Spherical Elements in the Affine Yokonuma-Hecke Algebra

Richard Martin Shaplin III

Thesis submitted to the Faculty of the
Virginia Polytechnic Institute and State University
in partial fulfillment of the requirements for the degree of

Master of Science
in
Mathematics

Daniel Orr, Chair
Peter Linnell
Mark Shimozono

May 13, 2020
Blacksburg, Virginia

Keywords: Affine Yokonuma–Hecke Algebras, Spherical Elements

Spherical Elements in the Affine Yokonuma-Hecke Algebra

Richard Martin Shaplin III

ABSTRACT

In Chapter 1 we introduce the Yokonuma-Hecke Algebra and a Yokonuma-Hecke Algebra-module. In Chapter 2 we determine that the possible eigenvalues of particular elements in the Yokonuma-Hecke Algebra acting on the module. In Chapter 3 we find determine module subspaces and eigenspaces that are isomorphic. In Chapter 4 we determine the structure of the q-eigenspace. In Chapter 5 we determine the spherical elements of the module.

Spherical Elements in the Affine Yokonuma-Hecke Algebra

Richard Martin Shaplin III

GENERAL AUDIENCE ABSTRACT

The Yokonuma-Hecke Algebra-module is a vector space over a particular field. Acting on vectors from the module by any element of the Yokonuma-Hecke Algebra corresponds to a linear transformation. Then, for each element we can find eigenvalues and eigenvectors. The transformations that we are considering all have the same eigenvalues. So, we consider the intersection of all the eigenspaces that correspond to the same eigenvalue. I.e. vectors that are eigenvectors of all of the elements. We find an algorithm that generates a basis for said vectors.

Contents

Chapter 1

Introduction

1.1 Definitions

We follow the setup and notation of [RS]. We set ring $\Bbbk = \mathbb{C}[q, q^{-1}]$ and $z = \frac{q-q^{-1}}{d}$. We consider all tensor products to be over $\Bbbk$. We call C_d the cyclic group of order d, generated by σ. We denote $\sigma_i = 1^{\otimes(i-1)} \otimes \sigma \otimes 1^{\otimes(n-i)} \in \Bbbk C_d^{\otimes n}$. Then for $n \geq 2$, $H_n^{\text{aff}}(\Bbbk C_d, z)$, the affine Yokonuma–Hecke algebra, is the free product of algebras,

$$\Bbbk[X_1^{\pm 1}, \cdots, X_n^{\pm 1}] \star \Bbbk C_d^{\otimes n} \star \langle T_i : 1 \leq i \leq n-1 \rangle$$

modulo the relations:

[RS, (2.2)]	$T_i T_j = T_j T_i,$	$1 \leq i, j \leq n-1, \lvert i-j \rvert > 1,$
[RS, (2.3)]	$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1},$	$1 \leq i \leq n-2,$
[RS, (2.4)]	$T_i^2 = z t_{i,i+1} T_i + 1,$	$1 \leq i \leq n-1,$
[RS, (2.5)]	$T_i a = s_i(a) T_i,$	$a \in \Bbbk C_d^{\otimes n}, 1 \leq i \leq n-1$
[RS, (2.8)]	$T_i X_j = X_j T_i,$	$1 \leq i \leq n-1, 1 \leq j \leq n, j \neq i, i+1,$
[RS, (2.9)]	$T_i X_i T_i = X_{i+1},$	$1 \leq i \leq n-1,$
[RS, (2.10)]	$X_i a = a X_i,$	$1 \leq i \leq n, a \in \Bbbk C_d^{\otimes n}.$

Where $t_{i,j} = \sum_{k=0}^{d-1} \sigma_i^k \sigma_j^{-k} \in \Bbbk C_d^{\otimes n}$. For $w \in S_n$, we define $T_w = T_{i_1} T_{i_2} \cdots T_{i_k}$, where $w = s_{i_1} s_{i_2} \cdots s_{i_k}$ is a reduced decomposition. Since the generators T_i satisfy the braid relations (2.2) and (2.3), this definition is independent of the choice of reduced decomposition. [RS, 3]

We also set $P_n = \Bbbk[X_1^{\pm 1}, \cdots, X_n^{\pm 1}]$, and $P_n(\Bbbk C_d) = P_n \otimes \Bbbk C_d^{\otimes n}$. Let $f \in P_n(\Bbbk C_d)$, $w \in S_n$.

Then define $w(f)$ as the permutation of the X_i and factors of $\Bbbk C_d^{\otimes n}$ in f. Define ${}^w(f)$ as the permutation of only the the X_i. And define the Demazure operator $\Delta_i(f) = \frac{1-{}^{s_i}f}{1-X_iX_{i+1}^{-1}}$.

Then for H free $\Bbbk$-module with basis $\{T_w : w \in S_n\}$, we see that $V = P_n(\Bbbk C_d) \otimes H$ is a tensor product of $\Bbbk$-modules. So by [RS, 3.8], V is an $H_n^{\mathrm{aff}}(\Bbbk C_d, z)$-module with action given by,

$$f(g \otimes T_w) = fg \otimes T_w$$

$$T_i(f \otimes T_w) = \begin{cases} s_i(f) \otimes T_{s_iw} + zt_{i,i+1}\Delta_i(f) \otimes T_w & \text{if } l(s_iw) > l(w) \\ s_i(f) \otimes T_{s_iw} + zt_{i,i+1}X_{i+1}^{-1}\Delta_i(X_{i+1}f) \otimes T_w & \text{if } l(s_iw) < l(w). \end{cases}$$

V has basis $B = \{p \otimes \sigma^e \otimes T_w \mid \rho = X^a = X_1^{a_1} \cdots X_n^{a_n} \in P_n, \sigma^e = \sigma^{e_1} \otimes \cdots \otimes \sigma^{e_n} \in \Bbbk C_d^{\otimes n}, w \in S_n\}$.

We also introduce our own definitions, $E_s = \{e \in \mathbb{Z}^n \mid \sum_{i=1}^n e_i \equiv s \pmod{d}\}$, and $K_s \subset V$ with basis $\{p \otimes \sigma^e \otimes T_w \mid p \otimes \sigma^e \otimes T_w \in B, e \in E_s\}$. Note: $K_s = K_t$ if $s \equiv t \pmod{d}$. Denote $r_i = 1^{\otimes(i-1)} \otimes \sigma^1 \otimes \sigma^{-1} \otimes 1^{\otimes(n-i+1)}$. Then, $t_{i,i+1} = \sum_{k=0}^{d-1} r_i^k$. Finally for $f \in V$, $f = \sum_{w \in S_n} f_w \otimes T_w$. So denote $[T_w]f = f_w$.

1.2 Overview

In Chapter 2 we determine that the only possible eigenvalues of the T_i's acting on V are $1, -1, q, -\frac{1}{q}$. In Chapter 3 we determine that the K_s's are isomorphic $H_n^{\mathrm{aff}}(\Bbbk C_d, z)$-modules. We also find that the 1 and -1 eigenspaces are isomorphic along with the q and $-\frac{1}{q}$ eigenspaces. In Chapter 4 we determine the structure of the q-eigenspace for any particular T_i. In Chapter 5 we determine the spherical elements of V.

Chapter 2

Eigenvalues of $\mathbf{T}_i$

Proposition 1. *The only T_i eigenvalues are $1, -1, q,$ and $-\frac{1}{q}$.*

Proof. Suppose $f \in V$ is an eigenvector of T_i. Then, $T_i f = \lambda f$ for $\lambda \in \Bbbk$. Next we observe (2.4) in [RS], $T_i^2 = z t_{i,i+1} T_i + 1$ So,

$$\begin{aligned} T_i^2 f &= (z t_{i,i+1} T_i + 1) f \\ \lambda^2 f &= z t_{i,i+1} \lambda f + f \\ (\lambda^2 - 1) f &= z \lambda t_{i,i+1} f. \end{aligned}$$

We also see that,

$$f = \sum_{p, \sigma^e, T_w} \alpha_{p \otimes \sigma^e \otimes T_w} (p \otimes \sigma^e \otimes T_w) \quad \alpha_{p \otimes \sigma^e \otimes T_w} \in \Bbbk.$$

So,

$$t_{i,i+1} f = \sum_{p, \sigma^e, T_w} \sum_{k=0}^{d-1} \alpha_{p \otimes \sigma^e \otimes T_w} (p \otimes r_i^k \sigma^e \otimes T_w)$$

$$= \sum_{p, \sigma^e, T_w} \sum_{k=0}^{d-1} \alpha_{p \otimes r_i^k \sigma^e \otimes T_w} (p \otimes \sigma^e \otimes T_w).$$

Then,

$$\begin{aligned} [p \otimes \sigma^e \otimes T_w](\lambda^2 - 1) f &= [p \otimes \sigma^e \otimes T_w] z \lambda t_{i,i+1} f \\ (\lambda^2 - 1) \alpha_{p \otimes \sigma^e \otimes T_w} &= z \lambda \sum_{k=0}^{d-1} \alpha_{p \otimes r_i^k \sigma^e \otimes T_w}. \end{aligned}$$

Case 1: $\forall p \otimes \sigma^e \otimes T_w \in B, \quad \sum_{k=0}^{d-1} \alpha_{p\otimes r_i^k \sigma^e \otimes T_w} = 0.$

Since $f \neq 0$, $\exists \alpha_{p\otimes\sigma^e\otimes T_w} \neq 0$ and we see,

$$\begin{aligned}
&(\lambda^2 - 1)\alpha_{p\otimes\sigma^e\otimes T_w} = 0 \\
&(\lambda^2 - 1) = 0 \\
&\lambda = 1, -1.
\end{aligned}$$

Case 2: $\exists p \otimes \sigma^e \otimes T_w \in B, \quad \sum_{k=0}^{d-1} \alpha_{p\otimes r_i^k \sigma^e \otimes T_w} \neq 0.$

Observe that $\forall x \in [0, d-1]$

$$\begin{aligned}
&(\lambda^2 - 1)\alpha_{p\otimes r_i^x \sigma^e \otimes T_w} \\
&= z\lambda \sum_{k=0}^{d-1} \alpha_{p\otimes r_i^{(x+k)} \sigma^e \otimes T_w} \\
&= z\lambda \sum_{k=0}^{d-1} \alpha_{p\otimes r_i^k \sigma^e \otimes T_w}.
\end{aligned}$$

So,

$$\begin{aligned}
&\sum_{x=0}^{d-1} (\lambda^2 - 1)\alpha_{p\otimes r_i^x \sigma^e \otimes T_w} = dz\lambda \sum_{k=0}^{d-1} \alpha_{p\otimes r_i^k \sigma^e \otimes T_w} \\
&(\lambda^2 - 1) \sum_{x=0}^{d-1} \alpha_{p\otimes r_i^x \sigma^e \otimes T_w} = dz\lambda \sum_{k=0}^{d-1} \alpha_{p\otimes r_i^k \sigma^e \otimes T_w} \\
&\lambda^2 - 1 = dz\lambda \\
&\lambda^2 - 1 = (q - q^{-1})\lambda \\
&(\lambda - q)(\lambda + q^{-1}) = 0 \\
&\lambda = q, -\frac{1}{q}.
\end{aligned}$$

□

Chapter 3

Isomorphisms Between Eigenspaces

3.1 Subspace Isomorphism

Proposition 2. *The K_j subspaces are invariant under T_i.*

Proof. In order to show invariance we first observe that for any basis element $p \otimes \sigma^e \otimes T_w \in K_j$,

$$\begin{aligned}
&t_{i,i+1} p \otimes \sigma^e \otimes T_w \\
&= \sum_{k=0}^{d-1} p \otimes r_i^k \sigma^e \otimes T_w \\
&= \sum_{k=0}^{d-1} p \otimes \sigma^{e_1} \otimes \cdots \otimes \sigma^{e_i+k} \otimes \sigma^{e_{i+1}-k} \otimes \cdots \otimes \sigma^{e_n} \otimes T_w \in K_j
\end{aligned}$$

since $(\sum_{i=1}^n e_i) + k - k \equiv j \pmod d$.

And,

$$s_i(p \otimes \sigma^e) \otimes T_w = s_i(p) \otimes \sigma^{e_1} \otimes \cdots \otimes \sigma^{e_{i+1}} \otimes \sigma^{e_i} \otimes \cdots \otimes \sigma^{e_n} \otimes T_w \in K_j.$$

So,

$$\begin{aligned}
&T_i(p \otimes \sigma^e \otimes T_w) \\
&= s_i(p \otimes \sigma^e) \otimes T_{s_i w} + z t_{i,i+1} \Delta_i (p \otimes \sigma^e) \otimes T_w \\
&= s_i(p \otimes \sigma^e) \otimes T_{s_i w} + z t_{i,i+1} \Delta_i (p) \otimes \sigma^e \otimes T_w.
\end{aligned}$$

Since $s_i(p \otimes \sigma^e) \otimes T_{s_i w} \in K_j$ and $t_{i,i+1} \Delta_i(p) \otimes \sigma^e \otimes T_w \in K_j$,

$T_i(p \otimes \sigma^e \otimes T_w) \in K_j$.

We conclude that K_j is invariant under T_i. □

Proposition 3. *The K_j subspaces are isomorphic $H_n^{\mathrm{aff}}(\Bbbk C_d, z)$-modules.*

Proof. It is sufficient to show

$$\forall j,\ K_j \cong K_{j+1}.$$

Let $\phi : K_j \to K_{j+1}$ be the linear V isomorphism given by

$$\phi(p \otimes \sigma^{e_1} \otimes \cdots \otimes \sigma^{e_n} \otimes T_w) = p \otimes \sigma^{e_1} \otimes \cdots \otimes \sigma^{e_{w(1)}+1} \otimes \cdots \otimes \sigma^{e_n} \otimes T_w.$$

We first show:

$$\phi T_i = T_i \phi.$$

We show this by considering ϕT_i on basis element $p \otimes \sigma^{e_1} \otimes \cdots \otimes \sigma^{e_n} \otimes T_w$.

Case 1: $l(w) < l(s_i w)$.

$$\begin{aligned}
&\phi T_i(p \otimes \sigma^{e_1} \otimes \cdots \otimes \sigma^{e_n} \otimes T_w) \\
&= \phi(s_i(p) \otimes \sigma^{e_1} \otimes \cdots \otimes \sigma^{e_{i+1}} \otimes \sigma^{e_i} \otimes \cdots \otimes \sigma^{e_n} \otimes T_{s_i w} \\
&\quad + z t_{i,i+1} \Delta_i(p) \otimes \sigma^{e_1} \otimes \cdots \otimes \sigma^{e_n} \otimes T_w).
\end{aligned} \tag{3.1}$$

Case 1a: $w(1) \neq i, i+1$.

$$\begin{aligned}
(3.1) &= T_i(p \otimes \sigma^{e_1} \otimes \cdots \otimes \sigma^{e_{w(1)}+1} \otimes \cdots \otimes \sigma^{e_n} \otimes T_w) \\
&= T_i(\phi(p \otimes \sigma^{e_1} \otimes \cdots \otimes \sigma^{e_n} \otimes T_w)).
\end{aligned}$$

Case 1b: $w(1) = i+1$ and $s_i w(1) = i$.

$$\begin{aligned}
(3.1) &= s_i(p) \otimes \sigma^{e_1} \otimes \cdots \otimes \sigma^{e_{i+1}+1} \otimes \sigma^{e_i} \otimes \cdots \otimes \sigma^{e_n} \otimes T_{s_i w} \\
&\quad + z t_{i,i+1} \Delta_i(p) \otimes \sigma^{e_1} \otimes \cdots \otimes \sigma^{e_i} \otimes \sigma^{e_{i+1}+1} \otimes \cdots \otimes \sigma^{e_n} \otimes T_w \\
&= s_i(p \otimes \sigma^{e_1} \otimes \cdots \otimes \sigma^{e_i} \otimes \sigma^{e_{i+1}+1} \otimes \cdots \otimes \sigma^{e_n}) \otimes T_{s_i w} \\
&\quad + z t_{i,i+1} \Delta_i(p) \otimes \sigma^{e_1} \otimes \cdots \otimes \sigma^{e_i} \otimes \sigma^{e_{i+1}+1} \otimes \cdots \otimes \sigma^{e_n} \otimes T_w \\
&= T_i(p \otimes \sigma^{e_1} \otimes \cdots \otimes \sigma^{e_{w(1)}+1} \otimes \cdots \otimes \sigma^{e_n} \otimes T_w) \\
&= T_i(\phi(p \otimes \sigma^{e_1} \otimes \cdots \otimes \sigma^{e_n} \otimes T_w)).
\end{aligned}$$

Case 1c: $w(1) = i$ and $s_i w(1) = i+1$.

$$
\begin{aligned}
(3.1) &= s_i(p) \otimes \sigma^{e_1} \otimes \cdots \otimes \sigma^{e_{i+1}} \otimes \sigma^{e_i+1} \otimes \cdots \otimes \sigma^{e_n} \otimes T_{s_i w} \\
&\quad + z t_{i,i+1} \Delta_i(p) \otimes \sigma^{e_1} \otimes \cdots \otimes \sigma^{e_i+1} \otimes \sigma^{e_{i+1}} \otimes \cdots \otimes \sigma^{e_n} \otimes T_w \\
&= s_i(p \otimes \sigma^{e_1} \otimes \cdots \otimes \sigma^{e_i+1} \otimes \sigma^{e_{i+1}} \otimes \cdots \otimes \sigma^{e_n}) \otimes T_{s_i w} \\
&\quad + z t_{i,i+1} \Delta_i(p) \otimes \sigma^{e_1} \otimes \cdots \otimes \sigma^{e_i+1} \otimes \sigma^{e_{i+1}} \otimes \cdots \otimes \sigma^{e_n} \otimes T_w \\
&= T_i(p \otimes \sigma^{e_1} \otimes \cdots \otimes \sigma^{e_{w(1)}+1} \otimes \cdots \otimes \sigma^{e_n} \otimes T_w) \\
&= T_i(\phi(p \otimes \sigma^{e_1} \otimes \cdots \otimes \sigma^{e_n} \otimes T_w)).
\end{aligned}
$$

Case 2: $l(w) > l(s_i w)$.

Similar to Case 1.

We next show,

$$\forall g \in P_n(\Bbbk C_d),\ \phi g = g\phi.$$

We show this by considering ϕg on basis element $p \otimes \sigma^{e_1} \otimes \cdots \otimes \sigma^{e_n} \otimes T_w$.

$$
\begin{aligned}
&\phi g(p \otimes \sigma^{e_1} \otimes \cdots \otimes \sigma^{e_n} \otimes T_w) \\
&= \phi(gp \otimes \sigma^{e_1} \otimes \cdots \otimes \sigma^{e_n} \otimes T_w) \\
&= gp \otimes \sigma^{e_1} \otimes \cdots \otimes \sigma^{e_{w(1)}+1} \otimes \cdots \otimes \sigma^{e_n} \otimes T_w \\
&= g(p \otimes \sigma^{e_1} \otimes \cdots \otimes \sigma^{e_{w(1)}+1} \otimes \cdots \otimes \sigma^{e_n} \otimes T_w) \\
&= g\phi(p \otimes \sigma^{e_1} \otimes \cdots \otimes \sigma^{e_n} \otimes T_w).
\end{aligned}
$$

So ϕ is a $H_n^{\text{aff}}(\Bbbk C_d, z)$-module isomorphism.

□

3.2 Eigenspace Isomorphisms

Proposition 4. *The $\lambda = q$ eigenspace of T_i is isomorphic to the $\lambda = -\frac{1}{q}$ eigenspace of T_i.*

Proof. Let $\varphi : V \to V$ be the V isomorphism given by,

$$\varphi(q) = -\frac{1}{q}$$

$$\varphi(p \otimes \sigma^e \otimes T_w) = p \otimes \sigma^e \otimes T_w \quad (\varphi = \varphi^{-1}).$$

Claim:

$$\varphi T_i = T_i \varphi.$$

We show this by considering φT_i on basis element $p \otimes \sigma^e \otimes T_w$.

Case 1: $l(w) < l(s_i w)$.

$$\begin{aligned}
&\varphi T_i(p \otimes \sigma^e \otimes T_w) \\
&= \varphi(s_i(p \otimes \sigma^e) \otimes T_{s_i w} + z t_{i,i+1} \Delta_i(p \otimes \sigma^e) \otimes T_{s_i}) \\
&= s_i(p \otimes \sigma^e) \otimes T_{s_i w} + z t_{i,i+1} \Delta_i(p \otimes \sigma^e) \otimes T_{s_i} && (\varphi(z) = z) \\
&= T_i(p \otimes \sigma^e \otimes T_w) \\
&= T_i \varphi(p \otimes \sigma^e \otimes T_w).
\end{aligned}$$

Case 2: $l(w) > l(s_i w)$.

Similar.

So for $T_i(v) = qv$,

$$\begin{aligned}
T_i(\varphi(v)) &= \varphi(T_i(v)) \\
&= \varphi(qv) \\
&= -\frac{1}{q}\varphi(v).
\end{aligned}$$

So $\varphi(v)$ is a $-\frac{1}{q}$ eigenvector of T_i. □

Proposition 5. *The $\lambda = 1$ eigenspace of T_i is isomorphic to the $\lambda = -1$ eigenspace of T_i.*

Proof. Let $\psi : V \to V$ be given by,

$$\psi(q) = \frac{1}{q}$$

$$\psi(p \otimes \sigma^n \otimes T_w) = (-1)^{l(w)} p \otimes \sigma^n \otimes T_w.$$

Claim:

$$\psi T_i = -T_i\psi.$$

We show this by considering ψT_i on basis element $p \otimes \sigma^e \otimes T_w$.

Case 1: $l(w) < l(s_i w)$.

$$\begin{aligned}
&= \psi T_i(p \otimes \sigma^e \otimes T_w) \\
&= \psi(s_i(p \otimes \sigma^e) \otimes T_{s_i w} + z t_{i,i+1}\Delta_i(p \otimes \sigma^e) \otimes T_w) \\
&= (-1)^{l(s_i w)} s_i(p \otimes \sigma^e) \otimes T_{s_i w} + (-1)^{l(w)}(-z) t_{i,i+1}\Delta_i(p \otimes \sigma^e) \otimes T_w) \\
&= (-1)^{l(w)+1}(s_i(p \otimes \sigma^e) \otimes T_{s_i w} + z t_{i,i+1}\Delta_i(p \otimes \sigma^e) \otimes T_w) \\
&= (T_i((-1)^{l(w)+1} p \otimes \sigma^e \otimes T_w) \\
&= T_i(-\psi(p \otimes \sigma^e \otimes T_w)) \\
&= -T_i\psi(p \otimes \sigma^e \otimes T_w).
\end{aligned}$$

Case 2: $l(s_i w) < l(w)$.

Similar.

So, for $T_i(v) = v$

$$\begin{aligned}
T_i(\psi(v)) &= -\psi(T_i(v)) \\
&= -\psi(v).
\end{aligned}$$

So $\psi(v)$ is a -1 eigenvector of T_i. □

Chapter 4

Structure of the q-Eigenspaces

Proposition 6. *f is a q eigenvector of T_i if and only if $f = \frac{1}{d}t_{i,i+1}f$ and for all w, s_iw pairs in S_n with $l(w) < l(s_iw)$ we have $qf_w = s_i(f_{s_iw}) + (q - q^{-1})\Delta_i(f_w)$.*

Proof. Suppose f is a q eigenvector of T_i. Then,

$$\begin{aligned}
&T_i^2 f = zt_{i,i+1}T_i f + f \\
&q^2 f = zqt_{i,i+1}f + f \\
&(q^2 - 1)f = \frac{q^2 - 1}{d}t_{i,i+1}f \\
&f = \frac{1}{d}t_{i,i+1}f.
\end{aligned}$$

The only components of f that contribute to $(T_if)_w \otimes T_w$ under T_i are $f_w \otimes T_w$ and $f_{s_iw} \otimes T_{s_iw}$. So,

$$\begin{aligned}
q(f_w \otimes T_w + f_{s_iw} \otimes T_{s_iw}) &= T_i(f_w \otimes T_w + f_{s_iw} \otimes T_{s_iw}) \\
&= s_i(f_w) \otimes T_{s_iw} + zt_{i,i+1}\Delta_i(f_w) \otimes T_w \\
&\quad + s_i(f_{s_iw}) \otimes T_w + zt_{i,i+1}X_{i+1}^{-1}\Delta_i(X_{i+1}f_{s_iw}) \otimes T_{s_iw}.
\end{aligned}$$

So,

$$\begin{aligned}
qf_w =& s_i(f_{s_iw}) + zt_{i,i+1}\Delta_i(f_w) \\
&= s_i(f_{s_iw}) + (q - q^{-1})\Delta_i(\frac{1}{d}t_{i,i+1}f_w) \\
&= s_i(f_{s_iw}) + (q - q^{-1})\Delta_i(f_w).
\end{aligned}$$

Now assume $f = \frac{1}{d}t_{i,i+1}f$ and for all w, s_iw pairs in S_n with $l(w) < l(s_iw)$ we have $qf_w = s_i(f_{s_iw}) + (q - q^{-1})\Delta_i(f_w)$.

$$
\begin{aligned}
&qf_w = s_i(f_{s_iw}) + (q - q^{-1})\Delta_i(f_w)\\
&(q + (q^{-1} - q)\Delta_i)f_w = s_i(f_{s_iw})\\
&(q^{-1} + (q - q^{-1})\Delta_i)(q + (q^{-1} - q)\Delta_i)f_w = (q^{-1} + (q - q^{-1})\Delta_i)s_i(f_{s_iw})\\
&(1 + (q^2 - 1)\Delta_i + (q^{-2} - 1)\Delta_i + (1 - q^{-2} - q^2 + 1)\Delta_i)f_w = (q^{-1} + (q - q^{-1})\Delta_i)s_i(f_{s_iw})\\
&f_w = (q^{-1} + (q - q^{-1})\Delta_i)s_i(f_{s_iw}).
\end{aligned}
$$

Then,

$$
\begin{aligned}
&s_i(f_w) + zt_{i,i+1}X_{i+1}^{-1}\Delta_i(X_{i+1}f_{s_iw}) - qf_{s_iw}\\
&= s_i((q^{-1} + (q - q^{-1})\Delta_i)s_i(f_{s_iw})) + (q - q^{-1})X_{i+1}^{-1}\Delta_i(X_{i+1}f_{s_iw}) - qf_{s_iw}\\
&= (q^{-1} - q)f_{s_iw} - (q - q^{-1})X_iX_{i+1}^{-1}\Delta_i(f_{s_iw}) + (q - q^{-1})X_{i+1}^{-1}\Delta_i(X_{i+1}f_{s_iw}) \quad \text{[RS, (3.6)]}\\
&= (q - q^{-1})(-f_{s_iw} - X_iX_{i+1}^{-1}\Delta_i(f_{s_iw}) + X_{i+1}^{-1}(X_i\Delta_i(f_{s_iw}) + f_{s_iw}\Delta_i(X_{i+1})) \quad \text{[RS, (3.3)]}\\
&= (q - q^{-1})(-f_{s_iw} + X_{i+1}^{-1}X_{i+1}f_{s_iw})\\
&= 0.
\end{aligned}
$$

Hence, $qf_{s_iw} = s_i(f_w) + zt_{i,i+1}X_{i+1}^{-1}\Delta_i(X_{i+1}f_{s_iw})$. So we see that,

$$
\begin{aligned}
T_i(f_w \otimes T_w + f_{s_iw} \otimes T_{s_iw}) &= s_i(f_w) \otimes T_{s_iw} + zt_{i,i+1}\Delta_i(f_w) \otimes T_w\\
&\quad + s_i(f_{s_iw}) \otimes T_w + zt_{i,i+1}X_{i+1}^{-1}\Delta_i(X_{i+1}f_{s_iw}) \otimes T_{s_iw}\\
&= q(f_w \otimes T_w + f_{s_iw} \otimes T_{s_iw}).
\end{aligned}
$$

For all w, s_iw pairs. Hence, f is an eigenvector of T_i.

□

Chapter 5

Spherical elements

We now consider $\cap_{i=0}^{n-1} Ker\{T_i - q\}$. We saw in the previous proof that $f_w = (q^{-1} + (q - q^{-1})\Delta_i)s_i(f_{s_iw})$ for $l(w) < l(s_iw)$. So denote the Demazure-Lusztig operator $A_i = (q^{-1} + (q - q^{-1})\Delta_i)s_i$. Then for $v \in S_n$ with $v = s_{i_a} \cdots s_{i_1} w_0$, and $l(s_{i_{b+1}} \cdots s_{i_1} w_0) < l(s_{i_b} \cdots s_{i_1} w_0)$ for all $1 \leq b \leq a$. Denote $A_{vw_0} = A_{i_a} \cdots A_{i_1}$. Note that A_v does not depend on the choice of s_{i_b} by Proposition 8.

Proposition 7. *$\cap_{i=0}^{n-1} Ker\{T_i - q\}$ has basis $\{\sum_{v \in S_n} A_{vw_0} f_{w_0} \otimes T_v \mid f_{w_0} = \rho \otimes \sum_{e \in E_j} \sigma^e$, $\rho = X^a$, $0 \leq j \leq d-1\}$.*

Proof. Let $f \in \cap_{i=0}^{n-1} Ker\{T_i - q\}$. By Proposition 6 it is sufficient to find a basis for f such that $f = \frac{1}{d} t_{i,i+1} f$ and for all w, s_iw pairs in S_n with $l(w) < l(s_iw)$ we have $qf_w = s_i(f_{s_iw}) + (q - q^{-1})\Delta_i(f_w)$.

We first show f is determined by f_{w_0} where w_0 is the maximal element of S_n in the Bruhat order. Let $v \in S_n$. Write $v = s_{i_a} \cdots s_{i_1} w_0$ so that $l(s_{i_{b+1}} \cdots s_{i_1} w_0) < l(s_{i_b} \cdots s_{i_1} w_0)$ for all $1 \leq b \leq a$. Then, $f_v = A_{vw_0} f_{w_0}$ provided that $A_{vw_0} f_{w_0}$ is well defined. And we observe that for v with $l(v) < l(s_iv)$,

$$\begin{aligned} & A_i f_{s_iv} \\ &= A_i A_{i_a} \cdots A_{i_1} f_{w_0} \\ &= A_{vw_0} f_{w_0} \\ &= f_v. \end{aligned}$$

Hence,

$$
\begin{aligned}
(q^{-1} + (q - q^{-1})\Delta_i)s_i(f_{s_i v}) &= f_v \\
q f_v = s_i(f_{s_i v}) &+ (q - q^{-1})\Delta_i(f_v).
\end{aligned}
$$

We also note for all i, j,

$$
\begin{aligned}
A_j \frac{1}{d} t_{i,i+1} &= (q^{-1} + (q - q^{-1})\Delta_j)s_j \frac{1}{d} t_{i,i+1} \\
&= \frac{1}{d} t_{i,i+1} (q^{-1} + (q - q^{-1})\Delta_j)s_j \\
&= \frac{1}{d} t_{i,i+1} A_j.
\end{aligned}
$$

So, if $\frac{1}{d} t_{i,i+1} f_{w_0} = f_{w_0}$, then for all $v \in S_n$ we have $\frac{1}{d} t_{i,i+1} f_v = f_v$. i.e. $\frac{1}{d} t_{i,i+1} f = f$. Hence, it is sufficient to find a basis for f_{w_0}.

$f_{w_0} = \sum_{p,\sigma^e} \alpha_{p\otimes\sigma^e}(p \otimes \sigma^e) \quad \alpha_{p\otimes\sigma^e} \in \Bbbk$.

So,

$$
\begin{aligned}
\frac{1}{d} t_{i,i+1} f_{w_0} &= \frac{1}{d} \sum_{p,\sigma^e} \sum_{k=0}^{d-1} \alpha_{p\otimes\sigma^e}(p \otimes r_i^k \sigma^e) \\
&= \frac{1}{d} \sum_{p,\sigma^e} \sum_{k=0}^{d-1} \alpha_{p\otimes r_i^k \sigma^e}(p \otimes \sigma^e).
\end{aligned}
$$

So,

$$
[p \otimes \sigma^e] f_{w_0} = \frac{1}{d} \sum_{k=0}^{d-1} \alpha_{p\otimes r_i^k \sigma^e}.
$$

And for $c \in \mathbb{Z}$,

$$
\begin{aligned}
[p \otimes r_i^c \sigma^e] f_{w_0} &= \frac{1}{d} \sum_{k=0}^{d-1} \alpha_{p\otimes r_i^{(k+c)} \sigma^e} \\
&= \frac{1}{d} \sum_{k=0}^{d-1} \alpha_{p\otimes r_i^k \sigma^e}.
\end{aligned}
$$

.

So we see,

$$\begin{aligned}[p \otimes \sigma^e] f_{w_0} &= [p \otimes \sigma^e] \frac{1}{d^{n-1}} t_{1,2} \cdots t_{n-2,n-1} f_{w_0} \\ &= \frac{1}{d^{n-1}} \sum_{i=1}^{n-1} \sum_{k_i=0}^{d-1} \alpha_{p \otimes r_i^{k_i} \sigma^e}.\end{aligned}$$

And for $c_i \in \mathbb{Z}$,

$$\begin{aligned}[p \otimes (\prod_{i=1}^{n-1} r_i^{c_i}) \sigma^e] f_{w_0} &= \frac{1}{d^{n-1}} \sum_{i=1}^{n-1} \sum_{k_i=0}^{d-1} \alpha_{p \otimes r_i^{(k_i+c_i)} \sigma^e} \\ &= \frac{1}{d^{n-1}} \sum_{i=1}^{n-1} \sum_{k_i=0}^{d-1} \alpha_{p \otimes r_i^{k_i} \sigma^e}.\end{aligned}$$

And we observe that for any choice of $e \in E_j$, $\{(\prod_{i=1}^{n-1} r_i^{c_i}) \sigma^e | c_i \in \mathbb{Z}\} = \{\sigma^g | g \in E_j\}$ So, f_{w_0} has the described basis.

□

Proposition 8. *Let w_0 be the maximal element of S_n in the Bruhat order. Let $v \in S_n$. If $v = s_{i_a} \cdots s_{i_1} w_0$ so that $l(s_{i_{b+1}} \cdots s_{i_1} w_0) < l(s_{i_b} \cdots s_{i_1} w_0)$ for all $1 \leq b \leq a$, then $f_v = A_{i_a} \cdots A_{i_1} f_{w_0}$ is well defined.*

Proof. It is sufficient to show that the A_i satisfy the braid relations.

First observe that for $|i - j| > 1$ we have,

$$\begin{aligned}A_i A_j &= (q^{-1} + (q - q^{-1})\Delta_i) s_i (q^{-1} + (q - q^{-1})\Delta_j) s_j \\ &= (q^{-1} + (q - q^{-1})\Delta_i)(q^{-1} + (q - q^{-1})\Delta_j) s_j s_i \\ &= (q^{-1} + (q - q^{-1})\Delta_j)(q^{-1} + (q - q^{-1})\Delta_i) s_j s_i \\ &= (q^{-1} + (q - q^{-1})\Delta_j) s_j (q^{-1} + (q - q^{-1})\Delta_i) s_i \\ &= A_j A_i.\end{aligned}$$

Next we consider $A_i A_{i+1} A_i - A_{i+1} A_i A_{i+1}$. We introduce notation $\Delta_{(i,i+2)} = \frac{1-(i,i+2)1}{1-X_i X_{i+2}}$, $f_{i,j} = \frac{1}{1-X_i X_j^{-1}}$. Then,

$$\begin{aligned}A_iA_{i+1}A_i &= (q^{-1}+(q-q^{-1})\Delta_i)s_i(q^{-1}+(q-q^{-1})\Delta_{i+1})s_{i+1}(q^{-1}+(q-q^{-1})\Delta_i)s_i\\ &= (q^{-1}+(q-q^{-1})\Delta_i)(q^{-1}+(q-q^{-1})\Delta_{(i,i+2)})s_is_{i+1}(q^{-1}+(q-q^{-1})\Delta_i)s_i\\ &= (q^{-1}+(q-q^{-1})\Delta_i)(q^{-1}+(q-q^{-1})\Delta_{(i,i+2)})(q^{-1}+(q-q^{-1})\Delta_{i+1})s_is_{i+1}s_i\\ &= (q^{-3}+q^{-2}(q-q^{-1})(\Delta_i+\Delta_{(i,i+2)}+\Delta_{i+1})\\ &\quad +q^{-1}(q-q^{-1})^2(\Delta_i\Delta_{(i,i+2)}+\Delta_{i+2}\Delta_{i+1}+\Delta_i\Delta_{i+1})\\ &\quad +(q-q^{-1})^3(\Delta_i\Delta_{(i,i+2)}\Delta_{i+1}))s_is_{i+1}s_i.\end{aligned}$$

Similarly,

$$\begin{aligned}A_{i+1}A_iA_{i+1} &= (q^{-3}+q^{-2}(q-q^{-1})(\Delta_{i+1}+\Delta_{(i,i+2)}+\Delta_i)\\ &\quad +q^{-1}(q-q^{-1})^2(\Delta_{i+1}\Delta_{(i,i+2)}+\Delta_{i+2}\Delta_i+\Delta_{i+1}\Delta_i)\\ &\quad +(q-q^{-1})^3(\Delta_{i+1}\Delta_{(i,i+2)}\Delta_i))s_{i+1}s_is_{i+1}.\end{aligned}$$

Noting that, $s_is_{i+1}s_i = s_{i+1}s_is_{i+1}$ we see,

$$\begin{aligned}&(q^{-3}+q^{-2}(q-q^{-1})(\Delta_i+\Delta_{(i,i+2)}+\Delta_{i+1})+q^{-1}(q-q^{-1})^2(\Delta_i\Delta_{(i,i+2)}+\Delta_{(i,i+2)}\Delta_{i+1}+\Delta_i\Delta_{i+1})\\ &+(q-q^{-1})^3(\Delta_i\Delta_{(i,i+2)}\Delta_{i+1}))s_is_{i+1}s_i-(q^{-3}+q^{-2}(q-q^{-1})(\Delta_{i+1}+\Delta_{(i,i+2)}+\Delta_i)\\ &+q^{-1}(q-q^{-1})^2(\Delta_{i+1}\Delta_{(i,i+2)}+\Delta_{i+2}\Delta_i+\Delta_{i+1}\Delta_i)+(q-q^{-1})^3(\Delta_{i+1}\Delta_{(i,i+2)}\Delta_i))s_{i+1}s_is_{i+1}\\ &=(q^{-1}(q-q^{-1})^2(\Delta_i\Delta_{(i,i+2)}+\Delta_{i+2}\Delta_{i+1}+\Delta_i\Delta_{i+1}-\Delta_{i+1}\Delta_{(i,i+2)}-\Delta_{(i,i+2)}\Delta_i-\Delta_{i+1}\Delta_i)\\ &+(q-q^{-1})^3(\Delta_i\Delta_{(i,i+2)}\Delta_{i+1}-\Delta_{i+1}\Delta_{(i,i+2)}\Delta_i))s_is_{i+1}s_i.\end{aligned}$$

Then,

$$\begin{aligned}&\Delta_i\Delta_{(i,i+2)}+\Delta_i\Delta_{i+1}+\Delta_{(i,i+2)}\Delta_{i+1}-\Delta_{i+1}\Delta_{(i,i+2)}-\Delta_{i+1}\Delta_i-\Delta_{(i,i+2)}\Delta_i\\ &=(f_{i,i+1}-f_{i,i+1}^{s_i})(f_{i,i+2}-f_{i,i+2}^{(i,i+2)})+(f_{i,i+1}-f_{i,i+1}^{s_i})(f_{i+1,i+2}-f_{i+1,i+2}^{s_{i+1}})+\\ &\quad +(f_{i,i+2}-f_{i,i+2}^{(i,i+2)})(f_{i+1,i+2}-f_{i+1,i+2}^{s_{i+1}})-(f_{i+1,i+2}-f_{i+1,i+2}^{s_{i+1}})(f_{i,i+2}-f_{i,i+2}^{(i,i+2)})\\ &\quad -(f_{i+1,i+2}-f_{i+1,i+2}^{s_{i+1}})(f_{i,i+1}-f_{i,i+1}^{s_i})-(f_{i,i+2}-f_{i,i+2}^{(i,i+2)})(f_{i,i+1}-f_{i,i+1}^{s_i})\\ &=(f_{i,i+1}f_{i,i+2}+f_{i,i+1}f_{i+1,i+2}+f_{i,i+2}f_{i+1,i+2}-f_{i+1,i+2}f_{i,i+2}-f_{i+1,i+2}f_{i,i+1}-f_{i,i+2}f_{i,i+1})\\ &\quad -(f_{i,i+1}f_{i+1,i+2}+f_{i,i+1}f_{i,i+2}-f_{i+1,i+2}f_{i,i+1}-f_{i,i+2}f_{i,i+1})^{s_i}\\ &\quad -(f_{i,i+1}f_{i+1,i+2}+f_{i,i+2}f_{i+1,i+2}-f_{i+1,i+2}f_{i,i+1}-f_{i+1,i+2}f_{i,i+2})^{s_{i+1}}\\ &\quad -(f_{i,i+1}f_{i,i+2}+f_{i,i+2}f_{i+1,i}-f_{i+1,i+2}f_{i,i+2}-f_{i,i+2}f_{i+2,i+1})^{(i,i+2)}\\ &\quad +(f_{i,i+1}f_{i+1,i+2}+f_{i,i+2}f_{i+1,i}-f_{i+1,i+2}f_{i,i+2})^{(i,i+2,i+1)}\\ &\quad +(f_{i,i+1}f_{i,i+2}-f_{i+1,i+2}f_{i,i+1}-f_{i,i+2}f_{i+2,i})^{(i,i+1,i+2)}\\ &=0.\end{aligned}$$

$$
\begin{aligned}
&\Delta_i\Delta_{(i,i+2)}\Delta_{i+1}\\
&=(f_{i,i+1}-f^{s_i}_{i,i+1})(f_{i,i+2}-f^{(i,i+2)}_{i,i+2})(f_{i+1,i+2}-f^{s_{i+1}}_{i+1,i+2})\\
&=f_{i,i+1}f_{i,i+2}f_{i+1,i+2}-f_{i,i+1}f_{i,i+2}f^{s_{i+1}}_{i+1,i+2}-f_{i,i+1}f^{(i,i+2)}_{i,i+2}f_{i+1,i+2}+f_{i,i+1}f^{(i,i+2)}_{i,i+2}f^{s_{i+1}}_{i+1,i+2}\\
&\quad-f^{s_i}_{i,i+1}f_{i,i+2}f_{i+1,i+2}+f^{s_i}_{i,i+1}f_{i,i+2}f^{s_{i+1}}_{i+1,i+2}+f^{s_i}_{i,i+1}f^{(i,i+2)}_{i,i+2}f_{i+1,i+2}-f^{s_i}_{i,i+1}f^{(i,i+2)}_{i,i+2}f^{s_{i+1}}_{i+1,i+2}\\
&=f_{i,i+1}f_{i,i+2}f_{i+1,i+2}-f_{i,i+1}f_{i,i+2}f^{s_{i+1}}_{i+1,i+2}-f_{i,i+1}f_{i,i+2}f^{(i,i+2)}_{i+1,i}+f_{i,i+1}f_{i,i+2}f^{(i,i+2,i+1)}_{i+1,i}\\
&\quad-f_{i,i+1}f_{i+1,i+2}f^{s_i}_{i,i+2}+f_{i,i+1}f_{i+1,i+2}f^{(i,i+1,i+2)}_{i,i+2}+f_{i,i+1}f_{i+1,i+2}f^{(i,i+2,i+1)}_{i,i+1}-f_{i,i+1}f_{i+1,i+2}f^{(i,i+2)}_{i,i+1}.
\end{aligned}
$$

$$
\begin{aligned}
&\Delta_{i+1}\Delta_{(i,i+2)}\Delta_i\\
&=(f_{i+1,i+2}-f^{s_{i+1}}_{i+1,i+2})(f_{i,i+2}-f^{(i,i+2)}_{i,i+2})(f_{i,i+1}-f^{s_i}_{i,i+1})\\
&=f_{i+1,i+2}f_{i,i+2}f_{i,i+1}-f_{i+1,i+2}f_{i,i+2}f^{s_i}_{i,i+1}-f_{i+1,i+2}f^{(i,i+2)}_{i,i+2}f_{i,i+1}+f_{i+1,i+2}f^{(i,i+2)}_{i,i+2}f^{s_i}_{i,i+1}\\
&\quad-f^{s_{i+1}}_{i+1,i+2}f_{i,i+2}f_{i,i+1}+f^{s_{i+1}}_{i+1,i+2}f_{i,i+2}f^{s_i}_{i,i+1}+f^{s_{i+1}}_{i+1,i+2}f^{(i,i+2)}_{i,i+2}f_{i,i+1}-f^{s_{i+1}}_{i+1,i+2}f^{(i,i+2)}_{i,i+2}f^{s_i}_{i,i+1}\\
&=f_{i+1,i+2}f_{i,i+2}f_{i,i+1}-f_{i+1,i+2}f_{i,i+2}f^{s_i}_{i,i+1}-f_{i+1,i+2}f_{i,i+2}f^{(i,i+2)}_{i+2,i+1}+f_{i+1,i+2}f_{i,i+2}f^{(i,i+1,i+2)}_{i+2,i+1}\\
&\quad-f_{i+1,i+2}f_{i,i+1}f^{s_{i+1}}_{i,i+2}+f_{i+1,i+2}f_{i,i+1}f^{(i,i+2,i+1)}_{i,i+2}+f_{i+1,i+2}f_{i,i+1}f^{(i,i+1,i+2)}_{i+1,i+2}-f_{i+1,i+2}f_{i,i+1}f^{(i,i+2)}_{i+1,i+2}.
\end{aligned}
$$

So,

$$
\begin{aligned}
&\Delta_i\Delta_{(i,i+2)}\Delta_{i+1}-\Delta_{i+1}\Delta_{(i,i+2)}\Delta_i\\
&=-(f_{i,i+1}f_{i,i+2}f_{i+1,i}+f_{i,i+1}f_{i+1,i+2}f_{i,i+1}-f_{i+1,i+2}f_{i,i+2}f_{i+2,i+1}-f_{i+1,i+2}f_{i,i+1}f_{i+1,i+2})^{(i,i+2)}\\
&\quad+(f_{i,i+1}f_{i+1,i+2}f_{i,i+2}-f_{i+1,i+2}f_{i,i+2}f_{i+2,i+1}-f_{i+1,i+2}f_{i,i+1}f_{i+1,i+2})^{(i,i+1,i+2)}\\
&\quad+(f_{i,i+1}f_{i,i+2}f_{i+1,i}+f_{i,i+1}f_{i+1,i+2}f_{i,i+1}-f_{i+1,i+2}f_{i,i+1}f_{i,i+2})^{(i,i+2,i+1)}\\
&=0.
\end{aligned}
$$

Hence, $A_iA_{i+1}A_i-A_{i+1}A_iA_{i+1}=0$ i.e. $A_iA_{i+1}A_i=A_{i+1}A_iA_{i+1}$.

□

Bibliography

[RS] D. Rosso and A. Savage, *Quantum Affine Wreath Algebras*, https://arxiv.org/abs/1902.00143, 2019.