EFFECT OF ELEVATED MASS CENTER ON THE GLOBAL STABILITY OF A SOLID SUPPORTED BY ELASTICA COLUMNS

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Abstract– Recently we advocated a new stability index, the global critical load, for very elastic structures. This index is extremely useful for flexible structures under large disturbances such as earthquakes. The present note determines this index for a two-dimensional rigid solid supported by two flexible columns. Using the nonlinear elastica equations the buckling and postbuckling problem is solved by a homotopy nonlinear system solver. The present results show the bifurcation curve is quite sensitive to the elevated mass center. The global buckling load is drastically reduced although the critical buckling load of linear stability analysis is the same. An explanation is given through the study of a solid supported by one column.

Keywords: frames, nonlinear stability, postbuckling, elastica, flexible structures, homotopy

1. INTRODUCTION

Consider a slender elastic column subjected to increasing compression end loads. The column would remain straight until a certain critical load is reached, whereby the column begins to bifurcate from the straight state. This bifurcation critical load has long been used as an index for elastic stability [see e.g., Timoshenko and Gere (1961)], i.e., if the force is below the bifurcation load the column would return to its undeformed state after removal of any added sufficiently *small* perturbations. We emphasize the word *small* here since the bifurcation load is obtained through a linearization about the trivial state.

At this stage we define the global critical load, F_{gc}, below which the column remains stable, or returns to its original state, after the removal of *any* applied disturbances, however large. It so happens that for the column buckling problem, the bifurcation load is also F_{gc}, since the bifurcation curve is monotonic, single-valued with a global minimum at the bifurcation point.

However, there are cases where F_{gc} is smaller than the bifurcation load. An example is the snap buckling of shallow arches under uniform pressure [Timoshenko and Gere (1961)]. The arch is stable only for small disturbance pressures. Application and subsequent removal of large disturbance pressures may cause snap through to another state. Recently Thacker *et. al.* (1993) studied the large deformations of a thin rigid platform supported by two flexible elastic columns. It was found that the global buckling load, in most cases, is smaller than the bifurcation load. The results are quite significant for flexible structures that are subjected to large disturbances such as earthquakes.

The present note studies the important case of a two-dimensional symmetric rigid solid supported by elastic columns. Specifically, we wish to investigate the effect of the height of the solid on the global critical load.

2. FORMULATION and METHOD

Figure 1(a) shows a rigid solid supported on two columns of equal length L. The columns are slender, perfectly elastic with negligible weight. The ends of the columns are fixed to the base and to the rigid solid. The distance between the ends of the two columns is βL. The total weight of the solid is W acting at the center of mass which in the undeformed state is $\alpha L + L$ from the base. We normalize all lengths by L and all forces by EI/L^2. Figure 1(b) shows the deformed state while Fig. 1(c) shows the forces and moments acting on each column. Here $F_1 + F_2 = WL^2/EI \equiv F$ and G is the horizontal reactive force.

The elastica equations for large deformations of slender columns are (Frisch-Fay, 1962)

$$\frac{d^2\theta_1}{ds^2} + F_1 \sin\theta_1 - G\cos\theta_1 = 0 \tag{1}$$

$$\frac{d^2\theta_2}{ds^2} + F_2 \sin\theta_2 + G\cos\theta_2 = 0 \tag{2}$$

where θ_i is the inclination of the ith column with the vertical and s is the arc length. The kinematic relations are

$$\frac{dx_i}{ds} = \cos\theta_i, \qquad \frac{dy_i}{ds} = \sin\theta_i, \qquad i = 1, 2 \tag{3}$$

The boundary conditions are

$$\theta_1(0) = \theta_2(0) = x_1(0) = x_2(0) = y_1(0) = 0 \tag{4}$$

$$y_2(0) = \beta \tag{5}$$

$$\theta_1(1) = \theta_2(1) \tag{6}$$

$$\beta\cos\theta_1(1) = y_2(1) - y_1(1) \tag{7}$$

$$\beta\sin\theta_1(1) = x_1(1) - x_2(1) \tag{8}$$

$$\frac{d\theta_1}{ds}(0) + \frac{d\theta_2}{ds}(0) + F_2\beta = F\left[\frac{y_1(1) + y_2(1)}{2} + \alpha\sin\theta_1(1)\right] \tag{9}$$

The last equation is from an overall moment balance on the whole structure about the origin. There are 10 equations and 10 unknowns.

Although the elastica equations (1, 2) have first integrals, the unknowns θ_i are implicit in the resulting elliptic integrals, making the boundary conditions impossible to apply. One can linearize about small θ_i and find the linear bifurcation load is $2\pi^2 EI/L^2$ due to side sway [Chan and Liu, (1987)]. However, since the global critical load results from large deformation phenomenon, the nonlinear equations must be integrated numerically. The techniques used are similar to those used in Thacker, *et. al.* (1993). Starting from a known solution with $\alpha = 0$, HOMPACK routines [Watson, *et. al.* (1987)] were used to find a family of nontrivial solutions for $\alpha = 0$ to $\alpha = 5$ with fixed F. One of these solutions was then used as a starting point to track the solution curve as F varies. A similar process was used to find the solution curves for different β values.

3. RESULTS AND DISCUSSION

We define displacement δ as the vertical (downward) translation of the center of mass after deformation

$$\delta = \alpha + 1 - \left[\frac{y_1(1) + y_2(1)}{2} + \alpha \cos\theta_1(1)\right] \tag{10}$$

Figure 2a shows the force-displacement curves for width $\beta = 1$ and different constant values of center of mass height α. When $\alpha = 0$ the results reduce to those of the thin platform studied by Thacker, *et. al.* (1993). For small deformations, all curves bifurcate from $2\pi^2$ which is the linear stability criterion for side sway independent of both α and β. As δ is increased, the corresponding load F first increases, then decreases, then increases again. Thus for a given load, there may be up to three equilibrium solutions. The increase in height α greatly magnifies this nonlinear behavior and thus drastically decreases F_{gc}, the maximum load below which the system would always spring back to the undeformed state. The effect of α is diminished for larger β as shown in Fig. 2b. For fixed α, the effect of β is to change the character of the curves. Figure 3a and 3b show when β is large not only does the global critical load becomes the bifurcation load $2\pi^2$, but the nonuniqueness may also disappear. An explanation of the bifurcation behavior is given in Appendix I where the case of a solid supported by a single column is studied.

The computed global critical load is shown in Fig. 4 for various β and α. The value of F_{gc} rises with β but is limited by the bifurcation load $2\pi^2$ which is independent of β. We see that the height α drastically decreases F_{gc}. Figure 4 should be quite useful in the design of solid bodies supported by elastic columns.

The current example shows, in certain cases, the linear stability index or bifurcation load is entirely erroneous in predicting the safe load under finite disturbances. The importance of using the nonlinear stability index F_{gc} for flexible structures cannot be overemphasized.

APPENDIX I. SOLID SUPPORTED ON ONE COLUMN

Consider a two dimensional symmetric solid of weight W supported on the plane of symmetry by a single elastic column which is fixed at both ends. In the unbuckled state the center of mass of the solid is αL from the top end of the column. Normalizing as before, we obtain the governing equations

$$\frac{d^2\theta}{ds^2} + F\sin\theta = 0, \qquad \frac{dx}{ds} = \cos\theta, \qquad \frac{dy}{dx} = \sin\theta \tag{11}$$

The boundary conditions are

$$\theta(0) = x(0) = 0 \tag{12}$$

$$\theta'(1) = F\alpha \sin\theta(1) \tag{13}$$

The bifurcation load is obtained by linearization and seeking non trivial solutions. It is governed by the equation

$$\tan(\sqrt{F_c}) = \frac{1}{\alpha\sqrt{F_c}} \tag{14}$$

The lowest values of F_c for $\alpha = 0, 1, 2, 3, 4, 5$ are 2.4674, 0.74017, 0.42676, 0.29938, 0.23049, 0.18735. The situation is similar to the buckling of a column composed of two sections [Timoshenko and

Gere (1961)] with the top section having infinite rigidity. For $F > F_c$ the postbuckling equations can be solved by one-parameter shooting. The vertical displacement of the mass center is

$$\delta = (1 + \alpha) - [x(1) + \alpha \cos \theta(1)] \tag{14}$$

Figure 5 shows the displacement δ as a function of F for various constant α. Notice all curves are monotonically increasing for $F > F_c$. Since the curves never bend back, the bifurcation load is also the global buckling load. Due to the fact that the distance α promotes bending, the system becomes lass stable when α increases.

Next consider the case where the solid is prevented from rotating but is free to translate. Equation (13) is then supplanted by

$$\theta(1) = 0 \tag{15}$$

This case is equivalent to the classical buckling of a fixed-fixed column. The critical buckling load is π^2, independent of α. The bifurcation curve is shown as a dashed curve in Fig. 5.

Now consider a solid supported by two columns. When the load is just above critical each column is like the above fixed-fixed column, since the solid is prevented from rotating when deflections are small. However if the load becomes large the solid topples over. The height of the mass center now greatly increases the moment at the top of the columns, similar to the free solid supported by a single column. Thus the bifurcation curve for the solid supported by two columns behaves initially as though along the dashed curve of Fig. 5 then, depending on α, turns to follow one of the solid curves.

We see the nonlinear buckling of frames is quite different from that of single columns.

APPENDIX II. REFERENCES

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FIGURE CAPTIONS

Fig. 1. a) Configuration before deformation.

Fig. 1. b) Configuration after deformation.

Fig. 1. c) Forces and moments on each column.

Fig. 2. a) Displacement δ as a function of F, $\beta = 1$

Fig. 2. b) Displacement δ as a function of F, $\beta = 2$

Fig. 3. a) Displacement δ as a function of F, $\alpha = 1$

Fig. 3. b) Displacement δ as a function of F, $\alpha = 5$

Fig. 4. The global critical load

Fig. 5. Bifurcation of solid supported by a single column. Solid curves- the load is free to rotate; dashed curve- the load is prevented from rotating.