Plane Permutations and their Applications to Graph Embeddings and Genome Rearrangements

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(ABSTRACT)

Maps have been extensively studied and are important in many research fields. A map is a 2-cell embedding of a graph on an orientable surface. Motivated by a new way to read the information provided by the skeleton of a map, we introduce new objects called plane permutations. Plane permutations not only provide new insight into enumeration of maps and related graph embedding problems, but they also provide a powerful framework to study less related genome rearrangement problems. As results, we refine and extend several existing results on enumeration of maps by counting plane permutations filtered by different criteria. In the spirit of the topological, graph theoretical study of graph embeddings, we study the behavior of graph embeddings under local changes. We obtain a local version of the interpolation theorem, local genus distribution as well as an easy-to-check necessary condition for a given embedding to be of minimum genus. Applying the plane permutation paradigm to genome rearrangement problems, we present a unified simple framework to study transposition distances and block-interchange distances of permutations as well as reversal distances of signed permutations. The essential idea is associating a plane permutation to a given permutation or signed permutation to sort, and then applying the developed plane permutation theory.

Plane Permutations and their Applications to Graph Embeddings and Genome Rearrangements

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(GENERAL AUDIENCE ABSTRACT)

This work is mainly concerned with studying two problems. The first problem starts with a graph G consisting of vertices and lines (called edges) linking some pairs of vertices. Intuitively, if the graph G can not be drawn on the sphere without crossing edges, it may be possibly drawn on a torus (i.e., the surface of a doughnut) without crossing edges; if it is still impossible, it may be possible to draw the graph G on the surface obtained by "gluing" several tori together. Once a graph G is drawn on a surface without crossing edges, there is a cyclic order of those edges incident to each vertex of the graph. Suppose you are not satisfied with how the edges around a vertex are cyclically arranged, and you want to arrange them differently. A question that arises naturally would be: is the adjusted drawing still cross-free on the original surface, or do we need to glue more (or fewer) tori in order for it to be cross-free? The second problem stems from genome rearrangements. In bioinformatics, people try to understand evolution (of species) by comparing the genome sequences (e.g., DNA sequences) of different species. Certain operations on genome sequences are believed to be potential ways of how species evolve. The operations studied in this work are transpositions, block-interchanges and reversals. For example, a transposition is such an operation that swaps two consecutive segments on the given genome sequence. As a candidate indicator of how far away one species is from another from an evolutionary perspective, we can compute how many transpositions are required to transform the genome sequence of one species to that of the other. In this work, we propose a plane permutation framework, which works effectively on solving the above mentioned two problems. In addition, plane permutations themselves are interesting objects to study and are studied as well.

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Chapter 1

Introduction and Background

In this chapter, we will introduce basic notation and will review topics related to this work.

1.1 Notation regarding permutations

Permutations are fundamental objects in many fields of mathematics, and will be used a lot in this work. In the following, we first introduce some notation regarding permutations.

Let $\mathcal{S}_n$ denote the group of permutations, i.e., the group of bijections, from $[n] = \{1, \ldots, n\}$ to $[n]$, where the multiplication is the composition of maps. The following three representations of a permutation π on $[n]$ will be used:

Two-line form: The top line lists all elements in $[n]$, following the natural order. The bottom line lists the corresponding images of the elements on the top line, i.e.

$$\pi = \begin{pmatrix} 1 & 2 & 3 & \cdots & n-2 & n-1 & n \\ \pi(1) & \pi(2) & \pi(3) & \cdots & \pi(n-2) & \pi(n-1) & \pi(n) \end{pmatrix}.$$

One-line form: π is represented as a sequence $\pi = \pi(1)\pi(2)\cdots\pi(n-1)\pi(n)$.

Cycle form: Regarding $\langle\pi\rangle$ as a cyclic group, we represent π by its collection of orbits (cycles). The set consisting of the lengths of these disjoint cycles is called the *cycle-type* of π. We can encode this set into a non-increasing integer sequence $\lambda = \lambda_1\lambda_2\cdots$, where $\sum_i \lambda_i = n$, or as $\lambda = 1^{a_1}2^{a_2}\cdots n^{a_n}$, where we have a_i cycles of length i. The number of disjoint cycles of π will be denoted by $C(\pi)$. A cycle of length k will be called a k-cycle. A cycle of odd or even length will be called an *odd* or *even* cycle, respectively. For a permutation γ having only one cycle, i.e., cycle-type n^1, we will abuse the term by just calling it a cycle. It is well known that all permutations of the same cycle-type form a conjugacy class of $\mathcal{S}_n$.

It is well known (e.g., Stanley [58]) that the number q^λ of permutations on $[n]$ in the conjugacy class of cycle-type $\lambda = 1^{a_1}2^{a_2}\cdots n^{a_n}$ is given by

$$q^\lambda = \frac{n!}{1^{a_1}2^{a_2}\cdots n^{a_n}a_1!a_2!\ldots a_n!}.$$

1.2 Graph embeddings, maps and fatgraphs

Graph embedding is one of the most important topics in topological graph theory. In particular, 2-cell embeddings of graphs (loops and multiple edges allowed) have been widely studied. A 2-*cell embedding* of a given graph G on a closed surface of genus g, S_g, is an embedding on S_g such that every face is homeomorphic to an open disk. An embedding is also called a *map*. (People use the terms graph embedding or map, depending on the specific topics they are working on. We will not differentiate between the two names and will use them interchangeably.) The closed surfaces could be either orientable or unorientable. In this work, we restrict ourselves to the orientable case.

Assume the graph G has e edges and v vertices, and that G is embedded in the surface S_g via the embedding ϵ. In view of *Euler's characteristic formula*, we have

$$v - e + f = 2 - 2g \quad \Longleftrightarrow \quad 2g = \beta(G) + 1 - f, \tag{1.1}$$

where $f \geq 1$ is the number of faces of ϵ and $\beta(G)$ is the *Betti number* of G.

We now introduce the combinatorial counterpart of maps, i.e., fatgraphs [24]. A *fatgraph* is a graph with a specified cyclic order of the ends of edges incident to each vertex of the graph. Intuitively, the corresponding fatgraph of a map is the remaining skeleton after deleting all faces (without boundaries) of the map. In this work, we will mainly work on fatgraphs, although the results may be stated in terms of graph embeddings and maps.

A fatgraph of n edges can be encoded into a triple of permutations (α, β, γ) on $[2n] = \{1, 2, \cdots 2n\}$, where α is a fixed-point free involution (i.e., cycle-type 2^n). This is obtained as follows: Given a fatgraph F, we firstly call the two ends of an edge *half-edges*. Label all half-edges using the labels from the set $[2n]$ so that each label appears exactly once. Then we immediately obtain two permutations α and β, where α is an involution without fixed points such that each α-cycle consists of the labels of the two half-edges of an edge and each cycle in β is the counterclockwise cyclic arrangement of all half-edges incident to a vertex. The third permutation $\gamma = \alpha\beta$, and the cycles of γ can be interpreted as the set of boundary components (or faces) of the fatgraph F. If γ has k cycles, the fatgraph has k boundary components. A *boundary component* of the fatgraph is obtained as follows: Starting from some half-edge, and every time when we meet a half-edge we next go to the half-edge paired with the counterclockwise neighbor of the current half-edge until we meet the starting half-edge again; the obtained cycle is a boundary component of the fatgraph and corresponds to a cycle in γ. Starting from a half-edge which does not appear in the previously obtained boundary component (or components) and continuing this traveling process, we can obtain all boundary components of the fatgraph.

An example of a fatgraph is illustrated in Figure 1.1; its corresponding triple of permutations are

$$\alpha = (1,4)(2,5)(3,6), \quad \beta = (1,5,3)(4,2,6), \quad \gamma = (1,2,3,4,5,6).$$

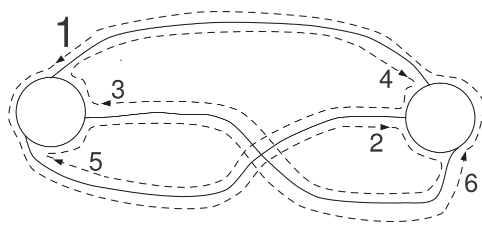

Figure 1.1: A fatgraph with 6 half-edges, where the dashed curve represents its boundary component.

From a triple of permutations (α, β, γ) representing a fatgraph, the genus of the map represented by the fatgraph (or just the genus of the fatgraph) is determined by

$$C(\beta) - C(\alpha) + C(\gamma) = 2 - 2g. \tag{1.2}$$

With regard to graph embeddings and maps, the following problems have been studied:

- Enumeration of one-face maps. See for instance [1, 10, 12, 13, 16, 17, 24, 31, 32, 33, 41, 42, 64, 65, 68] and the references therein.

- Determining the genus distribution of all possible embeddings of a given graph [11, 29, 46, 63, 66].

- Determining the minimum (resp. maximum) genus [10, 22, 35, 44, 47, 52, 53, 55, 61, 67] and constructing minimal (resp. maximal) embeddings of a given graph [26, 50, 62].

Next, we will introduce the most relevant existing results obtained in these studies.

Enumeration of one-face maps

The enumeration of maps with one face (i.e., one-face maps) has been particularly extensively studied. For the purpose of enumeration, we consider *rooted* one-face maps, i.e., the starting point of the boundary (when making a tour) will be marked and called the root. Distinguishing a root facilitates the enumeration of one-face maps, as it somehow breaks the symmetry (i.e., homeomorphic copy). (Enumerating non-homeomorphic structures is always a hard problem.) For convenience, we always label the root with the label 1.

Now given two rooted one-face maps which are respectively encoded into the triples (α, β, γ) and $(\alpha', \beta', \gamma')$, they will be viewed as equivalent if there exists a permutation π such that

$$\alpha' = \pi\alpha\pi^{-1}, \quad \beta' = \pi\beta\pi^{-1}, \quad \pi(1) = 1, \tag{1.3}$$

i.e., one is just a root preserving, relabeling of the other. Certainly, if the two are equivalent, then $\gamma' = \pi\gamma\pi^{-1}$ automatically.

It is not hard to see that two different triples (α, β, γ) and $(\alpha', \beta', \gamma)$ with γ being a cycle must represent two different (i.e., unequivalent) rooted one-face maps, as in this case, there is no such relabeling π to make one into the other. Because the only π satisfying

$$\gamma = \pi\gamma\pi^{-1}, \quad \pi(1) = 1$$

is the identity permutation since γ is a cycle, and the identity permutation can not conjugate α into a different permutation α'.

Also note that each factorization of the cycle γ into a fixed-point free involution α and a permutation β determines a triple representing a one-face map. Hence, the number of different rooted one-face maps of n edges (up to equivalence class) equals the number of factorizations of the cycle $(1, 2, \ldots, 2n)$ into a fixed-point free involution and another permutation. It is trivial to obtain the total number of rooted one-face maps to be $(2n-1)!! = (2n-1)\cdot(2n-3)\cdots 1$.

Next, we can see from Eq. (1.2), that the number of genus g one-face maps is equal to the number of ways of writing the cycle $(1, 2, \ldots, 2n)$ as the product of α and β, where α is a fixed-point free involution and β has $n + 1 - 2g$ cycles.

Let $A(n, g)$ denote the number of rooted one-face maps (up to equivalence) of genus g having n edges and let $A_n(x) = \sum_{g\geq 0} A(n, g)x^{n+1-2g}$ be the corresponding generating function. Four decades ago, Walsh and Lehman [64, Eq. (13)], using a direct recursive method and formal power series, obtained an explicit formula for $A(n, g)$ which can be reformulated as follows:

$$A(n, g) = \sum_{\lambda \vdash g} \frac{(n + 1)n \cdots (n + 2 - 2g - \ell(\lambda))}{2^{2g} \prod_i c_i!(2i + 1)^{c_i}} \frac{(2n)!}{(n + 1)!n!}, \tag{1.4}$$

where the summation is taken over all partitions λ of g, c_i is the number of parts i in λ, and $\ell(\lambda)$ is the total number of parts.

More than a decade later, Harer and Zagier [41] obtained in the context of computing the virtual Euler characteristics of a curve that

$$A(n, g) = \frac{(2n)!}{(n + 1)!(n - 2g)!}[x^{2g}] \left(\frac{x/2}{\tanh x/2}\right)^{n+1}, \tag{1.5}$$

where $[x^k]f(x)$ denotes the coefficient of x^k in the expansion of the function $f(x)$. Considering the relation between the RHS of Eq. (1.5) and its derivatives, they obtained the following three-term recurrence, known as the Harer-Zagier recurrence:

$$(n+1)A(n,g) = 2(2n-1)A(n-1,g) + (2n-1)(n-1)(2n-3)A(n-2,g-1). \tag{1.6}$$

They furthermore obtained the so-called Harer-Zagier formula:

$$A_n(x) = \frac{(2n)!}{2^n n!} \sum_{k\geq 1} 2^{k-1} \binom{n}{k-1} \binom{x}{k}. \tag{1.7}$$

There is a body of work on how to derive these results [16, 17, 31, 32, 42]. A direct bijection for the Harer-Zagier formula was given in [32]. Combinatorial arguments to obtain the Lehman-Walsh formula and the Harer-Zagier recurrence were recently given in [17]. One of the most recent advances is a new recurrence for $A(n,g)$ obtained by Chapuy [16] via a bijective approach:

$$2gA(n,g) = \sum_{k=1}^{g} \binom{n+1-2(g-k)}{2k+1} A(n,g-k). \tag{1.8}$$

In our work, we will see that we can refine almost all of these results and generalize some of them.

Studies on conventional graph embeddings

The genus is one of the most important topological characteristics of a graph embedding and surface. The minimum (resp. the maximum) genus g such that there exists an embedding of G on the surface S_g of genus g is denoted by $g_{min}(G)$ (resp. $g_{max}(G)$).

In Duke [22], an interpolation theorem is proved, which says that for any given graph G, there exists an embedding of genus g for any $g_{min}(G) \leq g \leq g_{max}(G)$. Later, people are interested in the problems of determining $g_{min}(G)$, $g_{max}(G)$, determining the genus distribution and constructing embeddings with genera prescribed in advance.

It is proved in Thomassen [61] that determining whether $g_{min}(G) \leq k$ is NP-Complete. However, for $g_{max}(G)$, there are explicit formulas to compute it in Xuong [67] and Nebesky [53]. In addition, in Furst et al. [26] and Glukhov [30] polynomial-time algorithms for determining the maximum genus of an arbitrary graph are devised independently.

The genus distribution problem is essentially counting the number of embeddings of genus g of a given graph for all possible g. It is conjectured in Gross et al. [29] that for any graph G, the genus distribution polynomial, i.e.,

$$w(x) = \sum_{g} (\#\text{of embeddings of genus } g)\ x^g,$$

is log-concave. This conjecture has been confirmed for some special graphs, see for instance [63, 66].

In Thomassen [62], there is a polynomially bounded algorithm to find a minimum genus embedding for a specific class of graphs. Later, in Mohar [50], it is shown that for each fixed integer g, there is a linear-time algorithm that, for a given graph G, either constructs an embedding of genus g for G or reports that no such an embedding exists.

The above two problems that the enumeration of one-face maps and the conventional graph embeddings can be viewed as two angles of studying maps:

- On the enumeration of one-face maps, the number of faces and the number of edges are fixed; from Euler's characteristic formula, maps of different genus correspond to maps having different number of vertices. In this case, the underlying graph may change.

- On the conventional graph embeddings, the number of edges and the number of vertices are fixed; from Euler's characteristic formula, embeddings of different genus correspond to embeddings having different number of faces. So in this case, the underlying graph will not change.

For one-face maps and graph embeddings, the main problems we will address are as follows:

- For enumeration aspects of one-face maps, we will enumerate a generalized version of maps, that is triples (α, β, γ) where $\gamma = \alpha\beta$ and α is not necessarily a fix-point free involution. These generalized maps are sometimes called *hypermaps*.

- For graph embeddings, we will study how the genus changes if we reembed a vertex in a given graph embedding, i.e., rearrange the half-edges around the vertex. More specifically, we will consider what is the minimum (resp. the maximum) genus can be achieved under reembeddings; and we will compute the genus polynomial under reembeddings. These problems can be viewed as local analogues of the problems we introduced above.

Studying these problems is facilitated by our proposed new objects, called plane permutations. In the upcoming section, we will first share the motivation of plane permutations.

1.3 Motivation for plane permutations

Plane permutations is a new object, added to the equivalence family of maps, graph embeddings and fatgraphs when restricted to certain subclass. A plane permutation is basically a two-line array, motivated by a new way to "read" a fatgraph as follows: Given a (half-edge) labelled fatgraph, we start from a half-edge, record the half-edge (i.e., its label) and record its counterclockwise neighbor right below it. Diagonally, we record the half-edge paired with the counterclockwise neighbor, namely we put the half-edge paired with the counterclockwise neighbor at the second entry on the top line of the array. Record the counterclockwise neighbor of the latter right below it, and go diagonally, and iterate the process until coming to a half-edge whose paired half-edge is the starting half-edge. When coming to that point, we start with a half-edge which has not been recorded yet, and iterate the process again. Iterating this process will eventually give us a two-line array, from which the given fatgraph can be reconstructed.

Here we give an example of a fatgraph with one boundary component in Figure 1.2. The two presentations (in the one on the RHS, edges are fattened into ribbons) are showing the same one-face map, whose corresponding two-line array reads

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 6 & 7 & 8 & 3 & 4 & 5 & 2 \end{pmatrix}.$$

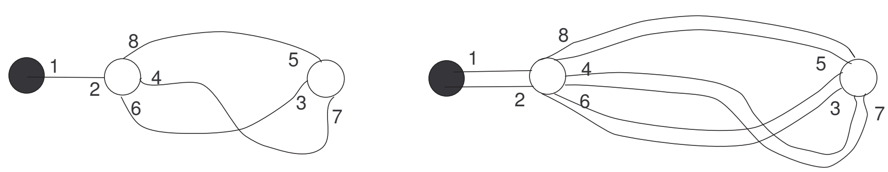

Figure 1.2: A one-face map with 4 edges.

It is not hard to see, that the two-line array will give us three permutations: the upper-horizontal, the vertical and the diagonal. In the last example, the upper-horizontal gives us $(1, 2, 3, 4, 5, 6, 7, 8)$, the vertical gives $(1)(2, 6, 4, 8)(3, 7, 5)$, and the diagonal gives us $(1, 2)(3, 6)(4, 7)(5, 8)$. In particular, we can see that the diagonal is a fixed-point free involution.

To reconstruct the fatgraph, it is easy if we use the following observations:

- The cycles of the vertical determine the vertices of the fatgraph.
- The cycles of the diagonal determine the edges of the fatgraph.
- The cycles of the upper-horizontal represent the boundary components.

Due to the equivalence relation, studying maps can be translated into studying plane permutations. We will see that working with two-line arrays provide new insights into studying maps. In particular, a diagonal transpose action (more generally, diagonal rearrangements) serves as the corner stone of the entire work.

In order for better illustrating the power of the diagonal transpose action and the connection to later genome rearrangements, let's briefly describe it here first. A *diagonal transpose* on a plane permutation is essentially just a swapping of two diagonal blocks. A *diagonal block* is a block of consecutive diagonal pairs. For example,

$$\begin{array}{cccc} & 3 & 4 & 5 \\ 6 & 7 & 8 & \end{array}$$

is a diagonal block in the above example. Clearly, the outcome after a diagonal transpose action is still a plane permutation. For example, by swapping the two diagonal blocks induced by the segments $3, 4, 5$ and $6, 7$, we obtain the following plane permutation:

$$\begin{pmatrix} 1 & 2 & \mathbf{6} & \mathbf{7} & 3 & 4 & 5 & 8 \\ 1 & \mathbf{3} & \mathbf{4} & 6 & 7 & 8 & 5 & 2 \end{pmatrix}.$$

We will compare the two plane permutations before and after the diagonal transpose:

- The upper-horizontals differ by a swapping of two segments.
- The diagonals are the same.

- The verticals only differ at the at most 4 boundary positions determined by the two involved diagonal blocks.

The first two items are easy to see. For the last item, let's look at our examples above. We can see that only the vertical images of the elements 2, 5 and 7 are different.

Following from the last item, the number of cycles in the verticals before and after a diagonal transpose action may be different. Note cycles in the vertical of plane permutations correspond to vertices in maps. Hence, in terms of maps, after a diagonal transpose action, we may end up with a map of different genus (because the number of faces and the number of edges are fixed). This implies that we can transform plane permutations (i.e., maps) of different genus back and forth via diagonal transposes, which may eventually allow us to determine the relation between the number of maps of different genus. This is indeed true as we will see later, so that we can enumerate maps of different genus recursively.

The cycle-type of the diagonal is not essential in the above analysis, so instead of a fixed-point free involution the reasoning above works for any cycle-type. Let $p_k^\lambda(n)$ denote the number of plane permutations having k cycles in the vertical and diagonals being a fixed permutation of cycle-type λ. A bijection induced by certain diagonal transposes will eventually allow us to give a recurrence for the numbers $p_k^\lambda(n)$. By restricting λ to the cycle-type 2^n, we will refine Chapuy's recursion mentioned before.

In addition, it turns out that our graph embedding problems are related to a more general operation of diagonal blocks. We will see in Chapter 3 that a vertex will divide a plane permutation into diagonal blocks, then reembedding the vertex is just equivalent to rearranging these diagonal blocks (not just swapping two of them). Anyway, both of them are related to the concept of diagonal blocks, and can be treated similarly. In fact, we will see that the numbers $p_k^\lambda(n)$ count the local genus distribution of a given graph embedding. From there, by studying these numbers using our obtained recurrence for them, we can determine the local minimum/maximum genus, log-concavity, etc. These are the connections between plane permutations and enumeration of maps as well as graph embeddings.

1.4 Genome rearrangements

In the following, we will look at another topic to which the plane permutation framework can be applied.

In bioinformatics, people try to understand the evolution of species by comparing their genome sequences. It was noticed that the genome sequence of one species might be just a rearrangement of that of the other by certain operations. In particular, the problem of determining the minimum number of certain operations required to transform one of two given genome sequences into the other, has been extensively studied. Combinatorially, this problem can be formulated as sorting a given permutation (or sequence) to the identity permutation by certain operations, in a minimum number of steps. This minimum number constitutes a distance of permutations. [5, 8, 19, 25, 48] study transpositions, [7, 19, 20, 37, 49] block-interchanges, and reversals are analyzed in[3, 6, 9, 15, 39, 40].

In what follows, we will introduce these three operations one by one.

Transposition distances

Given a sequence (one-line permutation) on $[n]$

$$s = a_1 \cdots a_{i-1} a_i \cdots a_j a_{j+1} \cdots a_k a_{k+1} \cdots a_n,$$

a *transposition* action on s means changing s into

$$s' = a_1 \cdots a_{i-1} a_{j+1} \cdots a_k a_i \cdots a_j a_{k+1} \cdots a_n$$

by swapping the two adjacent continuous segments $a_i \dots a_j$ and $a_{j+1} \dots a_k$ for some $1 \leq i \leq j < k \leq n$. Let $e_n = 123 \cdots n$. The transposition distance of a sequence s on $[n]$ is the minimum number of transpositions needed to sort s into e_n. Denote this distance as $td(s)$.

The cycle-graph model was firstly proposed by Bafna and Pevzner [8] to study transposition distances. Given a permutation $s = s_1 s_2 \cdots s_n$ on $[n]$, the *cycle graph* $G(s)$ of s is obtained as follows: Add two additional elements $s_0 = 0$ and $s_{n+1} = n + 1$. The elements in $[n+1]^*$ give the vertices of $G(s)$. Draw a directed black edge from i to $i + 1$, and draw a directed gray edge from s_{i+1} to s_i, we then obtain $G(s)$.

For example, the cycle graph for the permutation $s = 31458276$ is illustrated in Figure 1.3.

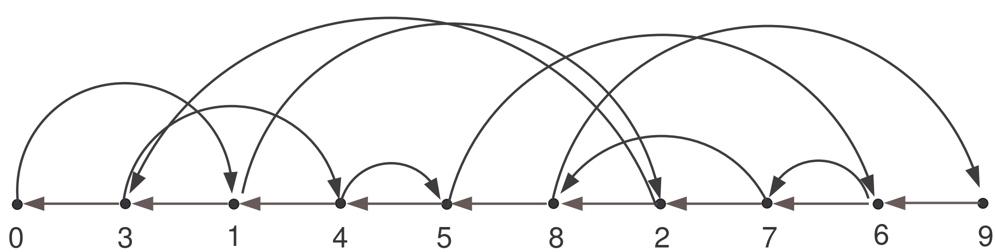

Figure 1.3: The cycle graph for the permutation $s = 31458276$.

An *alternating cycle* in $G(s)$ is a directed cycle, where its edges alternate in color. An alternating cycle is called *odd* if the number of black edges in the cycle is odd. Bafna and Pevzner obtained lower bounds for $td(s)$ in terms of the number of cycles and odd cycles of

$G(s)$ [8], which are respectively,

$$td(s) \geq \frac{n+1-C(G(s))}{2}, \tag{1.9}$$

$$td(s) \geq \frac{n+1-C_{odd}(G(s))}{2}, \tag{1.10}$$

where $C(G(s))$ and $C_{odd}(G(s))$ denote the number of cycles and odd cycles in $G(s)$, respectively. These lower bounds followed from observations that each transposition increases the number of cycles (and resp. odd cycles) in the cycle graph by at most 2 and there are $n+1$ cycles in the cycle graph of the identity permutation with all of them being odd.

Algorithms of various efficiency for sorting permutations by transpositions were studied in [8, 25] and references therein.

Block-interchange distances

A more general transposition problem, where the involved two segments are not necessarily adjacent, was firstly studied in Christie [20]. It is referred to as the *block-interchange* distance problem. The minimum number of block-interchanges needed to sort s into e_n is accordingly called the block-interchange distance of s and denoted as $bid(s)$. Christie [20] obtained an exact formula to compute the block-interchange distance of any given permutation s, based on the cycle-graph model. The formula is

$$bid(s) = \frac{n+1-C(G(s))}{2}. \tag{1.11}$$

Algorithms for sorting permutations by block-interchanges were studied in [19, 37].

Reversal distances

Reversals can be defined for both ordinary permutations and signed permutations. We mainly consider reversal distances for signed permutations in this work. A *signed permutation* on $[n]$ is a pair (a, w) where a is a sequence on $[n]$ while w is a word of length n on the alphabet set $\{+, -\}$.

Usually, a signed permutation is represented by a single sequence $a_w = a_{w,1}a_{w,2} \cdots a_{w,n}$ where $a_{w,k} = w_k a_k$, i.e., each a_k carries a sign determined by w_k.

Given a signed permutation $a = a_1a_2 \cdots a_{i-1}a_ia_{i+1} \cdots a_{j-1}a_ja_{j+1} \cdots a_n$ on $[n]$, a *reversal* $\varrho_{i,j}$ acting on a will change a into

$$a' = \varrho_{i,j} \diamond a = a_1a_2 \cdots a_{i-1}(-a_j)(-a_{j-1}) \cdots (-a_{i+1})(-a_i)a_{j+1} \cdots a_n$$

by reversing the segment $s_i \ldots s_j$ and flipping the signs of the entries there at the same time. The reversal distance $d_r(a)$ of a signed permutation a on $[n]$ is the minimum number of reversals needed to sort a into $e_n = 12 \cdots n$.

For example, the signed permutation $a = -5 + 1 - 3 + 2 + 4$ needs at least 4 steps to be sorted by reversals as illustrated below:

$$-5 \quad +1 \quad \underline{-3 \quad +2} \quad +4$$

$$-5 \quad +1 \quad \underline{-2} \quad +3 \quad +4$$

$$\underline{-5 \quad +1 \quad +2 \quad +3 \quad +4}$$

$$\underline{-4 \quad -3 \quad -2 \quad -1} \quad +5$$

$$+1 \quad +2 \quad +3 \quad +4 \quad +5$$

Let $[n]^- = \{-1, -2, \ldots, -n\}$.

The most common graph model used to study reversal distance is *breakpoint graph* proposed by Bafna and Pevzner [9]. The breakpoint graph for a given signed permutation $a = a_1a_2\cdots a_n$ on $[n]$ can be obtained as follows: Replacing a_i with $(-a_i)a_i$, and adding 0 at the beginning of the obtained sequence while adding $-(n+1)$ at the end of the obtained sequence, in this way we obtain a sequence $b = b_0b_1b_2\cdots b_{2n}b_{2n+1}$ on $[n]^* \cup [n+1]^-$. Draw a black edge between b_{2i} and b_{2i+1}, as well as a grey edge between i and $-(i+1)$ for $0 \leq i \leq n$. The obtained graph is the breakpoint graph $BG(a)$ of a.

The breakpoint graph $BG(a)$ for the signed permutation $a = -5+1-3+2+4$ is illustrated in Figure 1.4.

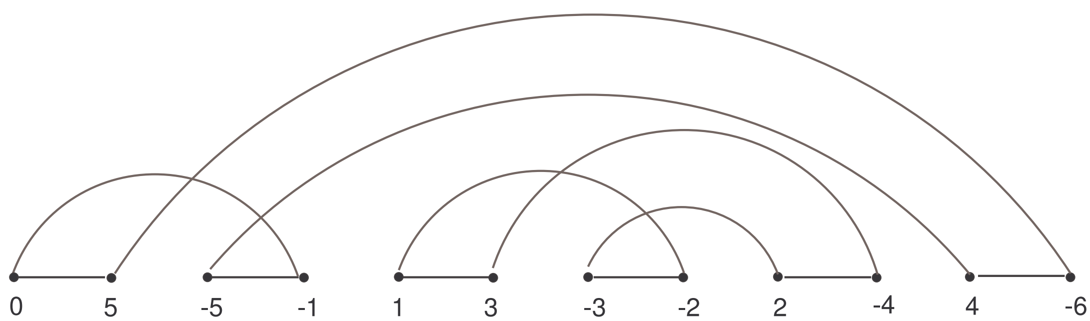

Figure 1.4: The breakpoint graph $BG(a)$ for the signed permutation $a = -5+1-3+2+4$.

Note that each vertex in $BG(a)$ has degree two so that it can be decomposed into disjoint cycles. Denote the number of cycles in $BG(a)$ as $C_{BG}(a)$. Then, the lower bound [9] via the breakpoint graph is given by

$$d_r(a) \geq n + 1 - C_{BG}(a). \tag{1.12}$$

Later, an exact formula for computing the reversal distance of any signed permutation and corresponding polynomial time algorithm were presented in Hannenhalli and Pevzner [40].

Connection to plane permutations

At first sight, there is no clear connection between genome rearrangements and maps or plane permutations. However, the introduction of the diagonal transpose action on plane permutations motivates the idea of associating a plane permutation to a given permutation to sort. Because, in a sense, diagonal transposes on plane permutations are just block-interchanges on "fattened" sequences. To be more specific, the permutation to sort induces the upper-horizontal of the plane permutation, while the vertical or the diagonal can be chosen as needed. Furthermore, we know that diagonal transposes may change the number of cycles in the vertical, so the variation of the number of cycles in the vertical provides a natural statistic partially reflecting the number of diagonal transposes applied (hence the number of block-interchanges on the upper-horizontal).

In the following, we present our first main theorem for better illustrating what we meant.

For a sequence on $[n]$ $s = a_1 a_2 \ldots a_n$, we denote $\bar{s} = (0\ a_1\ a_2\ \cdots\ a_n)$. Also we denote

$$\bar{e}_n = (0\ 1\ 2\ 3\ \cdots\ n), \quad p_t = (n\ n-1\ \ldots\ 1\ 0).$$

Let $C(\pi)$, $C_{odd}(\pi)$ and $C_{ev}(\pi)$ denote the number of cycles, the number of odd cycles and the number of even cycles in π, respectively. Furthermore, let $[n]^* = \{0, 1, \ldots, n\}$. Then for the transposition distance, we have the following general lower bound:

$$td(s) \geq \max_{\gamma} \left\{ \frac{\max\{|C(p_t\bar{s}\gamma) - C(\gamma)|, |C_{odd}(p_t\bar{s}\gamma) - C_{odd}(\gamma)|, |C_{ev}(p_t\bar{s}\gamma) - C_{ev}(\gamma)|\}}{2} \right\}, \tag{1.13}$$

where γ ranges over all permutations on $[n]^*$.

Here, we associated the plane permutation $(\bar{s}, \gamma)$ to s, where $\bar{s}$ specifies the upper-horizontal and γ specifies the vertical. When s is transformed into e_n by transpositions, the plane permutation (s, γ) will be transformed into the plane permutation $(\bar{e}_n, \gamma\bar{s}^{-1}\bar{e}_n)$ by induced diagonal transposes. We will later show that each diagonal transpose changes the number of cycles (resp. odd cycles and even cycles) by at most 2, then the respective differences between the starting vertical γ and the final vertical $\gamma\bar{s}^{-1}\bar{e}_n$ over 2 will give us lower bounds. Since the argument holds for any chosen γ, we can take the maximum over all options.

Obviously, by plugging in a specific γ, we may obtain more concrete lower bounds. It turns out Bafna-Pevzner's lower bounds are equivalent to the evaluation at a special γ.

Regarding reversal distances of signed permutations, we propose a way of translating reversals into block-interchanges in this work, so that we can use the above idea. Full details will be provided in Chapter 4.

1.5 Organization of the dissertation

An outline of the following chapters is as follows. In Chapter 2, we define and study plane permutations. The study of a transpose action on diagonals of plane permutations is the key to most of the rest of the dissertation. First, it allows us to enumerate plane permutations filtered by different criteria, so that we can refine and extend several known enumerative results on maps, e.g., Chapuy's recursion, a Zagier-Stanley result, etc. This chapter is based on the papers "*[10] Plane permutations and applications to a result of Zagier–Stanley and distances of permutations*[1], SIAM J. Discrete Math. 30(3) (2016) pp. 1660–1684" and "*[12] New formulas counting one-face maps and Chapuy's recursion*, Australa. J. Combin., in revision."

In Chapter 3, applying the plane permutation framework, we study graph embeddings. Specifically, we study the following problems: (i) Given an embedding of a graph, how will the genus change if we reembed one or more vertices of the graph? (ii) What is the local genus distribution induced by reembeddings? Studying the former problem gives us a local version of Duke's interpolation theorem and an easy-to-check necessary condition for an embedding to be of minimum genus; studying the latter gives us a log-concavity result of the local genus polynomials. This chapter is based on the paper "*[11] On the local genus distribution of graph embeddings*[2], J. Combin. Math. Combin. Comput. 101 (2017), pp. 157–173."

In Chapter 4, we provide a unified framework to study the transposition distances and the block-interchange distances of permutations, as well as the reversal distances of signed permutations, employing plane permutations. We refine and generalize many results initially obtained in many separate papers, e.g., the Bafna-Pevzner lower bounds and the Christie formula based on cycle graphs, and propose some open problems. This Chapter is based on the papers "*[10] Plane permutations and applications to a result of Zagier–Stanley and distances of permutations*, SIAM J. Discrete Math. 30(3) (2016) pp. 1660–1684" and "*[2] On a lower bound for sorting signed permutations by reversals, arXiv:1602.00778 [math.CO]*".

Chapter 2

The Theory of Plane Permutations

In this chapter, we will develop the theory of plane permutations.

2.1 Definition and basic properties

Let's look at some definitions at first.

Definition 2.1 (Cyclic plane permutation)**.** A *cyclic plane permutation* on $[n]$ is a pair $\mathfrak{p} = (s, \pi)$ where $s = (s_i)_{i=0}^{n-1}$ is an n-cycle and π is an arbitrary permutation on $[n]$.

In this chapter, for the purpose of simplicity, we call cyclic plane permutations just plane permutations.

Given $s = (s_0\ s_1\ \cdots\ s_{n-1})$, a plane permutation $\mathfrak{p} = (s, \pi)$ is represented by a two-row array:

$$\mathfrak{p} = \begin{pmatrix} s_0 & s_1 & \cdots & s_{n-2} & s_{n-1} \\ \pi(s_0) & \pi(s_1) & \cdots & \pi(s_{n-2}) & \pi(s_{n-1}) \end{pmatrix}. \tag{2.1}$$

The permutation $D_{\mathfrak{p}}$ induced by the diagonal-pairs (cyclically) in the array, i.e., $D_{\mathfrak{p}}(\pi(s_{i-1})) = s_i$ for $0 < i < n$, and $D_{\mathfrak{p}}(\pi(s_{n-1})) = s_0$, is called the *diagonal* of $\mathfrak{p}$.

Observation: $D_{\mathfrak{p}} = s\pi^{-1}$.

In a permutation π on $[n]$, i is called an *exceedance* if $i < \pi(i)$ following the natural order and an *anti-exceedance* otherwise. Note that s induces a partial order $<_s$, where $a <_s b$ if a appears before b in s from left to right (with the left most element s_0). These concepts then can be generalized for plane permutations as follows:

Definition 2.2. For a plane permutation $\mathfrak{p} = (s, \pi)$, an element s_i is called an *exceedance* of $\mathfrak{p}$ if $s_i <_s \pi(s_i)$, and an *anti-exceedance* if $s_i \geq_s \pi(s_i)$.

In the following, we mean by "the cycles of $\mathfrak{p} = (s, \pi)$" the cycles of π and any comparison of elements in s, π and $D_{\mathfrak{p}}$ references the partial order $<_s$.

Obviously, each $\mathfrak{p}$-cycle contains at least one anti-exceedance as it contains a minimum, s_i, for which $\pi^{-1}(s_i)$ will be an anti-exceedance. We call these trivial anti-exceedances and refer to a *non-trivial anti-exceedance* as an NTAE. Furthermore, in any cycle of length greater than one, its minimum is always an exceedance.

Example 2.3. For the plane permutation

$$\mathfrak{p} = \begin{pmatrix} 1 & 3 & 6 & 2 & 5 & 4 \\ 5 & 4 & 1 & 3 & 6 & 2 \end{pmatrix}, \tag{2.2}$$

3 is an exceedance, 5 is an anti-exceedance and also an NTAE.

The number of exceedances of $\mathfrak{p}$ does not depend on how we write s in the top row in the two-row representation of $\mathfrak{p}$ although the set of exceedances may vary according to different cyclic shift of s. The reason is that if we cyclically shift one position, say shifting s_0 to the end of the top row, then $\pi^{-1}(s_0)$ will become an exceedance while s_0 itself will become an anti-exceedance. However, it is clear that $\pi^{-1}(s_0)$ was an anti-exceedance while s_0 itself was an exceedance before the shifting. Thus, the total number of exceedances does not depend on the way of putting s on the top row.

Let $Exc(\mathfrak{p})$ and $AEx(\mathfrak{p})$ denote the number of exceedances and anti-exceedances of $\mathfrak{p}$, respectively. For $D_{\mathfrak{p}}$, the quantities $Exc(D_{\mathfrak{p}})$ and $AEx(D_{\mathfrak{p}})$ are defined in reference to $<_s$. The following lemma which relates the number of exceedances in the vertical and anti-exceedances on the diagonal will be very useful later.

Lemma 2.4. *For a plane permutation $\mathfrak{p}=(s,\pi)$, we have*

$$Exc(\mathfrak{p}) = AEx(D_{\mathfrak{p}}) - 1. \tag{2.3}$$

Proof. By construction of the diagonal permutation $D_{\mathfrak{p}}$, we have

$$\forall\, 0 \le i < n-1, \quad s_i <_s \pi(s_i) \qquad \Longleftrightarrow \qquad \pi(s_i) \ge_s D_{\mathfrak{p}}(\pi(s_i)) = s_{i+1}.$$

Note that s_{n-1} is always an anti-exceedance of $\mathfrak{p}$ since $s_{n-1} \ge \pi(s_{n-1})$, and that $\pi(s_{n-1})$ is always an anti-exceedance of $D_{\mathfrak{p}}$ since $D_{\mathfrak{p}}(\pi(s_{n-1})) = s(s_{n-1}) = s_0$ and $\pi(s_{n-1}) \ge s_0$. Thus we have

$$Exc(\mathfrak{p}) = AEx(D_{\mathfrak{p}}) - 1,$$

whence the lemma. □

As immediate applications of the above lemma, we have the following two results.

Proposition 2.5. *For a plane permutation $\mathfrak{p}=(s,\pi)$ on $[n]$, the sum of the number of cycles in π and in $D_{\mathfrak{p}}$ is smaller than $n+2$.*

Proof. Since each cycle has at least one anti-exceedance, we have $AEx(\mathfrak{p}) \geq C(\pi)$ and $AEx(D_\mathfrak{p}) \geq C(D_\mathfrak{p})$. Using Lemma 2.4,

$$AEx(\mathfrak{p}) = n - Exc(\mathfrak{p}) = n + 1 - AEx(D_\mathfrak{p}) \geq C(\pi).$$

Therefore,

$$n + 1 \geq C(\pi) + AEx(D_\mathfrak{p}) \geq C(\pi) + C(D_\mathfrak{p}),$$

whence the proposition. □

In fact, based on Proposition 2.5, we will prove in Chapter 3 that the maximum $n+1$ is attained for any given π, i.e., there exists $\mathfrak{p} = (s, \pi)$ such that $C(\pi) + C(D_\mathfrak{p}) = n + 1$.

Lemma 2.6. *There are $2g$ NTAEs in any plane permutation on $[2n]$ with $n+1-2g$ cycles and a fix-point free involution as its diagonal.*

Proof. This can be easily seen in the following way: given a plane permutation $\mathfrak{p} = (s, \pi)$, its diagonal $D_\mathfrak{p}$ has always n exceedances and n anti-exceedances irrespective of $<_s$ since it is an involution without fixed points. By Lemma 2.4, $\mathfrak{p}$ has $n+1$ anti-exceedances. Therefore, $\mathfrak{p}$ has $(n+1) - (n+1-2g) = 2g$ NTAEs since π has $n+1-2g$ cycles. □

We remark that Lemma 2.6 immediately implies the trisection lemma in Chapuy [16] which is the corner stone of that work.

Proposition 2.7. *For a plane permutation $\mathfrak{p} = (s, \pi)$ on $[n]$, the quantities $C(\pi)$ and $C(D_\mathfrak{p})$ satisfy*

$$C(\pi) + C(D_\mathfrak{p}) \equiv n - 1 \pmod 2. \tag{2.4}$$

Proof. In view of $s = D_\mathfrak{p}\pi$, the parity of both sides are equal. Since a k-cycle can be written as a product of $k-1$ transpositions, the parity of the LHS is the same as $n-1$ while the parity of the RHS is the same as $(n - C(\pi)) + (n - C(D_{\mathfrak{p}}))$, whence the proposition. $\square$

2.2 The diagonal transpose action

In this section, we will introduce an action on the diagonals of plane permutations, which serves as the corner stone of almost this entire work.

Given a plane permutation (s, π) on $[n]$ and a sequence $h = (i, j, k, l)$, such that $i \le j < k \le l$ and $\{i, j, k, l\} \subset [n-1]$, let

$$s^h = (s_0\ s_1\ \ldots\ s_{i-1}\ \underline{s_k\ \ldots\ s_l}\ s_{j+1}\ \ldots\ s_{k-1}\ \underline{s_i\ \ldots\ s_j}\ s_{l+1}\ \ldots),$$

i.e., the n-cycle obtained by transposing the blocks $[s_i, s_j]$ and $[s_k, s_l]$ in s. Note that in case of $j+1 = k$, we have

$$s^h = (s_0\ s_1\ \ldots\ s_{i-1}\ \underline{s_k\ \ldots\ s_l}\ \underline{s_i\ \ldots\ s_j}\ s_{l+1}\ \ldots).$$

Let furthermore

$$\pi^h = D_{\mathfrak{p}}^{-1} s^h,$$

that is, the derived plane permutation, (s^h, π^h), can be represented as

$$\left(\begin{array}{cccccccccc} \cdots s_{i-1} & s_k & \cdots & s_l & s_{j+1} & \cdots & s_{k-1} & s_i & \cdots & s_j & s_{l+1}\cdots \\ \downarrow & & & \downarrow & & & \downarrow & & & \downarrow & \\ \cdots \pi(s_{k-1}) & \pi(s_k) & \cdots & \pi(s_j) & \pi(s_{j+1}) & \cdots & \pi(s_{i-1}) & \pi(s_i) & \cdots & \pi(s_l) & \pi(s_{l+1})\cdots \end{array}\right).$$

We write $(s^h, \pi^h) = \chi_h \circ (s, \pi)$. Note that the bottom row of the two-row representation of

(s^h, π^h) is obtained by transposing the blocks $[\pi(s_{i-1}), \pi(s_{j-1})]$ and $[\pi(s_{k-1}), \pi(s_{l-1})]$ of the bottom row of (s, π). In the following, we refer to general χ_h as block-interchange and for the special case of $k = j + 1$, we refer to χ_h as transposition. As a result, we observe

Lemma 2.8. *Let (s, π) be a plane permutation on $[n]$ and $(s^h, \pi^h) = \chi_h \circ (s, \pi)$ for $h = (i, j, k, l)$. Then, $\pi(s_r) = \pi^h(s_r)$ if $r \in \{0, 1, \ldots, n-1\} \setminus \{i-1, j, k-1, l\}$. Moreover, for $j + 1 < k$*

$$\pi^h(s_{i-1}) = \pi(s_{k-1}), \quad \pi^h(s_j) = \pi(s_l), \quad \pi^h(s_{k-1}) = \pi(s_{i-1}), \quad \pi^h(s_l) = \pi(s_j),$$

and for $j = k - 1$, we have

$$\pi^h(s_{i-1}) = \pi(s_j), \quad \pi^h(s_j) = \pi(s_l), \quad \pi^h(s_l) = \pi(s_{i-1}).$$

We shall proceed by analyzing the induced changes of the π-cycles when passing to π^h. By Lemma 2.8, only the π-cycles containing s_{i-1}, s_j, s_l will be affected so that only these changes will be explicitly displayed.

Lemma 2.9. *Let $(s^h, \pi^h) = \chi_h \circ (s, \pi)$, where $h = (i, j, j+1, l)$. Then there exist the following six possible scenarios for the pair (π, π^h):*

Case 1	π	$(s_{i-1}\ v_1^i\ \ldots v_{m_i}^i)(s_j\ v_1^j\ \ldots v_{m_j}^j)(s_l\ v_1^l\ \ldots v_{m_l}^l)$
	π^h	$(s_{i-1}\ v_1^j\ \ldots v_{m_j}^j\ s_j\ v_1^l\ \ldots v_{m_l}^l\ s_l\ v_1^i\ \ldots v_{m_i}^i)$
Case 2	π	$(s_{i-1}\ v_1^i\ \ldots v_{m_i}^i\ s_l\ v_1^l\ \ldots\ v_{m_l}^l\ s_j\ v_1^j\ \ldots\ v_{m_j}^j)$
	π^h	$(s_{i-1}\ v_1^j\ \ldots\ v_{m_j}^j)(s_j\ v_1^l\ \ldots\ v_{m_l}^l)(s_l\ v_1^i\ \ldots\ v_{m_i}^i)$
Case 3	π	$(s_{i-1}\ v_1^i\ \ldots\ v_{m_i}^i\ s_j\ v_1^j\ \ldots\ v_{m_j}^j\ s_l\ v_1^l\ \ldots\ v_{m_l}^l)$
	π^h	$(s_{i-1}\ v_1^j\ \ldots\ v_{m_j}^j\ s_l\ v_1^i\ \ldots\ v_{m_i}^i\ s_j\ v_1^l\ \ldots\ v_{m_l}^l)$
Case 4	π	$(s_{i-1}\ v_1^i\ \ldots\ v_{m_i}^i\ s_j\ v_1^j\ \ldots\ v_{m_j}^j)(s_l\ v_1^l\ \ldots\ v_{m_l}^l)$
	π^h	$(s_{i-1}\ v_1^j\ \ldots\ v_{m_j}^j)(s_j\ v_1^l\ \ldots\ v_{m_l}^l\ s_l\ v_1^i\ \ldots\ v_{m_i}^i)$
Case 5	π	$(s_{i-1}\ v_1^i\ \ldots\ v_{m_i}^i)(s_j\ v_1^j\ \ldots\ v_{m_j}^j\ s_l\ v_1^l\ \ldots\ v_{m_l}^l)$
	π^h	$(s_{i-1}\ v_1^j\ \ldots\ v_{m_j}^j\ s_l\ v_1^i\ \ldots\ v_{m_i}^i)(s_j\ v_1^l\ \ldots\ v_{m_l}^l)$
Case 6	π	$(s_{i-1}\ v_1^i\ \ldots\ v_{m_i}^i\ s_l\ v_1^l\ \ldots\ v_{m_l}^l)(s_j\ v_1^j\ \ldots\ v_{m_j}^j)$
	π^h	$(s_{i-1}\ v_1^j\ \ldots\ v_{m_j}^j\ s_j\ v_1^l\ \ldots\ v_{m_l}^l)(s_l\ v_1^i\ \ldots\ v_{m_i}^i)$

Proof. We shall only prove Case 1 and Case 2, the remaining four cases can be shown analogously. For Case 1, the π-cycles containing s_{i-1}, s_j, s_l are

$$(s_{i-1}\ v_1^i\ \ldots\ v_{m_i}^i), (s_j\ v_1^j\ \ldots\ v_{m_j}^j), (s_l\ v_1^l\ \ldots\ v_{m_l}^l).$$

Lemma 2.8 allows us to identify the new cycle structure by inspecting the critical points s_{i-1}, s_j and s_l. Here we observe that all three cycles merge and form a single π^h-cycle

$$\begin{aligned}(s_{i-1}\ \pi^h(s_{i-1})\ (\pi^h)^2(s_{i-1})\ \ldots) &= (s_{i-1}\ \pi(s_j)\ \pi^2(s_j)\ \ldots) \\ &= (s_{i-1}\ v_1^j\ \ldots\ v_{m_j}^j\ s_j\ v_1^l\ \ldots\ v_{m_l}^l\ s_l\ v_1^i\ \ldots\ v_{m_i}^i).\end{aligned}$$

For Case 2, the π-cycle containing s_{i-1}, s_j, s_l is

$$(s_{i-1}\ v_1^i\ \ldots\ v_{m_i}^i\ s_l\ v_1^l\ \ldots\ v_{m_l}^l\ s_j\ v_1^j\ \ldots\ v_{m_j}^j).$$

We compute the π^h-cycles containing s_{i-1}, s_j and s_l in π^h as

$$\begin{aligned}
(s_j\ \pi^h(s_j)\ (\pi^h)^2(s_j)\ \ldots) &= (s_j\ \pi(s_l)\ \pi^2(s_l)\ \ldots) = (s_j\ v_1^l\ \ldots\ v_{m_l}^l) \\
(s_l\ \pi^h(s_l)\ (\pi^h)^2(s_l)\ \ldots) &= (s_l\ \pi(s_{i-1})\ \pi^2(s_{i-1})\ \ldots) = (s_l\ v_1^i\ \ldots\ v_{m_i}^i) \\
(s_{i-1}\ \pi^h(s_{i-1})\ (\pi^h)^2(s_{i-1})\ \ldots) &= (s_{i-1}\ \pi(s_j)\ \pi^2(s_j)\ \ldots) = (s_{i-1}\ v_1^j\ \ldots\ v_{m_j}^j)
\end{aligned}$$

whence the lemma. □

If we wish to express which cycles are impacted by a transposition of scenario k acting on a plane permutation, we shall say "the cycles are acted upon by a Case k transposition".

We next observe

Lemma 2.10. *Let $\mathfrak{p}^h = \chi_h \circ \mathfrak{p}$ where χ_h is a transposition. Then the difference of the number of cycles of $\mathfrak{p}$ and $\mathfrak{p}^h$ is even. Furthermore the difference of the number of cycles, odd cycles, even cycles between $\mathfrak{p}$ and $\mathfrak{p}^h$ is contained in $\{-2, 0, 2\}$.*

Proof. Lemma 2.9 implies that the difference of the numbers of cycles of π and π^h is even. As for the statement about odd cycles, since the parity of the total number of elements contained in the cycles containing s_{i-1}, s_j and s_l is preserved, the difference of the number of odd cycles is even. Consequently, the difference of the number of even cycles is also even whence the lemma. □

Suppose we are given $h = (i, j, k, l)$, where $j + 1 < k$, i.e., the two diagonal blocks are not adjacent. Then using the strategy of the proof of Lemma 2.9, we have the upcoming lemma.

Lemma 2.11. *Let* $(s^h, \pi^h) = \chi_h \circ (s, \pi)$, *where* $h = (i, j, k, l)$ *and* $j + 1 < k$. *Then, the difference of the numbers of* π*-cycles and* π^h*-cycles is contained in* $\{-2, 0, 2\}$. *Furthermore, the scenarios, where the number of* π^h*-cycles increases by* 2, *are given by:*

Case a	π	$(s_{i-1}\ v_1^i\ \ldots\ v_{m_i}^i\ s_j\ v_1^j\ \ldots\ v_{m_j}^j\ s_l\ v_1^l\ \ldots\ v_{m_l}^l\ s_{k-1}\ v_1^k\ \ldots\ v_{m_k}^k)$
	π^h	$(s_{i-1}\ v_1^k\ \ldots\ v_{m_k}^k)(s_j\ v_1^l\ \ldots\ v_{m_l}^l\ s_{k-1}\ v_1^i\ \ldots\ v_{m_i}^i)(s_l\ v_1^j\ \ldots\ v_{m_j}^j)$
Case b	π	$(s_{i-1}\ v_1^i\ \ldots\ v_{m_i}^i\ s_{k-1}\ v_1^k\ \ldots\ v_{m_k}^k\ s_j\ v_1^j\ \ldots\ v_{m_j}^j\ s_l\ v_1^l\ \ldots\ v_{m_l}^l)$
	π^h	$(s_{i-1}\ v_1^k\ \ldots\ v_{m_k}^k\ s_j\ v_1^l\ \ldots\ v_{m_l}^l)(s_{k-1}\ v_1^i\ \ldots\ v_{m_i}^i)(s_l\ v_1^j\ \ldots\ v_{m_j}^j)$
Case c	π	$(s_{i-1}\ v_1^i\ \ldots\ v_{m_i}^i\ s_{k-1}\ v_1^k\ \ldots\ v_{m_k}^k\ s_l\ v_1^l\ \ldots\ v_{m_l}^l\ s_j\ v_1^j\ \ldots\ v_{m_j}^j)$
	π^h	$(s_{i-1}\ v_1^k\ \ldots\ v_{m_k}^k\ s_l\ v_1^j\ \ldots v_{m_j}^j)(s_{k-1}\ v_1^i\ \ldots\ v_{m_i}^i)(s_j\ v_1^l\ \ldots\ v_{m_l}^l)$
Case d	π	$(s_{i-1}\ v_1^i\ \ldots\ v_{m_i}^i\ s_l\ v_1^l\ \ldots\ v_{m_l}^l\ s_j\ v_1^j\ \ldots\ v_{m_j}^j\ s_{k-1}\ v_1^k\ \ldots\ v_{m_k}^k)$
	π^h	$(s_{i-1}\ v_1^k\ \ldots\ v_{m_k}^k)(s_j\ v_1^l\ \ldots\ v_{m_l}^l)(s_{k-1}\ v_1^i\ \ldots\ v_{m_i}^i\ s_l\ v_1^j\ \ldots\ v_{m_j}^j)$
Case e	π	$(s_{i-1}\ v_1^i\ \ldots\ v_{m_i}^i\ s_{k-1}\ v_1^k\ \ldots\ v_{m_k}^k)(s_j\ v_1^j\ \ldots\ v_{m_j}^j\ s_l\ v_1^l\ \ldots\ v_{m_l}^l)$
	π^h	$(s_{i-1}\ v_1^k\ \ldots\ v_{m_k}^k)(s_j\ v_1^l\ \ldots\ v_{m_l}^l)(s_{k-1}\ v_1^i\ \ldots\ v_{m_i}^i)(s_l\ v_1^j\ \ldots\ v_{m_j}^j)$

These tables in Lemma 2.9 and 2.11, as we have seen, are easy to obtain. However, they provide quite a lot information and are very useful. For example, Case 1 and Case 2 in Lemma 2.9 are kind of inverse to each other in appearance. A careful investigation could translate this into a bijection, which will facilitate certain enumeration problems as we will discuss in the next section.

Similar to the idea of equivalent fatgraphs, we define equivalent plane permutations.

Definition 2.12. Two plane permutations (s, π) and (s', π') on $[n]$ are *equivalent* if there exists a permutation α on $[n]$ such that

$$s = \alpha s' \alpha^{-1}, \quad \pi = \alpha \pi' \alpha^{-1}.$$

Lemma 2.13. *For two equivalent plane permutations* $\mathfrak{p} = (s, \pi)$ *and* $\mathfrak{p}' = (s', \pi')$*, we have*

$$Exc(\mathfrak{p}) = Exc(\mathfrak{p}'). \tag{2.5}$$

Proof. Assume $s = \alpha s' \alpha^{-1}, \pi = \alpha \pi' \alpha^{-1}$ for some α. Since conjugation by α is equivalent to relabeling according to α, $a <_{s'} b$ implies $\alpha(a) <_s \alpha(b)$. Therefore, an exceedance of $\mathfrak{p}'$ will uniquely correspond to an exceedance of $\mathfrak{p}$, whence the lemma. □

2.3 Enumerative results

Let us first set up for what we want to enumerate. Let U_D denote the set of plane permutations having D as diagonals for some fixed permutation D on $[n]$ of cycle-type λ. In this section, we will divide this set into subsets by different criteria and enumerate the subsets correspondingly.

Let q^λ denote the number of permutations being of cycle-type λ. Given a permutation γ with cycle-type λ, denote $W^\lambda_{\mu,\eta}$ the number of different ways of writing γ as a product of α and β, i.e., $\gamma = \alpha\beta$, where α is of cycle-type μ and β is of cycle-type η. Clearly, this number only depends on λ instead of the specific choice of γ. In addition, We have the following lemma reflecting certain symmetry of these numbers.

Lemma 2.14.

$$W^\lambda_{\mu,\eta} = W^\lambda_{\eta,\mu}, \quad q^\lambda W^\lambda_{\mu,\eta} = q^\mu W^\mu_{\lambda,\eta} = q^\eta W^\eta_{\lambda,\mu}. \tag{2.6}$$

Proof. For any factorization $\gamma = \alpha\beta$, we have $\gamma^{-1} = \beta^{-1}\alpha^{-1}$. Furthermore, γ and γ^{-1} are certainly the same cycle-type λ. Then, the first relation follows.

Since for fixed γ, there are $W^{\lambda}_{\mu,\eta}$ triples (α, β, γ) where $\gamma = \alpha\beta$, if we let γ ranges over all permutations of cycle-type λ, the total number of triples will be $q^{\lambda}W^{\lambda}_{\mu,\eta}$. Next, among all these triples, we want to know the number of triples where the first coordinate is a fixed permutation of cycle-type μ. Since there will be q^{μ} permutations of cycle-type μ in the first coordinate and each of them gives the same number, the desired number would be $\frac{q^{\lambda}W^{\lambda}_{\mu,\eta}}{q^{\mu}}$, i.e.,

$$W^{\mu}_{\lambda,\eta} = \frac{q^{\lambda}W^{\lambda}_{\mu,\eta}}{q^{\mu}}.$$

The rest of the relations come in the same way and the proof is complete. □

Note $\mathfrak{p} = (s, \pi) \in U_D$ iff $D = D_{\mathfrak{p}} = s \circ \pi^{-1}$. Then, the number $|U_D|$ enumerates the ways to write D as a product of an n-cycle with another permutation. Due to the symmetry implied in Lemma 2.14, $|U_D|$ is also certain multiple of the number of factorizations of $(1\ 2\ \cdots\ n)$ into a permutation of cycle-type λ and another permutation, i.e., rooted hypermaps having one face. When D is a fixed-point free involution, $|U_D|$ is a multiple of rooted one-face maps. A *hypermap* is a triple of permutations $(\alpha, \beta_1, \beta_2)$, such that $\alpha = \beta_1\beta_2$. The cycles in α are called faces, the cycles in β_1 are called (hyper)edges, and the cycles in β_2 are called vertices.

2.3.1 Filtering by the number of cycles

First, let D be a fixed permutation of cycle-type λ. Recall that there is an enumeration problem of one-face maps of n edges and genus g in Chapter 1. By Euler's characteristic formula, here the number of edges and faces are fixed, so the genus g is completely determined by the number of vertices. In terms of plane permutations, that is the number of cycles in the vertical. This motivates our first criterion of partitioning the set U_D: We classify U_D into subsets such that plane permutations in the same subset have the same number of cycles. Our objective of this section is to enumerate the size of these subsets.

The following lemma will be presented here for later use.

Lemma 2.15. *Let C_1 and C_2 be two π-cycles of (s, π) such that $\min\{C_1\} <_s \min\{C_2\}$. Suppose we have a Case 2 transposition on C_2, splitting C_2 into the three π^h-cycles C_{21}, C_{22}, C_{23} in (s^h, π^h). Then*

$$\min\{C_1\} <_{s^h} \min\{\min\{C_{21}\}, \min\{C_{22}\}, \min\{C_{23}\}\}. \tag{2.7}$$

Proof. Note that any Case 2 transposition on C_2 will not change C_1. Furthermore, it will only impact the relative order of elements larger than $\min\{C_2\}$, whence the proof. □

Next, we will construct a bijection between the set Y_1 and $Y_2 \cup Y_3$, where Y_1 denotes the set of pairs $(\mathfrak{p}, \epsilon)$, where $\mathfrak{p} \in U_D$ has b cycles and ϵ is an NTAE in $\mathfrak{p}$, Y_2 denotes the set of $\mathfrak{p}' \in U_D$ in which there are 3 labeled cycles among the total $b+2$ $\mathfrak{p}'$-cycles and finally Y_3 denotes the set of plane permutations $\mathfrak{p}' \in U_D$ where there are 3 labeled cycles among the total $b+2$ $\mathfrak{p}'$-cycles and a distinguished NTAE contained in the labeled cycle that contains the largest minimal element. Note that the plane permutations in Y_1 and Y_2 (as well as Y_3) belong to different groups. Thus the bijection will eventually allow us to enumerate the sizes of these groups recursively.

The bijection is based on Case 1 and Case 2 of Lemma 2.9, and motivated by the gluing/slicing bijection of Chapuy [16] for maps (i.e., D is restricted to be an involution without fixed points). In fact, Case 1 corresponds to the gluing operation and Case 2 corresponds to the slicing operation. The difference is that in [16], operations were defined on vertices of maps first, then the corresponding transformations on the boundary were analyzed, while in our approach it is more natural to study the transposition action on the boundary (i.e., face) first and all possible transformations on vertices are immediately clear as in Lemma 2.9. Our results extend those of [16] to general permutations (or hypermaps) as Case 1/Case 2 can be employed irrespective of the cycle-type of the diagonal.

Proposition 2.16. *For any D, $|Y_1| = |Y_2| + |Y_3|$.*

Proof. Given $(\mathfrak{p}, \epsilon) \in Y_1$ where $\mathfrak{p} = (s, \pi)$. We consider the NTAE ϵ and identify a Case 2 transposition χ_h, $h = (i, j, j+1, l)$ as follows: assume ϵ is contained in the cycle

$$C = (s_{i-1}\ v_1^i\ \dots\ v_{m_i}^i\ s_l\ v_1^l\ \dots\ v_{m_l}^l\ s_j\ v_1^j\ \dots\ v_{m_j}^j),$$

where $s_{i-1} = \min\{C\}$, $v_{m_l}^l = \epsilon$, $s_j = \pi(\epsilon)$ and s_l has the property that s_l is the smallest in $\{v_1^i, \dots v_{m_i}^i, s_l, v_1^l, \dots v_{m_l}^l\}$ such that $s_j <_s s_l$. Such an element exists by construction and we have $s_{i-1} <_s s_j <_s s_l \leq_s \epsilon$.

Let $\mathfrak{p}^h = (s^h, \pi^h) = \chi_h \circ \mathfrak{p}$, we have

$$(s, \pi) = \begin{pmatrix} \cdots & s_{i-1} & s_i & \cdots & s_j & s_{j+1} & \cdots & s_l & \cdots & \epsilon & \cdots \\ & \downarrow & & & \downarrow & & & \downarrow & & & \\ \cdots & \pi(s_{i-1}) & \pi(s_i) & \cdots & \pi(s_j) & \pi(s_{j+1}) & \cdots & \pi(s_l) & \cdots & \pi(\epsilon) & \cdots \end{pmatrix},$$
$$(s^h, \pi^h) = \begin{pmatrix} \cdots & s_{i-1} & s_{j+1} & \cdots & s_l & s_i & \cdots & s_j & \cdots & \epsilon & \cdots \\ & \downarrow & & & \downarrow & & & \downarrow & & & \\ \cdots & \pi(s_j) & \pi(s_{j+1}) & \cdots & \pi(s_{i-1}) & \pi(s_i) & \cdots & \pi(s_l) & \cdots & \pi(\epsilon) & \cdots \end{pmatrix}.$$

Then, $s_{i-1} <_{s^h} s_l <_{s^h} s_j$. According to Lemma 2.9, s_{i-1}, s_j, s_l will be contained in three distinct cycles of π^h, namely

$$(s_{i-1}\ v_1^j\ \dots\ v_{m_j}^j), \quad (s_j\ v_1^l\ \dots\ v_{m_l}^l), \quad (s_l\ v_1^i\ \dots\ v_{m_i}^i).$$

It is clear that s_{i-1} is still the minimum element w.r.t. $<_{s^h}$ in its cycle. By construction we have

$$\{v_1^i, \dots v_{m_i}^i\} \subset\,]s_{i-1}, s_j[\, \cup\,]s_l, s_n] \quad \text{and} \quad \{v_1^l, \dots v_{m_l}^l\} \subset\,]s_{i-1}, s_j[\, \cup\,]s_l, s_n]$$

in s. After transposing $[s_i, s_j]$ and $[s_{j+1}, s_l]$, all elements contained in $]s_{i-1}, s_j[$ will be larger than s_l in s^h and all elements of $]s_l, s_n]$ remain in s^h to be larger than s_l. This implies that all elements in the segment $v_1^i \dots v_{m_i}^i$ will be larger than s_l in s^h. Accordingly, s_l is the minimum element in the cycle $(s_l\ v_1^i\ \dots\ v_{m_i}^i)$.

It remains to inspect $(s_j\ v_1^l\ \dots\ v_{m_l}^l)$. We find two scenarios:

1. If s_j is the minimum (w.r.t. $<_{s^h}$), then $v_1^l \dots v_{m_l}^l$ contains no element of $]s_{i-1}, s_j[$ in s. We claim that in this case there is a bijection between the pairs $(\mathfrak{p}, \epsilon)$ and the set Y_2. It suffices to specify the inverse: given an Y_2-element, $\mathfrak{p}' = (s', \pi')$ with three labeled cycles $(s'_{i-1}\ u_1^i\ \dots\ u_{m_i}^i)$, $(s'_j\ u_1^j\ \dots\ u_{m_j}^j)$ and $(s'_l\ u_1^l\ \dots\ u_{m_l}^l)$ we consider a Case 1 transposition determined by the three minimum elements, $s'_{i-1} <_{s'} s'_j <_{s'} s'_l$ in the respective three cycles. This generates a plane permutation (s, π) together with a distinguished NTAE, ϵ, obtained as follows: after transposing, the three cycles merge into
$$C = (s'_{i-1}\ u_1^j\ \dots\ u_{m_j}^j\ s'_j\ u_1^l\ \dots\ u_{m_l}^l\ s'_l\ u_1^i\ \dots\ u_{m_i}^i),$$
where $s'_{i-1} <_s s'_l <_s s'_j$. Since elements contained in $u_1^l \dots u_{m_l}^l$ are by construction larger than s'_l w.r.t. $<_{s'}$ and these elements will not be moved by the transpose, $u_{m_l}^l >_s s'_l$, i.e., $\epsilon = u_{m_l}^l$ is the NTAE. In case of $\{u_1^l, \dots u_{m_l}^l\} = \varnothing$ we have $\epsilon = s'_j$. The following diagram illustrates the situation
$$\begin{tikzcd}
s_{i-1} < s_j < s_l \le \epsilon = v_{m_l}^l \arrow[d] & s'_{i-1} < s'_l < s'_j \le \epsilon = u_{m_l}^l \arrow[l, "s_j = s'_l"', "s_{i-1}=s'_{i-1}, s_l = s'_j"] \\
(s_{i-1}\ \cdots_{v^i}\ s_l\ \cdots_{v^l}\ s_j\ \cdots_{v^j}) \arrow[d, "Case\ 2"] & (s'_{i-1}\ \cdots_{u^j}\ s'_j\ \cdots_{u^l}\ s'_l\ \cdots_{u^i}) \arrow[l, "v^j = u^i"', "v^i = u^j, v^l = u^l"] \arrow[u] \\
(s_{i-1}\ \cdots_{v^j})(s_j\ \cdots_{v^l})(s_l\ \cdots_{v^i}) \arrow[r, "v^i = u^j, v^l = u^l", "v^j = u^i"'] \arrow[d] & (s'_{i-1}\ \cdots_{u^i})(s'_j\ \cdots_{u^j})(s'_l\ \cdots_{u^l}) \arrow[u, "Case\ 1"] \\
s_{i-1} < s_l < s_j \arrow[r, "s_{i-1}=s'_{i-1}, s_l = s'_j", "s_j = s'_l"'] & s'_{i-1} < s'_j < s'_l \arrow[u]
\end{tikzcd}$$

where $\cdots_{v^i}$ denotes the sequence $v_1^i \dots v_{m_i}^i$.

2. If s_j is not the minimum, then $\{v_1^l, \dots v_{m_l}^l\} \neq \varnothing$ and $\epsilon = v_{m_l}^l$. Since by construction, $\epsilon \in]s_l, s_n]$ in s, it will not be impacted by the transposition and we have $s_j <_{s^h} \epsilon$. Therefore, ϵ persists to be an NTAE in $\mathfrak{p}^h$. We furthermore observe

$$\epsilon >_{s^h} s_j >_{s^h} \min\{s_j, v_1^l, \dots v_{m_l}^l\} >_{s^h} s_l >_{s^h} s_{i-1},$$

where $\min\{s_j, v_1^l, \dots v_{m_l}^l\} >_{s^h} s_l$ due to the fact that, after transposing $[s_i, s_j]$ and $[s_{j+1}, s_l]$, all elements in $\{v_1^l, \dots v_{m_l}^l\} \subset]s_{i-1}, s_j[\cup]s_l, s_n]$ will be larger than s_l following $<_{s^h}$. We claim that there is a bijection between such pairs $(\mathfrak{p}, \epsilon)$ and the set Y_3. To this end we specify its inverse: given an element in Y_3, $\mathfrak{p}' = (s', \pi')$ with three labeled cycles

$$(s'_{i-1}\ u_1^i\ \dots\ u_{m_i}^i), \quad (s'_j\ u_1^j\ \dots\ u_{m_j}^j), \quad (s'_l\ u_1^l\ \dots\ u_{m_l}^l),$$

where $\epsilon = u_{m_l}^l$ is the distinguished NTAE. Then a Case 1 transposition w.r.t. the two minima s'_{i-1} and s'_j, and s'_l generates a plane permutation, $\mathfrak{p}$, in which ϵ remains as a distinguished NTAE.

This completes the proof of the proposition. $\square$

Example 2.17. Here we look at an example to illustrate the bijection. Consider the plane permutation with 2 cycles:

$$\mathfrak{p} = \begin{pmatrix} 3 & 5 & 1 & 4 & 8 & 7 & 2 & 6 \\ 8 & 6 & 3 & 5 & 4 & 2 & 7 & 1 \end{pmatrix}, \quad \text{where} \quad \pi = (3\ 8\ 4\ 5\ 6\ 1)(2\ 7).$$

Clearly, both 8 and 6 are NTAEs. For $(\mathfrak{p}, 8)$, we find $3,\ 4,\ 8$ to determine a Case 2 transposition. After the transposition, we obtain

$$\mathfrak{p}' = \begin{pmatrix} 3 & \mathbf{8} & 5 & 1 & 4 & 7 & 2 & 6 \\ \mathbf{5} & 8 & 6 & 3 & 4 & 2 & 7 & 1 \end{pmatrix}, \quad \text{where} \quad \pi' = (3\ 5\ 6\ 1)(\mathbf{8})(4)(2\ 7),$$

and that $3,\ 4,\ 8$ are all the minimal elements in their respective cycles in π', i.e., we have scenario 1. For the pair $(\mathfrak{p}, 6)$, we find $3,\ 1$ and 4 (the smallest in $\{8, 4, 5, 6\}$ which is larger than 1) to determine a Case 2 transposition. After the transposition, we obtain

$$\mathfrak{p}' = \begin{pmatrix} 3 & \mathbf{4} & 5 & 1 & 8 & 7 & 2 & 6 \\ \mathbf{3} & 8 & 6 & 5 & 4 & 2 & 7 & 1 \end{pmatrix}, \quad \text{where} \quad \pi' = (3)(4\ 8)(5\ \mathbf{6}\ 1)(2\ 7),$$

and that $3,\ 4$ are the minimum in their respective cycles in π'. However, the NTAE 6 persists, i.e., scenario 2. This NTAE needs to be distinguished for the purpose of constructing the reverse map of the bijection.

Note that if a pair in Y_1 leads to an element in Y_3, e.g., the second case in the right above example, we can apply the transposition action w.r.t. the resulting plane permutation and the distinguished NTAE again. Iterating in this manner, each plane permutation in U_D with k cycles and a distinguished NTAE eventually leads to a plane permutation in U_D having $2i+1$ labeled cycles among its total $k+2i$ cycles for some $i > 0$. Lemma 2.15 guarantees that there is an unambiguous path to reverse the process, since the last transposition action will always lead to the three labeled cycles with the largest minimum elements.

Therefore, combining Lemma 2.15 and Proposition 2.16, we can conclude that each plane permutation in U_D with k cycles and a distinguished NTAE is in one-to-one correspondence with a plane permutation in U_D having $2i+1$ labeled cycles among its total $k+2i$ cycles for some $i > 0$. Then, we can obtain our first recurrence.

Theorem 2.18. *Let $p_k^\lambda(n)$ denote the number of $\mathfrak{p} \in U_D$ having k cycles where D is of cycle-type λ. Let $p_{a,k}^\lambda(n)$ denote the number of $\mathfrak{p} \in U_D$, where $\mathfrak{p}$ has k cycles, $Exc(\mathfrak{p}) = a$ and D is of type λ. Then,*

$$\sum_{a\geq 0}(n-a-k)p_{a,k}^\lambda(n) = \sum_{i\geq 1}\binom{k+2i}{k-1}p_{k+2i}^\lambda(n). \tag{2.8}$$

Proof. Using the notation of Proposition 2.16 and recursively applying Lemma 2.15 as well as Proposition 2.16, we have

$$\begin{aligned}
|Y_1| &= \sum_{a\geq 0}(n-a-k)p_{a,k}^\lambda(n)\\
&= |Y_2| + |Y_3| = \binom{k+2}{3}p_{k+2}^\lambda(n) + |Y_3|\\
&= \binom{k+2}{3}p_{k+2}^\lambda(n) + \binom{k+4}{5}p_{k+4}^\lambda(n) + \cdots
\end{aligned}$$

whence the theorem. ☐

Remark 2.19. Following from Proposition 2.5, the exact number of terms on the RHS of Eq. (2.8) depends on the number of parts in λ.

Although Eq. (2.8) is not simple for general λ, it can be immediately simplified by applying Lemma 2.6 if D is a fixed-point free involution which leads to Chapuy's recursion [16].

Corollary 2.20 (Chapuy's recursion [16])**.** *The numbers $A(n,g)$ of rooted one-face maps having n edges and genus g satisfy the recursion*

$$2gA(n,g) = \sum_{k=1}^{g}\binom{n+1-2(g-k)}{2k+1}A(n,g-k). \tag{2.9}$$

Proof. Setting $\lambda = 2^n$ in Eq. (2.8), and noticing that $n-a-k$ is the number of NTAEs which is always $2g$ for maps of genus g based on Lemma 2.6, Eq. (2.8) reduces to

$$2gp^{\lambda}_{n+1-2g}(2n) = \sum_{i\geq 1} \binom{n+1-2g+2i}{n-2g} p^{\lambda}_{n+1-2g+2i}(2n),$$

where $\lambda = 2^n$. Using Lemma 2.14, we know $A(n,g) = \frac{q^{2^n}}{q^{(2n)^1}} p^{2^n}_{n+1-2g}(2n)$. The last recurrence then implies Chapuy's recursion. $\square$

2.3.2 Filtering by the cycle-type in the vertical

The recurrence Eq. (2.8) is not very useful as both sides are summations. We proceed to enumerate a more refined classification of U_D, i.e., classifying by the cycle-type in the vertical, which will allow us to obtain more elegant recurrences.

Let μ, η be integer partitions of n. We write $\mu \rhd_{2i+1} \eta$ if μ can be obtained by splitting one η-part into $(2i+1)$ non-zero parts. Let furthermore $\kappa_{\mu,\eta}$ denote the number of different ways to obtain η from μ by merging $\ell(\mu) - \ell(\eta) + 1$ μ-parts into one, where $\ell(\mu)$ and $\ell(\eta)$ denote the number of blocks in the partitions μ and η, respectively.

Let U^{η}_{λ} denote the set of plane permutations, $\mathfrak{p} = (s, \pi) \in U_D$, where D is a fixed permutation of cycle-type λ and the vertical π has cycle-type η. The size of this set is given by the following theorem:

Theorem 2.21. *Let $f_{\eta,\lambda}(n) = |U^{\eta}_{\lambda}|$. For $\ell(\eta) + \ell(\lambda) < n + 1$, we have*

$$f_{\eta,\lambda}(n) = \frac{q^{\lambda} \sum_{i\geq 1} \sum_{\mu \rhd_{2i+1} \eta} \kappa_{\mu,\eta} f_{\mu,\lambda}(n) + q^{\eta} \sum_{i\geq 1} \sum_{\mu \rhd_{2i+1} \lambda} \kappa_{\mu,\lambda} f_{\mu,\eta}(n)}{q^{\lambda}[n+1-\ell(\eta)-\ell(\lambda)]}. \tag{2.10}$$

Proof. Let $f_{\eta,\lambda}(n,a)$ denote the number of $\mathfrak{p} \in U^{\eta}_{\lambda}$ having a exceedances. Note that every plane permutation has at least one exceedance. Thus $0 \leq a \leq n-1$.

*C*laim.

$$\sum_{a\geq 0}(n-a-\ell(\eta))f_{\eta,\lambda}(n,a)=\sum_{i\geq 1}\sum_{\mu\rhd_{2i+1}\eta}\kappa_{\mu,\eta}f_{\mu,\lambda}(n). \tag{2.11}$$

Given $\mathfrak{p}=(s,\pi)$ where the cycle-type of π is η, a Case 2 transposition will result in $\mathfrak{p}^h=(s^h,\pi^h)$ such that π^h has cycle-type μ and $\mu\rhd_3\eta$. Refining the proof of Proposition 2.16, we observe that each pair $(\mathfrak{p}=(s,\pi),\epsilon)$ for which $\mathfrak{p}\in U_\lambda^\eta$ and ϵ is an NTAE, uniquely corresponds to a plane permutation $\mathfrak{p}^h=(s^h,\pi^h)\in U_\lambda^\mu$ with $2i+1$ labeled cycles for some $i>0$, and $\mu\rhd_{2i+1}\eta$. Conversely, suppose we have $\mathfrak{p}^h=(s^h,\pi^h)\in U_\lambda^\mu$ with $\mu\rhd_{2i+1}\eta$. If there are $\kappa_{\mu,\eta}$ ways to obtain η by merging $2i+1$ μ-parts into one, then we can label $2i+1$ cycles of $\mathfrak{p}^h$ in $\kappa_{\mu,\eta}$ different ways, which correspond to $\kappa_{\mu,\eta}$ pairs $(\mathfrak{p}=(s,\pi),\epsilon)$ where the cycle-type of π is η and this implies the Claim.

Immediately, we have

$$\sum_{a\geq 0}(n-a-\ell(\lambda))f_{\lambda,\eta}(n,a)=\sum_{i\geq 1}\sum_{\mu\rhd_{2i+1}\lambda}\kappa_{\mu,\lambda}f_{\mu,\eta}(n). \tag{2.12}$$

*C*laim.

$$q^\lambda f_{\eta,\lambda}(n,a)=q^\eta f_{\lambda,\eta}(n,n-1-a). \tag{2.13}$$

Note that any $\mathfrak{p}=(s,\pi)\in U_\lambda^\eta$ satisfies $s=D\pi$. Taking the inverse to "reflect" the equation, we uniquely obtain $s^{-1}=\pi^{-1}D^{-1}$. The latter can be transformed into an equivalent plane permutation $\mathfrak{p}'=(s',\pi')\in U_\eta^\lambda$ by conjugation, where elements in U_η^λ have a fixed permutation D' of cycle-type η as diagonal. Namely, for some γ, we have

$$s'=\gamma s^{-1}\gamma^{-1}=\gamma(s_0\ s_{n-1}\ \cdots\ s_1)\gamma^{-1},\quad \pi'=\gamma D^{-1}\gamma^{-1},\quad D_{\mathfrak{p}'}=D'=\gamma\pi^{-1}\gamma^{-1}.$$

Next, we will show that if $\mathfrak{p}$ has a exceedances, the plane permutation (s^{-1},D^{-1}) has $n-1-a$

exceedances, so that $\mathfrak{p}'$ has $n-1-a$ exceedances according to Lemma 2.13. Indeed, if $\mathfrak{p}$ has a exceedances, Lemma 2.4 guarantees that D has $n-(a+1)=n-1-a$ exceedances w.r.t. $<_s$. Since an exceedance in D is a strict anti-exceedance (i.e., strictly decreasing) in D^{-1}, D^{-1} has $n-1-a$ strict anti-exceedances w.r.t. $<_s$. However, following the linear order $\hat{s}=s_0 s_{n-1} s_{n-2}\cdots s_1$ (induced by s^{-1}), any strict anti-exceedance w.r.t. $<_s$ of D^{-1} the image of which is not s_0, will become an exceedance. It remains to distinguish the following two situations: if s_0 is not the image of a strict anti-exceedance, s_0 must be a fixed point, so D^{-1} has $n-1-a$ exceedances; if s_0 is not a fixed point, the strict anti-exceedance having s_0 as image remains as a strict anti-exccedance in D^{-1}. Furthermore, s_0 must be an exceedance of D^{-1} (w.r.t. $<_s$), and it remains to be an exceedance w.r.t. $<_{\hat{s}}$. In this case, there are also $(n-1-a-1)+1=n-1-a$ exceedances in D^{-1}. Finally, due to the one-to-one correspondence, $f_{\eta,\lambda}(n,a)$ plane permutations (s,π) imply that $f_{\eta,\lambda}(n,a)$ plane permutations (s^{-1},D^{-1}) have $n-1-a$ exceedances. Following the same argument as Lemma 2.14, the cardinality of the latter set is also equal to $\frac{q^\eta}{q^\lambda}f_{\lambda,\eta}(n,n-1-a)$, whence the claim.

Therefore,

$$
\begin{aligned}
&\sum_{a\geq 0}(n-a-\ell(\eta))q^\lambda f_{\eta,\lambda}(n,a)+\sum_{a\geq 0}(n-a-\ell(\lambda))q^\eta f_{\lambda,\eta}(n,a)\\
=&\sum_{a\geq 0}(n-a-\ell(\eta))q^\lambda f_{\eta,\lambda}(n,a)+(n-(n-1-a)-\ell(\lambda))q^\eta f_{\lambda,\eta}(n,n-1-a)\\
=&\sum_{a\geq 0}(n-a-\ell(\eta))q^\lambda f_{\eta,\lambda}(n,a)+(n-(n-1-a)-\ell(\lambda))q^\lambda f_{\eta,\lambda}(n,a)\\
=&(n+1-\ell(\eta)-\ell(\lambda))\sum_{a\geq 0}q^\lambda f_{\eta,\lambda}(n,a)\\
=&(n+1-\ell(\eta)-\ell(\lambda))q^\lambda f_{\eta,\lambda}(n).
\end{aligned}
$$

Multiplying q^λ and q^η on both sides of Eq. (2.11) and Eq. (2.12), respectively, and summing up the LHS and the RHS of Eq. (2.11) and Eq. (2.12), respectively, completes the proof. □

In order to obtain a recurrence in terms of $p_k^\lambda(n)$, we sum over all η with $\ell(\eta) = k$.

Corollary 2.22. *For $\ell(\lambda) < n+1-k$, we have*

$$p_k^\lambda(n) = \frac{\sum_{i\geq 1}\binom{k+2i}{k-1}p_{k+2i}^\lambda(n)q^\lambda + \sum_{i\geq 1}\sum_{\mu\rhd_{2i+1}\lambda}\kappa_{\mu,\lambda}p_k^\mu(n)q^\mu}{q^\lambda[n+1-k-\ell(\lambda)]}. \tag{2.14}$$

Proof. For any μ with $\ell(\mu) = k+2i$, merging any $2i+1$ parts leads to some η with $\ell(\eta) = k$ and $\mu \rhd_{2i+1} \eta$. Also note, if $\mu \rhd_{2i+1} \eta$ does not hold, $\kappa_{\mu,\eta} = 0$. Thus, for any μ with $\ell(\mu) = k+2i$, $\sum_{\eta,\ell(\eta)=k}\kappa_{\mu,\eta} = \binom{k+2i}{2i+1}$. Furthermore, $\sum_{\mu,\ell(\mu)=k+2i} f_{\mu,\lambda}(n) = p_{k+2i}^\lambda(n)$. Therefore,

$$\begin{aligned}\sum_{\substack{\eta,\\ \ell(\eta)=k}}\sum_{i\geq 1}\sum_{\mu\rhd_{2i+1}\eta}\kappa_{\mu,\eta}f_{\mu,\lambda}(n)q^\lambda &= \sum_{i\geq 1}\sum_{\substack{\mu,\\ \ell(\mu)=k+2i}}\sum_{\substack{\eta,\\ \ell(\eta)=k}}\kappa_{\mu,\eta}f_{\mu,\lambda}(n)q^\lambda\\ &= \sum_{i\geq 1}\binom{k+2i}{k-1}p_{k+2i}^\lambda(n)q^\lambda.\end{aligned}$$

We also have

$$\begin{aligned}\sum_{\substack{\eta,\\ \ell(\eta)=k}}\sum_{i\geq 1}\sum_{\mu\rhd_{2i+1}\lambda}\kappa_{\mu,\lambda}f_{\mu,\eta}(n)q^\eta &= \sum_{i\geq 1}\sum_{\mu\rhd_{2i+1}\lambda}\kappa_{\mu,\lambda}\sum_{\substack{\eta,\\ \ell(\eta)=k}}f_{\mu,\eta}(n)q^\eta\\ &= \sum_{i\geq 1}\sum_{\mu\rhd_{2i+1}\lambda}\kappa_{\mu,\lambda}p_k^\mu(n)q^\mu,\end{aligned}$$

whence the corollary. □

Note that $p_1^\lambda(n)$ also counts the number of ways of writing a permutation of cycle-type λ into two n-cycles. In Stanley [58], an explicit formula for $p_1^\lambda(n)$ was given as, if $\lambda =$

$(1^{a_1}, 2^{a_2}, \ldots, n^{a_n})$, then

$$p_1^\lambda(n) = \sum_{i=0}^{n-1} \frac{i!(n-1-i)!}{n} \sum_{r_1,\ldots,r_i} \binom{a_1-1}{r_1}\binom{a_2}{r_2}\cdots\binom{a_i}{r_i}(-1)^{r_2+r_4+r_6+\cdots},$$

where $r_1, \ldots, r_i$ ranges over all non-negative integer solutions of the equation $\sum_j jr_j = i$. As a quick application of Corollary 2.22, we obtain a recurrence for $p_1^\lambda(n)$ from which we can obtain simple closed formulas for some particular cases which seems not obvious from Stanley's explicit formula.

Proposition 2.23. *For any $\lambda \vdash n$ and $n - \ell(\lambda)$ even, we have*

$$p_1^\lambda(n) = \frac{(n-1)! + \sum_{i\geq 1}\sum_{\substack{\mu,\\ \ell(\mu)=2i+\ell(\lambda)}} \kappa_{\mu,\lambda} p_1^\mu(n)\frac{q^\mu}{q^\lambda}}{n+1-\ell(\lambda)}. \tag{2.15}$$

In particular, for λ having only small parts, we have

$$p_1^{1^{a_1}2^{a_2}} = \frac{(n-1)!}{n+1-a_1-a_2}, \tag{2.16}$$

$$p_1^{1^{a_1}2^{a_2}3^1} = \frac{(n-1)![2(n-a_1-a_2)-3]}{2(n-a_1-a_2)(n-2-a_1-a_2)}, \tag{2.17}$$

$$p_1^{1^{a_1}2^{a_2}4^1} = \frac{(n-1)!(n-1-a_1-a_2)}{(n-2-a_1-a_2)(n-a_1-a_2)}. \tag{2.18}$$

Proof. Setting $k=1$ in Eq. (2.14), we have

$$[n-\ell(\lambda)]p_1^\lambda(n) = \sum_{i\geq 1}\binom{1+2i}{0}p_{1+2i}^\lambda(n) + \sum_{i\geq 1}\sum_{\mu \rhd_{2i+1}\lambda} \kappa_{\mu,\lambda}p_1^\mu(n)\frac{q^\mu}{q^\lambda}.$$

Note, for $n-\ell(\lambda)$ even, a permutation of cycle-type λ can be written as a product of any n-cycle and a permutation with $2i+1$ cycles for some $i \geq 0$. Thus, $\sum_{i\geq 0}\binom{1+2i}{0}p_{1+2i}^\lambda(n) = (n-1)!$ whence the recursion.

For the particular cases, we will only show the second one since the other two follow analogously. For $\lambda = 1^{a_1}2^{a_2}3^1$, we observe $\kappa_{\mu,\lambda} \neq 0$ iff $\mu = 1^{a_1+3}2^{a_2}$. In this case, $\kappa_{\mu,\lambda} = \binom{a_1+3}{3}$. Then, using Eq. (2.15) for $1^{a_1+3}2^{a_2}$, $q^\lambda = \frac{n!}{1^{a_1}2^{a_2}3^1 a_1! a_2! 1!}$ and $q^\mu = \frac{n!}{1^{a_1+3}2^{a_2}(a_1+3)! a_2!}$, and we obtain the second formula. □

2.4 Refining a Zagier-Stanley result

Zagier [69] and Stanley [59] studied the following problem: how many permutations ω from a fixed conjugacy class of $\mathcal{S}_n$ such that the product $\omega(1\ 2\ \cdots\ n)$ has exactly k cycles?

Both authors employed the character theory of the symmetric group in order to obtain certain generating polynomials. Then, by evaluating these polynomials at specific conjugacy classes, Zagier obtained an explicit formula for the number of rooted one-face maps (i.e., the conjugacy class consists of involutions without fixed points), and both, Zagier as well as Stanley, obtained the following surprisingly simple formula for the conjugacy class n^1: the number $\xi_{1,k}(n)$ of ω for which $\omega(1\ 2\ \cdots\ n)$ has exactly k cycles is 0 if $n-k$ is odd, and otherwise $\xi_{1,k}(n) = \frac{2C(n+1,k)}{n(n+1)}$ where $C(n,k)$ is the unsigned Stirling number of the first kind, i.e., the number of permutations on $[n]$ with k cycles. Stanley asked for a combinatorial proof for this result [59]. Such proofs were later given in [27] and in [18].

In this section, we will study the above problem and refine the Zagier-Stanley result, combinatorially, using the framework of plane permutations.

First, it is obvious that $\xi_{1,k}(n) = 0$ if $n-k$ is odd from Proposition 2.7. In addition, note that $\xi_{1,k}(n) = p_k^\lambda(n)$ when $\lambda = n^1$. Then, from Corollary 2.22, we obtain

Corollary 2.24. *For $k \geq 1$, $n \geq 1$ and $n-k$ is even,*

$$(n+1-k)\xi_{1,k}(n) = \sum_{i\geq 1} \binom{k+2i}{k-1} \xi_{1,k+2i}(n) + C(n,k). \tag{2.19}$$

Proof. Inspecting Eq. (2.14), it suffices to show that

$$\sum_{i\geq 1} \sum_{\mu \rhd_{2i+1} \lambda} \kappa_{\mu,\lambda} p_k^{\mu}(n) \frac{q^{\mu}}{q^{n^1}} = C(n-k) - \xi_{1,k}(n).$$

To this end, we first observe that for $\lambda = n^1$, $\mu \rhd_{2i+1} \lambda$ iff $\ell(\mu) = 2i+1$. And $\kappa_{\mu,\lambda} = 1$ then. Also, by symmetry the number of ways of writing the n-cycle $(1\ 2\ \cdots\ n)$ into a product of a permutation with k cycles and a permutation of cycle-type μ equals to $\frac{q^{\mu}}{q^{n^1}} p_k^{\mu}(n)$. On the other hand, it is easy to see that ranging over all $\mu \vdash n$, the total number of ways is exactly $C(n,k)$. Furthermore, if $n-k$ is even, Proposition 2.7 implies that $(1\ 2\ \cdots\ n)$ can be only factorized into a permutation with k cycles and a permutation with j cycles for some odd j, i.e., only $\ell(\mu) = 2i+1$ matter. Thus,

$$\sum_{i\geq 1} \sum_{\mu \rhd_{2i+1} \lambda} \kappa_{\mu,\lambda} p_k^{\mu}(n) \frac{q^{\mu}}{q^{n^1}} = \sum_{i\geq 1} \sum_{\substack{\mu, \\ \ell(\mu)=2i+1}} p_k^{\mu}(n) \frac{q^{\mu}}{q^{n^1}} = C(n-k) - \xi_{1,k}(n),$$

completing the proof. □

Our idea to prove $\xi_{1,k}(n) = \frac{2}{n(n+1)} C(n+1,k)$ for $n-k$ even is to show both sides satisfy the same recurrence and initial conditions. To this end, we will relate the obtained results in terms of exceedances of plane permutations and exceedances of (ordinary) permutations.

Obviously, exceedances of a plane permutation of the form (ε_n, π) is the same as exceedances of the ordinary permutation π. Let $\mathfrak{p} = (s, \pi) \in U_D$, where $\mathfrak{p}$ has a exceedances and k cycles. Assume $\gamma s \gamma^{-1} = \varepsilon_n = (1\ 2\ \cdots\ n)$. Then, the plane permutation $(\varepsilon_n, \gamma\pi\gamma^{-1})$ has a

exceedances and k cycles according to Lemma 2.13. Furthermore, its diagonal is equal to $\gamma D\gamma^{-1}$ which is of cycle-type λ.

Observation: viewing ordinary permutations π as plane permutations of the form (ε_n, π) provides a new way to classify permutations, i.e., by the diagonals.

Lemma 2.25. *Let $\hat{p}^{\lambda}_{a,k}(n)$ denote the number of ordinary permutations having k cycles, a exceedances and λ as the cycle-type of diagonals. Let $\hat{p}^{\lambda}_{k}(n)$ denote the number of ordinary permutations having k cycles and λ as the cycle-type of diagonals. Then,*

$$q^{\lambda}p^{\lambda}_{a,k}(n) = q^{n^1}\hat{p}^{\lambda}_{a,k}(n) = (n-1)!\hat{p}^{\lambda}_{a,k}(n), \quad q^{\lambda}p^{\lambda}_{k}(n) = (n-1)!\hat{p}^{\lambda}_{k}(n).$$

Proof. Let S be a set of plane permutations (s, π) on $[n]$ having k cycles, a exceedances and λ as the cycle-type of diagonals. Clearly, for any fixed s, the number of plane permutations of the form (s, π) is the same as the number of plane permutations of the form (ε_n, π) there. Thus, $|S| = q^{n^1}\hat{p}^{\lambda}_{a,k}(n) = (n-1)!\hat{p}^{\lambda}_{a,k}(n)$. Similarly, the number of plane permutations having a fixed permutation of cycle-type λ as diagonal does not depend on specific choice of the permutation. Hence, $|S| = q^{\lambda}p^{\lambda}_{a,k}(n)$, completing the proof of the first equation. The same reasoning leads to the second equation. □

Proposition 2.26. *Let $p_{a,k}(n)$ denote the number of permutations on $[n]$ containing a exceedances and k cycles. Then,*

$$\sum_{a\geq 0}(n-a-k)p_{a,k}(n) = \sum_{i=1}^{\lfloor\frac{n-k}{2}\rfloor}\binom{k+2i}{k-1}C(n, k+2i). \tag{2.20}$$

In particular, $p_{0,n}(n) = 1$, $p_{1,n-1}(n) = \binom{n}{2}$.

Proof. According to Lemma 2.25, we have

$$
\begin{aligned}
p_{a,k}(n) &= \sum_{\lambda} \hat{p}^{\lambda}_{a,k}(n) = \sum_{\lambda} \frac{q^{\lambda}}{(n-1)!} p^{\lambda}_{a,k}(n),\\
C(n,k) &= \sum_{\lambda} \hat{p}^{\lambda}_{k}(n) = \sum_{\lambda} \frac{q^{\lambda}}{(n-1)!} p^{\lambda}_{k}(n).
\end{aligned}
$$

Multiplying $\frac{q^{\lambda}}{(n-1)!}$ on both sides of Eq. (2.8) and summing over all possible cycle-types λ gives the proposition. $\square$

Clearly, summing over all possible number of exceedances, we get all permutations with k cycles, i.e., we have $\sum_a p_{a,k}(n) = C(n,k)$, and furthermore $\sum_a a p_{a,k}(n)$ counts the total number of exceedances in all permutations with k cycles. Hence, reformulating Eq. (2.20), we have the following corollary:

Corollary 2.27. *The total number of exceedances in all permutations on $[n]$ with k cycles is given by*

$$
\sum_a a p_{a,k}(n) = (n-k)C(n,k) - \sum_{i=1}^{\lfloor \frac{n-k}{2} \rfloor} \binom{k+2i}{k-1} C(n,k+2i). \tag{2.21}
$$

However, it is easy to compute the total number of exceedances as shown below.

Proposition 2.28. *The total number of exceedances in all permutations on $[n]$ with k cycles is $\binom{n}{2}C(n-1,k)$.*

Proof. Note the total number of exceedances in all permutations on $[n]$ with k cycles is equal to the size of the set X of permutations π on $[n]$ with k cycles and with one pair $(i,\pi(i))$ distinguished, where i is an exceedance in π. Let Y denote the set of pairs (τ,α), where τ is a subset of $[n]$ having 2 elements and α is a permutation on $[n-1]$ having k cycles. We will show that there is a bijection between X and Y. Given $(\pi,(i,\pi(i))) \in X$, we obtain

$(\tau, \alpha) \in Y$ as follows: set $\tau = \{i, \pi(i)\}$ and α' on $[n] \setminus \{\pi(i)\}$ as $\alpha'(j) = \pi(j)$ if $j \neq i$ while $\alpha'(i) = \pi^2(i)$. Now we obtain α from α' by substituting $x-1$ for every number $x > \pi(i)$. Conversely, given $(\tau, \alpha) \in Y$, where $\tau = \{a, b\}$ and $a < b$. Define α' from α by substituting $x+1$ for every number $x \geq b$. Next we define π from α' in the following way: $\pi(j) = \alpha'(j)$ if $j \neq a, b$ while $\pi(a) = b$ and $\pi(b) = \alpha'(a)$. Note that by construction a is an exceedance in π and clearly, $|Y| = \binom{n}{2} C(n-1, k)$, whence the proposition. □

Corollary 2.27 and Proposition 2.28 give rise to a new recurrence for the unsigned Stirling numbers of the first kind $C(n, k)$.

Theorem 2.29. *For $n \geq 1, k \geq 1$, we have*

$$C(n+1, k) = \sum_{i \geq 1} \binom{k+2i}{k-1} \frac{C(n+1, k+2i)}{n+1-k} + \binom{n+1}{2} \frac{C(n, k)}{n+1-k}. \tag{2.22}$$

Reformulating Eq. (2.22), we obtain

$$\frac{2C(n+1, k)}{n(n+1)} = \sum_{i \geq 1} \binom{k+2i}{k-1} \frac{1}{n+1-k} \frac{2C(n+1, k+2i)}{n(n+1)} + \frac{C(n, k)}{n+1-k}. \tag{2.23}$$

Comparing Eq. (2.19) and Eq. (2.23), we observe that $\frac{2}{n(n+1)} C(n+1, k)$ and $\xi_{1,k}(n)$ satisfy the same recurrence. Furthermore, the initial value $\xi_{1,n}(n)$ is equal to the number of different ways to factorize an n-cycle into an n-cycle and a permutation with n cycles. Since only the identity map has n cycles, we have $\xi_{1,n}(n) = 1$. On the other hand, $C(n+1, n)$ is the number of permutations on $[n+1]$ with n cycles. Such permutations have cycle-type $1^{n-1}2^1$. It suffices to determine the 2-cycle, which is equivalent to selecting 2 elements from $[n+1]$. Therefore, the initial value $\frac{2}{n(n+1)} C(n+1, n) = \frac{2}{n(n+1)} \binom{n+1}{2} = 1$. Thus, $\frac{2}{n(n+1)} C(n+1, k)$ and $\xi_{1,k}(n)$ agree on the initial values. So, we have

Proposition 2.30 (Zagier [69], Stanley [59]). *For $k \geq 1$, $n \geq 1$ and $n-k$ even, we have*

$$\xi_{1,k}(n) = \frac{2}{n(n+1)} C(n+1,k). \tag{2.24}$$

2.5 From the Lehman-Walsh formula to Chapuy's recursion

Recall that the number of plane permutations is a multiple of the number of rooted one-face maps. In 70's, Walsh and Lehman [64, Eq. (13)], using a direct recursive method and formal power series, obtained an explicit formula for $A(n,g)$ (hence the number of plane permutations) which can be reformulated as follows:

$$A(n,g) = \sum_{\lambda \vdash g} \frac{(n+1)n\cdots(n+2-2g-\ell(\lambda))}{2^{2g}\prod_i c_i!(2i+1)^{c_i}} \frac{(2n)!}{(n+1)!n!}, \tag{2.25}$$

where the summation is taken over partitions λ of g, c_i is the number of parts i in λ, and $\ell(\lambda)$ is the total number of parts.

More than a decade later, Harer and Zagier [41] obtained the following three-term recurrence, known as the Harer-Zagier recurrence:

$$(n+1)A(n,g) = 2(2n-1)A(n-1,g) + (2n-1)(n-1)(2n-3)A(n-2,g-1). \tag{2.26}$$

They furthermore obtained the so-called Harer-Zagier formula:

$$A_n(x) = \frac{(2n)!}{2^n n!} \sum_{k \geq 1} 2^{k-1} \binom{n}{k-1} \binom{x}{k}. \tag{2.27}$$

There is a body of work on how to derive these results [16, 17, 31, 32, 42]. A direct bijection for the Harer-Zagier formula was given in [32]. Combinatorial arguments to obtain the Lehman-Walsh formula and the Harer-Zagier recurrence were recently given in [17]. One of the most recent advances is a new recurrence for $A(n, g)$ obtained by Chapuy [16] via a bijective approach, which was refined in our Corollary 2.20.

In this section, we will show that from the earliest Lehman-Walsh formula, we can actually derive the Harer-Zagier results and Chapuy's recursion, which connects all these formulas spanning over 40 years.

2.5.1 New formulas counting one-face maps

In the following, we will first prove a new formula for $A(n, g)$ by constructing two involutions on pairs of permutations.

We call a cycle of a permutation odd and even if it contains an odd and even number of elements, respectively. Let $O(n+1, g)$ denote the number of permutations on $[n+1]$ which consist of $n+1-2g$ odd (disjoint) cycles. For readers familiar with the formula for the number of permutations of a specific cycle type, see Section 1.1, it may be immediately realized that the Lehman-Walsh expression can be rewritten as

$$\sum_{\lambda \vdash g} \frac{(n+1)n\cdots(n+2-2g-\ell(\lambda))}{2^{2g}\prod_i c_i!(2i+1)^{c_i}} \frac{(2n)!}{(n+1)!n!} = \frac{(2n)!}{(n+1)!n!2^{2g}} O(n+1, g). \tag{2.28}$$

Then, we have more fundamental objects (permutations instead of maps) to work with. First, we combinatorially obtain an explicit formula for $O(n, g)$:

Theorem 2.31. *For $n, l \geq 0$, we have*

$$\sum_{k=0}^{n} \binom{n}{k} \sum_{i+j=l} C(n-k+1,i)(-1)^{k+1-j} C(k+1,j) = 2^{l-2} O(n+1, \frac{n+2-l}{2}). \tag{2.29}$$

Proof. Let $S = \{a, 1, 2, \ldots n, b\}$. Let $\mathcal{T}_{A,l}$ denote the set of pairs (α, β) where α is a permutation on $A \subset S$ while β is a permutation on $S \setminus A$, such that the sum of the number of α- and β-cycles equals l. Let

$$\mathcal{T}_l = \bigcup_{A \subset S} \mathcal{T}_{A,l},$$

where the union is taken over all $A \subset S$ such that $a \in A$ and $b \notin A$. For each pair $(\alpha, \beta) \in \mathcal{T}_l$, we denote the difference between $|S \setminus A|$ and the number of β-cycles as $d(\beta)$ and set $W[(\alpha, \beta)] = (-1)^{d(\beta)}$. Then, it is clear that

Claim 1.

$$\sum_{(\alpha,\beta) \in \mathcal{T}_l} W[(\alpha, \beta)] = \sum_{k=0}^{n} \binom{n}{k} \sum_{i+j=l} C(n-k+1,i)(-1)^{k+1-j} C(k+1,j).$$

Next, let $\mathcal{T}' \subset \mathcal{T}_l$ consist of pairs (α, β) where $\alpha(a) = a$ and b is contained in an odd cycle.

Claim 2.

$$\sum_{(\alpha,\beta) \in \mathcal{T}_l} W[(\alpha, \beta)] = \sum_{(\alpha,\beta) \in \mathcal{T}'} W[(\alpha, \beta)].$$

Given $(\alpha, \beta) \in \mathcal{T}_l$, write both α and β in their cycle decompositions and denote the length of the cycle containing b as B. Define a map $\phi : \mathcal{T}_l \to \mathcal{T}_l$ as follows:

- Case 1: if $(\alpha, \beta) \in \mathcal{T}'$, then $\phi : (\alpha, \beta) \mapsto (\alpha, \beta)$;
- Case 2: if $(\alpha, \beta) \notin \mathcal{T}'$, we distinguish two scenarios:
  - if B is odd and $\alpha(a) \neq a$, then $\phi : (\alpha, \beta) \mapsto (\alpha', \beta')$, where α' is obtained by deleting $\alpha(a)$ from the cycle decomposition of α while β' is obtained by inserting

$\alpha(a)$ between b and $\beta(b)$ in the cycle containing b and if $b = \beta(b)$, we map the cycle (b) to $(b, \alpha(a))$;

- if B is even then $b \neq \beta(b)$. Define $\phi : (\alpha, \beta) \mapsto (\alpha', \beta')$, where α' is obtained by inserting $\beta(b)$ between a and $\alpha(a)$ and for $a = \alpha(a)$, we map (a) to $(a, \beta(b))$. β' is obtained by deleting $\beta(b)$ from the corresponding β-cycle.

It is not hard to check that ϕ is an involution whose fixed points are exactly all Case 1-pairs and the two elements in any length-2 cycle carry opposite signs, whence Claim 2.

Now, let $\mathcal{T}''$ be the set of pairs $(\alpha, \beta) \in \mathcal{T}'$ such that all cycles in α and β are odd.

Claim 3.

$$\sum_{(\alpha,\beta)\in\mathcal{T}'} W[(\alpha, \beta)] = \sum_{(\alpha,\beta)\in\mathcal{T}''} W[(\alpha, \beta)].$$

This can be proved by observing that the following defined map $\varphi : \mathcal{T}' \to \mathcal{T}'$ is a sign-reversing involution which has elements in $\mathcal{T}''$ as fixed points.

- Case 1: if $(\alpha, \beta) \in \mathcal{T}''$, then $\varphi : (\alpha, \beta) \mapsto (\alpha, \beta)$;

- Case 2: if $(\alpha, \beta) \notin \mathcal{T}''$, there is at least one even cycle in the collection of cycles from both α and β. Obviously, there is a unique even cycle, denoted by C, containing the minimal element among the union of elements from all even cycles. Let $\varphi : (\alpha, \beta) \mapsto (\alpha', \beta')$, where

  - if C is a cycle in α, then $\alpha' = \alpha \setminus C$ and $\beta' = \beta \cup C$;

  - otherwise $\alpha' = \alpha \cup C$ and $\beta' = \beta \setminus C$.

Based on these claims above, the LHS of Eq. (2.29) equals $\sum_{(\alpha,\beta)\in\mathcal{T}''} W[(\alpha, \beta)]$. Since in $\mathcal{T}''$ all cycles are odd, the number of elements and the number of cycles in β have the same parity. Thus, for any $(\alpha,\beta)\in\mathcal{T}''$, $W[(\alpha,\beta)]=1$, i.e the total weight over $\mathcal{T}''$ equals the total number of elements in $\mathcal{T}''$.

Since a is a fixed point in α for any $(\alpha,\beta)\in\mathcal{T}''$, each pair $(\alpha,\beta)\in\mathcal{T}''$ can be viewed as a partition of all $l-1$ odd cycles of a permutation on $[n]\cup\{b\}$, except the cycle containing b, into two ordered parts. Conversely, given a permutation on $[n]\cup\{b\}$ with $l-1$ odd cycles, there are 2^{l-2} different ways to partition all cycles except the one containing b into two ordered parts. Therefore, we have

$$\sum_{(\alpha,\beta)\in\mathcal{T}''} W[(\alpha,\beta)] = |\mathcal{T}''| = 2^{l-2}O(n+1,\frac{n+2-l}{2}),$$

completing the proof. □

Setting $l=n+2-2g$ in Eq. (2.29) and using Eq. (2.28), we obtain a new explicit formula for $A(n,g)$ which is more symmetric than Lehman-Walsh's explicit fomula:

Theorem 2.32. *$A(n,g)=\frac{(2n)!}{2^n n!(n+1)!}\bar{A}(n,g)$ where*

$$\bar{A}(n,g)=\sum_{k=0}^{n}\binom{n}{k}\sum_{i+j=n+2-2g} C(n-k+1,i)(-1)^{k+1-j}C(k+1,j). \tag{2.30}$$

Next, we immediately obtain

Theorem 2.33. *The generating functions $A_n(x)$ for $n\geq 0$ satisfy*

$$\sum_{g\geq 0} A(n,g)x^{n+1-2g} = \frac{(2n)!}{2^n n!}\sum_{k\geq 0}\binom{n}{k}\binom{x+n-k}{n+1}. \tag{2.31}$$

Proof. It is wellknown that

$$x(x+1)(x+2)\cdots(x+n-1)=\sum_{k\geq 1}C(n,k)x^k,$$
$$x(x-1)(x-2)\cdots(x-n+1)=\sum_{k\geq 1}(-1)^{n-k}C(n,k)x^k.$$

For instance, a short constructive proof can be found in [14]. From these facts and Eq. (2.30), we immediately obtain Eq. (2.31). □

Remark 2.34. Clearly, depending on n, $A_n(x)$ represents, either an odd or even function, which is not obvious from Harer-Zagier's formula. Our new formula on the RHS of Eq. (2.31) makes this feature immediately evident: let $(a)_n$ denote the falling factorial $a(a-1)\cdots(a-n+1)$. Then,

$$\begin{aligned}\binom{n}{k}\frac{(x+n-k)_{n+1}}{(n+1)!}&=(-1)^{n+1}\binom{n}{k}\frac{[-(x+n-k)][-(x+n-k-1)]\cdots[-(x-k)]}{(n+1)!}\\&=(-1)^{n+1}\binom{n}{n-k}\frac{(-x+n-(n-k))_{n+1}}{(n+1)!},\end{aligned}$$

which implies $A_n(x)=(-1)^{n+1}A_n(-x)$.

Remark 2.35. The proof of the three-term Harer-Zagier recurrence in [17] is based on a combinatorial isomorphism. The recurrence

$$O(n+1,g)=O(n,g)+n(n-1)O(n-1,g-1) \tag{2.32}$$

offers an additional indirect combinatorial proof, which can be obtained by considering the length of the cycle containing the element 1.

2.5.2 Chapuy's recursion refined again

In this part, by reformulating our expression for $A_n(x)$ in terms of the backward shift operator $\mathrm{E} : f(x) \to f(x-1)$ and proving a property satisfied by polynomials of the form $p(\mathrm{E})f(x)$, we easily establish Chapuy's recursion once again.

First, our new formula of $A_n(x)$ implies

$$\sum_{g\geq 0} A(n,g)x^{n+1-2g} = \frac{(2n)!}{2^n n!(n+1)!}(1+\mathrm{E})^n (x+n)_{n+1}. \tag{2.33}$$

We proceed by showing that any polynomial of the form $p(\mathrm{E})(x+n)_{n+1}$ satisfies:

Theorem 2.36. *Let $p(t) = \sum_{k=0}^{n} a_k t^k$ and $F(x) = p(\mathrm{E})(x+n)_{n+1}$. If $\frac{a_1}{a_0} = \frac{a_{n-1}}{a_n}$ and $ka_k + (n-k+2)a_{k-2} = \frac{a_1}{a_0}a_{k-1}$, for $2 \leq k \leq n$, then*

$$(n+2+\frac{a_1}{a_0})F(x) = x(F(x+1) - F(x-1)). \tag{2.34}$$

Moreover, let $b_k = [x^k]p(\mathrm{E})(x+n)_{n+1}$, then we have

$$\left(\frac{n+2+\frac{a_1}{a_0}}{2} - k\right) b_k = \sum_{j\geq 1} \binom{k+2j}{2j+1} b_{k+2j}. \tag{2.35}$$

Proof. Note that by assumption, the RHS of Eq. (2.34) is equal to

$$\sum_{k=0}^{n} \{a_k x(x+1+n-k)_{n+1} - a_k x(x-1+n-k)_{n+1}\}.$$

Clearly, we have

$$
\begin{aligned}
a_k x(x+1+n-k)_{n+1} &= a_k(x-k+k)(x+1+n-k)_{n+1}\\
&= (x+1+n-k)a_k(x+n-k)_{n+1} + k a_k(x+1+n-k)_{n+1},\\
a_k x(x-1+n-k)_{n+1} &= a_k[x+(n-k)-(n-k)](x-1+n-k)_{n+1}\\
&= (x-k-1)a_k(x+n-k)_{n+1} - (n-k)a_k(x-1+n-k)_{n+1}.
\end{aligned}
$$

Then, to obtain Eq. (2.34), it suffices to show that the difference between the respective sums of the RHS of the last two equations equals the LHS of Eq. (2.34). This follows from the following computations:

$$
\begin{aligned}
&\sum_{k=0}^{n}(x+1+n-k)a_k(x+n-k)_{n+1} - \sum_{k=0}^{n}(x-k-1)a_k(x+n-k)_{n+1}\\
=&(n+2)\sum_{k=0}^{n} a_k(x+n-k)_{n+1} = (n+2)F(x), \quad \text{and}
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{k=0}^{n} k a_k(x+1+n-k)_{n+1} + \sum_{k=0}^{n}(n-k)a_k(x-1+n-k)_{n+1}\\
=&\sum_{k=2}^{n}\left\{k a_k(x+1+n-k)_{n+1} + [n-(k-2)]a_{k-2}(x-1+n-(k-2))_{n+1}\right\} +\\
&\qquad a_1(x+1+n-1)_{n+1} + [n-(n-1)]a_{n-1}(x-1+n-(n-1))_{n+1}\\
=&\sum_{k=2}^{n}\left[k a_k + (n-k+2)a_{k-2}\right](x+n-(k-1))_{n+1} + a_1(x+n)_{n+1} + a_{n-1}(x)_{n+1}\\
=&\frac{a_1}{a_0}\sum_{k=1}^{n-1} a_k(x+n-k)_{n+1} + \frac{a_1}{a_0}a_0(x+n)_{n+1} + \frac{a_1}{a_0}a_n(x)_{n+1} = \frac{a_1}{a_0}F(x).
\end{aligned}
$$

Next, the polynomial $F(x)$ is analytic and has thus a power series expansion everywhere. In particular, we have

$$F(x+1)=\sum_{k\geq 0}\frac{F^{(k)}(x)}{k!}(x+1-x)^k,\quad F(x-1)=\sum_{k\geq 0}\frac{F^{(k)}(x)}{k!}(x-1-x)^k.$$

Then,

$$\frac{n+2+\frac{a_1}{a_0}}{2}F(x)=\frac{x(F(x+1)-F(x-1))}{2}=\sum_{k\geq 0}\frac{xF^{(2k+1)}(x)}{(2k+1)!},$$

which can be reformulated as

$$\frac{n+2+\frac{a_1}{a_0}}{2}F(x)-xF'(x)=\sum_{k\geq 0}\left(\frac{n+2+\frac{a_1}{a_0}}{2}-k\right)b_kx^k=\sum_{j\geq 1}\frac{xF^{(2j+1)}(x)}{(2j+1)!}.$$

Comparing the coefficients of the last equation based on the fact that

$$\frac{xF^{(2j+1)}(x)}{(2j+1)!}=\sum_{i\geq 0}\frac{(i)_{2j+1}}{(2j+1)!}b_ix^{i-2j},$$

we obtain Eq. (2.35) and the proof of the theorem is complete. $\square$

As a consequence of Theorem 2.36, we immediately obtain Chapuy's recurrence.

Corollary 2.37 (Chapuy's recursion)**.**

$$2gA(n,g)=\sum_{k=1}^{g}\binom{n+1-2(g-k)}{2k+1}A(n,g-k). \tag{2.36}$$

Proof. For $A_n(x)$, Eq. (2.33) gives us $p(t)=\sum_{k=0}^{n}a_kt^k$ where $a_k=\frac{(2n)!}{2^nn!(n+1)!}\binom{n}{k}$. It is obvious that $\frac{a_1}{a_0} = \frac{a_{n-1}}{a_n} = n$. Furthermore,

$$\begin{aligned} ka_k + (n-k+2)a_{k-2} &= \frac{(2n)!}{2^n n!(n+1)!}\left[k\binom{n}{k} + [n-(k-2)]\binom{n}{k-2}\right] \\ &= \frac{(2n)!}{2^n n!(n+1)!}\left[n\binom{n-1}{k-1} + n\binom{n-1}{k-2}\right] \\ &= \frac{(2n)!}{2^n n!(n+1)!} n\binom{n}{k-1} = na_{k-1}. \end{aligned}$$

Hence, we can apply Theorem 2.36 to $A_n(x)$ and obtain

$$\begin{aligned} 2gA(n,g) &= \left(\frac{n+2+n}{2} - (n+1-2g)\right)[x^{n+1-2g}]A_n(x) \\ &= \sum_{j\geq 1}\binom{n+1-2g+2j}{2j+1}[x^{n+1-2g+2j}]A_n(x) \\ &= \sum_{k=1}^{g}\binom{n+1-2(g-k)}{2k+1}A(n,g-k), \end{aligned}$$

which is Chapuy's recurrence. □

2.6 Conclusion

In this chapter, we introduced (cyclic) plane permutations. It allowed us to enumerate maps and hypermaps in a new insightful framework. From the enumeration of plane permutations which is facilitated by the diagonal transpose action on plane permutations, we obtained several recurrences. These recurrences applied to maps and hypermaps automatically, by just a simple coefficient compensation. As consequences, we refined and extended several existing results, e.g., Chapuy's recursion and the Zagier-Stanley result. These are the content in Sections 2.1–2.4, which is based on the paper "*[10] Plane permutations and applications to a result of Zagier–Stanley and distances of permutations*, SIAM J. Discrete Math. 30(3)

(2016) pp. 1660–1684". We also proposed a way to connect different formulas counting one-face maps, i.e., the Lehman-Walsh formula (1972), Harer-Zagier's formulas (1986) and Chapuy's recursion (2011), for the first time, in Section 2.5. This latter part is based on the paper "*[12] New formulas counting one-face maps and Chapuy's recursion,* Australa. J. Combin., in revision."

Chapter 3

Application to Graph Embeddings

In this chapter, we will apply our plane permutation framework to graph embedding problems. Let's first review a little bit regarding the topic.

Graph embedding is one of the most important topics in topological graph theory. In particular, 2-cell embeddings of graphs (loops and multiple edges allowed) have been widely studied. A 2-*cell embedding* or *map* of a given graph G on a closed surface of genus g, S_g, is an embedding on S_g such that every face is homeomorphic to an open disk. The closed surfaces could be either orientable or unorientable. In this work, we restrict ourselves to the orientable case.

Let $g_{min}(G)$ and $g_{max}(G)$ denote the minimum and the maximum genus g of the embeddings of G, respectively. There are many studies on determining these quantities and related problems [22, 26, 35, 44, 45, 46, 47, 50, 52, 53, 55, 61, 62, 63, 66, 67]. Assume G has e edges and v vertices, and that G is embedded in S_g via ϵ. In view of Euler's characteristic formula,

$$v - e + f = 2 - 2g \quad \Longleftrightarrow \quad 2g = \beta(G) + 1 - f, \tag{3.1}$$

where $f \geq 1$ is the number of faces of ϵ and $\beta(G)$ is the Betti number of G. Thus, the largest possible value of g is $\lfloor \frac{\beta(G)}{2} \rfloor$.

We already know that any embedding of G in a closed orientable surface can be equivalently represented by a fatgraph, generated by G [24, 55]. A fatgraph generated by G is the graph G with a specified cyclic order of edges around (i.e., incident to) each vertex of G, i.e., the topological properties of the embedding are implied in the cyclic orderings of edges. Any variation of the local topological structure around a vertex, i.e., the cyclic order of edges around the vertex, may change the topological properties of the whole embedding, e.g., the genus of the embedding. This is basically what we are going to study: the behavior of graph embeddings under local changes, i.e., local reembeddings.

3.1 Localization and inflation

We first introduce two essential concepts in this section: localization and inflation w.r.t. a vertex of an embedding.

For a permutation π on $[n]$, we denote Par_π the partition of $[n]$ induced by the cycles of π, i.e., every set of elements in a same cycle of π contributes a part (or block) in Par_π. For a cyclic plane permutation $\mathfrak{p} = (s, \pi)$ with $s = (s_0 s_1 \cdots s_{n-1})$, we assume $s_0 = 1$ and refer to the blocks in Par_π as $\mathfrak{p}$-vertices or vertices for short, and elements in a vertex half-edges.

Clearly, a fatgraph induces a unique graph, i.e., the underlying graph, by ignoring the cyclic orders around vertices. Suppose a fatgraph is encoded into a triple (α, β, γ). Then, its induced graph G can be recorded as the pair (α, Par_β), where each block in Par_β corresponds to a G-vertex and each α-cycle determines a G-edge.

Recall that hypermaps represent a generalization of maps by allowing hyper-edges, i.e., triples (α, β, γ), where $\gamma = \alpha\beta$ and α is not necessarily fixed point free. We will also call the pair $G = (\alpha, Par_\beta)$ the underline graph of the hypermap although it does not induce a conventional graph. Furthermore, any hypermap $(\alpha', \beta', \gamma')$ having G as the underlying graph is called an embedding of G.

Definition 3.1 (Localization)**.** Given a cyclic plane permutation $\mathfrak{p} = (s, \pi)$ on $[n]$ and a $\mathfrak{p}$-vertex ν, the *localization* of $\mathfrak{p}$ at ν, $\mathrm{loc}_\nu(\mathfrak{p}) = (s_\nu, \pi_\nu)$ is the cyclic plane permutation

$$(s_\nu, \pi_\nu) = \begin{pmatrix} s_{i1} & s_{i2} & \cdots & s_{i(k-1)} & s_{ik} \\ \pi(s_{i1}) & \pi(s_{i2}) & \cdots & \pi(s_{i(k-1)}) & \pi(s_{ik}) \end{pmatrix},$$

which is obtained by deleting all columns not containing half-edges incident to ν in the two-line representation of $\mathfrak{p} = (s, \pi)$.

Let D_ν denote the diagonal of $\mathrm{loc}_\nu(\mathfrak{p})$, i.e., $D_\nu = s_\nu \circ \pi_\nu^{-1}$. Note that even if (s, π) is a map, D_ν is not necessarily a fixed-point free involution.

Example 3.2. For example, given (s, π)

$$\begin{pmatrix} 1 & 3 & 2 & 5 & 7 & 4 & 6 & 9 & 8 & 10 & 11 & 12 \\ 5 & 8 & 3 & 4 & 7 & 10 & 12 & 2 & 6 & 1 & 9 & 11 \end{pmatrix},$$

where $\pi = (1, 5, 4, 10)(2, 3, 8, 6, 12, 11, 9)(7)$, let $\nu = \{2, 3, 8, 6, 12, 9, 11\}$. Then,

$$\mathrm{loc}_\nu(\mathfrak{p}) = \begin{pmatrix} 3 & 2 & 6 & 9 & 8 & 11 & 12 \\ 8 & 3 & 12 & 2 & 6 & 9 & 11 \end{pmatrix},$$

we arrive at $D_\nu = (2, 8)(3, 6, 11)(9, 12)$.

A set of consecutive diagonal-pairs in $\mathfrak{p} = (s, \pi)$ is called a *diagonal block*. In above example,

$$\begin{array}{ccccc} & 2 & 5 & 7 & 4 \\ 8 & 3 & 4 & 7 & \end{array}$$ is a diagonal block. It is completely determined by its *corners*, in this case, the lower left corner, 8, as well as the upper right corner, 4. The diagonal block is denoted by $< 8, 4 >$.

Given a cyclic plane permutation $\mathfrak{p} = (s, \pi)$ on $[n]$ and a sequence $h = h_1 h_2 \cdots h_{n-1}$ on $[n-1]$, let $s^h = (s_0, s_{h_1}, s_{h_2}, \ldots s_{h_{n-1}})$, i.e. s is acted upon by h via translation its indices, and $\pi^h = D_{\mathfrak{p}}^{-1} \circ s^h$. This induces the new cyclic plane permutation (s^h, π^h) having by construction the same diagonal as (s, π). Equivalently, the two-line representation of (s^h, π^h) can be obtained by permuting the diagonal-pairs of (s, π). In the following, a cyclic plane permutation written in the form like (s^h, π^h), always means that it is obtained from (s, π) by permuting diagonal-pairs by h.

Lemma 3.3 (Localization lemma)**.** *Let $(s, \pi), (s', \pi') = (s^H, \pi^H)$ be cyclic plane permutations such that $Par_\pi = Par_{\pi'}$ and π and π' exclusively differ at the vertex ν. Then, there exists some h such that $(s'_\nu, \pi'_\nu) = (s^h_\nu, \pi^h_\nu)$ and furthermore $(D_\nu, Par_{\pi_\nu}) = (D'_\nu, Par_{\pi'_\nu})$.*

Proof. Assume $\mathfrak{p} = (s, \pi)$ and $\mathfrak{p}' = (s', \pi')$ are respectively

$$\begin{pmatrix} \cdots s_{i_0} & s_{i_0+1} & \cdots & & s_{i_1} & \cdots & s_{i_{k-2}} & \cdots & s_{i_{k-1}} & \cdots \\ \cdots \pi(s_{i_0}) & \cdots & \pi(s_{i_1-1}) & \pi(s_{i_1}) & \cdots & \pi(s_{i_{k-2}}) & & \cdots & \pi(s_{i_{k-1}}) \cdots & \end{pmatrix},$$

$$\begin{pmatrix} \cdots s'_{i_0} & s'_{i_0+1} & \cdots & & s'_{i_1} & \cdots & s'_{i_{k-2}} & \cdots & s'_{i_{k-1}} & \cdots \\ \cdots \pi'(s'_{i_0}) & \cdots & \pi'(s'_{i_1-1}) & \pi'(s'_{i_1}) & \cdots & \pi'(s'_{i_{k-2}}) & & \cdots & \pi'(s'_{i_{k-1}}) \cdots & \end{pmatrix},$$

where we assume $\nu = \{s_{i_0}, s_{i_1}, \ldots s_{i_{k-1}}\} = \{s'_{i_0}, s'_{i_1}, \ldots s'_{i_{k-1}}\}$ and $s'_0 = s_0$. Since by assumption π and π' only differ at the vertex ν, we have $s_j = s'_j$ for $0 \le j \le i_0$. Furthermore $Par_\pi = Par_{\pi'}$ implies $Par_{\pi_\nu} = Par_{\pi'_\nu}$.

Claim. $D_\nu = D'_\nu$.

To show this we observe that for fixed j, each diagonal block $< \pi'(s'_{i_j}), s'_{i_{j+1}} >$ equals the diagonal block $< \pi(s_{i_l}), s_{i_{l+1}} >$ for some $l(j)$, i.e.,

$$\begin{pmatrix} & s'_{i_j+1} & s'_{i_j+2} & \cdots & s'_{i_{j+1}} \\ \pi'(s'_{i_j}) & \pi'(s'_{i_j+1}) & \cdots & \pi'(s'_{i_{j+1}-1}) & \end{pmatrix} = \begin{pmatrix} & s_{i_l+1} & s_{i_l+2} & \cdots & s_{i_{l+1}} \\ \pi(s_{i_l}) & \pi(s_{i_l+1}) & \cdots & \pi(s_{i_{l+1}-1}) & \end{pmatrix}$$

To prove this, we observe that by construction for fixed j there exists some index l such that $\pi(s_{i_l}) = \pi'(s'_{i_j})$ holds. This implies,

$$s'_{i_j+1} = D_\mathfrak{p} \circ \pi'(s'_{i_j}) = D_\mathfrak{p} \circ \pi(s_{i_l}) = s_{i_l+1}.$$

In case of $s'_{i_j+1} \notin \nu$, we have $\pi'(s'_{i_j+1}) = \pi(s'_{i_j+1}) = \pi(s_{i_l+1})$ and derive

$$s'_{i_j+2} = D_\mathfrak{p} \circ \pi'(s'_{i_j+1}) = D_\mathfrak{p} \circ \pi(s_{i_l+1}) = s_{i_l+2}.$$

Iterating we arrive at $s'_{i_{j+1}} = s_{i_{l+1}}$, whence the two diagonal blocks are equal, the Claim follows and the proof of the lemma is complete. $\square$

Definition 3.4. Given a cyclic plane permutation $\mathfrak{p} = (s, \pi)$ on $[n]$ and its localization at ν, $\mathrm{loc}_\nu(\mathfrak{p}) = (s_v, \pi_v)$. Suppose (s^h_ν, π^h_ν) is such that $Par_{\pi_\nu} = Par_{\pi^h_\nu}$. Then the *inflation* of (s^h_ν, π^h_ν) w.r.t. $\mathfrak{p}$ is the cyclic plane permutation $\mathrm{inf}_\mathfrak{p}((s^h_\nu, \pi^h_\nu))$, obtained from (s^h_ν, π^h_ν) by substituting each diagonal-pair with the diagonal block in $\mathfrak{p}$ having the diagonal-pair as its corners.

Example 3.5. Let $(s^h_\nu, \pi^h_\nu) = \begin{pmatrix} 3 & 9 & 8 & 11 & 2 & 6 & 12 \\ 12 & 2 & 6 & 8 & 3 & 9 & 11 \end{pmatrix}$, then the inflation of (s^h_ν, π^h_ν)

w.r.t. (s, π) is

$$\inf_{\mathfrak{p}}((s^h_\nu, \pi^h_\nu)) = \begin{pmatrix} 1 & 3 & 9 & 8 & 10 & 11 & 2 & 5 & 7 & 4 & 6 & 12 \\ 5 & 12 & 2 & 6 & 1 & 8 & 3 & 4 & 7 & 10 & 9 & 11 \end{pmatrix}.$$

Lemma 3.6 (Inflation lemma)**.** *Let* $\inf_{\mathfrak{p}}((s^h_\nu, \pi^h_\nu)) = (s', \pi')$. *Then we have*

$$(D_{\mathfrak{p}}, Par_\pi) = (D_{\mathfrak{p}'}, Par_{\pi'})$$

and π' *differs from* π *only at the vertex* ν*.*

Proof. By construction, $D_{\mathfrak{p}'} = D_{\mathfrak{p}}$. Any half-edge not contained in ν, is located inside the respective diagonal blocks, whence π and π' are equal on these half-edges. All ν-half-edges contribute exactly one block in Par_π and $Par_{\pi'}$, since $Par_{\pi_\nu} = Par_{\pi^h_\nu}$. Accordingly, we have $Par_\pi = Par_{\pi'}$, completing the proof of the lemma. □

Combining Lemma 3.3 and 3.6, we obtain

Theorem 3.7. *Let* $\mathfrak{p} = (s, \pi)$ *be a cyclic plane permutation. Let* $X = \{H \mid Par_\pi = Par_{\pi^H} \ \wedge \ \pi(i) = \pi^H(i),\ i \notin \nu\}$ *and let* $Y = \{h \mid Par_{\pi_\nu} = Par_{\pi^h_\nu}\}$. *Then there is a bijection between* X *and* Y *and we have the commutative diagram:*

$$\begin{tikzcd}
(s, \pi) \arrow[r, "H:\quad Par_\pi = Par_{\pi^H}"] \arrow[d, "\mathrm{loc}_\nu"] & (s', \pi') \\
(s_\nu, \pi_\nu) \arrow[r, "h:\quad Par_{\pi_\nu} = Par_{\pi^h_\nu}"'] & (s^h_\nu, \pi^h_\nu) \arrow[u, "\inf_{\mathfrak{p}}"']
\end{tikzcd}$$

3.2 Reembedding one-face graph embeddings

In this section, we discuss locally reembedding one-face graph embeddings.

Lemma 3.8. *Let $\mathfrak{p} = (s, \pi)$ be a cyclic plane permutation with the underline graph $G = (D_{\mathfrak{p}}, Par_{\pi})$. Then, $\mathfrak{p}' = (s', \pi')$ is an embedding of G iff $(s', \pi') = (s^h, \pi^h)$ for some h and $(D_{\mathfrak{p}'}, Par_{\pi'}) = (D_{\mathfrak{p}}, Par_{\pi})$.*

Proof. If $\mathfrak{p}' = (s', \pi')$ is an embedding of $G = (D_{\mathfrak{p}}, Par_{\pi})$, then, by definition, $(D_{\mathfrak{p}'}, Par_{\pi'}) = (D_{\mathfrak{p}}, Par_{\pi})$. Thus, $D_{\mathfrak{p}} = D_{\mathfrak{p}'} = s' \circ \pi'^{-1}$. Clearly, there exists some h such that $s' = s^h$. By construction we have $\pi^h = D_{\mathfrak{p}}^{-1} \circ s^h = D_{\mathfrak{p}'}^{-1} \circ s' = \pi'$. Hence, $(s', \pi') = (s^h, \pi^h)$ for some h. The converse is clear, whence the lemma. □

This lemma shows that any one-face embedding is originated by the action of some h on a cyclic plane permutation. Explicitly, a new embedding is obtained by permuting diagonal-pairs based on a fixed one-face embedding (s, π) of the graph such that $Par_{\pi} = Par_{\pi^h}$.

Corollary 3.9. *Let $\mathfrak{p} = (s, \pi)$ be a one-face map with the underline graph $G = (D_{\mathfrak{p}}, Par_{\pi})$. Fixing the (local) embedding of all vertices of G as in $\mathfrak{p}$ but the vertex ν, each local embedding of ν leading to a one-face map $\mathfrak{p}' = (s', \pi')$ corresponds to a H such that $(s', \pi') = (s^H, \pi^H)$ and π' differs from π only at the vertex ν.*

According to the bijection between H and h in Theorem 3.7, the number of different embeddings of ν keeping one face is equal to the number of different h such that (s_ν^h, π_ν^h) and (s_ν, π_ν) having the same underlying graph. We denote this number by R_ν. Moreover, the half-edges contained in ν split the cyclic plane permutation into $|\nu|$ diagonal blocks. We can view these diagonal blocks as they are arranged in a circular fashion, as displayed in Figure 3.1. To reembed ν means to permute these diagonal blocks circularly.

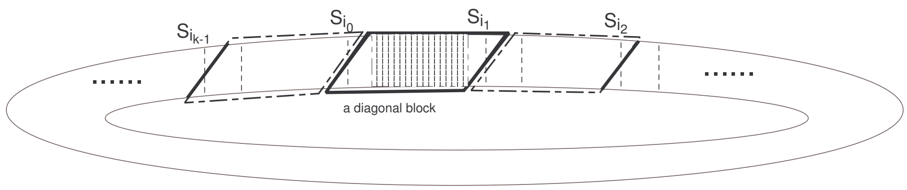

Figure 3.1: Circular arrangement of diagonal blocks determined by the vertex v.

Note that the localizations (s_ν^h, π_ν^h) and (s_ν, π_ν) have the same underlying hyper-graph G, iff both of them belong to U_D where $D = D_\nu$ and $C(\pi_\nu^h) = C(\pi_\nu) = 1$. Therefore, if D_ν has cycle-type λ, following our notation in Chapter 2, $R_\nu = p_1^\lambda(|\nu|)$. As a ressult we obtain

Theorem 3.10. *Let ϵ be a one-face embedding of G, and ν be a vertex of G with deg$(\nu) \geq 4$. Then there exists at least one additional way to reembed ν such that the obtained embedding ϵ' has the same genus as ϵ.*

Proof. Assume $d \geq 4$ and ϵ is localized at ν

$$(s_\nu, \pi_\nu) = \begin{pmatrix} v_1 & v_2 & \cdots & v_{d-1} & v_d \\ v_{i,1} & v_{i,2} & \cdots & v_{i,d-1} & v_{i,d} \end{pmatrix},$$

where $\pi_\nu = (v_1, V_2, \cdots, V_{d-1}, V_d)$. Firstly, if $V_l = v_p, V_m = v_q$ and $1 < l < m \leq d, 1 < p < q \leq d$, i.e. Case 3 in Lemma 2.9, then there exists at least one additional way to reembed ν preserving genus. Otherwise, we have $\pi_\nu = (v_1, v_d, v_{d-1}, \ldots, v_2)$. In this case,

$$(s_\nu, \pi_\nu) = \begin{pmatrix} v_1 & v_2 & \cdots & v_{d-1} & v_d \\ v_d & v_1 & \cdots & v_{d-2} & v_{d-1} \end{pmatrix},$$

whence

$$D_\nu = \begin{cases} (v_1, v_3, \ldots, v_d, v_2, v_4, \ldots, v_{d-1}), & d \in odd, \\ (v_1, v_3, \ldots, v_{d-1})(v_2, v_4, \ldots, v_d), & d \in even. \end{cases}$$

It remains to show that if $d \geq 4$ we have $R_\nu \geq 2$ in all cases. To this end, we apply a formula for $p_1^\lambda(k)$ due to Stanley [58]. If $\lambda = (1^{a_1}, 2^{a_2}, \ldots, k^{a_k})$, then

$$p_1^\lambda(k) = \sum_{i=0}^{k-1} \frac{i!(k-1-i)!}{k} \sum_{<r_1,\ldots,r_i>} \binom{a_1 - 1}{r_1}\binom{a_2}{r_2} \cdots \binom{a_i}{r_i}(-1)^{r_2+r_4+r_6+\cdots}, \tag{3.2}$$

where $< r_1, \ldots, r_i >$ ranges over all non-negative integer solutions of the equation $\sum_j jr_j = i$. Applying Stanley's formula we can compute R_ν, if $d \in odd$ as

$$R_\nu = \frac{(d-1)!}{d} \sum_{i=0}^{d-1} (-1)^i \binom{d-1}{i}^{-1} = \frac{2(d-1)!}{d+1}.$$

The simplification of the summation is stems from the following formula [60]

$$\sum_{i=0}^{n} (-1)^i \binom{x}{i}^{-1} = \frac{x+1}{x+2}(1 + (-1)^n \binom{x+1}{n+1}^{-1}).$$

It is not hard to see that $R_\nu \geq 2$ if $d \geq 4$. Similarly, if $4|d$ and $d \geq 4$, we have

$$\begin{aligned} R_\nu &= \sum_{i=0}^{\frac{d}{2}-1} (-1)^i \frac{i!(d-1-i)!}{d} + \sum_{i=\frac{d}{2}}^{d-1} (-1)^i \frac{i!(d-1-i)!}{d} [(-1)^i + (-1)^{i-\frac{d}{2}} \binom{2}{1}(-1)] \\ &= \frac{2(d-1)!}{d+1}(1 - \binom{d}{\frac{d}{2}}^{-1}). \end{aligned}$$

If $d \in even$ and $4 \nmid d$, we have

$$\begin{aligned} R_\nu &= \sum_{i=0}^{\frac{d}{2}-1}(-1)^i\frac{i!(d-1-i)!}{d} + \sum_{i=\frac{d}{2}}^{d-1}(-1)^i\frac{i!(d-1-i)!}{d}[(-1)^i + (-1)^{i-\frac{d}{2}}\binom{2}{1}] \\ &= \frac{2(d-1)!}{d+1}(1+\binom{d}{\frac{d}{2}}^{-1}). \end{aligned}$$

In both cases, if $d \geq 4$, it is straightforward to show that $R_\nu \geq 2$, since both $\frac{2(d-1)!}{d+1}$ and $(1-\binom{d}{\frac{d}{2}}^{-1})$ are increasing functions of d. Accordingly, in all cases, if $d \geq 4$, then $R_\nu \geq 2$, completing the proof. □

Example 3.11. Given a plane permutation

$$\begin{pmatrix} 1 & 2 & \cdots & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 \\ 1 & 7 & \cdots & 17 & 14 & 5 & 20 & 16 & 6 & 9 & 19 & 12 & 8 & 4 & 15 & 11 & 2 \end{pmatrix},$$

the corresponding one-face map of which is shown on the left in Figure 3.2. Consider the vertex $v = \begin{pmatrix} 8 & 11 & 14 & 16 & 19 \\ 14 & 16 & 19 & 8 & 11 \end{pmatrix}$,

$$\begin{aligned} D_v &= \begin{pmatrix} 8 & 19 & 16 & 14 & 11 \end{pmatrix} \\ &= \begin{pmatrix} 8 & 11 & 14 & 16 & 19 \end{pmatrix}\begin{pmatrix} 8 & 16 & 11 & 19 & 14 \end{pmatrix} \\ &= \begin{pmatrix} 8 & 14 & 19 & 11 & 16 \end{pmatrix}\begin{pmatrix} 8 & 14 & 19 & 11 & 16 \end{pmatrix}. \end{aligned}$$

Rearranging the half-edges around the vertex v following the second factorization of D_v, we obtain another one-face map as shown on the right hand side in Figure 3.2, where after relabeling the boundary is $(1', 2', \ldots, 20')$.

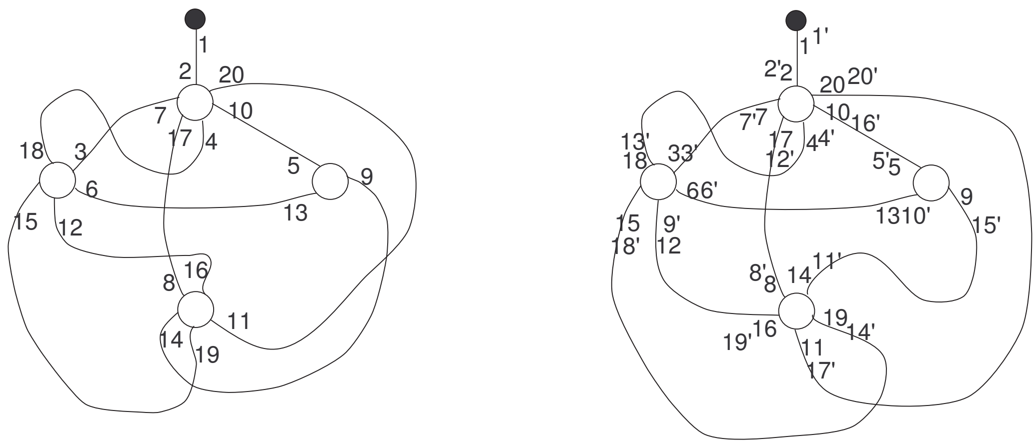

Figure 3.2: A one-face map with 10 edges (left) and rearranging half edges around one of its vertices (right).

Note in case of a vertex ν having degree 1 or 2, the situation is clear. Thus it remains to consider the case $deg(\nu) = 3$. For such a vertex, a reembedding preserving genus can be impossible. For example, $D_\nu = (132) = (123)(312)$ is the unique decomposition of D_ν, which means that the genus is going to change for any way of reembedding the vertex ν.

Corollary 3.12. *Any even permutation on $[n]$ with $n \geq 4$ has at least two different factorizations into two n-cycles.*

Proof. Since $D_\nu = s_\nu \circ \pi_\nu^{-1}$ and both s_ν as well as π_ν have only one cycle, D_ν is an even permutation. Theorem 3.10 implies that D_ν has at least 2 factorizations into two n-cycles, completing the proof. $\square$

Corollary 3.13. *Let G be a graph having m vertices of degree no less than 4. If there exists a one-face embedding of G, then there are at least 2^m one-face embeddings of G.*

Proof. If we start with a one-face embedding of G, and reembed these vertices of degree larger than 4 sequentially such that we keep one-face property at each vertex, we will obtain a new one-face embedding. Since at each vertex, there are two different ways to do that according to Corollary 3.12, we can obtain at least 2^m different one-face embeddings. □

Since we know how many ways of reembedding will keep one-face, we can certainly compute the probability of a random reembedding at a vertex keeping one-face. Accordingly, we have

Theorem 3.14. *Let (s, π) be a one-face embedding of G and (s_ν, π_ν) its localization at ν. Suppose D_ν has cycle-type $\lambda = (1^{a_1}, 2^{a_2}, \ldots, k^{a_k})$ where $k = deg(\nu)$, then the probability $prob_1(\nu)$ of a reembedding of ν to be one-face satisfies*

$$\frac{2}{deg(\nu) - a_1 + 2} \leq prob_1(\nu) \leq \frac{2}{deg(\nu) - a_1 + \frac{19}{29}}. \tag{3.3}$$

In particular, for any vertex ν, $prob_1(\nu) \geq \frac{2}{deg(\nu)+2}$.

Proof. In Zagier [69], it was proved that

$$\frac{2(k-1)!}{k - a_1 + 2} \leq p_1^\lambda(k) \leq \frac{2(k-1)!}{k - a_1 + \frac{19}{29}}.$$

Since there are $(k-1)!$ different ways to embed ν, Eq. (3.3) immediately follows in view of $p_1^\lambda(k) = R_\nu$. Clearly, we have $\frac{2}{deg(\nu)-a_1+2} \geq \frac{2}{deg(\nu)+2}$, whence the second assertion. □

The following corollary gives us the probability of a random embedding of a graph to be one-face. Clearly, if the given graph has no one-face embedding at all, the probability would be 0. So, we will assume the given graph has at least one-face embedding.

Corollary 3.15. *If there exists a one-face embedding of G, then the probability of a random embedding of G to have one face is at least $\prod_{\nu \in V(G)} \frac{2}{deg(\nu)+2}$.*

Proof. First, it should be clear that the probability of a random embedding to be one-face is equal to the probability of a random reembedding of a fixed one-face embedding of the graph preserving one-face. The latter is bounded below by the probability of sequentially random reembedding vertices of a fixed one-face embedding and keeping one-face at each step. Since we know from the last theorem the probability of keeping one-face at each step, and these steps are independent, the overall probability must be bounded by their products. This completes the proof. □

3.3 Multiple-face graph embeddings

In this section, we generalize cyclic plane permutations to general plane permutations. This puts us in position to study graph embeddings having k faces. Although it is hard to determine $g_{min}(G)$ and $g_{max}(G)$, as well as the genus distribution for a given graph G, we will show that locally these quantities can be more easily obtained.

Definition 3.16. A *plane permutation* on $[n]$ is a pair, $\mathfrak{p}$, of permutations s and π on $[n]$. The permutation $D_{\mathfrak{p}} = s \circ \pi^{-1}$ is called the *diagonal* of $\mathfrak{p}$. If s has k cycles, we write $\mathfrak{p} = (s, \pi)_k$.

Assume $s = (s_{11}, \dots s_{1m_1})(s_{21}, \dots s_{2m_2}) \cdots (s_{k1}, \dots s_{km_k})$, where $\sum_i m_i = n$. A plane permutation $(s, \pi)_k$ can be represented by two aligned rows:

$$\left(\begin{array}{ccccccccccc} \boxed{s_{11}} & s_{12} & \cdots & s_{1m_1} & \boxed{s_{21}} & \cdots & s_{2m_2} & \cdots & \boxed{s_{k1}} & \cdots & s_{km_k} \\ \pi(s_{11}) & \pi(s_{12}) & \cdots & \boxed{\pi(s_{1m_1})} & \pi(s_{21}) & \cdots & \boxed{\pi(s_{2m_2})} & \cdots & \pi(s_{k1}) & \cdots & \boxed{\pi(s_{km_k})} \end{array} \right).$$

$D_{\mathfrak{p}}$ can be defined as follows:

- For $1 \le i \le k$, $D_{\mathfrak{p}}(\pi(s_{ij})) = s_{i(j+1)}$ if $j \ne m_i$.
- For $1 \le i \le k$, $D_{\mathfrak{p}}(\pi(s_{im_i})) = s_{i1}$.

We call blocks

$$\begin{pmatrix} \boxed{s_{i1}} & s_{i2} & \cdots & s_{im_i} \\ \pi(s_{i1}) & \pi(s_{i2}) & \cdots & \boxed{\pi(s_{im_i})} \end{pmatrix}$$

the cycles of the plane permutation. If the face $(s_{i1}, \ldots, s_{im_i})$ is incident to a $\mathfrak{p}$-vertex ν, the corresponding cycle is said to be incident to ν. Since every embedding having k faces can be represented by a triple (α, β, γ), where $\gamma = \alpha\beta$ and γ has k cycles, any embedding can be expressed via a plane permutation $(\gamma, \beta)_k$.

3.3.1 Local genus polynomial is always log-concave

Although the conjecture of the genus polynomial for any graph being log-concave is still open, we can conclude in this section that its local version can be confirmed.

Let $H(f)$ denote the set of half-edges contained in the face f. The upcoming lemma tells us that reembedding a vertex ν has only to do with the faces incident to ν.

Lemma 3.17. *Let ν be a vertex of the graph G and ϵ be an embedding of G, where ν is incident to q faces, f_i, for $1 \leq i \leq q$. Let ϵ' be an embedding, obtained by reembedding ν such that ν is incident to q' faces, f'_i, for $1 \leq i \leq q'$. Then we have*

$$\bigcup_{i=1}^{q} H(f_i) = \bigcup_{i=1}^{q'} H(f'_i), \qquad q \equiv q' \pmod 2.$$

Proof. Let ϵ, ϵ' be two embeddings represented by $\mathfrak{p} = (s, \pi)_k$ and $\mathfrak{p}' = (s', \pi')_{k'}$, respectively, such that $D_{\mathfrak{p}} = D_{\mathfrak{p}'}$ and $Par_{\pi} = Par_{\pi'}$. Note that ϵ and ϵ' only differ w.r.t. the cyclic order of the half-edges around ν. Thus, for $z \notin \nu$, we have $\pi(z) = \pi'(z)$. Clearly, any face f of ϵ can be expressed as the sequence $(D_{\mathfrak{p}}\pi(z), (D_{\mathfrak{p}}\pi)^2(z), \ldots)$ for any $z \in H(f)$. The lemma is implied by the following

Claim. Any face f' of ϵ' either intersects some ϵ-face f_i for $1 \leq i \leq q$ and is incident to ν or it coincides with an ϵ-face f not incident to ν.

Suppose f' does not intersect any f_i for $1 \leq i \leq q$. For any $z \in H(f')$, $z \notin \nu$ holds and by construction $\pi'(z) = \pi(z)$. As a result, $D_{\mathfrak{p}}(\pi(z)) = D_{\mathfrak{p}}(\pi'(z)) = D_{\mathfrak{p}'}(\pi'(z))$, i.e. f' coincides with an ϵ-face f that is not incident to ν.

If f' intersects some ϵ-face f_i for $1 \leq i \leq q$, we shall prove that f' is incident to ν. Assume the half-edge u is contained in the ϵ-face f_j as well as in the face f' of ϵ'. Then,

$$\begin{aligned} f_j &= (D_{\mathfrak{p}}\pi(u), (D_{\mathfrak{p}}\pi)^2(u), \ldots, v_i, D_{\mathfrak{p}}(\pi(v_i)), \ldots) \\ f' &= (D_{\mathfrak{p}'}\pi'(u), (D_{\mathfrak{p}'}\pi')^2(u), \ldots), \end{aligned}$$

where v_i is the first half-edge of ν that appears in f_j. Since $\pi(z) = \pi'(z)$ if $z \notin \nu$, we have $D_{\mathfrak{p}}\pi(u) = D_{\mathfrak{p}}\pi'(u)$, whence the entire subsequence from $D_{\mathfrak{p}}(\pi(u))$ to v_i in f_j appears also in f'. In particular we have $v_i \in H(f')$, which means that f' is incident to ν and the Claim follows. $\square$

Let ϵ be an embedding of the graph G and ν be a vertex of G, where ν is incident to q faces in ϵ. Assume ϵ is represented by $\mathfrak{p} = (s, \pi)_k$. Similar to the situation of one-face maps, we can define the localization at ν which is a plane permutation having q cycles, $(s_\nu, \pi_\nu)_q$, and that is obtained as follows: the q cycles of $(s_\nu, \pi_\nu)_q$ are obtained from the q cycles of $\mathfrak{p}$ incident to ν by deleting all columns having no half-edges of ν. Let D_ν denote the diagonal of $(s_\nu, \pi_\nu)_q$. By construction, we have $s_\nu = D_\nu \circ \pi_\nu$, having q cycles.

Given a plane permutation $(s'_\nu, \pi'_\nu)_{q'}$, where $(D'_\nu, Par_{\pi'_\nu}) = (D_\nu, Par_{\pi_\nu})$, we can inflate w.r.t.

$\mathfrak{p}$ into an embedding of G as in the case of cyclic plane permutations. Namely, we substitute each diagonal-pair with the corresponding diagonal block in $\mathfrak{p}$ and then add any $\mathfrak{p}$-cycles containing half-edges not incident to ν.

Fix an embedding ϵ, represented by the plane permutation $(s, \pi)_k$, of genus $g(\epsilon)$. We compute in the following the distribution of genera resulting from reembedding the vertex ν. We call the distribution *the local genus distribution* (w.r.t. ν) and we can define the corresponding *local genus polynomial* as

$$\sum_{\Delta g} R_\nu(\Delta g) z^{\Delta g},$$

where $R_\nu(\Delta g)$ denote the number of different embeddings, ϵ', coming from reembedding ν such that $g(\epsilon') = g(\epsilon) + \Delta g$.

We proceed to give a formula to compute $R_\nu(\Delta g)$. Denote the cycle-type of D_ν as $\lambda(D_\nu)$. Then we have

Theorem 3.18.

$$R_\nu(\Delta g) = p_{q+2\Delta g}^{\lambda(D_\nu)}(deg(\nu)), \tag{3.4}$$

Proof. Let $(s, \pi)_k$ represent ϵ and $(s', \pi')_{k+2\Delta g}$ represent ϵ', respectively. Here the index $k + 2\Delta g$ stems from $g(\epsilon') = g(\epsilon) + \Delta g$ which implies that ϵ' differs by $2\Delta g$ faces from ϵ.

According to Lemma 3.17, we have the following situation: $\bigcup_i H(f_i)$ is reorganized into $q + 2\Delta g$ ϵ'-faces, $f'_1, \dots, f'_{q+2\Delta g}$ and any other ϵ'-face coincides with some ϵ-face not incident to ν.

Let $(s'_\nu, \pi'_\nu)_{q+2\Delta g}$ be the localization of $(s', \pi')_{k+2\Delta g}$, having the diagonal D'_ν. By definition, $s'_\nu = D'_\nu \circ \pi'_\nu$ has $(q + 2\Delta g)$ cycles.

Claim 1. Given ϵ represented by $(s, \pi)_k$, any reembedding of ν, ϵ' represented by $(s', \pi')_{k+2\Delta g}$

satisfies $D'_\nu = D_\nu$.

Suppose ν is incident to q ϵ-faces, $f_1, \dots f_q$. Furthermore, suppose the ϵ'-cycle of face f'_i reads:

$$\begin{pmatrix} \boxed{v'_{i1}} & x_1 & \cdots & v'_{i2} & x_2 & \cdots & v'_{i3} & \cdots & v'_{it_i} & \cdots & y \\ v'_{ij_1} & \cdots & x'_1 & v'_{ij_2} & \cdots & x'_2 & v'_{ij_3} & \cdots & v'_{ij_{t_i}} & \cdots & \boxed{z} \end{pmatrix},$$

where $v'_{ik}, v'_{ij_k} \in \nu \ \wedge \ v'_{ij_k} = \pi'_\nu(v'_{ik})$. Then, by the same argument as in the proof for the Lemma 3.3, the diagonal block

$$\begin{matrix} & x_l & \cdots & v'_{i(l+1)} \\ v'_{ij_l} & \cdots & x'_l & \end{matrix}$$

is also a diagonal block in ϵ, which in turn implies $D'_\nu = D_\nu$.

Claim 2. Suppose $(s'_\nu, \pi'_\nu)_{q+2\Delta g}$ is a localization such that $D'_\nu = D_\nu$ and $C(\pi'_\nu) = 1$. Then $(s'_\nu, \pi'_\nu)_{q+2\Delta g}$ can be inflated into an embedding ϵ' such that $g(\epsilon') = g(\epsilon) + \Delta g$, holds.

Suppose $(s'_\nu, \pi'_\nu)_{q+2\Delta g}$ is given by:

$$\begin{pmatrix} \boxed{v'_{11}} & \cdots & v'_{1t_1} & \cdots & \boxed{v'_{(q+2\Delta g)1}} & \cdots & v'_{(q+2\Delta g)t_{q+2\Delta g}} \\ \pi'_\nu(v'_{11}) & \cdots & \boxed{\pi'_\nu(v'_{1t_1})} & \cdots & \pi'_\nu(v'_{(q+2\Delta g)1}) & \cdots & \boxed{\pi'_\nu(v'_{(q+2\Delta g)t_{q+2\Delta g}})} \end{pmatrix}.$$

Inflating every diagonal-pair into a diagonal block w.r.t. ϵ and adding the ϵ-cycles which are not incident to ν, we obtain an embedding ϵ' with $2\Delta g$ more faces than ϵ, i.e., $g(\epsilon') = g(\epsilon) + \Delta g$. By construction, ϵ and ϵ' only differ by cyclic rearrangement of the half-edges around ν. □

It is conjectured in Gross et al. [29] that for any graph G, the genus distribution polynomial, i.e., $w(x) = \sum_g \#$of embeddings of genus $g\ x^g$, is log-concave. The conjecture is still open. However, it is implied in the paper of Stanley [59] that the sequence

$$\ldots, p^{\lambda}_{k-2}(n), p^{\lambda}_{k}(n), p^{\lambda}_{k+2}(n), \ldots$$

is log-concave for any λ, n. Thus, based on Theorem 3.18, we can conclude that for any embedding of any graph G and a vertex of G, the local genus polynomial w.r.t. the vertex is log-concave.

3.3.2 Local minimum and maximum genus

We proceed by studying the range of Δg in Theorem 3.18, i.e. the set $\{k | p^{\lambda}_{k}(n) \neq 0\}$. Firstly, from Proposition 2.5, we have

$$\max\{k | p^{\lambda}_{k}(n) \neq 0\} \leq n + 1 - \ell(\lambda).$$

Next we show that the maximum can be always achieved.

Proposition 3.19. *Let $\lambda \vdash n$ and $n \geq 1$. Then,*

$$\max\{k | p^{\lambda}_{k}(n) \neq 0\} = n + 1 - \ell(\lambda). \tag{3.5}$$

Proof. For $n = 1$, the assertion is clear, whence we can assume w.l.o.g. $n \geq 2$. We apply induction on the number of parts $\ell(\lambda)$ in λ. For any permutation α on $[n]$ of cycle type λ and $\ell(\lambda) = 1$, we have $\alpha = \alpha e_n$ where e_n is the identity permutation on $[n]$ which obviously has n cycles. Therefore, in case of $\ell(\lambda) = 1$, $\max\{k|p_k^\lambda(n) \neq 0\} = n = n + 1 - \ell(\lambda)$. Suppose for any λ with $1 \leq \ell(\lambda) = m < n$ holds

$$\max\{k|p_k^\lambda(n) \neq 0\} = n + 1 - m.$$

Let α' be a permutation on $[n]$ of cycle type λ' and $\ell(\lambda') = m + 1$. Since $m + 1 \geq 2$, we can always find a and b such that a and b are in different cycles of α'. Let $\alpha = \alpha'(a, b)$. Thus, α must be of cycle type μ for some μ such that $\ell(\mu) = m$. By assumption, there exists a relation $\alpha = s\pi$ such that s has only one cycle and π has $n + 1 - m$ cycles. Then,

$$\alpha' = \alpha(a, b) = s\pi(a, b).$$

Note $\pi(a, b)$ has the number of cycles either $n + 1 - m - 1$ or $n + 1 - m + 1$. The latter is impossible because it would contradict the bound established in Proposition 2.5. Hence, for any λ' with $\ell(\lambda') = m + 1$,

$$\max\{k|p_k^{\lambda'}(n) \neq 0\} = n + 1 - m - 1 = n + 1 - \ell(\lambda'),$$

which completes the proof of the proposition. □

Now we are ready to prove the local version of the interpolation theorem. At the same time, we can determine the local minimum genus and maximum genus.

Corollary 3.20 (Local interpolation theorem)**.** *Let ϵ be a fixed embedding of the graph G. Then for any vertex ν with localization $(s_\nu, \pi_\nu)_q$, there exists for any*

$$-\lfloor \frac{deg(\nu) + 1 - \ell(\lambda(D_\nu)) - q}{2} \rfloor \leq \Delta g \leq \lfloor \frac{q-1}{2} \rfloor.$$

an embedding ϵ' of G such that $g(\epsilon') = g(\epsilon) + \Delta g$.

Proof. According to Corollary 2.22, we have $p^{\lambda}_{k}(n) \neq 0$ as long as $p^{\lambda}_{k+2i}(n) \neq 0$ for some $i > 0$ holds, since all coefficients there are positive. Furthermore, Proposition 3.19 implies the number $p^{\lambda}_{deg(\nu)+1-\ell(\lambda(D_\nu))}(deg(\nu)) \neq 0$. Therefore, for any

$$1 \leq d \leq deg(\nu) + 1 - \ell(\lambda(D_\nu)), \quad d \equiv q \pmod 2,$$

reembedding ν can lead to an embedding where ν is incident to d faces. Accordingly, Euler's characteristic formula, implies

$$-\lfloor \frac{deg(\nu) + 1 - \ell(\lambda(D_\nu)) - q}{2} \rfloor \leq \Delta g \leq \lfloor \frac{q-1}{2} \rfloor,$$

completing the proof of the corollary. □

This result is similar to the result in Duke [22], where it was shown for any $g_{min}(G) \leq g \leq g_{max}(G)$, there exists an embedding of G on S_g. However, while it is very hard to obtain $g_{min}(G)$ and $g_{max}(G)$, we obtained easily the local minimum and the local maximum.

Suppose we are given two vertices, such that there exists no face of ϵ incident to both, then we call these two vertices ϵ-face disjoint. In view of Lemma 3.17, Corollary 3.20 has the following implication.

Corollary 3.21. *Let ϵ be an embedding of the graph G. If the vertices $\nu_i = (s_{\nu_i}, \pi_{\nu_i})_{q_i}$, $1 \leq i \leq m$, are mutually ϵ-face disjoint, then there exists an embedding ϵ' of G for any*

$$\sum_{i=1}^{m} -\lfloor \frac{deg(\nu_i) + 1 - \ell(\lambda(D_{\nu_i})) - q_i}{2} \rfloor \leq \Delta g \leq \sum_{i=1}^{m} \lfloor \frac{q_i - 1}{2} \rfloor.$$

such that $g(\epsilon') = g(\epsilon) + \Delta g$.

The following corollary provides a necessary condition for an embedding of G to be of maximum genus as well as an easy-to-check necessary condition for an embedding of G to be of minimum genus.

Corollary 3.22. *If ϵ is an embedding of the graph G with genus $g_{max}(G)$, then every vertex is incident to at most 2 faces in ϵ. Furthermore, if ϵ is an embedding of the graph G with genus $g_{min}(G)$, and $(s_\nu, \pi_\nu)_{q_\nu}$ is a localization at ν, then*

$$\ell(\lambda(D_\nu)) + q_v = deg(\nu) + 1. \tag{3.6}$$

Proof. The assertions are implied by Corollary 3.20. □

The fact that if there exists a vertex incident to at least 3 faces in an embedding, an embedding with higher genus always exists, is well known, see e.g., in [45, 67]. However, to the best of our knowledge, given an embedding ϵ, there is no simple characterization in order to determine if there exists an embedding of lower genus. Corollary 3.22 gives a sufficient condition, i.e., if $\ell(\lambda(D_\nu))+q_v \neq deg(\nu)+1$ for some vertex v, then there exists an embedding of lower genus. Of course, it can be used as a necessarily condition as well.

3.4 Concrete local moves and their impacts on genus

In the following, we present analogues of Case 3 (and Case $4, 5, 6$), Case 1 and Case 2 in Lemma 2.9, which shows what local moves can increase the genus by 0, 1 and -1, respectively. It is well known that applying Poincaré dual interchanges vertices and faces. In the derivation, we will apply a kind of local Poincaré dual.

Proposition 3.23. *Let $\mathbb{E}$ be an embedding of the graph G and a vertex $v = (s_v, \pi_v)_q$, where*

$$\pi_v = (s_{i-1}, v_1^i, \ldots v_{m_i}^i, s_j, v_1^j, \ldots v_{m_j}^j, s_l, v_1^l, \ldots v_{m_l}^l).$$

If in $\mathbb{E}$, there exists a face of the form $(s_{i-1}, \ldots s_j, \ldots s_l, \ldots)$, or two faces of the form

$$(s_{i-1}, \ldots s_j, \ldots)(s_l, \ldots),$$

then rearranging the set of half-edges $H(v)$ according to the cyclic order

$$(s_{i-1}, v_1^j, \ldots v_{m_j}^j, s_l, v_1^i, \ldots v_{m_i}^i, s_j, v_1^l, \ldots v_{m_l}^l)$$

will lead to the embedding $\mathbb{E}'$ with $g(\mathbb{E}') = g(\mathbb{E})$.

Proof. Since $v = (s_v, \pi_v)_q$, we have $s_v = D_v \circ \pi_v$, where s_v has q cycles and π_v has only one cycle. This is equivalent to $\pi_v = D_v^{-1} \circ s_v$ which corresponds to a plane permutation (π_v, s_v) with diagonal D_v^{-1}, i.e., a kind of local Poincaré dual interchanging vertices and faces. Now the given conditions in the proposition either agree with Case 3 or one of $\{\text{Case } 4, \text{Case } 5, \text{Case } 6\}$ in Lemma 2.9. Namely, if we transpose π_v into

$$(s_{i-1}, v_1^j, \ldots v_{m_j}^j, s_l, v_1^i, \ldots v_{m_i}^i, s_j, v_1^l, \ldots v_{m_l}^l),$$

we obtain a new plane permutation (π_v', s_v') where the number of cycles in s_v' equals to the number of cycles in s_v. That is, rearranging $H(v)$ according to the cyclic order

$$(s_{i-1}, v_1^j, \ldots v_{m_j}^j, s_l, v_1^i, \ldots v_{m_i}^i, s_j, v_1^l, \ldots v_{m_l}^l)$$

will not change the number of faces of the embedding. Therefore, the resulting embedding

$\mathbb{E}'$ satisfies $g(\mathbb{E}') = g(\mathbb{E})$. □

Proposition 3.24. *Let $\mathbb{E}$ be an embedding of the graph G and a vertex $v = (s_v, \pi_v)_q$, where*

$$\pi_v = (s_{i-1}, v_1^i, \dots v_{m_i}^i, s_j, v_1^j, \dots v_{m_j}^j, s_l, v_1^l, \dots v_{m_l}^l).$$

If s_{i-1}, s_j and s_l are contained respectively in three faces in $\mathbb{E}$, then rearranging $H(v)$ according to the cyclic order

$$(s_{i-1}, v_1^j, \dots v_{m_j}^j, s_l, v_1^i, \dots v_{m_i}^i, s_j, v_1^l, \dots v_{m_l}^l)$$

will lead to the embedding $\mathbb{E}'$ with $g(\mathbb{E}') = g(\mathbb{E}) + 1$.

Proof. After applying the "local Poincaré dual", the given conditions in the proposition agree with Case 1 in Lemma 2.9 whence the proposition. □

Proposition 3.25. *Let $\mathbb{E}$ be an embedding of the graph G and a vertex $v = (s_v, \pi_v)_q$, where*

$$\pi_v = (s_{i-1}, v_1^i, \dots v_{m_i}^i, s_j, v_1^j, \dots v_{m_j}^j, s_l, v_1^l, \dots v_{m_l}^l).$$

If in $\mathbb{E}$, there exists a face of the form $(s_{i-1}, \dots s_l, \dots s_j, \dots)$, then rearranging $H(v)$ according to the cyclic order

$$(s_{i-1}, v_1^j, \dots v_{m_j}^j, s_l, v_1^i, \dots v_{m_i}^i, s_j, v_1^l, \dots v_{m_l}^l)$$

will lead to the embedding $\mathbb{E}'$ with $g(\mathbb{E}') = g(\mathbb{E}) - 1$.

Proof. After applying the "local Poincaré dual", the given conditions in the proposition agree with Case 2 in Lemma 2.9 whence the proposition. □

3.5 Reembed more vertices simultaneously

In the following, we slightly generalize above results by considering changing the local structure around more vertices of the underlying graph and their local embeddings.

Firstly, we study rearrangement of half-edges around $m \geq 1$ vertices simultaneously and independently, i.e., the underlying graph is not changed. Let $(s, \pi)_k$ be a k-cyc plane permutation which corresponds to an embedding E of genus g of the graph G into k faces in total and $V_1, \ldots, V_m$ are m vertices of G, i.e., m cycles in π, which are incident to k' faces in total. Similar as the case of single vertex, we can represent all these vertices by the k'-cyc plane permutation $V_{1-m} = (s_{1-m}, \pi_{1-m})_{k'}$, where s_{1-m} is obtained from s by keeping only half edges in $V_1, \ldots, V_m$ (and the induced cycle structure) and π_{1-m} is the restriction of π to these half-edges.

Denote Dsh_{1-m} the number of different ways of simultaneous rearrangement of half-edges around V_i, $(1 \leq i \leq m)$, respectively, such that the resulting embedding has genus $g + \Delta g$.

Theorem 3.26. *Dsh_{1-m} is equal to the number of different ways to factor $D_{V_{1-m}}$ into $\gamma\sigma$, where γ has $k' + 2\Delta g$ disjoint cycles while σ has m disjoint cycles and each cycle is on the set of half-edges of V_i, respectively.*

Proof. Applying the same idea of diagonal blocks rearrangement as in the case of single vertex completes the proof. □

Now for a plane permutation (i.e., one-face embedding) (s, π) and m vertices $V_1, \ldots, V_m$ in π, if the half-edges belonging to one of these vertices are allowed to attach to another vertex among these m vertices, i.e., change the incident relation of these vertices and half-edges around them, how many different ways to keep one-face? Assume the degree distribution of these m vertices is encoded by the partition μ. Let $Le(V_1, \ldots, V_m; \mu)$ denote the number of different variations (including both local incident relation and local embedding) of these vertices to preserve the degree distribution and preserve one-face. Note the degree of a single vertex may change, but as a whole the degree distribution will not change. Even further, what if these m vertices become m' vertices after reattachment and reembedding. Let $Le_{m'}(V_1, \ldots, V_m)$ denote the number of different variations (including both local incident relation and local embedding) of these vertices to form m' vertices and keep one-face. Then, we have

Theorem 3.27. *Assume the cycle-type of $D_{V_{1-m}}$ is λ and the total number of half-edges around these m vertices are q. Then we have*

$$Le(V_1, \ldots, V_m; \mu) = f_{\mu,\lambda}(q), \tag{3.7}$$

$$Le_{m'}(V_1, \ldots, V_m) = p^{\lambda}_{m'}(q). \tag{3.8}$$

3.6 Conclusion

In this chapter, the plane permutation framework was used to study graph embeddings. This is based on the paper "*[11] On the local genus distribution of graph embeddings*, J. Combin. Math. Combin. Comput. 101 (2017), pp.157–173". In particular, the behavior of graph embeddings under local variation or reembedding was studied. Compared to the global graph embeddings problems, we have seen that their local version is easier to handle. To be specific, the local minimum and maximum genus can be exactly determined while the global version is NP-hard (at least for the minimum); the local genus distribution can be explicitly obtained and the local genus polynomial can be shown to be log-concave, although the conjecture for the global version is still open.

Furthermore, studying the local behavior of graph embeddings may provide some insights for the global problems. For instance, via studying the local structure, we obtained an easy-to-check necessary condition for an embedding to be of minimum. As to future research, studying the interaction between local behavior and global properties could be one direction.

Chapter 4

Application to Genome Rearrangements

In this chapter, we will present a unified simple framework for studying genome rearrangements using plane permutations.

4.1 Background and state of the art

In bioinformatics, comparative study of genome sequences is an important tool to understand evolution. In particular, the problem of determining the minimum number of certain operations required to transform one of two given genome sequences into the other, has been extensively studied. Combinatorially, this problem can be formulated as sorting a given permutation (or sequence) to the identity permutation by certain operations, in a minimum number of steps. The operations we will look at are transpositions [5, 8, 19, 25, 48], block-interchanges [7, 19, 20, 37, 49] and reversals [3, 6, 9, 15, 39, 40].

There are two main approaches to study these distance problems, the first being graph-based, e.g., cycle- and breakpoint-graphs [8, 9, 20, 40]. The second approach is based on permutation group theory [28, 37, 38, 48, 51]. Generally, the results obtained in these two different approaches are equivalent.

In this chapter, for a sequence (one-line permutation) on $[n]$ $s = a_1a_2\dots a_n$, we denote $\bar{s} = (0\ a_1\ a_2\ \cdots\ a_n)$. Also we denote

$$\bar{e}_n = (0\ 1\ 2\ 3\ \cdots\ n), \quad p_t = (n\ n-1\ \dots\ 1\ 0).$$

Transposition distances

Definition 4.1. Given a sequence (one-line permutation) on $[n]$

$$s = a_1 \cdots a_{i-1}a_i \cdots a_j a_{j+1} \cdots a_k a_{k+1} \cdots a_n,$$

a *transposition* action on s means to change s into

$$s' = a_1 \cdots a_{i-1}a_{j+1} \cdots a_k a_i \cdots a_j a_{k+1} \cdots a_n$$

by swapping the two continuous segments $a_i \dots a_j$ and $a_{j+1} \dots a_k$ for some $1 \leq i \leq j < k \leq n$.

Let $e_n = 123\cdots n$ be the identity permutation on $[n]$. The *transposition distance* of a sequence s on $[n]$ is the minimum number of transpositions needed to sort s into e_n. Denote this distance as $td(s)$.

The cycle-graph model was firstly proposed by Bafna and Pevzner [8] to study transposition distances. Given a permutation $s = s_1s_2\cdots s_n$ on $[n]$, the *cycle graph* $G(s)$ of s is constructed as follows: The vertices of $G(s)$ are the elements in the set $[n+1]^*$; for $0 \le i < n+1$, draw a directed black edge from i to $i+1$, and draw a directed gray edge from s_{i+1} to s_i, where we assume $s_0 = 0$ and $s_{n+1} = n+1$, we then obtain $G(s)$.

Example 4.2. The cycle graph for the permutation $s = 68134725$ is illustrated in Figure 4.1.

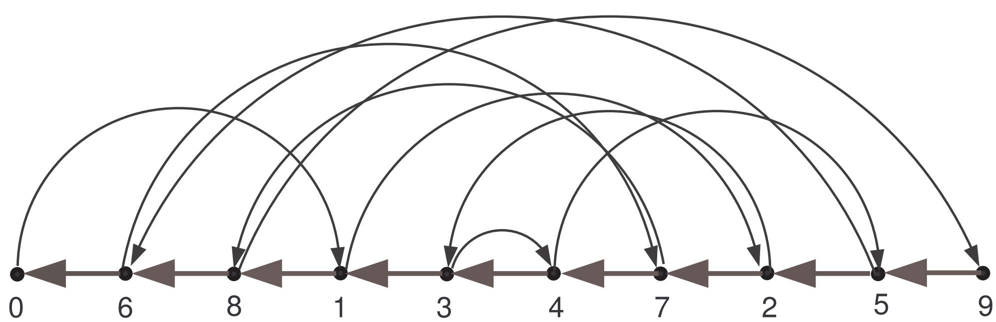

Figure 4.1: The cycle graph $G(s)$ for the permutation $s = 68134725$.

An *alternating cycle* in $G(s)$ is a directed cycle, where its edges alternate in color. An alternating cycle is called *odd* if the number of black edges in the cycle is odd. Bafna and Pevzner obtained lower bounds for $td(s)$ in terms of the number of cycles and odd cycles of $G(s)$ [8]. The lower bounds are respectively,

$$td(s) \ge \frac{n+1-C(G(s))}{2}, \tag{4.1}$$

$$td(s) \ge \frac{n+1-C_{odd}(G(s))}{2}, \tag{4.2}$$

where $C(G(s))$ and $C_{odd}(G(s))$ denote the number of cycles and odd cycles in $G(s)$, respectively. These lower bounds followed from key observations that each transposition increases the number of cycles (and resp. odd cycles) in the cycle graph by at most 2 and there are $n+1$ cycles in the cycle graph of the identity permutation. Obviously, all these cycles must be of length 1 hence being odd cycles, as there are only $n+1$ vertices in total.

In the permutation-group based approach, see for instance [48], a transposition can be treated as multiplying a 3-cycle on the left of $p_t\bar{s}$. Thus, a lower bound can be obtained by computing the minimum number of 3-cycles that $p_t\bar{s}$ can be factored into.

There is no exact formula to compute the transposition distances, and actually sorting by transpositions was proved to be NP-hard [5]. Algorithms of various efficiency for sorting permutations by transpositions were studied in [8, 25] and references therein.

Block-interchange distances

A more general transposition problem, where the involved two blocks are not necessarily adjacent, was firstly studied in Christie [20]. It is referred to as the *block-interchange* distance problem. The minimum number of block-interchanges needed to sort s into e_n is accordingly called the *block-interchange distance* of s and denoted as $bid(s)$. Christie [20] obtained an exact formula to compute the block-interchange distance of any given permutation s, based on the cycle-graph model. The formula is

$$bid(s) = \frac{n+1-C(G(s))}{2}. \tag{4.3}$$

Algorithms for sorting permutations by block-interchanges were studied in [19, 37].

Reversal distances

Definition 4.3. A *signed permutation* on $[n]$ is a pair (a, w) where a is a sequence on $[n]$ while w is a word of length n on the alphabet set $\{+, -\}$.

Usually, a signed permutation is represented by a single sequence $a_w = a_{w,1}a_{w,2}\cdots a_{w,n}$ where $a_{w,k} = w_k a_k$, i.e., each a_k carries a sign determined by w_k.

Definition 4.4. Given a signed permutation $a = a_1a_2\cdots a_{i-1}a_ia_{i+1}\cdots a_{j-1}a_ja_{j+1}\cdots a_n$ on $[n]$, a *reversal* $\varrho_{i,j}$ acting on a will change a into

$$a' = \varrho_{i,j} \diamond a = a_1a_2\cdots a_{i-1}(-a_j)(-a_{j-1})\cdots(-a_{i+1})(-a_i)a_{j+1}\cdots a_n.$$

The *reversal distance* $d_r(a)$ of a signed permutation a on $[n]$ is the minimum number of reversals needed to sort a into $e_n = 12\cdots n$.

Example 4.5. The signed permutation $a = -5+1-3+2+4$ needs at least 4 steps to be sorted as illustrated below:

$$\begin{array}{c}
-5 \quad +1 \quad \underline{-3 \quad +2} \quad +4 \\
-5 \quad +1 \quad \underline{-2} \quad +3 \quad +4 \\
\underline{-5 \quad +1 \quad +2 \quad +3 \quad +4} \\
\underline{-4 \quad -3 \quad -2 \quad -1} \quad +5 \\
+1 \quad +2 \quad +3 \quad +4 \quad +5
\end{array}$$

Let $[n]^- = \{-1, -2, \ldots, -n\}$.

The most common graph model used to study reversal distance is *breakpoint graph* proposed by Bafna and Pevzner [9]. The breakpoint graph for a given signed permutation $a = a_1a_2 \cdots a_n$ on $[n]$ can be obtained as follows: Replacing a_i with $(-a_i)a_i$, and adding 0 at the beginning of the obtained sequence while adding $-(n+1)$ at the end of the obtained sequence, in this way we obtain a sequence $b = b_0b_1b_2 \cdots b_{2n}b_{2n+1}$ on $[n]^* \cup [n+1]^-$. Draw a black edge between b_{2i} and b_{2i+1}, as well as a grey edge between i and $-(i+1)$ for $0 \leq i \leq n$. The obtained graph is the breakpoint graph $BG(a)$ of a.

Example 4.6. The breakpoint graph $BG(a)$ for the signed permutation $a = +4-2-5+1-3$ is illustrated in Figure 4.2.

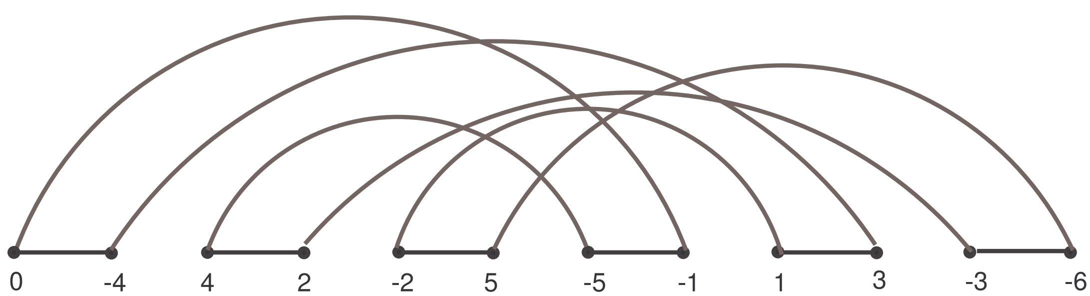

Figure 4.2: The breakpoint graph $BG(a)$ for the signed permutation $a = +4-2-5+1-3$.

Note that each vertex in $BG(a)$ has degree two so that it can be decomposed into disjoint cycles. Denote the number of cycles in $BG(a)$ as $C_{BG}(a)$. Then, the lower bound for the reversal distance of a [9] via the break point graph is given by

$$d_r(a) \geq n + 1 - C_{BG}(a). \tag{4.4}$$

In reality, the above lower bound is quite good as it actually gives the exact reversal distance for most of signed permutations. Later, by identifying additional "motifs" in breakpoint graphs, that are those can not be captured by only cycles, an exact formula for computing the reversal distance of any signed permutation and corresponding polynomial time algorithm were presented in Hannenhalli and Pevzner [40].

On the permutation group theory side, we refer the readers to [28, 51] for discussion of the reversal distance.

In the following, we will show that (cyclic) plane permutations provide a powerful tool to study these above defined distances. The idea comes from the observation that the diagonal transpose action on plane permutations is a kind of transposition on "fattened" sequences. Thus, in order to study distances of permutations, we can associate a plane permutation to a given permutation to sort and apply the results on the diagonal transpose action of the plane permutation.

4.2 The transposition distance

Let $C(\pi)$, $C_{odd}(\pi)$ and $C_{ev}(\pi)$ denote the number of cycles, the number of odd cycles and the number of even cycles in π, respectively. Furthermore, let $[n]^* = \{0, 1, \ldots, n\}$.

Our first result is the following general lower bound:

Theorem 4.7 (General lower bound).

$$td(s) \geq \max_{\gamma} \left\{ \frac{\max\{|C(p_t\bar{s}\gamma) - C(\gamma)|, |C_{odd}(p_t\bar{s}\gamma) - C_{odd}(\gamma)|, |C_{ev}(p_t\bar{s}\gamma) - C_{ev}(\gamma)|\}}{2} \right\}, \tag{4.5}$$

where γ ranges over all permutations on $[n]^*$.

Proof. For an arbitrary permutation γ on $[n]^*$, $\mathfrak{p} = (\bar{s}, \gamma)$ is a plane permutation. By construction, each transposition on the sequence s induces a transposition on $\mathfrak{p}$. (The auxiliary element 0 is used to handle the case where a_1 is contained in the first block of the transpositions, because the bottom row of a transposition action on a plane permutation is forward shifted.) If s changes to e_n by a series of transpositions, we have, for some β, that $\mathfrak{p}$ changes into the plane permutation $(\bar{e}_n, \beta)$. By construction, we have

$$D_{\mathfrak{p}} = \bar{s}\gamma^{-1} = \bar{e}_n\beta^{-1},$$

and accordingly

$$\beta = \gamma\bar{s}^{-1}\bar{e}_n.$$

Since each transposition changes the number of cycles by at most 2 according to Lemma 2.9, at least $\frac{|C(\gamma\bar{s}^{-1}\bar{e}_n)-C(\gamma)|}{2} = \frac{|C(p_t\bar{s}\gamma^{-1})-C(\gamma^{-1})|}{2}$ transpositions are needed from γ to β. The same argument also applies to deriving the lower bounds in terms of odd and even cycles, respectively. Note that γ can be arbitrarily selected and the proof follows. $\square$

Inspecting this general formulation of Theorem 4.7, setting $\gamma = (p_t\bar{s})^{-1}$ we immediately obtain

Corollary 4.8.

$$td(s) \geq \frac{n+1-C(p_t\bar{s})}{2}, \tag{4.6}$$

$$td(s) \geq \frac{n+1-C_{odd}(p_t\bar{s})}{2}. \tag{4.7}$$

By examining the cycle graph model $G(s)$ of a permutation s, it turns out the cycle graph $G(s)$ is actually the directed graph representation of the product $\bar{s}^{-1}p_t^{-1}$, if we identify the two auxiliary points 0 and $n+1$. The directed graph representation of a permutation π is the directed graph by drawing an directed edge from i to $\pi(i)$. If we color the directed edge of $\bar{s}^{-1}$ gray and the directed edge of p_t^{-1} black, an alternating cycle then determines a cycle of the permutation $\bar{s}^{-1}p_t^{-1}$ (thus $p_t\bar{s}$). Therefore, the number of cycles and odd cycles in $p_t\bar{s}$ is equal to the number of cycles and odd cycles in $G(s)$, respectively. As a result, the lower bounds in Corollary 4.8 are exactly the same as the lower bounds obtained by Bafna and Pevzner [8] and this relation was also derived in [21, 48]. In particular, in [48], this lower bound was obtained using permutations and by translating the transposition distance of s into the minimum number of 3-cycles, $p_t\bar{s}$ can be factored into.

In view of Theorem 4.7 we next ask: is it possible by employing an appropriate γ, to improve the lower bounds of in Corollary 4.8? Namely, given a permutation π, what is the maximum number of $|C(\pi\gamma) - C(\gamma)|$ (resp. $|C_{odd}(\pi\gamma) - C_{odd}(\gamma)|$, $|C_{ev}(\pi\gamma) - C_{ev}(\gamma)|$), when γ ranges over a set of permutations.

More generally, we can study the distribution functions

$$\sum_{\gamma\in A} z^{C(\pi\gamma)-C(\gamma)}, \quad \sum_{\gamma\in A} z^{C_{odd}(\pi\gamma)-C_{odd}(\gamma)}, \quad \sum_{\gamma\in A} z^{C_{ev}(\pi\gamma)-C_{ev}(\gamma)}, \tag{4.8}$$

where A is a set of permutations, e.g., a conjugacy class or all permutations.

We shall compute $\max_\gamma\{|C(\pi\gamma) - C(\gamma)|\}$ for an arbitrary permutation π and prove that its maximum is achieved for $\gamma = \pi^{-1}$ or γ being the identity permutation. For the cases of odd and even cycles, we have not established a general result. Here we provide an example showing that an analogue does not hold, i.e. for even cycles, the maximum is not necessarily achieved by $\gamma = \pi^{-1}$ or $\gamma =$identity.

Example 4.9. Suppose $p_t = (4,3,2,1,0)$ and $\bar{s} = (0,1,2,4,3)$. Consider $|C_{ev}(p_t\bar{s}\gamma) - C_{ev}(\gamma)|$.

- For $\gamma = (p_t\bar{s})^{-1}$ or $\gamma =$identity, $p_t\bar{s} = (0)(1)(2,3,4)$ whence $|C_{ev}(p_t\bar{s}\gamma) - C_{ev}(\gamma)| = 0$;

- For $\gamma = (0)(1,3)(2,4)$, $p_t\bar{s}\gamma = (0)(1,4,3)(2)$ whence $|C_{ev}(p_t\bar{s}\gamma) - C_{ev}(\gamma)| = 2$.

We remark an idea on how to obtain better lower bounds is to fix γ and to analyze the "unavoidable" transpositions which do not increase (or decrease) the number of cycles (or odd, or even cycles) from γ to $\gamma\bar{s}^{-1}\hat{e}_n$. In particular, by setting $\gamma = p_t\bar{s}$, it is not hard to analyze the number of "hurdles" as in Christie [19] using plane permutations. We shall not go into details here.

4.3 The block-interchange distance

Using Lemma 2.11 and Theorem 4.7, we immediately obtain

$$bid(s) \geq \frac{\max_\gamma\{|C(p_t\bar{s}\gamma) - C(\gamma)|\}}{2}, \tag{4.9}$$

where γ ranges over all permutations on $[n]^*$. Christie [20] proved an exact formula for the block-interchange distance which implies that the maximum of the RHS of Eq. (4.9) is achieved by $\gamma = (p_t\bar{s})^{-1}$. This follows immediately via plane permutations as follows:

Lemma 4.10. *Let $\mathfrak{p} = (\bar{s}, \pi)$ be a plane permutation on $[n]^*$ where $D_\mathfrak{p} = p_t^{-1}$ and $\bar{s} \neq \bar{e}_n$. Then, there exist $\bar{s}_{i-1} <_{\bar{s}} \bar{s}_j <_{\bar{s}} \bar{s}_{k-1} \leq_{\bar{s}} \bar{s}_l$ such that*

$$\pi(\bar{s}_{i-1}) = \bar{s}_{k-1}, \quad \pi(\bar{s}_l) = \bar{s}_j.$$

Proof. Since $\bar{s} \neq \bar{e}_n$, there exists $x \in [n]$ such that $x+1 <_{\bar{s}} x$. Assume $x = \bar{s}_{k-1}$ is the largest such integer and let $\bar{s}_i = x+1$. Then, $\pi(\bar{s}_{i-1}) = x = \bar{s}_{k-1}$ since $D_\mathfrak{p}(\pi(\bar{s}_{i-1})) = \pi(\bar{s}_{i-1}) + 1 = x+1$. Between $\bar{s}_{i-1}$ and x, find the largest integer which is larger than x. Since $x+1$ lies between $\bar{s}_{i-1}$ and x, this maximum exists and we denote it by y. Then we have by construction

$$\bar{s}_{i-1} <_{\bar{s}} \bar{s}_j = y <_{\bar{s}} x = \bar{s}_{k-1} <_{\bar{s}} y+1 = \bar{s}_{l+1}.$$

Therefore, $\pi(\bar{s}_l) = D_{\mathfrak{p}}^{-1}(y+1) = y = \bar{s}_j$, whence the lemma. □

Then, we obtain

Theorem 4.11.

$$bid(s) = \frac{n+1-C(p_t\bar{s})}{2} = \frac{n+1-C(G(s))}{2}. \tag{4.10}$$

Proof. Let $\mathfrak{p} = (\bar{s}, \pi)$ be a plane permutation on $[n]^*$ where $D_{\mathfrak{p}} = p_t^{-1}$ and $\bar{s} \neq \bar{e}_n$. According to Lemma 4.10, we either have $\bar{s}_{i-1} <_{\bar{s}} \bar{s}_j <_{\bar{s}} \bar{s}_{k-1} <_{\bar{s}} \bar{s}_l$ such that we either have π-cycle

$$(\bar{s}_{i-1}\ \bar{s}_{k-1}\ \ldots\ \bar{s}_l\ \bar{s}_j\ \ldots) \quad \text{or} \quad (\bar{s}_{i-1}\ \bar{s}_{k-1}\ \ldots)(\bar{s}_l\ \bar{s}_j\ \ldots),$$

or $\bar{s}_{i-1} <_{\bar{s}} \bar{s}_j <_{\bar{s}} \bar{s}_{k-1} =_{\bar{s}} \bar{s}_l$ such that we have the π-cycle $(\bar{s}_{i-1}\ \bar{s}_{k-1}\ \bar{s}_j\ \ldots)$. For the former case, the determined χ_h is either Case c or Case e of Lemma 2.9. For the latter case, the determined χ_h is Case 2 transposition of Lemma 2.11. Therefore, no matter which case, we can always find a block-interchange to increase the number of cycles by 2. Then, arguing as in Theorem 4.7 completes the proof. □

Theorem 4.11 was also proved in [49] using permutations by translating the block-interchange distance of s into the minimum number of pairs of 2-cycles the permutation $p_t\bar{s}$ can be factored into.

Furthermore, Zagier and Stanley's result refined in Chapter 2 implies that

Corollary 4.12. *Let $bid_k(n)$ denote he number of sequences s on $[n]$ such that $bid(s) = k$.*

Then,

$$bid_k(n) = \frac{2C(n+2, n+1-2k)}{(n+1)(n+2)}. \tag{4.11}$$

Proof. Let

$$k = bid(s) = \frac{n+1-C(p_t\bar{s})}{2}.$$

The number of s such that $bid(s) = k$ is equal to the number of permutation $\bar{s}$ such that $C(p_t\bar{s}) = n+1-2k$. Then, applying Zagier and Stanley's result completes the proof. □

We note that the corollary above was also used by Bona and Flynn [7] to compute the average number of block-interchanges needed to sort permutations.

In view of the general lower bound for block-interchanges Eq. (4.9) and Theorem 4.11, we are now in position to answer one of the optimization problems mentioned earlier.

Theorem 4.13. *Let α be a permutation on $[n]$ and $n \geq 1$. Then we have*

$$\max_{\gamma}\{|C(\alpha\gamma) - C(\gamma)|\} = n - C(\alpha), \tag{4.12}$$

where γ ranges over all permutations on $[n]$.

Proof. First, from Eq. (4.9) and Theorem 4.11, we have: for arbitrary s,

$$\max_{\gamma}\{|C(p_t\bar{s}\gamma) - C(\gamma)|\} = n+1-C(p_t\bar{s}), \tag{4.13}$$

where γ ranges over all permutations on $[n]^*$.

We now use the fact that any even permutation α' on $[n]^*$ has a factorization into two $(n+1)$-cycles. Assume $\alpha' = \beta_1\beta_2$ where β_1, β_2 are two $(n+1)$-cycles, and $p_t = \theta\beta_1\theta^{-1}$. Then, we have

$$\begin{aligned}\max_{\gamma}\{|C(\alpha'\gamma)-C(\gamma)|\} &= \max_{\gamma}\{|C(\theta\beta_1\beta_2\gamma\theta^{-1})-C(\theta\gamma\theta^{-1})|\}\\ &= \max_{\gamma}\{|C(p_t\theta\beta_2\theta^{-1}\theta\gamma\theta^{-1})-C(\theta\gamma\theta^{-1})|\}\\ &= n+1-C(p_t\theta\beta_2\theta^{-1})\\ &= n+1-C(\beta_1\beta_2)=n+1-C(\alpha').\end{aligned}$$

So the theorem holds for even permutations. Next we assume that α is an odd permutation. If $C(\alpha) < n$, then we can always find a transposition τ (i.e., a cycle of length 2) such that $\alpha = \alpha'\tau$, where α' is an even permutation and $C(\alpha) = C(\alpha') - 1$. Thus,

$$\begin{aligned}\max_{\gamma}\{|C(\alpha\gamma)-C(\gamma)|\} &= \max_{\gamma}\{|C(\alpha'\tau\gamma)-C(\gamma)|\}\\ &= \max_{\gamma}\{|C(\alpha'\tau\gamma)-C(\tau\gamma)+C(\tau\gamma)-C(\gamma)|\}\\ &\le \max_{\gamma}\{|C(\alpha'\tau\gamma)-C(\tau\gamma)|+|C(\tau\gamma)-C(\gamma)|\}\\ &= [n-C(\alpha')]+1=n-C(\alpha).\end{aligned}$$

Note that $|C(\alpha I) - C(I)| = n - C(\alpha)$, where I is the the identity permutation. Hence, we conclude that $\max_\gamma\{|C(\alpha\gamma) - C(\gamma)|\} = n - C(\alpha)$. When $C(\alpha) = n$, i.e., $\alpha = I$, it is obvious that $\max_\gamma\{|C(I\gamma) - C(\gamma)|\} = 0 = n - C(\alpha)$. Hence, the theorem holds for odd permutations as well, completing the proof. $\square$

4.4 The reversal distance

In this section, we consider the reversal distance for signed permutations, a problem extensively studied in the context of genome evolution [3, 9, 40] and the references therein.

Lower bounds for the reversal distance based on the breakpoint graph model were obtained in [9, 39, 40].

In our framework the reversal distance problem can be expressed as a block-interchange distance problem. A lower bound can be easily obtained in this point of view, and the lower bound will be shown to be the exact reversal distance for most of signed permutations.

For the given signed permutation a, we associate the sequence $s = s(a)$ as follows

$$s = s_0 s_1 s_2 \cdots s_{2n} = 0 a_1 a_2 \cdots a_n (-a_n)(-a_{n-1}) \cdots (-a_2)(-a_1),$$

i.e., $s_0 = 0$ and $s_k = -s_{2n+1-k}$ for $1 \leq k \leq 2n$. Furthermore, such sequences will be referred to as *skew-symmetric* sequences since we have $s_k = -s_{2n+1-k}$. A sequence s is called *exact* if there exists $s_i < 0$ for some $1 \leq i \leq n$.

The following lemma is obvious.

Lemma 4.14. *The reversal distance of a is equal to the block-interchange distance of $s(a)$ into*

$$e_n^\natural = 012 \cdots n(-n)(-n+1) \cdots (-2)(-1),$$

where only certain block-interchanges are allowed, i.e., only the actions χ_h, $h = (i, j, 2n + 1 - j, 2n + 1 - i)$ are allowed where $1 \leq i \leq j \leq n$.

Hereafter, we will denote these particular block-interchanges referred to in the right above lemma on s as reversals, $\varrho_{i,j}$.

Let

$$\begin{aligned}
\tilde{s} &= (s) = (0\ a_1\ a_2\ \ldots\ a_{n-1}\ a_n\ -a_n\ -a_{n-1}\ \ldots\ -a_2\ -a_1), \\
p_r &= (-1 \quad -2 \quad \cdots \quad -n+1 \quad -n \quad n \quad n-1 \quad \cdots \quad 2 \quad 1 \quad 0).
\end{aligned}$$

A plane permutation of the form $(\tilde{s}, \pi)$ will be called skew-symmetric. In view of Lemma 4.14, we immediately obtain the following lower bound:

Theorem 4.15.

$$d_r(a) \geq \frac{2n+1-C(p_r\tilde{s})}{2}. \tag{4.14}$$

Proof. Since reversals are restricted block-interchanges, the reversal distance will be bounded by the block-interchange distance without restriction. Theorem 4.11 then implies Eq. (4.14). □

Our approach gives rise to the question of how potent the restricted block-interchanges are. Is it difficult to find a block-interchange increasing the number of cycles by 2 that is a reversal (i.e., 2-reversal)?

We will call a plane permutation $(\tilde{s}, \pi)$ exact, skew-symmetric if $\tilde{s}$ is exact and skew-symmetric. The following lemma will show that there is almost always a 2-reversal.

Lemma 4.16. *Let $\mathfrak{p} = (\tilde{s}, \pi)$ be exact and skew-symmetric on $[n]^* \cup [n]^-$, where $D_{\mathfrak{p}} = p_r^{-1}$. Then, there always exist $i-1$ and $2n-j$ such that*

$$\pi(s_{i-1}) = s_{2n-j}, \tag{4.15}$$

where $0 \leq i-1 \leq n-1$ and $n+1 \leq 2n-j \leq 2n$. Furthermore, we have the following cases

(a) If $s_{i-1} <_s s_j <_s s_{2n-j} <_s s_{2n+1-i}$, then

$$\pi(s_{i-1}) = s_{2n-j}, \quad \pi(s_j) = s_{2n+1-i}. \tag{4.16}$$

(b) If $s_j <_s s_{i-1} <_s s_{2n+1-i} <_s s_{2n-j}$, then

$$\pi(s_{i-1}) = s_{2n-j}, \quad \pi(s_j) = s_{2n+1-i}. \tag{4.17}$$

Proof. We firstly prove the former part. Assume s_i is the smallest negative element among the subsequence $s_1 s_2 \cdots s_n$. If $s_i = -n$, then we have $s_{2n+1-i} = -s_i = n$ by symmetry. Since $D_{\mathfrak{p}} = p_r^{-1}$, for any k, $s_{k+1} = p_r^{-1}(s_k) = s_k + 1$ where $n+1$ is interpreted as $-n$. Thus, $\pi(s_{i-1}) = D_{\mathfrak{p}}^{-1}(s_i) = D_{\mathfrak{p}}^{-1}(-n) = n = s_{2n+1-i}$. Let $2n - j = 2n + 1 - i$, then $2n - j \geq n + 1$ and we are done. If $s_i > -n$, then we have $\pi(s_{i-1}) = D_{\mathfrak{p}}^{-1}(s_i) = s_i - 1 \geq -n$. Since s_i is the smallest negative element among s_t for $1 \leq t \leq n$, if $s_{2n-j} = s_i - 1 < s_i$, then $2n - j \geq n+1$, whence the former part.

Using $D_{\mathfrak{p}} = p_r^{-1}$ and the skew-symmetry $s_k = -s_{2n+1-k}$, we have in case of (a) the following situation in $\mathfrak{p}$ (only relevant entries are illustrated)

$$\left(\begin{array}{ccccccccc} i-1 & i & \cdots & j & j+1 & \cdots & 2n+1-j & \cdots & 2n+1-i \\ \hline s_{i-1} & (s_{2n-j}+1) & \cdots & s_j & -s_{2n-j} & \cdots & -s_j & \cdots & (-s_{2n-j}-1) \\ s_{2n-j} & \diamondsuit & \cdots & (-s_{2n-j}-1) & \diamondsuit & \cdots & \diamondsuit & \cdots & \diamondsuit \end{array}\right).$$

Therefore, we have

$$\pi(s_{i-1}) = s_{2n-j}, \quad \pi(s_j) = -s_{2n-j} - 1 = s_{2n+1-i}.$$

Analogously we have in case of (b) the situation

$$\left(\begin{array}{ccccccccc} j & j+1 & \cdots & i-1 & i & \cdots & 2n+1-i & 2n+2-i & \cdots & 2n-j \\ \hline s_j & -s_{2n-j} & \cdots & s_{i-1} & s_{2n-j}+1 & \cdots & -s_{2n-j}-1 & -s_{i-1} & \cdots & s_{2n-j} \\ -s_{2n-j}-1 & \diamondsuit & \cdots & s_{2n-j} & \diamondsuit & \cdots & \diamondsuit & \diamondsuit & \cdots & \diamondsuit \end{array}\right).$$

Therefore, we have

$$\pi(s_{i-1}) = s_{2n-j}, \quad \pi(s_j) = -s_{2n-j} - 1 = s_{2n+1-i}.$$

This completes the proof. $\square$

Remark 4.17. The pair s_{i-1} and s_{2n-j} such that $\pi(s_{i-1}) = s_{2n-j}$ is not unique. For instance, assume the positive integer k, $1 \leq k \leq n-1$, is not in the subsequence $s_1 s_2 \cdots s_n$ but $k+1$ is, then $\pi^{-1}(k)$ and $k = D_{\mathfrak{p}}^{-1}(k+1)$ form such a pair.

Inspection of Lemma 2.11 and Lemma 4.16 shows that there is almost always a 2-reversal for signed permutations. The only critical cases, not covered in Lemma 4.16, are

- The signs of all elements in the given signed permutation are positive.
- Exact signed permutation which for $1 \leq i \leq n$ and $n+1 \leq 2n-j$, $\pi(s_{i-1}) = s_{2n-j}$ iff $2n-j = 2n+1-i$.

We proceed to analyze the latter case. Since $\pi(s_{i-1}) = s_{2n+1-i} = -s_i$, we have

$$\begin{pmatrix} s_{i-1} & s_i & \cdots & s_n & -s_n & \cdots & s_{2n+1-i} & s_{2n+2-i} \\ \pi(s_{i-1}) & \diamondsuit & \cdots & \diamondsuit & \diamondsuit & \cdots & \diamondsuit & \diamondsuit \end{pmatrix}$$
$$= \begin{pmatrix} s_{i-1} & s_i & \cdots & s_n & -s_n & \cdots & -s_i & -s_{i-1} \\ -s_i & \diamondsuit & \cdots & \diamondsuit & \diamondsuit & \cdots & \diamondsuit & \diamondsuit \end{pmatrix}.$$

Due to $D_{\mathfrak{p}}$, $D_{\mathfrak{p}}(-s_i) = s_i = -s_i + 1$ (note that $n+1$ is interpreted as $-n$). The only situation satisfying this condition is that $s_i = -n$, i.e., the sign of n in the given signed permutation is negative. Then, we have $\pi(s_{i-1}) = s_{2n-j} = s_{2n+1-i} = n$. We believe that in this case Lemma 2.9 (instead of Lemma 2.11) provides a 2-reversal. Namely, s_{i-1} (i.e., the preimage of n), s_n and $n = s_{2n+1-i}$ will form a Case 2 transposition in Lemma 2.9, which will be true if n and s_n are in the same cycle of π, i.e., π has a cycle $(s_{i-1}\ s_{2n+1-i}\ \ldots\ s_n\ \ldots)$. In order to illustrate this we consider

Example 4.18.

$$\begin{pmatrix} 0 & -3 & 1 & 2 & -4 & 4 & -2 & -1 & 3 \\ -4 & 0 & 1 & 4 & 3 & -3 & -2 & 2 & -1 \end{pmatrix} \Longrightarrow \pi = (0\ -4\ 3\ -1\ 2\ 4\ -3)(1)(-2)$$

$$\begin{pmatrix} 0 & 2 & -4 & -1 & 3 & -3 & 1 & 4 & -2 \\ 1 & 4 & -2 & 2 & -4 & 0 & 3 & -3 & -1 \end{pmatrix} \Longrightarrow \pi = (0\ 1\ 3\ -4\ -2\ -1\ 2\ 4\ -3)$$

We inspect, that in the first case $s_{i-1} = 2$, $s_n = -4$ and $n = 4$ form a Case 2 transposition of Lemma 2.9. In the second case $s_{i-1} = 2$, $s_n = 3$ and $n = 4$ form again a Case 2 transposition of Lemma 2.9.

In the next section, we will show that this speculation is correct. As a consequence, we can conclude that for a random signed permutation, it is likely to be possible to transform s into $e_n^\natural$ via a sequence of 2-reversals. In fact, many examples, including Braga [6, Table 3.2], indicate that the lower bound of Theorem 4.15 gives the exact reversal distances.

4.5 Compare our lower bound and the Bafna-Pevzner lower bound

Note that the lower bound obtained in [9, 39] via breakpoint graphs also provides the exact reversal distance for most of signed permutations. It is a question worthy of being addressed that which lower bound is better, ours in Theorem 4.15 or the Bafna-Pevzner lower bound

Eq. (4.4). It turns out that they are actually equal. This section is devoted to show this result.

Algebraically, we can express Eq. (4.4) in a form similar to our lower bound. Let θ_1, θ_2 be the two involutions (without fixed points) determined by the black edges and grey edges in the breakpoint graph, respectively, i.e.,

$$\theta_1 = (b_0\ b_1)(b_2\ b_3)\cdots(b_{2n}\ b_{2n+1}),$$
$$\theta_2 = (0\ \ -1)(1\ \ -2)\cdots(n\ \ -n-1).$$

It is not hard to observe that $C_{BG}(a) = \frac{C(\theta_1\theta_2)}{2}$. Therefore, we have

Proposition 4.19.

$$d_r(a) \geq \frac{2n+2-C(\theta_1\theta_2)}{2}. \tag{4.18}$$

To show the equality of our lower bound in Theorem 4.15 and the Bafna-Pevzner lower bound Eq. (4.4), it suffices to show

$$C(p_r\tilde{s}) = C(\theta_1\theta_2) - 1 = 2C_{BG}(a) - 1.$$

Definition 4.20. Let σ be a permutation on the set $[n]^{\pm} = \{-n, \dots, -1, 0, 1, \dots, n\}$. We associate to σ the matrix $A_\sigma = [a_{ij}]$,

$$a_{ij} = \begin{cases} 1 & \text{if } i = \sigma(j) \\ 0 & \text{if } i \neq \sigma(j) \end{cases}$$

where $i, j \in [n]^{\pm}$, index rows and columns, following the order $-n, -n+1, \dots -1, 0, 1, \dots, n$, respectively. A_σ is the permutation matrix associated to σ and this is denoted by $\sigma \sim A_\sigma$.

Recall now, that $\sigma \sim A_\sigma \Leftrightarrow \sigma^{-1} \sim A_\sigma^T$. Furthermore if $\tau \sim A_\tau$ then $\sigma\tau = A_\sigma A_\tau$. And finally for id, the identity permutation on $[n]^\pm$, we have that $id \sim A_{id} = I_{2n+1}$ (i.e. the identity matrix).

Lemma 4.21. *Let*

$$p_r = (0, -1, -2, \dots - n, n, n-1, \dots 1) \sim A_p = P,$$
$$\tilde{s} = (0, a_1, a_2, \dots a_n, -a_n, -a_{n-1}, \dots - a_1) \sim A_{\tilde{s}} = S.$$

Let $R = [r_{ij}]$ be the $(2n+1) \times (2n+1)$ unitary anti-diagonal matrix, also known as the exchange matrix. Namely,

$$r_{ij} = \begin{cases} 1 & \text{if } j = 2n - i + 2 \\ 0 & \text{if } j \neq 2n - i + 2 \end{cases}$$

where $i, j \in [2n+1]$ index rows and columns of R respectively.
Then

$$PS = (PR)(RS), \tag{4.19}$$

Proof. It suffices to check that $R = R^T$. Then, since R is a permutation matrix we have $R^2 = RR^T = I_{2n+1}$. □

We next show that both permutations PR and RS are involutions with a unique fixed point.

Lemma 4.22. *The permutation corresponding to the matrix PR is the involution*

$$p_{invo} = (-n, n-1)(-n+1, n-2) \cdots (-1, 0)(n).$$

Proof. We compute

$$PR = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & \dots 0 & 1 & 0 \\ 0 & 0 & 0 & \dots 1 & 0 & 0 \\ 0 & 0 & 0 & \dots 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1 & 0 & \dots 0 & 0 & 0 \\ 1 & 0 & 0 & \dots 0 & 0 & 0 \\ 0 & 0 & 0 & \dots 0 & 0 & 1 \end{bmatrix}$$

Converting the resulting matrix into its associated permutation completes the proof. □

Lemma 4.23. *The permutation corresponding to the matrix RS is the involution*

$$s_{invo} = (0, -a_1)(a_1, -a_2) \cdots (a_{n-1}, -a_n)(a_n)$$

Proof. It is easy to check that left multiplication by the exchange matrix R reverses the order on the rows of the multiplied matrix $S = [x_{ij}]$, $i, j \in [n]^{\pm}$. Since $x_{i,j} = 1 \Longleftrightarrow i = s(j)$, when multiplied by the matrix R, the row indexed by i, in the matrix S, is sent to the row indexed by $-i$. This is due to the symmetry of $[n]^{\pm}$. However, this in turn means that in the permutation $s_{invo} \sim RS$, we have $j \xrightarrow{s_{invo}} -i$ for any $j \in [n]^{\pm}$ such that $\tilde{s}(j) = i$. But now by virtue of the structure of the cycle $\tilde{s} = (0, a_1, a_2, \dots a_n, -a_n, -a_{n-1}, \dots - a_1)$, we have

$$s_{invo} = (0, -a_1)(a_1, -a_2) \cdots (a_{n-1}, -a_n)(a_n)$$

and the lemma follows. □

So far we have shown that both PR and RS are matrices corresponding to involutions. We have thus exhibited an involution decomposition of $p\tilde{s}$. Furthermore, we note that each of the two involutions has a unique fixed point, namely n and s_n respectively. We proceed now to prove some results regarding the product of two involutions.

Lemma 4.24. *Let σ_1, σ_2 be two fixed-point free involutions on a set T, with $|T| = 2n$ for $n \geq 1$. Then, there does not exist $k \geq 1$ such that $\sigma_1(\sigma_2\sigma_1)^k(x) = x$.*

Proof. Assume by contradiction that there exists a $k \geq 1$ such that $\sigma_1(\sigma_2\sigma_1)^k(x) = x$. Then, since σ_1 is an involution, we have $\sigma_1(x) = (\sigma_2\sigma_1)^k(x)$.

Now if $k = 1$, we have $\sigma_1(x) = \sigma_2\sigma_1(x)$, implying that $\sigma_1(x)$ is a fixed point of σ_2, which is a contradiction.

If $k > 1$, since σ_2 is an involution, we have

$$\sigma_2\sigma_1(x) = (\sigma_1\sigma_2)^{k-1}(\sigma_1(x)) = \sigma_1(\sigma_2\sigma_1)^{k-1}(x).$$

Now, if $k - 1 = 1$, we have that $\sigma_2\sigma_1(x)$ has to be a fixed point of σ_1, which is again a contradiction. Otherwise, we can set $y = \sigma_2\sigma_1(x)$ and obtain $y = \sigma_1(\sigma_2\sigma_1)^{k-1}(y)$, and iterate the previous argument. In this way, we eventually obtain a fixed point, either for σ_1 or σ_2, which contradicts our assumption, hence the lemma follows. $\square$

Lemma 4.25. *Let σ_1, σ_2 be two involutions on a set T such that each of them has a unique fixed point, a and b respectively. Then $\sigma_2\sigma_1$ has a cycle which contains both a and b.*

Proof. There is nothing to prove if $a = b$, so we will assume $a \neq b$ in the following. Let now

$$\sigma_1 = (p_1, p_2)(p_3, p_4) \cdots (p_{2t-1}, p_{2t})(a),$$
$$\sigma_2 = (q_1, q_2)(q_3, q_4) \cdots (q_{2t-1}, q_{2t})(b),$$
$$\sigma_1' = (p_1, p_2)(p_3, p_4) \cdots (p_{2t-1}, p_{2t})(a, x),$$
$$\sigma_2' = (q_1, q_2)(q_3, q_4) \cdots (q_{2t-1}, q_{2t})(b, x),$$

where σ_1' and σ_2' are involutions on $T \cup \{x\}$, where $x \notin T$. We now compare the following two iterations:

$$a \to \sigma_1(a) \to \sigma_2\sigma_1(a) \to \sigma_1\sigma_2\sigma_1(a) \to (\sigma_2\sigma_1)^2(a) \cdots (\sigma_2\sigma_1)^{k_1}(a) = a,$$
$$x \to \sigma_1'(x) \to \sigma_2'\sigma_1'(x) \to \sigma_1'\sigma_2'\sigma_1'(x) \to (\sigma_2'\sigma_1')^2(x) \cdots \sigma_1'(\sigma_2'\sigma_1')^{k_2-1}(x) \to (\sigma_2'\sigma_1')^{k_2}(x) = x.$$

Note that $\sigma_1(a) = \sigma_1'(x) = a$, and that σ_1 and σ_1', excluding x, differ only at the image of a. Similarly, σ_2 and σ_2', excluding x, differ only at the image of b. Thus, the iterations starting with $\sigma_1(a)$ and $\sigma_1'(x)$ agree with each other until reaching a or b.

Claim 1. The iteration starting with $x \to \sigma_1'(x) = a$ will not reach a for a second time.
This is because, otherwise, there must exists some $k \geq 1$, such that $\sigma_1'(\sigma_2'\sigma_1')^k(x) = a$ or $(\sigma_2'\sigma_1')^k(x) = a$. The former case can not happen, otherwise $(\sigma_2'\sigma_1')^k(x) = x$, which will close the iteration instead of continuing to a. By Lemma 4.24, the latter case, $(\sigma_2'\sigma_1')^k(x) = \sigma_2'(\sigma_1'\sigma_2')^{k-1}(\sigma_1'(x)) = \sigma_2'(\sigma_1'\sigma_2')^{k-1}(a) = a$ cannot happen either. Hence, Claim 1 follows.

Claim 2. The iteration starting with $\sigma_1'(x) = a$ will reach b at least once.
This is obvious since $\sigma_1'(\sigma_2'\sigma_1')^{k_2-1}(x) = b$.

Now consider the first time the iteration starting with $\sigma_1'(x)$ reaches b. This must also be the first time the iteration starting with $\sigma_1(a)$ reaches b. There are two cases: either

$(\sigma_2\sigma_1)^k(a) = b$ or $\sigma_1(\sigma_2\sigma_1)^k(a) = b$ for some $k \geq 1$. For the former case, we already have a and b as being in the same cycle of $\sigma_2\sigma_1$;

For the latter case, we have $(\sigma_2\sigma_1)^{k+1}(a) = \sigma_2(b) = b$, which also implies that a and b are in the same cycle of $\sigma_2\sigma_1$. This completes the proof. □

Remark 4.26. Lemma 4.25 can be alternatively proved in the following approach: first, we show that there is a way to assign signs '+' and '-' to elements in the set T such that in both σ_1 and σ_2, every 2-cycle has exactly one '+' element and one '-' elelment while a and b are positive, see a dihedral group action argument as in the Intersection-Theorem [56]. Then, we apply the Garsia-Milne Involution Principle [34] to explain that a and b are in the same cycle of the product $\sigma_1\sigma_2$.

Lemma 4.27. *Let σ_1, σ_2, σ'_1, σ'_2 be defined as in the proof of Lemma 4.25. Then,*

$$C(\sigma'_1\sigma'_2) - 1 = C(\sigma_1\sigma_2). \tag{4.20}$$

Proof. Following the discussion in the proof of Lemma 4.25, any cycle, of $\sigma'_1\sigma'_2$, not containing a or b or x is also a cycle of $\sigma_1\sigma_2$. Thus, the difference $C(\sigma'_1\sigma'_2) - C(\sigma_1\sigma_2)$ equals the difference of the number of cycles containing a, b and x in $\sigma'_1\sigma'_2$ and the number of cycles containing a and b in $\sigma_1\sigma_2$.

On the one hand, we have already shown that a and b are in the same cycle of $\sigma_1\sigma_2$. On the other hand, it is clear that $\sigma'_1\sigma'_2(a) = b$; and by Claim 1 in the proof of Lemma 4.25, x and a are not in the same cycle of $\sigma'_1\sigma'_2$. Hence, the difference is exactly 1, completing the proof. □

Based on these lemmas above, we can easily prove

Theorem 4.28. *For any given signed permutation a, we have*

$$C(p_r\tilde{s}) = C(\theta_1\theta_2) - 1. \tag{4.21}$$

Proof. By construction, the relation between the pair s_{invo}, p_{invo} and θ_1, θ_2 is exactly the same as the pair σ_1, σ_2 and σ_1', σ_2'. Applying Lemma 4.27, we have

$$C(\theta_1\theta_2) - 1 = C(p_{invo}s_{invo}) = C(p\tilde{s}),$$

completing the proof. □

Accordingly, our lower bound in Theorem 4.15 equals the Bafna-Pevzner lower bound Eq. (4.4) through breakpoint graphs [9, 40].

In addition, we confirmed our speculation in the last section that

Theorem 4.29. *For any signed permutation a, the elements n and a_n are in the same cycle of the product $p_r\tilde{s}$.*

Proof. Applying Lemma 4.25 to the involution decomposition

$$p_r\tilde{s} = p_{invo}s_{invo}$$

the theorem follows. □

4.6 Conclusion and open problems

Here we provided a unified framework for the transposition and block-interchange distance of permutations as well as the reversal distance of signed permutations. This plane permutation framework augments the graph-based approaches and permutation based approaches. Specifically, we obtained general lower bounds for the transposition distance and the block-interchange distance, that give rise to novel optimization problems in terms of a new free parameter. The lower bounds obtained by Bafna-Pevzner [8], Christie [20], Lin et al. [49], Huang et al. [37], Labarre [48] can be refined in terms of this parameter. As for the reversal distance of signed permutations, plane permutations allowed us to connect with the block-interchange distances of skew-symmetric sequences and to immediately obtained a lower bound. This lower bound for the reversal distance was proved to be actually equal to the wellknown Bafna-Pevzner lower bound. Sections 4.2–4.4 are based on the paper "*[10] Plane permutations and applications to a result of Zagier–Stanley and distances of permutations*, SIAM J. Discrete Math. 30(3) (2016) pp. 1660–1684" while Section 4.5 is based on "*[2] On a lower bound for sorting signed permutations by reversals, arXiv:1602.00778 [math.CO]*".

As for future directions and outlook, we propose the following open problem: Solve the optimization problem for odd and even cycles w.r.t. Theorem 4.7, i.e., obtaining a result similar to Theorem 4.13. This is not only important in the context of the cycle-graph model or its variations, but also interesting as a purely combinatorial problem. Furthermore, it is also meaningful to examine if we can have more efficient algorithms for sorting permutations based on our plane permutation framework.

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