DRAG CONSIDERATIONS FOR FLIGHT IN ATMOSPHERIC TURBULENCE

by

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Thesis submitted to the Faculty of the
Virginia Polytechnic Institute and State University
in partial fulfillment of the requirements for the degree of
Master of Science
in
Aerospace Engineering

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December, 1989

Blacksburg, Virginia

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(ABSTRACT)

The distribution of lift between the wing and tail surfaces of a conventional aircraft is examined in order to determine the combination that would produce the minimum drag for a given lift. Further, the center of gravity (CG) position which gives the desired lift distribution and at the same time, maintains aircraft trim is determined.

Furthermore, a classic set of non-linear equations of motion for longitudinal flight is reduced to a set of linear equations by linearization. The location of the CG of the aircraft is then changed and a linear feedback control law is used to retain the dynamic characteristic (flying qualities) of the airplane. The response of the aircraft to an external disturbance such as a gust (modeled with a stochastic process) is studied in terms of drag versus CG position.

Finally, it is shown that the position of the CG for minimum drag should be determined with consideration of the expected atmospheric turbulence.

Acknowledgements

I am indebted to Dr. F. H. Lutze for all the help and guidance he has given me over the past two years. Thanks are also due to Dr. E. M. Cliff and Dr. J. F. Rossi for serving on my committee.

I appreciate the coordinators of the exchange program between Virginia Polytechnic Institute and the University of Technology of Compiegne, France for providing me the opportunity to study at Virginia Tech and to work on this research.

Finally, I take this opportunity to thank all my friends for their help and advice.

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List of Symbols

a_c	Speed-of-sound
AR	Aspect ratio
b_w	Wing span
$\bar{c}$	Mean aerodynamic chord (MAC)
C_D	Drag coefficient
C_L	Lift coefficient
C_m	Pitching moment coefficient
C_T	Thrust coefficient
D	Drag
ΔC_D	Variation of Drag coefficient
$\sigma\,[\Delta\delta_e]$	Root mean square of $\Delta\delta_e$ (rms)
E_t	Tail factor $= \dfrac{\eta_t S_t}{S}$
g	Acceleration due to gravity
h	Distance in chord lengths from leading edge of wing to CG

h_{nwb}	Distance in chord lengths from leading edge of wing to wing-body aerodynamic center
h_t	Distance in chord lengths from leading edge of wing to tail aerodynamic center
I	Pitching moment of inertia
K	Induced drag factor
L	Lift
m	Mass
M	Pitching moment
M_a	Mach number
q	Pitching rate
$\bar{q}$	Dynamic pressure
S	Wing Area
T	Thrust
U	Vector of Controls
V	Speed
W	Aircraft Weight
X	Vector of states

Greek Symbols.

α	Angle-of-attack
$\bar{\alpha}$	$\alpha - \alpha_{owb}$
α_{owb}	Angle-of-attack for zero lift wing-body
δ_e	Elevator setting
δ_t	Throttle setting
ε	Downwash angle at tail
ε_o	Downwash angle at zero lift wing-body
γ	Flight-path angle
η_t	Tail efficiency factor
ρ	Air density
σ	Root mean square value
θ	Pitch angle

Subscripts

$(\ )_r$	Reference flight condition
$(\ )_{ref}$	Value at original position of CG
$(\ )_{wb}$	Specific to wing-body
$(\ )_t$	Specific to tail

$(\ )_0$ ....Specific at zero lift

$(\ )_i$ ....Derivative with respect to i at i = constant

for i = V, α, q, δ_t, δ_e, M

Superscripts

$(\dot{\ })$ ....Time derivative

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1.0 Introduction

The continuing interest in fuel conservation as well as the desire to improve aircraft performance leads to the problem of reducing aircraft drag. To be more precise, it is of interest to reduce the trimmed drag or the drag of the vehicle in pitch equilibrium since the aircraft is operated primarily in this mode. Typically a reduction in wing-body or tail profile drag directly reduces the trimmed drag. Additional reductions, although small (1-5 %) can be made by paying careful attention to the lift distribution between the wing-body and the tail surfaces (maintaining zero pitching moment) so as to obtain the proper tradeoffs between static stability and induced drag which minimize the overall drag [Ref. 5, 13].

One result which seems uniformly accepted is that the overall drag on an aft-tail vehicle can be reduced by moving the center of gravity aft, causing a reduced download on the tail or possibly even an upload [Ref. 4, 6]. Such a design reduces the longitudinal stability considerably and is feasible only if the desired level of stability can be maintained by a reliable control system. In fact, when the airplane encounters atmospheric turbulence, the control activity to retain the dynamic characteristics (flying qualities) is increased. This activity creates an additional drag.

The purpose of this study, therefore, is to calculate the CG position for minimum drag in calm atmosphere (chapter 2) and the CG position for minimum drag in atmospheric turbulence (chapter 3). Furthermore, if certain assumptions are made about the expected duration of flight in calm and turbulent atmosphere, a CG position for minimum drag for the overall flight can be established.

2.0 Minimum drag consideration - calm atmosphere

The following calculations are based on a study previously done in [Ref. 9]. The distribution of lift between the wing and the tail surfaces of a conventional aircraft is examined in order to determine the combination that would produce the minimum drag for a given lift. The assumptions made are that the aerodynamic coefficients are linear and can be represented as indicated. In this chapter the drag coefficient is calculated in two parts : the contribution of the wing-body and the contribution of the tail. The contribution to the total aircraft drag coefficient can therefore be written as :

$$C_D = C_{Dwb} + C_{Dtail} \tag{2.1}$$

The drag coefficients due to the wing-body and the tail are :

$$C_{Dwb} = C_{Dowb} + K_{wb}\, C_{Lwb}^2$$

$$C_{Dtail} = (\, C_{Dot} + K_t\, C_{Lt}^2 + C_{Lt}\, \varepsilon\,)\, E_t$$

where

C_{Do} is the drag coefficient at zero lift,

C_L is the lift coefficient,

ε is the downwash angle at the tail

K is the induced drag factor.

Additional relations are given by :

$$\varepsilon = \frac{\partial \varepsilon}{\partial \alpha}\, \bar{\alpha} + \varepsilon_o$$

$$\bar{\alpha} = \frac{C_{Lwb}}{C_{L\alpha wb}}$$

$$E_t = \frac{\eta_t\, S_t}{S}$$

where ε_o is the downwash angle at $C_{Lwb} = 0$,

and $\bar{\alpha}$ is the wing-body angle of attack measured to the wing-body zero lift line.

It is to be noticed that the coefficients C_{Dowb} and C_{Dot} are assumed to be constant and won't be calculated here. Only $C_D - C_{Do}$ (where $C_{Do} = C_{Dowb} + C_{Dot}\, E_t$) is of interest.

The lift coefficients C_{Lt} and C_{Lwb} due to the tail and the wing-body can be determined from the following equations :

$$C_m = C_{mowb} + C_{mwb} + C_{mt} = 0 \tag{2.2}$$

$$C_{mwb} = C_{Lwb}\,(h - h_{nwb}) \tag{2.3}$$

$$C_{mt} = C_{Lt}\,(h - h_t)\,E_t \tag{2.4}$$

$$C_L = C_{Lwb} + C_{Lt}\,E_t \tag{2.5}$$

where C_m represents the pitching moment coefficient.

These equations insure a moment and force balance ($L = W$ and $M = 0$) and assumed that the tail is symmetric ($C_{mot} = 0$).

Eliminating C_{Lt} or C_{Lwb} with (2.5) and substituting (2.3) and (2.4) into (2.2) :

$$C_{mowb} + C_{Lwb}\,(h - h_{nwb}) + (C_L - C_{Lwb})\,(h - h_t) = 0$$

$$C_{mowb} + (C_L - C_{Lt}\,E_t)\,(h - h_{nwb}) + C_{Lt}\,E_t\,(h - h_t) = 0$$

Solving for C_{Lwb} or for C_{Lt} , the respective lift coefficients are :

$$C_{Lwb} = \frac{(h_t - h)\,C_L - C_{mowb}}{h_t - h_{nwb}} \tag{2.6}$$

$$C_{Lt} = \frac{(h - h_{nwb})\,C_L + C_{mowb}}{(h_t - h_{nwb})\,E_t} \tag{2.7}$$

The other constant K_t, K_{wb}, h_t, h_{nwb} are properties of the geometry of the aircraft and expressions for them can be found in Appendix A.

For a given speed and corresponding lift coefficient, the drag coefficient is now a function of the position h of the center of gravity. Equation (2.1) can be written as :

$$C_D = (C_{Dowb} + C_{Dot} E_t) + K_{wb} C_{Lwb}^2 (h) + (K_t C_{Lt}^2 (h) + C_{Lt} (h) \varepsilon) E_t \qquad (2.8)$$

with C_{Lwb} and C_{Lt} functions of h as in (2.6) and (2.7).

It can be seen that C_D is a quadratic function of h . The curve of the drag coefficient versus the CG position obtained from equation (2.8) is then a parabola and it can be established that there exists a position of the CG for a minimum drag.

3.0 Minimum drag consideration - turbulent atmosphere

3.1 Overview

The nominal design of the aircraft is such that the CG is located so that the airplane is statically stable in pitch. As the location of the CG is moved aft, the airplane becomes increasingly less stable and as such, requires a feedback control system to maintain its flying qualities. When the airplane flies in atmospheric turbulence , the activity of the control required to maintain the original level of stability creates an additional drag. The variation of drag can be represented with the variation of the drag coefficient ΔC_D . This variation can be calculated through the change in elevator deflection $\Delta\delta_e$, away from the value for balance in calm atmosphere. If it is assumed that any deviation of this elevator from trim position contributes to an additional drag, then the contribution to the drag due to turbulence is given by :

$$\Delta C_D = C_{D\delta e}\, \sigma\, [\Delta\delta_e] \qquad (3.1.1)$$

where $\sigma\,[\Delta\delta_e]$ is the rms value of the elevator deflection.

In this chapter, the equation of motion for the vehicle, the associated linearized matrix system, a linear feedback control law, and a model for atmospheric turbulence are determined from which one can calculate the root mean square value of $\Delta\delta_e$ ($\sigma\,[\Delta\delta_e]$).

3.2 Matrix System

3.2.1 Equations-of-motion

In this model the following assumption are made :

- Flat earth
- Constant gravity
- Rigid body
- Thrust along the velocity vector
- Symmetric flight

To describe the motion of an airplane it is necessary to define a suitable coordinate system for formulation of the equations-of-motion. In atmospheric flight, the wind axes system is commonly used. It has its origin fixed to the airplane's center of gravity. It is oriented so that the X_w axis is directed along the velocity V of the vehicle relative to the atmosphere. The Z_w axis lies in the plane of symmetry of the vehicle. The forces acting on an airplane in flight consist of aerodynamic, thrust, and gravitational forces. These forces are illustrated in Fig.1.

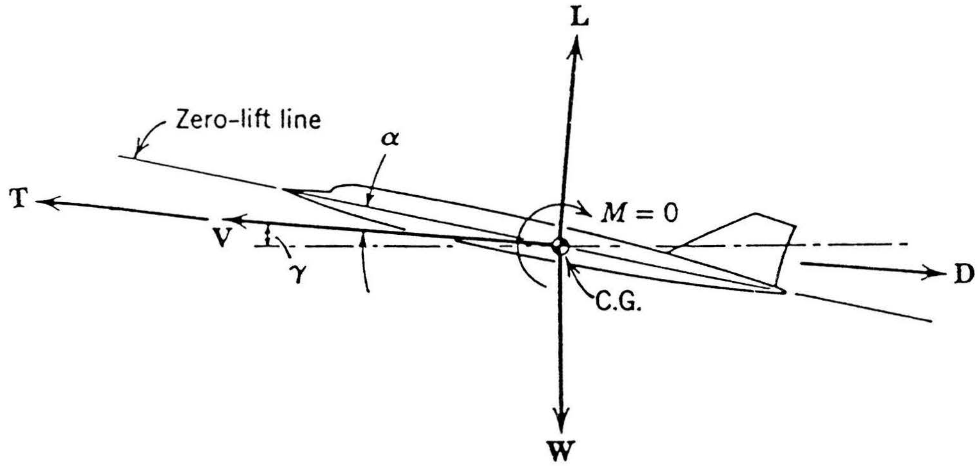

Figure 1. Forces acting on the airplane.

The force equations-of-motion result from the application of Newton's Laws of motion to the aircraft. The force equations-of-motion for symmetric longitudinal flight written in wind axes are :

$$T - D - m g \sin \gamma = m \dot{V}$$

$$L - m g \cos \gamma = m V \dot{\gamma} \tag{3.2.1}$$

$$M = I \ddot{\theta}$$

$$q = \dot{\theta}$$

The aerodynamic and thrust terms are assumed to have the functional forms :

$$T = T(V, \delta_T)$$

$$D = D(V, \alpha, \delta_e)$$

$$L = L(V, \alpha, q, \delta_e)$$

$$M = M(V, \alpha, \dot{\alpha}, q, \delta_e)$$

where α is the aircraft angle-of-attack and is related to the flight path angle γ and the pitch altitude angle θ by :

$$\gamma = \theta - \alpha$$

In order to perform a control system analysis, it is convenient to linearize the equation of motion about a steady state, straight and level flight reference flight condition. If V, α , q , θ are selected as state variables and δ_e and δ_T are selected as control variables, the linearized set of equations can be determined by methods shown in appendix B. The resulting system is given by :

$$\dot{X} = A\,X + B\,U \qquad (3.2.2)$$

where

$$X = \begin{bmatrix} V \\ \alpha \\ q \\ \theta \end{bmatrix} \qquad U = \begin{bmatrix} \delta_T \\ \delta_e \end{bmatrix}$$

and

$$A = \begin{bmatrix} \dfrac{T_V - D_V}{m} & \dfrac{T_\alpha - D_\alpha + mg\cos\gamma}{m} & \dfrac{D_q}{m} & -g\cos\gamma \\[2ex] -\dfrac{L_V}{mV} & -\dfrac{L_\alpha - mg\sin\gamma}{mV} & 1 - \dfrac{L_q}{mV} & -\dfrac{g\sin\gamma}{V} \\[2ex] -\dfrac{L_V M_{\dot\alpha}}{mVI} + \dfrac{M_V}{I} & -(L_\alpha - mg\sin\gamma)\dfrac{M_{\dot\alpha}}{mVI} + \dfrac{M_\alpha}{I} & -\dfrac{L_q M_{\dot\alpha}}{mVI} + \dfrac{M_q}{I} + \dfrac{M_{\dot\alpha}}{I} & \dfrac{g\sin\gamma}{IV} M_{\dot\alpha} \\[2ex] 0 & 0 & 1 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} \dfrac{T_{\delta T}}{m} & -\dfrac{D_{\delta e}}{m} \\[2ex] -\dfrac{T_{\delta T}}{mV} & -\dfrac{L_{\delta e}}{mV} \\[2ex] -\dfrac{T_{\delta T} M_{\dot\alpha}}{mVI} & -\dfrac{L_{\delta e} M_{\dot\alpha}}{mVI} + \dfrac{M_{\delta e}}{I} \\[2ex] 0 & 0 \end{bmatrix}$$

The elements of the matrices A and B contain derivatives of the thrust, and aerodynamic forces and moments with respect to the various state and control variables. These forces and moments are generally expressed in terms of force and moment coefficients. Information for determining these coefficients are presented in the next section.

3.2.2 Aerodynamic coefficients

It is assumed that the nominal values of the aerodynamic coefficients are provided for the reference flight condition of interest. However, for the study to be performed here, it is necessary to know how their values change with a change in the CG position. It is important then to see how the aerodynamic coefficients can be written as function of the non-dimensional CG position , h .

For the purpose of this study, the engine control will be the thrust coefficient. Hence,

$$C_T = \delta_T$$

The equation for the moment coefficient is :

$$C_m = C_{moL} + C_{m\alpha}\,\overline{\alpha} + C_{mq}\,\hat{q} + C_{m\dot{\alpha}}\,\dot{\alpha} + C_{m\delta e}\,\delta_e$$

where :

$$C_{m\alpha} = C_{L\alpha}\,(\,h - h_n\,)$$

$$C_{mq} = (C_{mq})_{ref} + C_{Lq}\,(\,h - h_{ref})$$

$$C_{m\dot{\alpha}} = (C_{m\dot{\alpha}})_{ref} + C_{L\dot{\alpha}}\,(\,h - h_{ref})$$

$$C_{m\delta e} = C_{L\delta e}\,(\,h - h_t\,)$$

C_{moL} is the pitching moment coefficient when $\overline{\alpha}$, $\hat{q}$, $\dot{\alpha}$ and δ_e are zero.

The equation for the lift coefficient has the form :

$$C_L = C_{L\alpha}\,\overline{\alpha} + C_{L\delta e}\,\delta_e$$

where $C_{L\alpha}$ and $C_{L\delta e}$ are not affected by the CG movement.

The equation for the drag coefficient is :

$$C_D = C_{Do} + C_{D\delta e}\,\delta_e$$

where :

C_{Do} is the drag coefficient in calm atmosphere (chapter 2),

$$C_{D\delta e} = 2\,K\,C_L\,C_{L\delta e}$$

By redefining C_{moL} and the α_{oL} in $\overline{\alpha}$, the elevator δ_e, at the reference flight condition and original CG position, can be assigned the value of zero. Thus the previous equation can be written as :

$$(C_m)_{ref} = C_{moL} + (C_{m\alpha})_{ref}\,\overline{\alpha} = 0$$

$$C_L = C_{L\alpha}\,\overline{\alpha}$$

where C_{moL} and α_{oL} are defined to satisfy these equations and are subsequently held constant.

The constants $\overline{\alpha}_{ref}$ and C_{moL} are then redefined accordingly :

$$\overline{\alpha}_{ref} = \frac{C_L}{C_{L\alpha}}$$

$$C_{moL} = -\frac{C_L\,(C_{m\alpha})_{ref}}{C_{L\alpha}}$$

3.2.3 Angle-of-attack and elevator deflection at trim

As the location of the CG is changed, $\overline{\alpha}$ and δ_e take new values at the reference flight condition which need to be calculated. For all CG positions, the reference flight condition gives $M = 0$, i.e $C_m = 0$. This leads to the system :

$$C_L = C_{L\alpha}\,\alpha + C_{L\delta e}\,\delta_e$$

$$0 = C_{moL} + C_{m\alpha}\,\overline{\alpha} + C_{m\delta e}\,\delta_e$$

solving for $\overline{\alpha}$ and δ_e , the angle of attack and the elevator deflection at trim are:

$$\alpha_{trim} = \frac{C_L\,C_{M\delta e} + C_{L\delta e}\,C_{m0L}}{C_{L\alpha}\,C_{m\delta e} - C_{L\delta e}\,C_{m\alpha}}$$

$$\delta_{etrim} = -\frac{C_{L\alpha}\,C_{m0L} + C_L\,C_{m\alpha}}{C_{L\alpha}\,C_{m\delta e} - C_{L\delta e}\,C_{m\alpha}}$$

Note that for the original CG position $\delta_{etrim} = 0$.

3.2.4 Summary

The necessary ingredients for computing the changes in the elements of the A and B matrices, in the system $\dot{X} = A\,X + B\,U$, as the CG position is moved, are now entirely known.

3.3 Linear feedback control

3.3.1 Overview

The system $\dot{X} = A\,X + B\,U$ is stable for the initial position of the CG.

For static stability, a positive disturbance in α (nose up disturbance) produces a nose down or a negative pitching moment, or in symbols $C_{m\alpha} < 0$.

As seen before the pitching moment coefficient is $C_{m\alpha} = C_{L\alpha}\,(\,h - h_n\,)$

where h is the position of the CG and h_n the aerodynamic center of the aircraft.

$C_{L\alpha}$ being positive, it requires $h < h_n$.

In other words, by moving the CG aft h becomes greater than h_n and the airplane becomes unstable.

But it is possible to stabilize the system by feeding back the complete state $X(t)$. Another words, by setting the control as a function of the states X, the value of the control will be dictated by the deviation of the state. A linear feedback control law which links the control to the state will then be discussed here.

3.3.2 Linear feedback control

Consider the linear time-invariant system with state differential equation :

$$\dot{X}(t) = A\,X(t) + B\,U(t) \tag{3.3.1}$$

If it is supposed that the complete state can be accurately measured at all times, it is possible to implement a linear control law of the form :

$$U(t) = -\,F\,X(t)$$

where F is a time-invariant feedback gain matrix.

If this control law is connected to the system (3.3.1), the closed-loop system is described by the state differential equation :

$$\dot{X}(t) = [A - B\,F]\,X(t)$$

Since the characteristics of the aircraft behaviour are contained in the A matrix, one way to insure a behaviour close to the original behaviour is to pick the feedback matrix F so that :

$$A - B\,F = A_{ref}$$

where A_{ref} is the original matrix for the stable system.

The feedback matrix F can be calculated from :

$$F = B^s\,(A - A_{ref}) \tag{3.3.2}$$

where the speudo-inverse matrix B^s is used for B (B being 4×2) :

$$B^s = (B^T B)^{-1} B^T$$

The system becomes :

$$\dot{X} = A_{new} X \tag{3.3.3}$$

with

$$A_{New} = A - B F$$

$$F = ((B^T B)^{-1} B^T)\,(A - A_{ref})$$

3.4 Atmospheric turbulence

The effect of atmospheric turbulence on the airplane is seen through its velocity field [Ref.3]. In this chapter, the velocity field is modeled as a linear system driven by white noise because the power spectral density of the atmosphere and the power spectral density of a linear system driven by white noise can be easily matched.

3.4.1 Input to the airplane

The velocity field of the atmosphere is used to describe atmospheric turbulence and is defined with a vector g (g for " gust ") . To simplify the expression of g, the airplane is considered as a point. This approximation gives useful results of good accuracy for large scale turbulence and small airplanes [Ref.7].
The longitudinal response being studied here, the coordinate of interest is the vertical component of the gust W_g . The effect of W_g is essentially on α and q .

Therefore the state components $\begin{bmatrix} V \\ \alpha \\ q \\ \theta \end{bmatrix}$ become $\begin{bmatrix} V \\ \alpha + \alpha_g \\ q + q_g \\ \theta \end{bmatrix}$

The previous system become :

$$\begin{bmatrix} \dot{V} \\ \dot{\alpha} \\ \dot{q} \\ \dot{\theta} \end{bmatrix} = A_{New} \begin{bmatrix} V \\ \alpha + \alpha_g \\ q + q_g \\ \theta \end{bmatrix}$$

$$= A_{New}\begin{bmatrix} V \\ \alpha \\ q \\ \theta \end{bmatrix} + A_{New}\begin{bmatrix} 0 \\ \alpha_g \\ q_g \\ 0 \end{bmatrix}$$

The system can be transformed to :

$$\dot{X} = A_{New}\, X + A_g\, X_g \qquad (3.4.1)$$

with A_g a 4×2 matrix, composed of the 2^{nd} and the 3^{rd} column of A_{New}

and $X_g = \begin{bmatrix} \alpha_g \\ q_g \end{bmatrix}$.

3.4.2 Modeling of α_g and q_g

Since turbulence is a random process that can not be described by explicit functions of time, only a statistical approach can be taken.

Also, since

$$\alpha_g = \frac{W_g}{V}$$

$$q_g = \frac{\partial W_g}{\partial X} = \frac{\partial W_g}{\partial t}\ \frac{\partial t}{\partial X} = \frac{\dot{W}_g}{V}$$

only W_g need to be modeled as a stochastic process.

A very useful modeling of a stochastic process is a linear differential system driven by white noise :

$$W_g(t) = (\, v_1 \,,\; v_2 \,)\; Y(t)$$

$$\dot{Y}(t) = A_w\, Y(t) + B_w\, w(t) \tag{3.4.2}$$

where :

w is a white noise of intensity σ_w^2 ,

$$A_W = \begin{bmatrix} 0 & 1 \\ -\alpha_0 & -\alpha_1 \end{bmatrix}$$

$$B_W = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

For convenience, the system has been chosen in phase canonical form.

Hence, W_g is modeled with a linear system driven by white noise and its power spectral density will be matched with that of the atmospheric model in order to determine the constants α_0 , α_1 , v_1 , v_2 in the system.

The model for α_g and q_g is then

$$\alpha_g = \left(\frac{v_1}{V} \ , \ \frac{v_2}{V} \right) Y(t)$$

$$q_g = \left(\frac{v_1}{V} \ , \ \frac{v_2}{V} \right) \dot{Y}(t)$$

$$= \frac{1}{V} [v_1 \, , \, v_2] \begin{bmatrix} 0 & 1 \\ -\alpha_0 & -\alpha_1 \end{bmatrix} Y(t) + \frac{1}{V} [v_1 \, , \, v_2] \begin{bmatrix} 0 \\ 1 \end{bmatrix} w$$

$$= \left[-\frac{v_2 \alpha_0}{V} \ , \ \frac{v_1 - v_2 \alpha_1}{V} \right] Y(t) + \frac{v_2}{V} w$$

Hence,

$$\begin{bmatrix} \alpha_g \\ q_g \end{bmatrix} = \frac{1}{V} \begin{bmatrix} v_1 & v_2 \\ -v_2 \alpha_0 & v_1 - v_2 \alpha_1 \end{bmatrix} Y(t) + \begin{bmatrix} 0 \\ \frac{v_2}{V} \end{bmatrix} w$$

or

$$X_g(t) = A_d \, Y(t) + B_d \, w$$

A_d and B_d being the corresponding matrix.

The unknowns are now α_0 , α_1 , v_1 , v_2 .

3.4.3 Power spectral density of W_g

The convenience of using the phase canonical form for the stochastic process is the ease to calculate its power spectral density.

The model for W_g is :

$$W_g(t) = (v_1 \, , \, v_2) \, Y(t)$$

$$\dot{X}(t) \; = \begin{bmatrix} 0 & 1 \\ -\alpha_0 & -\alpha_1 \end{bmatrix} X(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w(t)$$

The transfer function of the system linking w and W_g is then :

$$\begin{aligned} H(S) \; &= \; [\, v_1 \, , \, v_2 \,] \, [\, S\, I - A \,]^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \; \frac{v_1 + v_2 \, S}{S^2 + \alpha_1 \, S + \alpha_0} \end{aligned}$$

Or,

$$H(i\,\omega\,) \; = \; \frac{v_1 + v_2 \, i\,\omega}{(i\,\omega)^2 + \alpha_1 \, i\,\omega + \alpha_0}$$

Furthermore, the power spectral density of W_g is defined as follows [Ref. 11] :

$$\Phi_{Wg}\,(\,\omega\,) = H(\,i\,\omega\,)\,\Phi_w\,(\,\omega\,)\,H^T\,(\,-\,i\,\omega\,)$$

Let w be a white noise with a constant intensity σ_w^2 .

Its power spectral density is then :

$$\Phi_w(\omega) = \sigma_w^2$$

Hence,

$$\begin{aligned}\Phi_{Wg}(\omega) &= H(i\omega)\,\sigma_w^2(\omega)\,H^T(-i\omega)\\ &= \sigma_w^2\,|H(i\omega)|^2\\ &= \sigma_w^2\,\frac{v_1^2 + v_2^2\,\omega^2}{(\alpha_0 - \omega^2)^2 + \alpha_1^2\,\omega^2}\end{aligned}$$

The power spectral density for W_g is then :

$$\Phi_{Wg}(\omega) = \sigma_w^2\,\frac{v_1^2 + v_2^2\,\omega^2}{\alpha_0^2 + (\alpha_1^2 - 2\alpha_0)\,\omega^2 + \omega^4} \tag{3.4.3}$$

3.4.4 Atmospheric model

The total velocity field of the atmosphere is variable in both space and time, composed of a mean value and variations from it. The mean wind is not of interest here and is eliminated by choosing a reference frame, the atmosphere fixed frame relative to which the mean motion is zero. The correlation matrix and the spectrum function of the velocity of the air relative to this frame are then used to describe the needed statistic of the turbulence. From them, all the pertinent results can be derived by making some simplifying assumptions.

The turbulence is assumed to be :

- a stationary process (no dependance on time)
- homogeneous (no dependance on position)
- isotropic (no dependance on orientation of axes)
- frozen (the speed of the airplane is much bigger than the turbulent velocities and their rate of changes; therefore the vehicle can traverse the turbulence so fast that the turbulent velocities have not had time to change very much).

Under this assumption, the one dimensional spectrum that best fits the data for the vertical component of turbulence is the von Karman spectrum [Ref. 3, 7] :

$$\Phi_w(\Omega) = \frac{\sigma_w^2 L}{\pi} \frac{1 + \frac{8}{3}(1.339\, L\, \Omega)^2}{\left[1 + (1.339\, L\, \Omega)^2\right]^{\frac{11}{6}}}$$

Another representation often used for its convenience is a simplified version of the von Karman spectrum [Ref.7]:

$$\Phi_w(\Omega) = \frac{\sigma_w^2 L}{\pi} \frac{1 + 3\, L^2 \Omega^2}{(1 + L^2 \Omega^2)^2}$$

σ_w^2	mean square value of vertical turbulence velocity
L	scale of turbulence
Ω	spatial frequency defined by $\frac{2\pi}{\lambda}$, where λ is the wave length of a sinusoidal component.

This latter representation will be used here.

Let's put $\Phi_w(\Omega)$ in the form of the power spectral density of W_g (eq. 3.4.3). Since $\Omega = \frac{\omega}{V}$,

$$\Phi_w(\Omega) = \frac{\sigma_w^2 L}{\pi} \frac{1 + 3L^2(\frac{\omega}{V})^2}{\left[1 + L^2(\frac{\omega}{V})^2\right]^2}$$

$$= \frac{\sigma_w^2 V^2}{\pi L} \frac{\frac{V^2}{L^2} + 3\omega^2}{\frac{V^4}{L^4} + \frac{2V^2\omega^2}{L^2} + \omega^4}$$

If the power spectral density of W_g and the power spectral density of the turbulence are compared, the parameters required for the linear model approximation are :

$$v_1 = \frac{V^2}{\sqrt{\pi L^2}}$$

$$v_2 = \frac{3\, V}{\sqrt{3\, \pi\, L}} \tag{3.4.4}$$

$$\alpha_0 = \frac{V^2}{L}$$

$$\alpha_1 = \frac{2\, V}{L}$$

3.4.5 Summary

The atmospheric turbulence is then described by the system :

$$X_g(t) = A_d\, Y(t) + B_d\, w \tag{3.4.5}$$

where :

$$X_g = \begin{bmatrix} \alpha_g \\ q_g \end{bmatrix}$$

$$A_d = \begin{bmatrix} v_1 & v_2 \\ -\, v_2\, \alpha_0 & v_1 - v_2\, \alpha_1 \end{bmatrix}$$

$$B_d = \begin{bmatrix} 0 \\ \frac{v_2}{V} \end{bmatrix}$$

and α_0 , α_1 , v_1 , v_2 as in (3.4.4) .

3.5 Complete system

It is now possible to assemble the equation representing the aircraft with a feedback controller and the linear system driven by white noise representing the properties of the vertical gusts into a single linear constant coefficient system driven by white noise from which statistical properties of the states are easily determined.

From previous calculation :

$$\dot{X} = A_{New} X + A_g X_g \qquad (\textit{idem as } 3.4.1)$$

$$X_g = A_d Y + B_d w \qquad (\textit{idem as } 3.4.5)$$

$$\dot{Y} = A_w Y + B_w w \qquad (\textit{idem as } 3.4.2)$$

The complete system is then :

$$\begin{bmatrix} \dot{X} \\ \dot{Y} \end{bmatrix} = [A_{Aug}] \begin{bmatrix} X \\ Y \end{bmatrix} + [B_{Aug}]\, w \tag{3.5.1}$$

with

$$A_{Aug} = \begin{bmatrix} A_{New} & A_g A_d \\ 0 & A_w \end{bmatrix}$$

$$B_{Aug} = \begin{bmatrix} A_g B_d \\ B_w \end{bmatrix}$$

Or :

$$\begin{bmatrix} \dot{V} \\ \dot{\alpha} \\ \dot{q} \\ \dot{\theta} \\ \dot{Y}_1 \\ \dot{Y}_2 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} & & & \\ & A - B\,F & & \\ & & & \end{bmatrix} & \begin{bmatrix} & \\ A_g A_d & \\ & \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 1 \\ -\alpha_0 & -\alpha_1 \end{bmatrix} \end{bmatrix} \begin{bmatrix} V \\ \alpha \\ q \\ \theta \\ Y_1 \\ Y_2 \end{bmatrix} + \begin{bmatrix} \begin{bmatrix} \\ A_g B_d \\ \\ \end{bmatrix} \\ 0 \\ 1 \end{bmatrix} w$$

3.6 Standard deviation of the elevator deflection

As seen previously the variation of the drag coefficient is of the form :

$$\Delta C_D = C_{D\delta e}\, \sigma\, [\, \Delta\delta_e]$$

The rms value $\sigma\,[\,\Delta\delta_e\,]$ need then to be calculated, in order to get ΔC_D.

As the control is $U = \begin{bmatrix} \delta_T \\ \delta_e \end{bmatrix}$, the mean square value of $\Delta\delta_e{}^2$ can be obtained

from the variance matrix of U, C_u . C_u is of the form $C_u = E\,[\,U\,U^T\,]$

As C_u is a 2×2 matrix, the root mean square of $\Delta\delta_e$ is :

$$\sigma\,[\,\Delta\delta_e] = \sqrt{C_u(\,2\,,\,2\,)} \qquad (3.6.1)$$

where $C_U(\,2\,,\,2\,)$ is the coefficient on the second row, second column of C_U.

Furthermore, since $U = -FX$,

$$\begin{aligned} C_u &= E\,[\,F\,X\,X^T\,F^T\,] \\ &= F\,E\,[\,X\,X^T\,]\,F^T \end{aligned}$$

The mean square value $E\,[\,X\,X^T\,]$ can be calculated through the variance matrix of $Z(t)$, where $Z(t)$ is the solution of the previous augmented system (eq. 3.5.1):

$$\dot{Z} = A_{Aug} Z + B_{Aug} w \qquad (3.6.2)$$

For the steady state case, the variance matrix of $Z(t)$ tends to the matrix $\overline{Q}$, the solution of the Lyapunov equation [Ref. 11] :

$$A_{Aug} \overline{Q} + \overline{Q} A_{Aug}^{T} + B_{Aug} V B_{Aug}^{T} = 0 \qquad (3.6.3)$$

3.7 Summary

For each CG position, the unstable matrix system (eq. 3.2.2) is calculated, a feedback control law (eq. 3.3.2) is designed to retain the dynamic response properties of the airplane, the stabilized system (eq 3.3.3) is assembled with the linear system driven by white noise modeling the atmospheric turbulence (eq. 3.4.2) into a single linear system driven by white noise (eq. 3.5.1) ; the solution of the Lyapunov equation (eq. 3.6.3) is then computed for this augmented system, from which the rms value of the elevator deflection can be obtained (eq. 3.6.1). Finally, the variation of the drag coefficient due to atmospheric turbulence is calculated with the formula :

$$\Delta C_D = C_{D\delta e} \, \sigma \, [\, \Delta\delta_e]$$

4.0 Numerical Results

4.1 Overview

All the necessary formulas are now set in order to get the drag coefficient in calm atmosphere (Chapter 2) and its variation in atmospheric turbulence (Chapter 3), both versus CG position.

This chapter discusses the computational method used and the numerical results obtained.

4.2 Model of Airplane

The foregoing equations presented in Chapter 2 and Chapter 3 were evaluated for 6 different types of airplane whose data were given in references [8-10, 14]. Complete data for the aerodynamic coefficients and the wing-body and tail configurations were available for some nominal CG position and reference flight condition. The geometry and aerodynamic data are given in Table 1.

The 6 airplanes are :

- General aviation airplane : Navion
- Fighter aircraft : F 104-A
- Fighter aircraft : A4-D
- Business Jet : Jetstar
- Transport aircraft : Convair 880
- Transport aircraft : Boeing 747

4.3 Procedure

The static margin rather than the CG position will be used as a reference. The static margin is defined as follows :

$$SM = -(h - h_n)$$

where :

h distance in chord lengths from leading edge of wing
mean aerodynamic chord (MAC) to CG

h_n distance in chord lengths from leading edge of wing MAC
to aerodynamic center of the airplane .

The calculations (eq. 2.8 in chapter 2 and the matrix A and B (eq. 3.2.2) in chapter 3) are determined first with the aerodynamic properties given for the nominal static margin. Then, starting with this value of the static margin, it is decreased in small increments. For each new static margin, the previous calculation are repeated. The calculations for the calm atmosphere (eq. 2.8) are straight forward while these for the turbulent atmosphere require several steps. The A and B matrix (eq. 3.2.2) are recalculated for the new static margin. The feedback control matrix F must be determined from (eq. 3.3.2) and used to transform this A matrix into the A_{new} matrix (eq. 3.3.3) (which has approximately the same properties as the nominal A matrix). Then the system $\dot{X} = A_{new} X$ is augmented with the linear system describing the gust properties (eq. 3.4.2) resulting in a linear system driven by white noise $\dot{Z} = A_{aug} Z + B_{aug} w$ (eq. 3.6.2). The response of this system to white noise is calculated through the steady state solution of the Lyapunov equation (eq. 3.6.3). The rms value of the elevator deflection $\sigma\,[\,\Delta\delta_e\,]$ is obtained from this solution using (eq. 3.6.1). Finally, the variation of the drag coefficient in atmospheric turbulence is calculated with (eq. 3.1.1). This variation is added to the drag coefficient in calm atmosphere to obtain the drag coefficient in atmospheric turbulence.

Next, a curve fitting procedure (using a Least square method) is applied for the two sets of data points and the minimum of each curve is calculated.

In addition to these calculations, the eigenvalues of each A matrix and each A_{new} matrix are determined. The root loci for both are then drawn. The first root loci is to show that as the static margin decreases, at least 2 eigenvalues become real and move along the real axes. When one of them moves into the right half plane, this is the sign that the airplane becomes unstable. The second root loci is to show that the eigenvalues of the controlled A matrix are close to the eigenvalues of the nominal A matrix, verifying that the simple controller retains the related flying qualities.

4.4 Results

The intensity of the gust σ_w is choosen to be 1. It could range from 1 for clear air to 12 for thunderstorm.

The curves shown in the figures represent for each airplane :

- The relative drag coefficient ($C_D - C_{D0}$) represented by a dashed line (---) corresponding to a flight in calm atmosphere.
- The relative drag coefficient ($C_D - C_{D0}$) represented by a solid line (—) corresponding to the variation of the relative drag coefficient due to atmospheric turbulence added to the initial relative drag coefficient above.
- The star represents the minimum for each curve.
- The extreme left of the curve (---) corresponds to the nominal CG position associated with the nominal aerodynamic coefficients. (Ref. 10).

As supposed previously, the relative drag coefficient in atmospheric turbulence is slightly increased. The location of the CG for minimum relative drag coefficient in atmospheric turbulence is not as far aft as for the minimum in calm atmosphere. In fact, the further aft the CG, the more unstable the airplane and the bigger the relative drag coefficient.

Table 2 shows for each airplane :

- The static margin corresponding to the minimum relative drag coefficient calculated in calm atmosphere (SM_{cal}) and in atmospheric turbulence (SM_{tur}).
- The relative drag coefficient in calm atmosphere calculated with the CG placed at the position required for minimum relative drag coefficient in calm atmosphere (C_{Dcal} (SM_{calm})) and in atmospheric turbulence (C_{Dcal} (SM_{tur})).
- The relative drag coefficient in atmospheric turbulence calculated with the CG placed at the position required for minimum relative drag coefficient in calm atmosphere (C_{Dtur} (SM_{calm})) and in atmospheric turbulence (C_{Dtur} (SM_{tur})).
- The reduction of the relative drag coefficient for flight in atmospheric turbulence and the increase of the relative drag coefficient for flight in calm atmosphere both using the CG position for atmospheric turbulence.
- The total improvement of the relative drag coefficient if the flight is considered half of the time in turbulence and half of the time in calm atmosphere.

The values for the static margin are non dimensional and have to be multiplied by the mean aerodynamic chord $\bar{c}$ in order to get the actual distance from the CG position to the neutral point. Furthermore, the values for the relative drag coefficients have to be read as the value in the table times 10^{-2} . These values correspond to $C_D - C_{Do}$. Therefore the % given in the table are calculated based on $C_D - C_{Do}$, not on C_D , and the results are called " relative drag coefficient " .

Let's take the Navion as an example. Results can be found in table 2 and in figure 14. The minimum relative drag coefficients in turbulence and calm atmosphere can be read as $.9337 \times 10^{-2}$ and $.7707 \times 10^{-2}$, respectively; the relative drag coefficient in turbulent atmosphere is then increased over the drag in calm atmosphere by 20 %. Furthermore, the relative drag coefficient in turbulent atmosphere with the CG placed at the minimum for calm atmosphere is $.9929 \times 10^{-2}$; the reduction of the relative drag coefficient in turbulent atmosphere from the CG in calm atmosphere to the CG in turbulent atmosphere is then 6.3 %. But the relative drag coefficient in calm atmosphere with the CG placed at the minimum for turbulent atmosphere is $.8186 \times 10^{-2}$ and is then increased over the relative drag coefficient with the CG placed at the minimum for calm atmosphere by 5.3 %. Therefore, if it is assumed that the airplane flies 50 % of the time in atmospheric turbulence, by placing the CG for the minimum relative drag coefficient in turbulent atmosphere, 1 % of the relative drag coefficient could be saved.

If it is assumed a C_{Do} of 0.03, then the above numbers for the Navion yield a 4.3 % increase in total drag coefficient in atmospheric turbulence over the drag coefficient in calm atmosphere; a 1.5 % reduction in total drag coefficient in turbulent atmosphere and a 1.2 % increase in total drag coefficient in calm atmosphere using the CG for turbulent atmosphere instead of the CG for calm atmosphere; and a 0.3 % reduction in total drag coefficient assuming 50 % of the time in both atmosphere.

5.0 Conclusion

The results of the study show that moving the CG aft doesn't always reduce the drag coefficient and therefore the drag. In fact when the airplane encounters atmospheric turbulence, the control activity is such that its effective drag may increases. According to the study, by calculating the difference between the minimum drag coefficient in calm atmosphere and the minimum drag coefficient in turbulent atmosphere, the relative drag coefficient due to turbulence can increase from 5 % (heavy airplane (HA)) to 40 % (light airplane (LA)).

When an airplane flies in atmospheric turbulence, .8 % (HA) to 9.6 % (LA) of the relative drag coefficient can be saved by placing the CG at the position for minimum drag in atmospheric turbulence rather than the position for minimum drag in calm atmosphere. But if the airplane flies in calm atmosphere with this position .75 % (HA) to 9 % (LA) would be lost . If it is assumed that the airplane flies 50 % of the time in atmospheric turbulence, .02 % (HA) to 1 % (LA) of the relative drag coefficient could then be saved. Furthermore, if it assumed a C_{Do} of 0.03, the total drag coefficient in turbulent atmosphere is increased by 1.8 % (HA) to 8.2 % (LA) over the drag coefficient in calm atmosphere; using the CG for turbulent atmosphere instead of the CG for calm atmosphere, the total drag coefficient in turbulent atmosphere is reduced by .3 % (HA) to 2.5 % (LA) while the total drag coefficient in calm atmosphere is increased by .28 % (HA) to 1.9 % (LA). Again, if the airplane flies 50 % of the time in turbulent atmosphere, by positioning the CG for turbulent atmosphere, .02 % (HA) to .6 % (LA) of the total drag coefficient could be saved. Therefore the best CG position to improve the drag is close to the position for minimum drag in atmospheric turbulence. It is also to be noticed that the best improvement can be made on the light airplane.

Further test have been done with different ε_0, with increased aspect ratio (by 10 %), with increased $C_{D\delta e}$ (by 5 %) and with increased velocity ($M = .9$ instead of $M = .25$) for the Boeing 747. The improvement in drag coefficient is also of the same order. An increase of the intensity of the turbulence yields to an increase of the drag coefficient in atmospheric turbulence and to a better improvement of the total drag coefficient (in % saved).

Further investigation for supersonic and hypersonic flight still remain to be done.

Table 1. Data

	Navion	F 104	A4-D	Jetstar	Convair	Boeing
M_c	.158	.257	.400	.200	.250	.250
W	2750	16300	17578	38200	126000	6366000
h_{ref}	.295	.070	.250	.250	.250	.250
I	3000	58511	25900	135869	2450000	33100000
S	184	196.1	260	542.5	2000	5500
b_w	33.4	21.94	27.5	53.75	120	195.68
$\bar{c}$	5.70	9.55	10.8	10.93	18.94	27.31
b_t	13.20	11.11	11.35	24.90	40.00	72.90
S_t	28.95	48.20	48.85	117.8	395.0	1470.
C_L	.410	.735	.280	.737	.680	1.11
C_D	.050	.263	.030	.095	.080	.102
$C_{L\alpha}$	4.44	3.44	3.45	5.00	4.52	5.70
$C_{D\alpha}$	.330	.450	.300	.750	.270	.660
$C_{m\alpha ref}$	-.683	-.640	-.380	-.800	-.903	-1.26
$C_{L\dot{\alpha}}$	0.00	0.00	.72	0.00	2.70	6.70
$C_{m\dot{\alpha} ref}$	-4.36	-1.60	-1.10	-3.00	-4.13	-3.20
C_{Lq}	3.80	0.00	0.00	0.00	7.72	5.40
C_{mqref}	-9.96	-5.80	-3.60	-8.00	-12.1	-20.8
C_{LMa}	0.00	0.00	0.00	0.00	0.00	-0.81
C_{DMa}	0.00	0.00	0.00	0.00	0.00	0.00
C_{mMa}	0.00	0.00	0.00	-.05	0.00	0.27
$C_{L\delta e}$	.355	.680	.360	.400	.213	.338
$C_{m\delta eref}$	-.923	-1.46	-.50	-.81	-.637	-1.34
h_{nwb}	.250	.250	.250	.257	.250	.250
C_{mowb}	-.099	0.00	-.099	-.040	0.00	-.099
$C_{L\alpha w}$	5.56	5.56	5.56	6.02	5.56	5.56
ε_o	0.04	0.04	0.04	0.04	0.04	0.04
η	1.00	1.00	1.00	1.00	1.00	1.00
L	5000	5000	5000	5000	5000	5000

Table 2. Results

	Navion	F 104	A4-D	Jetstar	Convair	Boeing
SM_{cal}	-.390	-.390	-.441	-.251	-.092	-.342
SM_{tur}	-.163	-.279	-.285	-.117	-.012	-.227
$C_{Dcal}(SM_{cal})$	.7707	5.626	.7396	2.698	1.851	4.977
$C_{Dcal}(SM_{tur})$	.8186	5.734	.8131	2.779	1.865	5.016
$C_{Dtur}(SM_{cal})$	.9929	6.549	1.150	3.159	1.955	5.325
$C_{Dtur}(SM_{tur})$	.9337	6.426	1.049	3.062	1.940	5.283
$\% inc\ C_D$	5.3	1.9	9	2.9	.75	.78
$\% dec\ C_D$	6.3	1.9	9.6	3.2	.8	.8
$\% saved\ C_D$	1	0	.6	0.3	.05	.02

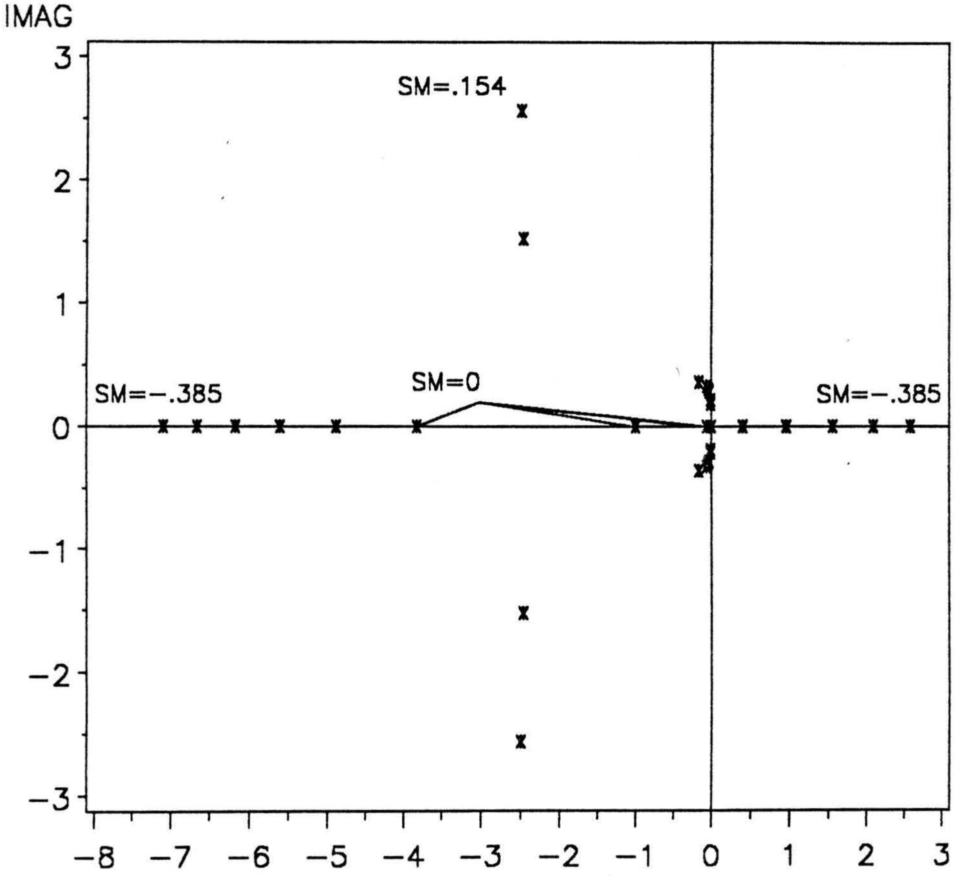

Figure 2. Root loci for NAVION.

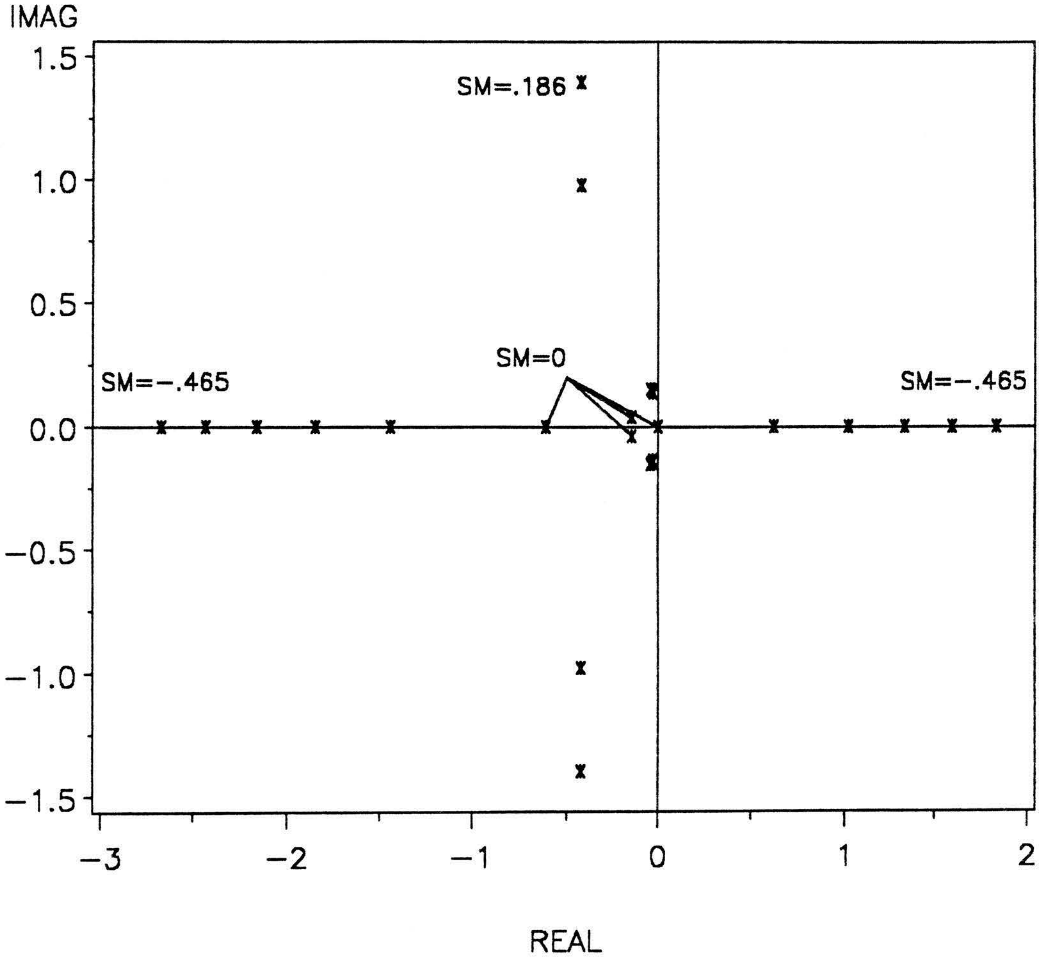

Figure 3. Root loci for F 104.

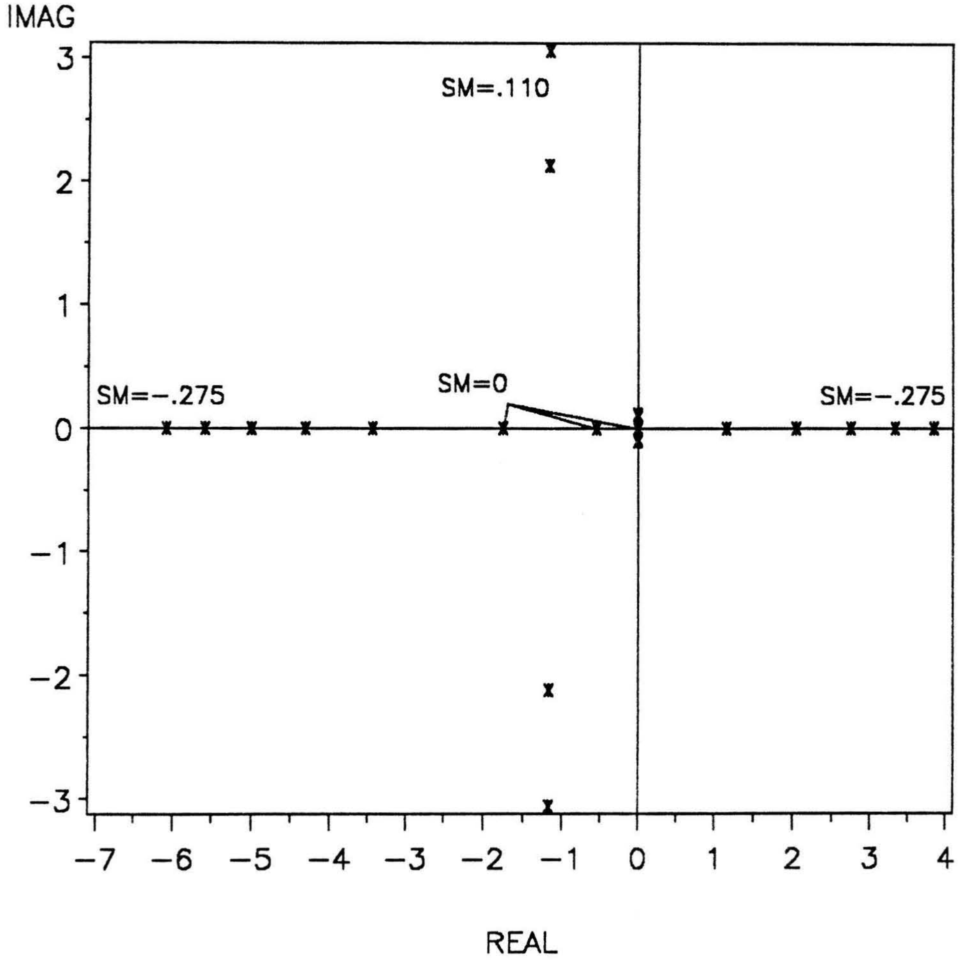

Figure 4. Root loci for A4-D.

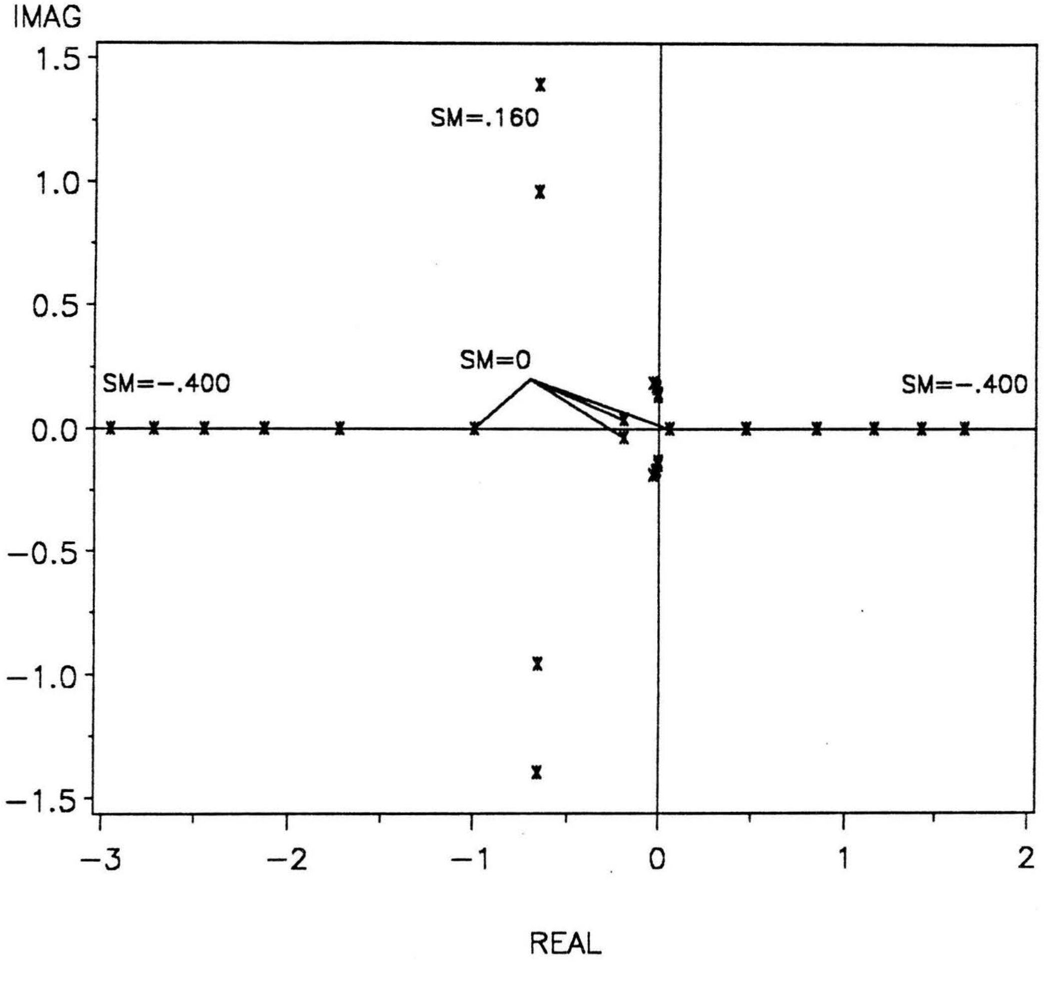

Figure 5. Root loci for JETSTAR.

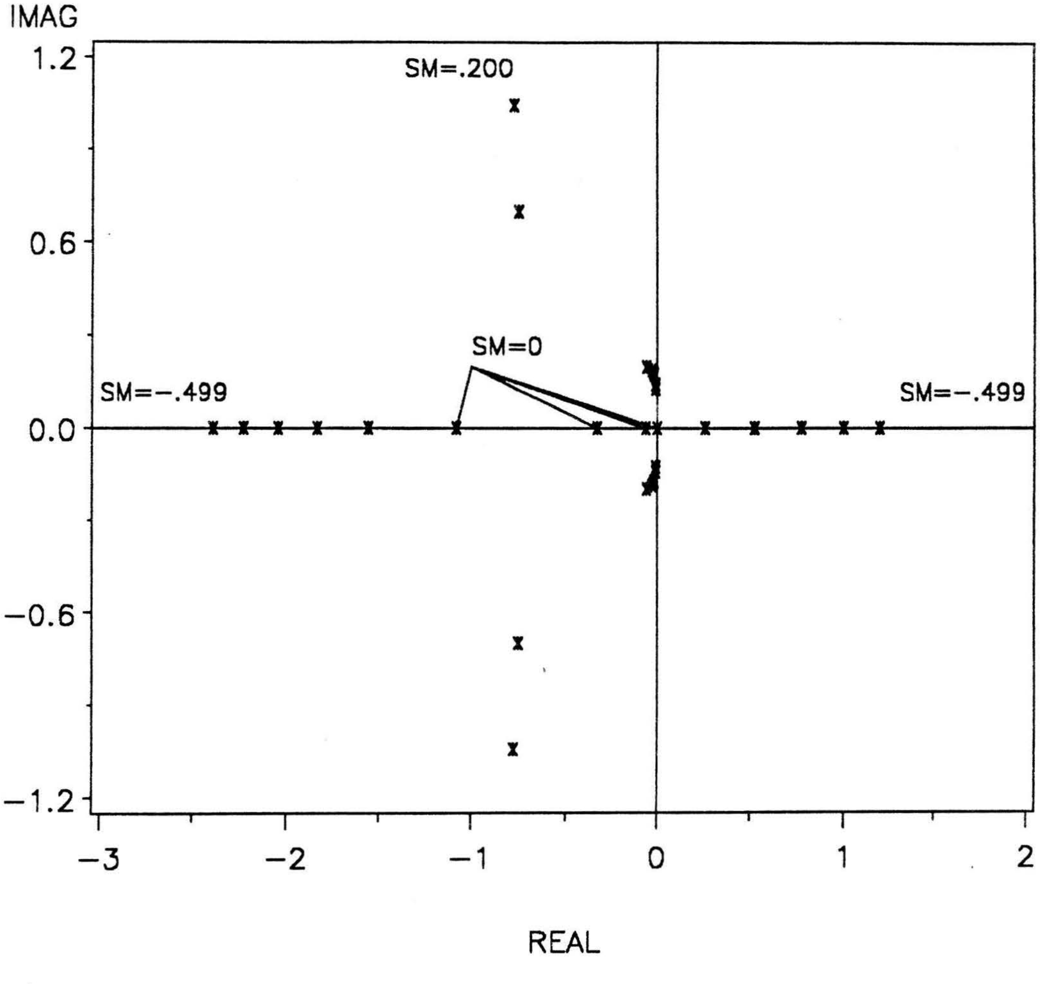

Figure 6. Root loci for CONVAIR.

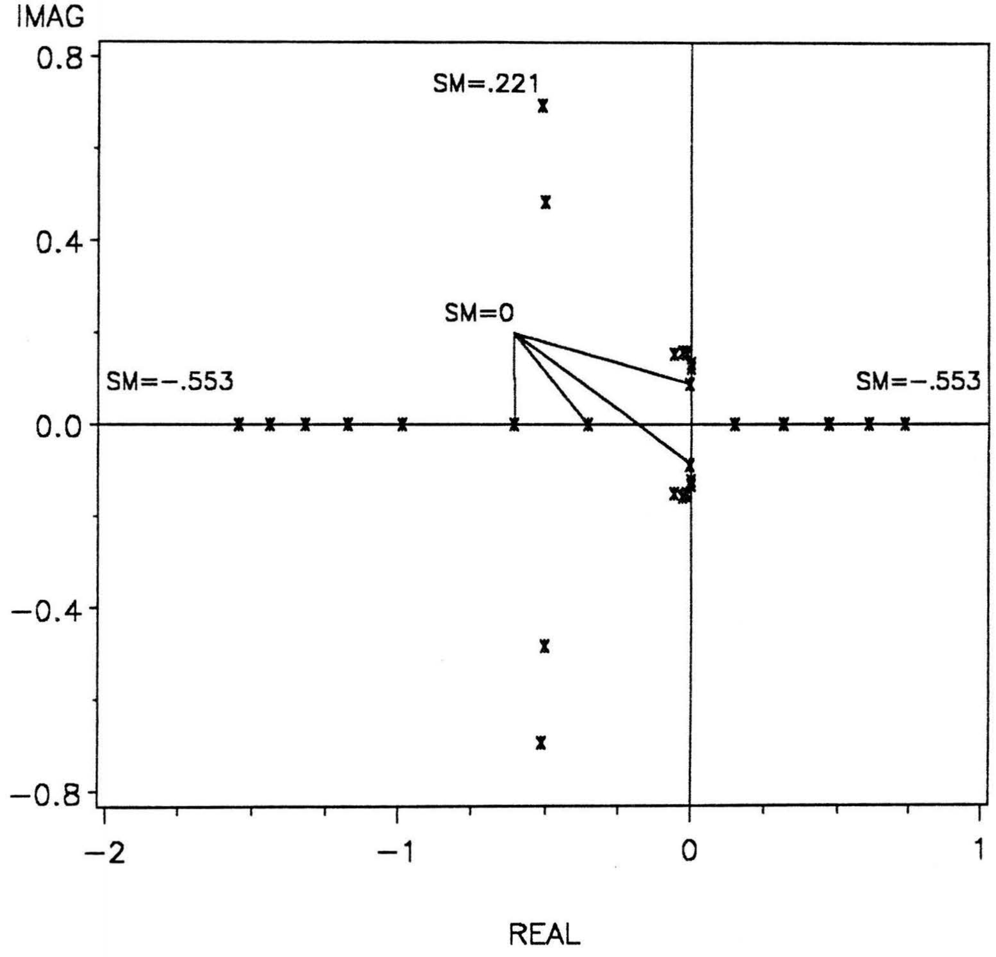

Figure 7. Root loci for BOEING.

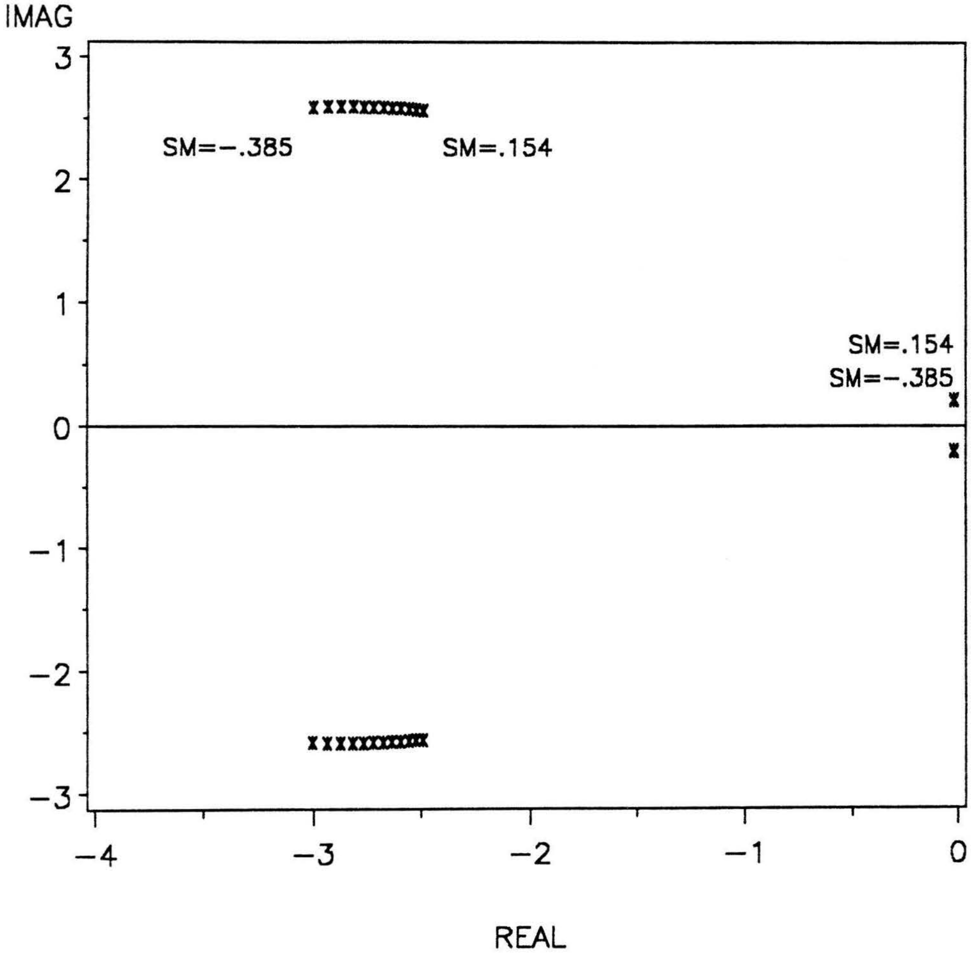

Figure 8. Eigenvalues of the A matrices for NAVION.

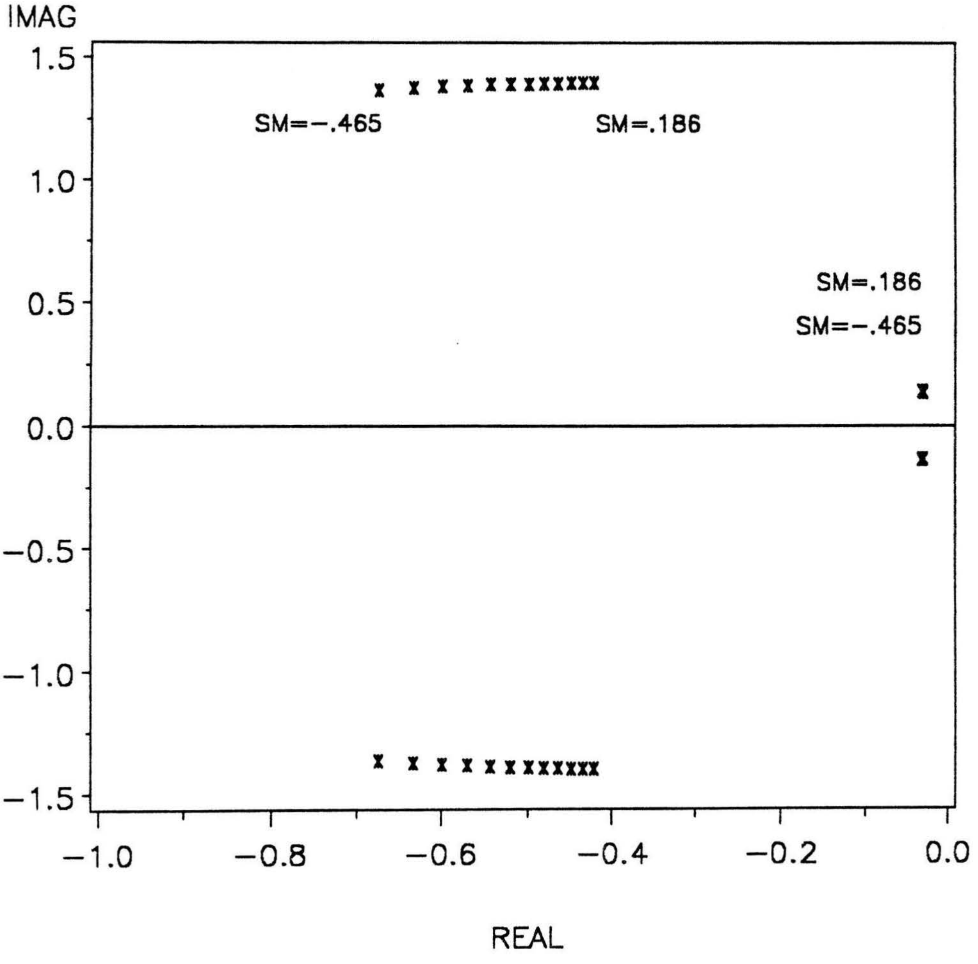

Figure 9. Eigenvalues of the A matrices for F 104.

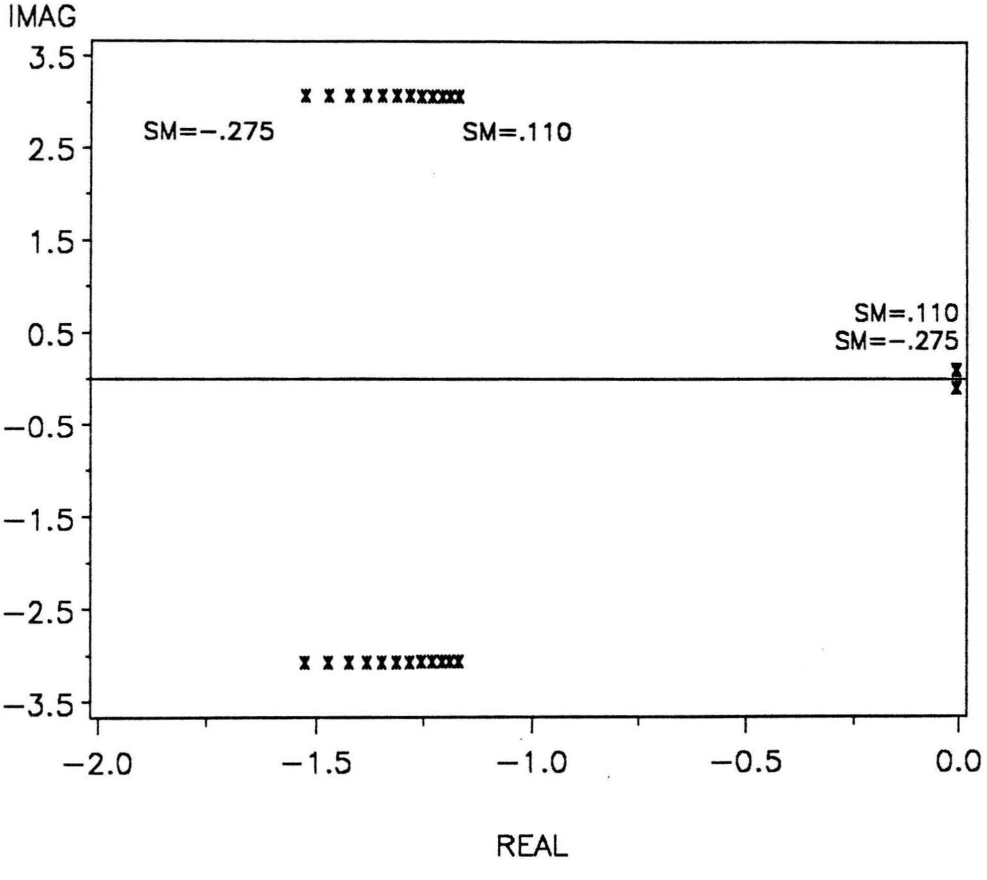

Figure 10. Eigenvalues of the A matrices for A4-D.

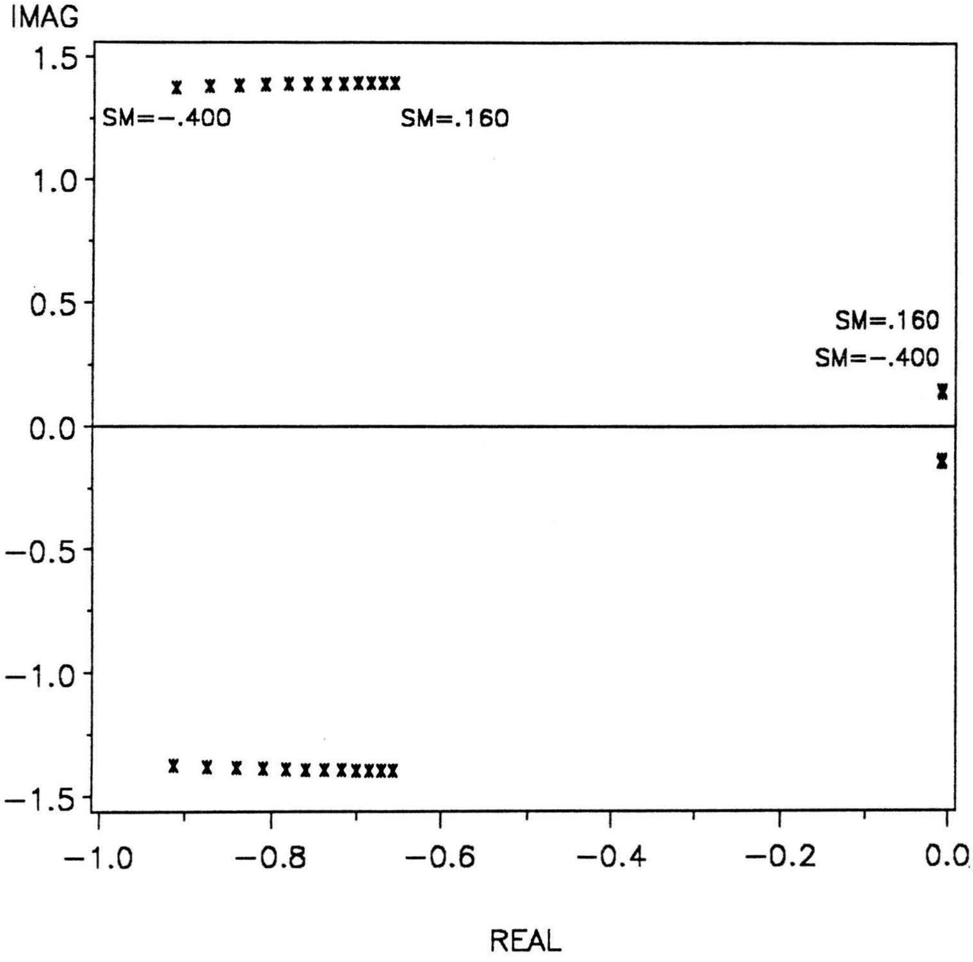

Figure 11. Eigenvalues of the A matrices for JETSTAR.

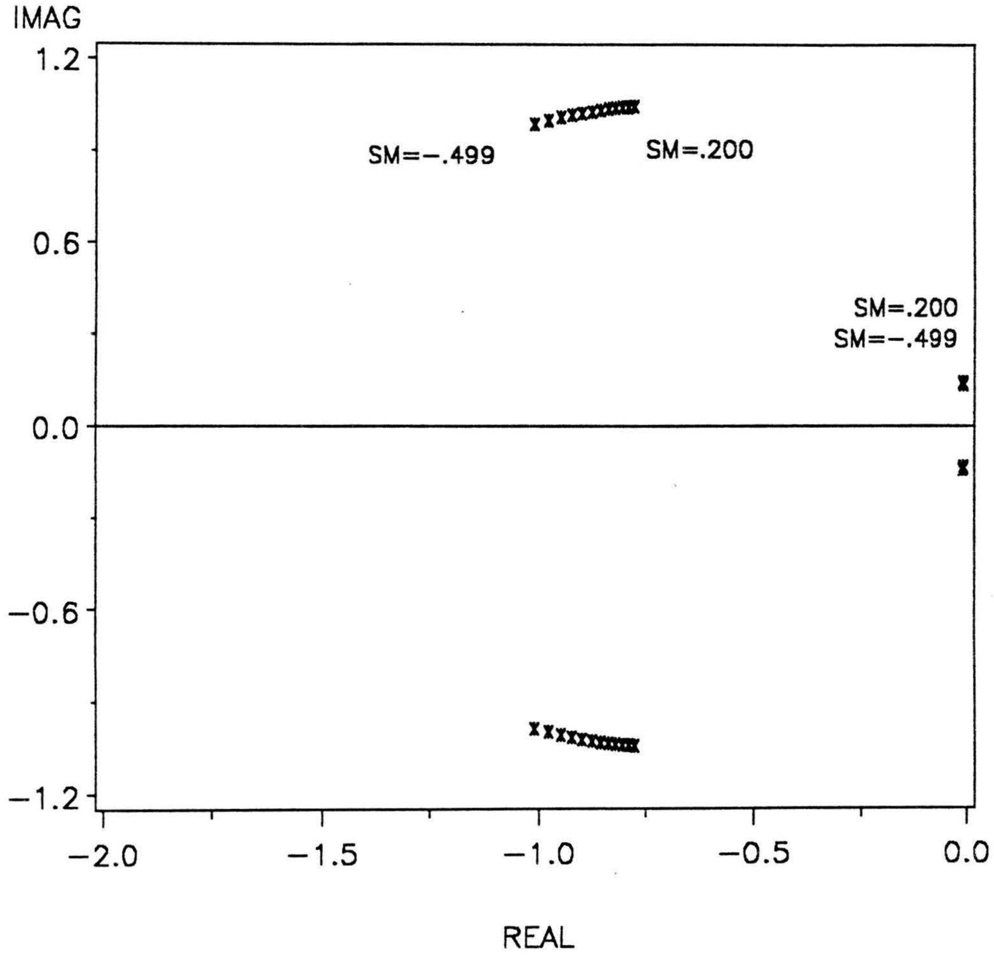

Figure 12. Eigenvalues of the A matrices for CONVAIR.

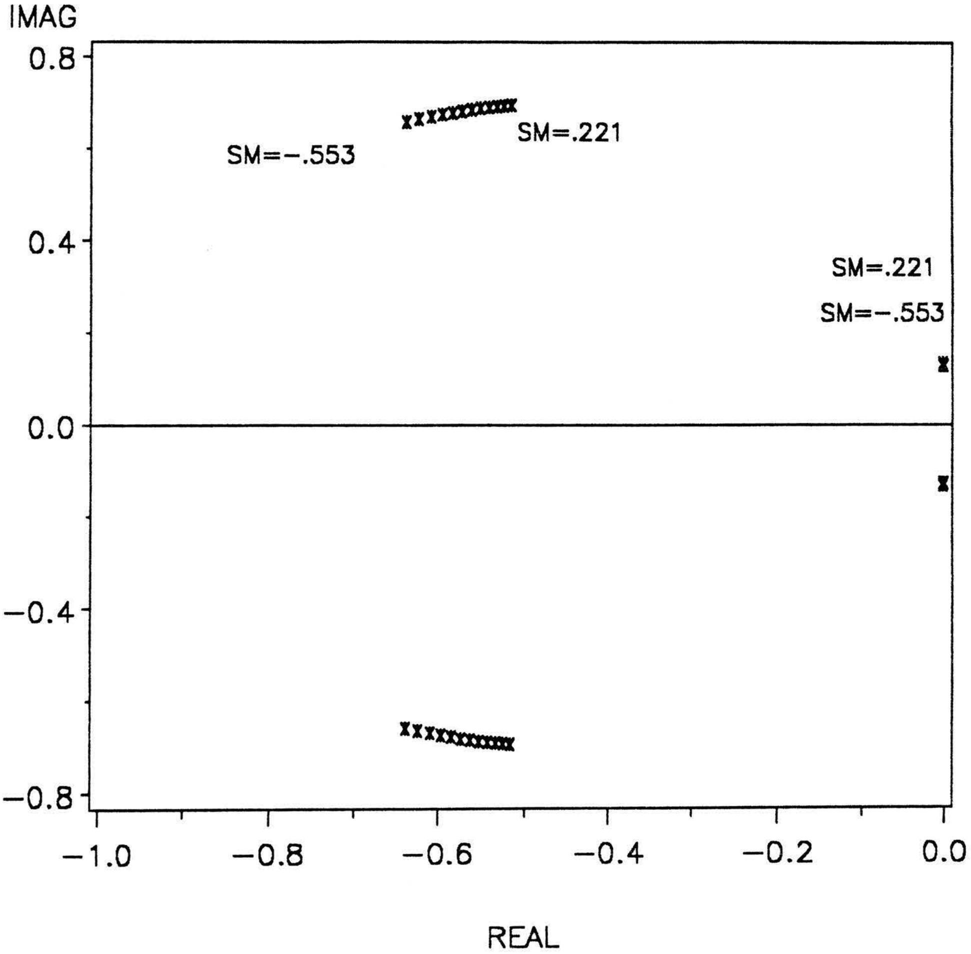

Figure 13. Eigenvalues of the A matrices for BOEING.

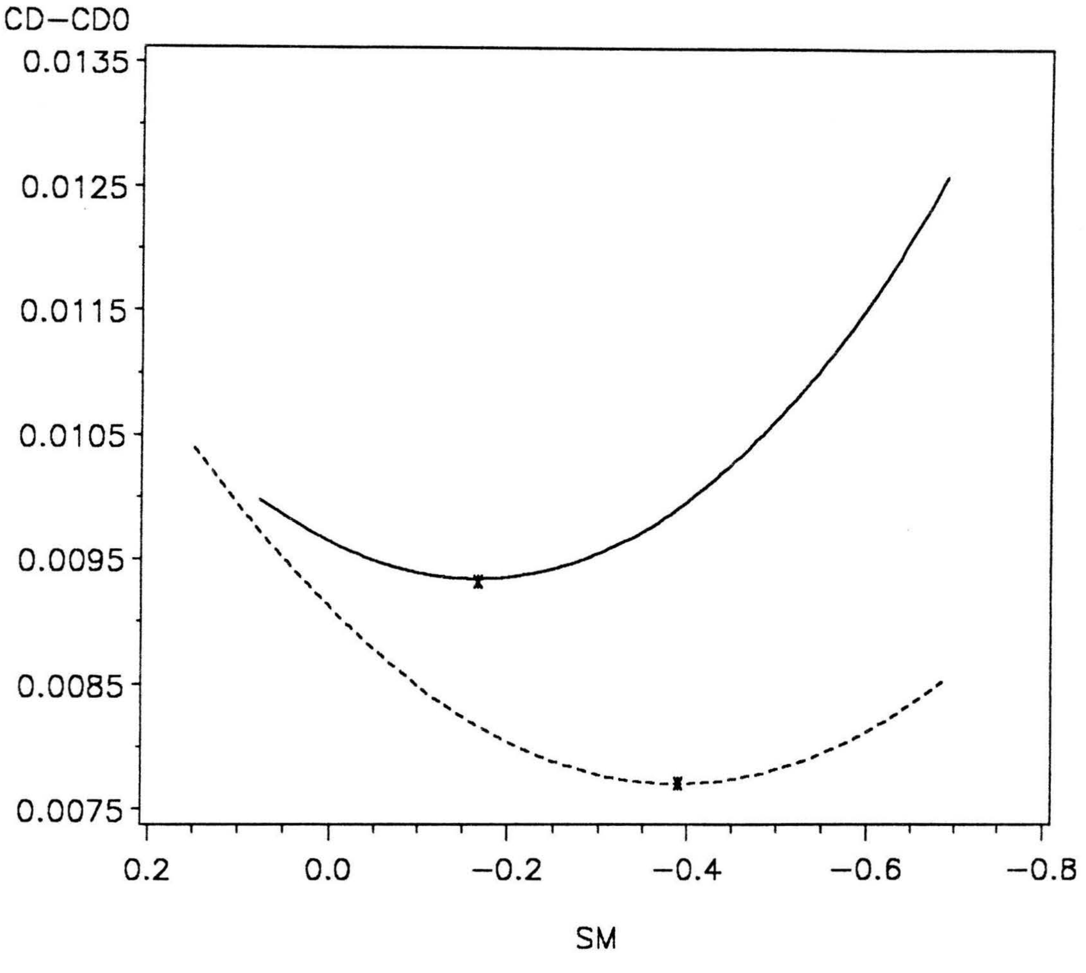

Figure 14. Drag-Coefficient versus static margin for NAVION.

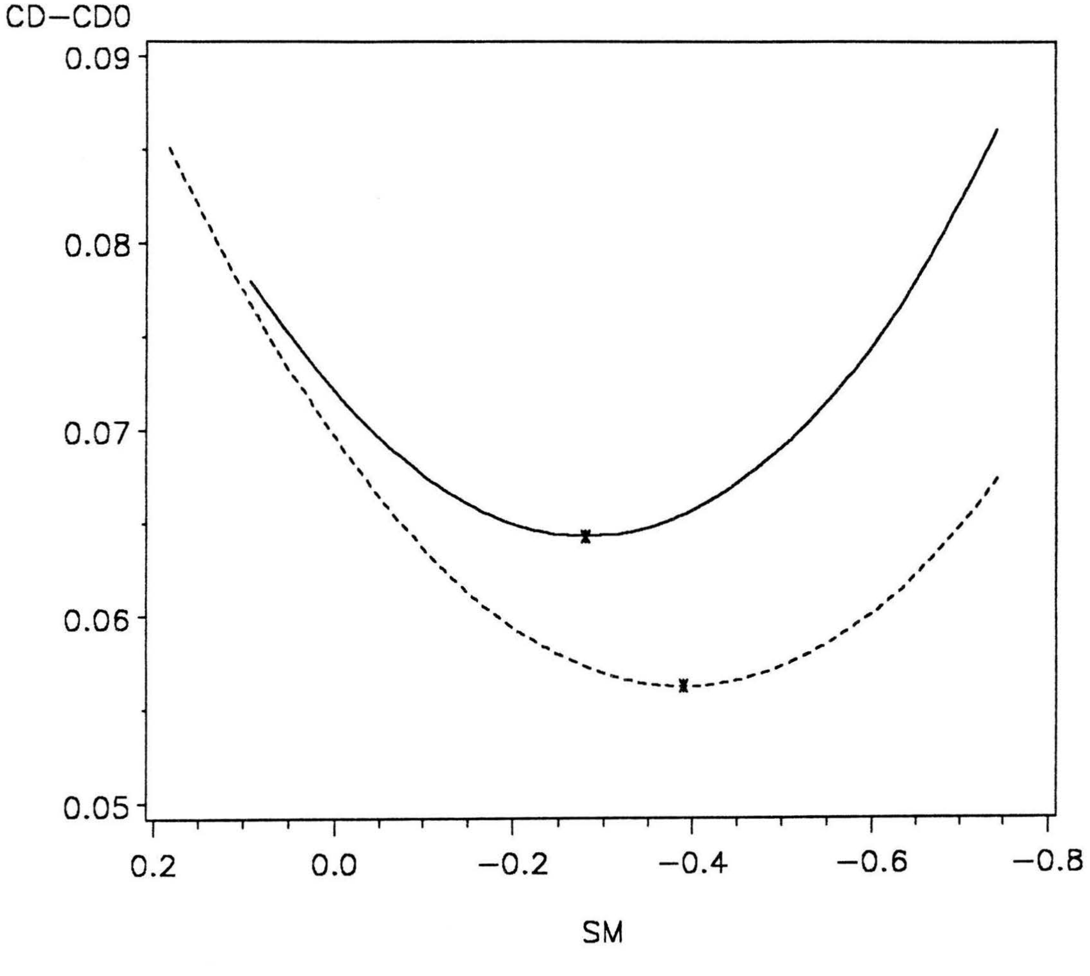

Figure 15. Drag-Coefficient versus static margin for F 104.

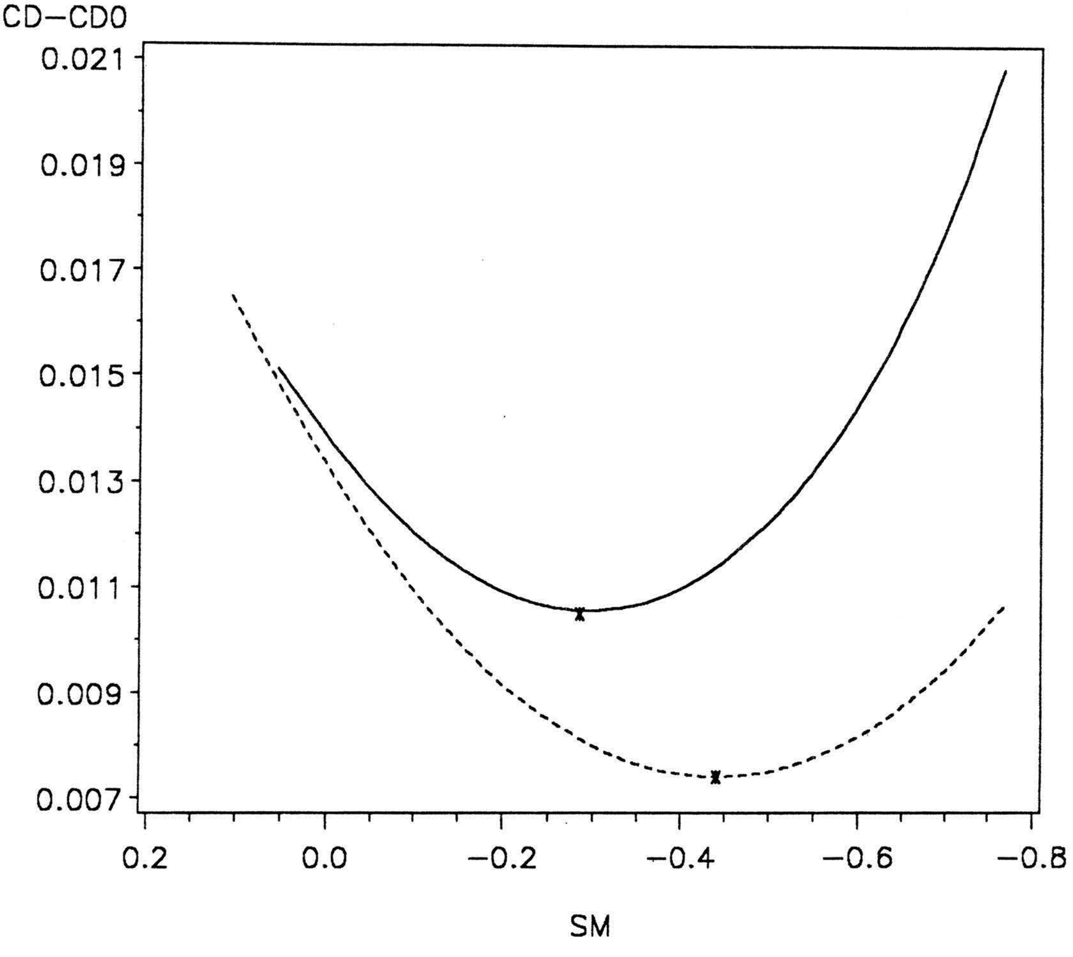

Figure 16. Drag-Coefficient versus static margin for A4-D.

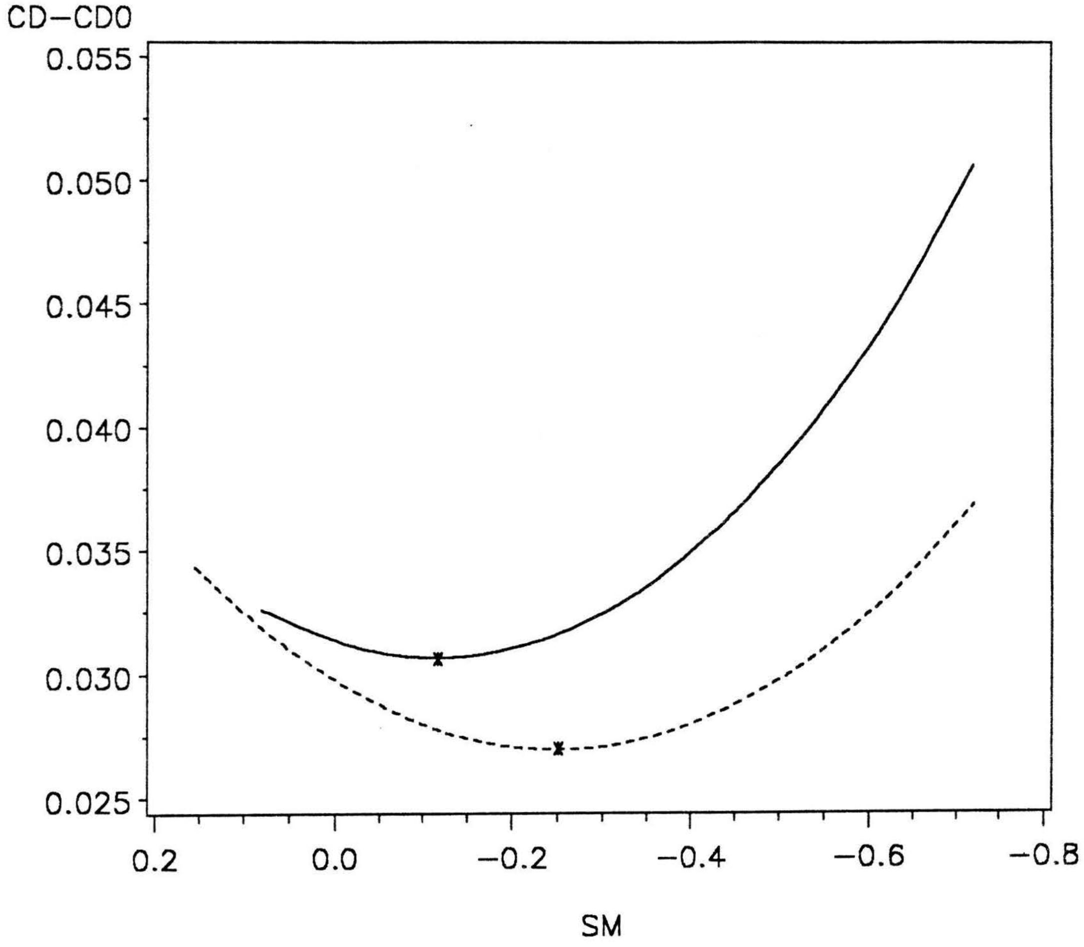

Figure 17. Drag-Coefficient versus static margin for JETSTAR.

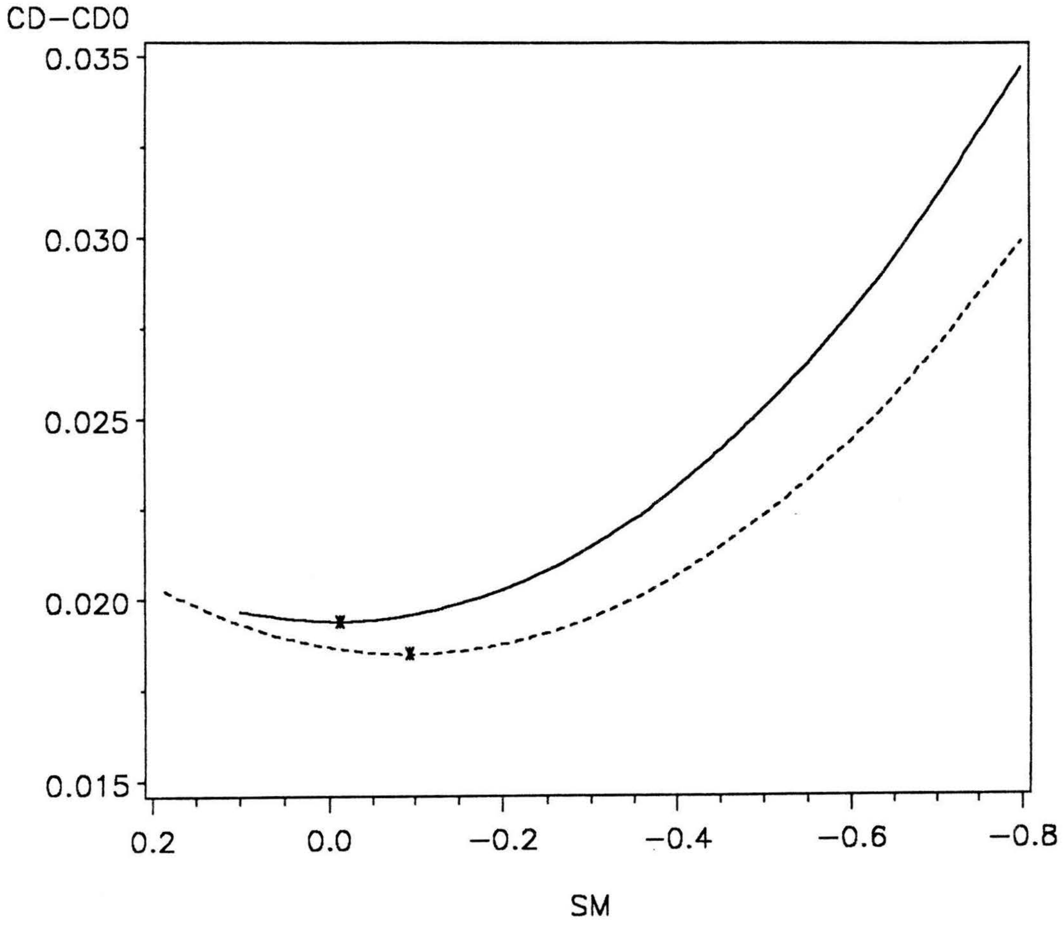

Figure 18. Drag-Coefficient versus static margin for CONVAIR.

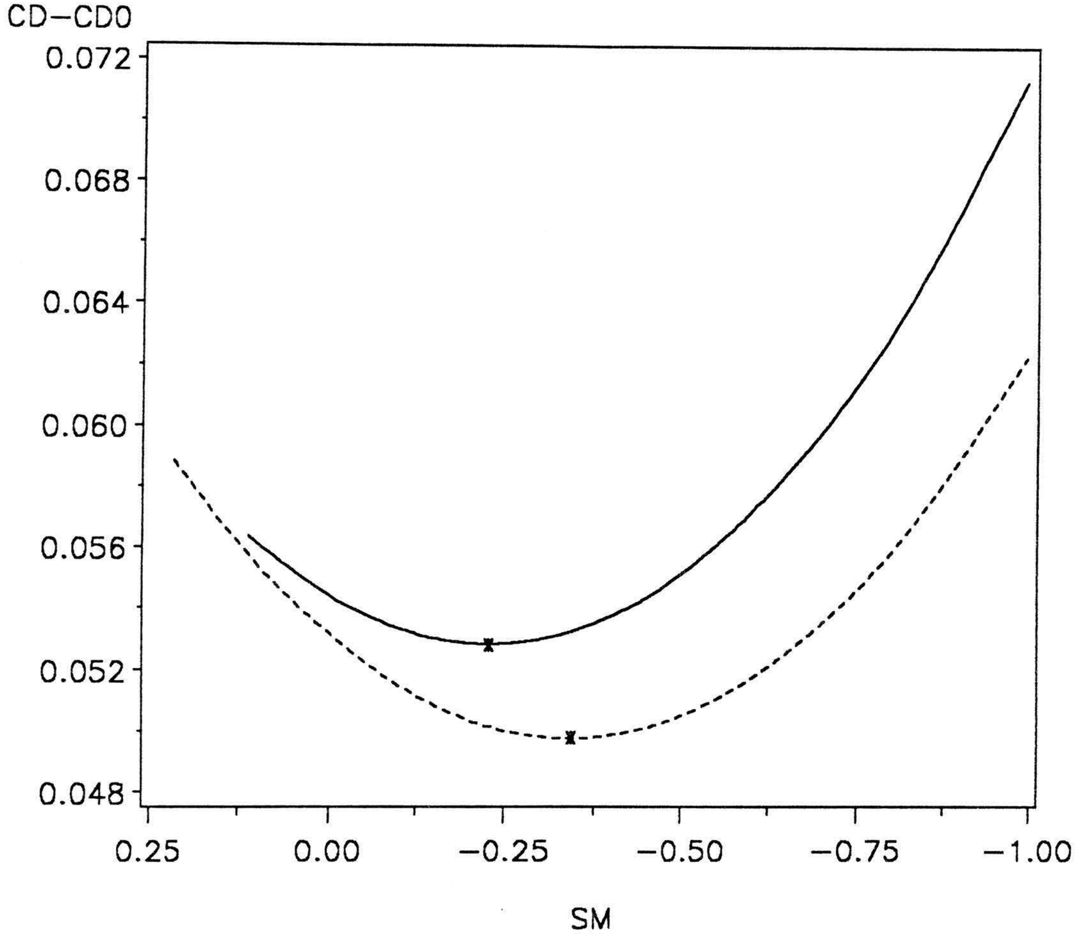

Figure 19. Drag-Coefficient versus static margin for BOEING.

Appendix A. Additional relations and parameters

A1. Additional relations

$$h_t = h_{ref} - \frac{(C_{m\delta e})_{ref}}{C_{L\delta e}}$$

$$h_n = h_{ref} - \frac{(C_{m\alpha})_{ref}}{C_{L\alpha}}$$

$$K_{wb} = \frac{1}{\pi AR_{wb}}$$

$$K_t = \frac{1}{\pi AR_t}$$

$$AR_{wb} = \frac{b_w^2}{S}$$

$$AR_t = \frac{b_t^2}{S_t}$$

$$m = \frac{W}{g}$$

$$\bar{q} = \frac{1}{2} \rho V^2$$

$$V = M_a \ a_c$$

A2. Parameters

$\rho \ = \ 0.002377$

$g \ = \ 32.174$

$a_c \ = \ 1116.4$

$\gamma \ = \ 0.0$

Appendix B. Linearization.

B1. Linearization

The governing equations-of-motion are of the form :

$$f_i(X, \dot{X}, U, t) = 0 \qquad i = 1, \dots, 4 \qquad (B.1)$$

where

X is a 4 dimensional state vector

U is a 2 dimensional control vector

A reference solution is such that :

$$f_i(X, \dot{X}, U, t)_r = 0 \qquad i = 1, \dots, 4 \qquad (B.2)$$

For small disturbances the variables can be written as :

$$X = X_r + \Delta X$$
$$\dot{X} = \dot{X}_r + \Delta \dot{X}$$
$$U = U_r + \Delta U$$

(B.1) can be written in the form :

$$f_i(\, X_r + \Delta X,\ \dot{X}_r + \Delta \dot{X},\ U_r + \Delta U,\ t\,) = 0 \qquad i = 1,\ldots,4$$

Expanding this equation in a multivariable Taylor series about the reference trajectory :

$$\begin{aligned} f(\, X_r + \Delta X,\ \dot{X}_r + \Delta \dot{X},\ U_r + \Delta U,\ t\,) &= f(\, X,\ \dot{X},\ U,\ t) \\ &\quad + \left(\frac{\partial f}{\partial X}\right)_r \Delta X + \left(\frac{\partial f}{\partial \dot{X}}\right)_r \Delta \dot{X} + \left(\frac{\partial f}{\partial U}\right)_r \Delta U + \ldots \\ &= 0 \end{aligned}$$

But from (B.2) the first term in the right hand side must be zero. Hence, by neglecting higher order terms :

$$\left(\frac{\partial f}{\partial X}\right)_r \Delta X + \left(\frac{\partial f}{\partial \dot{X}}\right)_r \Delta \dot{X} + \left(\frac{\partial f}{\partial U}\right)_r \Delta U = 0$$

Then,

$$\Delta \dot{X} = A\, \Delta X + B\, \Delta U$$

with,

$$A = -\left(\frac{\partial f}{\partial X}\right)^{-1}\left(\frac{\partial f}{\partial \dot{X}}\right)_r$$

$$B = -\left(\frac{\partial f}{\partial X}\right)^{-1}\left(\frac{\partial f}{\partial U}\right)_r$$

Applied to the problem, using eqn. (3.2.1) :

$$f_1 = T - D - mg \sin \gamma - m\dot{V} = 0$$

$$f_2 = L - mg \cos \gamma - m\dot{V}\,\dot{\gamma} = 0$$

$$f_3 = M - I\,\dot{q} = 0$$

$$f_4 = q - \dot{\theta} = 0$$

The $\dot{V}$, $\dot{\alpha}$, $\dot{q}$, $\dot{\theta}$ derivatives are

f_1 :

$$\frac{\partial f_1}{\partial \dot{V}} = -m$$
$$\frac{\partial f_1}{\partial \dot{\alpha}} = \frac{\partial f_1}{\partial \dot{q}} = \frac{\partial f_1}{\partial \dot{\theta}} = 0$$

f_2 :

$$\frac{\partial f_2}{\partial \dot{V}} = 0$$

$$\frac{\partial f_2}{\partial \dot{\alpha}} = \frac{\partial f_2}{\partial \dot{\gamma}} \frac{\partial \dot{\gamma}}{\partial \dot{\alpha}} = mV$$

$$\frac{\partial f_2}{\partial \dot{q}} = 0$$

$$\frac{\partial f_2}{\partial \dot{\theta}} = \frac{\partial f_2}{\partial \dot{\gamma}} \frac{\partial \dot{\gamma}}{\partial \dot{\theta}} = -mV$$

f_3:

$$\frac{\partial f_3}{\partial \dot{V}} = 0$$

$$\frac{\partial f_3}{\partial \dot{\alpha}} = \frac{\partial M}{\partial \dot{\alpha}}$$

$$\frac{\partial f_3}{\partial \dot{q}} = -I$$

$$\frac{\partial f_3}{\partial \dot{\theta}} = \frac{\partial M}{\partial \dot{\alpha}} \frac{\partial \dot{\alpha}}{\partial \dot{\theta}} = \frac{\partial M}{\partial \dot{\alpha}}$$

$f_4 =$

$$\frac{\partial f_4}{\partial \dot{V}} = \frac{\partial f_4}{\partial \dot{\alpha}} = \frac{\partial f_4}{\partial \dot{q}} = 0$$

$$\frac{\partial f_4}{\partial \dot{\theta}} = -1$$

The Matrix is then

$$\frac{\partial f}{\partial \dot{X}} = \begin{bmatrix} -m & 0 & 0 & 0 \\ 0 & mV & 0 & -mV \\ 0 & M_{\dot{\alpha}} & -I & M_{\dot{\alpha}} \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

The Inverse Matrix is then

$$\left(\frac{\partial f}{\partial \dot{X}}\right)^{-1} = \begin{bmatrix} -\frac{1}{m} & 0 & 0 & 0 \\ 0 & \frac{1}{mV} & 0 & -1 \\ 0 & \frac{M_{\dot{\alpha}}}{mVI} & -\frac{1}{I} & \frac{M_{\dot{\alpha}}}{I} \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

The V, α, q, θ, δ_T, δ_e derivatives are,

V:

$$\frac{\partial f_1}{\partial V} = T_V - D_V$$

$$\frac{\partial f_2}{\partial V} = L_V$$

$$\frac{\partial f_3}{\partial V} = M_V$$

$$\frac{\partial f_4}{\partial V} = 0$$

α :

$$\frac{\partial f_1}{\partial \alpha} = T_\alpha - D_\alpha + mg \cos \gamma$$
$$\frac{\partial f_2}{\partial \alpha} = L_\alpha - mg \sin \gamma$$
$$\frac{\partial f_3}{\partial \alpha} = M_\alpha$$
$$\frac{\partial f_4}{\partial \alpha} = 0$$

q :

$$\frac{\partial f_1}{\partial q} = - D_q$$
$$\frac{\partial f_2}{\partial q} = L_q$$
$$\frac{\partial f_3}{\partial q} = M_q$$
$$\frac{\partial f_4}{\partial q} = 1$$

θ :

$$\frac{\partial f_1}{\partial \theta} = - mg \cos \gamma$$
$$\frac{\partial f_2}{\partial \theta} = mg \sin \gamma$$
$$\frac{\partial f_3}{\partial \theta} = 0$$
$$\frac{\partial f_4}{\partial \theta} = 0$$

δ_T :

$$\frac{\partial f_1}{\partial \delta_T} = T_{\delta T}$$

$$\frac{\partial f_2}{\partial \delta_T} = T_{\delta T}$$

$$\frac{\partial f_3}{\partial \delta_T} = \frac{\partial f_4}{\partial \delta_T} = 0$$

δ_e :

$$\frac{\partial f_1}{\partial \delta_e} = -\, D_{\delta e}$$

$$\frac{\partial f_2}{\partial \delta_e} = L_{\delta e}$$

$$\frac{\partial f_3}{\partial \delta_e} = M_{\delta e}$$

$$\frac{\partial f_4}{\partial \delta_e} = 0$$

The Matrices are

$$\frac{\partial f}{\partial X} = \begin{bmatrix} T_V - D_V & T_\alpha - D_\alpha + mg\cos\gamma & -\, D_q & -\, mg\cos\gamma \\ L_V & L_\alpha - mg\sin\gamma & L_q & mg\sin\gamma \\ M_V & M_\alpha & M_q & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

and

$$\frac{\partial f}{\partial U} = \begin{bmatrix} T_{\delta T} & -D_{\delta e} \\ T_{\delta T} & L_{\delta e} \\ 0 & M_{\delta e} \\ 0 & 0 \end{bmatrix}$$

The state Matrix is then

$$A = \begin{bmatrix} \frac{T_V - D_V}{m} & \frac{T_\alpha - D_\alpha + mg\cos\gamma}{m} & \frac{D_q}{m} & -g\cos\gamma \\ -\frac{L_V}{mV} & -\frac{L_\alpha - mg\sin\gamma}{mV} & 1 - \frac{L_q}{mV} & -\frac{g\sin\gamma}{V} \\ -\frac{L_V M_{\dot\alpha}}{mVI} + \frac{M_V}{I} & -(L_\alpha - mg\sin\gamma)\frac{M_{\dot\alpha}}{mVI} + \frac{M_\alpha}{I} & -\frac{L_q M_{\dot\alpha}}{mVI} + \frac{M_q}{I} + \frac{M_{\dot\alpha}}{I} & \frac{g\sin\gamma}{IV} M_{\dot\alpha} \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

The Control Matrix is

$$B = \begin{bmatrix} \frac{T_{\delta T}}{m} & -\frac{D_{\delta e}}{m} \\ -\frac{T_{\delta T}}{mV} & -\frac{L_{\delta e}}{mV} \\ -\frac{T_{\delta T} M_{\dot{\alpha}}}{mVI} & -\frac{L_{\delta e} M_{\dot{\alpha}}}{mVI} + \frac{M_{\delta e}}{I} \\ 0 & 0 \end{bmatrix}$$

B2. Dimensional Stability Derivatives

V derivatives :

$$T_V = 0$$

$$D_V = \frac{2\bar{q}S}{V} C_D + \frac{\bar{q}S}{V} M_a C_{DMa}$$

$$L_V = \frac{2\bar{q}S}{V} C_L + \frac{\bar{q}S}{V} M_a C_{LMa}$$

$$M_V = \frac{2\bar{q}S}{V} \bar{c} C_m + \frac{\bar{q}S}{V} \bar{c} M_a C_{mMa}$$

α derivatives :

$$T_\alpha = 0$$

$$D_\alpha = \bar{q}SC_{D\alpha}$$

$$L_\alpha = \bar{q}SC_{L\alpha}$$

$$M_\alpha = \bar{q}S\,\bar{c}\,C_{L\alpha}$$

q derivatives :

$$T_q = 0$$

$$D_q = 0$$

$$L_q = \frac{\bar{q}S}{2V}\,C_{Lq}$$

$$M_q = \frac{\bar{q}S}{2V}\,\bar{c}^2\,C_{mq}$$

$\dot{\alpha}$ derivatives :

$$T_{\dot{\alpha}} = 0$$

$$D_{\dot{\alpha}} = 0$$

$$L_{\dot{\alpha}} = 0$$

$$M_{\dot{\alpha}} = \frac{\bar{q}S}{2V}\,\bar{c}^2\,C_{m\dot{\alpha}}$$

δ_T derivatives :

$T_{\delta T} = \bar{q} S$

$D_{\delta T} = 0$

$L_{\delta T} = 0$

$M_{\delta T} = 0$

δ_e derivatives :

$T_{\delta e} = 0$

$D_{\delta e} = \bar{q} S C_{D\delta e}$

$L_{\delta e} = \bar{q} S C_{L\delta e}$

$M_{\delta e} = \bar{q} S \, \bar{c} \, C_{m\delta e}$

Appendix C. Program

```
C    PROGRAM TEST
C= = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = =
C PURPOSE : CD VERSUS STATIC MARGIN
C          WITH CONTROL DEFLECTION DUE TO DISTURBANCES
C= = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = =
C                     V A R I A B L E S
C= = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = =
      IMPLICIT REAL*8(A-H,L,M,O-W)
      DIMENSION BTB(2,2),B(4,2),BTBINV(2,2),BSI(2,4),
     *        AREF(4,4),A(4,4),ADIF(4,4),F(2,4),
     *        BF(4,4),AAUG(6,6),BAUG(6),CAUG(6,6),Q(6,6),
     *        RR(4),RI(4),E(2,2),SM(20),DE(20),CD(20),CDD(20),CDT(20),
     *        AAAUG(6,6),C(6,6),X(20),Y(20),R(20,20),T(20),G(20),
     *        GD(20)
C
      CHARACTER*15 FLNM1, FLNM2
      COMMON/INOU/KIN,KOUT
      COMMON/MAIN1/DUM1(6,6),NDIM
      COMMON/MAIN2/DUM2(6,6)
      COMMON/MAIN3/DUM3(6,6)
C
      NAMELIST/DATA/   MAC,W,HOREF,MI,S,BW,CBAR,BT,ST,
     *              CL,CDREF,CLA,CDA,CMAREF,
     *              CLADOT,CMADOTREF,CLQ,CMQREF,
     *              CLM,CDM,CMM,CLDE,CMDEREF,
     *              HNWB,CMOWB,CLAW,
     *              EPSOBR,ETA,L,
     *              Z,ZZ,ZZZ,ZZZZ
C
      DATA RHO,GR,GA,AC/
     *  .002377,32.174,0.0,1116.4/
C
      KIN=1
      KOUT=2
      TOL=.00000001
C
C     Z: =1 FOR COMPLETE RESULTS
C     ZZ: PACE OF ITERATION
C     ZZZ: DEGREE OF POLYNOMIAL FOR CURVE FITTING IN SUBROUTINE MIN
C     ZZZZ: NUMBER OF ITERATION
C= = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = =
C                  I N S T R U C T I O N S
```

```
C= = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = =
C
      READ(1,DATA)
      PI=DACOS(-1.D0)
      M= W/GR
      V= MAC*AC
      QBAR= RHO*V*V/2
      AR=BW*BW/S
      AKWB=1/PI/AR
      ART=BT*BT/ST
      AKT=1/PI/ART
      HN= HOREF-CMAREF/CLA
      HT= HOREF-CMDEREF/CLDE
      CMOL= -CL*CMAREF/CLA
      P=-CMAREF/CLA/ZZ
C
      CALL DROPT(HN,HT,DEN,HNWB,ET,ETA,ST,S,AKTPRM,AKT,HOPT,
     *        HLT,CLWB,CMOWB,CL,CLT,ALFBAR,CLAWB,EPS,DEPSDA,
     *        EPSOBR,CDWB,AKWB,CDTAIL,CDOPT,SMOPT,PI,AR,CLAW,
     *        H1,H2,H3,H4,H5,HDEN)
C
       HO=HOREF-P
      DO 1 J=1,ZZZZ
        HO=HO+P
        SM(J)=-(HO-HN)
C
      CALL DR(HT,DEN,HNWB,ET,ETA,ST,S,HO,HLT,CLWB,CMOWB,CL,CLT,ALFBAR,
     *      CLAWB,EPS,DEPSDA.EPSOBR,CDWB,AKT,
     *      AKWB,CDTAIL,CD,J,PI,AR,CLAW)
      CALL COEF(ALPHA,DELTAE,CMQ,CMADOT,CMA,CMDE,CM,
     *       CLDE,HO,HT,CLA,HN,CL,CMOL,DEN,
     *       CMQREF,CLQ,HOREF,CMADOTREF,CLADOT,AKWB,CDDE)
      CALL DERIV(TV,DV,LV,MV,QBAR,S,CDREF,V,M,CDM,CL,CLM,CM,CBAR,
     *        MAC,CMM,TA,DA,LA,MA,CDA,CLA,CMA,DQ,LQ,MQ,CLQ,CMQ,
     *        MADOT,CMADOT,TDT,DDE,LDE,MDE,CLDE,CMDE,CDDE)
      CALL MATA(A,TV,DV,M,TA,DA,W,DQ,GR,GA,LV,V,LA,LQ,MADOT,MV,MI,MA,MQ)
      CALL MATB(B,TDT,M,DDE,V,LDE,MADOT,MI,MDE)
        NDIM=4
      CALL MEIGV(4,A,RR,RI)
C
      IF(J.EQ.1) THEN
C
      DO 2 I=1,4
      DO 3 K=1,4
3       AREF(I,K)=A(I,K)
2      CONTINUE
C
      ELSE
      CALL MATF(J,BTB,B,BTBINV,BSI,AREF,A,ADIF,F,DET)
      CALL MAAUG(BF,B,F,AAUG,A,V,L,BAUG,CAUG,PI,AAAUG,C)
        NDIM=6
      CALL MLINEQ(6,AAAUG,C,Q,TOL)
      CALL VAR(Q,F,EP,E,DE,CDD,J,CDDE)
      CALL CDTOT(CD,CDD,CDT,J)
      ENDIF
C
      IF(Z.EQ.1) THEN
C
      WRITE(2,100) SM(J)
C
      DO 4 I=1,4
4      WRITE(2,101) (A(I,K),K=1,4)
C
      WRITE(2,110)
      DO 5 I=1,4
5      WRITE(2,111) (B(I,K),K=1,2)
```

```
C
      WRITE(2,120)
      DO 6 I=1,4
6     WRITE(2,121) RR(I),RI(I)
      ENDIF
C
      DO 40 I=1,4
40     WRITE(3,125) RR(I),RI(I)
C
      IF(J.GT.1.AND.Z.EQ.1) THEN
C
      WRITE(2,130)
      DO 7 I=1,2
7      WRITE(2,131) (F(I,K),K=1,4)
C
      WRITE(2,140)
      DO 8 I=1,6
8     WRITE(2,141) (AAUG(I,K),K=1,6)
C
      WRITE(2,150)
      WRITE(2,151) (BAUG(I),I=1,6)
C
      WRITE(2,160)
      DO 9 I=1,6
9     WRITE(2,161) (CAUG(I,K),K=1,6)
C
      WRITE(2,170)
      DO 10 I=1,6
10     WRITE(2,171) (Q(I,K),K=1,6)
C
      WRITE(2,180)
      DO 11 I=1,2
11     WRITE(2,181) (E(I,K),K=1,2)
C
      WRITE(2,190) DE(J)
C
      ENDIF
C
1     CONTINUE
C
      IF(J.GT.1.AND.Z.EQ.1) THEN
C
      WRITE(2,200)
      DO 12 I=2,ZZZZ
12     WRITE(2,201) SM(I),DE(I)
C
      ENDIF
C
      CALL MIN(SM,CD,N,ZZZ,ZZZZ,T,R,G,SUM1,SUM2,SUM,PIK,MN,MNY)
      CALL MIN(SM,CDT,N,ZZZ,ZZZZ,T,R,GD,SUM1,SUM2,SUM,PIK,MND,MNYD)
C
      IF(Z.EQ.2) THEN
C
      WRITE(2,100) SM(1)
C
      DO 22 I=1,4
22     WRITE(2,101) (AREF(I,K),K=1,4)
      ENDIF
C
      WRITE(2,300)
      DO 20 I=1,ZZZZ
20     WRITE(2,301) SM(I),CD(I)
C
C      WRITE(2,500)
C      WRITE(2,501) SMOPT,CDOPT
C
```

```
C    WRITE(2,600)
C    DO 15 I=1,ZZZ+1
C5    WRITE(2,601) G(I)
C
C    WRITE(2,210)
C    DO 13 I=2,ZZZZ
C3    WRITE(2,211) SM(I),CDD(I)
C
     WRITE(2,400)
     DO 30 I=2,ZZZZ
30    WRITE(2,401) SM(I),CDT(I)
C
C    WRITE(2,600)
C    DO 16 I=1,ZZZ+1
C6    WRITE(2,601) GD(I)
C
     WRITE(2,500)
     WRITE(2,501) MN,MNY
C
     WRITE(2,700)
     WRITE(2,701) MND,MNYD
     STOP
C===============================================
C                    F O R M A T S
C===============================================
100  FORMAT(//,71('='),
   &       /,24X,'S Y S T E M   M A T R I X',
   &       /,71('='),
   &       //,27X,'STATIC MARGIN = ',F5.3,
   &       // 33X,'MATRIX A',
   &       /,33X,'--------')
101  FORMAT(5X,4E15.4)
110  FORMAT(// 33X,'MATRIX B',
   &       /,33X,'--------')
111  FORMAT(20X,2E15.4)
C
120  FORMAT(//,27X,'MATRIX A  EIGENVALUES',
   &       /,27X,'---------------------')
121  FORMAT(16X,'RP  =',E15.4,5X,'IP  =',E15.4)
C
125  FORMAT(16X,E15.4,5X,E15.4)
C
130  FORMAT(// 33X,'MATRIX F',
   &       /,33X,'--------')
131  FORMAT(5X,4E15.4)
C
140  FORMAT(// 32X,'MATRIX AAUG',
   &       /,32X,'-----------')
141  FORMAT(6X,6F10.4)
C
150  FORMAT(// 32X,'MATRIX BAUG',
   &       /,32X,'-----------')
151  FORMAT(6X,6F10.4)
C
160  FORMAT(// 34X,'MATRIX C',
   &       /,34X,'--------')
161  FORMAT(6X,6F10.4)
C
170  FORMAT(// 34X,'MATRIX Q',
   &       /,34X,'--------')
171  FORMAT(6X,6F10.4)
C
180  FORMAT(// 34X,'MATRIX E',
   &       /,34X,'--------')
181  FORMAT(21X,2E15.4)
190  FORMAT(//,19X,'EXPECTED VALUE OF DELTAE = ',E10.4)
```

```
200  FORMAT(//,22X,'STATIC MARGIN',10X,'DELTAE',
   &       /,22X,'-------------',10X,'------')
201  FORMAT(26X,F5.3,12X,E10.4)
210  FORMAT(//,23X,'STATIC MARGIN',7X,'VAR. CD-CDO',
   &       /,23X,'-------------',7X,'-----------')
211  FORMAT(26X,F5.3,12X,E10.4)
300  FORMAT(/,71('='),
   &      /,19X,'DRAG  COEFFICIENT  VS  STATIC MARGIN',
   &      /,71('='),
   &      //,22X,'STATIC MARGIN',7X,'   CD-CDO  ',
   &       /,22X,'-------------',7X,' ---------')
301  FORMAT(26X,F5.3,12X,E10.4)
400  FORMAT(//,22X,'STATIC MARGIN',7X,'CD-CDO TOTAL',
   &       /,22X,'-------------',7X,'------------')
401  FORMAT(26X,F5.3,12X,E10.4)
500  FORMAT( //,23X,'  (SM)OPT   ',7X,'(CD-CD0)OPT ',
   &        /,23X,' ---------  ',7X,'----------- ')
501  FORMAT(26X,F5.3,12X,E10.4)
600  FORMAT(//,16X,'THE COEFFICIENTS ARE :')
601  FORMAT(/,27X,F15.6)
700  FORMAT( //,23X,'  (SM)OPT   ',6X,'(CD-CD0)TOTAL OPT ',
   &        /,23X,' ---------  ',6X,'----------------- ')
701  FORMAT(26X,F5.3,12X,E10.4)
C
     END
C
     SUBROUTINE DR(HT,DEN,HNWB,ET,ETA,ST,S,HO,HLT,CLWB,CMOWB,CL,CLT,
    *        ALFBAR,CLAWB,EPS,DEPSDA,EPSOBR,CDWB,AKT,
    *        AKWB,CDTAIL,CD,J,PI,AR,CLAW)
C= = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = =
C PURPOSE : CD VERSUS STATIC MARGIN - NO CONTROL DEFLECTION
C= = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = =
     IMPLICIT REAL*8(A-I,M,O-Z)
     DIMENSION CD(20)
C
     DEN=HT-HNWB
     ET=ETA*ST/S
     CLAWB=CLAW/(1+CLAW/PI/AR)
     DEPSDA=2*CLAWB/PI/AR
C
     HLT=HT-HO
     CLWB=(-CMOWB+CL*HLT)/DEN
     CLT=(CMOWB+CL*(HO-HNWB))/DEN/ET
     ALFBAR=CLWB/CLAWB
     EPS=(DEPSDA*ALFBAR+EPSOBR)/57.3
     CDWB=AKWB*CLWB*CLWB
     CDTAIL=(AKT*CLT*CLT+CLT*EPS)*ET
     CD(J)=CDWB+CDTAIL
C
     RETURN
     END
C
     SUBROUTINE DROPT(HN,HT,DEN,HNWB,ET,ETA,ST,S,AKTPRM,AKT,HOPT,
    *          HLT,CLWB,CMOWB,CL,CLT,ALFBAR,CLAWB,EPS,DEPSDA,
    *          EPSOBR,CDWB,AKWB,CDTAIL,CDOPT,SMOPT,PI,AR,CLAW,
    *          H1,H2,H3,H4,H5,HDEN)
C= = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = -
C PURPOSE : COMPUTE CD OPTIMUM - NO CONTROL DEFLECTION
C= = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = =
     IMPLICIT REAL*8(A-I,M,O-Z)
C
     DEN=HT-HNWB
     ET=ETA*ST/S
     AKTPRM=AKT/ET
     CLAWB=CLAW/(1+CLAW/PI/AR)
     DEPSDA=2*CLAWB/PI/AR
```

```
C
C     COMPUTE OPTIMUM CG POSITION
C
        H1 = (2.0*AKWB*CLAWB*57.3-DEPSDA)
        H2 = (2.0*AKTPRM*CLAWB*57.3-DEPSDA)
        H3 = (H1 + H2)*CMOWB
        H4 = H3 + EPSOBR*DEN*CLAWB
        H5 = H1*HT + H2*HNWB
        HDEN = H1 + H2
        HOPT = (H5*CL-H4)/(HDEN*CL)
        SMOPT = -(HOPT-HN)
C
C     COMPUTE CD OPTIMUM
C
        HLT = HT-HOPT
        CLWB = (-CMOWB + CL*HLT)/DEN
        CLT = (CMOWB + CL*(HOPT-HNWB))/DEN/ET
        ALFBAR = CLWB/CLAWB
        EPS = (DEPSDA*ALFBAR + EPSOBR)/57.3
        CDWB = AKWB*CLWB*CLWB
        CDTAIL = (AKT*CLT*CLT + CLT*EPS)*ET
        CDOPT = CDWB + CDTAIL
      RETURN
      END
C
      SUBROUTINE COEF(ALPHA,DELTAE,CMQ,CMADOT,CMA,CMDE,CM,CLDE,HO,HT,
     * CLA,HN,CL,CMOL,DEN,CMQREF,CLQ,HOREF,CMADOTREF,CLADOT,AKWB,CDDE)
C= = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = =
C PURPOSE : COMPUTE ALPHA,DELTAE,CMQ,CMADOT,CMA,CMDE,CM
C= = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = =
      IMPLICIT REAL*8(A-I,L-W)
C
      CDDE = 2*AKWB*CL*CLDE
      CMDE =  CLDE*(HO-HT)
      CMA =  CLA*(HO-HN)
      DEN =  CLA*CMDE-CLDE*CMA
      ALPHA =  (CL*CMDE + CLDE*CMOL)/DEN
      DELTAE =  -(CLA*CMOL + CL*CMA)/DEN
      CM =  CMOL + CMA*ALPHA + CMDE*DELTAE
      CMQ =  CMQREF + CLQ*(HO-HOREF)
      CMADOT =  CMADOTREF + CLADOT*(HO-HOREF)
C
      RETURN
      END
C
      SUBROUTINE DERIV(TV,DV,LV,MV,QBAR,S,CDREF,V,M,CDM,CL,CLM,CM,CBAR,
     *           MAC,CMM,TA,DA,LA,MA,CDA,CLA,CMA,DQ,LQ,MQ,CLQ,
     *           CMQ,MADOT,CMADOT,TDT,DDE,LDE,MDE,CLDE,CMDE,CDDE)
C= = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = =
C PURPOSE : COMPUTE THE FORCE DERIVATIVE (TV,DV,...)
C= = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = =
      IMPLICIT REAL*8(A-I,L-W)
C
      TV =  0.0
      DV =  2.0*QBAR*S*CDREF/V + QBAR*S*MAC*CDM/V
      LV =  2.0*QBAR*S*CL/V + QBAR*S*MAC*CLM/V
      MV =  2.0*QBAR*S*CBAR*CM/V + QBAR*S*CBAR*MAC*CMM/V
C
      TA =  0.0
      DA =  QBAR*S*CDA
      LA =  QBAR*S*CLA
      MA =  QBAR*S*CBAR*CMA
C
      DQ =  0.0
      LQ =  QBAR*S*CBAR*CLQ/2.0/V
      MQ =  QBAR*S*CBAR*CBAR*CMQ/2.0/V
```

```
C
      MADOT= QBAR*S*CBAR*CBAR*CMADOT/2.0/V
C
      TDT= QBAR*S
C
      DDE= QBAR*S*CDDE
      LDE= QBAR*S*CLDE
      MDE= QBAR*S*CBAR*CMDE
C
      RETURN
      END
C
      SUBROUTINE MATA(A,TV,DV,M,TA,DA,W,DQ,GR,GA,LV,V,LA,LQ,MADOT,MV,
     * MI,MA,MQ)
C= = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = =
C PURPOSE : COMPUTE THE MATRIX A
C= = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = =
      IMPLICIT REAL*8(A-I,L-W)
      DIMENSION A(4,4)
C
      A(1,1)= (TV-DV)/M
      A(1,2)= (TA-DA+W*DCOS(GA))/M
      A(1,3)= DQ/M
      A(1,4)= -GR*DCOS(GA)
      A(2,1)= -LV/(M*V)
      A(2,2)= -(LA-W*DSIN(GA))/(M*V)
      A(2,3)= 1.0-LQ/(M*V)
      A(2,4)= -GR*DSIN(GA)/V
      A(3,1)= (-LV*MADOT/(M*V)+MV)/MI
      A(3,2)= (-(LA-W*DSIN(GA))*MADOT/(M*V)+MA)/MI
      A(3,3)= (-LQ*MADOT/(M*V)+MQ+MADOT)/MI
      A(3,4)= GR*DSIN(GA)*MADOT/(MI*V)
      A(4,1)=  0.0
      A(4,2)=  0.0
      A(4,3)=  1.0
      A(4,4)=  0.0
C
      RETURN
      END
C
      SUBROUTINE MATB(B,TDT,M,DDE,V,LDE,MADOT,MI,MDE)
C= = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = =
C PURPOSE : COMPUTE THE MATRIX B
C= = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = =
      IMPLICIT REAL*8(A-I,L-W)
      DIMENSION B(4,2)
C
      B(1,1)= TDT/M
      B(1,2)= -DDE/M
      B(2,1)= -TDT/(M*V)
      B(2,2)= -LDE/(M*V)
      B(3,1)= -TDT*MADOT/(M*V*MI)
      B(3,2)= (-LDE*MADOT/(M*V)+MDE)/MI
      B(4,1)= 0.0
      B(4,2)= 0.0
C
      RETURN
      END
C
      SUBROUTINE MATF(J,BTB,B,BTBINV,BSI,AREF,A,ADIF,F,DET)
C= = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = =
C PURPOSE : COMPUTE THE GAIN MATRIX F
C= = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = =
      IMPLICIT REAL*8(A-I,L-W)
      DIMENSION BTB(2,2),B(4,2),BTBINV(2,2),BSI(2,4),
     *        AREF(4,4),A(4,4),ADIF(4,4),F(2,4)
```

```
C
      DO 1 X=1,4
      DO 2 Y=1,4
2       ADIF(X,Y)=A(X,Y)-AREF(X,Y)
1      CONTINUE
C
      DO 3 X=1,2
      DO 4 Y=1,2
        BTB(X,Y)=0.0
      DO 5 Z=1,4
5       BTB(X,Y)=BTB(X,Y)+B(Z,X)*B(Z,Y)
4      CONTINUE
3      CONTINUE
C
        DET=BTB(1,1)*BTB(2,2)-BTB(1,2)*BTB(2,1)
        BTBINV(1,1)=BTB(2,2)/DET
        BTBINV(1,2)=-BTB(1,2)/DET
        BTBINV(2,1)=-BTB(2,1)/DET
        BTBINV(2,2)=BTB(1,1)/DET
C
      DO 6 X=1,2
      DO 7 Y=1,4
        BSI(X,Y)=0.0
      DO 8 Z=1,2
8       BSI(X,Y)=BSI(X,Y)+BTBINV(X,Z)*B(Y,Z)
7      CONTINUE
6      CONTINUE
C
c     DO 9 X=1,2
      DO 10 Y=1,4
        F(X,Y)=0.0
      DO 11 Z=1,4
11       F(X,Y)=F(X,Y)+BSI(X,Z)*ADIF(Z,Y)
10     CONTINUE
9      CONTINUE
C
      RETURN
      END
C
      SUBROUTINE MAAUG(BF,B,F,AAUG,A,V,L,BAUG,CAUG,PI,AAAUG,C)
C=========================================================
C PURPOSE : COMPUTE THE AUGMENTED MATRIX IN : XDOT  =  AAUG*X+BAUG*W
C         COMPUTE THE C-MATRIX IN THE LYAPUNOV EQ. : AQ+QAT+C=0
C=========================================================
      IMPLICIT REAL*8(A-I,L-W)
      DIMENSION BF(4,4),B(4,2),F(2,4),AAUG(6,6),A(4,4),BAUG(6),
     &       CAUG(6,6),AD(6,6),AAAUG(6,6),C(6,6)
C
      DO 1 X=1,4
      DO 2 Y=1,4
        BF(X,Y)=0.0
      DO 3 Z=1,2
3       BF(X,Y)=BF(X,Y)+B(X,Z)*F(Z,Y)
        AAUG(X,Y)=A(X,Y)-BF(X,Y)
2      CONTINUE
1      CONTINUE
C
      AAUG(5,6)=1.0
      AAUG(6,5)=-V*V/L/L
      AAUG(6,6)=-2.0*V/L
C
      AD(2,5)=V/(PI*L*L*L)**.5
      AD(2,6)=3.0/(3.0*PI*L)**.5
      AD(3,5)=AD(2,6)*AAUG(6,5)
      AD(3,6)=AD(2,5)+AD(2,6)*AAUG(6,6)
C
```

```
      DO 4 X=1,4
      DO 5 Y=5,6
        AAUG(X,Y)=0.0
      DO 6 Z=2,3
6         AAUG(X,Y)=AAUG(X,Y)+AAUG(X,Z)*AD(Z,Y)
5      CONTINUE
4      CONTINUE
C
      DO 7 X=1,4
7         BAUG(X)=AAUG(X,3)*AD(2,6)
        BAUG(6)=1.0
c
      DO 8 X=1,6
      DO 9 Y=1,6
9         CAUG(X,Y)=BAUG(X)*BAUG(Y)
8      CONTINUE
9         CAUG(X,Y)=BAUG(X)*BAUG(Y)
C
      DO 12 X=1,6
      DO 13 Y=1,6
13        C(X,Y)=BAUG(X)*BAUG(Y)
12     CONTINUE
C
      DO 10 X=1,6
      DO 11 Y=1,6
11         AAAUG(X,Y)=AAUG(Y,X)
10     CONTINUE
C
      RETURN
      END
C
      SUBROUTINE VAR(Q,F,EP,E,DE,CDD,J,CDDE)
C= = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = =
C PURPOSE : COMPUTE THE EXPECTED VALUE OF U
C= = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = =
      IMPLICIT REAL*8(A-I,L-W)
      DIMENSION EP(4,2),Q(6,6),F(2,4),E(2,2),DE(20),CDD(20)
C
      DO 1 X=1,4
      DO 2 Y=1,2
        EP(X,Y)=0.0
      DO 3 Z=1,4
3         EP(X,Y)=EP(X,Y)+Q(X,Z)*F(Y,Z)
2      CONTINUE
1      CONTINUE
C
      DO 4 X=1,2
      DO 5 Y=1,2
        E(X,Y)=0.0
      DO 6 Z=1,4
6         E(X,Y)=E(X,Y)+F(X,Z)*EP(Z,Y)
5      CONTINUE
4      CONTINUE
C
      DE(J)=(DABS(E(2,2)))**.5
C
      CDD(J)=CDDE*DE(J)
      RETURN
      END
C
      SUBROUTINE CDTOT(CD,CDD,CDT,J)
C= = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = =
C PURPOSE : COMPUTE CD + ITS VARIATION
C= = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = = =
      IMPLICIT REAL*8(A-I,L-W)
      DIMENSION CD(20),CDD(20),CDT(20)
```

```
      CDT(J)=CD(J)+CDD(J)
      RETURN
      END
C
      SUBROUTINE MIN(X,Y,N,ZZZ,ZZZZ,T,R,G,SUM1,SUM2,SUM,PIK,MN,MNY)
C=================================================================
C PURPOSE : CURVE FITTING FOR DATA KNOTS + MINIMUM
C=================================================================
      IMPLICIT REAL*8 (A-H,M,O-Z)
      DIMENSION X(20),Y(20),T(20),R(20,20),G(20)
      N=ZZZZ-1
C
        T(1)=Y(2)
      DO 1 I=3,N
1         T(1)=T(1)+Y(I)
        R(1,1)=N-1
      DO 3 J=2,ZZZ+1
        SUM2=X(2)**(J-1)
      DO 4 K=3,N
        SUM2=SUM2+(X(K))**(J-1)
4      CONTINUE
        R(1,J)=SUM2
3      CONTINUE
      DO 2 I=2,ZZZ+1
        SUM1=Y(2)*(X(2))**(I-1)
      DO 18 K=3,N
18        SUM1=SUM1+Y(K)*(X(K))**(I-1)
      DO 10 J=1,ZZZ+1
        SUM2=X(2)**(I+J-2)
      DO 11 K=3,N
        SUM2=SUM2+(X(K))**(I+J-2)
11      CONTINUE
        R(I,J)=SUM2
10      CONTINUE
        T(I)=SUM1
2      CONTINUE
C
C      GAUSSIAN ELIMINATION
C
      DO 6 K=1,ZZZ
      DO 5 I=K+1,ZZZ+1
        PIK=R(I,K)/R(K,K)
      DO 7 J=K+1,ZZZ+1
        R(I,J)=R(I,J)-PIK*R(K,J)
7      CONTINUE
        T(I)=T(I)-PIK*T(K)
5      CONTINUE
6      CONTINUE
C
      G(ZZZ+1)=T(ZZZ+1)/R(ZZZ+1,ZZZ+1)
      DO 9 I=ZZZ,1,-1
        SUM=0
      DO 8 K=I+1,ZZZ+1
        SUM=SUM+R(I,K)*G(K)
8      CONTINUE
        G(I)=(T(I)-SUM)/R(I,I)
9      CONTINUE
C
        MN=-G(ZZZ)/G(ZZZ+1)/2.0
        MNY=G(1)+G(2)*MN+G(3)*MN*MN
C
      RETURN
      END
C
C=================================================================
C      NEXT IS DR. LUTZE PROGRAM TO SOLVE THE LYAPUNOV EQ.
```

```
C=================================================================
C
C-----------------------------------------------------------------------
      SUBROUTINE MLINEQ(N,A,C,X,TOL)
C-----------------------------------------------------------------------
C
C           SOLVES A'X + XA + C = 0
C           A AND X CAN BE IN SAME LOCATION IF DESIRED
C           ANSWER RETURNED IN C AND X
C
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION A(1),C(1),X(1),RR(30),RI(30)
      COMMON/MAIN1/F(36),NDIM
      COMMON/MAIN2/Y(1)
      COMMON/INOU/KIN,KOUT
      NDIM1 = NDIM + 1
      DT = .5
      DT1 = 0.
      NN = N*NDIM
      DO 5 II = 1,NN,NDIM1
    5 DT1 = DT1 + DABS(A(II))
      DT1 = DT1/N
      IF(DT1.GT.4.0) DT = DT*4.0/DT1
      II = 1
      DO 20 I = 1,N
      DO 15 JJ = I,NN,NDIM
   15 Y(JJ) = DT*A(JJ)
      Y(II) = Y(II)-.5
   20 II = II + NDIM1
      CALL GMINV(N,N,Y,F,MR,1)
      CALL EQUATE(N,N,Y,A)
      IF(MR.EQ.N) GO TO 21
      IT = 0
      DO 18 I = 1,NN,NDIM1
   18 C(I) = 1.E25
      GO TO 95
   21  CALL MMUL(C,F,N,N,N,X)
C      INITIALIZATION OF X,F
      I = 1
      DO 40 II = 1,NN,NDIM
      J = II
      IF(I.EQ.1) GO TO 30
      DO 25 JJ = I,II,NDIM
      C(J) = C(JJ)
   25 J = J + 1
   30 ID = J
      DO 35 JJ = II,NN,NDIM
      C(J) = DT*DOT(N,F(II),X(JJ))
   35 J = J + 1
      F(ID) = F(ID) + 1.0
   40 I = I + 1
   50 ADV = TOL*1.E-7
      DO 90 IT = 1,30
      NEZ = 0
      SIZE = 0.0
      CALL MMUL(C,F,N,N,N,X)
      I = 1
      II = 1
      J = 1
      GO TO 70
   60 J = II
      DO 65 JJ = I,II,NDIM
      C(J) = C(JJ)
   65 J = J + 1
   70 ID = J
      DT1 = C(J)
```

```
      DO 75 JJ = II,NN,NDIM
      C(J) = C(J) + DOT(N,F(II),X(JJ))
   75 J = J + 1
      J = J-1
      DO 80 JJ = II,J
   80 X(JJ) = F(JJ)
      IF(DABS(C(ID)-DT1).LT.(ADV + TOL*DABS(C(ID)))) NEZ = NEZ + 1
      I = I + 1
      II = II + NDIM
      SIZE = SIZE + DT1
      IF(I.LE.N) GO TO 60
      IF(NEZ.EQ.N) GO TO 150
      IF(DABS(SIZE).GT.1.E25) GO TO 95
      CALL MMUL(X,X,N,N,N,F)
   90 CONTINUE
   95 WRITE(KOUT,100)IT
  100 FORMAT(33H0LIN EQN ALGORITHM NON-CONVERGENT,I3,10HITERATIONS)
      WRITE(KOUT,110)
  110 FORMAT(35H0A MATRIX EIGENVALUES..REAL    IMAJ)
      CALL MEIGV(N,Y,RR,RI)
      WRITE(KOUT,120)(RR(I),RI(I),I = 1,N)
  120 FORMAT(18X,1P,2E12.3)
  150 CONTINUE
      CALL EQUATE(N,N,X,C)
      RETURN
      END
C-------------------------------------------------------------------
      SUBROUTINE GMINV(NR,NC,A,U,MR,MT)
C-------------------------------------------------------------------
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION A(1),U(1),S(30)
      COMMON/MAIN1/XX(36),NDIM
      COMMON/INOU/KIN,KOUT
      NDIM1 = NDIM + 1
      TOL = 1.E-14
      ADV = 1.E-24
      MR = NC
      NRM1 = NR-1
      TOL1 = 0.
      JJ = 1
      DO 10 J = 1,NC
      S(J) = DOT(NR,A(JJ),A(JJ))
      IF(S(J).GT.TOL1)TOL1 = S(J)
   10 JJ = JJ + NDIM
      TOL1 = ADV*TOL1
      ADV = TOL1
      JJ = 1
      DO 100 J = 1,NC
      FAC = S(J)
      JM1 = J-1
      JRM = JJ + NRM1
      JCM = JJ + JM1
      DO 20 I = JJ,JCM
   20 U(I) = 0.
      U(JCM) = 1.0
      IF(J.EQ.1) GO TO 54
      KK = 1
      DO 30 K = 1,JM1
      IF(S(K).EQ.1.0) GO TO 30
      TEMP = -DOT(NR,A(JJ),A(KK))
      CALL VADD(K,TEMP,U(JJ),U(KK))
   30 KK = KK + NDIM
      DO 50 L = 1,2
      KK = 1
      DO 50 K = 1,JM1
      IF(S(K).EQ.0.) GO TO 50
```

```
      TEMP=-DOT(NR,A(JJ),A(KK))
      CALL VADD(NR,TEMP,A(JJ),A(KK))
      CALL VADD(K,TEMP,U(JJ),U(KK))
   50 KK=KK+NDIM
      TOL1=TOL*FAC+ADV
      FAC=DOT(NR,A(JJ),A(JJ))
   54 IF(FAC.GT.TOL1) GO TO 70
      DO 55 I=JJ,JRM
   55 A(I)=0.
      S(J)=0.
      KK=1
      DO 65 K=1,JM1
      IF(S(K).EQ.0.) GO TO 65
      TEMP=-DOT(K,U(KK),U(JJ))
      CALL VADD(NR,TEMP,A(JJ),A(KK))
   65 KK=KK+NDIM
      FAC=DOT(J,U(JJ),U(JJ))
      MR=MR-1
      GO TO 75
   70 S(J)=1.0
      KK=1
      DO 72 K=1,JM1
      IF(S(K).EQ.1.) GO TO 72
      TEMP=-DOT(NR,A(JJ),A(KK))
      CALL VADD(K,TEMP,U(JJ),U(KK))
   72 KK=KK+NDIM
   75 FAC=1./SQRT(FAC)
      DO 80 I=JJ,JRM
   80 A(I)=A(I)*FAC
      DO 85 I=JJ,JCM
   85 U(I)=U(I)*FAC
  100 JJ=JJ+NDIM
      IF(MR.EQ.NR.OR.MR.EQ.NC) GO TO 120
      IF(MT.NE.0)WRITE(KOUT,110)NR,NC,MR
  110 FORMAT(I3,1HX,I2,8H M: RANK,I2)
  120 NEND=NC*NDIM
      JJ=1
      DO 135 J=1,NC
      DO 125 I=1,NR
      II=I-J
      S(I)=0.
      DO 125 KK=JJ,NEND,NDIM
  125 S(I)=S(I)+A(II+KK)*U(KK)
      II=J
      DO 130 I=1,NR
      U(II)=S(I)
  130 II=II+NDIM
  135 JJ=JJ+NDIM1
      RETURN
      END
C-----------------------------------------------------------------
      SUBROUTINE MEIGV(N,A,RR,RI)
C-----------------------------------------------------------------
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION A(1),RR(1),RI(1),C(31),TEMP(30)
      COMMON/MAIN1/ X(36),NDIM
      NDIM1=NDIM+1
      NN=N*NDIM
      DO 1 I=1,N
      DO 1 J=1,NN,NDIM
    1 X(J)=A(J)
      C(N+1)=1.0
      L=1
    5 C1=0.0
      DO 10 I=1,NN,NDIM1
   10 C1=C1-X(I)
```

```
      C1=C1/DFLOAT(L)
      I=N+1-L
      C(I)=C1
      DO 15 I=1,NN,NDIM1
   15 X(I)=X(I)+C1
      IF(L.EQ.N) GO TO 50
      DO 40 I=1,N
      JJ=1
      DO 35 J=I,NN,NDIM
      C1=0.
      KK=J-I
      DO 25 K=I,NN,NDIM
      KK=KK+1
   25 C1=C1+X(K)*A(KK)
      TEMP(JJ)=C1
   35 JJ=JJ+1
      JJ=1
      DO 40 J=I,NN,NDIM
      X(J)=TEMP(JJ)
   40 JJ=JJ+1
      L=L+1
      GO TO 5
   50 CALL POLRT(C,N,RR,RI)
      RETURN
      END
C-----------------------------------------------------------------------
       SUBROUTINE POLRT(A,N,U,V)
C-----------------------------------------------------------------------
C     DETERMINES THE ROOTS OF AN N-TH ORDER POYNOMIAL
C       X**N + A(N)*X**(N-1) + ... + A(1) = 0
C     WHERE:   U(N) WILL CONTAIN THE ROOT REAL PARTS
C              V(N) WILL CONTAIN ROOT IMAGINARY PARTS
C              A(N) WILL BE DESTROYED DURING COMPUTATION
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION A(20),U(20),V(20)
    1  NR = N
   10  IF (NR-2) 61,71,11
   11  IF (A(1)) 20,12,20
   12  U(NR) = 0.0
       V(NR) = 0.0
       NR = NR-1
      CALL VECTEQ(NR,A(2),A(1))
       GO TO 10
   20  EMIN = 1.
       TOL = .1
       V4 = 1.0
       P = 0.
       Q = 0.
       R = 0.0
   30  U(NR) = A(NR) - P
       U(NR-1) = A(NR-1) - P*U(NR) - Q
       V(NR) = U(NR) - P
       V(NR-1) = U(NR-1) - P*V(NR) - Q
       I = NR-2
   35  U(I) = A(I) - (P*U(I+1) + Q*U(I+2))
       V(I) = U(I) - (P*V(I+1) + Q*V(I+2))
       I = I-1
       IF (I.GT.0) GO TO 35
   40  IF (A(2)) 42,41,42
   41  E = U(2)/A(1)
       GO TO 43
   42  E = U(2)/A(2)
   43  E = DMAX1(DABS(E),1.0D-6)*DMAX1(DABS(U(1)/A(1)),1.0D-6)
       IF (E.LE.1.E-12) GO TO 70
       IF (E.GE.EMIN) GO TO 44
C        THIS FORCES EMIN TO HOLD STEADY FOR 5 ITERATIONS
```

```
      EMIN = E
      TOL = EMIN*0.7
      GO TO 45
C        THIS WILL ALLOW AN ERROR X*EMIN ONLY AFTER N ITERATIONS
C          WHERE  X = (1.1)**N
   44 IF (E.LT.TOL) GO TO 70
   45 CBAR = V(2) - U(2)
      IF (NR.GT.3) V4 = V(4)
      D = V(3)**2 - CBAR*V4
      IF (D) 47,46,47
   46 P = P - 2.0
      Q = Q*(Q+1.0)
      GO TO 50
   47 P = P + (U(2)*V(3) - U(1)*V4)/D
      Q = Q + (-U(2)*CBAR + U(1)*V(3))/D
   50 U(NR) = A(NR) + R
      V(NR) = U(NR) + R
      I = NR - 1
   55 U(I) = A(I) + R*U(I+1)
      V(I) = U(I) + R*V(I+1)
      I = I-1
      IF (I.GT.0) GO TO 55
      E = DABS(U(1)/A(1))
      IF (E.LE.1.E-12) GO TO 60
      IF (E.GE.EMIN) GO TO 56
      EMIN = E
      TOL = EMIN*0.7
      GO TO 57
   56 IF (E.LT.TOL) GO TO 60
   57 IF (V(2).NE.0) GO TO 58
      R = R+1.
      GO TO 59
   58 R = R - U(1)/V(2)
   59 TOL = TOL*1.1
      GO TO 30
C        STORE A SINGLE REAL ROOT
   60 CALL VECTEQ(NR-1,U(2),A)
      GO TO 62
   61 R = -A(1)
   62 U(NR) = R
      V(NR) = 0.0
      NR = NR-1
      GO TO 80
C        STORE A PAIR OF ROOTS
   70 CALL VECTEQ(NR-2,U(3),A)
      GO TO 72
   71 P = A(2)
      Q = A(1)
   72 P = (-0.5)*P
      D = P*P - Q
      IF (D) 75,78,78
   75 U(NR) = P
      U(NR-1) = P
      V(NR) = -SQRT(-D)
      V(NR-1) = -V(NR)
      GO TO 79
   78 V(NR) = 0.0
      V(NR-1) = 0.0
      D = DABS(P) + SQRT(D)
      IF (P.LT.0.0) D = -D
      U(NR) = D
      U(NR-1) = Q/D
   79 NR = NR-2
   80 IF (NR.GT.0) GO TO 10
      RETURN
      END
```

```
C-----------------------------------------------------------------
      SUBROUTINE EQUATE(NR,NC,A,B)
C-----------------------------------------------------------------
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION A(1),B(1)
      COMMON/MAIN1/XX(36),NDIM
      NN=NC*NDIM
      NR1=NR-1
      DO 1 J=1,NN,NDIM
      II=J+NR1
      DO 1 IJ=J,II
      A(IJ)=B(IJ)
    1 CONTINUE
      RETURN
      END
C-----------------------------------------------------------------
      SUBROUTINE MMUL(X,Y,N1,N2,N3,Z)
C-----------------------------------------------------------------
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION X(1),Y(1),Z(1)
      COMMON/MAIN1/XX(36),NDIM
      NEND3=NDIM*N3
      NEND2=NDIM*N2
      DO 1 I=1,N1
      DO 1 J=I,NEND3,NDIM
      TM=0.
      K=I
      KK=J-I
    5 KK=KK+1
      TM=TM+X(K)*Y(KK)
      K=K+NDIM
      IF(K.LE.NEND2) GO TO 5
    1 Z(J)=TM
      RETURN
      END
C-----------------------------------------------------------------
      FUNCTION DOT(NR,A,B)
C-----------------------------------------------------------------
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION A(1),B(1)
      DOT=0.
      DO 1 I=1,NR
    1 DOT=DOT+A(I)*B(I)
      RETURN
      END
C-----------------------------------------------------------------
      SUBROUTINE VADD(N,C1,A,B)
C-----------------------------------------------------------------
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION A(1),B(1)
      DO 1 I=1,N
    1 A(I)=A(I)+C1*B(I)
      RETURN
      END
C-----------------------------------------------------------------
      SUBROUTINE VECTEQ(N,X,Y)
C-----------------------------------------------------------------
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION X(1),Y(1)
      DO 10 I=1,N
   10 Y(I)=X(I)
      RETURN
C
C=========================================================================
C     END OF DR.LUTZE PROGRAM
C=========================================================================
```

```
    END
```

EXAMPLE OF THE INPUT FOR NAVION :

```
&DATA
 MAC= .158, W= 2750.0, HOREF= .295, MI= 3000.0,
 S= 184.0, BW= 33.4, CBAR= 5.7, BT= 13.2, ST= 28.95,
 CL= .41, CDREF= .05, CLA= 4.44, CDA= .33, CMAREF= -.683,
 CLADOT= 0.0, CMADOTREF= -4.36, CLQ= 3.8, CMQREF= -9.96,
 CLM= 0.0, CDM= 0.0, CMM= 0.0, CLDE= .355, CMDEREF= -.923,
 HNWB=.250, CMOWB= -.099, CLAW= 5.56
 EPSOBR= 0.04, ETA= 1.0,
 L= 5000,
 Z= 2, ZZ= 2, ZZZ= 2, ZZZZ= 12
&END
```

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