CHAPTER IV. One-Dimensional Analysis

1. Formulation of the Problem

1.1 Problem Statement

The objective of this work is to determine the temperature as a function of position and time of a rod of known properties as it travels with a steady velocity through a microwave cavity. The section that passes through the cavity is being heated volumetrically by microwave energy that is modeled by

$$\dot{q}(T) = 2\pi f\, \varepsilon_o \varepsilon''(T) |E|^2 \tag{1.1}$$

where ε_o is the permittivity of free space, ε'' is the dielectric loss coefficient of the material being heated, f is the frequency, and E is the applied electric field. Heat is being transferred through the rod by conduction, and heat loss occurs at the surface due to convection and radiation. The cylindrical rod initially starts at ambient temperature of 25^oC, and the cavity wall is constantly cooled to maintain a temperature also at the ambient. Our goal is to calculate the time-dependent temperature distribution in the rod after application of the microwave field. Figure 1.1 shows a simple diagram of the system.

1.2 Problem Setup

If a cylindrical rod is very thin, it can be assumed that the temperature through the diameter is constant. This yields a one dimensional analysis with conduction along the length in the x-direction only, and radiation and convection account for the heat loss at the

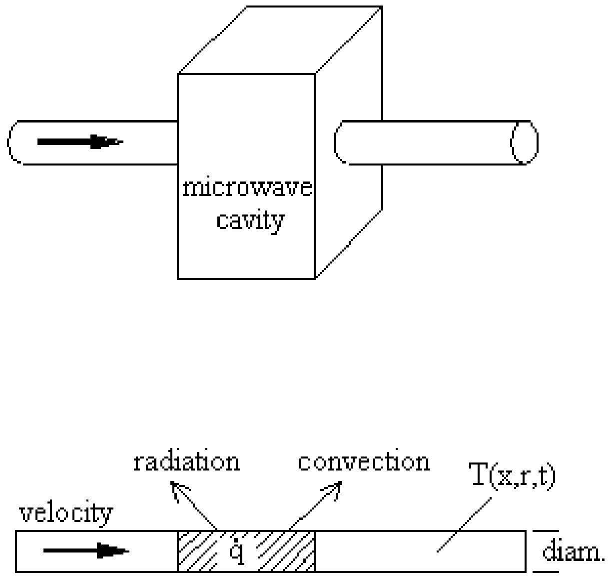

Figure 1.1 Diagram of Problem Statement

surface. Applying the conservation of energy principle to a differential cylindrical element, we obtain

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\dot{q}-\rho v\frac{\partial}{\partial x}\left(C_p T\right)-\frac{2h}{R}\left(T-T_\infty\right)-\frac{2h_r}{R}\left(T-T_{wall}\right)=\rho C_p\frac{\partial T}{\partial t} \quad (1.2)$$

where

$\dot{q}$ = volumetric heating source from microwave energy,

$T = T(x,r,t)$ = temperature,

T_∞ = ambient temperature,

T_{wall} = microwave cavity temperature,

k = temperature dependent thermal conductivity,

C_p = temperature dependent specific heat,

h = convection heat transfer coefficient,

h_r = radiation heat transfer coefficient,

v = velocity of the rod in the x-direction,

and

t = time.

The boundary and initial conditions are

$$T(0,t) = T(L,t) = T(x,0) = T_\infty\ . \tag{1.3}$$

A numerical approach is used to solve the model equation due to the nonlinear nature of the equations and the temperature dependent properties. The implicit finite difference method is chosen [13], central in x and backward in time. The resulting finite difference form of the equations is

$$-T_{m-1}^{(p+1)}\left[\frac{k^-}{\Delta x^2}+\frac{r\ VC_p^-}{2\Delta x}\right] + T_m^{(p+1)}\left[\frac{k^+ + k^-}{\Delta x^2}+\frac{2(h+h_r)}{R}+\frac{r\ C_p}{\Delta t}\right]$$

$$-T_{m+1}^{(p+1)}\left[\frac{k^+}{\Delta x^2}-\frac{r\ VC_p^+}{2\Delta x}\right] = \frac{r\ C_p}{\Delta t}T_m^{(p)} + \dot{q} + \frac{2h}{R}T_\infty + \frac{2h_r}{R}T_{wall}\,, \tag{1.4}$$

with boundary and initial conditions

$$T_0^{(p)} = T_M^{(p)} = T_i^{(0)} = 25^o\,C\ . \tag{1.5}$$

$T^{(p)}$ is the notation used to refer to the temperature at a given time, and it is known from either the initial condition or from previous calculations at full time step earlier. $T^{(p+1)}$ is the temperature at one time step after the same given time. These are the unknowns in Eq. 1.4.

C_p^+ and k^+ are the specific heat and conductivity evaluated at an average temperature of T_m and T_{m+1}. C_p^- and k^- correspond to evaluated values at an average temperature of T_m and T_{m-1}.

The resulting algebraic equations are solved by a tridiagonal matrix equation solver to obtain the temperature distribution along a rod at any given time. The one dimensional model is used for comparison purposes to the two dimensional model.

CHAPTER V.   Two Dimensional Analysis

1. Formulation of the Problem

As thicker rods are heated, the one-dimensional model becomes less accurate since this model assumes that the temperature gradient along the radius is zero. Intuition would tell us that as a thick rod is volumetrically heated, the inside is hotter than the outside as the result of convection and radiation at the surface. This second-dimensional effect needs to be taken into account, so a two-dimensional model was developed.

The first step in developing this model is to derive the differential equation that describes the temperature distribution in the rod. To do this, we analyze a differential ring element as displayed in Fig. 1.2.

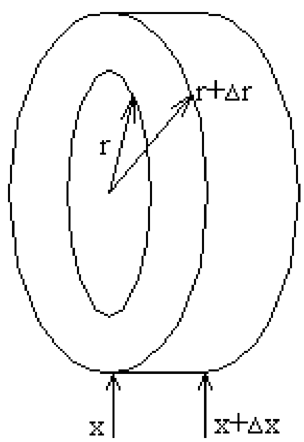

Figure 1.2 Differential Control Volume for Two-Dimensional Conduction

Allowing conduction in both x- and r- directions, we apply the conservation of energy principle to the element shown in Fig 1.2:

$$\dot{E}_{in} - \dot{E}_{out} + \dot{E}_{gen} = \dot{E}_{storage} \tag{1.6}$$

where

$$\dot{E}_{in} = -kA\left.\frac{\partial T}{\partial x}\right|_x + \rho C_p vAT\Big|_x - kA_r\left.\frac{\partial T}{\partial r}\right|_r ,$$

$$\dot{E}_{out} = -kA\left.\frac{\partial T}{\partial x}\right|_{x+\Delta x} + \rho C_p vAT\Big|_{x+\Delta x} - kA_r\left.\frac{\partial T}{\partial r}\right|_{r+\Delta r} ,$$

$$\dot{E}_{gen} = \dot{q}V ,$$

and

$$\dot{E}_{storage} = \rho C_p V\frac{\partial T}{\partial t},$$

and where

V = volume of the differential ring,

A = cross-sectional area of the differential ring,

and

A_r = radial surface area at r.

First, we divide Eq 1.6 by Δx and take the limit as Δx approaches zero, then divide by A and take the limit as Δr approaches zero. These steps lead to

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) - \rho v\frac{\partial}{\partial x}(C_p T) + \frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right) + \dot{q} = \rho C_p\frac{\partial T}{\partial t} . \tag{1.7}$$

The differential equation (1.3) is not solvable without four boundary conditions and one initial condition. Since heating is only applied to a small segment of the rod inside the microwave cavity, far from the cavity where the heat source is not present and the axial conduction has no effect at all, the temperature can be fixed at the ambient temperature. For the left axial boundary condition, we fix the axial temperatures at the ambient air temperature at a distance of two cavity lengths from the left side of the cavity, and the right axial boundary temperature is also fixed at the ambient temperature at two cavity lengths from the right side of the cavity. There are also two boundary conditions for the radial direction. In the center of the rod where the radius is zero, we can invoke symmetry and state that the temperature gradient is equal to zero. The final boundary condition, applied on the radial surface of the rod, is that heat conducted in the rod to the outer surface must be equal to the heat loss by convection and radiation. We assume the initial temperature is the ambient temperature. These boundary conditions and initial condition are summarized in Table 1.1, and the location of the boundary conditions are shown in Fig 1.3.

2. Finite Difference Form of the Equations

2.1 Finite Difference Form of the Heat Conduction Equation

Since the equation is non-linear due to the temperature dependent properties, a numerical method for the solution is required. The finite difference method is chosen over the finite element method because of the simple geometry of the problem.

In order to account for temperature-dependence properties, the control volume integration approach is used. The first step in deriving the finite difference equations is to

Table 1.1 Summary of Two-Dimensional Heat Conduction Equation with Boundary Conditions and Initial Condition

$\frac{\partial}{\partial x}(k\frac{\partial T}{\partial x}) - \rho v \frac{\partial}{\partial x}(c_p T) + \frac{1}{r}\frac{\partial}{\partial r}(kr\frac{\partial T}{\partial r}) + \dot{q} = \rho\ c_p \frac{\partial T}{\partial t}$	
$T(x,r,t) = T_\infty$	@ x=0
$T(x,r,t) = T_\infty$	@ x=L
$-k\frac{\partial T}{\partial r} = 0$	@ r=0
$-k\frac{\partial T}{\partial r} = h_c(T - T_\infty) + h_r(T - T_{wall})$	@ r=R
$T(x,r,t) = T\infty$	@ t=0

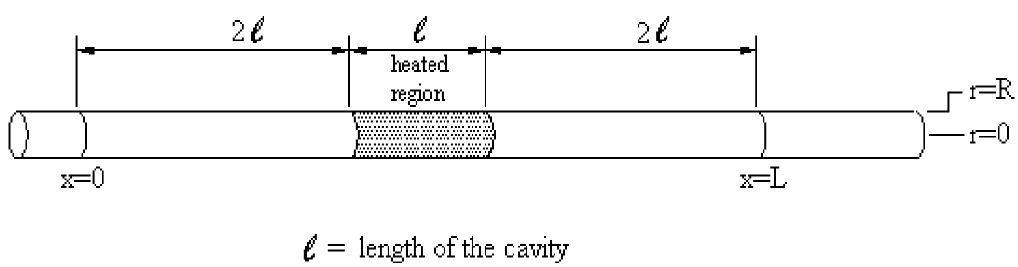

Figure 1.3 Boundary Condition Locations

integrate the differential equation (Eq. 1.2) over a control volume over x from $(x-\frac{\Delta x}{2})$ to $(x+\frac{\Delta x}{2})$ and over r from $(r-\frac{\Delta r}{2})$ to $(r+\frac{\Delta r}{2})$ as follows:

$$\int_{r-\Delta r/2}^{r+\Delta r/2}\int_{x-\Delta x/2}^{x+\Delta x/2}\frac{\P}{\P x}(k\frac{\P T}{\P x})\, r\ dxdr - \int_{r-\Delta r/2}^{r+\Delta r/2}\int_{x-\Delta x/2}^{x+\Delta x/2} r\ v\frac{\P}{\P x}(C_p T)\, r\ dxdr +$$

$$\int_{r-\Delta r/2}^{r+\Delta r/2}\int_{x-\Delta x/2}^{x+\Delta x/2}\frac{1}{r}\frac{\P}{\P r}(kr\frac{\P T}{\P r})\ dxdr + \int_{r-\Delta r/2}^{r+\Delta r/2}\int_{x-\Delta x/2}^{x+\Delta x/2}\dot{q}\ dxdr =$$

$$\int_{r-\Delta r/2}^{r+\Delta r/2}\int_{x-\Delta x/2}^{x+\Delta x/2} r\ C_p\frac{\P T}{\P t}\ dxdr. \qquad (1.8)$$

Evaluating the integrals in Eq. 1.4 we and grouping the terms we obtain the finite difference form of the equation. For example, the first term in Eq. 1.8 is evaluated as follows

$$\int_{r-\Delta r/2}^{r+\Delta r/2}\int_{x-\Delta x/2}^{x+\Delta x/2}\frac{\P}{\P x}(k\frac{\P T}{\P x})\, r\ dxdr = \int_{r-\Delta r/2}^{r+\Delta r/2} r\left[k\frac{\P T}{\P x}\bigg|_{x+\frac{\Delta x}{2}} - k\frac{\P T}{\P x}\bigg|_{x-\frac{\Delta x}{2}}\right] dr$$

$$= \int_{r-\Delta r/2}^{r+\Delta r/2} r\left[k^{+}\left(\frac{T_{m+1,n}^{(p+1)} - T_{m,n}^{(p+1)}}{\Delta x}\right) - k^{-}\left(\frac{T_{m,n}^{(p+1)} - T_{m-1,n}^{(p+1)}}{\Delta x}\right)\right] dr\ =$$

$$\left[k^{+}\left(\frac{T_{m+1,n}{}^{(p+1)} - T_{m,n}{}^{(p+1)}}{\Delta x}\right) - k^{-}\left(\frac{T_{m,n}{}^{(p+1)} - T_{m-1,n}{}^{(p+1)}}{\Delta x}\right)\right] \frac{\left[\left(r + \frac{\Delta r}{2}\right)^{2} - \left(r - \frac{\Delta r}{2}\right)^{2}\right]}{2}$$

$$= \left[k^{+}\left(\frac{T_{m+1,n}{}^{(p+1)} - T_{m,n}{}^{(p+1)}}{\Delta x}\right) r\Delta r - k^{-}\left(\frac{T_{m,n}{}^{(p+1)} - T_{m-1,n}{}^{(p+1)}}{\Delta x}\right) r\Delta r\right].$$

Integrating over all of the terms in Eq. 1.8 and grouping the temperature terms, we obtain

$$k\,\Delta x\left(\frac{1}{\Delta r} - \frac{1}{2r}\right) T_{m,n-1}{}^{(p+1)} + \left(\frac{k^{-}\Delta r}{\Delta x} + \frac{r\ vC_p{}^{-}\Delta r}{2}\right) T_{m-1,n}{}^{(p+1)} +$$

$$\left(-\frac{k^{-}\Delta r}{\Delta x} - \frac{k^{+}\Delta r}{\Delta x} - 2\frac{k\Delta x}{\Delta r} - \frac{r\ C_p \Delta x \Delta r}{\Delta t}\right) T_{m,n}{}^{(p+1)} +$$

$$k\,\Delta x\left(\frac{1}{\Delta r} + \frac{1}{2r}\right) T_{m,n+1}{}^{(p+1)} + \left(\frac{k^{+}\Delta r}{\Delta x} - \frac{r\ vC_p \Delta r}{2}\right) T_{m+1,n}{}^{(p+1)} =$$

$$-\dot{q}\,\Delta x \Delta r - \frac{r\ C_p \Delta x \Delta r}{\Delta t} T_{m,n}{}^{(p)} \qquad (1.9)$$

where

$T_{m,n-1}{}^{(p+1)}$ = Temperature at a ($-\Delta r$) step from the point of evaluation at time $t+\Delta t$

$T_{m-1,n}{}^{(p+1)}$ = Temperature at a ($-\Delta x$) step from the point of evaluation at time $t+\Delta t$

$T_{m,n}{}^{(p+1)}$ = Temperature at the point of evaluation at time $t+\Delta t$

$T_{m,n+1}{}^{(p+1)}$ = Temperature at a ($+\Delta r$) step from the point of evaluation at time $t+\Delta t$

$T_{m+1,n}{}^{(p+1)}$ = Temperature at a ($+\Delta x$) step from the point of evaluation at time $t+\Delta t$

$T_{m,n}{}^{(p)}$ = Temperature at the point of evaluation at time t.

An illustration of the finite difference element is shown in Fig. 1.4.

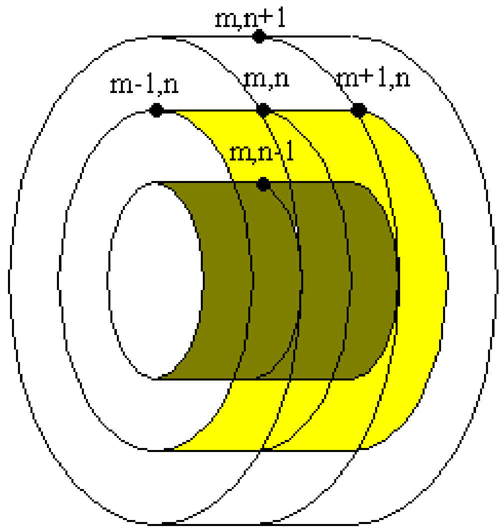

Figure 1.4 Finite Difference Element

2.2 Finite Difference Form of the Axial Boundary Conditions

A different approach is used to determine the finite difference equations for the boundary conditions. Since the axial endpoints are fixed, we can simply state

$$T_{1,n}^{(p+1)} = T_{M+1,n}^{(p+1)} = T_{\infty} \tag{1.10}$$

where

$$M = \frac{5 \times \text{cavity length}}{\Delta x} \tag{1.11}$$

2.3 Finite Difference of the Radial, Outer Boundary Condition

The finite difference form of the radial boundary conditions are not as simple to determine as are the axial ones. An analysis of an individual boundary node is done by applying the first law of thermodynamics to the finite, differential element. Figure 1.5 illustrates this element.

Now, we apply the first law of thermodynamics in the form

$$\Sigma \dot{E}_{in} + \dot{E}_{gen} = \dot{E}_{st} \ . \tag{1.12}$$

After substitution of the energy rate terms, use of Fourier's Law, and dividing by 2πR, Eq. 1.8 becomes

$$k\,\Delta x\left(\frac{1}{\Delta r} - \frac{1}{2R}\right)\left(T_{m,n-1}^{(p+1)} - T_{m,n}^{(p+1)}\right) + k\frac{1}{2\Delta x}\left(\Delta r - \frac{\Delta r^2}{4R}\right)\left(T_{m-1,n}^{(p+1)} - T_{m,n}^{(p+1)}\right) +$$

$$k\frac{1}{2\Delta x}\left(\Delta r - \frac{\Delta r^2}{4R}\right)\left(T_{m+1,n}^{(p+1)} - T_{m,n}^{(p+1)}\right) + \qquad h\,\Delta x\left(T_\infty - T_{m,n}^{(p+1)}\right) +$$

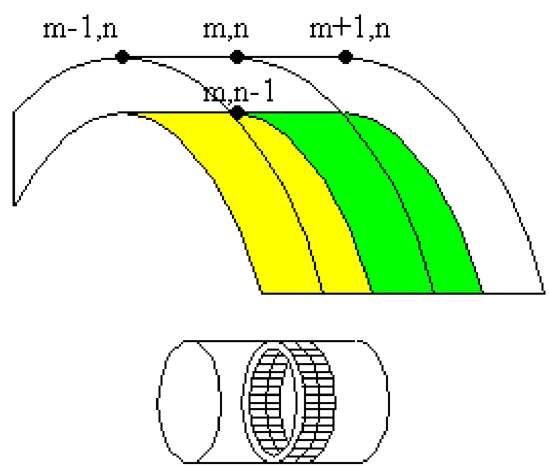

Figure 1.5 Finite Difference on Outer Boundary Elements

$$h_r \Delta x(T_{wall} - T_{m,n}^{(p+1)}) \quad + \quad \dot{q}\,\frac{\Delta x}{2}(\Delta r - \frac{\Delta r^2}{4R}) \quad +$$

$$r\ C_p\,\frac{v}{4}(\Delta r - \frac{\Delta r^2}{4R})(T_{m-1,n}^{(p+1)} + T_{m,n}^{(p+1)}) \quad - \ r\ C_p\,\frac{v}{4}(\Delta r - \frac{\Delta r^2}{4R})(T_{m+1,n}^{(p+1)} + T_{m,n}^{(p+1)}) \quad =$$

$$r\ C_p\,\frac{\Delta x}{2\Delta t}(\Delta r - \frac{\Delta r^2}{4R})(T_{m,n}^{(p+1)} - T_{m,n}^{p}) \,. \tag{1.13}$$

Finally, the temperatures in Eq 1.9 can be grouped together to obtain

$$k\,\Delta x\,(\frac{1}{\Delta r} - \frac{1}{2R})T_{m,n-1}^{(p+1)} \quad + \quad (\Delta r - \frac{\Delta r^2}{4R})\left(\frac{k}{2\Delta x} + r\ C_p\,\frac{v}{4}\right)T_{m-1,n}^{(p+1)} \quad +$$

$$\left[-k\Delta x\,(\frac{1}{\Delta r} - \frac{1}{2R}) - \frac{k}{\Delta x}(\Delta r - \frac{\Delta r^2}{4R}) - h_c\Delta x - h_r\Delta x - r\ C_p\,\frac{\Delta x}{2\Delta t}(\Delta r - \frac{\Delta r^2}{4R})\right]T_{m,n}^{(p+1)}$$

$$+ \qquad (\Delta r - \frac{\Delta r^2}{4R})\left(k\frac{1}{2\Delta x} - r\ C_p\,\frac{v}{4}\right)T_{m+1,n}^{(p+1)} \quad = \quad -h_c\Delta x T_\infty - h_r\Delta x T_{wall} \ -$$

$$(\Delta r - \frac{\Delta r^2}{4R})\ \dot{q}\,\frac{\Delta x}{2} \ - \ r\ C_p(\Delta r - \frac{\Delta r^2}{4R})\frac{\Delta x}{2\Delta t}T_{m,n}^{p} \,. \tag{1.14}$$

2.4 Finite Difference Form of the Radial, Inner Boundary Condition

Figure 1.6 displays the symmetric boundary at the center of the rod. Again, we apply the conservation of energy principle to the element and divide by $\pi\Delta r$ to obtain

$$4k\frac{\Delta x}{\Delta r}(T_{m,n+1}^{(p+1)} - T_{m,n}^{(p+1)}) \quad + \quad k\frac{\Delta r}{\Delta x}(T_{m-1,n}^{(p+1)} - T_{m,n}^{(p+1)}) \quad + \quad k\frac{\Delta r}{\Delta x}(T_{m+1,n}^{(p+1)} - T_{m,n}^{(p+1)})$$

$$+ \quad \dot{q}\ \Delta x\Delta r \quad + \quad r\ C_p v\frac{\Delta r}{2}(T_{m-1,n}^{(p+1)} + T_{m,n}^{(p+1)}) \ -$$

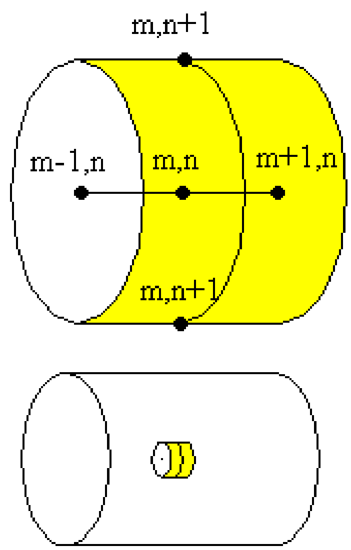

Figure 1.6 Finite Difference on Inner Boundary Element

$$r\ C_p v \frac{\Delta r}{2}\left(T_{m+1,n}^{(p+1)} + T_{m,n}^{(p+1)}\right) \;=\; r\ C_p \frac{\Delta r \Delta x}{\Delta t}\left(T_{m,n}^{(p+1)} - T_{m,n}^{p}\right). \qquad (1.15)$$

By arranging the equations so that the temperatures are grouped together, we have

$$\left(k\frac{\Delta r}{\Delta x} + r\ C_p v\frac{\Delta r}{2}\right)T_{m\text{-}1,n}^{(p+1)} \;+$$

$$\left(-4k\frac{\Delta x}{\Delta r} - k\frac{\Delta r}{\Delta x} - r\ C_p\frac{\Delta x \Delta r}{\Delta t}\right)T_{m,n}^{(p+1)} \;+\; 4k\frac{\Delta x}{\Delta r}T_{m,n+1}^{(p+1)} \;+$$

$$\left(k\frac{\Delta r}{\Delta x} - r\ C_p v\frac{\Delta r}{2}\right)T_{m+1,n}^{(p+1)} \;=\; -\dot{q}\,\Delta x \Delta r \;-\; r\ C_p\frac{\Delta x \Delta r}{\Delta t}T_{m,n}^{p}. \qquad (1.16)$$

3. Matrix Equation Solver

Now we have the governing equation, the four boundary condition equations, and the initial condition equation all in finite difference form. The next step is to solve the system of equations using the necessary material properties. It is noted that the number of equations will be $\left[\left(\frac{5\times\text{cavity length}}{\Delta x}+1\right)\left(\frac{\text{Radius}}{\Delta r}+1\right)\right]$. These equations are put in matrix form before solution.

3.1 Gaussian Elimination

One way to solve this set of equations may be to use Gaussian elimination. However, this method is slow and requires much more memory than needed, the reason being that the coefficients of the matrix are mostly zeros. This method requires that all of the zeros be stored and participate in the calculations. A modification of this method using sparse matrix techniques by eliminating most of the zeros reduces memory requirements but is still slow.

3.2 Line-by-Line Method

In the one-dimensional analysis, a tridiagonal matrix was derived from the governing heat transfer equations; hence, a tridiagonal matrix equation solver, a special case of Gaussian elimination, was used to solve the set of equations. This specific solver is very quick and uses less memory than a complete Gaussian elimination or some other type of matrix equation solver. The matrix for the two-dimensional case consists of five non-zero diagonal elements, thus precluding application of a tridiagonal solver; however, the line-by-line method, sometimes called the Alternating Direction Implicit (ADI) method

[14], can use the tridiagonal matrix equation solver in combination with iteration. This method allows the quickness and low memory requirements that result from applying the tridiagonal solver and still arrives at accurate results for the two-dimensional case.

Instead of analyzing all of the axial finite difference nodes with all of the radial finite difference nodes at the same time, this method requires that we examine only a row of axial finite difference nodes at a given radius at a time. By analyzing the temperatures at any radius and guessing the temperatures at the bordering radial elements, the tridiagonal matrix equation solver can be used. This, however, allows for predicted axial temperatures only at one value of radius. By repeating this process for different radii or by "sweeping" through the different rows at different radii, we have a good estimate for all of the unknown temperatures. Since this process involves iteration, we must repeat the process using the newly determined values as the "guessed" values. This iteration is repeated until a desired convergence is achieved.

An improvement to this method is to alternate the direction of the sweep. In other words, we sweep in the x-direction for one iteration and sweep in the r-direction in the next iteration. The computer model included in the appendix implements this method.

4. Computer Model

A computer program was created to solve the earlier described sets of equations by using the line-by-line method. The computer model requires 2 major iterations. First an iteration is required to find the temperature profile at a given time using the line-by-line method. The program uses "old" values of temperature and calculates "new" values of temperature until a convergence criteria is satisfied. The second iteration is to determine whether steady state is reached. A simplified flow chart describing how the computer model runs is shown in Fig. 1.7.