_k \leq C_{n,\epsilon} \|f\|_k. \tag{5.35}$$

Then, using $\tau_k - \tau_{k-1} = \epsilon$ and (5.35) and (4.18), one obtains the following inequalities for the left side ($\equiv H$) of (5.33):

$$H \leq \epsilon^2 C_{n,\epsilon}'^2 \|g_\lambda^k\|_4 \leq \epsilon^2 C_{n,\epsilon}'^2 a_4 \|f\|_4 \equiv \epsilon^2 C_{n,\epsilon} \|f\|_4. \tag{5.36}$$

Therefore, (5.33) is proved. ∎

The following two lemmas are basically the same as Lemma 2 and Lemma 3 in [26]. The only difference is that the constant $C$ there is replaced by our constant $C_{n,\epsilon}$. So we state these lemmas without proof.

**Lemma 5.2.5** *For $f \in C^3$, $k = 1, 2, ..., \tilde{m}$,*

$$\| \int_{\tau_{k-1}/\epsilon^2}^{\tau_k/\epsilon^2} \int_\sigma^{\tau_k/\epsilon^2} \int_\lambda^{\tau_k/\epsilon^2} E\{V(\epsilon^2\sigma)V(\epsilon^2\lambda)V(\epsilon^2\nu)f\} \, d\nu d\lambda d\sigma \|_0 \leq \frac{C_{n,\epsilon}}{\epsilon} \|f\|_3. \tag{5.37}$$

**Lemma 5.2.6** *For $f \in C^4$, $k = 1, 2, ..., \tilde{m}$,*

$$\| \int_{\tau_{k-1}/\epsilon^2}^{\tau_k/\epsilon^2} \int_{\sigma}^{\tau_k/\epsilon^2} \int_{\lambda}^{\tau_k/\epsilon^2} \int_{\lambda}^{\nu} E\{V(\epsilon^2\sigma)V(\epsilon^2\lambda)U^\epsilon(\epsilon^2\lambda, \epsilon^2\mu) \cdot$$
$$\cdot V(\epsilon^2\mu)V(\epsilon^2\nu)f\} \, d\mu d\nu d\lambda d\sigma\|_0 \leq \frac{C_{n,\epsilon}}{\epsilon^2}\|f\|_4. \tag{5.38}$$

From the above Lemmas 5.2.3–5.2.6, we have the following estimate for $I_{n,k}^{11}(f)$ by replacing $f$ in these lemmas by $h_k$; inequality (5.23) becomes

$$I_{n,k}^{11}(f) \leq \epsilon^2 C_{n,\epsilon}(\|h_k\|_2 + \|h_k\|_4 + \|h_k\|_3 + \|h_k\|_4)$$
$$\leq 4\epsilon^2 C_{n,\epsilon}\|h_k\|_4 \leq 4\epsilon^2 C_{n,\epsilon} a_4 \|f\|_4, \tag{5.39}$$

where condition (4.18) was used. Therefore, from the notational convention about $C_{n,\epsilon}$, the desired estimate for $I_{n,k}^{11}(f)$ is obtained;

$$I_{n,k}^{11}(f) \leq \epsilon^2 C_{n,\epsilon}\|f\|_4. \tag{5.40}$$

Next, we estimate $I_{n,k}^{12}(f)$ (defined by (5.20)). Using propagator properties (5.3)-(5.4), we obtain the equalities

$$I_{n,k}^{12}(f) \equiv \|E\{U^\epsilon(\tau_0, \tau_{k-1})U^\epsilon(\tau_{k-1}, \tau_k)h_k\} - E\{U^\epsilon(\tau_0, \tau_{k-1})E\{U^\epsilon(\tau_{k-1}, \tau_k)h_k\}\}\|_0$$
$$= \frac{1}{\epsilon^2}\|E\{\int_{\tau_0}^{\tau_{k-1}} U^\epsilon(\tau_0, \sigma)V(\sigma)d\sigma \int_{\tau_{k-1}}^{\tau_k} V(\lambda)U^\epsilon(\lambda, \tau_k)h_k d\lambda\}$$
$$-E\{\int_{\tau_0}^{\tau_{k-1}} U^\epsilon(\tau_0, \sigma)V(\sigma)d\sigma E\{\int_{\tau_{k-1}}^{\tau_k} V(\lambda)U^\epsilon(\lambda, \tau_k)h_k d\lambda\}\}\|_0$$
$$= \frac{1}{\epsilon^2}\| \int_{\tau_0}^{\tau_{k-1}} \int_{\tau_{k-1}}^{\tau_k} [E\{U^\epsilon(\tau_0, \sigma)V(\sigma)V(\lambda)U^\epsilon(\lambda, \tau_k)h_k\}$$
$$-E\{U^\epsilon(\tau_0, \sigma)V(\sigma)E\{V(\lambda)U^\epsilon(\lambda, \tau_k)h_k\}\}]d\lambda d\sigma\|_0. \tag{5.41}$$

Note that $U^\epsilon(\tau_0, \sigma)V(\sigma)$ is $\mathcal{F}_0^{\sigma/\epsilon^2}$ measurable and $V(\lambda)U^\epsilon(\lambda, \tau_k)$ is $\mathcal{F}_{\lambda/\epsilon^2}^{\infty}$ measurable and thus a gap $(= \frac{\lambda - \sigma}{\epsilon^2})$ has been created. We use Lemma 5.2.2 to obtain the estimate

$$I_{n,k}^{12}(f) \leq \frac{1}{\epsilon^2} \int_{\tau_0}^{\tau_{k-1}} \int_{\tau_{k-1}}^{\tau_k} 2\rho(\frac{\lambda - \sigma}{\epsilon^2}) d\lambda d\sigma \cdot$$
$$\cdot \sup_{\tau_0 \leq \sigma \leq \tau_{k-1} \leq \lambda \leq \tau_k} \sup_{\omega, \omega'} \|U^\epsilon(\tau_0, \sigma)V(\sigma)V(\lambda)U^\epsilon(\lambda, \tau_k)h_k\|_0. \tag{5.42}$$

Using the contraction property of $U^\epsilon(\tau_0,\sigma)$, (4.12), (5.25) and (4.18), we obtain

$$\begin{aligned}\sup_{\tau_0\le\sigma\le\tau_{k-1}\le\lambda\le\tau_k}\ \sup_{\omega,\omega'}\|U^\epsilon(\tau_0,\sigma)V(\sigma)V(\lambda)U^\epsilon(\lambda,\tau_k)h_k\|_0 \\ \le C_{n,\epsilon}^{\prime 2}(\beta_2 2e^{\alpha_2})a_2\|f\|_2 \equiv C_{n,\epsilon}\|f\|_2,\end{aligned} \tag{5.43}$$

where the uniform boundedness of $C_{n,\epsilon}$ and the notational convention about these quantities were also used. Thus inequality (5.42) becomes

$$I_{n,k}^{12}(f) \le \frac{2}{\epsilon^2}C_{n,\epsilon}\|f\|_2\int_{\tau_0}^{\tau_{k-1}}\int_{\tau_{k-1}}^{\tau_k}\rho(\frac{\lambda-\sigma}{\epsilon^2})d\lambda d\sigma. \tag{5.44}$$

Using the change of variables

$$\nu\equiv\lambda-\sigma, \qquad \mu\equiv\sigma, \tag{5.45}$$

for the above double integral in (5.44), we have the inequality

$$\begin{aligned}&\int_{\tau_0}^{\tau_{k-1}}\int_{\tau_{k-1}}^{\tau_k}\rho(\frac{\lambda-\sigma}{\epsilon^2})d\lambda d\sigma \\ &\le \int_0^{\tau_{k-1}-\tau_0}\int_{\tau_{k-1}-\nu}^{\tau_{k-1}}\rho(\frac{\nu}{\epsilon^2})d\mu d\nu + \int_{\tau_{k-1}-\tau_0}^{\tau_k-\tau_0}\int_{\tau_0}^{\tau_k-\nu}\rho(\frac{\nu}{\epsilon^2})d\mu d\nu \\ &= \int_0^{(k-1)\epsilon}\nu\rho(\frac{\nu}{\epsilon^2})d\nu + \int_{(k-1)\epsilon}^{k\epsilon}(k\epsilon-\nu)\rho(\frac{\nu}{\epsilon^2})d\nu.\end{aligned} \tag{5.46}$$

Note that

$$\int_0^{(k-1)\epsilon}\nu\rho(\frac{\nu}{\epsilon^2})d\nu = \epsilon^4\int_0^{(k-1)/\epsilon}\tilde{\nu}\rho(\tilde{\nu})d\tilde{\nu}, \tag{5.47}$$

$$\int_{(k-1)\epsilon}^{k\epsilon}(k\epsilon-\nu)\rho(\frac{\nu}{\epsilon^2})d\nu \le \epsilon\int_{(k-1)\epsilon}^{k\epsilon}\rho(\frac{\nu}{\epsilon^2})d\nu \le \epsilon^3\rho^{1/2}(\frac{1}{\epsilon})\int_{(k-1)/\epsilon}^{k/\epsilon}\rho^{1/2}(\tilde{\nu})d\tilde{\nu}, \tag{5.48}$$

where $k\ge 2$ (Note that the left side of (5.46) vanishes if $k=1$). Since the mixing rate condition (4.3) is assumed and $\int_0^\infty s\rho(s)ds<\infty$ and $\frac{1}{\epsilon}\rho^{1/2}(\frac{1}{\epsilon})$ is uniformly bounded for $\epsilon\in(0,1]$, (5.46)-(5.48) implies

$$\int_{\tau_0}^{\tau_{k-1}}\int_{\tau_{k-1}}^{\tau_k}\rho(\frac{\lambda-\sigma}{\epsilon^2})d\lambda d\sigma \sim O(\epsilon^4). \tag{5.49}$$

Therefore, from (5.44), we obtain the estimate

$$I_{n,k}^{12}(f) \le \epsilon^2 C_{n,\epsilon}\|f\|_2. \tag{5.50}$$

Now, we substitute two estimates (5.50) and (5.40) into (5.20) and use the fact that $\tilde{m} \sim O(\epsilon^{-1})$. This yields to the desired estimate (5.18) of Lemma 5.2.1;

$$I_n^1(f) \leq \sum_{k=1}^{\tilde{m}} \left( I_{n,k}^{11}(f) + I_{n,k}^{12}(f) \right)$$
$$\leq \epsilon^2 C_{n,\epsilon} \sum_{k=1}^{\tilde{m}} (\|f\|_4 + \|f\|_2) \leq 2\epsilon^2 C_{n,\epsilon} \sum_{k=1}^{\tilde{m}} \|f\|_4 \equiv \epsilon C_{n,\epsilon} \|f\|_4. \quad (5.51)$$

## 5.3 $\mathbf{I}_n^2$–estimation

In this section, we derive the desired estimate for $I_n^2(f)$ which is defined by (5.14). The desired estimate is (5.16). The main tool for the $I_n^2$–estimate is the p-norm version of mixing lemma (cf. [7] or [27]). We first introduce this mixing lemma without proof and then show that it can be applied to our problem for the $I_n^2$–estimate.

**Lemma 5.3.1** *Let $F(\omega', \omega)$ be a function on $\Omega \times \Omega$ such that for fixed $\omega$ $F(\cdot, \omega)$ is $\mathcal{F}_0^s$ measurable and for fixed $\omega'$ $F(\omega', \cdot)$ is $\mathcal{F}_{s+t}^{\infty}$ measurable and $|F(\omega', \omega)| \leq \phi(\omega')\psi(\omega), E\{\phi^q\} < \infty$ and $E\{\psi^p\} < \infty$ with $1/q+1/p=1$ and $p, q > 1$. Let $\mathcal{F}_s^t$ and $P$ satisfy the hypotheses in Section 4.1 and set*

$$\overline{F}(\omega') \equiv E\{F(\omega', \cdot)\} = \int_{\Omega} F(\omega', \omega) P(d\omega). \quad (5.52)$$

*Then*

$$|E\{F\} - E\{\overline{F}\}| \ \leq \ 2\rho^{1/q}(t) E^{1/q}\{\phi^q\} E^{1/p}\{\psi^p\}. \quad (5.53)$$

We note that, as in Lemma 5.2.2, there must be a gap between two measurable ranges of the random function $F$ in order to profitably use inequality (5.53). If one uses the backward propagator property (5.4) for $U(\sigma_{n-1}, \sigma_n)$, then from (5.14) one obtains

$$I_n^2(f) \equiv \|E\{U^{\epsilon}(0, \sigma_{n-1}) U^{\epsilon}(\sigma_{n-1}, \sigma_n) g_n\} - E\{U^{\epsilon}(0, \sigma_{n-1}) E\{U^{\epsilon}(\sigma_{n-1}, \sigma_n) g_n\}\}\|_0$$
$$= \sup_{x \in \mathcal{R}_c} \frac{1}{\epsilon} \left| \int_{\sigma_{n-1}}^{\sigma_n} [E\{ U^{\epsilon}(0, \sigma_{n-1}) V(s) U^{\epsilon}(s, \sigma_n)\} \right.$$
$$\left. - E\{U^{\epsilon}(0, \sigma_{n-1}) E\{V(s) U^{\epsilon}(s, \sigma_n)\}\}] g_n(x) ds \right|$$
$$\equiv \sup_{x \in \mathcal{R}_c} I_n^2(f, x), \quad (5.54)$$

where $g_n = A(\sigma_n, \tau)f$. Note that a gap ($= \frac{s-\sigma_{n-1}}{\epsilon^2}$) has been created between $U^\epsilon(0, \sigma_{n-1})$, which is $\mathcal{F}_0^{\sigma_{n-1}/\epsilon^2}$ measurable, and $V(s)U^\epsilon(s, \sigma_n)$, which is $\mathcal{F}_{s/\epsilon^2}^{\infty}$ measurable. This allows us to apply Lemma 5.3.1 to the integral of (5.54) if corresponding $\phi$ and $\psi$ in Lemma 5.3.1 can be found appropriately. Let us take the term $U^\epsilon(0, \sigma_{n-1})V(s)U^\epsilon(s, \sigma_n)g_n(x)$ as the one corresponding to $F$ in Lemma 5.3.1. If one writes explicitly this random field ($\equiv F_n(\omega', \omega)$),

$$\begin{aligned} F_n(\omega', \omega) &\equiv U^\epsilon(0, \sigma_{n-1})V(s)U^\epsilon(s, \sigma_n)g_n(x) \\ &= F(s, \frac{s}{\epsilon^2}, x^\epsilon(\sigma_{n-1}, 0, x, \omega'), \omega)\, g_n'(x^\epsilon(\sigma_n, s, x^\epsilon(\sigma_{n-1}, 0, x, \omega'), \omega)) \cdot \\ &\cdot \frac{\partial x^\epsilon}{\partial x}(\sigma_n, s, x^\epsilon(\sigma_{n-1}, 0, x, \omega'), \omega). \end{aligned} \tag{5.55}$$

Let us define

$$\overline{F}_n(\omega') \equiv E\{F_n(\omega', \cdot)\}. \tag{5.56}$$

Then, from (5.54), $I_n^2(f, x)$ becomes

$$I_n^2(f, x) = \frac{1}{\epsilon} \left| \int_{\sigma_{n-1}}^{\sigma_n} [E\{F_n\} - E\{\overline{F}_n\}] ds \right|. \tag{5.57}$$

We define $\psi_n(\omega)$ and $\phi_n(\omega')$ as follows:

$$\phi_n(\omega') \equiv \sup_{s \in I_n, x \in \mathcal{R}_c} \sup_{\omega \in \Omega} |F(s, \frac{s}{\epsilon^2}, x, \omega)\, g_n'(x^\epsilon(\sigma_n, s, x, \omega))|, \tag{5.58}$$

$$\psi_n(\omega) \equiv \sup_{s \in I_n, x \in \mathcal{R}_c} |\frac{\partial x^\epsilon}{\partial x}(\sigma_n, s, x, \omega)|. \tag{5.59}$$

$$(I_n \equiv [\sigma_{n-1}, \sigma_n])$$

Notice that $\phi_n(\omega')$ does not actually depend on $\omega'$. We wish to control the random variables $\phi_n$ and $\psi_n$ (see Lemma 5.3.2). Anticipating Lemma 5.3.2, we can apply Lemma 5.3.1 to each integrand of (5.57) and get the following inequality:

$$|E\{F_n\} - E\{\overline{F}_n\}| \leq 2\, \rho^{1/q}(\frac{s - \sigma_{n-1}}{\epsilon^2}) \cdot (1 + C_{n,\epsilon}) \cdot C_{n,\epsilon} \|g_n\|_1, \tag{5.60}$$

leading to the estimate

$$I_n^2(f, x) \leq 2(1 + C_{n,\epsilon}) \frac{C_{n,\epsilon}}{\epsilon} \|g_n\|_1 \int_{\sigma_{n-1}}^{\sigma_n} \rho^{1/q}(\frac{s - \sigma_{n-1}}{\epsilon^2})\, ds$$

$$\leq 2\epsilon(1+C_{n,\epsilon})C_{n,\epsilon}\|g_n\|_1 \int_0^\infty \rho^{1/q}(u)du$$

$$\leq 2\epsilon(1+C_{n,\epsilon})C_{n,\epsilon}a_1\|f\|_1 \int_0^\infty \rho^{1/q}(u)du. \tag{5.61}$$

Then, for $p=q=2$, the assumed mixing rate condition (4.3) implies that

$$I_n^2(f) \equiv \sup_{x\in\mathcal{R}_c} I_n^2(f,x) \leq \epsilon C_{n,\epsilon}\|f\|_1, \tag{5.62}$$

where the notational convention about $C_{n,\epsilon}$ was used. Thus the desired estimate for $I_n^2(f)$, i.e., (5.16), holds.

To complete the $I_n^2$-estimate, therefore, we are left to prove the following lemma. In Lemma 5.3.2, we use general notation $\sigma$ and $\tau$ instead of $s$ and $\sigma_n$, respectively, used in (5.58)-(5.59).

**Lemma 5.3.2** *Let $x^\epsilon(\tau,\sigma,x)$ be the solution of (4.5)-(4.6). Then we obtain the estimate*

$$\sup_{\sigma_{n-1}\leq\sigma\leq\tau\leq\sigma_n} \sup_{x\in\mathcal{R}_c} E^{1/p}\{\left|\frac{\partial x^\epsilon}{\partial x}(\tau,\sigma,x)\right|^p\} \leq 1 + C_{n,\epsilon} \tag{5.63}$$

*and thus the $p^{th}$ moment of $\frac{\partial x^\epsilon}{\partial x}$ is bounded uniformly in $n$ by a constant. Random functions $\phi_n$ and $\psi_n$ defined by (5.58)-(5.59) satisfy*

$$E^{1/p}\{\psi_n^p(\omega)\} \leq 1 + C_{n,\epsilon}, \tag{5.64}$$

$$E^{1/q}\{\phi_n^q(\omega')\} \leq C_{n,\epsilon}\|g_n\|_1. \tag{5.65}$$

**Proof**: The proof of this lemma consists of several lemmas (Lemmas 5.3.3–5.3.8) following in the rest of this section. Inequality (5.65) holds from (4.12) because condition (4.12) is equivalent to the inequality

$$\sup_{\sigma_{n-1}\leq\tau\leq\sigma_n} |\partial_{x..x}^k F(\tau,\tau/\epsilon^2,x)| \leq C_{n,\epsilon} \quad a.e., \qquad 0\leq k\leq 3, \tag{5.66}$$

in view of definition of the operator $V(\tau)$ given by (4.8). Using the dominated convergence theorem, inequality (5.64) will follow from (5.63) once (5.63) is proved;

$$E\{\psi_n^p(\omega)\} = E\{\sup_{s\in I_n, x\in\mathcal{R}_c}\left|\frac{\partial x^\epsilon}{\partial x}\right|^p\} = \sup_{s\in I_n, x\in\mathcal{R}_c} E\{\left|\frac{\partial x^\epsilon}{\partial x}\right|^p\} \leq (1+C_{n,\epsilon})^p, \tag{5.67}$$

where for the second equality we used meaning of supremum and the dominated convergence theorem.

From now on, therefore, we focus on the estimate of the p$^{th}$ moment of $\frac{\partial x^\epsilon}{\partial x}$. In order to estimate the p$^{th}$ moment of $\frac{\partial x^\epsilon}{\partial x}(\tau,\sigma,x)$ on $O(1)$ interval $[\sigma_{n-1},\sigma_n]$, we use the variational argument. First we observe from (4.5)-(4.6) that

$$\frac{d}{d\tau}\frac{\partial x^\epsilon}{\partial x}(\tau,\sigma,x) = \frac{\partial}{\partial x}\frac{d}{d\tau}x^\epsilon(\tau,\sigma,x) = \frac{\partial}{\partial x}\epsilon^{-1}F(\tau,\frac{\tau}{\epsilon^2},x^\epsilon(\tau,\sigma,x))$$

$$= \epsilon^{-1}F'(\tau,\frac{\tau}{\epsilon^2},x^\epsilon(\tau,\sigma,x))\frac{\partial x^\epsilon}{\partial x}(\tau,\sigma,x), \tag{5.68}$$

$$\frac{\partial x^\epsilon}{\partial x}(\sigma,\sigma,x) = 1. \tag{5.69}$$

Based on this observation, it is useful to define the following (vector) notation:

$$\underline{x}^\epsilon \equiv \begin{bmatrix} x_1^\epsilon \\ x_2^\epsilon \end{bmatrix}, \tag{5.70}$$

$$\underline{F} \equiv \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}, \tag{5.71}$$

where

$$x_1^\epsilon(\tau,\sigma,\underline{x}_0) \equiv x^\epsilon(\tau,\sigma,x), \tag{5.72}$$

$$x_2^\epsilon(\tau,\sigma,\underline{x}_0) \equiv \frac{\partial x^\epsilon}{\partial x}(\tau,\sigma,x), \tag{5.73}$$

$$\underline{x}_0 \equiv \begin{bmatrix} x \\ 1 \end{bmatrix}, \tag{5.74}$$

$$F_1(\tau,\frac{\tau}{\epsilon^2},\underline{x}^\epsilon(\tau,\sigma,\underline{x}_0)) \equiv F(\tau,\frac{\tau}{\epsilon^2},x^\epsilon(\tau,\sigma,x)), \tag{5.75}$$

$$F_2(\tau,\frac{\tau}{\epsilon^2},\underline{x}^\epsilon(\tau,\sigma,\underline{x}_0)) \equiv F'(\tau,\frac{\tau}{\epsilon^2},x^\epsilon(\tau,\sigma,x))\frac{\partial x^\epsilon}{\partial x}(\tau,\sigma,\underline{x}_0). \tag{5.76}$$

Then stochastic initial value problem (4.5)-(4.6) with (5.68)-(5.69) becomes

$$\frac{d}{d\tau}\underline{x}^\epsilon(\tau,\sigma,\underline{x}_0) = \frac{1}{\epsilon}\underline{F}(\tau,\frac{\tau}{\epsilon^2},\underline{x}^\epsilon(\tau,\sigma,\underline{x}_0)), \tag{5.77}$$

$$\underline{x}^\epsilon(\sigma,\sigma,\underline{x}_0) = \underline{x}_0. \tag{5.78}$$

Now, using the above new notation, we write (5.68)-(5.69) in the integral form

$$x_2^\epsilon(\tau,\sigma,\underline{x}_0) = 1 + \frac{1}{\epsilon}\int_\sigma^\tau F_2(s,\frac{s}{\epsilon^2},\underline{x}^\epsilon(s,\sigma,\underline{x}_0))\,ds. \tag{5.79}$$

Then one can get the following local estimate.

**Lemma 5.3.3** *If* $\sigma_{n-1} \le \sigma \le \tau \le \sigma_n$ *and* $0 \le \tau - \sigma \le \epsilon$, *then*

$$|x_2^\epsilon(\tau,\sigma,\underline{x}_0)| \le e^{C_{n,\epsilon}}, \qquad |\partial_{x_1} x_2^\epsilon(\tau,\sigma,\underline{x}_0)| \le C_{n,\epsilon}. \tag{5.80}$$

**Proof**: From (5.79), (5.76) and (5.66), we obtain the inequalities

$$\begin{aligned} |x_2^\epsilon(\tau,\sigma,\underline{x}_0)| &\le 1 + \frac{1}{\epsilon}\int_\sigma^\tau |F_2(s,\frac{s}{\epsilon^2},\underline{x}^\epsilon(s,\sigma,\underline{x}_0))|\,ds, \\ &\le 1 + \frac{C_{n,\epsilon}}{\epsilon}\int_\sigma^\tau |x_2^\epsilon(s,\sigma,\underline{x}_0)|\,ds. \end{aligned} \tag{5.81}$$

If we apply the Gronwall inequality to the above inequality (5.81) and use $0 \le \tau - \sigma \le \epsilon$,

$$|x_2^\epsilon(\tau,\sigma,\underline{x}_0)| \le e^{\frac{C_{n,\epsilon}}{\epsilon}(\tau-\sigma)} \le e^{C_{n,\epsilon}}, \tag{5.82}$$

which is the first part of (5.80). From (5.79) and (5.76), we have

$$\partial_{x_1} x_2^\epsilon(\tau,\sigma,\underline{x}_0) = \frac{1}{\epsilon}\int_\sigma^\tau [\partial^2_{x_1x_1}F\cdot x_2^\epsilon + \partial_{x_1}F\cdot\partial_{x_1}x_2^\epsilon]\,ds. \tag{5.83}$$

Then, using (5.66) and (5.82) and the Gronwall inequality,

$$|\partial_{x_1}x_2^\epsilon| \le C_{n,\epsilon}e^{C_{n,\epsilon}} + \frac{C_{n,\epsilon}}{\epsilon}\int_\sigma^\tau |\partial_{x_1}x_2^\epsilon|ds \le C_{n,\epsilon}e^{2C_{n,\epsilon}} \equiv C_{n,\epsilon}, \tag{5.84}$$

which is the second part of (5.80). ∎

**To continue proving Lemma 5.3.2**: Let us rewrite (5.63) in terms of the new (vector) notation by using (5.73);

$$\sup_{\sigma_{n-1}\le\sigma\le\tau\le\sigma_n}\ \sup_{x\in\mathcal{R}_c} E^{1/p}\{\,|x_2^\epsilon(\tau,\sigma,\underline{x}_0)|^p\} \le 1 + C_{n,\epsilon}. \tag{5.85}$$

Note that, once one can show the above inequality (5.85) for any even integer $2p$, the same result is true for any positive integer $p$ by the Cauchy-Schwartz inequality;

$$E^{1/p}\{\,|x_2^\epsilon(\tau,\sigma,\underline{x}_0)|^p\} \le E^{1/2p}\{\,|x_2^\epsilon(\tau,\sigma,\underline{x}_0)|^{2p}\}\,E^{1/2p}\{\,1^{2p}\} \le 1 + C_{n,\epsilon}. \tag{5.86}$$

So we consider only the case of even integer power of $|x_2^\epsilon(\tau,\sigma,\underline{x}_0)|$ to prove Lemma 5.3.2.

Let us first divide the interval $[\sigma,\tau] \subseteq [\sigma_{n-1},\sigma_n]$ into $\epsilon$-length segments such that

$$\sigma \equiv \tau_0 < \tau_1 < \tau_2 < \cdots < \tau_k < \cdots < \tau_{r-1} < \tau_r \equiv \tau, \quad \tau_k \equiv \sigma + k\epsilon, \tag{5.87}$$

where $\epsilon$ can be chosen such that $r = \frac{\tau-\sigma}{\epsilon}$ is an integer without loss of generality. If one takes the $2p$-power of (5.79) and differentiates with respect to $\tau$ and integrates it from $\sigma$ to $\tau$, then one can get

$$\begin{aligned}&[x_2^\epsilon(\tau,\sigma,\underline{x}_0)]^{2p} - [x_2^\epsilon(\sigma,\sigma,\underline{x}_0)]^{2p}\\ &= \frac{2p}{\epsilon}\int_\sigma^\tau \left[1+\frac{1}{\epsilon}\int_\sigma^s F_2(t,\frac{t}{\epsilon^2},\underline{x}^\epsilon(t,\sigma,\underline{x}_0))\,dt\right]^{2p-1} F_2(s,\frac{s}{\epsilon^2},\underline{x}^\epsilon(s,\sigma,\underline{x}_0))\,ds,\end{aligned} \tag{5.88}$$

which, from (5.69) and (5.79), leads to

$$[\,x_2^\epsilon(\tau,\sigma,\underline{x}_0)\,]^{2p} = 1 + \frac{2p}{\epsilon}\int_\sigma^\tau F_2(s,\frac{s}{\epsilon^2},\underline{x}^\epsilon(s,\sigma,\underline{x}_0))\,[\,x_2^\epsilon(s,\sigma,\underline{x}_0)\,]^{2p-1}\,ds. \tag{5.89}$$

We decompose the integral on the right side of (5.89) into the integrals on the $\epsilon$-length segments $[\tau_k,\tau_{k+1}]$:

$$[\,x_2^\epsilon(\tau,\sigma,\underline{x}_0)\,]^{2p} = 1 + \frac{2p}{\epsilon}\sum_{k=0}^{r-1}\int_{\tau_k}^{\tau_{k+1}} F_2(s,\frac{s}{\epsilon^2},\underline{x}^\epsilon(s,\sigma,\underline{x}_0))\,[\,x_2^\epsilon(s,\sigma,\underline{x}_0)\,]^{2p-1}\,ds. \tag{5.90}$$

For convenience, let us use the following symbol for the integral of (5.90):

$$J_k^\epsilon(\underline{x}_0) \equiv \int_{\tau_k}^{\tau_{k+1}} F_2(s,\frac{s}{\epsilon^2},\underline{x}^\epsilon(s,\sigma,\underline{x}_0))\,[\,x_2^\epsilon(s,\sigma,\underline{x}_0)\,]^{2p-1}\,ds, \quad k = 0,1,...,r. \tag{5.91}$$

For $k = 0$, in particular, the following estimate from (5.76), (5.66) and Lemma 5.3.3 holds:

$$|J_0^\epsilon(\underline{x}_0)| \le \epsilon\, C_{n,\epsilon} e^{2pC_{n,\epsilon}} \equiv \epsilon C_{n,\epsilon} \quad a.e., \tag{5.92}$$

where the uniform boundedness of $\{C_{n,\epsilon} : n = 1,2,..,m_0\}$ and the notational convention for $C_{n,\epsilon}$ were used. Combining (5.90), (5.91) and (5.92), one obtains

$$E\{\,|x_2^\epsilon(\tau,\sigma,\underline{x}_0)|^{2p}\} \le 1 + C_{n,\epsilon} + \frac{2p}{\epsilon}\sum_{k=1}^{r-1}|E\{\,J_k^\epsilon(\underline{x}_0)\}|. \tag{5.93}$$

This inequality will have the form to which the discrete version of the Gronwall inequality (cf. [28]) can be used if the following inequality holds:

$$|E\{\, J_k^\epsilon(\underline{x}_0)\}| \le \epsilon^2\, C_{n,\epsilon}\, E\{\, [\, x_2^\epsilon(\tau_{k-1}, \sigma, x_0)\,]^{2p}\}, \qquad k = 1, 2, ..., r-1. \tag{5.94}$$

Suppose this is true ((5.94) will be proven in Lemma 5.3.4). Then we can get estimate (5.85) with even integer $2p$ in the following way. Using the notation

$$E_k \equiv |E\{\, [\, x_2^\epsilon(\tau_k, \sigma, \underline{x}_0)\,]^{2p} - 1\}|, \qquad k = 0, 1, ..., r, \tag{5.95}$$

for convenience, one can obtain from assumption (5.94) that (5.93) becomes

$$E_r \le C_{n,\epsilon} + \epsilon\, C_{n,\epsilon} \sum_{k=1}^{r-1} (1 + E_{k-1}) \le C_{n,\epsilon} + \epsilon\, C_{n,\epsilon} \sum_{k=1}^{r-1} E_k, \tag{5.96}$$

where the fact that $r = O(\epsilon^{-1})$ and $E_0 = 0$ and the notational convention for $C_{n,\epsilon}$ were used. With the notation $S_j$ defined by

$$S_0 \equiv\, 0, \quad S_j \equiv \sum_{k=1}^{j} E_k, \qquad j = 1, 2, ..., r, \tag{5.97}$$

(5.96) can be expressed as

$$S_r \le C_{n,\epsilon} + (1 + \epsilon\, C_{n,\epsilon}) S_{r-1}, \tag{5.98}$$

because $E_r = S_r - S_{r-1}$. Now we apply the discrete version of the Gronwall inequality to (5.98);

$$S_r \le C_{n,\epsilon} \frac{(1 + \epsilon\, C_{n,\epsilon})^r - 1}{\epsilon\, C_{n,\epsilon}} \le \frac{e^{r\epsilon C_{n,\epsilon}} - 1}{\epsilon}. \tag{5.99}$$

By substituting this inequality (corresponding to $r-1$ instead of $r$) into (5.98), we have

$$E_r = S_r - S_{r-1} \le C_{n,\epsilon} + \epsilon\, C_{n,\epsilon} \frac{e^{(r-1)\epsilon C_{n,\epsilon}} - 1}{\epsilon} \equiv C_{n,\epsilon}, \tag{5.100}$$

where we used $r \sim O(\epsilon^{-1})$. Since $1 + C_{n,\epsilon} \le (1 + C_{n,\epsilon})^{2p}$, therefore, (5.85) follows for even integer $2p$.

To complete the proof of Lemma 5.3.2, we are left to show inequality (5.94). Let us explicitly restate this inequality as follows by substituting (5.91) into (5.94):

**Lemma 5.3.4** *For $\sigma_{n-1} \le \sigma \le \cdots \le \tau_k \le \tau_{k+1} \le \cdots \le \tau \le \sigma_n$ and $\tau_k$'s defined by (5.87),*

$$|E\{\int_{\tau_k}^{\tau_{k+1}} F_2(s, \frac{s}{\epsilon^2}, \underline{x}^\epsilon(s,\sigma,\underline{x}_0))\,[\,x_2^\epsilon(s,\sigma,\underline{x}_0)\,]^{2p-1} ds\}| \le \epsilon^2\, C'_{n,\epsilon}\, E\{\,[\,x_2^\epsilon(\tau_{k-1},\sigma,\underline{x}_0)\,]^{2p}\}. \tag{5.101}$$

**Proof**: To derive estimate (5.101), we first introduce some useful identities. These identities provide the creation of gaps to allow a mixing lemma (Lemma 5.3.5) to be used. Let $\tau_k \le s \le \tau_{k+1}$. By differentiation and integration, one can get

$$[\,x_2^\epsilon(s,\sigma,\underline{x}_0)\,]^{2p-1} = [\,x_2^\epsilon(\tau_{k-1},\sigma,\underline{x}_0)\,]^{2p-1} + \frac{2p-1}{\epsilon}\int_{\tau_{k-1}}^{s} F_2(\nu, \frac{\nu}{\epsilon^2}, \underline{x}^\epsilon(\nu,\sigma,\underline{x}_0))\,[\,x_2^\epsilon(\nu,\sigma,\underline{x}_0)\,]^{2(p-1)} d\nu, \tag{5.102}$$

for any positive integer $p$. We have the following identity by the fundamental theorem of calculus and chain rule:

$$F_2(s,\frac{s}{\epsilon^2},\underline{x}^\epsilon(s,\sigma,\underline{x}_0)) = F_2(s,\frac{s}{\epsilon^2},\underline{x}^\epsilon(\tau_{k-1},\sigma,\underline{x}_0)) + \frac{1}{\epsilon}\int_{\tau_{k-1}}^{s}\left(\frac{\partial F_2}{\partial \underline{x}}(s,\frac{s}{\epsilon^2},\underline{x}^\epsilon(\nu,\sigma,\underline{x}_0)), \underline{F}(\nu,\frac{\nu}{\epsilon^2},\underline{x}^\epsilon(\nu,\sigma,\underline{x}_0))\right) d\nu, \tag{5.103}$$

where ( , ) denotes the inner product. From the above two identities (5.102) and (5.103), the integral on the left side of inequality (5.101) becomes

$$\begin{aligned}
&\int_{\tau_k}^{\tau_{k+1}} F_2(s,\frac{s}{\epsilon^2},\underline{x}^\epsilon(s,\sigma,\underline{x}_0))\,[\,x_2^\epsilon(s,\sigma,\underline{x}_0)\,]^{2p-1} ds\\
&= \int_{\tau_k}^{\tau_{k+1}} F_2(s,\frac{s}{\epsilon^2},\underline{x}^\epsilon(\tau_{k-1},\sigma,\underline{x}_0))\,[\,x_2^\epsilon(\tau_{k-1},\sigma,\underline{x}_0)\,]^{2p-1} ds\\
&+\frac{2p-1}{\epsilon}\int_{\tau_k}^{\tau_{k+1}}\int_{\tau_{k-1}}^{s} F_2(s,\frac{s}{\epsilon^2},\underline{x}^\epsilon(\tau_{k-1},\sigma,\underline{x}_0))\, F_2(\nu,\frac{\nu}{\epsilon^2},\underline{x}^\epsilon(\nu,\sigma,\underline{x}_0))\,[\,x_2^\epsilon(\nu,\sigma,\underline{x}_0)\,]^{2(p-1)} d\nu ds\\
&+\frac{1}{\epsilon}\int_{\tau_k}^{\tau_{k+1}}\int_{\tau_{k-1}}^{s}\left(\frac{\partial F_2}{\partial \underline{x}}(s,\frac{s}{\epsilon^2},\underline{x}^\epsilon(\nu,\sigma,\underline{x}_0)), \underline{F}(\nu,\frac{\nu}{\epsilon^2},\underline{x}^\epsilon(\nu,\sigma,\underline{x}_0))\right)[\,x_2^\epsilon(\tau_{k-1},\sigma,\underline{x}_0)\,]^{2p-1} d\nu ds\\
&+\frac{2p-1}{\epsilon^2}\int_{\tau_k}^{\tau_{k+1}}\int_{\tau_{k-1}}^{s}\int_{\tau_{k-1}}^{s}\left(\frac{\partial F_2}{\partial \underline{x}}(s,\frac{s}{\epsilon^2},\underline{x}^\epsilon(\nu,\sigma,\underline{x}_0)), \underline{F}(\nu,\frac{\nu}{\epsilon^2},\underline{x}^\epsilon(\nu,\sigma,\underline{x}_0))\right)\cdot\\
&\cdot F_2(\mu,\frac{\mu}{\epsilon^2},\underline{x}^\epsilon(\mu,\sigma,\underline{x}_0))\,[\,x_2^\epsilon(\mu,\sigma,\underline{x}_0)\,]^{2(p-1)}\, d\mu d\nu ds.
\end{aligned} \tag{5.104}$$

There are, therefore, four terms to be estimated for the proof of Lemma 5.3.4. We will need the following version of mixing lemma from [7] and three lemmas (Lemmas 5.3.6–5.3.8) for the estimation. We state and prove these lemmas first and then perform the estimation of (5.104) using these lemmas.

**Lemma 5.3.5** *Let $F(\omega,\omega')$ be a function on $\Omega \times \Omega$ such that, for fixed $\omega'$, $F(\cdot,\omega')$ is $\mathcal{F}_{t+s}^{\infty}$ measurable and, for fixed $\omega$, $F(\omega,\cdot)$ is $\mathcal{F}_0^s$ measurable and $|F(\omega,\omega')| \le \phi(\omega')$. Let $\mathcal{F}_s^t$ and $P$ satisfy the hypotheses of Section 4.1 and set*

$$\overline{F}(\omega') \equiv E\{F(\cdot,\omega')\} = \int_{\Omega} F(\omega,\omega')P(d\omega). \tag{5.105}$$

*Then*

$$|E\{F(\cdot,\omega')|\mathcal{F}_0^s\} - \overline{F}(\omega')| \le 2\rho(t)\phi(\omega'). \tag{5.106}$$

**Lemma 5.3.6** *For $\sigma_{n-1} \le \sigma \le \cdots \le \tau_{k-1} \le \nu \le \tau_{k+1} \le \cdots \le \tau \le \sigma_n$,*

$$|x_2^{\epsilon}(\nu,\sigma,\underline{x}_0)| \;\le\; e^{2C_{n,\epsilon}}|x_2^{\epsilon}(\tau_{k-1},\sigma,\underline{x}_0)|. \tag{5.107}$$

**Proof**: From (5.102) with $p = 1$, we have the inequality

$$|x_2^{\epsilon}(\nu,\sigma,\underline{x}_0)| \le |x_2^{\epsilon}(\tau_{k-1},\sigma,\underline{x}_0)| + \frac{1}{\epsilon}\int_{\tau_{k-1}}^{\nu} |F_2(t,\frac{t}{\epsilon^2},\underline{x}^{\epsilon}(t,\sigma,\underline{x}_0))|\,dt. \tag{5.108}$$

When (5.76) and (5.66) are used, the above (5.108) becomes

$$|x_2^{\epsilon}(\nu,\sigma,\underline{x}_0)| \le |x_2^{\epsilon}(\tau_{k-1},\sigma,\underline{x}_0)| + \frac{C_{n,\epsilon}}{\epsilon}\int_{\tau_{k-1}}^{\nu} |x_2^{\epsilon}(t,\sigma,\underline{x}_0)|\,dt. \tag{5.109}$$

So the Gronwall inequality gives

$$|x_2^{\epsilon}(\nu,\sigma,\underline{x}_0)| \le |x_2^{\epsilon}(\tau_{k-1},\sigma,\underline{x}_0)|\,e^{\frac{C_{n,\epsilon}}{\epsilon}(s-\tau_{k-1})} \le |x_2^{\epsilon}(\tau_{k-1},\sigma,\underline{x}_0)|\,e^{2C_{n,\epsilon}}, \tag{5.110}$$

where the fact that $\nu - \tau_{k-1} \le \tau_{k+1} - \tau_{k-1} = 2\epsilon$ was used. ■

**Lemma 5.3.7** *For $\sigma_{n-1} \le \sigma \le \cdots \le \tau_k \le \tau_{k+1} \le \cdots \le \tau \le \sigma_n$,*

$$\int_{\tau_k}^{\tau_{k+1}}\int_{\tau_{k-1}}^{s} \rho(\frac{s-\nu}{\epsilon^2})\,d\nu ds \le \epsilon^3\int_0^{\infty}\rho(t)dt. \tag{5.111}$$

**Proof**: In order to prove this lemma, we consider the transformation from $(\nu, s)$-coordinate to $(u, v)$-coordinate defined by

$$u \equiv s - \nu, \tag{5.112}$$

$$v \equiv \nu. \tag{5.113}$$

In terms of these new coordinates, the left side of (5.111) becomes

$$\begin{aligned}
&\int_{\tau_k}^{\tau_{k+1}} \int_{\tau_{k-1}}^{s} \rho(\frac{s-\nu}{\epsilon^2})\, d\nu ds \\
&= \int_0^{\epsilon} \int_{\tau_k - u}^{\tau_{k+1}-u} \rho(\frac{u}{\epsilon^2})\, dv du + \int_{\epsilon}^{2\epsilon} \int_{\tau_{k-1}}^{\tau_{k+1}-u} \rho(\frac{u}{\epsilon^2})\, dv du \\
&= \epsilon \int_0^{\epsilon} \rho(\frac{u}{\epsilon^2})\, du + \int_{\epsilon}^{2\epsilon} (2\epsilon - u) \rho(\frac{u}{\epsilon^2})\, du,
\end{aligned} \tag{5.114}$$

where the fact that $\tau_k - \tau_{k-1} = \epsilon$ was used. Then, using the change of variable $t = \frac{u}{\epsilon^2}$,

$$\begin{aligned}
&\int_{\tau_k}^{\tau_{k+1}} \int_{\tau_{k-1}}^{s} \rho(\frac{s-\nu}{\epsilon^2})\, d\nu ds \\
&\leq \epsilon^3 \int_0^{1/\epsilon} \rho(t)\, dt + \epsilon^3 \int_{1/\epsilon}^{2/\epsilon} \rho(t)\, dt = \epsilon^3 \int_0^{2/\epsilon} \rho(t)\, dt.
\end{aligned} \tag{5.115}$$

Therefore, Lemma 5.3.7 is proved. ■

**Lemma 5.3.8** *For* $\sigma_{n-1} \leq \sigma \leq \cdots \leq \tau_{k-1} \leq \nu \leq \mu \leq s \leq \tau_{k+1} \leq \cdots \leq \tau \leq \sigma_n$ *and* $\tau_k \leq s \leq \tau_{k+1}$,

$$\begin{aligned}
&|E\{ \left( \frac{\partial F_2}{\partial \underline{x}}(s, \frac{s}{\epsilon^2}, \underline{x}^{\epsilon}(\nu,\sigma,\underline{x}_0)), \underline{F}(\nu, \frac{\nu}{\epsilon^2}, \underline{x}^{\epsilon}(\nu,\sigma,\underline{x}_0)) \right) | \mathcal{F}_0^{\mu/\epsilon^2} \}| \\
&\leq C_{n,\epsilon} \Big[ |E\{ \partial^2_{x_1 x_1} F_1(s, \frac{s}{\epsilon^2}, \underline{x}^{\epsilon}(\nu,\sigma,\underline{x}_0)) | \mathcal{F}_0^{\mu/\epsilon^2} \}| \\
&+ |E\{ \partial_{x_1} F_1(s, \frac{s}{\epsilon^2}, \underline{x}^{\epsilon}(\nu,\sigma,\underline{x}_0)) | \mathcal{F}_0^{\mu/\epsilon^2} \}| \Big] |x_2^{\epsilon}(\nu,\sigma,\underline{x}_0)|.
\end{aligned} \tag{5.116}$$

**Proof**: Let us use definition of inner product and (5.76). Then

$$\begin{aligned}
\left( \frac{\partial F_2}{\partial \underline{x}}, \underline{F} \right) &= \frac{\partial F_2}{\partial x_1} F_1 + \frac{\partial F_2}{\partial x_2} F_2 \\
&= (\partial^2_{x_1 x_1} F_1) x_2^{\epsilon} F_1 + (\partial_{x_1} F_1)(\partial_{x_1} F_1) x_2^{\epsilon}.
\end{aligned} \tag{5.117}$$

Now we take conditional expectation of (5.117) and use the fact that $x_2^\epsilon(\nu,\sigma,\underline{x}_0)$ and $F_1(\nu,\frac{\nu}{\epsilon^2}, \underline{x}^\epsilon(\nu,\sigma,\underline{x}_0))$ and $\partial_{x_1}F_1(\nu,\frac{\nu}{\epsilon^2},\underline{x}^\epsilon(\nu,\sigma,\underline{x}_0))$ are $\mathcal{F}_0^{\mu/\epsilon^2}$ measurable ($\nu \le \mu$). Then

$$E\{\left(\frac{\partial F_2}{\partial \underline{x}}(s,\frac{s}{\epsilon^2},\underline{x}^\epsilon(\nu,\sigma,\underline{x}_0)),\ \underline{F}(\nu,\frac{\nu}{\epsilon^2},\underline{x}^\epsilon(\nu,\sigma,\underline{x}_0))\right)|\mathcal{F}_0^{\mu/\epsilon^2}\} \tag{5.118}$$
$$= E\{\partial^2_{x_1x_1}F_1(s,\frac{s}{\epsilon^2},\underline{x}^\epsilon(\nu,\sigma,\underline{x}_0))|\mathcal{F}_0^{\mu/\epsilon^2}\}\,x_2^\epsilon(\nu,\sigma,\underline{x}_0)\,F_1(\nu,\frac{\nu}{\epsilon^2},\underline{x}^\epsilon(\nu,\sigma,\underline{x}_0))$$
$$+\,E\{\partial_{x_1}F_1(s,\frac{s}{\epsilon^2},\underline{x}^\epsilon(\nu,\sigma,\underline{x}_0))|\mathcal{F}_0^{\mu/\epsilon^2}\}\,\partial_{x_1}F_1(\nu,\frac{\nu}{\epsilon^2},\underline{x}^\epsilon(\nu,\sigma,\underline{x}_0))\,x_2^\epsilon(\nu,\sigma,\underline{x}_0).$$

Upon using (5.66), we obtain (5.116) immediately. ■

**To continue proving Lemma 5.3.4**: With the above four lemmas, we are ready to estimate the four terms on the right side of (5.104). We call the absolute value of the expectation of each term the *11*, *12*, *21* and *22*-term from now on and estimate these individually.

(i) *11*-term:

$$|E\{\int_{\tau_k}^{\tau_{k+1}} F_2(s,\frac{s}{\epsilon^2},\underline{x}^\epsilon(\tau_{k-1},\sigma,\underline{x}_0))\,[\,x_2^\epsilon(\tau_{k-1},\sigma,\underline{x}_0)\,]^{2p-1}ds\,\}|$$
$$\le E\{\int_{\tau_k}^{\tau_{k+1}}|E\{\partial_{x_1}F_1(s,\frac{s}{\epsilon^2},\underline{x}^\epsilon(\tau_{k-1},\sigma,\underline{x}_0))|\mathcal{F}_0^{\tau_{k-1}/\epsilon^2}\}|\,|x_2^\epsilon(\tau_{k-1},\sigma,\underline{x}_0)|^{2p}ds\}$$
$$\le E\{\int_{\tau_k}^{\tau_{k+1}} 2\rho(\frac{s-\tau_{k-1}}{\epsilon^2})ds\,C_{n,\epsilon}\,|x_2^\epsilon(\tau_{k-1},\sigma,\underline{x}_0)|^{2p}\}$$
$$\le C_{n,\epsilon}\,\epsilon\,\rho(\frac{1}{\epsilon})E\{|x_2^\epsilon(\tau_{k-1},\sigma,\underline{x}_0)|^{2p}\}$$
$$\le \epsilon^2\,C_{n,\epsilon}\,E\{|x_2^\epsilon(\tau_{k-1},\sigma,\underline{x}_0)|^{2p}\},$$

where we used (5.76), a property of conditional expectation, Lemma 5.3.5 (a mixing lemma) with (5.66), the monotonicity of the mixing rate, $\tau_{k+1}-\tau_k=\epsilon$ and the uniform boundedness of $\frac{1}{\epsilon}\,\rho(\frac{1}{\epsilon})$ in $\epsilon \in (0,1]$, sequentially. Also the notational convention for $C_{n,\epsilon}$ was used.

(ii) *12*-term:

$$|E\{\frac{2p-1}{\epsilon}\int_{\tau_k}^{\tau_{k+1}}\int_{\tau_{k-1}}^{s} F_2(s,\frac{s}{\epsilon^2},\underline{x}^\epsilon(\tau_{k-1},\sigma,\underline{x}_0))\,F_2(\nu,\frac{\nu}{\epsilon^2},\underline{x}^\epsilon(\nu,\sigma,\underline{x}_0))\cdot$$
$$\cdot[\,x_2^\epsilon(\nu,\sigma,\underline{x}_0)\,]^{2(p-1)}d\nu ds\,\}|$$

$$= |E\{ \frac{2p-1}{\epsilon} \int_{\tau_k}^{\tau_{k+1}} \int_{\tau_{k-1}}^{s} \partial_{x_1} F_1(s, \frac{s}{\epsilon^2}, \underline{x}^\epsilon(\tau_{k-1}, \sigma, \underline{x}_0))\, \partial_{x_1} F_1(\nu, \frac{\nu}{\epsilon^2}, \underline{x}^\epsilon(\nu, \sigma, \underline{x}_0)) \cdot$$

$$\cdot x_2^\epsilon(\tau_{k-1}, \sigma, \underline{x}_0)\, [\, x_2^\epsilon(\nu, \sigma, \underline{x}_0)\,]^{2p-1} d\nu ds \,\}|$$

$$\leq E\{ \frac{2p-1}{\epsilon} \int_{\tau_k}^{\tau_{k+1}} \int_{\tau_{k-1}}^{s} |E\{\, \partial_{x_1} F_1(s, \frac{s}{\epsilon^2}, \underline{x}^\epsilon(\tau_{k-1}, \sigma, \underline{x}_0))\, | \mathcal{F}_0^{\nu/\epsilon^2} \}| \cdot$$

$$\cdot |\partial_{x_1} F_1(\nu, \frac{\nu}{\epsilon^2}, \underline{x}^\epsilon(\nu, \sigma, \underline{x}_0))|\, |x_2^\epsilon(\tau_{k-1}, \sigma, \underline{x}_0)|\, |x_2^\epsilon(\nu, \sigma, \underline{x}_0)|^{2p-1} d\nu ds \,\}|$$

$$\leq E\{ \frac{2p-1}{\epsilon} \int_{\tau_k}^{\tau_{k+1}} \int_{\tau_{k-1}}^{s} 2\rho(\frac{s-\nu}{\epsilon^2}) d\nu ds\, C_{n,\epsilon}^2 e^{2(2p-1)C_{n,\epsilon}} |x_2^\epsilon(\tau_{k-1}, \sigma, \underline{x}_0)|^{2p} \}$$

$$\leq \epsilon^2 C_{n,\epsilon}\, E\{\, |x_2^\epsilon(\tau_{k-1}, \sigma, \underline{x}_0)|^{2p} \},$$

where we used (5.76), a property of conditional expectation, Lemma 5.3.5 with (5.66), Lemma 5.3.6 and Lemma 5.3.7, sequentially, with the notational convention for $C_{n,\epsilon}$.

(iii) *21*-term:

$$|E\{ \frac{1}{\epsilon} \int_{\tau_k}^{\tau_{k+1}} \int_{\tau_{k-1}}^{s} \left( \frac{\partial F_2}{\partial \underline{x}}(s, \frac{s}{\epsilon^2}, \underline{x}^\epsilon(\nu, \sigma, \underline{x}_0)), \underline{F}(\nu, \frac{\nu}{\epsilon^2}, \underline{x}^\epsilon(\nu, \sigma, \underline{x}_0)) \right) \cdot$$

$$\cdot [\, x_2^\epsilon(\tau_{k-1}, \sigma, \underline{x}_0)\,]^{2p-1} d\nu ds \,\}|$$

$$\leq E\{ \frac{1}{\epsilon} \int_{\tau_k}^{\tau_{k+1}} \int_{\tau_{k-1}}^{s} |E\{ \left( \frac{\partial F_2}{\partial \underline{x}}(s, \frac{s}{\epsilon^2}, \underline{x}^\epsilon(\nu, \sigma, \underline{x}_0)), \underline{F}(\nu, \frac{\nu}{\epsilon^2}, \underline{x}^\epsilon(\nu, \sigma, \underline{x}_0)) \right) | \mathcal{F}_0^{\nu/\epsilon^2} \}| \cdot$$

$$\cdot |x_2^\epsilon(\tau_{k-1}, \sigma, \underline{x}_0)|^{2p-1} d\nu ds \,\}$$

$$\leq E\{ \frac{1}{\epsilon} \int_{\tau_k}^{\tau_{k+1}} \int_{\tau_{k-1}}^{s} C_{n,\epsilon} \; [\, |E\{\, \partial^2_{x_1 x_1} F_1(s, \frac{s}{\epsilon^2}, \underline{x}^\epsilon(\nu, s, \underline{x}_0))\, | \mathcal{F}_0^{\nu/\epsilon^2} \}| + |E\{\, \partial_{x_1} F_1(s, \frac{s}{\epsilon^2}$$

$$, \underline{x}^\epsilon(\nu, \sigma, \underline{x}_0))\, | \mathcal{F}_0^{\nu/\epsilon^2} \}|\, ]\, |x_2^\epsilon(\nu, \sigma, \underline{x}_0)|\, |x_2^\epsilon(\tau_{k-1}, \sigma, \underline{x}_0)|^{2p-1} d\nu ds \,\}$$

$$\leq \frac{C_{n,\epsilon}}{\epsilon} 2 \int_{\tau_k}^{\tau_{k+1}} \int_{\tau_{k-1}}^{s} 2\rho(\frac{s-\nu}{\epsilon^2}) d\nu ds\, C_{n,\epsilon} e^{2C_{n,\epsilon}} E\{\, |x_2^\epsilon(\tau_{k-1}, \sigma, \underline{x}_0)|^{2p} \}$$

$$\leq \epsilon^2 C_{n,\epsilon}\, E\{\, |x_2^\epsilon(\tau_{k-1}, \sigma, \underline{x}_0)|^{2p} \},$$

where we used a property of conditional expectation, Lemma 5.3.8, Lemma 5.3.5 with (5.66), Lemma 5.3.6 and Lemma 5.3.7, sequentially, and the notational convention for $C_{n,\epsilon}$.

(iv) *22*-term:

$$|E\{ \frac{2p-1}{\epsilon^2} \int_{\tau_k}^{\tau_{k+1}} \int_{\tau_{k-1}}^{s} \int_{\tau_{k-1}}^{s} \left( \frac{\partial F_2}{\partial \underline{x}}(s, \frac{s}{\epsilon^2}, \underline{x}^\epsilon(\nu, \sigma, \underline{x}_0)), \underline{F}(\nu, \frac{\nu}{\epsilon^2}, \underline{x}^\epsilon(\nu, \sigma, \underline{x}_0)) \right) \cdot$$

$$
\begin{aligned}
&\cdot F_2(\mu, \frac{\mu}{\epsilon^2}, \underline{x}^\epsilon(\mu,\sigma,\underline{x}_0))\,[\,x_2^\epsilon(\mu,\sigma,\underline{x}_0)\,]^{2(p-1)} d\mu d\nu ds\,\}| \\
\le\ & E\{\frac{2p-1}{\epsilon^2}\int_{\tau_k}^{\tau_{k+1}}\int_{\tau_{k-1}}^{s}\int_{\tau_{k-1}}^{\nu} |E\{\Big(\frac{\partial F_2}{\partial \underline{x}}(s,\frac{s}{\epsilon^2},\underline{x}^\epsilon(\nu,\sigma,\underline{x}_0)), \underline{F}(\nu,\frac{\nu}{\epsilon^2},\underline{x}^\epsilon(\nu,\sigma,\underline{x}_0))\Big)\,|\mathcal{F}_0^{\nu/\epsilon^2}\}|\cdot \\
&\cdot|\partial_{x_1}F_1(\mu,\frac{\mu}{\epsilon^2},\underline{x}^\epsilon(\mu,\sigma,\underline{x}_0))|\,|x_2^\epsilon(\mu,\sigma,\underline{x}_0)|^{2p-1} d\mu d\nu ds\,\} \\
&+E\{\frac{2p-1}{\epsilon^2}\int_{\tau_k}^{\tau_{k+1}}\int_{\tau_{k-1}}^{s}\int_{\tau_{k-1}}^{\mu} |E\{\Big(\frac{\partial F_2}{\partial \underline{x}}(s,\frac{s}{\epsilon^2},\underline{x}^\epsilon(\nu,\sigma,\underline{x}_0)), \underline{F}(\nu,\frac{\nu}{\epsilon^2},\underline{x}^\epsilon(\nu,\sigma,\underline{x}_0))\Big)\,|\mathcal{F}_0^{\mu/\epsilon^2}\}|\cdot \\
&\cdot|\partial_{x_1}F_1(\mu,\frac{\mu}{\epsilon^2},\underline{x}^\epsilon(\mu,\sigma,\underline{x}_0))|\,|x_2^\epsilon(\mu,\sigma,\underline{x}_0)|^{2p-1} d\nu d\mu ds\,\} \\
\le\ & E\{\frac{2p-1}{\epsilon^2}\int_{\tau_k}^{\tau_{k+1}}\int_{\tau_{k-1}}^{s}\int_{\tau_{k-1}}^{\nu} C_{n,\epsilon}\Big[|E\{\,\partial^2_{x_1x_1}F_1(s,\frac{s}{\epsilon^2},\underline{x}^\epsilon(\nu,\sigma,\underline{x}_0))\,|\mathcal{F}_0^{\nu/\epsilon^2}\}| \\
&+|E\{\,\partial_{x_1}F_1(s,\frac{s}{\epsilon^2},\underline{x}^\epsilon(\nu,\sigma,\underline{x}_0))\,|\mathcal{F}_0^{\nu/\epsilon^2}\}|\Big]\,|x_2^\epsilon(\nu,\sigma,\underline{x}_0)|\,C_{n,\epsilon}\,|x_2^\epsilon(\mu,\sigma,\underline{x}_0)|^{2p-1} d\mu d\nu ds\,\} \\
&+E\{\frac{2p-1}{\epsilon^2}\int_{\tau_k}^{\tau_{k+1}}\int_{\tau_{k-1}}^{s}\int_{\tau_{k-1}}^{\mu} C_{n,\epsilon}\Big[|E\{\,\partial^2_{x_1x_1}F_1(s,\frac{s}{\epsilon^2},\underline{x}^\epsilon(\mu,\sigma,\underline{x}_0))\,|\mathcal{F}_0^{\mu/\epsilon^2}\}| \\
&+|E\{\,\partial_{x_1}F_1(s,\frac{s}{\epsilon^2},\underline{x}^\epsilon(\nu,\sigma,\underline{x}_0))\,|\mathcal{F}_0^{\mu/\epsilon^2}\}|\Big]\,|x_2^\epsilon(\nu,\sigma,\underline{x}_0)|\,C_{n,\epsilon}\,|x_2^\epsilon(\mu,\sigma,\underline{x}_0)|^{2p-1} d\nu d\mu ds\,\} \\
\le\ & \frac{C_{n,\epsilon}}{\epsilon^2}\Big(2\int_{\tau_k}^{\tau_{k+1}}\int_{\tau_{k-1}}^{s}\int_{\tau_{k-1}}^{\nu} 2\rho(\frac{s-\nu}{\epsilon^2})d\mu d\nu ds + 2\int_{\tau_k}^{\tau_{k+1}}\int_{\tau_{k-1}}^{s}\int_{\tau_{k-1}}^{\mu} 2\rho(\frac{s-\mu}{\epsilon^2})d\nu d\mu ds\Big)\cdot \\
&\cdot C'^2_{n,\epsilon}e^{2C_{n,\epsilon}}e^{2(2p-1)C_{n,\epsilon}}E\{\,|x_2^\epsilon(\tau_{k-1},\sigma,\underline{x}_0)|^{2p}\,\} \\
\le\ & \epsilon^2\,C_{n,\epsilon}\,E\{\,|x_2^\epsilon(\tau_{k-1},\sigma,\underline{x}_0)|^{2p}\,\},
\end{aligned}
$$

where we used a property of conditional expectation, (5.76), Lemma 5.3.8, Lemma 5.3.5 together with (5.66), Lemma 5.3.6, the fact that $\int_{\tau_{k-1}}^{\nu}\rho(\frac{s-\nu}{\epsilon^2})d\mu \le 2\epsilon\rho(\frac{s-\nu}{\epsilon^2})$ for $\tau_{k-1}\le\nu\le\tau_{k+1}$, Lemma 5.3.7, sequentially, and the notational convention for $C_{n,\epsilon}$.

In summary of this section, from the above estimates (i)-(iv), Lemma 5.3.4, i.e., inequality (5.94), is proved. Inequality (5.94) allows us to use the discrete version of the Gronwall inequality to show the uniform boundedness of the p$^{th}$ moment of $\frac{\partial x^\epsilon}{\partial x}$ on the entire interval $0\le\tau\le\eta_0$ (see Lemma 5.3.2). This boundedness in turn gives the applicability of a mixing lemma (Lemma 5.3.1) to the estimation of $I_n^2(f)$.

# Chapter 6

# PROBLEM WITH RAPIDLY VARYING DETERMINISTIC PART

## 6.1 Problem Formulation below the Turning Point

We have formulated in Chapter 3 the problem of interest above the turning point in the medium as a stochastic initial value problem for the phase of the reflection coefficient. The asymptotic stochastic behavior of this problem is characterized by the extended limit theorem established in Chapter 4. For the problem of interest below the turning point, however, the scattering variables defined in Chapter 3 are not appropriate. The error term in the Lynn and Keller approximant is not "small" at all below the turning point; it grows exponentially in the outer region below the turning point. Below the turning point, therefore, we need different scattering variables for the same basic problem, i.e., (3.9).

First let us introduce the following dependent variables:

$$v_1(z) \equiv P(z), \tag{6.1}$$

$$v_2(z) \equiv \rho_0 c_0 U_3(z). \tag{6.2}$$

With these dependent variables, our basic system (3.9) becomes

$$\frac{d}{dz}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = i\frac{k}{\epsilon}\begin{bmatrix} 0 & \hat{\rho} \\ \hat{K}^{-1} - \hat{\rho}^{-1}\sin^2\theta & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}, \tag{6.3}$$

where $\hat{\rho}$, $\hat{c}$ and $\hat{K}$ are defined as $\rho/\rho_0$, $c/c_0$ and $\hat{\rho}\hat{c}^2$, respectively, and $k \equiv \omega/c_0$.

The above system (6.3) becomes the following second order differential equation if $\hat{\rho}$ and $\hat{K}$ are constants:

$$\frac{d^2 v_1}{dz^2} + \frac{k^2}{\epsilon^2}\left(\hat{c}^{-2} - \sin^2\theta\right) v_1 = 0, \tag{6.4}$$

$$v_2 = \frac{\epsilon}{ik}\hat{\rho}^{-1}\frac{dv_1}{dz}. \tag{6.5}$$

When, in particular, $\hat{\rho} = 1$ and $\hat{c} = 1$ (corresponding to the upper homogeneous medium above the layer), the solutions of equations (6.4)-(6.5) are

$$v_1(z) = V_1\, e^{-i\frac{k}{\epsilon}z\cos\theta} + V_2\, e^{i\frac{k}{\epsilon}z\cos\theta}, \tag{6.6}$$

$$v_2(z) = -\cos\theta\,[V_1\, e^{-i\frac{k}{\epsilon}z\cos\theta} - V_2\, e^{i\frac{k}{\epsilon}z\cos\theta}], \tag{6.7}$$

where $V_1$ and $V_2$ are constants to be determined.

Motivated by (6.6) and (6.7), we consider another formulation of (6.3) by introducing the following dependent variables as in [13]:

$$a(z) \equiv \frac{1}{2}[v_1(z) + v_2(z)\sec\theta], \tag{6.8}$$

$$b(z) \equiv \frac{1}{2}[v_1(z) - v_2(z)\sec\theta]. \tag{6.9}$$

In terms of these scattering variables, (6.3) becomes

$$\frac{d}{dz}\begin{bmatrix} a \\ b \end{bmatrix} = \frac{ik}{2\epsilon}\begin{bmatrix} A_{11}(z) & A_{12}(z) \\ -A_{12}(z) & -A_{11}(z) \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix}, \tag{6.10}$$

$$A_{1i}(z) \equiv (-1)^{i-1}\hat{\rho}\cos\theta + (\hat{K}^{-1} - \hat{\rho}^{-1}\sin^2\theta)\sec\theta, \quad i = 1, 2. \tag{6.11}$$

Here the dependent variable $a$ represents an upward-propagating wave, while $b$ corresponds to a downward-propagating wave.

Within the region of interest, i.e., $-L \leq z \leq z_T$, scaling condition (3.11)-(3.12) is expressed as follows in terms of the new notation:

$$\hat{\rho}(z, z/\epsilon^2) = \hat{\alpha}(z)[1 + \eta(z, z/\epsilon^2)], \tag{6.12}$$

$$\hat{K}^{-1}(z, z/\epsilon^2) - \hat{\rho}^{-1}(z, z/\epsilon^2)\sin^2\theta = \hat{\beta}(z)[1 + \nu(z, z/\epsilon^2)], \tag{6.13}$$

where $\hat{\alpha}$ and $\hat{\beta}$ are mean values of the left side of (6.12) and (6.13), respectively, i.e.,

$$\hat{\alpha} \equiv E\{\hat{\rho}(z, z/\epsilon^2)\}, \quad \hat{\beta} \equiv E\{\hat{K}^{-1}(z, z/\epsilon^2) - \hat{\rho}^{-1}(z, z/\epsilon^2)\sin^2\theta\}, \tag{6.14}$$

and functions $\eta$ and $\nu$ are zero mean, unit correlation length random functions of their second arguments. Note that $\hat{\alpha} = \rho_0^{-1}\alpha$ and $\hat{\beta} = \rho_0 c_0^2 \beta$ compared with old notation $\alpha$ and $\beta$ defined by (3.13). We define

$$\hat{\alpha}_1(z, z/\epsilon^2) \equiv \hat{\alpha}(z)\eta(z, z/\epsilon^2), \quad \hat{\beta}_1(z, z/\epsilon^2) \equiv \hat{\beta}(z)\nu(z, z/\epsilon^2). \tag{6.15}$$

Then, using the following notation

$$\overline{A}_{1i}(z) = (-1)^{i-1}\hat{\alpha}(z)\cos\theta + \hat{\beta}(z)\sec\theta, \quad i = 1, 2, \tag{6.16}$$

$$\mathcal{A}_{1i}(z, z/\epsilon^2) = (-1)^{i-1}\hat{\alpha}_1(z, z/\epsilon^2)\cos\theta + \hat{\beta}_1(z, z/\epsilon^2)\sec\theta, \tag{6.17}$$

the coefficient matrix on the right side of (6.10) is expressed as the sum of an average value and a zero mean random fluctuation:

$$A_{1i}(z, z/\epsilon^2) = \overline{A}_{1i}(z) + \mathcal{A}_{1i}(z, z/\epsilon^2), \quad -L \le z \le z_T. \tag{6.18}$$

To obtain the initial value problem in the region of interest, let us first define the following quantity called the reflection coefficient $r$:

$$r \equiv \frac{a}{b}. \tag{6.19}$$

Recall that the total reflection at $z = -L$ was assumed in Chapter 3;

$$a(-L) = \Gamma_{-L} b(-L), \quad |\Gamma_{-L}| = 1, \tag{6.20}$$

where $\Gamma_{-L} = r(-L)$. Here we note that although the total reflection at $z = -L$ was assumed in terms of the different scattering variables (i.e., $A$ and $B$ in Chapter 3), the above identity for the scattering variables $a$ and $b$ still holds from relation (6.36) described in next section. With a (upward) scaled measure

$$\tau \equiv z + L \tag{6.21}$$

from the end point $z = -L$ of the slab, (6.10) together with (6.20) becomes the Riccati initial value problem for the reflection coefficient:

$$\frac{d}{d\tau} r = \frac{ik}{2\epsilon}\left[\,2rA_{11} + (1 + r^2)A_{12}\,\right], \quad 0 \le \tau \le \tau_0 \equiv z_T + L, \tag{6.22}$$

$$r(0) = \Gamma_{-L}, \quad |\Gamma_{-L}| = 1. \tag{6.23}$$

We focus on the evolution of the reflection coefficient from now on. The problem reduces to a consideration of the phase of the reflection coefficient due to the next lemma.

**Lemma 6.1.1** *Under assumption (6.23), the reflection coefficient $r$ is unimodular, i.e.,*

$$|r(\tau)| = \frac{|a(\tau)|}{|b(\tau)|} = 1, \quad \tau \geq 0. \tag{6.24}$$

**Proof**: The structure of (6.10) and the initial condition (6.23) establish the lemma as follows. For simplicity, let $A \equiv \frac{ik}{2\epsilon} A_{11}$. A direct calculation gives

$$\frac{d}{d\tau}(|a|^2 - |b|^2) = (A + A^*)(|a|^2 - |b|^2). \tag{6.25}$$

Since $A$ is pure imaginary, the right side of the above identity vanishes; $|a|^2 - |b|^2$ is a constant. This constant is, however, zero at $\tau = 0$ because of (6.23). Hence, $|a| = |b|$ throughout the region $\tau \geq 0$. ■

Due to Lemma 6.1.1, the problem reduces to a consideration of the phase $\psi$ (of the unimodular reflection coefficient $r$) defined by

$$r \equiv e^{-i\psi} \tag{6.26}$$

Then, from (6.22), one obtains the corresponding initial value problem for the phase $\psi$:

$$\frac{d}{d\tau}\psi = \frac{1}{\epsilon}F_0(\tau, \psi) + \frac{1}{\epsilon}F_1(\tau, \tau/\epsilon^2, \psi), \quad 0 \leq \tau \leq \tau_0 \tag{6.27}$$

$$\psi(0) = \psi_0, \tag{6.28}$$

where $F_0$ and $F_1$ are the following deterministic and mean-zero random fields, respectively:

$$F_0(\tau, \psi) \equiv -k\,[\,\overline{A}_{11}(\tau) + \overline{A}_{12}(\tau)\cos\psi\,], \tag{6.29}$$

$$F_1(\tau, \tau/\epsilon^2, \psi) \equiv -k\,[\,\mathcal{A}_{11}(\tau, \tau/\epsilon^2) + \mathcal{A}_{12}(\tau, \tau/\epsilon^2)\cos\psi\,]. \tag{6.30}$$

Recall that $\overline{A}_{1i}$ and $\mathcal{A}_{1i}$, i=1,2, are assumed to be bounded. We also note that $\overline{A}_{12}$ is negative throughout the region of present interest.

## 6.2 Relation between Problems above and below the Turning Point

Our basic stochastic problem in the random slab $-L \leq z \leq 0$ consists of equation (3.9) with scaling condition (3.11)-(3.12) and the assumed total reflection at $z = -L$. It has been dealt with in two divided regions, i.e., the region above the turning point and the region below the turning point. For each region, the structure of the equations and the total reflection at $z = -L$ lead to considering unimodular reflecton coefficients. Two nonlinear stochastic initial value problems for these quantities are formulated corresponding to these two regions. They are (3.74)-(3.77) and (6.27)-(6.30), respectively.

We are, therefore, required to consider how the transformation between the two problems occurs at the point where they adjoin, i.e., the turning point $z_T$. Suppose that $\psi^\epsilon(z_T, -L, \psi_0)$ ($\equiv \psi_T$ in brief) is the random phase evaluated at the turning point for the problem below the turning point, i.e., (6.27)-(6.30). From (6.26),

$$r(z_T) = \frac{a(z_T)}{b(z_T)} = e^{-i\psi_T}. \tag{6.31}$$

For the dependent variables $v_1$ and $v_2$ defined by (6.1)-(6.2), we have the following expression in terms of the scattering variables $a$ and $b$: From (6.8)-(6.9), $v_1(z_T) = a(z_T) + b(z_T)$ and $v_2(z_T) = \cos\theta(a(z_T) - b(z_T))$. So, from (6.1)-(6.2), $P(z_T) = a(z_T) + b(z_T)$ and $U_3(z_T) = \frac{\cos\theta}{\rho_0 c_0}(a(z_T) - b(z_T))$. The pressure and velocity at the turning point obtained by solving the problem in the region $-L \leq z \leq z_T$ are, therefore, given by

$$P(z_T) = b(z_T)(e^{-i\psi_T} + 1), \tag{6.32}$$

$$U_3(z_T) = b(z_T)\frac{\cos\theta}{\rho_0 c_0}(e^{-i\psi_T} - 1). \tag{6.33}$$

Now, consider the problem above the turning point. The scattering variables $A$ and $B$ at the turning point are given by

$$A(z_T) = \frac{1}{2}\left(P(z_T)\phi_{22}(z_T) - U_3(z_T)\phi_{12}(z_T)\right), \tag{6.34}$$

$$B(z_T) = \frac{1}{2}\left(-P(z_T)\phi_{21}(z_T) + U_3(z_T)\phi_{11}(z_T)\right), \tag{6.35}$$

from (3.21), where $\phi_{ij}$'s are given by (3.28)-(3.31). Let $\Psi_T$ denote the phase of the corresponding reflection coefficient at $z = z_T$. Then, from the continuity of the fields at the turning point, we obtain the following algebraic relation between $\Psi_T$ and $\psi_T$ from (6.32)-(6.33), (6.34)-(6.35) and (3.73):

$$e^{-i\Psi_T} = \frac{A(z_T)}{B(z_T)} = \frac{(e^{-i\psi_T}+1)\phi_{22}(z_T) - \frac{\cos\theta}{\rho_0 c_0}(e^{-i\psi_T}-1)\phi_{12}(z_T)}{-(e^{-i\psi_T}+1)\phi_{21}(z_T) + \frac{\cos\theta}{\rho_0 c_0}(e^{-i\psi_T}-1)\phi_{11}(z_T)}. \tag{6.36}$$

Recall that the limit theory established in Chapter 4 provides uniform approximate statistics for the stochastic initial value problem for the phase $\psi^\epsilon(z, z_T, \Psi_T)$ in the region $z_T \le z \le 0$ above the turning point. For arbitrary $f \in C^4$, the conditional expectation of $f(\psi^\epsilon(0, z_T, \Psi_T))$ is asymptotically approximated by the solution $u(z_T, 0, \Psi_T; f)$ of the backward Kolmogorov equation (4.14)-(4.16) uniformly in $\Psi_T$ according to Theorem 1;

$$E\{\, f(\psi^\epsilon(0, z_T, \Psi_T))\,|\mathcal{F}_0^{z_T/\epsilon^2}\} \;\approx\; u(z_T, 0, \Psi_T; f). \tag{6.37}$$

The above relation (6.36) defines $\Psi_T$ as a (multi-valued) function of $\psi_T$. We call this function $h^T$, i.e.,

$$\Psi_T \equiv h^T(\psi_T). \tag{6.38}$$

Then obviously we have

$$u(z_T, 0, \Psi_T; f) = u(z_T, 0, h^T(\psi_T); f) \equiv g(\psi_T), \tag{6.39}$$

which is a function of $\psi_T$ $(= \psi^\epsilon(z_T, -L, \psi_0))$. Note that $g \in C^4$ because it solves a parabolic differential equation. Then, from (6.37) and (6.39), the expected value of $f(\psi^\epsilon(0, z_T, \Psi_T))$ becomes

$$E\{f(\psi^\epsilon(0, z_T, \Psi_T))\} = E\{E\{f(\psi^\epsilon(0, z_T, \Psi_T))|\mathcal{F}_0^{z_T/\epsilon^2}\}\} \approx E\{g(\psi^\epsilon(z_T, -L, \psi_0))\}, \tag{6.40}$$

where $\psi_0$ is the initial value given at $z = -L$.

For the final estimate in the whole region $-L \le z \le 0$, therefore, we are required to develop another limit theory which can characterize asymptotically the stochastic initial value problem in the region $-L \le z \le z_T$. This will be carried out in the following section.

## 6.3 A Limit Theorem in Banach Space – Theorem 2

In this section, we develop an abstract limit theorem for the stochastic initial value problem that models (6.27)-(6.30). We introduce function spaces and evolution operators as in Chapter 4 and define differential operators relevant to deterministic and random parts of the right-hand side of (6.27).

We start with introducing some preliminary terminology and hypotheses as we did in Chapter 4. Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\mathcal{F}_s^t$, $0 \leq s \leq t \leq \infty$, be a family of $\sigma$-algebras contained in $\mathcal{F}$ such that

$$\mathcal{F}_{s_1}^{t_1} \subset \mathcal{F}_{s_2}^{t_2}, \quad 0 \leq s_2 \leq s_1 \leq t_1 \leq t_2 \leq \infty. \tag{6.41}$$

We assume that $P$ satisfies the following strong mixing condition:

$$\sup_{s \geq 0} \sup_{A \in \mathcal{F}_{s+t}^{\infty}, B \in \mathcal{F}_0^s} |P(A|B) - P(A)| \equiv \rho(t) \downarrow 0, \quad as\ t \uparrow \infty. \tag{6.42}$$

This monotonically decreasing nonnegative function $\rho$ is called the mixing rate and assumed to satisfy

$$\int_0^{\infty} \rho^{1/2}(s)\, ds < \infty. \tag{6.43}$$

The conditional probabilities relative to $\mathcal{F}_0^s$, $0 \leq s \leq \infty$, are assumed to have a regular version so that we are able to have the following representation almost everywhere:

$$E\{\,\cdot\,|\mathcal{F}_0^s\} = \int_{\Omega} \cdot\, P_s(d\omega|\omega'). \tag{6.44}$$

Let $F(\tau, \nu, x, \omega)$ be a function from $[0, \tau_0] \times [0, \infty) \times \mathcal{R} \times \Omega$ into $\mathcal{R}$, where $\tau_0$ is a fixed positive number and $\mathcal{R}$ denotes the set of real numbers. The random field $F$ is assumed to be jointly measurable with respect to its arguments and $\mathcal{F}_\nu^\nu$ measurable as a function of $\omega \in \Omega$ for fixed $\tau$, $\nu$ and $x$.

We introduce the one-point compactification of $\mathcal{R}$, denoted by $\mathcal{R}_c$, and $\mathcal{C}^0$ denotes the space of bounded continuous real valued functions on $\mathcal{R}_c$ with the supremum norm $\|\cdot\|_0$. Let $\mathcal{C}^k$ denote the space of real valued functions on $\mathcal{R}_c$ with bounded continuous derivatives up to order $k$ with $\|\cdot\|_k$ the sum of the supremum norm of the function and its derivatives up to order $k$. The spaces $\mathcal{C}^k$ are separable Banach spaces and dense subspaces of $\mathcal{C}^0$ such that $\mathcal{C}^k \subset \mathcal{C}^{k-1}$ and $\|f\|_{k-1} \leq \|f\|_k$ for arbitrary $f \in \mathcal{C}^k$.

With a positive parameter $\epsilon > 0$, we consider the following stochastic initial value problem which is a general form of our model problem (6.27)-(6.30):

$$\frac{d}{d\tau}x^\epsilon(\tau,\sigma,x) = \frac{1}{\epsilon}F(\tau,\tau/\epsilon^2,x^\epsilon(\tau,\sigma,x),\omega), \quad \tau > \sigma, \tag{6.45}$$

$$x^\epsilon(\sigma,\sigma,x) = x, \tag{6.46}$$

where the solution $x^\epsilon(\tau,\sigma,x)$ is $\mathcal{F}_{\sigma/\epsilon^2}^{\tau/\epsilon^2}$ measurable as a function of $\omega$ for any fixed $x$. Also we think of the following deterministic initial value problem:

$$\frac{d}{d\tau}x^0(\tau,\sigma,x) = \frac{1}{\epsilon}F_0(\tau,x^0(\tau,\sigma,x)), \quad \tau > \sigma, \tag{6.47}$$

$$x^0(\sigma,\sigma,x) = x, \tag{6.48}$$

which is an averaged problem in the sense that the zero-mean random fluctuation part $F_1$ is suppressed in the context of (6.27)-(6.30).

To develop a limit theory for the problem of interest, it is convenient to introduce evolution operators associated with (6.45)-(6.46) and (6.47)-(6.48), respectively; we define random propagators $U^\epsilon(\sigma,\tau)$ and deterministic propagators $U^0(\sigma,\tau)$ by

$$(U^\epsilon(\sigma,\tau)f)(x) \equiv f(x^\epsilon(\tau,\sigma,x)), \tag{6.49}$$

$$(U^0(\sigma,\tau)f)(x) \equiv f(x^0(\tau,\sigma,x)), \tag{6.50}$$

for any $f \in \mathcal{C}^0$. These are contraction operators on $\mathcal{C}^0 \to \mathcal{C}^0$. $U^\epsilon(\sigma,\tau)f$, $f \in \mathcal{C}^0$, is strongly $\mathcal{F}_{\sigma/\epsilon^2}^{\tau/\epsilon^2}$ measurable. Note that $U^0(\sigma,\tau)$ depends on $\epsilon$, though not shown explicitly.

It is also useful to define the following random differential operators $V(\tau)$ and $V_1(\tau)$ and deterministic differential operator $V_0(\tau)$: Define

$$(V(\tau)f)(x) \equiv F(\tau,\tau/\epsilon^2,x)\,\partial_x f(x), \tag{6.51}$$

$$(V_0(\tau)f)(x) \equiv F_0(\tau,x)\,\partial_x f(x), \tag{6.52}$$

for arbitrary $f \in C^1$. The operator $V_1(\tau)$ is defined by the following which corresponds to the random part of the random field $F$:

$$V_1(\tau)f \equiv V(\tau)f - V_0(\tau)f, \tag{6.53}$$

for arbitrary $f \in C^1$.

Both operators $U^\epsilon(\sigma, \tau)$ and $U^0(\sigma, \tau)$ satisfy the finite propagator property and the infinitesimal forward and backward propagator properties already described in Chapter 5 (see Lemma 5.1.1).

Now we are ready to state and prove our main result of this chapter.

**Theorem 2** *Let $U^\epsilon(\sigma, \tau)$, $U^0(\sigma, \tau)$, $V(\tau)$, $V_0(\tau)$ and $V_1(\tau)$, $0 \le \sigma \le \tau \le \tau_0$, be the operators corresponding to stochastic initial value problem (6.45)-(6.46) or deterministic initial value problem (6.47)-(6.48). Suppose mixing rate condition (6.43) is satisfied. Let us assume the following conditions (i), (ii) and (iii) hold:*
*(i) For $f \in C^1$,*

$$E\{\, V_1(\tau)f\,\} = 0. \tag{6.54}$$

*(ii) There are positive constants $\tilde{\alpha}_k$, $\alpha_k$, $\tilde{\beta}_k$ and $\beta_k$ independent of $\sigma$, $\tau$ and $\epsilon$ such that for arbitrary $f \in C^k$,*

$$||U^\epsilon(\sigma,\tau)f||_k \le \tilde{\beta}_k\{1 + \frac{\tau-\sigma}{\epsilon} + \cdots + (\frac{\tau-\sigma}{\epsilon})^{k-1}\}e^{\tilde{\alpha}_k\frac{\tau-\sigma}{\epsilon}}||f||_k \;\; a.e., k = 1, 2 \tag{6.55}$$

$$||U^0(\sigma,\tau)f||_k \le \beta_k\{1 + \frac{\tau-\sigma}{\epsilon} + \cdots + (\frac{\tau-\sigma}{\epsilon})^{k-1}\}e^{\alpha_k\frac{\tau-\sigma}{\epsilon}}||f||_k, \quad 1 \le k \le 4. \tag{6.56}$$

*(iii) There are positive constants $\tilde{c}_k$ and $c_k$ independent of $\tau$ and $\epsilon$ such that for arbitrary $f \in C^k$*

$$||V(\tau)f||_{k-1} \le \tilde{c}_k||f||_k \;\; a.e., \qquad k = 1, 2, \tag{6.57}$$

$$||V_1(\tau)f||_{k-1} \le c_k||f||_k \;\; a.e., \qquad 1 \le k \le 4. \tag{6.58}$$

*Let us define operator $W^\epsilon(\sigma)$ on $C^{k+1} \to C^{k-1}$, $k \ge 1$, by*

$$W^\epsilon(\sigma)f \equiv \int_0^{1/\epsilon} E\{V_1(\sigma)U^0(\sigma, \sigma + \epsilon^2 t)V_1(\sigma + \epsilon^2 t)U^0(\sigma + \epsilon^2 t, \sigma)f\}\, dt \tag{6.59}$$

*and let $(B^\epsilon(\sigma,\tau)f)(x)$ denote the solution $u^\epsilon(\sigma,\tau,x;f)$ of parabolic final value problem*

$$\frac{\partial}{\partial\sigma}u(\sigma,\tau,x)+\mathcal{L}^\epsilon_\sigma u(\sigma,\tau,x)=0, \quad \sigma<\tau, \tag{6.60}$$

$$u(\tau,\tau,x)=f(x), \tag{6.61}$$

*where infinitesimal generator $\mathcal{L}^\epsilon_\sigma$ is defined on $\mathcal{C}^2\to\mathcal{C}^0$ as the sum of $\frac{1}{\epsilon}V_0(\sigma)$ and $W^\epsilon(\sigma)$ given by (6.52) and (6.59), respectively, i.e.,*

$$\mathcal{L}^\epsilon_\sigma \equiv \frac{1}{\epsilon}V_0(\sigma)+W^\epsilon(\sigma). \tag{6.62}$$

*Then, for arbitrary $f\in\mathcal{C}^4$, we obtain the estimate*

$$\sup_{0\le\tau\le\tau_0}\|E\{U^\epsilon(0,\tau)f\}-B^\epsilon(0,\tau)f\|_0\le\epsilon\, C(f;\tau_0), \tag{6.63}$$

*where $C(f;\tau_0)$ denotes a positive constant depending on $f$ and its derivatives up to order 4 and $\tau_0$ but independent of $\epsilon$.*

**Remark.** From conditions (6.56) and (6.58) and a mixing lemma (Lemma 5.2.2), it is not difficult to see that for arbitrary $f\in\mathcal{C}^{k+1}$, $1\le k\le 3$,

$$\|W^\epsilon(\sigma)f\|_{k-1}\le\gamma_{k+1}\|f\|_{k+1} \tag{6.64}$$

for some positive constant $\gamma_{k+1}$ independent of $\sigma$ and $\epsilon$ (see the proof of Lemma 6.4.4). So, the infinitesimal generator $\mathcal{L}^\epsilon_\sigma$ is the sum of a singularly perturbed drift term and a bounded diffusion term. Theorem 2, therefore, can characterize the solutions of stochastic differential equations with rapidly varying deterministic behavior plus random noise. This type of theory can be found also in [29]; the semigroup approach using projection operator was used. There, however, the scaling is different from our case.

## 6.4 Proof of Theorem 2

The proof of Theorem 2 will consist of several lemmas including some useful properties about the operators $U^\epsilon(\sigma,\tau)$, $U^0(\sigma,\tau)$ and $B^\epsilon(\sigma,\tau)$. We prove the following lemmas before we proceed with the proof of Theorem 2.

**Lemma 6.4.1** *The operators $U^\epsilon(\sigma,\tau)$, $U^0(\sigma,\tau)$ and $V_1(\tau)$ defined by (6.49), (6.50) and (6.53), respectively, satisfy the following variation of constants formulas:*

$$U^\epsilon(\sigma,\tau) = U^0(\sigma,\tau) + \frac{1}{\epsilon}\int_\sigma^\tau U^0(\sigma,\eta)V_1(\eta)U^\epsilon(\eta,\tau)\,d\eta, \tag{6.65}$$

$$U^\epsilon(\sigma,\tau) = U^0(\sigma,\tau) + \frac{1}{\epsilon}\int_\sigma^\tau U^\epsilon(\sigma,\eta)V_1(\eta)U^0(\eta,\tau)\,d\eta. \tag{6.66}$$

**Proof**: If we differentiate $U^0(\sigma,\eta)U^\epsilon(\eta,\tau)$ with respect to $\eta$ using the product rule, then

$$\begin{aligned}
\partial_\eta U^0(\sigma,\eta)U^\epsilon(\eta,\tau) &= (\partial_\eta U^0(\sigma,\eta))U^\epsilon(\eta,\tau) + U^0(\sigma,\eta)\partial_\eta U^\epsilon(\eta,\tau)\\
&= \frac{1}{\epsilon}U^0(\sigma,\eta)V_0(\eta)U^\epsilon(\eta,\tau) - \frac{1}{\epsilon}U^0(\sigma,\eta)V(\eta)U^\epsilon(\eta,\tau)\\
&= -\frac{1}{\epsilon}U^0(\sigma,\eta)V_1(\eta)U^\epsilon(\eta,\tau).
\end{aligned} \tag{6.67}$$

Here the forward propagator property for $U^0(\sigma,\eta)$, the backward propagator property for $U^\epsilon(\eta,\tau)$ and (6.53) were used. When one integrates (6.67) from $\sigma$ to $\tau$, (6.65) can be obtained. We can derive (6.66) as we did in the proof of Lemma 5.1.1. ∎

**Lemma 6.4.2** *The operators $B^\epsilon(\sigma,\tau)$ defined in Theorem 2 satisfy the following properties:*

$$\partial_\sigma B^\epsilon(\sigma,\tau) + \mathcal{L}^\epsilon_\sigma B^\epsilon(\sigma,\tau) = 0, \tag{6.68}$$

$$B^\epsilon(\sigma,\eta)B^\epsilon(\eta,\tau) = B^\epsilon(\sigma,\tau), \quad B^\epsilon(\sigma,\sigma) = I, \tag{6.69}$$

$$B^\epsilon(\sigma,\tau) = U^0(\sigma,\tau) + \int_\sigma^\tau U^0(\sigma,\eta)W^\epsilon(\eta)B^\epsilon(\eta,\tau)\,d\eta. \tag{6.70}$$

**Proof**: Equation (6.68) comes from equation (6.60) directly. The uniqueness of the solution of final value problem (6.60)-(6.61) leads to (5.69). To prove (6.70), we differentiate $U^0(\sigma,\eta)B^\epsilon(\eta,\tau)$ with respect to $\eta$ using the product rule:

$$\begin{aligned}
\partial_\eta U^0(\sigma,\eta)B^\epsilon(\eta,\tau) &= (\partial_\eta U^0(\sigma,\eta))B^\epsilon(\eta,\tau) + U^0(\sigma,\eta)\partial_\eta B^\epsilon(\eta,\tau)\\
&= \frac{1}{\epsilon}U^0(\sigma,\eta)V_0(\eta)B^\epsilon(\eta,\tau) - U^0(\sigma,\eta)\mathcal{L}^\epsilon_\eta B^\epsilon(\eta,\tau)\\
&= -U^0(\sigma,\eta)W^\epsilon(\eta)B^\epsilon(\eta,\tau).
\end{aligned} \tag{6.71}$$

Here the infinitesimal forward property for $U^0(\sigma,\eta)$ and (6.68) and definition (6.62) were used. When (6.71) is integrated from $\sigma$ to $\tau$, (6.70) can be obtained. ∎

Let us first decompose the interval $[0,\tau]$ into $[k\epsilon,(k+1)\epsilon]$, $k=0,1,...,m-1$, where, without loss of generality, $\epsilon$ is such that $\tau=m\epsilon$. Then $m\sim O(\epsilon^{-1})$ and for arbitrary $f\in C^4$

$$
\begin{aligned}
&E\{U^\epsilon(0,\tau)f\}-B^\epsilon(0,\tau)f \\
&\quad=\sum_{k=0}^{m-1}E\{U^\epsilon(0,(k+1)\epsilon)B^\epsilon((k+1)\epsilon,\tau)f-U^\epsilon(0,k\epsilon)B^\epsilon(k\epsilon,\tau)f\} \\
&=\sum_{k=0}^{m-1}E\{U^\epsilon(0,k\epsilon)\,[E\{U^\epsilon(k\epsilon,(k+1)\epsilon)g_k\}-B^\epsilon(k\epsilon,(k+1)\epsilon)g_k]\} \\
&+\sum_{k=0}^{m-1}E\{U^\epsilon(0,k\epsilon)\,[U^\epsilon(k\epsilon,(k+1)\epsilon)g_k-E\{U^\epsilon(k\epsilon,(k+1)\epsilon)g_k\}]\},
\end{aligned}
\tag{6.72}
$$

where $g_k\equiv B^\epsilon((k+1)\epsilon,\tau)f$. Since $U^\epsilon(0,k\epsilon)$ are contraction operators on $C^0\to C^0$ and $g_k$ is a solution of parabolic differential equation (6.60)-(6.61), the estimate (i.e., $\|\cdot\|_0$) of the quantities in the bracket of (6.72) determines the estimate of (6.72);

$$
\begin{aligned}
&\|E\{U^\epsilon(0,\tau)f\}-B^\epsilon(0,\tau)f\|_0 \\
&\leq\sum_{k=0}^{m-1}\|E\{U^\epsilon(k\epsilon,(k+1)\epsilon)g_k\}-B^\epsilon(k\epsilon,(k+1)\epsilon)g_k\|_0 \\
&+\sum_{k=0}^{m-1}\|E\{U^\epsilon(0,k\epsilon)U^\epsilon(k\epsilon,(k+1)\epsilon)g_k\}-E\{U^\epsilon(0,k\epsilon)E\{U^\epsilon(k\epsilon,(k+1)\epsilon)g_k\}\}\|_0 \\
&\equiv\sum_{k=0}^{m-1}\left(I_{1,k}(f)+I_{2,k}(f)\right).
\end{aligned}
\tag{6.73}
$$

Let us estimate $I_{1,k}(f)$ first. If we iterate the variation of constants formulas (6.65) twice, (6.66) twice, and (6.70) twice, then we can obtain the following expansion

$$
\begin{aligned}
&E\{U^\epsilon(k\epsilon,(k+1)\epsilon)g_k\}-B^\epsilon(k\epsilon,(k+1)\epsilon)g_k \\
&=\Big[\frac{1}{\epsilon^2}\int_{k\epsilon}^{(k+1)\epsilon}\int_\sigma^{(k+1)\epsilon}E\{U^0(k\epsilon,\sigma)V_1(\sigma)U^0(\sigma,\lambda)V_1(\lambda)U^0(\lambda,(k+1)\epsilon)g_k\}\,d\lambda d\sigma \\
&\qquad-\int_{k\epsilon}^{(k+1)\epsilon}U^0(k\epsilon,\sigma)W^\epsilon(\sigma)U^0(\sigma,(k+1)\epsilon)g_k\,d\sigma\Big] \\
&-\int_{k\epsilon}^{(k+1)\epsilon}\int_\sigma^{(k+1)\epsilon}U^0(k\epsilon,\sigma)W^\epsilon(\sigma)U^0(\sigma,\lambda)W^\epsilon(\lambda)B^\epsilon(\lambda,(k+1)\epsilon)g_k\,d\lambda d\sigma \\
&+\frac{1}{\epsilon^3}\int_{k\epsilon}^{(k+1)\epsilon}\int_\sigma^{(k+1)\epsilon}\int_\lambda^{(k+1)\epsilon}E\{U^0(k\epsilon,\sigma)V_1(\sigma)U^0(\sigma,\lambda)V_1(\lambda)U^0(\lambda,\nu)V_1(\nu)\cdot
\end{aligned}
$$

$$\cdot U^0(\nu,(k+1)\epsilon)g_k\}\,d\nu d\lambda d\sigma$$
$$+\frac{1}{\epsilon^4}\int_{k\epsilon}^{(k+1)\epsilon}\int_{\sigma}^{(k+1)\epsilon}\int_{\lambda}^{(k+1)\epsilon}\int_{\lambda}^{\nu} E\{U^0(k\epsilon,\sigma)V_1(\sigma)U^0(\sigma,\lambda)V_1(\lambda)U^\epsilon(\lambda,\mu)V_1(\mu)\cdot$$
$$\cdot U^0(\mu,\nu)V_1(\nu)U^0(\nu,(k+1)\epsilon)g_k\}\,d\mu d\nu d\lambda d\sigma, \tag{6.74}$$

where hypothesis (6.54) was used.

In the following lemmas, we proceed with the term-by-term estimation of (6.74). We use repeatedly the Banach space version of mixing lemma, i.e., Lemma 5.2.2 and the following inequalities from (6.55)-(6.56): If $0 \le \tau - \sigma \le \epsilon$, then for arbitrary $f \in \mathcal{C}^k$,

$$\|U^\epsilon(\sigma,\tau)f\|_k \le \tilde{\beta}_k k e^{\tilde{\alpha}_k}\|f\|_k \quad a.e., \quad k=1,2, \tag{6.75}$$
$$\|U^0(\sigma,\tau)f\|_k \le \beta_k k e^{\alpha_k}\|f\|_k, \qquad 1 \le k \le 4. \tag{6.76}$$

Here $\alpha_k$, $\tilde{\alpha}_k$, $\beta_k$ and $\tilde{\beta}_k$ are independent of $\sigma$, $\tau$ and $\epsilon$. From now on, we will use $C_{\tau_0}$ to denote a positive constant depending on only $\tau_0$. The constant multiple of $C_{\tau_0}$ will be denoted also by the same $C_{\tau_0}$. We will use a general term $z$ instead of $k\epsilon$ throughout.

**Lemma 6.4.3** *For arbitrary $f \in \mathcal{C}^2$ and $0 \le z < \tau_0$,*

$$K \equiv \|\frac{1}{\epsilon^2}\int_z^{z+\epsilon}\int_\sigma^{z+\epsilon} E\{U^0(z,\sigma)V_1(\sigma)U^0(\sigma,\lambda)V_1(\lambda)U^0(\lambda,z+\epsilon)f\}\,d\lambda d\sigma$$
$$-\int_z^{z+\epsilon} U^0(z,\sigma)W^\epsilon(\sigma)U^0(\sigma,z+\epsilon)f\,d\sigma\|_0 \le \epsilon^2 C_{\tau_0}\|f\|_2, \tag{6.77}$$

**Proof**: From definition (6.59) and the change of variables $\lambda = \sigma + \epsilon^2 t$, we note that

$$W^\epsilon(\sigma)f = \frac{1}{\epsilon^2}\int_\sigma^{\sigma+\epsilon} E\{V_1(\sigma)U^0(\sigma,\lambda)V_1(\lambda)U^0(\lambda,\sigma)f\}\,d\lambda, \quad f \in \mathcal{C}^2. \tag{6.78}$$

We use Lemma 5.2.2 to obtain

$$K = \|-\frac{1}{\epsilon^2}\int_z^{z+\epsilon}\int_{z+\epsilon}^{\sigma+\epsilon} E\{U^0(z,\sigma)V_1(\sigma)U^0(\sigma,\lambda)V_1(\lambda)U^0(\lambda,z+\epsilon)f\}\,d\lambda d\sigma\|_0$$
$$\le \frac{1}{\epsilon^2}\int_z^{z+\epsilon}\int_{z+\epsilon}^{\sigma+\epsilon} 2\rho(\frac{\lambda-\sigma}{\epsilon^2})d\lambda d\sigma \sup_{z\le\sigma\le z+\epsilon\le\lambda\le z+2\epsilon}\ \sup_{\omega,\omega'} \|U^0(z,\sigma)V_1(\sigma)U^0(\sigma,\lambda)\cdot$$
$$\cdot V_1(\lambda)U^0(\lambda,z+\epsilon)f\|_0$$
$$\le \frac{2}{\epsilon^2}\int_0^\epsilon\int_{z+\epsilon-u}^{z+\epsilon} \rho(\frac{u}{\epsilon^2})\,dv du\cdot c_1(\beta_1 e^{\alpha_1})c_2(\beta_2 2e^{\alpha_2})\|f\|_2$$

$$\leq 4\epsilon^2 \int_0^{1/\epsilon} \acute{u}\rho(\acute{u})d\acute{u} \cdot c_1 c_2 \beta_1 \beta_2 e^{\alpha_1+\alpha_2} \|f\|_2$$
$$\leq \epsilon^2 C_{\tau_0} \|f\|_2, \tag{6.79}$$

for some constant $C_{\tau_0}$. Here (6.58), (6.76) and that $\int_0^\infty s\rho(s)ds < \infty$ were used. ■

**Lemma 6.4.4** *For arbitrary $f \in C^4$ and $0 \leq z < \tau_0$,*

$$J \equiv \| \int_z^{z+\epsilon} \int_\sigma^{z+\epsilon} U^0(z,\sigma)W^\epsilon(\sigma)U^0(\sigma,\lambda)W^\epsilon(\lambda)B^\epsilon(\lambda, z+\epsilon)f \, d\lambda d\sigma\|_0$$
$$\leq \epsilon^2 C_{\tau_0}\|f\|_4. \tag{6.80}$$

**Proof**: For arbitrary $f \in C^2$, using Lemma 5.2.2, one obtains

$$\|W^\epsilon(\sigma)f\|_0 \leq \int_0^{1/\epsilon} 2\rho(t)dt \sup_{0\leq t\leq 1/\epsilon} \sup_{\omega,\omega'} \|V_1(\sigma)U^0(\sigma+\epsilon^2 t) \cdot$$
$$\cdot V_1(\sigma+\epsilon^2 t)U^0(\sigma+\epsilon^2 t, \sigma)f\|_0. \tag{6.81}$$

From (6.58) and (6.76), the above inequality becomes

$$\|W^\epsilon(\sigma)f\|_0 \leq \int_0^\infty 2\rho(t)dt \cdot c_1(\beta_1 e^{\alpha_1})c_2(\beta_2 2e^{\alpha_2})\,\|f\|_2 \equiv \gamma_2\|f\|_2, \tag{6.82}$$

where $\gamma_2$ is a positive constant independent of $\sigma$ and $\epsilon$. Similarly, for arbitrary $f \in C^4$,

$$\|W^\epsilon(\lambda)f\|_2 \leq \int_0^\infty 2\rho(t)dt \cdot c_3(\beta_3 3e^{\alpha_3})c_4(\beta_4 4e^{\alpha_4})\,\|f\|_4 \equiv \gamma_4\|f\|_4. \tag{6.83}$$

From (6.76), therefore, we obtain the following inequality for the integrand of $J$:

$$\|U^0(z,\sigma)W^\epsilon(\sigma)U^0(\sigma,\lambda)W^\epsilon(\lambda)B^\epsilon(\lambda,z+\epsilon)f\|_0$$
$$\leq \gamma_2(\beta_2 2e^{\alpha_2})\gamma_4\|B^\epsilon(\lambda,z+\epsilon)f\|_4. \tag{6.84}$$

Since $B^\epsilon(\lambda, z+\epsilon)f$ solves parabolic differential equation (6.60)-(6.61), the above inequality leads to the inequality $J \leq \epsilon^2 C_{\tau_0}\|f\|_4$ for some positive constant $C_{\tau_0}$. ■

**Lemma 6.4.5** *For arbitrary $f \in C^3$ and $0 \leq z < \tau_0$,*

$$H \equiv \|\frac{1}{\epsilon^3}\int_z^{z+\epsilon}\int_\sigma^{z+\epsilon}\int_\lambda^{z+\epsilon} E\{U^0(z,\sigma)V_1(\sigma)U^0(\sigma,\lambda)V_1(\lambda)U^0(\lambda,\nu)V_1(\nu)\cdot$$
$$\cdot U^0(\nu, z+\epsilon)\, f\}\, d\nu d\lambda d\sigma\|_0 \leq \epsilon^2 C_{\tau_0}\|f\|_3. \tag{6.85}$$

**Proof**: Let us consider the following change of variables:

$$\tilde{\sigma} \equiv \sigma - z, \tag{6.86}$$

$$\tilde{\lambda} \equiv \lambda - \sigma = \lambda - (z + \tilde{\sigma}) \tag{6.87}$$

$$\tilde{\nu} \equiv \nu - \lambda = \nu - (z + \tilde{\sigma} + \tilde{\lambda}). \tag{6.88}$$

Then (6.85) becomes

$$\begin{aligned} H = &\|\frac{1}{\epsilon^3}\int_0^{\epsilon}\int_0^{\epsilon-\tilde{\sigma}}\int_0^{\epsilon-\tilde{\sigma}-\tilde{\lambda}} E\{U^0(z, z+\tilde{\sigma})V_1(z+\tilde{\sigma})U^0(z+\tilde{\sigma}, z+\tilde{\sigma}+\tilde{\lambda})\cdot \\ &\cdot V_1(z+\tilde{\sigma}+\tilde{\lambda})U^0(z+\tilde{\sigma}+\tilde{\lambda}, z+\tilde{\sigma}+\tilde{\lambda}+\tilde{\nu})V_1(z+\tilde{\sigma}+\tilde{\lambda}+\tilde{\nu})\cdot \\ &\cdot U^0(z+\tilde{\sigma}+\tilde{\lambda}+\tilde{\nu}, z+\epsilon)\, f\}\, d\tilde{\nu} d\tilde{\lambda} d\tilde{\sigma}\|_0. \end{aligned} \tag{6.89}$$

In this step, we need an inequality

$$\begin{aligned} &\int_0^{\epsilon-\tilde{\sigma}}\int_0^{\epsilon-\tilde{\sigma}-\tilde{\lambda}} (\cdot)\, d\tilde{\nu} d\tilde{\lambda} \\ \le\ &\{\int_0^{\epsilon/2}\int_0^{\tilde{\nu}} + \int_{\epsilon/2}^{\epsilon}\int_0^{\epsilon-\tilde{\nu}} (\cdot)\,\} d\tilde{\lambda} d\tilde{\nu} \ +\ \{\int_0^{\epsilon/2}\int_0^{\tilde{\lambda}} + \int_{\epsilon/2}^{\epsilon}\int_0^{\epsilon-\tilde{\lambda}} (\cdot)\,\} d\tilde{\nu} d\tilde{\lambda}, \end{aligned} \tag{6.90}$$

where the integrand $(\cdot)$ is nonnegative. We use (6.90) and Lemma 5.2.2 to obtain

$$\begin{aligned} H \le\ & c_1(\beta_1 e^{\alpha_1})c_2(\beta_2 2e^{\alpha_2})c_3(\beta_3 3e^{\alpha_3})\frac{1}{\epsilon^3}\int_0^{\epsilon}\ \Big[\int_0^{\epsilon/2}\int_0^{\tilde{\nu}} + \int_{\epsilon/2}^{\epsilon}\int_0^{\epsilon-\tilde{\nu}} 2\rho(\frac{\tilde{\nu}}{\epsilon^2})\, d\tilde{\lambda} d\tilde{\nu} \\ &+ \int_0^{\epsilon/2}\int_0^{\tilde{\lambda}} + \int_{\epsilon/2}^{\epsilon}\int_0^{\epsilon-\tilde{\lambda}} 2\rho(\frac{\tilde{\lambda}}{\epsilon^2})\, d\tilde{\nu} d\tilde{\lambda}\ \Big] d\tilde{\sigma}\ \|f\|_3 \\ \le\ & 24 c_1 c_2 c_3 \beta_1\beta_2\beta_3 e^{\alpha_1+\alpha_2+\alpha_3}\frac{1}{\epsilon^2}\,\Big[\int_0^{\epsilon/2}\tilde{\nu}\rho(\frac{\tilde{\nu}}{\epsilon^2})d\tilde{\nu} + \int_{\epsilon/2}^{\epsilon}(\epsilon-\tilde{\nu})\rho(\frac{\tilde{\nu}}{\epsilon^2})d\tilde{\nu}\Big]\|f\|_3. \end{aligned} \tag{6.91}$$

Here (6.58), (6.76) and the symmetric property between $\tilde{\nu}$ and $\tilde{\lambda}$ were used. Finally, from

$$\int_0^{\epsilon/2}\tilde{\nu}\rho(\frac{\tilde{\nu}}{\epsilon^2})d\tilde{\nu} = \epsilon^4\int_0^{1/2\epsilon}\acute{\nu}\rho(\acute{\nu})d\acute{\nu}, \tag{6.92}$$

$$\int_{\epsilon/2}^{\epsilon}(\epsilon-\tilde{\nu})\rho(\frac{\tilde{\nu}}{\epsilon^2})d\tilde{\nu} \le \frac{\epsilon}{2}\int_{\epsilon/2}^{\epsilon}\rho(\frac{\tilde{\nu}}{\epsilon^2})d\tilde{\nu} \le \epsilon^4\int_{1/2\epsilon}^{1/\epsilon}\acute{\nu}\rho(\acute{\nu})d\acute{\nu}, \tag{6.93}$$

we obtain the inequality $H \le \epsilon^2 C_{T_0}\|f\|_3$ for some positive constant $C_{T_0}$. ■

**Lemma 6.4.6** *For arbitrary* $f \in C^4$ *and* $0 \le z < \tau_0$,

$$L \equiv \|\frac{1}{\epsilon^4}\int_z^{z+\epsilon}\int_\sigma^{z+\epsilon}\int_\lambda^{z+\epsilon}\int_\lambda^{\nu} E\{U^0(z,\sigma)V_1(\sigma)U^0(\sigma,\lambda)V_1(\lambda)U^\epsilon(\lambda,\mu)V_1(\mu)\cdot$$
$$\cdot U^0(\mu,\nu)V_1(\nu)U^0(\nu,z+\epsilon)\,f\}\,d\mu d\nu d\lambda d\sigma\|_0 \le \epsilon^2 C_{\tau_0}\|f\|_4. \quad (6.94)$$

**Proof**: Let $g_1(\sigma,\lambda)$ denote the double integral with respect to $\mu$ and $\nu$ of $L$ so that

$$L = \|\frac{1}{\epsilon^4}\int_z^{z+\epsilon}\int_\sigma^{z+\epsilon} g_1(\sigma,\lambda)\,d\lambda d\sigma\|_0. \quad (6.95)$$

By the change of variables (6.86)-(6.87), the above $L$ becomes

$$L = \|\frac{1}{\epsilon^4}\int_0^{\epsilon}\int_0^{\epsilon-\tilde\sigma} g_1(z+\tilde\sigma, z+\tilde\sigma+\tilde\lambda)\,d\tilde\lambda d\tilde\sigma\|_0. \quad (6.96)$$

Now let $g_2(\mu,\nu)$ denote the integrand of the double integral $g_1$ so that

$$g_1(z+\tilde\sigma, z+\tilde\sigma+\tilde\lambda) = \int_{z+\tilde\sigma+\tilde\lambda}^{z+\epsilon}\int_{z+\tilde\sigma+\tilde\lambda}^{\nu} g_2(\mu,\nu)\,d\mu d\nu. \quad (6.97)$$

By the change of variables (6.88) (using $\mu$ instead of $\nu$ there) and

$$\tilde\nu \equiv \nu - \mu = \nu - (z+\tilde\sigma+\tilde\lambda+\tilde\mu), \quad (6.98)$$

the above double integral $g_1$ becomes

$$g_1(z+\tilde\sigma, z+\tilde\sigma+\tilde\lambda) = \int_0^{\epsilon-\tilde\sigma-\tilde\lambda}\int_0^{\epsilon-\tilde\sigma-\tilde\lambda-\tilde\nu} g_2(z+\tilde\sigma+\tilde\lambda+\tilde\mu, z+\tilde\sigma+\tilde\lambda+\tilde\mu+\tilde\nu)\,d\tilde\mu d\tilde\nu. \quad (6.99)$$

Substitute (6.99) into (6.96). Then (6.96) becomes

$$L = \|\frac{1}{\epsilon^4}\int_0^{\epsilon}\int_0^{\epsilon-\tilde\sigma}\int_0^{\epsilon-\tilde\sigma-\tilde\lambda}\int_0^{\epsilon-\tilde\sigma-\tilde\lambda-\tilde\nu} E\{U^0(z,z+\tilde\sigma)V_1(z+\tilde\sigma)U^0(z+\tilde\sigma,z+\tilde\sigma+\tilde\lambda)\cdot$$
$$\cdot V_1(z+\tilde\sigma+\tilde\lambda)U^\epsilon(z+\tilde\sigma+\tilde\lambda, z+\tilde\sigma+\tilde\lambda+\tilde\mu)V_1(z+\tilde\sigma+\tilde\lambda+\tilde\mu)U^0(z+\tilde\sigma+\tilde\lambda+\tilde\mu, z+\tilde\sigma$$
$$+\tilde\lambda+\tilde\mu+\tilde\nu)V_1(z+\tilde\sigma+\tilde\lambda+\tilde\mu+\tilde\nu)U^0(z+\tilde\sigma+\tilde\lambda+\tilde\mu+\tilde\nu, z+\epsilon)\,f\}\,d\tilde\mu d\tilde\nu d\tilde\lambda d\tilde\sigma\|_0 (6.100)$$

Now we use inequality (6.90) for the double integral in the middle of (6.100) and then use Lemma 5.2.2 to obtain

$$L \le c_1(\beta_1 e^{\alpha_1})c_2(\tilde\beta_2 2e^{\tilde\alpha_2})c_3(\beta_3 3e^{\alpha_3})c_4(\beta_4 4e^{\alpha_4})\cdot$$
$$\cdot\frac{1}{\epsilon^4}\int_0^{\epsilon}\Big[\{\int_0^{\epsilon/2}\int_0^{\tilde\nu} + \int_{\epsilon/2}^{\epsilon}\int_0^{\epsilon-\tilde\nu}\}\int_0^{\epsilon-\tilde\sigma-\tilde\lambda-\tilde\nu} 2\rho(\frac{\tilde\nu}{\epsilon^2})\,d\tilde\mu d\tilde\lambda d\tilde\nu$$
$$+\{\int_0^{\epsilon/2}\int_0^{\tilde\lambda} + \int_{\epsilon/2}^{\epsilon}\int_0^{\epsilon-\tilde\lambda}\}\int_0^{\epsilon-\tilde\sigma-\tilde\lambda-\tilde\nu} 2\rho(\frac{\tilde\lambda}{\epsilon^2})\,d\tilde\mu d\tilde\nu d\tilde\lambda\ \Big]\,d\tilde\sigma\,\|f\|_4. \quad (6.101)$$

Note that $\int_0^{\epsilon-\tilde{\sigma}-\tilde{\lambda}-\tilde{\nu}} \rho(\frac{\tilde{\nu}}{\epsilon^2})\, d\tilde{\mu} \leq \epsilon\rho(\frac{\tilde{\nu}}{\epsilon^2})$ and $\int_0^{\epsilon-\tilde{\sigma}-\tilde{\lambda}-\tilde{\nu}} \rho(\frac{\tilde{\lambda}}{\epsilon^2})\, d\tilde{\mu} \leq \epsilon\rho(\frac{\tilde{\lambda}}{\epsilon^2})$. Then, using the symmetric property between $\lambda$ and $\nu$, we have

$$
\begin{aligned}
L &\leq 96 c_1 c_2 c_3 c_4 \beta_1 \tilde{\beta}_2 \beta_3 \beta_4 \epsilon^{\alpha_1+\tilde{\alpha}_2+\alpha_3+\alpha_4} \frac{1}{\epsilon^3} \int_0^{\epsilon} \Big[ \int_0^{\epsilon/2} \int_0^{\tilde{\nu}} + \int_{\epsilon/2}^{\epsilon} \int_0^{\epsilon-\tilde{\nu}} \rho(\frac{\tilde{\nu}}{\epsilon^2}) d\tilde{\lambda} d\tilde{\nu} \Big] d\tilde{\sigma} \|f\|_4 \\
&\leq 96 c_1 c_2 c_3 c_4 \beta_1 \tilde{\beta}_2 \beta_3 \beta_4 \epsilon^{\alpha_1+\tilde{\alpha}_2+\alpha_3+\alpha_4} \cdot \\
&\quad \cdot \frac{1}{\epsilon^3} \int_0^{\epsilon} \Big[ \int_0^{\epsilon/2} \tilde{\nu}\rho(\frac{\tilde{\nu}}{\epsilon^2}) d\tilde{\nu} + \int_{\epsilon/2}^{\epsilon} (\epsilon-\tilde{\nu})\rho(\frac{\tilde{\nu}}{\epsilon^2}) d\tilde{\nu} \Big] d\tilde{\sigma} \|f\|_4. \qquad (6.102)
\end{aligned}
$$

Now we use (6.92) and (6.93). Then the desired estimate will follow immediately. ■

From the above Lemmas 6.4.3–6.4.6 (with $z = k\epsilon$ and $f = g_k$), we obtain that the supremum norm of (6.74) is bounded by $4\epsilon^2 C_{\tau_0} \|g_k\|_4$;

$$
I_{1,k}(f) \leq \epsilon^2 C_{\tau_0} (\|g_k\|_2 + \|g_k\|_4 + \|g_k\|_3 + \|g_k\|_4) \leq 4\epsilon^2 C_{\tau_0} \|g_k\|_4. \qquad (6.103)
$$

Therefore, the desired estimate for $I_{1,k}(f)$ is obtained since $g_k$ $(= B^{\epsilon}((k+1)\epsilon, \tau)f)$ solves parabolic differential equation (6.60)-(6.61) (cf. Lemma 4 in [7]); using the notational convention about $C_{\tau_0}$, we have

$$
I_{1,k}(f) \leq \epsilon^2 C_{\tau_0} \|f\|_4. \qquad (6.104)
$$

Next, we estimate $I_{2,k}(f)$ defined by (6.73). As we did in Section 5.2, using the propagator properties (5.3)-(5.4) again, we have the equality

$$
\begin{aligned}
I_{2,k}(f) &\equiv \|E\{U^{\epsilon}(0,k\epsilon)U^{\epsilon}(k\epsilon,(k+1)\epsilon)g_k\} - E\{U^{\epsilon}(0,k\epsilon)E\{U^{\epsilon}(k\epsilon,(k+1)\epsilon)g_k\}\}\|_0 \\
&= \frac{1}{\epsilon^2} \| \int_0^{k\epsilon} \int_{k\epsilon}^{(k+1)\epsilon} [E\{U^{\epsilon}(0,\sigma)V(\sigma)V(\lambda)U^{\epsilon}(\lambda,(k+1)\epsilon)g_k\} \\
&\qquad - E\{U^{\epsilon}(0,\sigma)V(\sigma)E\{V(\lambda)U^{\epsilon}(\lambda,(k+1)\epsilon)g_k\}\}] d\lambda d\sigma \|_0. \qquad (6.105)
\end{aligned}
$$

From Lemma 5.2.2, we obtain the inequality

$$
\begin{aligned}
I_{2,k}(f) &\leq \frac{1}{\epsilon^2} \int_0^{k\epsilon} \int_{k\epsilon}^{(k+1)\epsilon} 2\rho(\frac{\lambda-\sigma}{\epsilon^2}) d\lambda d\sigma \cdot \\
&\quad \cdot \sup_{0\leq\sigma\leq k\epsilon\leq\lambda\leq(k+1)\epsilon} \sup_{\omega,\omega'} \|U^{\epsilon}(0,\sigma)V(\sigma)V(\lambda)U^{\epsilon}(\lambda,(k+1)\epsilon)g_k\|_0. \qquad (6.106)
\end{aligned}
$$

We use the contraction property for $U^{\epsilon}(0,\sigma)$, (6.57) and (6.75) to obtain

$$
\sup_{0\leq\sigma\leq k\epsilon\leq\lambda\leq(k+1)\epsilon} \sup_{\omega,\omega'} \|U^{\epsilon}(0,\sigma)V(\sigma)V(\lambda)U^{\epsilon}(\lambda,(k+1)\epsilon)g_k\|_0 \leq \tilde{c}_1\tilde{c}_2(\tilde{\beta}_2 2e^{\tilde{\alpha}_2})\|g_k\|_2 \qquad (6.107)
$$

Thus (6.106) becomes

$$I_{2,k}(f) \leq \frac{2}{\epsilon^2}\tilde{c}_1\tilde{c}_2(\tilde{\beta}_2 2e^{\tilde{\alpha}_2})||g_k||_2 \int_0^{k\epsilon}\int_{k\epsilon}^{(k+1)\epsilon} \rho(\frac{\lambda-\sigma}{\epsilon^2})d\lambda d\sigma. \tag{6.108}$$

The above double integral in (6.108) was already estimated in Section 5.2 ($\tau_0$ and $\tau_{k-1}$ in (5.46) are 0 and $k\epsilon$ in the present case, respectively). From result (5.49), the above double integral is $O(\epsilon^4)$. Hence, we obtain the estimate

$$I_{2,k}(f) \leq \epsilon^2 C_{\tau_0}||f||_2, \tag{6.109}$$

for some positive constant $C_{\tau_0}$.

Now, by substituting two estimates (6.109) and (6.104) into (6.73), we have

$$\begin{aligned} &||E\{U^\epsilon(0,\tau)f\} - B^\epsilon(0,\tau)f||_0 \\ &\leq \epsilon^2 C_{\tau_0}\sum_{k=0}^{m-1}(||f||_4 + ||f||_2) \leq 2\epsilon^2 C_{\tau_0}\sum_{k=0}^{m-1}||f||_4 \equiv \epsilon C(f;\tau_0), \end{aligned} \tag{6.110}$$

where the fact that $m \sim O(\epsilon^{-1})$ was used. Here the positive constant $C(f;\tau_0)$ depends on $f$ and its derivatives up to order 4 and $\tau_0$ but is independent of $\epsilon$. Therefore, the desired estimate (6.63) in Theorem 2 is proved.

## 6.5 Application to Random Noise below the Turning Point

In Sections 6.3-6.4, we derived a theorem which can characterize stochastic initial value problems with a rapidly varying deterministic component. According to Theorem 2, the asymptotic behavior of our model problem (6.27)-(6.30) can be characterized by final value problem (6.60)-(6.61) with the infinitesimal generator $\mathcal{L}_\sigma^\epsilon$ defined by (6.62) if the necessary hypotheses are satisfied. We will check these conditions and subsequently represent explicitly the generator $\mathcal{L}_\sigma^\epsilon$ in the context of the problem formulated in Section 6.1.

The corresponding non-zero deterministic and random fields $F_0$ and $F_1$ are (6.29) and (6.30), respectively. The random field $F$ is the sum of these two fields. Since $F_1$ has zero mean, condition (6.54) obviously holds. In order to check conditions (6.55)-(6.56), we first note the boundedness of the fields $F_0$, $F_1$ and $F$ and their derivatives with respect to the phase variable. Since $\overline{A}_{1i}(\tau)$ and $\mathcal{A}_{1i}(\tau, \tau/\epsilon^2)$, $i = 1, 2$, were assumed (in Chapter 3) to be each bounded by a constant independent of $\tau$ and $\epsilon$ in the whole region, (6.29)-(6.30) imply that the fields $\partial^k_{\psi..\psi}F_0(\tau, \psi)$, $\partial^k_{\psi..\psi}F_1(\tau, \tau/\epsilon^2, \psi)$ and so $\partial^k_{\psi..\psi}F(\tau, \tau/\epsilon^2, \psi)$ are each bounded by a constant independent of $\tau$, $\epsilon$ and $\psi$.

From deterministic initial value problem (6.47)-(6.48) with $F_0$ defined by (6.29) (using $\psi$ instead of $x$),

$$\frac{\partial\psi^0}{\partial\psi}(\tau, \sigma, \psi) = e^{\frac{1}{\epsilon}\int_\sigma^\tau F_0'\,ds}, \tag{6.111}$$

$$\frac{\partial^2\psi^0}{\partial\psi^2}(\tau, \sigma, \psi) = e^{\frac{1}{\epsilon}\int_\sigma^\tau F_0'\,ds} \cdot \frac{1}{\epsilon}\int_\sigma^\tau F_0''\frac{\partial\psi^0}{\partial\psi}\,ds, \tag{6.112}$$

$$\frac{\partial^3\psi^0}{\partial\psi^3}(\tau, \sigma, \psi) = e^{\frac{1}{\epsilon}\int_\sigma^\tau F_0'\,ds}\ \Big\{ \left(\frac{1}{\epsilon}\int_\sigma^\tau F_0''\frac{\partial\psi^0}{\partial\psi}\,ds\right)^2 + \frac{1}{\epsilon}\int_\sigma^\tau \left(F_0'''(\frac{\partial\psi^0}{\partial\psi})^2 + F_0''\frac{\partial^2\psi^0}{\partial\psi^2}\right) ds\ \Big\}, \tag{6.113}$$

$$\begin{aligned}\frac{\partial^4\psi^0}{\partial\psi^4}(\tau, \sigma, \psi) = e^{\frac{1}{\epsilon}\int_\sigma^\tau F_0'\,ds}\ \Big\{ &\left(\frac{1}{\epsilon}\int_\sigma^\tau F_0''\frac{\partial\psi^0}{\partial\psi}\,ds\right)^3 \\ &+3\left(\frac{1}{\epsilon}\int_\sigma^\tau F_0''\frac{\partial\psi^0}{\partial\psi}\,ds\right)\left(\frac{1}{\epsilon}\int_\sigma^\tau [F_0'''(\frac{\partial\psi^0}{\partial\psi})^2 + F_0''\frac{\partial^2\psi^0}{\partial\psi^2}]\,ds\right) \\ &+\frac{1}{\epsilon}\int_\sigma^\tau\left(F_0''''(\frac{\partial\psi^0}{\partial\psi})^3 + 3F_0'''\frac{\partial\psi^0}{\partial\psi}\frac{\partial^2\psi^0}{\partial\psi^2} + F_0''\frac{\partial^3\psi^0}{\partial\psi^3}\right) ds\ \Big\}.\end{aligned} \tag{6.114}$$

$$\left(F_0^{(n)} \equiv \partial^n_{\psi..\psi}F_0(s, \psi^0(s, \sigma, \psi)), \quad 1 \le n \le 4\right)$$

Since $\partial^n_{\psi..\psi}F_0(s, \psi)$, $n \ge 0$, are each bounded by a constant independent of $s$ and $\psi$, there exist positive constants $\hat{\alpha}_n$ and $\hat{\beta}_n$ independent of $\tau$, $\sigma$, $\psi$ and $\epsilon$ such that from (6.111)-(6.114) we have the inequalities

$$\left|\frac{\partial^n\psi^0}{\partial\psi^n}\right| \le \hat{\beta}_n\{\frac{\tau-\sigma}{\epsilon} + \cdots + (\frac{\tau-\sigma}{\epsilon})^{n-1}\}e^{\hat{\alpha}_n\frac{\tau-\sigma}{\epsilon}}, \quad 1 \le n \le 4. \tag{6.115}$$

Using the chain rule for the derivatives of $U^0(\sigma, \tau)f$, we can obtain

$$\|\partial^n_{\psi..\psi}U^0(\sigma, \tau)f\|_0 \le \check{\beta}_n \sup_\psi \sum_{i_1+2i_2+\cdots+ni_n=n} \left|\frac{\partial\psi^0}{\partial\psi}\right|^{i_1}\left|\frac{\partial^2\psi^0}{\partial\psi^2}\right|^{i_2}\cdots\left|\frac{\partial^n\psi^0}{\partial\psi^n}\right|^{i_n}\|f\|_n,$$
$$1 \le n \le 4, \quad (\psi^0 \equiv \psi^0(\tau, \sigma, \psi)) \tag{6.116}$$

for some positive constant $\check{\beta}_n$ independent of $\tau$, $\sigma$ and $\epsilon$. We apply inequalities (6.115) to (6.116) and rearrange the result to obtain the right side of (6.116) as the form of the right side of (6.56) with $k = n$. Since $\|U^0(\sigma,\tau)f\|_k$ is simply the sum of $\|\partial^n_{\psi..\psi}U^0(\sigma,\tau)f\|_0$, $n = 1, 2, ..., k$, (6.56) can be obtained for some suitably chosen constants $\alpha_k$ and $\beta_k$. By the same procedure as done above, we can obtain condition (6.55) also for the operators $U^\epsilon(\sigma,\tau)$ due to the boundedness of the random field $F$ and its derivatives with respect to phase variable.

Since $\partial^k_{\psi..\psi}F(\tau,\tau/\epsilon^2,\psi)$ are each bounded by a constant independent of $\tau$, $\epsilon$ and $\psi$, condition (6.57) holds. In view of the definition (6.53) of $V_1(\tau)$, we have for any $f \in C^1$

$$V_1(\tau)f = F_1(\tau,\tau/\epsilon^2,\psi)\partial_\psi f. \tag{6.117}$$

Then condition (6.58) follows by the fact that $\partial^k_{\psi..\psi}F_1(\tau,\tau/\epsilon^2,\psi)$ are each bounded by a constant independent of $\tau$, $\epsilon$ and $\psi$.

Therefore, all the necessary hypotheses of Theorem 2 are satisfied for our stochastic initial value problem (6.27)-(6.30). The solutions of the problem are approximated by the corresponding solutions of (6.60)-(6.61) with the infinitesimal generator given by (6.62). We shall now write the explicit form of the generator of our stochastic model problem.

In view of the definition (6.59) (or (6.78)) of $W^\epsilon(\sigma)$, we have

$$\begin{aligned}&(W^\epsilon(\sigma)f)(\psi)\\&= \frac{1}{\epsilon^2}\int_\sigma^{\sigma+\epsilon} E\{F_1(\sigma,\sigma/\epsilon^2,\psi)\,\partial_\psi[\,F_1(\lambda,\lambda/\epsilon^2,\psi_1)\,\partial_{\psi_1}f(\psi_2)\,]\}\,d\lambda,\end{aligned} \tag{6.118}$$

where $\psi_1 \equiv \psi^0(\lambda,\sigma,\psi)$ and $\psi_2 \equiv \psi^0(\sigma,\lambda,\psi_1) = \psi$. The integrand ($\equiv I$) of the above integral (6.118) is the same as

$$\begin{aligned}I = E\{\,&F_1(\sigma,\sigma/\epsilon^2,\psi)F_1'(\lambda,\lambda/\epsilon^2,\psi_1) + F_1(\sigma,\sigma/\epsilon^2,\psi)F_1(\lambda,\lambda/\epsilon^2,\psi)\frac{\partial\psi_1}{\partial\psi}\frac{\partial^2\psi_2}{\partial\psi_1^2}\,\}\,\frac{\partial f}{\partial\psi}\\&+E\{\,F_1(\sigma,\sigma/\epsilon^2,\psi)F_1(\lambda,\lambda/\epsilon^2,\psi)\frac{\partial\psi_2}{\partial\psi_1}\,\}\,\frac{\partial^2 f}{\partial\psi^2}.\end{aligned} \tag{6.119}$$

Let us substitute the random field $F_1$ given by (6.30) into the above (6.119). Then the following expression for the generator $\mathcal{L}_\sigma^\epsilon$ can be obtained by direct calculation:

$$
\begin{aligned}
(\mathcal{L}_\sigma^\epsilon f)(\psi) &= \frac{1}{\epsilon}(V_0(\sigma)f)(\psi) \,+\, (W^\epsilon(\sigma)f)(\psi) \\
&= \Bigg[-\frac{k}{\epsilon}\{\overline{A}_{11}(\sigma) + \overline{A}_{12}(\sigma)\cos\psi\} + \frac{k^2}{\epsilon^2}\int_\sigma^{\sigma+\epsilon}\Big[\rho_{12}^s(\sigma,s) + \rho_{22}^s(\sigma,s)\cos\psi \\
&\qquad + \{\rho_{11}(\sigma,s) + \rho_{12}^c(\sigma,s) + \rho_{21}(\sigma,s)\cos\psi + \rho_{22}^c(\sigma,s)\cos\psi\}\cdot \\
&\qquad \cdot\frac{k}{\epsilon}\int_s^\sigma \overline{A}_{12}(t)\cos\psi^0(t,\sigma,\psi)\, e^{\frac{k}{\epsilon}\int_s^t \overline{A}_{12}(u)\sin\psi^0(u,\sigma,\psi)du}dt\,\Big]ds\Bigg]\frac{\partial f}{\partial\psi} \\
&+\Bigg[\frac{k^2}{\epsilon^2}\int_\sigma^{\sigma+\epsilon}\{\rho_{11}(\sigma,s) + \rho_{12}^c(\sigma,s) + \rho_{21}(\sigma,s)\cos\psi + \rho_{22}^c(\sigma,s)\cos\psi\}\cdot \\
&\qquad \cdot e^{\frac{k}{\epsilon}\int_s^\sigma \overline{A}_{12}(t)\sin\psi^0(t,\sigma,\psi)\,dt}\,ds\Bigg]\frac{\partial^2 f}{\partial\psi^2},
\end{aligned}
\tag{6.120}
$$

where the following notation was used:

$$\rho_{ij}(\sigma,s) \equiv E\{\mathcal{A}_{1i}(\sigma,\sigma/\epsilon^2)\mathcal{A}_{1j}(s,s/\epsilon^2)\}, \tag{6.121}$$

$$\rho_{ij}^s(\sigma,s) \equiv \rho_{ij}(\sigma,s)\sin\psi^0(s,\sigma,\psi), \qquad i,j = 1,2, \tag{6.122}$$

$$\rho_{ij}^c(\sigma,s) \equiv \rho_{ij}(\sigma,s)\cos\psi^0(s,\sigma,\psi). \tag{6.123}$$

# Chapter 7

# CONCLUSIONS

Our stochastic problem in the random slab has been dealt with in two divided regions, i.e., the region above the turning point and the region below the turning point. The two different scattering formulations were established corresponding to these two different regions. Theorem 1 in Chapter 4 and Theorem 2 in Chapter 6 are general statements that describe the asymptotic statistics of the reflection coefficient corresponding to the two regions, respectively. In particular, Theorem 1 provides a uniform description for both transition region and outer region above the turning point.

As shown in [13], the reflection process above the turning point undergoes rapid phase-wrapping, while the reflection process below the turning point (to leading order) decays exponentially. Significant diffusion (multiple scattering) occurs in the region above the turning point, while one expects little diffusion in the region below the turning point.

Using the properties of conditional expectation, we combine the above two statistics and derive the final approximation in the whole region. When Theorem 1 was applied to the problem above the turning point, the final value (evaluated at the turning point) of the Markov diffusion process was assumed to be known. We now replace this value by the random noise value coming from the characterization of the problem below the turning point. Final characterization of the problem of interest, then, follows by taking the expectation of the process containing the random noise. Theorem 2 provides the characterization of the problem below the turning point.

To understand explicitly the final characterization of the problem, let $\psi_a^\epsilon(\tau, \sigma, \psi)$ and $\psi_b^\epsilon(\tau, \sigma, \psi)$ denote the phases of the problems in the regions, respectively, above and below the turning point $z_T$. Also we use $\Psi^\epsilon(\tau, \sigma, \psi)$ to denote the "unified" phase of the problem throughout the whole region $-L \leq z \leq 0$, i.e.,

$$\Psi^\epsilon(\tau,\sigma,\psi) \equiv \begin{cases} \psi_a^\epsilon(\tau,\sigma,\psi) & \text{if } z_T \leq \sigma < \tau \\ \psi_a^\epsilon(\tau,z_T,h^T(\psi_b^\epsilon(z_T,\sigma,\psi))) & \text{if } \sigma < z_T < \tau \\ \psi_b^\epsilon(\tau,\sigma,\psi) & \text{if } \sigma < \tau \leq z_T \end{cases}, \tag{7.1}$$

where the function $h^T$, defined by (6.38), relates at the turning point the two phases of the reflection coefficients corresponding to the two regions. Note that we are using the $z$-scale instead of the stretched scale ($\eta$-scale) for the solution $\psi_a^\epsilon$ above the turning point; $\tau$ in the $z$-scale corresponds to $(\frac{\omega}{\epsilon})^{2/3}\phi(\tau)$ in the $\eta$-scale (cf. (3.51)) so that $z_T$ and 0 correspond to 0 and $(\frac{3\omega}{2\epsilon}\tau_T)^{2/3}$, respectively, where $\phi(\tau)$ and $\tau_T$ are defined by (3.32)-(3.33).

Then, by Theorem 1 and Theorem 2, we have the following final estimate for the whole region $-L \leq z \leq 0$. For arbitrary initial phase $\psi_0$ given at $z = -L$ and function $f \in C^4$,

$$\begin{aligned}
&E\{f(\Psi^\epsilon(0,-L,\psi_0))\} \\
&= E\{f(\psi_a^\epsilon(0,z_T,h^T(\psi_b^\epsilon(z_T,-L,\psi_0))))\} \\
&= E\{E\{f(\psi_a^\epsilon(0,z_T,h^T(\psi_b^\epsilon(z_T,-L,\psi_0))))|\mathcal{F}_0^{z_T/\epsilon^2}\}\} \\
&\qquad \approx E\{u(z_T,0,h^T(\psi_b^\epsilon(z_T,-L,\psi_0));f)\} \qquad \text{by Theorem 1} \\
&\qquad = E\{(A(z_T,0)f)(h^T(\psi_b^\epsilon(z_T,-L,\psi_0)))\} \\
&\qquad\qquad \approx u^\epsilon(-L,z_T,\psi_0;(A(z_T,0)f)\circ h^T) \qquad \text{by Theorem 2} \\
&\qquad\qquad = B(-L,z_T)((A(z_T,0)f)\circ h^T)(\psi_0).
\end{aligned} \tag{7.2}$$

Let $H(z_T)$ be an operator on $C^4 \to C^4$ defined by the following composition with the function $h^T$:

$$H(z_T)f \equiv f \circ h^T. \tag{7.3}$$

Conclusively, we can rewrite the result (7.2) in terms of this operator $H(z_T)$ and the evolution operators $B^\epsilon(-L,z_T)$ and $A(z_T,0)$ (see Theorem 2 and Theorem 1, respectively, for definition) which correspond to the regions below and above the turning point, respectively. If we use $U^\epsilon(\sigma,\tau)$ to denote the propagator corresponding to the unified solution (7.1), i.e.,

$(U^\epsilon(\sigma,\tau)f)(\psi) \equiv f(\Psi^\epsilon(\tau,\sigma,\psi))$, then, from (7.2), the evolution of the reflection coefficient statistics in the whole region $-L \leq z \leq 0$ can be described as the following product of the operators:

$$E\{U^\epsilon(-L,0)f\} \;\approx\; B^\epsilon(-L,z_T)H(z_T)A(z_T,0)f, \qquad f \in C^4, \tag{7.4}$$

where the error in the estimate is of order $O(\epsilon^{1/3})$. The propagator $B^\epsilon(-L,z_T)$ represents the Markov diffusion process with singular drift term below the turning point. The operator $H(z_T)$ transforms at the turning point the reflection coefficient from the region below the turning point into the reflection coefficient above the turning point. The propagator $A(z_T,0)$ characterizes the Markov diffusion process above the turning point.

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# VITA

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