Kevin L. Watson was awarded a posthumous doctoral degree in fall 2020

Students' Conceptions of Normalization

Kevin Lee Watson

Dissertation proposal submitted to the faculty of the Virginia Polytechnic Institute and State University towards the partial fulfillment of the requirements for the degree of

Doctor of Philosophy
In
Mathematics

Megan Wawro. Chair
Estrella M. S. Johnson. Member
Anderson H. Norton. Member
Jesse L. M. Wilkins. Member
John R. Thompson. Member

Preliminary exam successfully defended
December 04, 2018
Blacksburg, VA

Keywords: Normalization, norm, quantum physics, linear algebra, coordination class

LIST OF FIGURES

TABLE OF CONTENTS

Chapter 1: Introduction

Improving the learning and success of students in undergraduate science, technology, engineering, and mathematics (STEM) courses has become an increased focus of education researchers within the past decade. As part of these efforts, discipline-based education research (DBER) has emerged within STEM education as a way to address discipline-specific challenges for teaching and learning, by combining expert knowledge of the various STEM disciplines with knowledge about teaching and learning (Dolan et al., 2018; National Research Council, 2012). Particularly important to furthering DBER and improving STEM education are interdisciplinary studies that examine how the teaching and learning of specific concepts develop among and across various STEM disciplines. As the National Research Council (NRC) (2012) stated in their DBER report:

> Interdisciplinary studies are needed to examine cross-cutting concepts and cognitive processes. DBER scholars have no shortage of discipline-specific problems and challenges to study, but crosscutting concepts … and structural or conceptual similarities that underlie discipline-specific problems … also merit attention. Interdisciplinary studies could help to increase the coherence of students' learning experience across disciplines by uncovering areas of overlap and gaps in content coverage, and could facilitate an understanding of how to promote the transfer of knowledge from one setting to another. (p. 202)

Interdisciplinary studies are especially important for mathematics education researchers, as many of the students enrolled in mathematics courses are studying mathematics to use within their chosen STEM majors and careers. As mathematics education researchers examine how students learn and then use various mathematical concepts throughout their undergraduate studies, they can provide insights to mathematics instructors about their students and how they might teach those concepts effectively, helping students succeed now as well as in the future.

Mathematics is particularly relevant to physics as a powerful tool for modeling physical systems, yet a common lament of physics educators is the considerable struggle students have with the mathematical problem solving that is an integral part of this modeling (Tuminaro & Redish, 2007). In line with the NRC's call, interdisciplinary studies between mathematics education and physics education can be a useful endeavor for addressing the struggles students have in understanding and using mathematical concepts within both their mathematics and physics courses. Some examples of this interdisciplinary work include: (a) mathematics educators investigating the ways students understand and reason about the meaning of derivatives in physical contexts (Jones & Watson, 2018); (b) physics education researchers investigating multivariable calculus students' abilities to solve problems involving slopes and derivatives stripped of physics contexts (Christensen & Thompson, 2012); and (c) collaborations between mathematics and physics educators to improve students' understanding of multivariable calculus in both mathematics and physics (Gire, Wangberg, & Wangberg, 2017; Roundy et al., 2015; Wangberg & Johnson, 2013). These studies illustrate the important and fruitful investigations made possible through engaging in interdisciplinary educational research.

One of the more difficult undergraduate physics courses that relies heavily on mathematics, providing plentiful opportunities for interdisciplinary educational studies, is quantum mechanics. Physics education research on quantum mechanics has grown within the past two decades and has made progress in exploring experts' and students' understanding of, or difficulty with, quantum mechanical concepts, as well as effective ways to teach quantum mechanics. For instance, Singh and Marshman (2015) and Marshman and Singh (2015) delineate several difficulties that students have with understanding quantum mechanics, including the jarring "paradigm shift" students encounter as they move from classical mechanics to quantum mechanics. Passante, Emigh, and Shaffer (2015) specifically focus on the challenges students encounter when reasoning about energy measurements of quantum states, such as students' failure to understand the relationship between a wave function and the possible energy measurements, or students' failure to recognize the role of the Hamiltonian in determining the possible energy values for a quantum system. Additionally, Emigh, Passante, and Shaffer (2015) identify several difficulties underlying student errors in reasoning about the time dependence of quantum systems, including a tendency for students to misinterpret the mathematical formalism used to model time dependent quantum mechanical systems.

This is not to say research on students' understanding of quantum mechanics has only focused on student difficulties. For example, Gire and Price (2015) outlined several features of different quantum mechanical notations (matrix, Dirac, and wave function) that can help facilitate computations and interpretation when working on quantum mechanical problems. They further posited that Dirac notation can be a particularly useful notation to students. As another example, physics faculty at Oregon State University worked to improve students' overall experience in learning physics, including quantum mechanics, by completely reorganizing their upper-division physics curriculum (Manogue et al., 2001):

> [The] new curriculum for junior-year physics majors consists of a sequence of nine courses, each lasting about three weeks and meeting for seven hours per week. Each course is a case study involving a single physical situation or simple, conceptual principle. We call these case studies Paradigms. … The Paradigms are followed by six single-term Capstone courses that systematically present the usual deductive systems of

physics. (p. 979)

In quantum mechanics specifically, junior physics students are introduced to the "paradigm shift" from classical mechanics to quantum through a "spins-first" approach (McIntyre, 2012). These changes have made it easier for teachers to point out similarities and differences between classical and quantum concepts, and "students [have] indicated an improved comfort level with applications of mathematical tools" (Manogue et al., 2001, p. 988).

More directly relevant to undergraduate mathematics education, several studies have explored students' understanding of linear algebra concepts (such as eigentheory) as they are used within quantum mechanics. Dreyfus, Elby, Gupta, and Sohr (2017) posit that productive mathematical sense-making with the eigen-equations used within quantum mechanics (e.g., $\hat{H}|\psi\rangle = E|\psi\rangle$) requires a student to understand and employ two symbolic forms (Sherin, 2001), namely the transformation symbolic form (a matrix or linear operator acting on a vector "transforms" it into a new vector) and the eigenvector-eigenvalue symbolic form. Also investigating physics students' understanding of eigentheory, Wawro, Watson, and Christensen (2018a, 2018b) used a Resources perspective (Hammer, 2000)[1] to identify resources physics students activated as they reasoned about eigenvectors and eigenvalues of real 2x2 matrices, including how they thought about solutions to the equation $\begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 2\begin{bmatrix} x \\ y \end{bmatrix}$. They found that many students were able to recognize this equation as an instantiation of eigentheory and were able to find a solution, but some students were limited as they activated a resource that made them think the system of equations encapsulated by the matrix equation should only have a single solution. Hillebrand-Viljoen and Wheaton (2018) further found that students struggle to

[1] "A resource is a basic cognitive network that represents an element of student knowledge or a set of knowledge elements that the student tends to consistently activate together" (Sabella & Redish, 2007, p. 1018)

coordinate the "math-story" and "physics-story" spaces they have to navigate within quantum mechanics. They explain how solving complex quantum mechanics problems requires a facility with the mathematics (particularly linear algebra) involved in quantum mechanics, an understanding of the physical systems that are being modeled, and the ability to move back and forth between these two story spaces.

One mathematical concept that is essential to quantum mechanics is normalization of vectors, a crosscutting idea that appears in several other mathematics and physics courses. It is particularly important to directional derivatives in multivariable calculus, the development of orthonormal bases through the Gram-Schmidt process in linear algebra and numerical analysis, 2- or 3-dimensional motion in physics, and modeling quantum mechanical states in quantum physics. Despite students encountering normalization several times in their undergraduate study of mathematics and physics, students' understanding of normalization has been relatively uninvestigated, within both undergraduate mathematics education and physics education research. The purpose of this study is to explore students' various conceptions of normalization after encountering the concept in mathematics and physics courses, and to see how students make sense of the novel use of normalization for probabilistic modeling in quantum mechanics. More specifically, the research questions for this study are:

1. What are the various conceptions math and physics students have about normalization from math and physics courses in which the concept is taught?
2. What changes occur in physics students' conceptions of normalization while taking a quantum mechanics course?

The first research question aims to further and deepen our understanding of the ways students think about and conceptualize this important mathematical idea. Answering this first research question will provide necessary data for delineating the components and connections involved in a student's conception of normalization. These components include students' understanding of mathematical norms[2], vectors and vector spaces, and the reasons why normalization is important (Watson, 2017a, 2017b, 2018). To examine and analyze students' various conceptions of normalization, I make use of coordination class theory (diSessa & Sherin, 1998; diSessa & Wagner, 2005) which provides organization for looking at the mental structures and processes involved in constructing a concept such as normalization.

The emphasis on examining students' changing conceptions of normalization within quantum mechanics encapsulated in the second question has a two-fold purpose. First, normalization of the vectors representing quantum mechanical states is essential due to the probabilistic nature of quantum mechanical systems; hence, it provides a rich setting to investigate students' conceptions of normalization and their perceptions of its usefulness to mathematics and science. Second, the use of normalization in quantum mechanics could be very different from the uses of normalization students encounter prior to a quantum mechanics course. In particular, the use of new (to the students) notational systems (e.g., Dirac Notation), vector spaces (e.g., L^2-space or $\mathbb{C}^n$), and norms on these various vector spaces provide ample opportunity to explore changes that can occur in a student's conception of normalization. By investigating this rich setting, this study has the potential to inform both mathematics and physics educators of specific student conceptions of normalization that seem to be productive for understanding normalization in novel contexts.

To operationalize these research questions, I begin chapter 2 with the theoretical

[2] In mathematics education research, we sometimes talk of "social norms" and "sociomathematical norms" (e.g., Cobb & Yackel, 1996). In this paper, whenever the word "norm" is used, it specifically refers to the mathematical concept of norm, which is a function that assigns a strictly positive "length" or "size" to each vector of a specific vector space.

framework for this study. I first explain my chosen epistemological lens of constructivism. Next, I define what students' conceptions are, and how these conceptions grow and develop. In particular, I explain how students' ideas and conceptions consist of knowledge systems (diSessa, 1996) that have been constructed through their experience and have proved useful for making sense of the world. I also elaborate how students' constructed conceptions of normalization can be analyzed with coordination class theory (diSessa & Sherin, 1998; diSessa & Wagner, 2005). Chapter 2 additionally includes a literature review in which I explore research that has been done on students' understanding of vector, absolute value, norms and normalization, and related concepts. Furthermore, Chapter 2 contains a deeper explanation of the mathematical connections between linear algebra and quantum mechanics. Chapter 2 finishes with an explanation of the components involved in students' understanding of normalization that I have discovered through previous work (Watson, 2017a, 2017b, 2018), and how this can be combined with coordination class theory to analyze students' conceptions of normalization.

Chapter 3 is a thorough explanation of all data collection and analysis. As a brief overview, I will obtain the first data set (mainly focused on addressing the second research question) by giving a survey to physics students who are just beginning the course *Introduction to Quantum Mechanics*. This survey will assess students' understanding of normalization of vectors. Based on these surveys, I will choose a diverse set of six students in the course who are willing to participate in 30-45 minute interviews three times during the course: within the first week of the course, shortly after being introduced to normalization in quantum mechanical systems (probably during week 2 or 3), and then later in the course after they have studied quantum mechanics for a decent amount of time, and covered several concepts where normalization is important. These interviews will focus on their initial conceptions of normalization and will explore how their conceptions about normalization change and adapt as they learn the quantum mechanics material.

I will obtain the second data set (focused on addressing the first research question) by giving the same brief survey assessing students' conceptions of normalization to students in three different courses that physics students would likely take before taking quantum mechanics: *Introduction to Multivariable Calculus*, *Introduction to Linear Algebra*, and *Math Methods in Physics*. Additionally, this survey will be given to students in *Linear Algebra I*, a class that is not necessarily required for physics majors, but importantly explores vector spaces other than $\mathbb{R}^n$ and norms on these more abstract vector spaces. Based on these surveys, I will then select three students from each of the four classes who seem to have diverse conceptions of normalization and conduct single, 30-40 minute one-on-one interviews with each student. Data analysis will be a thematic analysis (Braun & Clarke, 2006) and will mainly consist of first and second levels of coding (Miles, Huberman, & Saldaña, 2014) to identify the various conceptions students hold about normalization, and what changes and adaptations occur in their conceptions while taking a quantum mechanics class. I now proceed to explain the theoretical framework for my study.

Chapter 2: Theory and Literature

In this chapter, I first explain the theoretical framework for this study that draws upon the epistemological lens of constructivism and explicates what students' conceptions are and how these can change with time and experience. This is followed by a review of literature relevant to students' understanding of norms and normalization, including research on student understanding of vectors, absolute value, and mathematical norms. This review includes literature from both physics and mathematics education research. Next, I explain more thoroughly the mathematical connections between quantum mechanics and linear algebra. I conclude this chapter by explaining how normalization can be thought of as a coordination class and how coordination class theory can be used to investigate students' conceptions of normalization.

2.1 Theoretical Framework

Constructivism is an epistemological lens centered on the idea that people construct their own knowledge based on their experience, building upon prior knowledge that they have already constructed. Knowledge is not simply acquired by placing known facts or truths inside our heads but is rather built up to make sense of the world. Constructivism could be considered as originating from the work of Jean Piaget (Piaget, 1952, 1954, 1970, 1977/2001) but has developed into a spectrum of different types of constructivism, depending on whether emphasis is placed on the individual or society. Examples include radical constructivism that focuses more on the individual (Steffe, 1991a, 1991b; von Glasersfeld, 1984, 1995, 2001), social constructionism that focuses more on society (Berger & Luckmann, 1967), and constructivist perspectives that balance a focus on the individual and society (Blumer, 1969; Cobb & Yackel, 1996; Vygotsky, 1978, 1986/1934).

The constructivist spectrum could be described as consisting of four main philosophical tenets (Doolittle & Hicks, 2003) with various branches of constructivism emphasizing these tenets in different ways:

1. Knowledge is not passively accumulated, but rather actively constructed by individuals through rational thought.
2. Cognition is always attempting to make sense of and organize one's experiences, and the knowledge constructed through this process cannot ever by a perfect reflection or accurate representation of some external reality; hence, knowledge is subjective.
3. Knowledge construction is an adaptive process that works to make individual's thoughts and actions more viable and useful for accomplishing goals within their experiential environments.
4. Knowledge is constructed through both individual psychological/neurological processes and interactions with society, culture, and language.

In this project, I adopt a type of constructivism that has a strong emphasis on the individual and is based on the work of Andrea diSessa and colleagues (diSessa & Sherin, 1998; Smith, diSessa, & Roschelle, 1993):

> Constructivism emphasizes the role of prior knowledge in learning. Students interpret tasks and instructional activities involving new concepts in terms of their prior knowledge. Errors are characteristic of initial phases of learning because students' existing knowledge is inadequate and supports only partial understandings. As their existing knowledge is recognized to be inadequate to explain phenomena and solve problems, students learn by transforming and refining that prior knowledge into more sophisticated forms. Substantial conceptual change does not take place rapidly, and

> relatively stable intermediate states of understanding often precede conceptual mastery. (Smith et al., 1993, p. 123)

In the sections that follow, I define what I mean by "conceptions" in this study, what this means for investigating conceptual change, and the different processes by which conceptual change occurs.

2.1.1 Concepts, conceptions, and coordination classes. Models of "concept" often take them to be simple, unitary mental structures, or small numbers of mental structures that are all connected to and associated with a "concept." For diSessa and Sherin (1998), and the perspective I adopt in this project, concepts are more of a knowledge system (diSessa, 1996) rather than nodes within a cognitive structure.

> As a consequence of adopting a system view, however, the boundaries of any concept become somewhat fuzzy, since many system parts are involved. Our contention is that this is not a disadvantage, but reflects a fundamental reality. Instead of stating that one either has or does not have a concept, we believe it is necessary to describe specific ways in which a learner's concept system behaves like an expert's – and the ways and circumstances in which it behaves differently. (diSessa & Sherin, 1998, p. 1170)

Hence, concepts (i.e., conceptions) are knowledge systems that are constructed by an individual to make sense of the world. Delineating what a concept is from this perspective must be done both structurally (what are the parts of the system?) and functionally (how does a concept perform?). For this purpose, I turn to *coordination class* theory.

diSessa and Sherin (1998) call specific types of concepts *coordination classes*, which are "systematically connected ways of getting information from the world" (p. 1171). Observation of the world is never a simple process of gathering pure data through our senses, but is rather afforded and constrained by what we attend to and how we interpret it through our current knowledge systems. While this process of getting information from the world can sometimes mean determining class membership with a category, this is not the limits of a coordination class, as information is used for many other purposes. To emphasize the coordination class perspective, diSessa and Sherin (1998) choose to use the verb "coordinate" to describe how an individual sees or determines information from a situation, which I will also adopt in this research project. For example, if an individual wants to determine the magnitude of a mathematical object, they must recognize and *coordinate* several pieces of information about that object, such as classifying the object (e.g., Is the object a vector? A geometric shape?), deciding which properties of the object are salient for determining its magnitude (e.g., Does the direction or orientation of the object affect its magnitude?), and choosing how to measure its magnitude.

Coordination classes contain two main components. *Readout strategies*, or strategies that "deal with the diversity of presentations of information to determine, for example, characteristic attributes of a concept exemplar in different situations" (diSessa & Sherin, 1998, p. 1176), constitute the first main component of a coordination class. These readout strategies are the way in which an individual more or less directly observes information from the world, and selects particular information to coordinate for a specific purpose (diSessa & Wagner, 2005). Coordination has a double meaning, and as such, a coordination class includes two types of coordination that are central to readout. First, within any given situation, an individual must coordinate multiple observations to determine the necessary information, which can be described as *integration*. Readout strategies therefore include ways to collect, select, and combine observations into the information required for accomplishing whatever task is at hand. Second, "across instances and situations, the knowledge that accomplishes readout of information must reliably determine the *same* information" (diSessa & Sherin, 1998, p. 1172), a type of coordination described as *invariance*. Thus, a coordination class also includes strategies for directing attention and "seeing" the same thing in multiple situations.

It might be tempting at this point to think of a coordination class as a way to categorize information that one is able to glean from the world around them, thus arriving at the idea that coordination classes are a fancy way to define "categories." However, not all coordination classes can be considered as categories, such as the coordination class of "location." Location is not category-like, since every physical object has a specific location, so objects cannot be placed into a bin or category generically labeled as "location." Instead of placing things into a category called "location," we most often use this coordination class to determine the location of something we are looking for, or to describe the location of a place. Furthermore, categorization is not the only goal or purpose of a coordination class. For instance, when assessing someone's personality, we are not determining what things in the world are personalities and which are not; rather, we are interested in gathering information about a person (diSessa & Sherin, 1998). "In general, complex attributes like location and personality don't make prototypical categories, although they are fine candidates for coordination classes" (p. 1173).

Coordination classes take time to develop. "An accumulation of a complex and broad set of strategies and understandings is characteristic of coordination classes, as opposed to, say, learning a rule or definition" (diSessa & Sherin, 1998, p. 1173). Part of this broad set of strategies and understandings includes a critical pool of inferential knowledge that allows an individual to move from directly observable information to unobservable information. diSessa and Sherin (1998) call this the *causal net*, and it is the second main component of a coordination class. As an example, a force can be considered to "cause" acceleration of an object, and by knowing either the force or the acceleration, one can infer the other quantity according to Newton's Second Law. This law could thus be a part of an individual's causal net in their individually constructed coordination class of force.

As a main focus of their theoretical paper on coordination classes, diSessa and Sherin (1998) argue that physical quantities, such as force, can be coordination classes. In particular, they note that one must develop readout strategies to "see" specific physical quantities in the world around them, and the causal net for the quantity must include inferential reasoning strategies that enable that person to determine the value of the physical quantity from specific observations in the world. Furthermore, with many physical quantities, equations can play a pivotal role in the causal net, especially for experts. For example, understanding the equation $F = ma$ in Newton's Second Law can be a powerful component in a person's causal net for their coordination class of force, enabling that person to infer or calculate either force or acceleration when the other quantity is known. In later work on coordination classes, diSessa and Wagner (2005) further posited that mathematical quantities could also be good candidates for coordination classes, which is particularly important to my own study.

Readout strategies and causal nets within a coordination class are very closely related. In order to coordinate observations in the world to "see" a coordination class, an individual's causal net of the coordination class influences and informs what readout strategies are used to look for necessary information. Conversely, if the readout strategies for a coordination class are relevant to a given situation, the causal net may allow the individual to make inferences about the situation beyond what is directly observable. Returning to the force example, if an individual wants to coordinate or see force in a given situation, the causal net that includes Newton's Second Law could drive that individual's readout strategies for observing mass and acceleration in the situation to be activated. On the other hand, if an individual's readout strategies recognize something in the world as a force, their causal net could enable them to make inferences about the acceleration. Although this close relationship exists between readout strategies and causal nets, it is important to note that an individual is not usually consciously aware of the readout strategies they activate or causal net they use to make inferences when they coordinate observations to obtain information from the world around them.

2.1.2 What is conceptual change from the coordination class perspective? To begin answering this question, I first explain how a coordination class is constructed. Because coordination class theory takes a knowledge system perspective,

> We should expect no sharp line between "having" and "not having" a concept. If very many elements and relations are involved, certainly a few may be missing or malformed, and yet the person could exhibit generally competent performance. Indeed, there is every reason to suspect either that no humans achieve complete or perfect construction, or that no such state is specifiable. … states of partial construction are much more important to describe than "has it, or not." (diSessa & Wagner, 2005, p. 126)

Hence, demarcating whether or not an individual has constructed a coordination class can be fuzzy. Still, diSessa and Wagner (2005) posit that early conceptions constructed by an individual are unlikely to constitute a coordination class; in particular, they explained two intrinsic difficulties for constructing a coordination class. First, there can be a wide variety of contexts and situations where a coordination class could be applicable, so an individual who has constructed a coordination class needs to have "conceptual resources adequate to cover a sufficiently wide range of contexts" (p. 128) which diSessa and Wagner call *span*. The second intrinsic difficulty is the complexity involved with coordinating different information in a variety of contexts and relating them all to the same information or concept (i.e. the coordination class) reliably and consistently. diSessa and Wagner call this difficulty *alignment*.

Although the early conceptions of beginning learners are unlikely to be constructed coordination classes, it is possible to investigate the ways students coordinate various kinds of information related to a specific concept across a variety of relevant situations. In other words, in investigating students' conceptions, students' readout strategies and causal nets they employ as they attempt to obtain information about the world and accomplish certain tasks can be a focus of study. Furthermore, these early conceptions can be framed as the initial stages of constructing a coordination class. In fact, many of the nascent ideas students have about a concept can be productive and could ultimately be incorporated into the cognitive structure of a well-formed coordination class for the individual.

With this in mind for my current study, I will refer to students with early conceptions and ideas as being "in the process of constructing a coordination class." This will allow me to investigate and explore the readout strategies and causal nets a student employs within a given situation, even though these strategies and nets may not necessarily constitute a well-formed coordination class. This collection of readout strategies, causal nets, and other cognitive operations that a person uses when applying his or her concept can collectively be referred to as their *concept projection* (diSessa & Wagner, 2005); this concept projection is analogous to an evoked concept image in the work of Tall and Vinner (1981) on concept image and concept definition.

Conceptual change, from the coordination class perspective, thus is a shift, change, or modification in either or both the readout strategies and causal nets employed by an individual in situations where their early conception or coordination class is applicable. "Shifting the means of seeing, a fortiori, is the core problem of conceptual change" (diSessa & Sherin, 1998, p. 1171). For example, suppose a student has constructed a naïve concept of force wherein they "see" or coordinate motion as being the result of force. This student has readout strategies for recognizing (i.e., coordinating) objects as being in motion and for "seeing" the source of that motion, which they coordinate as an instance of force. Their causal net could include an intuitive idea that if an object is moving in a certain direction, there must be a continuous force applied that "pushes" the object in that direction. Based on this student's early conception for force, they may believe that an object thrown into the air has a force continuously acting upon it that "pushes" it upwards until it reaches the apex of its trajectory, at which point another force begins to "pull" the object back down. Through further experience, the student could experience conceptual change in their conception for force, making progress in constructing a coordination class for force. More specifically, their causal net might change as they gain experience with Newton's Second Law, gaining an understanding that forces cause acceleration, a refinement of the idea that force and motion are related. With this shift in their causal net, the student's readout strategies could also change, as they recognize the need to coordinate acceleration within the force coordination class. Over time, and with further experience, the student might see how the concept of force applies in a variety of situations (*span*), and begin to use the concept of force in these various contexts reliably and consistently (*alignment*). At this point, it could be said that the student has constructed a coordination class of force in their cognitive structure.

diSessa and Wagner (2005) define two processes involved in this conceptual change and construction of a coordination class. *Incorporation* is the inclusion of old knowledge elements (e.g., readout strategies, inferential knowledge, causal nets, previously constructed coordination classes, facts about a situation) in the construction and operation of new ideas, concepts, or coordination classes. Returning to the force example, ideas about motion, pushing, pulling, and effort could be incorporated into an individual's coordination class of force. This relates to the learning paradox (Norton, 2009) which questions how an individual can learn or construct higher level concepts or ideas that they have no prior experience with. From a constructivist perspective, the development of more advanced conceptions and cognitive structures must be built from prior knowledge through some type of "bootstrapping" in order to overcome the learning paradox (Smith et al., 1993). Hence, from this constructivist standpoint, new coordination classes can be constructed by an individual through reorganizing and extending existing readout strategies and causal nets from known coordination classes and other prior knowledge (diSessa & Sherin, 1998). Sometimes no new readout strategies are needed to construct a coordination class, but existing readout strategies are reorganized and utilized in new ways; at other times, new readout strategies will be necessary. Similarly, causal nets may need to be reorganized and refined for a new coordination class, or new causal nets will need to be built up from ideas that are more primitive.

A complementary process to incorporation is *displacement*, or the recognition that previous ideas contained in an individual's early conception are not applicable or helpful in a particular context where the corresponding coordination class would be applicable. "Such displacement is understood to take place without prejudicing the value of the displaced ways of thinking in other contexts; that is, displacement is not replacement" (diSessa & Wagner, 2005, p. 130). In the force example given in the previous paragraph, the student's idea that a force must be continuously applied for motion to occur needs to be *displaced* before they can construct the coordination class of force. However, this idea does not need to be replaced, since continuous forces do cause a form of motion (an accelerating motion to be exact).

This view of conceptual change runs counter to the belief that students have "misconceptions" that must be replaced with productive, expert conceptions. In fact, as all knowledge is built from and upon prior knowledge, the term "misconception" can be deemed problematic. Concepts, particularly coordination classes, are complex knowledge systems that cannot be simply plucked out and replaced (Smith et al., 1993). Instead, conceptual change takes time, and consists of refining and reorganizing ideas:

> The examples suggest that mastery is achieved, in part, by using what you already know in more general and powerful ways and also by learning where and why pieces of knowledge that are conceptually correct may work only in more restricted contexts. … knowledge refinement [should be the] general description of conceptual change. Old ideas can combine (and recombine) in diverse ways with other old ideas and new ideas learned from instruction. (Smith et al., 1993, p. 137 & 147)

By taking this perspective, when students' concepts fail or are problematic, these conceptions cannot be thought of as misconceptions but rather extensions and applications of previously constructed knowledge to situations where that knowledge is either not applicable or insufficient. However, these early conceptions often have nascent potential, and through refinement, further development, and reorganization of the ideas therein, expert conceptions can develop. With coordination classes, these nascent ideas consist of readout strategies and causal nets that need to be refined, extended, reorganized, and further developed to construct a coordination class that fulfills the important requirements of *span* and *alignment*.

2.1.3 Summary of Coordination Class Theory. As a way of succinctly organizing the functionality and structure of coordination classes, Figure 2.1 adapts Table 1 from diSessa and Wagner (2005, p. 131), summarizing the core function of, intrinsic difficulties in constructing, and architecture of a coordination class. In sharing this, I remind the reader that diSessa and Wagner (2005) have pointed out that early conceptions are usually not coordination classes, but rather partial constructions of a coordination class. This begs the question, when can a coordination class be said to be completely constructed? As diSessa and Wagner explain, it cannot reasonably be expected that a coordination class is complete when the individual has

Core Function of a Coordination Class

A ***coordination class*** is a particular type of concept whose principle function is to allow people to read a particular class of information out of situations in the world.

Intrinsic Difficulties in Constructing a Coordination Class

Reading the same information out of a wide range of situations poses core problems for developing a coordination class.

> ***Span***: A learner must accumulate enough knowledge to "operate" the concept across the full range of contexts in which it is applicable.
>
> ***Alignment***: The information determined in different situations, possibly using different knowledge, must be the same information.

Architecture of a Coordination Class

Decomposition: The knowledge in a coordination class can be partitioned by function.

> ***Readout Strategies***: Readout strategies are the ways in which people focus their attention and read out (or coordinate) information from the world that are relevant to, but possibly not the same as, the defining information of the concept/coordination class.
>
> ***Causal Net***: The causal net is the set of inferences that people use to infer the defining information of the concept/coordination class from related kinds of information. Inferences can include: (a) recognizing the mere existence of the coordination class in a situation within the world; (b) identifying what is required or problematic about a situation; and (c) determining defining information about the coordination class from other information read out of that situation.

Construction Processes: Two very generic processes are involved in conceptual change and constructing a coordination class.

> ***Incorporation***: Incorporation is the process of recruiting elements of prior conceptualization into partial encoding of the new or changing concept.
>
> ***Displacement***: Displacement is the process of "dismissing" elements of prior conceptualization that may initially and inappropriately "take over" consideration of particular circumstances from a coordination class.

Figure 2.1. Main Elements of Coordination Classes. Adapted from diSessa and Wagner (2005, p. 131)

incorporated *all* contexts where the concept is applicable into their cognitive structures, as there are probably an infinite amount of context variations in which a concept could be applied. Furthermore, even experts may initially stumble when attempting to apply a coordination class to a novel context. diSessa and Wagner (2005) offer three possible solutions to this question. First, a coordination class may be considered completely constructed when an individual adequately coordinates the concept across a span of "the typical range of contexts" where the concept operates. Second, "completeness might be an inflection point in competence, where one has achieved the *main power* of the class" (p. 138), that is, a more generalized conception that is not necessarily tied to a specific context. Third, it might be that an individual "develops a layer of meta-knowledge that allows one to generate new concept projections adequate to all (or a selected set) of contexts" (p. 138). In any case, diSessa and Wagner explain this needs further empirical investigation to determine when a person can be said to have completed the construction of a coordination class, which is something I will work to address in this research project.

2.1.4 Examples of using coordination class theory in research. As a first example, I share the case study diSessa and Sherin (1998) present in their article on coordination classes. J was a student who had done well in high school physics and fairly well in an introductory university physics class. In the episodes of an extended clinical interview shared by diSessa and Sherin (1998), J attempted to apply her concept of force (i.e., conceptual projection) to a few different situations. An important aspect in J's causal net for force was the idea that motion is the result of an imbalance in forces, that is, when one force overcomes another. The interviewer and J were discussing how a person could push a book across a table, and J explained that the force exerted on the book must be greater than the frictional force of the table pushing back on the book. The interviewer reminded J of the equation $F = ma$ from Newton's Second Law, at which point J began to coordinate acceleration with the situation at hand, using readout strategies for "seeing" acceleration in the situation in relation to the forces involved. When the interviewer asked how this equation made sense with pushing the book across the table with a constant velocity, J was puzzled. She was able to coordinate constant velocity as an indication of no acceleration, but with no acceleration, there must not be a net force acting on the book according to Newton's Second Law. This contradicted her causal net that motion only occurs when a force overcomes another (in this case, the force from the hand overcoming the frictional force). While this episode may have provided an opportunity for J to modify, change, or *displace* this idea within her causal net in her conception of force, J actually concluded that Newton's Second Law and the equation $F = ma$ must not apply in the situation of a book being pushed across a table. Still, the authors importantly note that J was working on a critical component in developing a coordination class, namely the discriminating aspect of readout strategies that determine when ideas (such as Newton's Second Law) apply in specific situations.

Wittmann (2002) provides another example of using coordination classes to investigate student reasoning of physics. In his study, students were asked several questions about the propagation of waves through a medium (such as a string). Wittmann found that many of the students coordinated (i.e., "saw") waves as objects. Students' readout strategies included "objects as points" where they focused on the peak of the propagating wave(s) to discuss the movement and interaction of the wave pulse overall, and "waves as solid." Students' causal nets related to these readout strategies included a variety of reasoning resources that led to a variety of conclusions about wave propagation. For example, thinking of a wave as an object led many students to think that the speed of a wave pulse along a string could be increased simply by flicking the string faster, much like throwing a baseball faster requires a harder throw. The causal net in this case included the idea that exerted force determines the speed of an object. Another element of students' causal net in thinking about waves as objects was the idea of collisions of objects leading to a "bounce" of the objects away from each other. This led many students to infer that two waves travelling toward each other would collide and "bounce back," rather than pass through one another. While this was an interesting study, it was somewhat unclear if Wittmann was thinking of "object" as a coordination class that students were using to reason about waves, or if "wave" was considered a coordination class. An important part of my own study will be making sure to clearly explain what coordination class I am studying.

Two more examples of coordination class research can be found in the work of Thaden-Koch, Dufresne, and Mestre (2006) and Levrini and diSessa (2008). Exploring students' abilities to judge how realistic an animation of two balls travelling down two tracks was, Thaden-Koch et al. (2006) found that many students were able to use readout strategies to assess the speed of the balls (e.g., compare to fixed background, compare balls to each other) as well as elements of a causal net for speed expectations (e.g., steeper slope of track means ball should go faster, ball should slow down on uphill portion of track) to make fairly accurate judgments about the realism of the animations. Levrini and diSessa (2008) used coordination class theory to show how proper time in special relativity can be considered a coordination class, and explained how students made progress in their understanding (i.e., conceptual change) during a single classroom episode. In particular, as students were exposed to several different problem contexts where proper time was relevant, students expanded the *span* in their abilities to determine proper time within several contexts. Furthermore, these contexts fostered students' construction of important readout strategies (e.g., choosing a frame of reference) and causal nets (e.g., "the critical idea that a phenomenon has a locus, and that locus … is the 'home' for determining proper time" (p. 10)) necessary in constructing a coordination class of proper time.

Although the examples here are not related to vector normalization, they do provide examples of the usefulness of coordination class theory in exploring students' conceptions of complex ideas, and illustrate how conceptual change can be explored with this theoretical framework. Before making an argument for why normalization fits the definition of coordination class and how this theory will be useful in this current study, I turn to the literature that has already explored students' understanding of concepts related to normalization.

2.2 Review of Education Research Relevant to Student Understanding of Normalization

In reviewing the mathematics and physics education literature, I could not find any studies that specifically focused on students' understanding of normalization. However, conceptions of normalization inherently involve ideas about (a) vectors, vector spaces, and their representations, and (b) mathematical norms. As such, in this literature review I explore the research that has been conducted on student understanding of vectors and norms, as well as research that has touched upon the importance of understanding unit vectors in various contexts.

2.2.1 Research on student understanding of vectors, vector spaces, and representations. One of the first mathematical subjects students encounter involving a systematic building of theory, relying on definitions and formal proofs, is Linear Algebra (Hillel, 2000). Linear Algebra is a particularly useful branch of mathematics for much of the sciences, as well as higher mathematics courses, and, as such, many students are required to take linear algebra within the first two years of university studies (Aydin, 2014). Although linear algebra is a particularly powerful mathematical subject, many students struggle to make sense of some of the fundamental concepts in linear algebra including vector, span, linear independence/dependence, and basis (Carlson, 1993). Understanding these key ideas is fundamental to understanding the formal definition of a vector space. These ideas are also important for being able to "see" or coordinate other mathematical objects, such as functions, as elements of vector spaces.

To build up students' understanding of vector, span, linear independence/dependence, and basis, recommendations have been made to introduce students to linear algebra and the study of vectors and vector spaces through the use of $\mathbb{R}^n$ and matrix arithmetic (Carlson, 1993; Harel, 2000). Further refining this recommendation, Harel (2000) and Gueudet-Chartier (2006) have argued that geometrically exploring $\mathbb{R}, \mathbb{R}^2$, and $\mathbb{R}^3$ can be helpful, giving students concrete, conceptual entities (e.g., vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$ visualized as directed line segments) that can be used to explore notions, concepts, and theorems from linear algebra in a context where these vector space properties can be visualized and often appear self-evident. This prototype might then be used to promote students' understanding of the underlying mathematical structure of vector space (Gueudet-Chartier, 2006; Harel, 2000). However, both Harel (2000) and Gueudet-Chartier (2006) caution that students can become "stuck" in this geometric model of linear algebra and miss out on understanding the more general theory that applies to all vector spaces. To overcome this, students must see the geometric context as an example of the structure of vector space, not as the actual object of study (Harel, 2000). This can be accomplished by fostering an intellectual need to unify several mathematical domains under the vector space structure (for example, introducing students to function spaces) and by encouraging students to generalize the commonalities among these vector space contexts.

Identifying further complexity with geometric conceptions of vector, Hayfa (2006) discusses the role of language and its impact on students' understanding of vector. She explains how the language used in vector geometry textbooks often predisposes students to think of vectors as “tied vectors,” which are vectors bound to a specific place in the geometric plane that can be uniquely represented. Hayfa argues that students who are only able to conceptualize of vectors as “tied vectors” miss out on an equally important conceptualization, namely that of a “free vector,” which is determined by its direction and length rather than by its location in the geometric plane. For normalization of vectors in $\mathbb{R}^n$, this latter understanding of vectors having a direction and length or magnitude is critical.

Although cautions about using geometric ideas in teaching linear algebra have been given, success with introducing students to important linear algebra concepts through geometric notions can be found. One example is the Inquiry-Oriented Linear Algebra (IOLA) project (Wawro, Zandieh, & Rasmussen, 2013). This project is based on the instructional design theory of Realistic Mathematics Education (Gravemeijer, 1999) and uses experientially real contexts to give students the opportunity to explore and reinvent important linear algebra ideas. For example, Wawro, Rasmussen, Zandieh, and Larson (2012) explain how the concepts of span and linear independence can be reinvented by students through engaging with an instructional sequence called the Magic Carpet Ride sequence. In this sequence, students are presented with a rich, imaginary scenario of travelling using a hover-board and a magic carpet which travel in specific directions. This situation is used to introduce students to vectors and vector equations, and build on their intuitive understanding to lead them to formal definitions of span, linear dependence, and linear independence. Furthermore, the authors give evidence that the Magic Carpet Ride sequence helps develop students’ knowledge of and use of formal definitions to argue and justify claims, an essential skill in upper level mathematics. Furthermore, this skill of understanding and using formal definitions could prove particularly useful for understanding and applying the definition of vector space to more abstract spaces.

Illustrating the difficulty students can encounter with these more abstract vector spaces, Maracci (2006, 2008) shared examples of students struggling to make sense of questions related to linear combinations, linear independence, basis, and spanning set when the questions were asked within the context of an abstract vector space. For example, students struggled to see how a vector $\vec{u}$, written as the linear combination of five linearly independent vectors of V, could be contained in $U_1 + U_2$, the sum of two two-dimensional subspaces of V, but not in U_1 or U_2 alone. Most of the students thought this was impossible because $\vec{u}$ was written as five linearly independent vectors of V, and the sum of two two-dimensional subspaces would have at most four linearly independent vectors. This reasoning omitted the possibility that one of the basis vectors of U_1 or U_2 could be a sum of two of the vectors in the linear combination that composed $\vec{u}$. Maracci (2008) hypothesized that students may have thought of the five linearly independent vectors of V as a "canonical basis," similar to the canonical basis of $\mathbb{R}^n$, which led them to think that any linear combination should consist of a linear combination of these vectors. Furthermore, the students may have thought any subspaces of V should have a basis consisting of these individual "canonical basis" vectors.

As another possibility to explain students' difficulties with vector spaces, Maracci (2008) turned to Sfard (1991) and the process-object duality. In particular, Maracci explains how linear combinations of vectors can be thought of both operationally (as a process) and structurally (as an object), but students struggled to coordinate these two. This inability to conceptualize linear combination as an object may have contributed to students' difficulties with the questions posed about abstract vector spaces, such as their difficulties with linear dependence and spanning sets

Maracci's work demonstrates the complexity of understanding important ideas related to abstract vector spaces, and calls attention to the need for clear delineations of how students might develop or construct these concepts within these more abstract settings.

Proposing a possible pathway students might take in constructing the vector space concept, Parraguez and Oktaç (2010) employed the perspective of APOS theory (Arnon et al., 2014) to explicate the mental mechanisms and constructions that are necessary for learning this concept. In their proposed genetic decomposition for the vector space concept, they explain that a student must first activate the concepts of set and binary operation (which they would have previously constructed) to perform an *action* of applying the binary operation to two elements of the set and obtaining the resulting element. By thinking about what this binary operation does to all pairs of elements of the set, this action is interiorized into a process. Further reflection on this process encapsulates it into an object, which can then be assimilated with the axiom schema (which would need to have been previously constructed) to examine whether or not a set with a binary operation satisfies a set of axioms. This object of a set with a binary operation that satisfies a set of axioms is then coordinated with the object of a field. Parraguez and Oktaç (2010) then explain how the concept of vector space is constructed:

> The objects that are sets with two kinds of operations (addition and multiplication by a scalar) can be coordinated through the related processes and the vector space axioms that involve both operations, to give rise to a new object that can be called a vector space." (p. 2116)

The authors tested the viability of their genetic decomposition by preparing a questionnaire and semi-structured interview about the vector space concept, which they gave to several mathematics students. They concluded that students who lack the prerequisite constructions (such as binary operation and set) or who have weak conceptions of these concepts will have a difficult time constructing the vector space concept. The authors further suggest that students need to be given opportunities to explore sets and binary operations that are different from the usual operations they may have encountered in studying vector spaces like $\mathbb{R}^n$. Furthermore, activities need to be designed for helping students specifically coordinate the two operations (such as addition and scalar multiplication) contained within the axioms for vector spaces.

Taking a different theoretical approach to students' understanding of vector space, Dogan, Carrizales, and Beaven (2011) use metonymy as cognitive construct (Presmeg, 1998) to interpret the interview responses of a linear algebra student. They found that the student metonymically used "linear independence" in place of "linear combination," metonymically referred to matrices as sets (e.g., "the matrix is linearly dependent"), used a metonymy of the "identity form" of a matrix to check for linear independence, and used symbols of "x_i's" to metonymically refer to vectors. The authors explain that these metonymies are cognitive constructs that have specific meanings tied to them and are not simply strategies for recalling information; hence, careful attention should be paid to the metonymies students use in their thinking, especially concerning advanced mathematical concepts such as vector and vector spaces.

In concluding this discussion of literature on students' understanding of vector and vector space, it is important to note the complexity involved with understanding the concept of vector, particularly when thinking about vectors as elements of a vector space. In my study, it will be important to investigate the ways students' conceptions of vector and vector space afford or constrain their conceptions of normalization.

2.2.2 Research on student understanding of mathematical norms. Students' conceptions of normalization not only depend on their conceptions of vectors but also their conceptions of mathematical norms. I first remind the reader of the mathematical definition of a norm before discussing research on students' understanding of this concept.[3] For a vector space V over a scalar field $\mathbb{F}$ (with operations of vector addition and scalar multiplication), a *norm* (denoted by $\| \quad \|$) is a function $\| \quad \|: V \to \{0\} \cup \mathbb{R}^{+}$ that assigns a nonnegative "length," "size," or "magnitude" to each vector of the vector space. This *norm* must also satisfy the following three properties: For $a \in F$, and $\boldsymbol{u}, \boldsymbol{v} \in V$,

1. $\|a\boldsymbol{u}\| = |a|\|\boldsymbol{u}\|$
2. The Triangle Inequality: $\|\boldsymbol{u} + \boldsymbol{v}\| \leq \|\boldsymbol{u}\| + \|\boldsymbol{v}\|$
3. $\|\boldsymbol{u}\| \geq 0$, and $\|\boldsymbol{u}\| = 0$ if and only if $\boldsymbol{u} = \boldsymbol{0}$ (the zero vector).

The rest of this section expounds upon mathematics education research that has examined (a) student understanding of absolute value, (b) student understanding of norms and normed vector spaces, and (c) student understanding of unit vectors.

2.2.2.1 Absolute value: students' first introduction to the concept of norm. Although probably no K-12 teacher introduces absolute value as an example of a mathematical norm, the absolute value function really is students' first introduction to the concept of norm. More specifically, the absolute value is a norm for the vector space of the real numbers, $V = \mathbb{R}$, over the scalar field of the real numbers, $F = \mathbb{R}$, as it satisfies the following properties:

1. For any $a \in V$, and any $b \in F$, $|ba| = |b||a|$.
2. For any $a, b \in V$, $|a + b| \leq |a| + |b|$ (The Triangle Inequality)
3. For any $a \in V, |a| \geq 0$, and $|a| = 0$ if and only if $a = 0$.

In a broader, less technical sense, the absolute value assigns to each real number a "length,"

[3] This definition is based on that given by Dym (2013, p. 138)

namely its distance from the number zero.

There are dozens of articles written for practitioner journals about ways to teach the concept of absolute value in a meaningful way (e.g., Brumfiel, 1980; Kidd, 2007; Taylor & Mittag, 2015; Wade, 2012), with most advocating (at least in part) for some version of the "distance from zero" conception of absolute value. Looking at mathematics education research into students' understanding of absolute values, Wilhelmi, Godino, and Lacasta (2007) point out that the absolute value can be "partially defined" in a variety of ways: (a) *arithmetical*, namely "a rule that leaves the positive numbers unchanged and changes the negative numbers into positive ones" (p. 76); (b) *geometrical*, namely the "simple rule to 'delete the minus sign'" (p. 87); and (c) *analytical*, namely defining the absolute value as a piecewise function, in terms of a maximum ($|x| = \max\{x, -x\}$), or as a compound function ($|x| = \sqrt{x^2}$). Furthermore, they argue that understanding only the formal mathematical definition of the concept of absolute value is not as effective as having a global or holistic understanding of the various meanings and ways to define absolute value, since having this holistic understanding allows for "effective selection of meanings in each specific educational circumstance" (p. 88). To support this claim, they shared results from a questionnaire given to 55 pre-service secondary teachers requiring them to solve a variety of absolute value problems. As predicted, they found that students who relied on only one partial meaning for the absolute value struggled on the entire questionnaire, particularly if the reliance was upon the arithmetical or geometric meaning. On the other hand, those who were able to reason with the analytical meaning of the absolute value fared much better on the questionnaire. Surprisingly, Wilhelmi et al. (2007) did not mention anything about defining the absolute value as a magnitude of a number, or its distance from zero.

Other mathematics education research specifically examine the difficulties students have with solving inequalities involving absolute values and different approaches to improving students' understanding (Almog & Ilany, 2012; Sierpinska et al., 2011). Almog and Ilany (2012) examined students' solution strategies for solving absolute value inequality problems and found several commonalities (such as a dominance of thinking only in integer values or forgetting that the absolute value can be zero). More pertinent for this current study, there was some evidence of students thinking of absolute value as giving a magnitude or distance from zero, and that these "immediate solutions without algebraic manipulations" (p. 351) were more often correct than any of the other solution methods (e.g., algebraic manipulations). Sierpinska et al. (2011) conducted a teaching experiment in which they used three different approaches for solving absolute value inequalities: (a) the theoretical approach (based off of a formal definition of absolute value); (b) the procedural approach (i.e., make negative numbers positive); and (c) the visual approach (where the formal definition was given but supplemented with graphical ideas). Students in the visual approach group outperformed students in the other two groups, indicating that a graphical or visual approach to absolute value inequalities can be highly effective. Furthermore, and particularly relevant to this current work, Sierpinska et al. (2011) did briefly argue that

> Definitions based on the notion of distance are important in applications and in mathematical theory, in particular in generalizations of absolute value to norms in higher dimensions and general vector spaces, and in generalizations of limits and continuity in topology. (p. 280)

Still, neither article explored teaching absolute value explicitly as an example of a mathematical norm, focusing more on students' ability to solve absolute value inequality problems rather than their particular conceptions of absolute value.

Yet, this work on students' understanding of absolute value is useful and informative for my development of a model for students' understanding of norms and normalization. More specifically, since the conception of absolute value of a real number as representing its distance from zero is particularly useful and powerful for students' solutions of absolute value problems, it may be the case that thinking about mathematical norms as representing a vector's distance from the zero vector[4] is also particularly useful and powerful. Related to students' reliance on procedural conceptions of absolute value being problematic, conceptions of mathematical norms that are based only on a procedure might also be problematic in solving problems using norms, particularly when moving into unfamiliar vector spaces.

2.2.2.2 Research on student understanding of norms and normed vector spaces. Research on students' understanding of norms is sparse, but we can gain insight into how students might think about norms and normed vector spaces by considering research on students' understanding of the concept of "magnitude." Thompson, Carlson, Byerley, and Hatfield (2014) explain that:

> The idea of magnitude, at all levels, is grounded in the idea of a quantity's size. A quantity, however, is not something in the world. It is a person's conception of an object and an attribute of it, and a means by which to measure that attribute. Anyone's understanding of a quantity's size will be colored by his or her conception of the quantity being considered and by his or her understanding of how it might be measured. (p. 2)

[4] Recall that on a normed vector space V with norm $\| \quad \|$, the "distance" between two vectors $\boldsymbol{u}, \boldsymbol{v} \in V$ can be defined as $d(\boldsymbol{u}, \boldsymbol{v}) = \|\boldsymbol{u} - \boldsymbol{v}\|$, which is also known as the *norm induced metric* on V. Hence, we can think of the norm of a vector $\boldsymbol{u} \in V$ as $\|\boldsymbol{u}\| = \|\boldsymbol{u} - \boldsymbol{0}\|$, that is, $\boldsymbol{u}'s$ "distance" from the zero vector.

In other words, from a coordination class perspective, the readout strategies and causal nets an individual employs to determine a quantity's magnitude will determine what aspects of the quantity will be focused on and how they will be measured.

Assuming that a person can reliably determine what quantity they want to find the magnitude of, Thompson et al. (2014) explain five levels of meaning for a quantity's magnitude:

1. An awareness of size (e.g., ability to judge when one quantity is bigger/smaller than another),
2. Equating the measurement of a quantity with its magnitude,
3. Assessing the relative size of one quantity in comparison to another (e.g., quantity A is 7/3 times as large as quantity B; quantity B is 3/7 as times as large as A),
4. Anticipating that any measurement of a quantity with respect to an appropriate unit can be expressed in any other appropriate unit, but the quantity's magnitude will remain invariant; and
5. Understanding the magnitude of *intensive quantities*, that is, quantities composed of other quantities (e.g., rates of change, force), and how this magnitude, or the relationship between the magnitudes of the constituent quantities, is invariant with changes in units of measurement for either of the constituent quantities.

This last level is the most sophisticated and is often the most important for reasoning about the measurement of quantities in high-level scientific or mathematical contexts. Unfortunately, as illustrated by Thompson et al. (2014) through data from 112 mathematics teachers answering questions about measurement and magnitude, a large number of mathematics teachers are ill-prepared to teach students about magnitude, as their own thinking about magnitude is often not at the highest level. In my study on students' conceptions of normalization, it will be important to keep these levels of meaning in mind and investigate how students' understandings of magnitude affect their conceptions of mathematical norms and normalization.

Research on student understanding of norms and normed vector spaces is sparse; however, there has been recent research related to students' understanding of these concepts. For example, Reed (2018) investigated undergraduate students' abilities to generalize their knowledge and understanding of real analysis on $\mathbb{R}$, including their understanding of norms on $\mathbb{R}$, to more abstract vector spaces. He conducted two teaching experiments wherein students reinvented (Gravemeijer, 1999) the formal definition of a metric space through working with more abstract, normed vector spaces (sequence and function spaces). More specifically, students in his teaching experiment reinvented the concept of a metric using the norm and the structure afforded by the normed vector space, determining that the distance between two vectors in a normed vector space can be defined as $d(\boldsymbol{u}, \boldsymbol{v}) = \|\boldsymbol{u} - \boldsymbol{v}\|$. Reed's study showed that students can generalize their knowledge and understanding of the real metric space $\mathbb{R}^n$ to more abstract vector spaces and are able to use reflective abstraction to extend their knowledge of norms and metrics on $\mathbb{R}^n$ to sequence and function spaces. This is important, as my own study will be looking to see if students' can extend their understanding of norms and normalization to contexts of potentially unfamiliar vector spaces used in quantum mechanics, like $\mathbb{C}^n$ and L^2-space.

2.2.2.3 Research on student understanding of unit vectors. The scarcity of articles about students' understanding of vector norms and normalization necessitates looking deeper into other education literature where these ideas may still be relevant. To this end, I briefly touch on some of the research that has investigated students' understanding of unit vectors. Barniol and Zavala (2011) gave physics students at a Mexican university a question in which they were asked to draw a unit vector in the direction of a vector already drawn from the origin to the point (2,2) on the Cartesian coordinate plane. Only 22% of students gave a correct answer, while others drew a vector from the origin to the point $(1, 1)$ (25%), or drew the component vectors $2\hat{\boldsymbol{\imath}}$ and $2\hat{\boldsymbol{\jmath}}$ (14%), along with other incorrect ideas. In a later study, Barniol and Zavala (2014) developed a multiple choice test that assessed students' understanding of various ideas about vectors. On the question that asked students to find the unit vector in the direction of a given vector, more students were able to answer correctly (43%), but a large proportion of students still chose incorrect answers corresponding to those of the earlier study. This shows that finding a unit vector (i.e., normalizing) is not a trivial process.

Vega, Christensen, Farlow, Passante, and Loverude (2016) investigated students' abilities to draw unit vectors representing the motion of a particle moving in a two-dimensional plane, where the unit vectors were in terms of polar unit vectors (i.e., $\hat{r}$ and $\hat{\theta}$). Many students were unable to correctly answer this question, drawing vectors that do not satisfy the definition of a unit vector. More specifically, the authors explain that students need to understand four fundamental ideas when it comes to unit vectors in physics in order to correctly solve this problem: (a) unit vectors are vectors; (b) unit vectors have a length or magnitude of one; (c) unit vectors point in the increasing direction of the corresponding coordinate; and (d) unit vectors are dimensionless. Furthermore, several students drew vectors that were curved, so an important understanding is knowing that unit vectors are straight, directed arrows when within these physical motion contexts.

Knowing that unit vectors (and relatedly normalization) are also important for directional derivatives in multivariable calculus, I turn to research that has investigated student understanding of this concept. Martínez-Planell, Gaisman, and McGee (2015, 2017) used APOS theory to delineate the important concepts needed to understand directional derivatives and investigated students' conceptions of these ideas. In their preliminary genetic decomposition for the concept, the authors present a unique conceptualization of directional derivative that involves calculating the "vertical change" along a plane ($f_x(a,b)\Delta x + f_y(a,b)\Delta y$), and dividing this by the "horizontal change" which is the magnitude of the direction vector. By doing so, the authors posit that students may be able to extend their knowledge of slope from single variable calculus to talking about "slope" in the two-variable case. However, in interviewing students about their understanding of directional derivative, very few were able to discuss the concept competently. In fact, some evidence was shown that students may not necessarily see the need for the direction vector to be a unit (normalized) vector.

These articles demonstrate a need in my own study to explore and examine how students think about unit vectors and normalized vectors. In particular, I want to ask students questions that will provide an opportunity for them to explain why they think normalization is important, and how it is used in a variety of contexts. For example, a student may know that normalization is used in calculating directional derivatives, but have no idea why it is important within that context.

2.3 A Brief Overview of Quantum Mechanics and the Importance of Normalization Therein

Quantum mechanical systems are probabilistic in nature, and the mathematical modeling of these systems relies heavily on linear algebra concepts. In his introductory book on quantum mechanics, McIntyre (2012) summarizes the mathematical modeling of quantum systems through six postulates:

1. "The state of a quantum mechanical system, including all the information you can know about it, is represented mathematically by a normalized ket $|\psi\rangle$." (p. 5)

2. "A physical observable is represented mathematically by an operator A that acts on kets." (p. 34)
3. "The only possible result of a measurement of the physical observable is one of the eigenvalues a_n of the corresponding operator A." (p. 35)
4. "The probability of obtaining the eigenvalue a_n in a measurement of the observable A on the system in the state $|\psi\rangle$ is

$$P_{a_n} = |\langle a_n|\psi\rangle|^2$$

where $|a_n\rangle$ is the normalized eigenvector of A corresponding to the eigenvalue a_n."
5. "After a measurement of A that yields the result a_n, the quantum system is in a new state that is the normalized projection of the original system ket onto the ket (or kets) corresponding to the result of the measurement:

$$|\psi'\rangle = \frac{P_n|\psi\rangle}{\langle\psi|P_n|\psi\rangle}\text{" (p. 46)}$$

6. "The time evolution of a quantum system is determined by the Hamiltonian or total energy operator $H(t)$ through the Schrödinger equation:

$$i\hbar\frac{d}{dt}|\psi(t)\rangle = H(t)|\psi(t)\rangle.\text{" (p. 69)}$$

It is important to note that these kets, or state vectors, can be discrete or continuous depending on the quantum system in question. For example, in studying quantum spin, the kets can be represented by elements of $\mathbb{C}^n$, but in studying the location of quantum particles, the kets can be represented by wave functions (i.e., elements of L^2-space).

I share these postulates for three reasons. First, it provides a brief explanation of the relevance of mathematical concepts, such as eigentheory, normalization, and probability, to modeling quantum mechanical systems. Second, it illustrates the necessity of normalization in quantum mechanics, as state vectors must be normalized, otherwise the model will not be probabilistic. Third, it shows the rich setting quantum mechanics provides for studying students' conceptions of normalization, affording me the ability to ask students why normalization is so important in quantum mechanics, and how it relates to other contexts where they have normalized vectors in the past.

At this time, it is important to briefly familiarize the reader with the norms generally encountered in a quantum mechanics course. In particular, students will generally be working with vectors in $\mathbb{C}^n$ or elements of $\mathbb{C}^n$ or L^2-space. For vectors in $\mathbb{C}^n$, the norm used in quantum mechanics is the ℓ^2-norm, which is the square root of the sum of the squares of the complex moduli of the elements of a vector $\boldsymbol{x}$:

$$|\boldsymbol{x}| = \sqrt{\sum_{k=1}^{n} |x_k|^2}.$$

Alternatively, you can also consider this norm as the square root of the inner product of the vector $\boldsymbol{x}$ with itself

$$|\boldsymbol{x}| = \sqrt{\langle \boldsymbol{x}, \boldsymbol{x} \rangle} = \sqrt{\boldsymbol{x}^{\dagger}\boldsymbol{x}}$$

with the dagger meaning the conjugate transpose of $\boldsymbol{x}$. For example, the norm of the complex vector $\boldsymbol{v} = \begin{bmatrix} 3 \\ 3i \end{bmatrix}$ is:

$$|\boldsymbol{v}| = \sqrt{|3|^2 + |3i|^2} = \sqrt{9+9} = \sqrt{18} = \sqrt{\begin{bmatrix} 3 & -3i \end{bmatrix} \begin{bmatrix} 3 \\ 3i \end{bmatrix}} = \sqrt{\boldsymbol{v}^{\dagger}\boldsymbol{v}}$$

In the L^2-space used in quantum mechanics, the elements are complex valued functions, and the L^2-norm of a function ϕ is

$$|\phi| = \sqrt{\int_{-\infty}^{\infty} |\phi(x)|^2 \, dx} = \sqrt{\int_{-\infty}^{\infty} \phi^* \phi(x) dx} = \sqrt{\langle \phi, \phi \rangle}$$

which can be seen as analogous to or a continuous extension of the ℓ^2-norm. As an example, the function

$$f(x) = \begin{cases} \sec\left(\frac{\pi x}{4}\right) & -1 \leq x \leq 1 \\ 0 & x < -1 \text{ or } x > 1 \end{cases}$$

has L^2-norm

$$|f| = \sqrt{\int_{-\infty}^{\infty} |f(x)|^2 \, dx} = \sqrt{\int_{-1}^{1} \sec^2\left(\frac{\pi x}{4}\right) dx} = \sqrt{\frac{8}{\pi}}.$$

Note these are simple examples meant to illustrate the norms students would typically encounter in an introductory quantum mechanics course (based on quantum mechanics courses I have observed previous to this study), but other vectors and norms could possibly be encountered in other quantum courses.

2.4 Analytical Tool for Examining Students' Understanding of Norms and Normalization

In past work, I have been developing an analytical tool for examining students' understanding of mathematical norms and normalization (Watson, 2018) to analyze students' conceptions. This work was inspired and influenced by Zandieh's (2000) framework for student understanding of derivatives and Lockwood's (2013) model of students' combinatorial thinking. Similar to the work of Lockwood (2013), I used a *conceptual analysis* (von Glasersfeld, 1995) or "a detailed description of what is involved in knowing a particular (mathematical) concept" (Lockwood, 2013, p. 252) to create this tool for modeling students' understanding of norms and normalization. This conceptual analysis involved coordinating: (a) my own theoretical thinking about the constructs involved in understanding norms and normalization; (b) relevant literature, such as the literature discussed in the previous sections; (c) student interview data; and (d) feedback received during conference presentations of this research.

The organization of the analytical tool as it currently stands is presented in Figure 2.2. To model a student's conception of norms and normalization, a researcher fills out the "general understanding" column of the framework according to evidence gathered from the student's interview regarding the following ideas:

		General Understanding
Vectors	Vector Space What is a vector space? Examples of vector spaces	
	Vector Representations	
Norm	Norms: What is a norm? Examples of norms	
	Procedure(s) to Find Norm	
Normalizing	Metaphors for Normalizing	
	Procedure(s) to Normalize	
Normalized Vectors	Properties of Normalized Vectors	
	Reasons for Normalized Vectors	

Figure 2.2. Analytical Tool for Examining Students' Understanding of Norms and Normalization

1. Vector Spaces – What vector spaces does the student seem to know? Can they explain the formal definition of a vector space? What types of vector representations (e.g., graphical, symbolic, matrix) does the student have access to in their conception?
2. Norm – What mathematical norms does the student seem to know? Can they explain the formal definition of a norm? What procedures for finding the norm of a vector does the student have access to in their conception?

3. Normalizing – What imagery does a student use when they describe normalizing a vector (i.e., does normalizing *transform* the given vector to have a length of one, or does it *produce* a vector in the same direction as the given vector but with a length of one)? What procedures for normalizing a vector does the student have access to in their conception?
4. Normalized Vectors – What properties about normalized vectors does the student seem to know? What reasons does the student give as to why normalization of vectors is important?

At any time during a student's attempt to normalize a vector, they may access their general understanding to understand the problem context, especially when asked to normalize vectors that are unfamiliar to the student (such as a complex vector). Those with more robust,

		General Understanding
Vectors	Vector Space What is a vector space? Examples of vector spaces	$\mathbb{R}^n$ $\mathbb{C}$
	Vector Representations	Algebraic: $\vec{v}$ Matrix Notation: $\begin{bmatrix} a \\ b \end{bmatrix}$ Graphical: Arrow from Origin Argand Diagram
Norm	Norms: What is a norm? Examples of norms	Euclidean Norm for Real Vector Norm of Complex Number Gives the length or magnitude of vector Always real valued
	Procedure(s) to Find Norm	Pythagorean Theorem Square root of product of complex number with its conjugate Square root of sum of squares of components Square root of dot product of vector with its complex conjugate
Normalizing	Metaphors for Normalizing	Production Metaphor: Calls it a new vector and labels it w
	Procedure(s) to Normalize	Divide vector (or components of the vector) by the magnitude
Normalized Vectors	Properties of Normalized Vectors	Magnitude of one In same direction as original vector
	Reasons for Normalized Vectors	Only want direction, no magnitude

Figure 2.3: Example of Modeling a Students' Conception of Norms and Normalization

general understandings of norms and normalization will likely be able to access this understanding when working in a specific problem context to recognize mistakes and be successful at normalizing vectors in unfamiliar contexts (Watson, 2018). An example of how the framework can be used for recording and analyzing students' normalization understanding is shown in Figure 2.3.

2.4.1 How might normalization be a coordination class? In order to normalize a vector, a person must coordinate a number of pieces of information. First, there must be recognition that the object to normalize is a vector, or at least a mathematical object that has a specific magnitude or length that can be calculated; in other words, a person must have readout strategies for "seeing" the object to normalize as a vector or something with a magnitude/length. These readout strategies could include various ways of representing the vector, such as algebraically, graphically, or numerically. Second, calculations of the magnitude or norm of the vector, and the determination of the normalized vector (possibly by multiplying the vector by a factor of one over the magnitude), might make up the causal net for the coordination class of normalization. This causal net would allow a person to make inferences about (a) what unit vector is in the same direction as a given vector, or (b) what the "original" vector was, given a normalized vector and the magnitude of the original vector.

It is also important to look at the characteristics of *span* and *alignment* within the coordination class of normalization. For *span*, the types of vectors an individual is familiar with, and the contexts within which they have had to normalize vectors in the past can be explored. For *alignment*, the consistency with which an individual is able to apply their knowledge of normalization to normalize vectors in a variety of problem solving contexts and vector spaces must be determined.

As a particular example of the conceptual change I am interested in exploring, consider a student who enters quantum mechanics with readout strategies for vector that only sees elements of $\mathbb{R}^n$ as vectors. As students are asked to normalize complex vectors or wave functions, how do students' conceptions of normalization change? Do their readout strategies adapt to include these objects as vectors? Does their causal net change and adapt to include new ways of finding norms or magnitudes? Or, do the students compartmentalize (see Vinner & Dreyfus, 1989) normalization in quantum mechanics as a completely distinct concept from normalization of real vectors? Are students consistent in interpreting normalization of vectors across different contexts and vector spaces, that is, does their conception of normalization have the feature of *alignment*?

2.4.1.1 An ideal normalization coordination class. What might constitute an expert or "ideal" coordination class of normalization? To give a hypothetical answer to this question, I hypothesize the readout strategies, causal net, span, and alignment of an expert's coordination class for normalization. First, an expert would have readout strategies for seeing or coordinating the mathematical object to be normalized as a vector, specifically an element of a particular vector space. These readout strategies would include strategies for representing the vector in different ways (such as graphically or symbolically), cognitively situating the vector within its proper vector space, and drawing upon their more general or formal knowledge of vector spaces. The expert would also have readout strategies for seeing unit vectors as vectors with a length or magnitude of one. Additionally, the expert would have the ability to see or coordinate a normalized vector as a unit vector in the same direction as the original vector (for vectors in $\mathbb{R}^n$ or $\mathbb{C}^n$) or with the "same shape" as the original function (for elements of L^2-space).

Second, an expert would have an intricate *causal net* that would give them the ability to calculate the norm or magnitude of vectors from a variety of vector spaces and the ability to find the normalized vector. This causal net would include knowledge of different norms on vector spaces, properties of the norm of a vector (e.g., the norm of a nonzero vector is greater than zero), and different procedures for calculating those norms (e.g., using the Pythagorean theorem, using inner products). It would also include knowledge of different ways the normalized vector can be found (e.g., multiply by the reciprocal of the norm, find the "normalization constant"). Furthermore, the causal net would contain knowledge for the reversibility of normalization, that is, the ability to find an original vector given the normalized vector and the vector's norm/magnitude. Lastly, the causal net would include the inferential knowledge for the usefulness of normalization within different problem solving contexts. For instance, in working with directional derivatives, an expert would understand that the directional vector must be normalized so that the calculated derivative is a unit rate of change. An expert might also understand that vectors representing quantum mechanical states must be normalized so that they can be used for the probabilistic modeling of quantum systems.

For *span*, an expert must have seen normalization applied in a sufficient variety of contexts and vector spaces. That is, they must have experience with finding the norm of a variety of vectors from different vector spaces and finding the corresponding normalized/unit vector in that vector space. An expert would likely have experience normalizing vectors in $\mathbb{R}^n$, $\mathbb{C}^n$, and functions in L^p-space. For alignment, an expert would be able to see the commonality across the various instances of normalization, knowing that in each context one must have a vector, find the norm of that vector, and then multiply the original vector by a factor of the reciprocal of that norm to find the normalized vector. This would be consistent and accurate across all contexts where a vector is to be normalized for a specific purpose. This may even include the ability to apply this knowledge in novel contexts, such as normalizing a matrix from a vector space of matrices.

This normalization coordination class would have been constructed through the processes of *incorporation* and *displacement* resulting from multiple experiences with normalization over time. For instance, in the expert's early conceptions of normalization, they may have *incorporated* their knowledge of using the Pythagorean Theorem to find the length of vectors in $\mathbb{R}^2$ into their conception of normalization, specifically for normalizing vectors in $\mathbb{R}^2$. As an example of *displacement*, experts may have needed to *displace* from their early conception of normalization an idea that the magnitude of any vector is found by finding the square root of the sum of the squares of the components of the vector.

In this dissertation study, my goal is to explore the *readout strategies* and *causal nets* students employ as they work to normalize vectors in a variety of contexts. More specifically, I hope to identify: (a) the *readout strategies* students have in their conceptions of normalization; (b) the inferential knowledge students employ in working with normalization, that is, what is in the *causal net* of their conceptions of normalization; (c) the variations that exist among students' conceptions of normalization (e.g., looking at the variations that exist among students' readout strategies and causal nets for their conceptions of normalization); (d) the changes and development in physics students' conceptions for normalization as they take a quantum mechanics class (that is, what knowledge is *incorporated* or *displaced* from their conception of normalization?); and (e) when a student's conception of normalization can be considered a coordination class. Note that goals (a)-(c) are focused on my first research question, and goal (d) is focused on my second research question in this study. Furthermore, note that (e) will require me to specifically look for the features of *span* (what contexts and vector spaces can they use normalization) and *alignment* (how consistent are they in their use of normalization across various contexts) in students' conceptions of normalization. In the next chapter, I explain the methods I will use to accomplish these goals.

Chapter 3: Methods

In this chapter, I explain the qualitative research methods I will use to address the research questions of this study. As a reminder, the research questions are:

1. What are the various conceptions math and physics students have about normalization from math and physics courses in which the concept is taught?
2. What changes occur in physics students' conceptions of normalization while taking a quantum mechanics course?

However, with the theoretical framework of coordination class theory explained in chapter two, I refine these questions as follows:

1. What are the *readout strategies* and *causal nets* in the various conceptions of normalization students have from math and physics courses in which the concept is taught, and what are the *span* and *alignment* properties of those conceptions?
2. What changes occur, including *incorporation* and *displacement*, in students' conceptions of normalization while taking a quantum mechanics course?

I begin by explaining the data collection methods, who will be the participants for the study, the survey instrument I will use to explore multiple students' conceptions of normalization and for selecting interview participants, and the interview protocols for conducting semi-structured interviews (Bernard, 2006) intended to explore students' conceptions of normalization. Next, I explain the methods of data analysis, including how the interviews will be transcribed and how the interview and survey data will be analyzed through thematic analysis and two-levels of coding (Miles et al., 2014). In particular, I address how the first and second levels of coding will help to (a) address the research questions for this study and (b) determine when a student can be considered to have constructed a coordination class of normalization.

3.1 Data Collection

3.1.1 Participants. To address the first research question, I need to recruit student participants from a variety of mathematics and physics classes where normalization is important. I hope to recruit students from five different classes: *Introduction to Multivariable Calculus*, *Introduction to Linear Algebra*, *Linear Algebra I*, *Mathematical Methods in Physics*, and *Introduction to Quantum Mechanics*. To recruit students from the first four courses (the non-quantum courses), I will contact the instructors of these courses during the data collection semester, meet with them in person to explain my study, and ask for their help. I will ask the instructors when they expect to teach or use normalization in their class, and I will ask for permission to attend their class(es) shortly thereafter to solicit student participation. With the instructors' permission, I will briefly describe my study to the students during the first 5 minutes of the classes I attend before I give out consent forms to all students in the class. This consent form will further explain the study and give students the opportunity to indicate their willingness to participate in one or both parts of the study. First, students may indicate a willingness to allow me to analyze their written work on a 15-20 minute survey asking questions about their personal conception of normalization (described in more detail in Section 3.1.2). Second, students may indicate a willingness to participate in a single, 30-40 minute, semi-structured, problem-solving interview during which I will further explore their conceptions of normalization (described in more detail in Section 3.1.3). I will collect these consent forms at the end of the class period.

To recruit students in the quantum mechanics class, I will attend the first class of the semester and briefly describe my study to the students during the first 5 minutes of class, giving consent forms to all students in the class. This consent form will further explain the study and give students the opportunity to indicate their willingness to participate in one or both parts of the study. First, students may indicate a willingness to allow me to analyze their written work on a 15-20 minute survey asking questions about their personal conception of normalization. This survey will be identical to the aforementioned survey that will be given to the students in the other classes. Second, students may indicate a willingness to participate in three, 30-40 minute, semi-structured, problem-solving interviews during which I will further explore their conceptions of normalization and investigate how these conceptions change as they take the quantum mechanics course. I will collect these consent forms at the end of the class period.

The survey will be given to students in all five classes in one of two ways, dependent upon the instructor. First, and most preferably, the survey will be assigned to all students in the course as an assignment (for a grade or for extra credit). All students will then have an opportunity to take the survey, but I will only analyze the responses of students who gave me permission through the consent form described previously. Preferably the survey will be assigned the same day I attend class and collect consent forms from the students. Alternatively, if the instructors do not want to give the survey as an assignment, consenting students will be given the survey during the next class period. They will be instructed to answer the questions to the best of their ability and return the survey back to me during the next class period. In all five classes, this survey will need to be collected quickly to help with the selection of interview participants. Thus, if the survey is given as an assignment or extra credit, it will be due by the next class period, similar to classes where it is simply given as a survey to complete at home.

To select three interview participants from each of the four non-quantum courses (Introduction to Multivariable Calculus, Introduction to Linear Algebra, Linear Algebra I, Math Methods in Physics), I will examine the responses on the survey of those who indicated willingness to participate in both the survey and the interview. In examining these responses, I will specifically look for a wide range of ideas and conceptions that students demonstrated, choosing interview participants so I can investigate this variety in students' conceptions about normalization. In doing so, I will also strive to choose a diverse group of student interviewees, so that different genders and races have an opportunity to be heard in the ways they conceptualize the mathematical concept of normalization.

In order to address the second research question, a larger number of students will be recruited from the quantum mechanics class for the interviews. Similarly looking for a diverse group of students who demonstrated different ideas and conceptions on the survey, I will recruit 6 students from the quantum course, asking these students to participate in three 30-40 minute interviews during the course of the semester. By interviewing these students multiple times during the course, I will be able to examine any changes that occur in their conceptions of normalization as they proceed through the course.

3.1.2 Survey Instrument. The survey consists of 10 questions (see Appendix A) designed to explore the *readout strategies* and *causal nets* within students' conceptions of normalization, as well as the *span* and *alignment* of those conceptions. The first question asks the student to normalize a vector from $\mathbb{R}^2$, the second question asks the student to normalize the vector $\begin{bmatrix} 3 \\ 3i \end{bmatrix}$ from $\mathbb{C}^2$, and the third question asks the student to normalize a function in L^2-space (although students are not told the norm for any of the three vector spaces). These three questions will give me an opportunity to see elements of students' causal nets within their conceptions of normalization. More specifically, I can see the procedures students use to find the norm, length, or magnitude of a vector and the procedures they use to normalize a vector. Furthermore, the second question has previously been used in interviews for *Project LinAl-P* (NSF-DUE 1452889), where we have observed students experiencing a perturbation in calculating a norm, magnitude, or length of zero when trying to use the procedure of taking the square root of the sum of the squares of the components of the vector. This question and possible perturbation will allow me to observe if students understand that a norm for a vector should be greater than zero if the vector is not the zero vector, another possible component of their *causal net*. Furthermore, the third question gives students the opportunity to demonstrate knowledge of norms and normalization on a function space, giving me further ability to assess the *span* of their conception of normalization.

The fourth question asks the student to explain what normalization of vectors means to them, and the fifth question asks students to give some applications for normalization and why normalization is important in those applications. These will give me insight into students' understanding of the uses of normalization and why it is important, which I hypothesized in the previous chapter as being possible elements of their *causal net*. Questions 6 and 7 ask students to define what a vector is and give several examples of vectors and/or vector spaces. This will give me another opportunity to examine their understanding of vectors and to see if they are aware of vector spaces other than $\mathbb{R}^n$, which would inform me about their vector *readout strategies* within their conceptions of normalization. Questions 8 and 9 ask students if they have any geometrical or graphical understanding of vectors and normalization and to explain this understanding. This provides another opportunity for me to investigate students' understanding of vectors, vector representations, and normalization (i.e. *readout strategies* in their normalization conception). Lastly, question 10 provides students an opportunity to explain what a unit vector is. There is always a possibility that students will not know what "normalize" or "normalization" means, and this question provides these students an opportunity to still demonstrate some understanding of the important ideas involved in the normalization of vectors, providing me a further opportunity to assess students' constructed *readout strategies* for unit vectors they might be able to *incorporate* into their conception of normalization as they construct it.

The survey will be administered to the quantum mechanics students at the very start of the semester (the first day, if possible) to facilitate the selection of the 6 interviewees within the first few days of the semester so the initial interviews can be scheduled as early as possible. For the other four classes, the survey will be administered after students have learned or used normalization within the course, as determined through discussions with the instructors for the courses. Interviewees from these classes will then be selected and interviewed shortly thereafter.

3.1.3 Interviews with students not in quantum mechanics. Interviews for this study will be semi-structured: "Semistructured, or in-depth interviewing is a scheduled activity. A semistructured interview is open ended, but follows a general script and covers a list of topics" (Bernard, 2006, p. 210). When a researcher only has one opportunity to interview a participant, semi-structured interviews are usually best, as they allow the researcher to focus the interview on the topics of most interest. For my study, I will focus the interview on exploring students' personal conceptions of the normalization of vectors, including their conceptions of norm, vector, vector representations, and the procedures they use for normalizing vectors. In these interviews, I will have the opportunity to probe students' thinking, or "to stimulate a respondent to produce more information" (Bernard, 2006, p. 217), beyond what they demonstrated on the survey. Students' elaborations that will be produced through the interviews will provide me with a more-detailed look at their conceptions, giving me a better chance at answering the first of my research questions for this study.

As mentioned earlier, 12 students will be asked to participate in these single, one-on-one interviews from the four non-quantum classes. These interviews will take place in a private and quiet location on campus that is convenient for the student. These interviews will be scheduled after students have had a chance to participate in the survey portion of this study. Interviews will be video-recorded, and the video-recordings will be stored electronically on a password protected computer. By video-recording the interviews, I will have the ability to watch and analyze the students' work and expressed thoughts multiple times; this will allow me to characterize the students' conceptions of norms, vectors, and normalization as accurately as possible.

The 30-40 minute interviews will further explore and probe students' conceptions of normalization beyond what they demonstrated on the survey described above (see Appendix B for a sample interview script). I will first ask students about the mathematics courses they have taken, what their current major is, and how long they have been studying at the university. Next, I will ask the student some questions about their understanding of absolute value and see if they can solve an absolute value problem. Pairing this information with later questions on the interview where they compare across the different problems they solved will help me determine if students are able to recognize absolute value as an example of a mathematical norm during the interview. This gives me additional opportunity to assess students' *readout strategies* and *causal nets* by exploring their ability to see real numbers as vectors in a vector space and the absolute value as an example of a norm. It will also help me further explore the *span* of their conceptions of normalization, specifically students' awareness of various vector spaces and norms.

I will then ask clarifying questions about their responses to the survey specifically about normalization. First, I will ask them to elaborate on their personal understanding of vector normalization; I will then give them a hypothetical definition of normalization (that normalizing a vector "gets rid of" the vector's magnitude) and ask the student to comment on it. Second, I will point out their answers to the three vector normalization problems, probing the students to explain why they chose the particular procedure they used to normalize the vector. A crucial part of the interview includes asking specific questions to students who had decided that they could not normalize either of the vectors in Questions 2 or 3 (the complex vector or the function). These questions will seek to have the student elaborate why they encountered difficulty in trying to normalize those vectors. Possible responses include not knowing how to work with a complex vector or a function as a vector, or obtaining a length or magnitude of zero when trying to find the norm of the vector. If it is the latter, I will ask students if they believe the vector should have a norm of zero, and have them explain. I will also ask the student to explain how normalizing the vector in Question 3 (a function) is similar to normalizing the vectors in Questions 1-2 (the real-valued and complex-valued vectors). This will further assess the *causal nets* within their conceptions of normalization, as well as *readout strategies* for recognizing the mathematical objects to be normalized as vectors.

In the next part of the interview, I will ask students about their answers to Question 9 on the survey, specifically seeking further elaboration about their geometric or graphical ideas about normalization; this will allow me to further explore their *causal net* for normalization, specifically looking for the inferential knowledge the students can call upon to calculate the magnitude of a vector. This will be followed by asking students to elaborate and clarify their answers to Questions 6-8 from the survey, seeking to elicit their personal conceptions of vectors and vector spaces; this will allow me to further explore their *readout strategies* for vectors. This will also give me an opportunity to see if there are any other vector spaces the students are aware of that they did not write down when taking the survey.

Following students' elaboration about vectors, I will ask students if they see connections among the four different problem settings ($\mathbb{R}$, $\mathbb{R}^2$, $\mathbb{C}^2$, and L^2-space) they have worked within as part of this research project. This is to see if the students have a more abstract or general understanding of mathematical norm and to assess the *span* of their conception of normalization. I will then ask the students to explain why they think normalization is important, how normalization has been used in their class, and probe their answers to Question 5 on the survey that asks the students to give examples of applications of normalization. This will be followed by asking the students if they see any connections between normalization and unit vectors. This provides an additional opportunity to explore the interconnectedness of students' *readout strategies* and *causal nets* within their conceptions of normalization.

Lastly, if time permits, I will ask students a question about normalizing a wave function. I will not be asking students to solve this problem, but rather I will ask them what pieces of information they would need to know in order to carry out the normalization. This provides another opportunity to see if students could recognize the function as an example of a vector, understand a need for a norm, and abstractly explain some procedure for normalizing the function. Putting the problem into more of a physics context additionally provides another opportunity to explore the *span* of students' conceptions of normalization.

3.1.4 Interviews with students in quantum mechanics. Interviews with the quantum mechanics students will similarly be semi-structured, one-on-one interviews (Bernard, 2006). These interviews will also take place in a quiet and private location on campus that is convenient for the student. Interviews will be video-recorded, and the video-recordings will be stored electronically on a password protected computer. Again, this provides me an opportunity to analyze the interviews multiple times, as I seek to characterize students' conceptions of vector normalization.

The first interview will be identical to the interview with the other student participants in this study described in Section 3.1.2 (see Appendix B). This will help me establish the conceptions of normalization the students initially have as they enter the quantum mechanics course; this “base-line” conception will be critical for examining the changes that occur in these students’ conceptions as they learn about quantum mechanics.

The second and third interviews will engage students in further problem solving, during which they will be asked to normalize vectors from different vector spaces. In particular, I will choose normalization problems that are situated within a quantum mechanics context, and normalization problems that are situated in a purely mathematical context (i.e., problems typical within a mathematics course), to explore how their conception of normalization might be changing as they take the course. More specifically, (a) what changes occur in their *readout strategies* and *causal nets*, and (b) what knowledge seems to have been *incorporated into* or *displaced from* their normalization conception. These questions will also give me an opportunity to investigate possible moments of conceptual change within the interviews themselves, specifically looking for moments during which the student seems to be *incorporating* previously constructed knowledge into their conception of normalization or *displacing* elements from their conception of normalization when those elements prove unhelpful in a particular situation or context. Questions will include normalizing different wave functions, normalizing complex-valued vectors, and physics problems in which normalization is a step in the process of finding a solution. In addition to these problem solving tasks, I will ask students further questions about their conception of normalization, such as asking why normalization is important in the problem they are solving, or more generally, why normalization is important in their quantum mechanics course; this is an additional investigation of the *causal net* of students' conceptions of normalization.

An important part of these second and third interviews will be questions that ask students how their current conceptualization of normalization in the quantum mechanics course relates to their initial conceptions that they demonstrated at the beginning of the course. In particular, I will be asking students explicitly how instances of normalization within the quantum mechanics course relates to the normalization of vectors they were familiar with at the beginning of the course. This will provide me opportunities to look for instances where students see the normalization in quantum mechanics as separate or distinct from the normalization of vectors they have seen previously, versus instances where students are able to see (i.e., coordinate) normalization in different contexts as instantiations of the same concept. If students are able to consistently apply normalization ideas across a variety of contexts, this will be evidence of *span* and *alignment*, indicating that the student has possibly constructed a coordination class for normalization.

Because the material students see in class and the problems they are assigned in homework will inform these interviews, full interview protocols for these interviews will be developed during the course of the study. In Appendix C, I do provide some examples of the problems and questions I intend to ask within these interviews. In section 3.1.4, I describe the observation of the class, which will help to inform the development of these interview protocols.

The first interview will take place as early as possible in the course, hopefully within the first week. The second interview will be scheduled with the students after they have seen normalization within the course and have had a chance to solve a few problems where normalization is involved, either in class or on the homework. The third interview will be scheduled later in the course, after students have seen a fair amount of the material and have used normalization multiple times. Ideally, this third interview will occur after the students have seen quantum spin (Griffiths, 2005, pp. 171–190), because quantum spin presents a nice opportunity for students to see normalization of complex vectors within a physical context.

3.1.5 Quantum mechanics class observation. I will attend all of the quantum mechanics classes to observe the material students are learning within the course. I will keep detailed field notes of the content taught, making special note about any instances where normalization was used in class and how it was presented. These data will be used to inform the types of questions I ask in the interviews with the quantum students. The field notes will also help to contextualize students' phrases, procedures, or ways of conceptualizing normalization within the broader context of their quantum mechanics learning.

3.1.6 Written work in the quantum class. If the instructor and students' give their permission, I will make copies of any written assignments students turn in in which normalization is used and is integral to solving the problems. This additional data source will provide an opportunity for triangulation, supplying an additional setting where students could demonstrate their knowledge or understanding of normalization. This is important, as some students may not be comfortable with the survey or interview, and may do their best work on the homework or in-class assignments. Collecting this data will help in the categorization of students' conceptions of normalization, providing a check to the analyses that will be done on the survey and interview data.

3.2 Data Analysis

To analyze the data for this study, I will be using a thematic analysis (Braun & Clarke, 2006). Thematic analysis can be applied across a range of theoretical and epistemological perspectives, including the constructivist epistemology and coordination class theoretical framing I use within this study. “Thematic analysis is a method for identifying, analysing and reporting patterns (themes) within data. It minimally organizes and describes your data set in (rich) detail” (Braun & Clarke, 2006, p. 79). More specifically, this thematic analysis is a theoretical thematic analysis that will “be driven by the researcher’s theoretical or analytic interest in the area, and is thus more explicitly analyst driven” (p. 84). As I have already developed some sense of the important components of normalization conceptions (see Section 2.4), I will specifically be looking for the various ways students think about and conceptualize vectors, vector representations, mathematical norms, and normalization. In other words, I hope to be able to explore the *readout strategies* and *causal nets* of students’ conceptions of normalization, and investigate how students’ conceptions of normalization change as they learn quantum mechanics through processes of *incorporation* and *displacement*.

Thematic analyses generally consist of six steps (Braun & Clarke, 2006). First, I will familiarize myself with the data by transcribing all of the interviews, reading through all of the surveys, written work, and interview transcripts, and writing down initial ideas in analytic memos (Ely, Vinz, Downing, & Anzul, 1997). Second, I will generate an initial set of codes aimed to capture and categorize the *readout strategies* and *causal nets* within students’ conceptions of normalization.

> *Codes* are labels that assign symbolic meaning to the descriptive or inferential information compiled during a study. Codes usually are attached to data ‘chunks’ of varying size and can take the form of a straightforward, descriptive label or a more complex one (e.g., a metaphor). (Miles et al., 2014, pp. 71–72).

As such, these codes will label and describe the *readout strategies* and inferential knowledge in the *causal nets* that students seem to employ from their conceptions of normalization as they answer normalization questions. The codes will be attached to sections of the interview transcripts and written work (the survey and any other assignments collected) as one of the first steps in organizing and analyzing the data for this study.

The third step in my thematic analysis will consist of a second level of coding (Miles et al., 2014) that will consist of four different sets of thematic codes. The first set of second level codes will look to find themes in the data of particular elements of students' conceptions that are productive or powerful for making sense of normalization and solving problems that involve normalization. The second set of second level codes will look for patterns and differences across students' conceptions of normalization. The third set of second level codes will label instances when the *span* and *alignment* of students' conceptions of normalization is evident within the data. Lastly, the fourth set of second level codes will help me identify changes that seem to have occurred in students' conceptions of normalization, specifically looking for the *incorporation* of new ideas (e.g., *readout strategies*) or *displacement* of unproductive ideas in their conception of normalization. The first three sets of second level codes will help to address my first research question and will help with delineating when a person can be said to have constructed a coordination class for normalization. The fourth set of second level codes will focus on addressing my second research question, examining the changes that occur to students' conceptions of normalization as they take quantum mechanics.

Fourth, I will review the themes generated in the third step of the analysis and carefully check that instances of *readout strategies*, *causal nets*, *incorporation*, *displacement*, *span*, *alignment*, and moments when I consider a student to have constructed a coordination class of normalization have all been identified. This will involve making sure that the data for each student has been thoroughly coded and analyzed, and that codes given to students with similar thoughts and procedures are consistent. Additionally, the themes of productive elements in students' conceptions will also be reviewed across the data, specifically looking for productive ideas that students had that the themes might be missing. This will lead to the fifth step in the analysis, during which I will name the themes found, including commonalities across students' conceptions of normalization, the productive conceptions students demonstrated, and how students who have constructed a coordination class for normalization can be identified.

The final step of the thematic analysis will be a write up of the results. In writing up the results for both research questions, it is important to note that further analysis can and will occur as I engage in the writing process. As Ely et al. (1997) note, writing is a messy and chaotic process. As researchers try to generate and form their ideas into a coherent presentation, they often find that further and deeper ideas emerge and develop. Hence, I fully expect that the writing up of results may lead to further insights and knowledge about the variety of students' conceptions of normalization and the conceptual changes that occur in physics students' *readout strategies* and *causal nets* for normalization as they learn quantum mechanics. These additional insights might lead to a reshaping and reworking of the results through further data analysis, or they may lead to important topics and ideas that will be explored in the discussion chapter of this study.

3.3 Pilot Study – Example of the Coordination Class Analysis

During the Fall 2016 semester, the lead investigator of Project LinAl-P (NSF-DUE 1452889), Megan Wawro, conducted 1-hour long, one-on-one, semi-structured interviews with physics students at a medium public research university in the northeastern United States. These students were taking a senior-level, first semester quantum mechanics course. The interviews were focused on exploring how these students thought about and reasoned with linear algebra concepts in the quantum mechanics course. Some of the concepts that were explored through the interview questions include eigentheory, complex numbers, change of basis, and normalization of vectors. Eight students were interviewed at the beginning of the course, and nine were interviewed at the end of the course. This made for a total of ten students participating in at least one interview, with seven participating in both. Interviews were video-recorded and student work scanned, allowing for repeated examination and analysis of student work and answers to the interview questions.

As part of this interview was specifically focused on examining these students' understanding of normalization and its use in quantum mechanics (see Appendix D for the related questions), I now use one student and his pre-quantum interview from this data set to illustrate the analysis I will do with coordination class theory to explore students' conceptions of normalization. It should be noted that the normalization questions in these interviews only took around 10-15 minutes of the hour-long interview. As such, these interviews may not be as revealing or informative as the data I intend to collect for this study, but they still provide a good opportunity to demonstrate the analysis that is made possible by using coordination class theory.

Brett was a fourth-year, double-major in mathematics and physics who had taken an introductory linear algebra class about one year before taking quantum mechanics, as well as a math methods course offered through the Physics department. He was quite interested in how differential equations operated within physics contexts and was planning on going to graduate school for applied mathematics after graduating with his Bachelor's degree. He mentioned linear algebra as a weak point in his mathematical knowledge, explaining that it might be somewhat difficult to recall all of the topics in linear algebra, such as the Gram-Schmidt process. However, with some review of his notes, he thinks he can remember most of what he learned.

In the Pre-quantum interview, when Brett was given the initial normalization question ("Normalize the following vector: $\vec{v} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$."), he seemed to have *readout strategies* for recognizing $\vec{v}$ as a vector, talking about the vector as if he were familiar with real vectors. As he began to think about what he would need to do in order to normalize the vector, there seemed to be two *causal nets* that were activated in his cognitive structures:

> *Brett*: OK, so, if I were to normalize the vector… Wow, I'm drawing a blank. [Pause]. So, I have two definitions of normalize in my head right now. It's either … just … [pause]. Normalizing is either, I would assume, to make it into the unit vector, or something like that. And so… [laughs]. That's where I'm at right now.
>
> *Interviewer*: OK. OK. Let's see. What was, or, was there a competing one as well, or is that the one you're gonna go with?
>
> *Brett*: Well, normal, I think of, if something is normal to something, then, like, I think of orthogonality in a way, but I'm pretty sure that's not what this is.

The first *causal net* is a set of inferential knowledge and calculations that would help him make the given vector into a unit vector. The second *causal net* is a set of inferential knowledge and calculations that would lead to finding a vector normal/orthogonal to the given vector. Ultimately Brett decided the latter was not correct, and began an attempt to make the vector a unit vector. He explained that he would need to multiply the vector by some scalar, but struggled for a while to find that scalar. After fumbling around for a couple of minutes, Brett suddenly recalled important inferential knowledge from his *causal net* for normalization. This inferential knowledge included how to find the magnitude of the vector by squaring both components and taking the square root, and how to find the normalized vector by immediately multiplying the vector by the reciprocal of that magnitude (see Figure 3.1). Brett went on to explain how he thought of the 5 and 2 in the

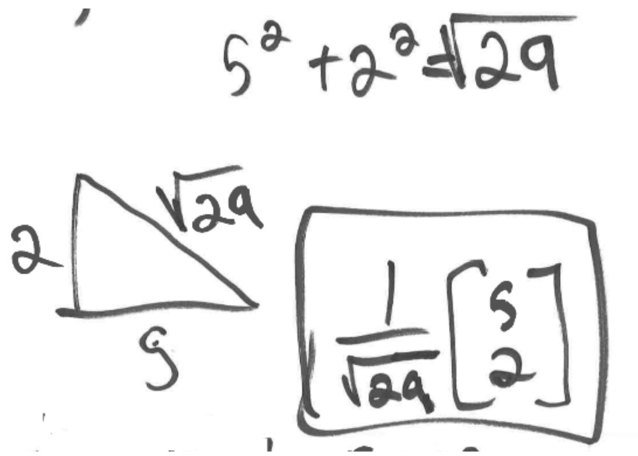

Figure 3.1: Brett's Normalization of a Real Vector

vector as directions, with 5 being the x-direction and 2 being the y-direction. He then proceeded to draw the labeled triangle shown in Figure 3.1, explaining how the hypotenuse can be found using the Pythagorean Theorem. This gives evidence that Brett had *readout strategies* for transforming the column vector into a graphical or geometrical representation; he further had the Pythagorean Theorem as a knowledge element in his *causal net* for normalization, specifically as a way to find the magnitude of the vector. However, it was not clear if Brett had completely constructed the inferential knowledge within his causal net that the square root of the sum of the squares of the components gives you the magnitude of the vector, never clearly calling this the magnitude or norm of the vector.

After normalizing the vector, Brett demonstrated that he had *readout strategies* for recognizing the original vector and the normalized vector as two different vectors, explaining that the normalized vector could not be labeled as $\vec{v}$ because it was a different vector than $\vec{v}$. The interviewer then explicitly asked Brett what it means to normalize:

Brett: Um, when you normalize a vector, you divide by, or multiply one over the square root of the dimensions squared—dimensions again might not be the correct term. But … what’s ever inside your vector, square all the terms, put it under a square root, put, like, a one over it…

Interviewer: OK.

Brett: And, the question is…

Interviewer: Any idea why that’s something you do?

Brett went on to give another way to normalize the vector (see Figure 3.2), and then goes on to say

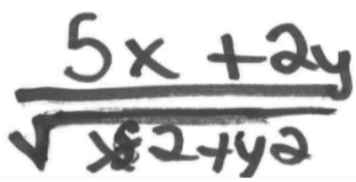

Figure 3.2: Brett’s Additional Calculation for Normalizing

the following:

Brett: I’m pretty sure it’s supposed to put it into its simplest form. Um, not its unit vector, ‘cause, this [pointing to the normalized vector] would have to come out to one. But—which is what I was saying before.

Here is further evidence that Brett’s *causal net* for normalization was not coherently constructed; this is because at first he thought normalizing would produce a unit vector, but later he seemed to think that normalizing the vector would put it into its “simplest form”. Furthermore, there is some evidence here that Brett did not have *readout strategies* for “seeing” or coordinating unit vectors as having a magnitude of one, but rather he seemed to think that a unit vector must “come out to one.” The interviewer then pressed further on this idea:

Interviewer: But you don't think that's, that's resulting in a vector whose magnitude is one?

Brett: Well I hope – you just have to – I don't know. Depends on your values for x.

After thinking out loud for thirty seconds more, Brett eventually said that maybe this process does make the vector's magnitude one, but explained that he did not really "see" that in the normalized vector. In fact, he thought this process would result in a vector whose components are $\frac{5}{\sqrt{29}}$ and $\frac{2}{\sqrt{29}}$, and said this "isn't one." Again, here is evidence that his *causal net* for normalizing a vector did not have a clear connection to producing a unit vector, and his *readout strategies* for unit vectors seemed to focus on the vector "being one" rather than having a magnitude of one.

The interviewer then gave Brett the complex vector $\vec{w} = \begin{bmatrix} 3 \\ 3i \end{bmatrix}$ to normalize. Brett immediately explained he would stick with the same process as he used previously, and began writing the work in Figure 3.3. He immediately recognized he would run into a problem in using

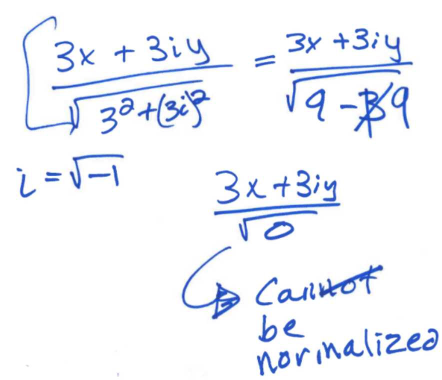

Figure 3.3: Brett's Initial Attempt to Normalize a Complex Vector

this process:

Brett: That's not going to be good. 'Cause I'm gonna get a square root of zero? If I'm not mistaken. [Begins writing the fraction after the equal sign in the first line of Figure 3.3] Nine, i-squared … So i is the square root of negative one, so i-squared is negative one

…

[After writing the fraction in the second line of Figure 3.3] Which is just bad news bears. Um, [mumbles while checking calculations]. Uh, yeah, so I would say it cannot be normalized, given that information.

This is further evidence that Brett's *causal net* for normalization was mainly made up of a rote procedure, namely dividing the vector by the square root of the sum of the components squared. He then goes on to hypothesize that this vector is in its "simplest" form, and cannot be reduced any further, providing additional evidence that Brett's *causal net* includes the idea that normalization helps to get a vector into its simplest form.

At this point in the interview, the interviewer asks Brett how confident he is in this procedure and his answer to normalizing the complex vector. Brett then provides an interesting response:

Brett: Um, I feel like it's right there. Um, like I just know that's what it is. Like, I know I've seen it before, and now that I've, like, worked through the problem more, I just remember, like I said, those examples where it would be, the vector would be [writes out the vector $\begin{bmatrix}1\\1\end{bmatrix}$] one, one, and to normalize that vector, you'd have to multiply by one over root two [writes $\frac{1}{\sqrt{2}}$ in front of $\begin{bmatrix}1\\1\end{bmatrix}$], because the square root of those two summed together squared is square root of two. So, that, that's in my head from like, when I hear, normalize the vector, and, like, you know, we did have those basis vectors, but this is definitely not making a unit vector, 'cause this [circles the $\begin{bmatrix}1\\1\end{bmatrix}$ vector] is a unit vector.

Interviewer: It *is* a unit vector? One-one?

Brett: Well, um, it's got a magnitude of wo—so it doesn't! Which is why you have to divide by the magnitude in order to make it one. So that makes sense.

In this exchange, there is a moment during which the previous idea that normalizing a vector does not produce a unit vector (part of his *causal net*) began to be *displaced* from Brett's conception of normalization. Brett then went on to explain how the vector $\begin{bmatrix}1\\1\end{bmatrix}$ actually has a magnitude of $\sqrt{2}$ by drawing a triangle and using the Pythagorean Theorem:

Brett: The overall magnitude would be the square root of two. Looking at the triangle again [see picture Brett draws in Figure 3.4]. So, this [pointing to the square root of 2] is its magnitude. So, in order to normalize the vector, you have to divide by its magnitude.

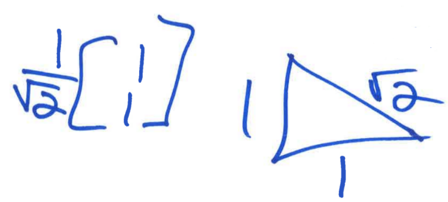

Figure 3.4: Brett explaining the magnitude of $\begin{bmatrix}1\\1\end{bmatrix}$.

Here it seemed Brett began to better *incorporate* his knowledge of the magnitude of vectors into his *causal net* for normalization, and recognized the need to divide a vector by its magnitude. With this incorporation, it seemed Brett was moving closer to the idea that normalization does result in a unit vector.

The next part of the interview provided even further opportunities for Brett to change and refine his conception of normalization when the interviewer asked Brett if he had a geometric

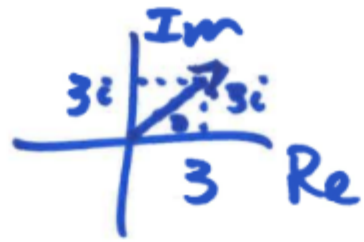

Figure 3.5: Brett's Graphical Representation of $\vec{w} = \begin{bmatrix} 3 \\ 3i \end{bmatrix}$

way to think about the complex vector $\vec{w}$. Brett drew the picture in Figure 3.5 (note that this drawing is actually more of a representation of the complex number $3 + 3i$, but the drawing was still a powerful tool for Brett in his thinking), leading the interviewer to ask the following:

> *Interviewer*: What about the notion of length? You had mentioned that before. Did you think about length at all when you were thinking about this w?
>
> *Brett*: Um, I mean, yeah. I mean, it's, it's similar I guess. The length of this vector [pointing to the vector drawn in Figure 3.5] would be that right there. Um, and I know it's, you're summing it the same, it's just something of three magnitude in the imaginary direction, and then three, uh…
>
> *Interviewer*: OK, does that show up in that calculation you did before?
>
> *Brett*: Um, yeah, it's … I think we're dividing by the length is what we're doing. It's, it's one over the length of the vector, so, which in this case is apparently zero. Or the square root of zero, which doesn't really make sense. So…
>
> *Interviewer*: So, do you trust your picture more, or your calculation more? Or do you think they can both be right?
>
> *Brett*: I'm trusting my picture more now! So… I think it'd just be three and three. The question is, do you square the absolute values of these things, and so the i goes away. So you'd want…now that I've seen the picture, um, I know that it does have some length,

which would be three squared plus three squared which is 18, so it would be square root of 18.

After debating a bit further over whether the length of $\vec{w}$ should be zero or $\sqrt{18}$, Brett eventually concluded that it should be $\sqrt{18}$, and further explained that the calculation to find the length should actually be the square root of the sum of the squares of the magnitudes of the components. With some prompting from the interviewer, Brett then pulled out a new sheet of paper, and redid the normalization problem, shown in Figure 3.6. When asked further about what

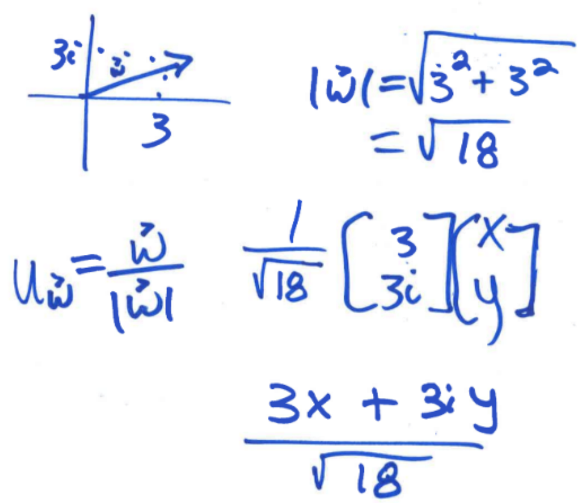

Figure 3.6: Brett's Second Attempt to Normalize a Complex Vector

changed his mind from his first answer that $\vec{w}$ cannot be normalized to his second answer, Brett said the following:

Brett: Um, drawing a picture helped me out to see that the vector clearly has a length. And then what I did [previously] just didn't make sense [laughs]. So, that's really just it.

In this excerpt the beginnings of a construction and incorporation of important inferential knowledge within Brett's *causal net* for normalization can be seen. More specifically that nonzero vectors always have a length or magnitude greater than zero, that is, they must have some "length".

While this interview only gave Brett two vectors to normalize, there are still some changes in the *span* and *alignment* of Brett's conception of normalization evident during this problem solving session. At first, Brett's conception of normalization did not *align* well over the two different vectors, leading Brett to conclude that the complex vector could not be normalized. This further indicated that the *span* of Brett's conception of normalization may have been limited to real vectors. However, through using his *readout strategies* for translating vectors into geometric representations, Brett was able to modify his *causal net* for normalization, concluding that the magnitude of a complex vector should be found by taking the square root of the sum of the squares of the magnitudes (or "absolute values") of the components. With this modification to his *causal net*, Brett increased the possibility of his conception of normalization having a greater amount of *alignment*, as this could also apply in the real vector case. The *span* of his conception of normalization likewise increased as he gained important successful experience with normalizing complex vectors.

This is not to say Brett had constructed a coordination class for normalization at this point. For instance, we did not assess his knowledge of normalization of functions, and he may have struggled to see how something like a function could be normalized. Furthermore, complex vectors such as $\begin{bmatrix} 3+2i \\ 4-6i \end{bmatrix}$ may have presented a new challenge, as these vectors cannot easily be thought about geometrically. Hence, the *span* of Brett's conception of normalization was probably not sufficient enough to be considered a coordination class at this point. Additionally, more time would probably be needed beyond this interview for Brett to solidify in his conception that normalization produces a unit vector in the same direction as the original vector. Understanding normalized vectors as unit vectors is an extremely important piece of knowledge to have in the *causal net* for normalization, particularly for understanding the important uses of normalization in various contexts, and it was not clear that he had constructed this understanding.

Postscript

Written by Megan Wawro, PhD

Kevin Watson's dissertation advisor

Kevin began collecting data for his dissertation, as outlined in these chapters, in the spring semester of 2019. Towards the end of the semester, Kevin fell ill. After more than a year of fighting an aggressive brain cancer, Kevin passed away peacefully on July 9, 2020.

The data collected during Spring 2019 consisted of nearly 200 written surveys, as detailed in Appendix A. The surveys were completed by students from all five of Kevin's target classes. He also conducted twelve individual interviews with six unique students: five students from Quantum Mechanics (two participated in three interviews, two in two interviews, and one in one interview). One student from Mathematical Methods participated in one interview. Kevin was able to begin his initial analyses of this data set. I anticipate that trusted colleagues will carry the mantle of continuing to analyze Kevin's data, striving to do it justice according to Kevin's vision and passion for his work.

Kevin was a valued member of the Virginia Tech community. His kindness, positive energy, and curiosity will be sorely missed. Kevin was from Springville, Utah. He was an active member of the Church of Jesus Christ of Latter Day Saints, and he loved his faith and family. He is survived by his wife Jennifer and his children Abby, Christian, Nicholas, and Ethan.

References

Arnon, I., Cottrill, J., Dubinsky, E., Oktaç, A., Fuentes, S. R., Trigueros, M., & Weller, K. (2014). *APOS Theory: A Framework for Research and Curriculum Development in Mathematics Education*. New York, NY: Springer.

Aydin, S. (2014). Using example generation to explore students' understanding of the concepts of linear dependence/independence in linear algebra. *International Journal of Mathematical Education in Science and Technology*. doi:10.1080/0020739X.2013.877606

Barniol, P., & Zavala, G. (2011). Students' difficulties with unit vectors and scalar multiplication of a vector. In N. S. Rebello, P. V. Engelhardt, & C. Singh (Eds.), *2011 PERC Proceedings* (pp. 115–118). Omaha, NE.

Barniol, P., & Zavala, G. (2014). Test of understanding of vectors: A reliable multiple-choice vector concept test. *Physical Review Special Topics – Physics Education Research, 10*(010121), 1–14.

Berger, P., & Luckmann, T. (1967). *The social construction of reality: A treatise in the sociology of knowledge*. New York: Anchor.

Bernard, H. R. (2006). *Research methods in cultural anthropology* (4th ed.). Lanham, MD: AltaMira Press.

Blumer, H. (1969). *Symbolic interactionism: Perspective and method*. Berkeley, CA: University of California Press.

Braun, V., & Clarke, V. (2006). Using thematic analysis in psychology. *Qualitative Research in Psychology, 3*, 77–101.

Carlson, D. (1993). Teaching linear algebra: Must the fog always roll in? *The College Mathematics Journal, 24*(1), 29–40.

Christensen, W. M., & Thompson, J. R. (2012). Investigating graphical representations of slope and derivative without a physics context. *Physical Review Special Topics - Physics Education Research, 8*(023101).

Cobb, P., & Yackel, E. (1996). Constructivist, emergent, and sociocultural perspectives in the context of developmental research. *Educational Psychologist, 31*(3-4), 175-190.

diSessa, A. (1996). What do 'just plain folk' know about physics? In D. R. Olson & N. Torrance (Eds.), *The Handbook of Education and Human Development: New Models of Learning, Teaching, and Schooling* (pp. 709–730). Oxford: Blackwell.

diSessa, A., & Sherin, B. (1998). What changes in conceptual change? *International Journal of Science Education, 20*(10), 1155–1191.

diSessa, A., & Wagner, J. F. (2005). What coordination has to say about transfer. In J. P. Mestre (Ed.), *Transfer of learning from a modern multidisciplinary perspective* (pp. 121–154). Greenwich, CT: Information Age Publishing.

Dogan, H., Carrizales, R., & Beaven, P. (2011). Metonymy and object formation: Vector space theory. In B. Ubuz (Ed.), *Proceedings of the 35th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 265–272). Ankara, Turkey: PME.

Dolan, E. L., Elliott, S. L., Henderson, C., Curran-Everett, D., St. John, K., & Ortiz, P. A. (2018). Evaluating Discipline-Based Education Research for promotion and tenure. *Innovative Higher Education, 43*(1), 31–39.

Doolittle, P. E., & Hicks, D. (2003). Constructivism as a theoretical foundation for the use of technology in social studies. *Theory and Research in Social Education, 31*(1), 72–104.

Dreyfus, B. W., Elby, A., Gupta, A., & Sohr, E. R. (2017). Mathematical sense-making in quantum mechanics: An initial peek. *Physical Review Physics Education Research, 13*(2), 020141. doi:10.1103/PhysRevPhysEducRes.13.020141

Dym, H. (2013). *Linear algebra in action* (2nd ed.). Providence, RI: American Mathematical Society.

Ely, M., Vinz, R., Downing, M., & Anzul, M. (1997). *On writing qualitative research: Living by words*. Bristol, PA: The Falmer Press.

Emigh, P. J., Passante, G., & Shaffer, P. S. (2015). Student understanding of time dependence in quantum mechanics. *Physical Review Special Topics - Physics Education Research, 11*(020112).

Gire, E., & Price, E. (2015). Structural features of algebraic quantum notations. *Physical Review Special Topics – Physics Education Research, 11*(2), 1–11.

Gire, E., Wangberg, A., & Wangberg, R. (2017). *Multiple tools for visualizing equipotential surfaces: Optimizing for instructional goals.* Paper presented at the Physics Education Research Conference 2017, Cincinnati, OH.

Gravemeijer, K. (1999). How emergent models may foster the constitution of formal mathematics. *Mathematical Thinking and Learning, 1*(155–177).

Griffiths, D. J. (2005). *Introduction to quantum mechanics*. Upper Saddle River, NJ: Pearson Education, Inc.

Gueudet-Chartier, G. (2006). Using geometry to teach and learn linear algebra. *Research in Collegiate Mathematics Education, Conference Board of the Mathematical Sciences, Issues in Mathematics Education, VI*, 171–195.

Hammer, D. (2000). Student resources for learning introductory physics. *American Journal of Physics, 68*(7), S52–S59.

Harel, G. (2000). Three principles of learning and teaching mathematics. In J.-L. Dorier (Ed.), *On the teaching of linear algebra*. Dordrecht, The Netherlands: Kluwer Academic Publishers.

Hayfa, N. (2006). Impact of language on conceptualization of the vector. *For the Learning of Mathematics, 26*(2), 36–40.

Hillebrand-Viljoen, C., & Wheaton, S. (2018). *How students apply linear algebra to quantum mechanics*. Poster presented at the 2018 Physics Education Research Conference (PERC), Washington, D.C. http://www.per-central.org/perc/2018/detail.cfm?ID=7154

Hillel, J. (2000). Modes of description and the problem of representation in linear algebra. In J.-L. Dorier (Ed.), *On the teaching of linear algebra* (pp. 191–207). Dordrecht, Netherlands: Kluwer.

Jones, S. R., & Watson, K. L. (2018). Recommendations for a "target understanding" of the derivative concept for first-semester calculus teaching and learning. *International Journal of Research in Undergraduate Mathematics Education, 4*(2), 199–227.

Levrini, O., & diSessa, A. (2008). How students learn from multiple contexts and definitions: Proper time as a coordination class. *Physical Review Special Topics - Physics Education Research, 4*(010107), 1–18.

Manogue, C. A., Siemens, P., Tate, J., Browne, K., Niess, M., & Wolfer, A. (2001). Paradigms in Physics: A new upper-division curriculum. *American Journal of Physics, 69*(9), 978–990.

Maracci, M. (2006). On students' conceptions in vector space theory. In J. Novotná, H. Maraová, M. Krátká, & N. Stehlíková (Eds.), *Proceedings of the 30th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 129–136). Prague: PME.

Maracci, M. (2008). Combining different theoretical perspectives for analyzing students' difficulties in vector spaces theory. *ZDM Mathematics Education, 40*, 265–276.

Marshman, E., & Singh, C. (2015). Framework for understanding the patterns of student difficulties in quantum mechanics. *Physical Review Special Topics - Physics Education Research, 11*(020119).

Martínez-Planell, R., Gaisman, M. T., & McGee, D. (2015). On students' understanding of the differential calculus of functions of two variables. *Journal of Mathematical Behavior, 38*, 57–86.

Martínez-Planell, R., Gaisman, M. T., & McGee, D. (2017). Students' understanding of the relation between tangent plane and directional derivatives of functions of two variables. *Journal of Mathematical Behavior, 46*, 13–41.

McIntyre, D. (2012). *Quantum Mechanics: A Paradigms Approach* (1st ed.). Chicago, IL: Addison-Wesley Longman.

Miles, M. B., Huberman, A. M., & Saldaña, J. (2014). Fundamentals of qualitative data analysis *Qualitative Data Analysis: A Methods Sourcebook* (3rd ed.). Thousand Oaks, CA: Sage Publications Inc.

National Research Council. (2012). *Discipline-Based Education Research: Understanding and Improving Learning in Undergraduate Science and Engineering* (S. R. Singer, N. R. Nielsen, & H. A. Schweingruber Eds.). Washington, DC: The National Academies Press.

Norton, A. H. (2009). Re-solving the learning paradox: Epistemological and ontological questions for radical constructivists. *For the Learning of Mathematics, 29*(2–7).

Parraguez, M., & Oktaç, A. (2010). Construction of the vector space concept from the viewpoint of APOS theory. *Linear Algebra and its Applications, 432*, 2112–2124.

Passante, G., Emigh, P. J., & Shaffer, P. S. (2015). Examining student ideas about energy measurements on quantum states across undergraduate and graduate levels. *Physical Review Special Topics - Physics Education Research, 11*(020111).

Piaget, J. (1952). *The origins of intelligence in the child* (M. Cook, Trans.). New York: W W Norton & Co.

Piaget, J. (1954). *The construction of reality in the child* (M. Cook, Trans.). New York: Basic Books.

Piaget, J. (1970). *Structuralism* (C. Maschler, Trans.). New York: Basic Books, Inc.

Piaget, J. (1977/2001). *Studies in reflecting abstraction* (R. L. Campbell, Trans.). New York, NY: Taylor and Francis Group.

Presmeg, N. C. (1998). Metaphoric and metonymic signification in mathematics. *Journal of Mathematical Behavior, 17*, 25–32.

Reed, Z. (2018). *Undergraduate students' generalizing activity in real analysis: Constructing a general metric*. (Unpublished doctoral dissertation). Oregon State University. Corvallis, OR.

Roundy, D., Weber, E., Dray, T., Bajracharya, R. R., Dorko, A., Smith, E. M., & Manogue, C. A. (2015). Experts' understanding of partial derivatives using the partial derivative machine. *Physical Review Special Topics - Physics Education Research, 11*(020126).

Sabella, M. S., & Redish, E. F. (2007). Knowledge organization and activation in physics problem solving. *American Journal of Physics, 75*(11), 1017–1029.

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics, 22*(1), 1–36.

Sherin, B. (2001). How students understand physics equations. *Cognition and Instruction, 19*(4), 479–541.

Singh, C., & Marshman, E. (2015). Review of student difficulties in upper-level quantum mechanics. *Physical Review Special Topics - Physics Education Research, 11*(020117). doi:10.1103/PhysRevSTPER.11.020117

Smith, J. P., III, diSessa, A., & Roschelle, J. (1993). Misconceptions reconceived: A constructivist analysis of knowledge in transition. *The Journal of the Learning Sciences, 3*(2), 115–163.

Steffe, L. P. (1991a). The Constructivist Teaching Experiment: Illustrations and Implications. In E. Glasersfeld (Ed.), *Radical Constructivism in Mathematics Education* (pp. 177-194). Dordrecht: Springer Netherlands.

Steffe, L. P. (1991b). The Learning Paradox: A Plausible Counterexample. In L. P. Steffe (Ed.), *Epistemological Foundations of Mathematical Experience* (pp. 26-44). New York, NY: Springer New York.

Tall, D. O., & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics, 12*, 151-169.

Thaden-Koch, T. C., Dufresne, R. J., & Mestre, J. P. (2006). Coordination of knowledge in judging animated motion. *Physical Review Special Topics - Physics Education Research, 2*(020107), 1–11.

Thompson, P. W., Carlson, M. P., Byerley, C., & Hatfield, N. (2014). Schemes for thinking with magnitudes: An hypothesis about foundational reasoning abilities in algebra. In K. C. Moore, L. P. Steffe, & L. L. Hatfield (Eds.), *Epistemic algebra students: Emerging models of students' algebraic knowing. WISDOMe Monographs* (Vol. 4, pp. 1–24). Laramie, WY: University of Wyoming.

Tuminaro, J., & Redish, E. F. (2007). Elements of a cognitive model of physics problem solving: Epistemic games. *Physical Review Special Topics - Physics Education Research, 3*(020101).

Vega, M., Christensen, W., Farlow, B., Passante, G., & Loverude, M. (2016). Student understanding of unit vectors and coordinate systems beyond cartesian coordinates in upper division physics courses. In D. L. Jones, L. Ding, & A. Traxler (Eds.), *2016 PERC Proceedings*. Sacramento, CA.

Vinner, S., & Dreyfus, T. (1989). Images and definitions for the concept of function. *Journal for Research in Mathematics Education, 20*(4), 356-366.

von Glasersfeld, E. (1984). An introduction to radical constructivism. In P. Watzlawick (Ed.), *The invented reality* (pp. 17–40). New York: Norton.

von Glasersfeld, E. (1995). *Radical constructivism: A way of knowing and learning*. London: Falmer Press.

von Glasersfeld, E. (2001). The radical constructivist view of science. *Foundations of Science, 6*(1), 31–43.

Vygotsky, L. S. (1978). Interaction between learning and development. In M. Cole, V. John-Steiner, S. Scribner, E. Souberman, & L. S. Vygotsky (Eds.), *Mind in society: The development of higher psychological processes* (pp. 79–91). Cambridge, MA: Harvard University Press.

Vygotsky, L. S. (1986/1934). *Thought and Language* (A. Koulin, Trans.). Boston, MA: MIT Press.

Wangberg, A., & Johnson, B. (2013). Discovering calculus on the surface. *PRIMUS: Problems, Resources, and Issues in Mathematics Undergraduate Studies, 23*(7), 627–639.

Watson, K. L. (2017a). *Framework for students' understanding of mathematical norms and normalization*. Paper presented at the Transforming Research in Undergraduate STEM Education (TRUSE) Conference, St. Paul, MN.

Watson, K. L. (2017b). *Quantum mechanics students' understanding of mathematical norms and normalization*. Paper presented at the Physics Education Research Conference, Cincinnati, OH.

Watson, K. L. (2018). *Framework for students' understanding of mathematical norms and normalization*. Paper presented at the 21st Annual Conference on Research on Undergraduate Mathematics Education, San Diego, CA.

Wawro, M., Rasmussen, C., Zandieh, M., & Larson, C. (2012). An inquiry-oriented approach to span and linear independence: The case of the magic carpet ride sequence. *PRIMUS: Problems, Resources, and Issues in Mathematics Undergraduate Studies, 22*(8), 577-599.

Wawro, M., Watson, K. L., & Christensen, W. (2018a). *Quantum physics students' reasoning about eigenvectors and eigenvalues*. Poster presented at the 21st annual Conference on Research in Undergraduate Mathematics Education, San Diego, CA. http://sigmaa.maa.org/rume/crume2018/Abstracts_Files/Submissions/292_Quantum_Physics_Students__Reasoning_about_Eigenvectors_and_Eigenvalues.pdf

Wawro, M., Watson, K. L., & Christensen, W. (2018b). *Student reasoning about eigenvectors and eigenvalues from a resources perspective*. Poster presented at the 2018 Physics Education Research Conference (PERC), Washington, D.C.

Wawro, M., Zandieh, M., & Rasmussen, C. (2013). *Inquiry oriented linear algebra: Course materials*. Available at http://iola.math.vt.edu. This work is licensed under a Creative Commons Attribution-NonCommercial-ShareAlike 4.0 International License.

Wittmann, M. C. (2002). The object coordination class applied to wavepulses: Analysing student reasoning in wave physics. *International Journal of Science Education, 24*(1), 97–118.

Appendix A: Survey Instrument

Thank you for your willingness to participate in this survey. Below you will find several questions about normalization of vectors. Please answer the questions to the best of your ability without using outside resources. If you do not know how to answer a question, please write "I do not know" as your response. This survey will take approximately 20 minutes to complete.

Question 1: Please normalize the following vector if possible:

$$\begin{bmatrix} 3 \\ -4 \end{bmatrix}$$

Question 2: Please normalize the following vector if possible (here $i = \sqrt{-1}$):

$$\begin{bmatrix} 3 \\ 3i \end{bmatrix}$$

Question 3: Please normalize the following vector if possible:

$$f(x) = \begin{cases} \sec\left(\frac{\pi x}{4}\right) & -1 \leq x \leq 1 \\ 0 & x < -1 \text{ or } x > 1 \end{cases}$$

Question 4: To you, what does it mean to normalize a vector?

Question 5: What are some applications of normalization, and why is normalization important in those applications?

Question 6: How do you define what a vector is?

Question 7: Please give some examples of vectors and/or vector spaces below.

Question 8: Do you have a graphical or geometric way to think about vectors? Please explain.

Question 9: Do you have a graphical or geometric way to think about normalization? Please explain.

Question 10: Please describe what a unit vector is.

Results from this survey will be used to select potential interview candidates. I am purposefully trying to select participants that have a variety of lived experiences, thus I am collecting the following data:

What is your gender identity?

What is your racial/ethnic identity?

Appendix B: Sample Interview Protocol (First Interview)

Interviewer _______________
Camera operator _______________
Date _______________

Introduction Prompt

"Thank you for agreeing to help me by participating in this interview. I expect this interview to take 30-40 minutes.

I am interested in finding out more about how students actually think about mathematical norms and normalization. I am curious to see how you personally think about these concepts. Please try to be as honest as possible about how you're thinking. Please keep in mind that I am NOT looking for particular answers, I really just want to understand how *you* think about the mathematical ideas.

Please feel free to use the paper and markers and write anything you wish during the interview. Do you have any questions before we begin?"

1. **Personal Information**
   a. "What university mathematics courses have you taken?"
   b. "What is your current major?"
   c. "How long have you been studying here?"

2. **Absolute Value**
   a. "People think about concepts in mathematics in many different ways. I am curious, how would you describe and define what absolute value is?"
      i. (*If needed*) Do you have any other ways you think about absolute value?
      ii. (*If needed*) What are some properties of absolute values?
      iii. (*If needed*) Can you give me a couple examples to illustrate your understanding?
      iv. (*If needed*) Do you have a graphical way to think about absolute value?
      v. "Please try to solve the following problem, and think out loud as you solve it."

$$|x - 7| = 3$$

3. **Norms and Normalization**
   "I have here your answers to the normalization survey you filled out recently. I would like to ask you some further questions to understand your thinking about these ideas.
   a. "You said here on question three that normalizing a vector means [insert student's description of normalization]. Can you elaborate on that?"
      i. Another student described normalization this way: "We normalize a vector to get rid of its magnitude, so we are only left with direction." What do you think of this statement?
   b. "In question 1, why did you choose that procedure to normalize the vector?"
      i. "Similarly, in question 2, why did you choose that particular procedure?"
      ii. (If applicable) "Similarly, in question 3, why did you choose that particular procedure?"
      iii. (If applicable) "You wrote that you did not know how to normalize the vector on (Question 2/Question 3). Do you have any ideas how you might do so?"
      iv. "Are there other ways you could have normalized those vectors?"

c. (*If applicable*) "It looks like you decided the complex vector could not be normalized. Why was that the case?"
d. (*If applicable*) "Why do you think you had trouble with the complex vector?"
e. (*If applicable*) "Do you think that complex vector should have a norm/length/magnitude of zero? Why or why not?"
f. (*If applicable*) "How is the normalization of that function similar to normalizing the vectors?"
g. "How would you describe and define a mathematical norm?"
    i. (If applicable) "That is alright that you do not know that term. What if I asked you to describe and define the magnitude of a vector?"
h. "On question eight, you said [insert student's graphical or geometric description of normalization]. Can you elaborate on that?"
    i. (*If needed*) "You said you do not have a geometric or graphical way to think about normalization. Why do you think that is the case?"
    ii. (*If needed*) "Do you have a graphical or geometric way of thinking about the norm of a vector?"
i. "Suppose I told you a vector has a length of $\sqrt{13}$ and its corresponding normalized vector is $\begin{bmatrix} -2/\sqrt{13} \\ -3/\sqrt{13} \end{bmatrix}$. Can you find the vector?"

4. Vectors and Vector Spaces

a. "On question five, you defined a vector as [insert student's definition here]. Is there anything you want to add to that definition?"
b. "You gave the following as examples of vectors/vector spaces on question six. Are there any other vectors or vector spaces you know of?"
c. "Are there any other ways you could represent those vectors?"

5. Properties of Norms and Normalized Vectors

a. "Looking at the problems you have solved in this interview and on the survey, what similarities or connections, if any, do you see between them?"
    i. "Do you think it makes sense to normalize a real number, say 5? Please explain."
b. "If you had to try and define what a mathematical norm is now, would your answer be different from before? Why or why not?"
c. "Why do you think mathematical norms are useful?"
d. "On question four, you described [insert student's applications here] as being (an) application(s) of normalization. Can you explain again why normalization is important for that application?"
    i. (*If needed*) "Has normalization come up in your class? In what context?"
    ii. (*If needed*) "Why do you think normalization was used there?"
e. "Do you see unit vectors as connected to normalization? How so?"

6. (If Time Permits) Normalizing a Wave Function

a. What would you need to know in order to find ψ_0 so that the wave function
$$\psi(x) = \psi_0 e^{-(x-x_0)^2/(4\sigma^2)}$$
is normalized?

Appendix C: Sample Questions for Interviews 2 and 3 with Quantum Students

Problem Solving Tasks

1. Problem 1.5 in Griffiths (2005, p. 14)
   "Consider the wave function
   $$\psi(x,t) = Ae^{-\lambda|x|}e^{-i\omega t},$$
   where A, λ, and ω are positive real constants.
   (a) Normalize ψ.
   (b) Determine the expectation values of x and x^2."
2. Problem 2.7 in Griffiths (2005, pp. 39-40)
   "A particle in the infinite square well has the initial wave function
   $$\psi(x,0) = \begin{cases} Ax, & 0 \le x \le a/2 \\ A(a-x), & a/2 \le x \le a \end{cases}$$
   (a) Sketch $\psi(x,0)$, and determine the constant A.
   (b) Find $\psi(x,t)$
   (c) What is the probability that a measurement of the energy would yield the value E_1?
   (d) Find the expectation value of the energy."
3. Normalize the following vector:
   $$v = \begin{bmatrix} 4i \\ 4 \end{bmatrix}$$
4. Problem 4.27 in Griffiths (2005, p. 177)
   "An electron is in the spin state
   $$\chi = A\begin{pmatrix} 3i \\ 4 \end{pmatrix}$$
   (a) Determine the normalization constant A.
   (b) Find the expectation values of S_x, S_y, and S_z."
5. What do you think it would mean to normalize the matrix $\begin{bmatrix} 1 & 2 \\ 2 & -3 \end{bmatrix}$?

Discussion Questions

1. Why is normalization important within this problem?
2. Why did you choose that procedure to normalize the wave function/quantum state? Are there other ways you could have done it?
3. Why is normalization important in quantum mechanics in general?
4. How does normalizing wave functions/quantum states in this class compare to normalization of vectors you have encountered in the past?
5. Do you see normalization in quantum mechanics as being similar to or different from normalizing a vector such as $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$?
6. Are there other applications of normalization (such as those you may have learned in previous classes) that you feel are just as important as normalization within quantum?

Appendix D: Pre- and Post-Quantum Interview Questions from Project LinAl-P Fall 2016 Data Collection

Pre-Quantum Interview

1. Normalization

Please read this out loud and then begin your work on this problem: (*Hand interviewee Student Page 1 which says "*Normalize the following vector*:* $\vec{v} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$.")

- What does it mean to normalize?
- Why did you choose that procedure to normalize $\vec{v}$?
- Do you have a geometric or graphical way to think about normalization?
- Normalize the vector: $\vec{w} = \begin{bmatrix} 3 \\ 3i \end{bmatrix}$. (*Write that vector on SP 1*)
    - Why did you choose that procedure to normalize w?
    - Do you have a geometric/graphical way to think about this problem?
    - (*If the student mentioned 'length' in regards to normalization*) You mentioned before that normalization is related to length. How do you make sense of that with this vector $\vec{w}$?

Post-Quantum Interview

Normalization & Probability

a. Please normalize a vector whose components are 3 and 2i.
    - How do you think about what it means to normalize a vector?
    - Do you have a geometric or graphical way you think about normalization?
    - You used [Dirac notation / matrix representation / both] in your solution. Tell me about that.

b. In general, what does normalization mean to you?
    - What if you were asked to think about other objects, like a vector with all complex components or a wave function. Do you think you think about normalization and normalized vectors or states in the same way as what you described?
    - [If relevant]: Are there any other ways you think of norm, normalization, and normalized vectors/states that are not based on length?