# Core Entropy of Finite Subdivision Rules

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(ABSTRACT)

The topological entropy of the subdivision map of a finite subdivision rule restricted to the 1-skeleton of its model subdivision complex, which we call **core entropy**, is examined. We consider core entropy for finite subdivision rules realizing quadratic Misiurewicz polynomials and matings of such polynomials. It is shown that for a non-restrictive class of finite subdivision rules realizing quadratic Misiurewicz polynomials, core entropy equals Thurston's core entropy. We also show that the core entropy of formal and degenerate matings of Misiurewicz polynomials is determined by Thurston's core entropy of the mated polynomials.

# Core Entropy of Finite Subdivision Rules


(GENERAL AUDIENCE ABSTRACT)

Imagine taking a programmable calculator, inputting a number, and repeatedly pushing one of the buttons which corresponds to one of the calculator's built-in functions. For example, starting by inputting 0.5 and hitting the "$x^2$" button over and over, or starting with 1.47 and repeatedly pressing the "$\sin(x)$" button. The calculator may eventually return numbers that get closer and closer to a specific value, it may repeatedly cycle through some collection of specific numbers, it may not exhibit a clear pattern at all. It is of interest to understand, in some average sense, when, how often, and in what manner these patterns are exhibited and, in a quantitative fashion, compare how complicated the patterns are for different buttons on the calculator corresponding to different functions. For example, is the "$x^2$" button, in some average sense, more or less "complex", in terms of the patterns exhibited by the above procedure, than the "$\sin(x)$" button? Modeling or simulating physical phenomena such as particle motion or the orbits of collections of celestial bodies often entails the use of computer programs. These computer programs carry out calculations which often involves repeated application of various pre-programmed functions. Repeatedly pushing a button on a calculator can be viewed as a simplified version of what goes on with the calculations that a computer carries out in simulating physical phenomena. Understanding how to compare the patterns exhibited by simple, fundamental collections of functions makes for a good starting point for understanding the models that represent various physical phenomena. This work contributes to this endeavor by investigating a quantity which measures the complexity of some fundamental functions.

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# Chapter 1

# Introduction

In this work, we examine a quantity which we call *core entropy* of a finite subdivision rule. Let $X$ be a topological space and let $f : X \to X$ be a continuous map. Given $x \in X$, one may consider the sequence of iterates

$$x, f(x), f^2(x), f^3(x), \ldots.$$

Such a sequence is called the **orbit** of $x$ under $f$. Broadly speaking, in **discrete topological dynamics** one studies the orbits of $x \in X$ and refers to the pair $(X, f)$ as a **discrete topological dynamical system**. The number of iterations is a discrete notion of *time*. It is of general interest to compare discrete dynamical systems by the complexity of their orbits. To do so in a quantitatively precise fashion, one uses the *topological entropy* of a system. Let $k \geq 0$ be a natural number and let $x \in X$. The finite sequence $x, f(x), \ldots, f^k(x)$ is called the **partial orbit** of $x$. Intuitively speaking, the *topological entropy* of $f$ is the limiting rate at which the number of distinguishable partial orbits grows as time (the length of the partial orbits) becomes arbitrarily large. The discrete topological dynamical systems of primary interest to us are those for which $X = \widehat{\mathbb{C}}$ is the Riemann sphere and $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a rational map; these are **complex dynamical systems**. Although there exist many additional structures present in this context of use in the study of these systems, our attention is focused primarily on topological properties.

*Topological entropy* has a rich history dating back to the birth of statistical mechanics circa 1877 with Ludwig Boltzmann's formulation of *entropy*. Since then, the quantity has appeared in multiple forms in multiple contexts; a few of the most popular settings that include a form of *entropy* are the realms of statistical mechanics, thermodynamics, and information theory. The modern mathematical theory of dynamical systems (which includes much more than discrete topological dynamical systems) provides one with an abstract lens through which to view each of the settings in which some form of *entropy* appears.

For complex dynamical systems, the following result by M. Lyubich in [17] determines the topological entropy for every single non-constant rational map.

**Theorem 1.0.1.** *Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a rational function that is not constant. Let $deg(f)$ be the degree of $f$. Then the topological entropy $h(f) = \ln(deg(f))$.*

This, in particular, suggests that topological entropy is not an especially informative quantity in the context of complex dynamical systems. For example, this result tells us that topological entropy is not sensitive to any of the distinct dynamical phenomena present in iterating quadratic polynomials as each quadratic polynomial map has the same topological entropy $\ln(2)$. This, however, does not end the story for topological entropy in complex dynamical systems.

W. Thurston developed a more interesting, variable notion of entropy in the setting of important classes of complex dynamical systems. A polynomial is said to be **post-critically finite** if the union of the forward orbits of its critical points is a finite set. All post-critically finite polynomials have an invariant topological tree in $\widehat{\mathbb{C}}$ that encodes a great deal of information about the dynamics of the polynomial map under iteration. This tree is called the *Hubbard tree* (see Section 2.3). Thurston considered the topological entropy of the polynomial map's restriction to its Hubbard tree and called it *core entropy*. This idea and the tools employed in analyzing core entropy can be regarded as a natural extension of the work done in [20] to the two-dimensional setting. In addition to defining a new notion of entropy for important classes of complex dynamical systems, Thurston developed an algorithm for calculating entropy for post-critically finite quadratic polynomials. A plot of core entropy as a function of a parametrization of post-critically finite quadratic polynomials is depicted in Figure 1.1. The prospect of having an informative entropy theory in the context of complex dynamical systems along with the wild appearance of the graph compels one to understand the nature of core entropy. Some results on core entropy include the continuity of the core entropy function whose graph is depicted above (see [26]) and extensions of Thurston's algorithm to post-critically finite polynomials of degree higher than 2 (see [14]).

Finite subdivision rules are essentially combinatorial objects. They are finite combinatorial rules for recursively subdividing tiled spaces. Finite subdivision rules are formalized using a finite CW complex called the *model subdivision complex* $S_{\mathcal{R}}$, a subdivision of the model subdivision complex $\mathcal{R}(S_{\mathcal{R}})$, and a *subdivision map* $\varphi_{\mathcal{R}} : \mathcal{R}(S_{\mathcal{R}}) \to S_{\mathcal{R}}$. Finite subdivision rules can be used to study the dynamics of rational maps on the Riemann sphere. In particular, one may look for tilings of the Riemann sphere which naturally realize the rational map under consideration as the subdivision map of a finite subdivision rule.

One may regard this idea as a manifestation of the general notion of *symbolic representation*. That is, given a *partition* of $\widehat{\mathbb{C}} = X_1 \sqcup X_2 \sqcup \cdots \sqcup X_m$, one may symbolically encode an orbit $x, f(x), f^2(x), \ldots$ by a sequence $X_{k_1}, X_{k_2}, X_{k_3}, \ldots$, where $k_i \in \{1, \ldots, m\}$ for all $i$ and $k_i = j$ if and only if $f^i(x) \in X_j$. One may then represent points in $x \in \widehat{\mathbb{C}}$ by the sequences which symbolically encode their orbits. This replaces a given complex dynamical system with a system $(X, f)$ where $X$ is a space of sequences and $f$ is a shift along those sequences. There are circumstances in which such a representation can be exploited to great advantage

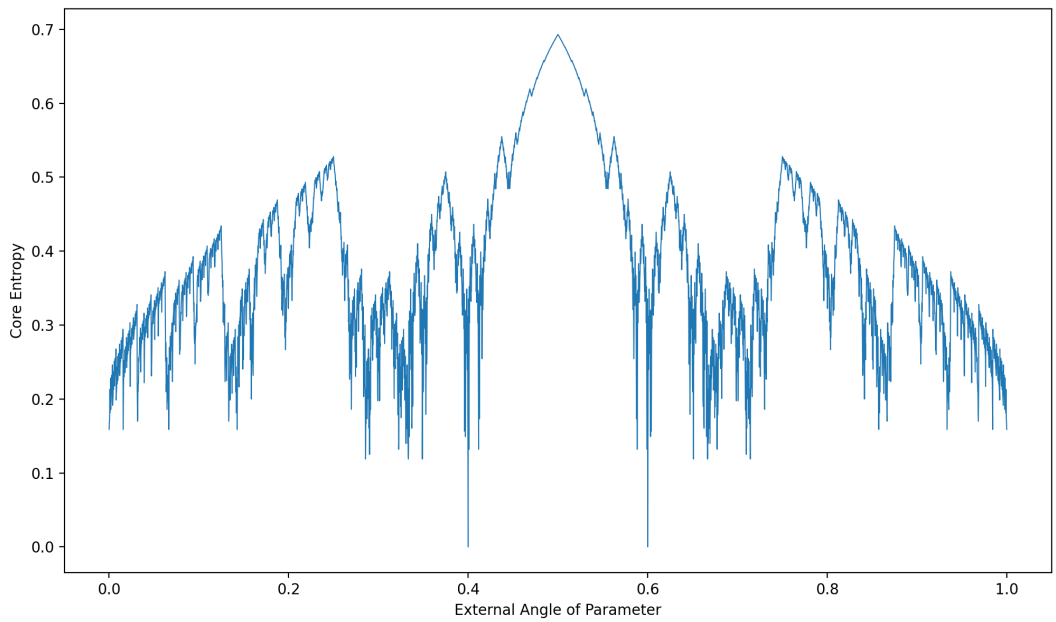


Figure 1.1: Core entropy $h = h(x)$ as a function of *external angle*. A polynomial defined by *external angle* $t$ is conjugate to the polynomial determined by angle $1 - t$ which explains the symmetery of the graph about the line $x = 1/2$.

enabling one to gain insights into systems which are difficult to understand using other classical methods. One particular advantage in such circumstances is that the topological entropy becomes algorithmically computable. Finite subdivision rules exist in abundance as the following result from [4] (Theorem 15.1) and [6] (Theorem 1) demonstrates.

**Theorem 1.0.2.** *Suppose $f$ is a post-critically finite rational map with no periodic critical points. If $n$ is a sufficiently large positive integer, then $f^{\circ n}$ is the subdivision map of a finite subdivision rule.*

This abundance, along with the the general utility of symbolic representation described above, lends credence to the idea that finite subdivision rules have a place in studying the dynamics of rational maps. We explore this idea to understand and extend the notion of *core entropy*.

We define *core entropy* to be the topological entropy of a subdivision map's restriction to the 1-skeleton of its model subdivision complex. This quantity is the namesake of a quantity defined by Thurston for post-critically finite quadratic polynomials and further developed and studied by Tiozzo, Gao, and others (see [26], [14]). We now outline this work.

## 1.1 Outline

Our main result is Theorem 3.1.1. This theorem states that, in the original setting of its namesake, core entropy coincides with the original core entropy or is 0. In this way, *core entropy* generalizes its namesake. We then examine *core entropy* in settings outside its origins by considering matings of post-critically finite quadratic polynomials realized by finite subdivision rules. We begin with a preliminary chapter which discusses:

- **Finite subdivision rules**: in which we define finite subdivision rules and consider how they appear in the study of complex dynamical systems;
- **Symbolic Dynamics and Topological Entropy**: in which we discuss the notion of a fruitful symbolic representation of a dynamical system. We show how finite subdivision rules provide the structure for a useful symbolic representation of the systems that define *core entropy*;
- **Complex Quadratic Dynamics**: in which we discuss basic content from the general theory of complex dynamics in the particular context of the dynamics of *quadratic Misiurewicz polynomials*;
- **Matings of Quadratic Misiurewicz Polynomials**: in which we define three main forms of a *mating*, define constructions found in [27] realizing matings by finite subdivision rules, and provide an example illustrating the constructions.

# Chapter 2

# Preliminaries

In this chapter, we discuss preliminary material. The four areas of well-established theory discussed in this chapter are finite subdivision rules, symbolic dynamics, complex quadratic dynamics, and matings of quadratic Misiurewicz polynomials. We rely heavily on employing archetypal examples in lieu of detailed developments of the machinery discussed in this chapter.

## 2.1 Finite Subdivision Rules

In this section, we define, describe, and provide examples of finite subdivision rules. We focus on examples of finite subdivision rules which are archetypal of the ones appearing in this work.

### 2.1.1 Definition and Notes

Cannon, Floyd, and Parry, in [7], introduce and study finite subdivision rules. These rules precisely designate a structure which enables one to recursively subdivide tilings. These rules were used to model recursive structures naturally encountered by the authors of [7] in their study of negatively curved groups. Their work in [7] was part of their endeavour to resolve Cannon's conjecture, a generalization of W. P. Thurston's, well-known and proven to be true, Geometrization conjecture.

One of the components of a finite subdivision rule structure, as we'll see below, is a space called the model subdivision complex. Before giving the definition, which admits a broad set of possibilities for these complexes, we inform the reader of the sorts of complexes to come in this work so as to narrow one's attention; we also specify the terms *cellular* and *subdivision*. There exists a general distinction between CW complexes and cell decompositions, though, in this work, the general distinction does not manifest itself, and so we will have no occasion to be wary of the distinction, thus we will use the terms interchangeably. The underlying space for all CW complexes encountered in this work is a topological 2-sphere. We refer the

reader to [15] for the general distinction between cell decompositions and CW complexes being referred to here and to [4] (Chapter 5) for properties of cell decompositions of spheres. All of the CW complexes appearing in this work are finite and 2-dimensional. That is, the underlying spheres are partitioned into finitely-many open cells of dimensions 0, 1, and 2.

Let $D$ and $\widetilde{D}$ be two cell decompositions of a space $X$, let $f : X \to X$ be a continuous map, and let $D^k$ and $\widetilde{D}^k$ be the $k$-skeletons of $D$ and $\widetilde{D}$, where $k \in \{0, 1, 2\}$. We say that $f$ is **cellular** (with respect to $D$ and $D'$) if $f(D^k) \subseteq \widetilde{D}^k$ for each $k \in \{0, 1, 2\}$.

A CW complex $Y$ is said to be a **subdivision** of a CW complex $X$ if $X = Y$ and each open cell of $Y$ is contained in an open cell of $X$. That is, $X$ and $Y$ have the same underlying space yet are distinguished by their individual cell decompositions. We will often abuse notation and use a single symbol to refer both to the collection of open cells into which the underlying space is partitioned and to the underlying space itself. In [7], the authors make the following definition.

**Definition.** A **finite subdivision rule** $\mathcal{R}$ consists of the following:

1. A finite 2-dimensional CW complex $S_{\mathcal{R}}$, called the **model subdivision complex**, with a fixed cell structure such that $S_{\mathcal{R}}$ is the union of its closed 2-cells. We assume that for each closed 2-cell $\tilde{s}$ of $S_{\mathcal{R}}$ there is a CW structure $s$ on a closed 2-disk such that the vertices and edges of $s$ are contained in $\partial s$ and the characteristic map $\psi_s : s \to S_{\mathcal{R}}$ which maps onto $\tilde{s}$ restricts to a homeomorphism on each open cell.

2. A finite 2-dimensional CW complex $\mathcal{R}(S_{\mathcal{R}})$ which is a subdivision of $S_{\mathcal{R}}$.

3. A continuous cellular map $\varphi_{\mathcal{R}} : \mathcal{R}(S_{\mathcal{R}}) \to S_{\mathcal{R}}$, called the **subdivision map**, whose restriction to every open cell is a homeomorphism onto an open cell.

In [7], the authors provide an intuitive vision for what the above formally defines: one has a finite collection of colors with which to color closed tiles and a finite collection of colors with which to color their edges. Additionally, one has tile prototypes $p_c$ and edge prototypes $e_k$ of each color. Lastly, one has subdivisions $X_c$ of each tile $p_c$ into colored tiles so that if $t$ is a tile of $X_c$ with color $c(t)$, then there is a cellular homeomorphism of $t$ to $p_{c(t)}$ and subdivisions of each edge $Y_k$ of each edge $e_k$ into colored subedges. The separate colorings for tiles and edges are to ensure compatibility along the shared boundaries of distinct tiles, thus, the subdivisions of the colored edges are those induced from the subdivisions of the colored tiles.

With this, given a tiled space, one can subdivide each tile according to the subdivisions of each associated $p_c$ and $e_k$. This, in turn, yields another tiling of the given space where each tile is again colored by colors among the same finite collection one started with. This describes the recursive subdivision process obtained from a finite subdivision rule. Through the

examples of this section, we hope to establish for the reader a sense for the main components and descriptive elements we will employ in discussing and analyzing the finite subdivision rules of interest in this work.

**Observations.** Let $\mathcal{R}$ be a finite subdivision rule with model subdivision complex $S_{\mathcal{R}}$, subdivided complex $R(S_{\mathcal{R}})$, and subdivision map $\varphi_{\mathcal{R}} : \mathcal{R}(S_{\mathcal{R}}) \to S_{\mathcal{R}}$.

1. Let $\sigma$ be an open cell of $S_{\mathcal{R}}$. Every open cell $\tau$ of $\mathcal{R}(S_{\mathcal{R}})$ that intersects $\sigma$ must be contained in $\sigma$. Indeed, suppose $\sigma \cap \tau \neq \varnothing$. There must exist an open cell $\sigma'$ of $S_{\mathcal{R}}$ which contains $\tau$ because $R(S_{\mathcal{R}})$ is a subdivision. Since the open cells of $S_{\mathcal{R}}$ partition the underlying space, this implies $\sigma = \sigma'$ and hence $\tau \subseteq \sigma$.

2. By observation 1, it follows that every open cell $\sigma$ of $S_{\mathcal{R}}$ is the union of all open cells $\tau$ of $\mathcal{R}(S_{\mathcal{R}})$ such that $\tau \subseteq \sigma$. Indeed, let $A$ be the set of open cells of $\mathcal{R}(S_{\mathcal{R}})$ contained in $\sigma$. The containment $\bigcup_{\tau \in A} \tau \subseteq \sigma$ is clear; for every $x \in \sigma$, since the open cells of $\mathcal{R}(S_{\mathcal{R}})$ partition the underlying space, there must be some cell $c$ of $\mathcal{R}(S_{\mathcal{R}})$ containing $x$ and hence there must exist an open cell $\sigma'$ of $S_{\mathcal{R}}$ which contains $c$ and thus non-trivially intersects $\sigma$. Therefore, $\sigma' = \sigma$, $c \in A$, $x \in \bigcup_{\tau \in A} \tau$, and $\bigcup_{\tau \in A} \tau = \sigma$.

3. The 1-skeleton of $S_{\mathcal{R}}$, which we denote by $S^1_{\mathcal{R}}$, is forward invariant under the subdivision map $\varphi_{\mathcal{R}}$. It follows from observation 2 that each cell $\sigma$ of $S_{\mathcal{R}}$ is the union of the cells of $\mathcal{R}(S_{\mathcal{R}})$ contained in $\sigma$, closed or open. Thus, each cell $\sigma$ of $S_{\mathcal{R}}$ inherits a CW complex structure from $\mathcal{R}(S_{\mathcal{R}})$, in particular, each 1-cell of $S_{\mathcal{R}}$ is comprised of a union of 1-cells of $\mathcal{R}(S_{\mathcal{R}})$. Since $\varphi_{\mathcal{R}}$ is cellular and restricts to a homeomorphism on open cells, it maps each open 1-cell of $\mathcal{R}(S_{\mathcal{R}})$ homeomorphically into $S^1_{\mathcal{R}}$. Therefore, the image of every 1-cell of $S_{\mathcal{R}}$ is contained in $S^1_{\mathcal{R}}$, hence $\varphi_{\mathcal{R}}(S^1_{\mathcal{R}}) \subseteq S^1_{\mathcal{R}}$.

4. If $\varphi_{\mathcal{R}}$ is surjective, every open cell of $S_R$ is the homeomorphic image of a cell of $\mathcal{R}(S_{\mathcal{R}})$.

We will invoke these observations below. Lastly, before moving on to examples, we define some useful terms. Any domain of a characteristic map for a 2-cell, as described in part 1, is called a **tile type**. One often describes or depicts a rule by illustrating the subdivisions of its tile types. From such depictions, one usually observes the induced subdivisions along edges and, similarly, refers to the domains of characteristic maps from closed 1-disks onto closed 1-cells of $S_{\mathcal{R}}$ as **edge types**. We will often use **vertices**, **edges**, and **tiles** interchangeably with 0-, 1-, and 2-cells, open or closed, respectively.

### 2.1.2 Examples

In the first example, we will use the Riemann sphere $\hat{\mathbb{C}}$ and a polynomial map to define our finite subdivision rule. We will only be appealing to the topological structure of the Riemann sphere as a 2-sphere and will not have occasions to exploit its additional structure as a complex manifold. We use the Riemann sphere mainly because the subdivision rules that realize a polynomial map are of particular interest in this work and because it is convenient to refer to the elements of a subdivision complex by their specific descriptions as subsets of $\hat{\mathbb{C}}$. The second example will be less specific in designating the elements of the finite subdivision rule. In Section 1.3 we will see that this second example of a finite subdivision rule realizes the quadratic polynomial given by $f(z) = z^2 + i$.

**Example 2.1.1.** We will depict both subdivisions of the tile type, the subdivision complex, and successive subdivisions of the subdivision complex for this example.

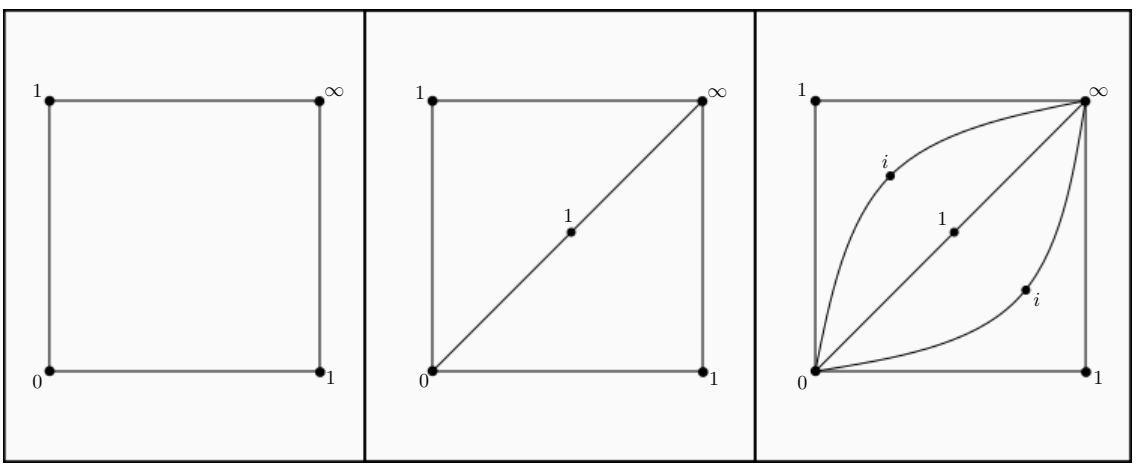


Figure 2.1: The subdivision of a tile type for a finite subdivision rule whose subdivision map is $f(z) = z^2$. The left-most panel shows the tile type, the center panel shows the first subdivision, and the right-most shows the second subdivision.

The 1-skeleton of the subdivision complex in this example, as depicted in Figure 2.2, is comprised of the vertices given by the points $\{0, 1, \infty\}$, the open edges given by the real intervals $(0, 1)$ and $(1, \infty)$. The complement $\hat{\mathbb{C}} \setminus S^1_{\mathcal{R}}$ is the sole tile of the subdivision complex.

Taking preimages of $S^1_{\mathcal{R}}$ under the map $f(z) = z^2$ yields a subdivided complex $\mathcal{R}(S_{\mathcal{R}})$ whose vertices are given by the points $\{0, \pm 1, \infty\}$ and whose open edges are given by the real intervals $(0, 1), (-1, 0), (1, \infty), (\infty, -1)$. The map $f(z) = z^2$, by construction, is cellular with respect to $\mathcal{R}(S_{\mathcal{R}})$ and $S_{\mathcal{R}}$. This example illustrates the archetypal construction process for the finite subdivision rules encountered in this work. That is, one starts their construction process with a topological graph that is a viable 1-skeleton for a finite subdivision rule to realize a map of interest and lets the natural decomposition of the complement of this viable

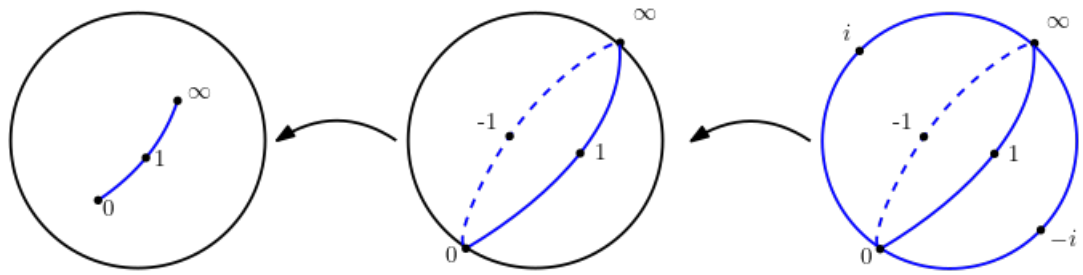


Figure 2.2: The subdivision complex for this rule along with two of its subdivisions in succession from left to right.

1-skeleton define the 2-cells of the subdivision complex. Using the preimage of this starting structure under the map of interest to determine the subdivided complex, one thereby makes the map of interest cellular with respect to these structures by construction and has all the necessary components of a finite subdivision rule. In this way, a finite subdivision rule $\mathcal{R}$ **realizes** a map $f : X \to X$ if there exist cell decompositions of $X$ that are identifiable with $S_{\mathcal{R}}$ and $\mathcal{R}(S_{\mathcal{R}})$ such that $f$ has the properties of a subdivision map with respect to these decompositions where by *identifiable*, we mean, there exist embeddings of the 1-skeleta of $S_{\mathcal{R}}$ and $\mathcal{R}(S_{\mathcal{R}})$ in $X$.

This work aims to understand core entropy for finite subdivision rules. We focus on maps given by quadratic polynomials whose critical orbits are strictly preperiodic and matings of such polynomials. In particular, this work will define core entropy as the topological entropy of the subdivision map's restriction to its 1-skeleton. The calculation of the entropy of such maps is accessible through the edge recursion matrices of these rules which we define below. In the former setting, the quantity of *core entropy* already exists and has been defined and studied in [26]. The namesake is naturally justified in Chapter 2 by showing that, in the original setting, core entropy of the finite subdivision rules realizing the polynomial maps of the original setting coincides with the original core entropy defined by Thurston or is 0 otherwise. This makes core entropy of finite subdivision rules a generalization of the original core entropy. We then apply this definition to a setting outside the setting to which Thurston's core entropy applied, namely, for matings of strictly preperiodic quadratic polynomials. We now depict the realization by a finite subdivision rule of a quadratic polynomial map whose critical point is strictly preperiodic.

**Example 2.1.2.** Similar to the example above, we depict the model subdivision complex along with the subdivided complex of the finite subdivision rule. We depict an abstract projected view of the Riemann sphere to make visualization clearer than if we were to depict the CW structure as we did in the previous example. The vertices of $S_{\mathcal{R}}$ are labeled $b, c_1, c_2, c_3$, and $\infty$. Together with the edges connecting the vertices depicted in Figure 2.3, we have a 1-skeleton for $S_{\mathcal{R}}$. This designates a decomposition of a 2-sphere into topological quadrilaterals. Unlike the first example, we have not designated a specific identification

between the vertices and edges of the subdivision complex and points and subsets of the Riemann sphere. In Section 1.3, we provide such an identification. This is, nonetheless, an example of a finite subdivision rule. The process of identifying this structure for a particular polynomial map is what constitutes *realization* by a finite subdivision rule.

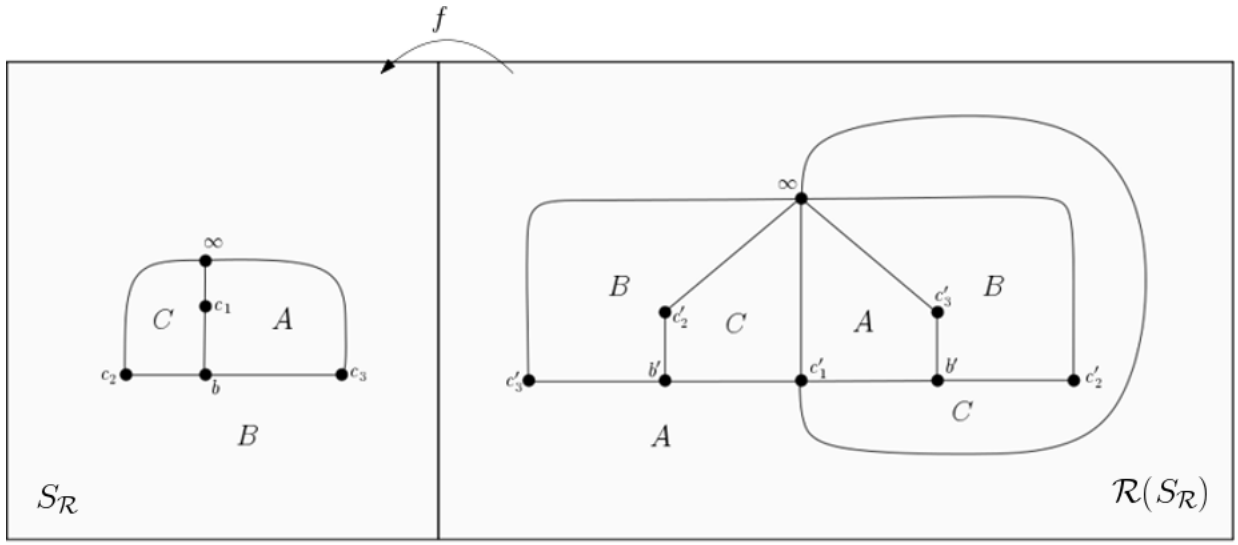


Figure 2.3: Projected views of the model subdivision complex $S_{\mathcal{R}}$ (left panel) and the subdivided complex $\mathcal{R}(S_{\mathcal{R}})$ (right panel). The subdivision map is $f(z) = z^2 + i$.

The vertices of $\mathcal{R}(S_{\mathcal{R}})$ have been labeled according to where they map under the subdivision map. That is, $x'$ maps to $x$ under $\varphi_{\mathcal{R}}$ for each vertex $x$ of $S_{\mathcal{R}}$. We let $\varphi_{\mathcal{R}}$ map each open edge $(x', y')$ homeomorphically onto its image edge $(x, y)$. Similarly, we let $\varphi_{\mathcal{R}}$ map each open 2-cell defined by a quadruple of vertices $x', y', z', w'$ homeomorphically onto the corresponding open two cell with vertices $x, y, z, w$. With this, by construction, we've formed another archetypal example of the finite subdivision rules in focus for this work. We will make explicit identifications between the CW structure depicted in Figure 2.3 with their embedded counterparts after introducing some constructions from complex dynamics.

We depict the subdivisions of the tile types in Figure 2.4.

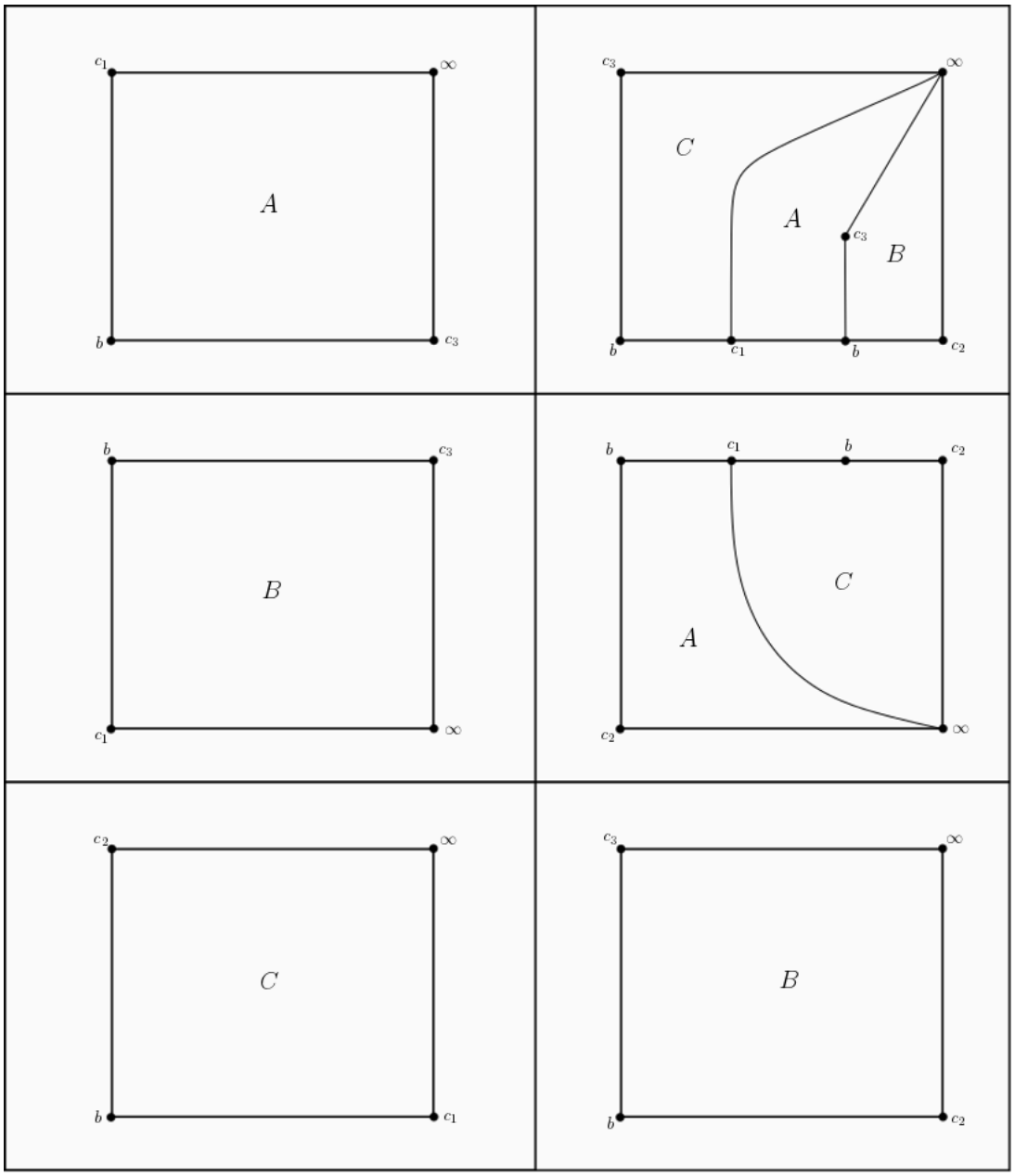


Figure 2.4: The subdivision of each quadrilateral tile type. Here the ambient space of the tile is just ambient space. Each tile type is depicted on the left panel while the subdivded tile type is depicted on the right panel.

### 2.1.3 The Edge Recursion Matrix

In this subsection, we define a matrix called the *edge recursion matrix* of a finite subdivision rule. We will see in the next section how this matrix often enables one to calculate the core entropy of a finite subdivision rule. We start with an example.

**Example 2.1.3.** Let $\mathcal{R}$ be the finite subdivision rule defined in Example 2.1.2 with subdivision complex $S_{\mathcal{R}}$, subdivided complex $\mathcal{R}(S_{\mathcal{R}})$, 1-skeleton of subdivided complex $\mathcal{R}(S_{\mathcal{R}})^1$, and subdivision map $\varphi_{\mathcal{R}}$. There are six edges in $S_{\mathcal{R}}$: three edges with $\infty$ as a vertex and three edges that connect the points $c_1, c_2, c_3$ to the vertex labeled $b$. We label these edges $e_1, \ldots, e_6$. Each edge $e$ of $S_{\mathcal{R}}$ inherits a CW structure from $\mathcal{R}(S_{\mathcal{R}})$ which we call its **subdivision**, denoted $\mathcal{R}(e)$. In particular, for each edge $e$ of $S_{\mathcal{R}}$ the cells of $\mathcal{R}(S_{\mathcal{R}})^1 \cap e$, equivalently, the cells of $\mathcal{R}(S_{\mathcal{R}})$ contained in $e$, decompose $e$ into a collection of 1-cells of $\mathcal{R}(S_{\mathcal{R}})$. We depict the subdivisions of each of the six edges of $S_{\mathcal{R}}$ below. The *edge recursion matrix* for this example is the $9 \times 9$ matrix $A_{\mathcal{R}} = (a_{ij})$ given by

$$a_{ij} = \text{the number of open edges in } \mathcal{R}(S_{\mathcal{R}})^1 \cap e_j \text{ which map to } e_i \text{ under } \varphi_{\mathcal{R}}$$

for all $i, j \in \{1, \ldots, 6\}$. The edge recursion matrix for this example is given below the figure for the edge subdivisions.

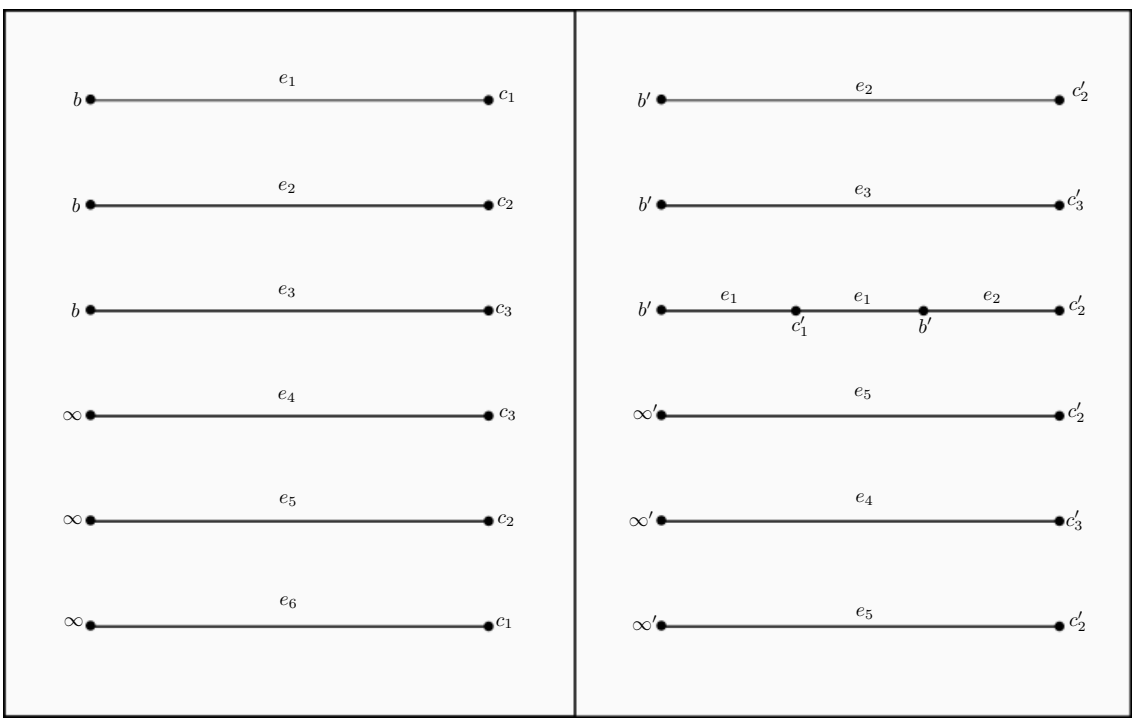


Figure 2.5: The edge subdivisions for rule $\mathcal{R}$ of Example 1.1.2. The left panel depicts the six edge types and the right panel depicts their subdivisions as seen in the subdivided complex $\mathcal{R}(S_{\mathcal{R}})$.

$$A_{\mathcal{R}} = \begin{bmatrix} 0 & 0 & 2 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

This matrix is defined up to the choice of an enumeration of the edges of $S_{\mathcal{R}}$. Distinct enumerations of the edges of $S_{\mathcal{R}}$ yield matrices that relate to each other through products of permutation matrices as one may obtain one matrix from another through a permutation of its row and column indices. There is no preferred choice of labeling as the quantity of interest with this matrix is its Perron eigenvalue which doesn't depend on the particular permutation of row and column indices. We now give the general definition of an edge recursion matrix of a finite subdivision rule.

**Definition (Edge recursion matrix).** Given a finite subdivision rule $\mathcal{R}$ with subdivision map $\varphi_{\mathcal{R}}$, let $e_1, e_2, \ldots, e_m$ be an enumeration of the open edges of $S_{\mathcal{R}}$. The **edge recursion matrix** is the $m \times m$ matrix $A_{\mathcal{R}} = (a_{ij})$ with entry $a_{ij}$ equal to the number of open edges in $\mathcal{R}(S_{\mathcal{R}}) \cap e_j$ which map to $e_i$ under $\varphi_{\mathcal{R}}$.

## 2.2 Symbolic Dynamics, Topological Entropy, and Core Entropy of FSRs

In this section, we discuss standard notions from symbolic dynamics that will be used later and topological entropy in the context of the symbolic systems of interest in this work. We do not reproduce standard theory in its full generality but instead introduce the terms and concepts in particular forms best suited for the context of this work. We define full shift spaces on $m$ symbols and the shift map, discuss the idea of symbolic representation of dynamical systems, define core entropy of finite subdivision rules, and discuss the connection between the edge recursion matrix and the core entropy of a finite subdivision rule.

### 2.2.1 Markov Representation

Throughout this work, the symbol $\mathbb{N}$ will denote the set of positive integers and $\mathbb{N}_0$ the set of non-negative integers. The sole ingredient in constructing a shift space is a finite set of symbols $\mathcal{A}$ called the **alphabet**. All of the alphabets we work with will be ordered and so we content ourselves with alphabets of the form $\mathcal{A} = [m] = \{1, 2, \ldots, m\}$ where $m \in \mathbb{N}$. A **full one-sided $m$-shift space** $\Sigma_m$ is the collection of all sequences of elements of $[m]$ indexed by $\mathbb{N}_0$, that is

$$\Sigma_m = \{(s_i)_{i\in\mathbb{N}_0} : s_i \in [m] \text{ for all } i \in \mathbb{N}_0\}.$$

The term "one-sided" is there because, in general, symbolic representations are often being made for systems defined by iteration of an invertible map. The systems of this work are all non-invertible so we focus on one-sided shifts. The term "space" in "full one-sided $m$-shift space" is suggestive of additional structure. There exist multiple metrics on $\Sigma_m$ endowing such spaces with equivalent topologies making them homeomorphic to a Cantor set. The **shift map** $\sigma : \Sigma_m \to \Sigma_m$ is defined by $t = \sigma(s)$ where $t = (t_i)_{i\in\mathbb{N}_0}$ is the sequence given by $t_i = s_{i+1}$ for each $i \in \mathbb{N}_0$ and each $s = (s_i)_{i\in\mathbb{N}_0} \in \Sigma_m$. That is, the shift map simply shifts any given sequence one symbol to the left. One of the original appearances of these objects in dynamics came from the idea of encoding orbits by way of a finite partition of the phase space. For the moment, let us take a *partition* of a space to be a finite collection of subsets of the space whose union is the space.

That is, let $(X, f)$ be a discrete dynamical system, so $X$ is a topological space and $f$ is a continuous self-map of $X$. Let $K = \{X_1, X_2, \ldots, X_m\}$ be a partition of $X$. The orbit of a sequence $s = (s_i)_{\mathbb{N}_0} \in \Sigma_m$ under the shift map $\sigma$ represents the orbit of some point $x = \pi(s)$ in $X$ in accordance with the partition $K$ if $f^n(x) \in X_{s_n}$ for all $n \in \mathbb{N}_0$. This is what's meant by the dynamics of $f$ being symbolically encoded by the dynamics of $\sigma$ and is depicted by the diagram below.

$$\begin{tikzcd} \Sigma_m \arrow[r, "\sigma"] \arrow[d, "\pi"'] & \Sigma_m \arrow[d, "\pi"] \\ X \arrow[r, "f"] & X \end{tikzcd}$$

The nature of this representation depends strongly, as one might expect, on the properties of the partition. Some natural considerations in this regard include whether the partition is such that the map $\pi$...

1. ... is well-defined,
2. ... makes the diagram commute,
3. ... preserves structure, e.g. topological, measure, etc. between $\Sigma_m$ and $X$,
4. ... preserves orbits,
5. ... is surjective, injective, both,
6. ... has a universal bound on the cardinality of preimages.

One rarely has occasion to employ the full one-sided $m$-shift in symbolically representing a dynamical system. Although we use a full one-sided $m$ shift in the example given below, this situation is uncommon. One is more often working with a closed shift-invariant subset $\Sigma \subset \Sigma_m$ and the restriction of $\sigma$ to such subsets. These are generally called **subshifts**. In this work, the particular kinds of subshifts we will work with are called **topological Markov shifts**, or **shifts of finite type**, or **edge shifts**. The latter name comes from how one obtains such spaces as the set of infinite walks along the edges of a finite directed graph. We define edge shifts below and introduce some terminology to be used later.

All graphs discussed in this section are finite and directed. Let $G$ be such a graph with vertex set $V$ and edge set $E = \{e_1, \ldots, e_m\}$. For each edge $e_j \in E$, let $i(e_j) \in V$ denote the initial vertex of $e$ and let $t(e_j) \in V$ denote its terminal vertex. A **path** $\pi = e_{s_0} \cdots e_{s_{k-1}}$ is a finite sequence of edges of $G$ such that $t(e_{s_j}) = i(e_{s_{j+1}})$ for all $0 \leq j \leq k-2$. The **length** of a path $\pi = e_{s_0} \cdots e_{s_{k-1}}$ is $|\pi| = k$. We say that a path $\pi = e_{s_0} \cdots e_{s_{k-1}}$ **starts at** vertex $i(\pi) = i(e_{s_0})$ and **ends** at vertex $t(\pi) = t(e_{s_{k-1}})$. A path whose starting and ending vertices coincide is a **cycle**. A cycle $\pi = e_{s_0} \cdots e_{s_{k-1}}$ is **simple** if $i(e_{s_0}), i(e_{s_1}), \ldots, i(e_{s_{k-1}})$ are all distinct. We call a graph **irreducible** if for every ordered pair of vertices $i$ and $j$, there exists a path in $G$ starting at $i$ and ending at $j$. A non-negative matrix is **irreducible** if for each ordered pair of indices $i, j$ there exists an $n \in \mathbb{N}$ such that $(A^n)_{ij} > 0$.

**Definition.** Let $A = (a_{ij})$ be an $m \times m$ matrix with non-negative integer entries. The **graph of** $A$ is the finite directed graph $G_A$ with vertex set $V(G_A) = \{1, \ldots, m\}$ and with $a_{ij}$ distinct

edges with initial vertex $i$ and terminal vertex $j$.

One can start with a finite directed graph $G$, label its verticies $1, \ldots, m$, and define a square matrix $A$ called its **adjacency matrix** with non-negative integer entries whose entry $a_{ij}$ is the number of edges starting at the vertex labeled $i$ and ending at the vertex labeled $j$ for all pairs $i, j \in \{1, \ldots, m\}$. These two constructions are inverses of one another, in a sense, and vary only when it comes to the choice of labeling of vertices. It is straight-forward to show that, for any graph $G$, the number of paths of length $n$ starting from vertex $i$ and ending at vertex $j$ is the $ij$-th entry of $A^n$ for each $n \in \mathbb{N}$. It is similarly straight-forward to see that a graph is irreducible if and only if its adjacency matrix is irreducible. We now define a shift space associated to any matrix or its associated graph.

**Definition. (Edge shifts)** Let $G$ be a graph with edge set $E = \{e_1, \ldots, e_m\}$ and adjacency matrix $A$. The **edge shift** $X_G$ or $X_A$ is the shift space over the alphabet $\mathcal{A} = E$ given by

$$X_G = X_A = \{(e_{s_j})_{j \in \mathbb{N}_0} : t(e_{s_j}) = i(e_{s_{j+1}}) \text{ for all } j \in \mathbb{N}_0\}.$$

Provided an enumeration of the edges of $G$ by the elements of $[m]$, it is clear how edge shifts correspond to subshifts of full one-sided $m$-shifts. We now elucidate the notions above with a classic example from symbolic dynamics.

**Example 2.2.1. Symbolic Dynamics of Doubling.**
There are many ways to manifest the dynamical system we represent symbolically in this example. We wish only to elucidate the notion of a reasonably *ideal*, or *satisfactory* symbolic representation obtained from a partition of the phase space, so we list here a few of the most popular ways of manifesting this system by providing the phase space $X$ along with the map $d : X \to X$ on the phase space whose iteration defines the dynamics.

| | $X$ | $d$ | Description |
|---|---|---|---|
| 1 | $S^1 = \{z \in \mathbb{C} : \lvert z\rvert = 1\}$ | $d(z) = z^2$ | Angle doubling on circle, units: radians |
| 2 | $\mathbb{R}$ | $d(x) = 2x$ | Lift of (1) to $\mathbb{R}$ |
| 3 | $\mathbb{R}/\mathbb{Z}$ | $d([x]) = [2x]$ | (1) with units of revolutions |
| 4 | $[0, 1)$ | $d(x) = 2x \mod 1$ | Choose fundamental domain for (3) |

We work with manifestation 4. Let $D = \{D_0, D_1\}$ be the pair of subsets of $X = [0, 1)$ where $D_0 = (0, 1/2)$ and $D_1 = (1/2, 1)$. We refer to $D$ as a **partition** of $X$ which we now take to mean a collection of pairwise disjoint open subsets, the union of whose closures equals $X$. This pair of subsets yields a natural symbolic representation of points $x \in X$ through their orbits under doubling as follows. Let $\Sigma_2$ be the full one-sided 2-shift space and let $\sigma : \Sigma_2 \to \Sigma_2$ be the shift map. Let $s = (s_j)_{j \in \mathbb{N}_0} \in \Sigma_2$. One associates to $s$ the point $\pi(s)$ in the set

$$\pi(s) \in \bigcap_{n=0}^{\infty} \overline{D_{s_0} \cap d^{-1}(D_{s_1}) \cap \cdots \cap d^{-n}(D_{s_n})}. \tag{2.1}$$

The idea is, if $x \in X$ and the orbit of $x$ is symbolically represented by the sequence $s$, then $x \in \overline{d^{-n}(D_{s_n})}$ for all $n \in \mathbb{N}_0$. We do not define $\pi(s)$ to be the point in the simpler-looking $\cap_{n=0}^{\infty}\overline{d^{-n}(D_{s_n})}$ because this set, with the exception of two sequences ($s = \overline{0}$ and $s = \overline{1}$), is of the form $\{x, 0\}$ for some $x \in (0, 1)$, and the amibiguity makes $\pi$ not well-defined. This is because the fixed point 0 is in the closure of all preimages of each partition element. This symbolic representation has its origins in the binary expansion of real numbers in $[0, 1)$. This representation is depicted in the diagram below.

$$\begin{tikzcd}
\Sigma_2 \arrow[r, "\sigma"] \arrow[d, "\pi"'] & \Sigma_2 \arrow[d, "\pi"] \\
{[0,1)} \arrow[r, "d"] & {[0,1)}
\end{tikzcd}$$

The map $\pi$ turns out to satisfy the following properties:

1. continuity (in the standard topologies on $\Sigma_2$ and $X$),
2. surjectivity,
3. preimages of points are uniformly bounded,
4. most points in $X$ have unique preimages,
5. $d \circ \pi = \pi \circ \sigma$

The map $\pi$ is not a homeomorphism and so does not provide a topological conjugacy between the two systems but can still be used to represent the system $(X, d)$ to a satisfactory extent, satisfaction being defined by the properties enumerated above. This brings about the following general question: What properties must a partition necessarily possess in order to represent a discrete dynamical system $(X, f)$ to a satisfactory extent? This question is addressed in [2]. Adler determines a property, known as the **Markov** property; partitions which possess this property are called **Markov partitions**. To any Markov partition $D = \{D_1, D_2, \ldots, D_m\}$ of the space $X$, one associates a finite directed graph $G$ called a **Markov graph** whose vertex set is $V = [m]$ and where there is an edge from vertex $i$ to vertex $j$ if and only if $D_j \cap f^{-1}(D_i) \neq \varnothing$.

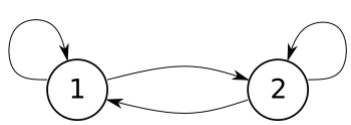


Figure 2.6: The Markov graph for the partition $D$

This is, indeed, where the name *topological Markov shift* came from and why one defines such shift spaces as we do above. We depict the Markov graph for the example of this section in Figure 1.5. The edge shift $X_G$, its shift map $\sigma$, and the map $\pi$ given by equation 1.1 comprise the representation

$$\begin{tikzcd} \Sigma_G \arrow[r, "\sigma"] \arrow[d, "\pi"'] & \Sigma_G \arrow[d, "\pi"] \\ X \arrow[r, "f"] & X \end{tikzcd}$$

which satisfies all the properties that the example above satisfies. We call $\pi$ a **Markov representation** of $(X, f)$. The idea behind the Markov property is remarkably simple to state. It goes like this: Let $b_n = D_{s_1} D_{s_2} \cdots D_{s_n}$ be a sequence of elements of your partition, call this an **$n$-block** over your partition. We think of this as a candidate sequence of partition elements potentially visited in order of iteration by some point. If, for every $n$-block that can conceivably occur as part of the orbit of some point, there exists a point in $X$ whose orbit realizes that sequence, then your partition is Markov and one has a representation with the above properties. Adler also provides a way to check that a partition satisfies this property that is easier than having to deal directly with the definition of the Markov property. We now summarize the observations motivating this work.

Let $\mathcal{R}$ be a finite subdivision rule with subdivision map $\varphi_{\mathcal{R}}$, let the 1-skeleton of the model subdivision complex $S_{\mathcal{R}}$ be $S^1_{\mathcal{R}}$, and let $\varphi = \varphi_{\mathcal{R}}|_{S_{\mathcal{R}^1}}$.

- $S^1_{\mathcal{R}}$ is forward-invariant and so $(S^1_{\mathcal{R}}, \varphi)$ defines a dynamical system by iteration of $\varphi$.
- The open 1-cells $\mathcal{E} = \{e_1, e_2, \ldots, e_m\}$ of $S_{\mathcal{R}}$ form a Markov partition of $S^1_{\mathcal{R}}$.
- Let $G$ be the finite directed graph associated to the Markov partition $\mathcal{E}$. The edge recursion matrix of $\mathcal{R}$ is the adjacency matrix for $G$ (up to a choice of edge labels).
- The topological entropy of $\varphi$ is then computable by determining the Perron eigenvalue of the edge recursion matrix.
- Most finite subdivision rules that have been studied have a property known as *mesh approaching 0* that captures a notion of expansion for the subdivision map. A finite subdivision rule is said to have **mesh approaching 0** if, given an open cover of the model subdivision complex $S_{\mathcal{R}}$, there exists an $n \in \mathbb{N}$ such that each tile of the $n^{\text{th}}$ subdivided complex $\mathcal{R}^n(S_{\mathcal{R}})$ is contained in one of the elements of the open cover. It can be shown that for any finite subdivision rule with mesh approaching 0, the dynamical system $(S^1_{\mathcal{R}}, \varphi)$ has a useful symbolic representation for the open edges of $\mathcal{R}(S_{\mathcal{R}})$ yield a *Markov partition* for $(S^1_{\mathcal{R}}, \varphi)$ in the sense given in Section 6.5 of [16]. Such symbolic representations, in particular, underlie the equivalence between the topological entropy and Perron eigenvalue of the edge recursion matrix. We discuss related notions below.

The main class of finite subdivision rules we consider, however, do not have mesh approaching 0. This is because the Fatou set is non-empty and since every post-critical point must be included as a vertex, we must include $\infty$ in $\widehat{\mathbb{C}}$ as a vertex. This makes it necessary for us to include edges in the Fatou set on which there will not be this expansive property. The topological entropy is still computable using the edge recursion matrix for such systems and we describe the relation before the proof of the main result of this work.

In the next subsection, we state a classical connection between Perron eigenvalues of square matrices $A$ with non-negative integer entries and the topological entropy of the edge shift systems $(X_A, \sigma)$ they define.

### 2.2.2 Topological Entropy and Markov Representations

We will start this section by defining topological entropy as given by [1]. We then state how this quantity is determined in the particular contexts we encounter in this work.

Let $X$ be a compact metric space, let $\varphi : X \to X$ be a continuous map, and let $\mathcal{U}$ be an open cover of $X$. For two covers, $\mathcal{U}$ and $\mathcal{B}$, let $\mathcal{U} \vee \mathcal{B} = \{A \cap B : A \in \mathcal{U}, B \in \mathcal{B}\}$ be the **join** of $\mathcal{U}$ and $\mathcal{B}$ and let $\varphi^{-1}(\mathcal{U}) = \{\varphi^{-1}(A) : A \in \mathcal{U}\}$ be the **preimage** or **pullback** of $\mathcal{U}$; both are open covers of $X$. The main components prefacing the definition of the topological entropy of $\varphi$ are

1. $N(\mathcal{U})$, defined to be the number of sets in a subcover of minimal cardinality. Compactness and well-ordering imply the existence of such a quantity for any given $\mathcal{U}$, and

2. $\mathcal{H}(\mathcal{U}) = \log(N(\mathcal{U}))$, called the **entropy** of the cover $\mathcal{U}$.

One can establish the existence of limits of the form:

$$h(\varphi,\mathcal{U}) = \lim_{n\to\infty} \frac{\mathcal{H}\Big(\bigvee_{k=0}^{n-1} \varphi^{-k}(\mathcal{U})\Big)}{n} = \lim_{n\to\infty} \frac{\mathcal{H}\big(\mathcal{U} \vee \varphi^{-1}(\mathcal{U}) \vee \cdots \vee \varphi^{-n+1}(\mathcal{U})\big)}{n}$$

using basic properties of $N(\mathcal{U})$ and $\mathcal{H}(N(\mathcal{U}))$ and their behavior with respect to the operations of joining and taking preimages (see [1]). We call $h(\varphi,\mathcal{U})$ the **entropy of $\varphi$ with respect to a cover $\mathcal{U}$**. One then defines topological entropy of the map as follows.

**Definition (Topological entropy).** The **entropy** $h(\varphi)$ of a mapping $\varphi$ is given by

$$h(\varphi) = \sup\{h(\varphi,\mathcal{U}) : \mathcal{U} \text{ is a cover of } X\}.$$

In this work, we are focused on the topological entropy of a subdivision map's restriction to the 1-skeleton of its model subdivision complex $\varphi : S^1_{\mathcal{R}} \to S^1_{\mathcal{R}}$. We now describe how Markov

representations manifest in this context.

For rational maps, the classical dichotomy decomposes the Riemann sphere into the *Fatou set* and its complement, the *Julia set*. For polynomials, it is natural to decompose the Riemann sphere into the *filled Julia set*, which consists of the Julia set along with the bounded components of the Fatou set, and the unbounded component of the Fatou set (the component containing $\infty$). For Misiurewicz polynomials, the Julia set coincides with the filled Julia set and the unbounded component of the Fatou set is the Fatou set. We will see that in the Fatou set, the dynamics is well-understood because it is conjugate to a well-understood system. That leaves the more interesting dynamical behavior concentrated in the filled Julia set. The finite subdivision rules that we consider which realize these complex dynamical systems naturally follow suit and are decomposed into pieces that are in the Fatou set and pieces that are in the filled Julia set. The pieces that are in the filled Julia set have Markov representations. The rest are pieces whose dynamical behavior is similarly well-understood as they conform to the well-understood dynamics of the Fatou set.

We will thus have a relation in the form of a Markov representation between the dynamics of the part of a model subdivision complex that is in the filled Julia set. The following proposition from [10] relates the entropy between such systems under such a relation.

**Proposition 2.2.2.** *Let $X$ and $Y$ by compact topological spaces, $f : X \to X$, $g : Y \to Y$, and $\pi : Y \to X$ continuous maps with $\pi$ surjective, and such that $f \circ \pi = \pi \circ g$. Then $h(f) \leq h(g)$. Suppose all fibers $\pi^{-1}(x)$ have uniformly bounded cardinality. Then $h(f) = h(g)$.*

This proposition will give us a way to compute $h(\varphi)$ by calculating $h(\sigma)$, the entropy of the shift system, $(X_{A_{\mathcal{R}}}, \sigma)$, which symbolically represent the part of the system $(S^1_{\mathcal{R}}, \varphi)$ that is of fundamental significance. We state a couple more propositions, see [1] and [10] for their proofs. We will invoke these propositions in the proof of Theorem 2.1. Lastly, we describe how the entropy of an edge shift is determined by the Perron eigenvalue of the adjacency matrix defining the shift and conclude this section with the definition of core entropy and state how one calculates this quantity.

**Proposition 2.2.3.** *Let $X$ be a compact topological space and let $\varphi : X \to X$ be a continuous map. Let $X_1$ and $X_2$ be two closed subsets of $X$ such that $X = X_1 \cup X_2$ and $\varphi(X_1) \subseteq X_1$ and $\varphi(X_2) \subseteq X_2$. Then*

$$h(\varphi) = max\{h(\varphi_1), h(\varphi_2)\}$$

*where $\varphi_1$ and $\varphi_2$ are the restrictions of $\varphi$ to $X_1$ and $X_2$, respectively.*

**Proposition 2.2.4.** *Let $X$ be a compact metric space and let $\varphi : X \to X$ be a continuous map. Let $Y$ be a closed subset of $X$ such that $\varphi(Y) \subseteq Y$. Suppose that, for any $x \in X$, the distance between $\varphi^n(x)$ and $Y$ tends to 0, uniformly on any compact set in $X \setminus Y$. Then $h(\varphi) = h(\varphi|_Y)$.*

Let $X_G$ be an edge shift for some finite directed graph $G$. The alphabet for the edge shift is $[m]$ where $m$ is the number of edges of $G$. Let $\mathcal{B}_n(X_G) = \{b_1 b_2 \cdots b_n : b_i \in [m] \text{ and } b_1 \cdots b_n \text{ occurs in some element of } X_G\}$. This is the set of all possible $n$-blocks or words formed over the alphabet $[m]$ that occur in sequences of $X_G$. We call such $n$-blocks **allowable**.

**Definition. (Entropy of Edge Shifts).** Let $G$ be a finite directed graph and let $X_G$ be its edge shift. Then the **entropy** of $X_G$ is defined by

$$h(X_G) = \lim_{n\to\infty} \frac{\log\left(|\mathcal{B}_n(X_G)|\right)}{n}.$$

This limit, which we are taking as the definition of entropy for edge shifts, is actually the topological entropy of the shift map on $X_G$. The reconciliation is not especially obvious, particularly because of the definition of topological entropy we use above. Including a proper reconciliation or deriving the equality above would require a lengthy digression. We content ourselves with a brief description of the reconciliation.

To start, there is another formulation of topological entropy set in terms of the metric on the phase space of one's dynamical system, due to Bowen [16]. Let $(X, f)$ be a dynamical system and let the **$n$-partial orbit** of a point $x$ be the set of points $\{x, f(x), f^2(x), \ldots, f^{n-1}(x)\}$. Fix an $\epsilon > 0$. The idea for this formulation of entropy is to consider the rate at which the number of $n$-partial orbits grows, where the distinction between orbits is made at a resolution of $\epsilon$, that is, one cannot distinguish points that are within $\epsilon$ of each other. The entropy is the limiting growth rate taken as the resolution $\epsilon$ goes to 0. A metric on shift spaces, alluded to earlier, measures distance between sequences by the distinction in their elements, that is, the distance between two sequences $s = (s_i)_{i\in\mathbb{N}_0}$ and $t = (t_i)_{i\in\mathbb{N}_0}$ depends on the number of $i$ such that $s_i \neq t_i$ and on where these differences occur (differences in $s_i$ and $t_i$ occurring at large $i$ contribute less to the metric than for small $i$). Fixing a resolution in the metric on an edge shift space $X_G$ is then essentially fixing a length $n$ of sequence indices. An $n$-partial orbit for a sequence $s$ in an edge shift space $X_G$ is clearly identified with allowable $n$-blocks or words formed over the alphabet of $X_G$. Therefore, for an edge shift space, the growth rate of $n$-partial orbits at a fixed resolution is equivalent to the growth rate of the number of possible allowable $n$-blocks over the alphabet of $X_G$. This is exactly what the limit above is determining.

An allowable $n$-block is clearly identified with a path of length $n$ in $G$. Conversely, paths of length $n$ in $G$ give rise to allowable $n$-blocks. Therefore, to calculate the entropy of an edge shift as given by the limit above, we must enumerate the number of paths of length $n$ in $G$. This is conveniently encoded in powers of the adjacency matrix. That is, if $A$ is the adjacency matrix of $G$, the entry $(A^n)_{ij}$ equals the number of paths of length $n$ from vertex

$i$ to vertex $j$. Thus,

$$|\mathcal{B}_n(X_G)| = \sum_{i,j=1}^{m} (A^n)_{ij}$$

and the growth rate we want becomes the growth rate of the sum of the entries of the adjacency matrix. This is where the Perron-Frobenius theory comes in. We state the Perron-Frobenius Theorem for reference. Proofs or references for proofs of any of the facts stated in the remainder of this section can be found in the reference for this section [16].

**Theorem 2.2.5** (Perron-Frobenius)**.** *Let $A \neq 0$ be an irreducible matrix. Then $A$ has a positive eigenvector $v_A$ with corresponding eigenvalue $\lambda_A > 0$ that is both geometrically simple, meaning the corresponding eigenspace of $\lambda_A$ is one-dimensional, and algebraically simple, meaning it is a simple root of the characteristic polynomial of $A$. If $\mu$ is another eigenvalue of $A$, then $|\mu| \leq \lambda_A$. Any positive eigenvector for $A$ is a positive multiple of $v_A$.*

We call the eigenvalue $\lambda_A$ the Perron eigenvalue of $A$. The following fact gives us our ticket to calculating core entropy of finite subdivision rules with mesh approaching 0 by determining the Perron eigenvalue of the edge recursion matrix.

**Proposition 2.2.6** (Entropy of Edge Shifts for Irreducible Graphs)**.** *Let $G$ be an irreducible graph, let $A$ be its adjacency matrix, and let $X_G$ be the associated edge shift. The entropy of $X_G$ satisfies*

$$h(X_G) = \log(\lambda_A),$$

*where $\lambda_A$ is the Perron eigenvalue of the adjacency matrix $A$.*

The results stated above were for irreducible matrices or graphs. We next describe a procedure applicable to arbitrary finite directed graphs and their associated adjacency matrices decomposing them into their so-called *irreducible components*. From there, we define Perron eigenvalues for any square matrix with nonnegative integer entries. Let $G$ be a finite directed graph with vertex set $V$. We define an equivalence relation $\sim$ on $V$. We say $u \sim v$ if and only if there exists a path in $G$ starting from $u$ ending at $v$ and a path starting from $v$ ending at $u$. This equivalence relation induces a relation on the graph itself by designating classes comprised of equivalence classes of vertices along with the edges that are included in paths two-way connecting members of a class. These classes are called **communication classes**. Decomposing $G$ in this way yields a finite number of irreducible subgraphs. We first illustrate the decomposition with an example.

**Example 2.2.7.** Let

$$A = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

.

The graph associated to $A$ is given in Figure 1.5. We can define a new graph $H$ from the communication classes as follows. Let the vertices of $H$ be the communication classes of vertices. Let there be an edge between any two classes $C$ and $D$, $C \neq D$, if and only if there is an edge between some state in $C$ and some state in $D$. It is straight-forward to see that $H$ cannot have cycles, $H$ must have *sources*, namely, vertices with no incoming edges, and $H$ must have sinks, vertices without outgoing edges. We depict the graph $H$ for the example below.

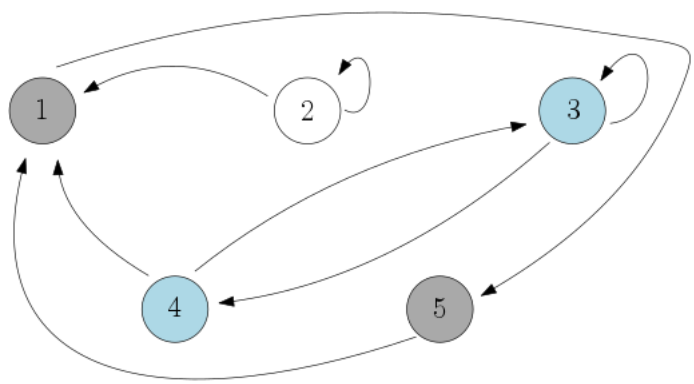


Figure 2.7: The graph along with its decomposition into communication classes. One class has its vertices shaded gray, another blue, the last white.

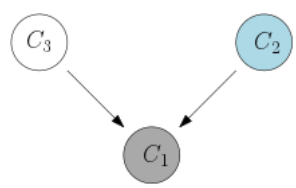


Figure 2.8: The associated graph $H$ to the example graph $G$.

We can label the vertices of $H$ $C_1, \ldots, C_k$ in any order which labels sinks first such that there can be a path in $H$ from $C_j$ to $C_i$ only if $j > i$. We can permute the row and column indices according to this new order. For the example above we get

$$A = \begin{array}{c} \\ 1 \\ 5 \\ 3 \\ 4 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 5 & 3 & 4 & 2 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

.

The **irreducible components** of $A$ are the adjacency matrices for the irreducible subgraphs obtained by decomposing the graph $G$ into communication classes. For our example, the irreducible components of $A$ are then

$$A_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, A_3 = \begin{bmatrix} 1 \end{bmatrix}.$$

The irreducible components appear as blocks along the diagonal of the transformed matrix $A$ and $A$ will always have a lower block-triangular form through permuting row and column indices in this way. The characteristic polynomial is then the product of characteristic polynomials of the irreducible components and thus the set of eigenvalues of the matrix is the union of the sets of eigenvalues of its irreducible components. We can now define Perron eigenvalues for arbitrary square matrices with nonnegative integer entries.

**Definition. (Perron eigenvalues)** Let $A$ be a square matrix with non-negative integer entries with irreducible components $A_1, \ldots, A_k$. The **Perron eigenvalue** $\lambda_A$ of $A$ is $\lambda_A = \max\{\lambda_{A_i} : 1 \leq i \leq k\}$.

What follows naturally is the classical result referenced above in a more general setting.

**Theorem 2.2.8** (Entropy of Edge Shifts)**.** *Let $G$ be a graph with adjacency matrix $A$ and associated edge shift $X_G$. Then $h(X_G) = \log(\lambda_A)$.*

In section 2.1, we will have occasion to consider a collection of edge recursion matrices whose Perron eigenvalues, and hence entropy, we wish to ascertain. The constructions and terminology introduced in this section will be of use in demonstrating that the Perron eigenvalues of this collection of matrices is 1. We now define core entropy for finite subdivision rules and state how they're calculated.

**Definition (Core entropy of Finite Subdivision Rules).** Let $\mathcal{R}$ be a finite subdivision rule, $S^1_{\mathcal{R}}$ the 1-skeleton of the model subdivision complex $S_{\mathcal{R}}$, $\varphi : S^1_{\mathcal{R}} \to S^1_{\mathcal{R}}$ the restriction of the subdivision map to $S^1_{\mathcal{R}}$, and $A$ the edge recursion matrix of $\mathcal{R}$. The **core entropy** of $\mathcal{R}$ is the topological entropy of $\varphi$, $h(\varphi)$. Core entropy of $\mathcal{R}$ is given by

$$h(\varphi) = \log(\lambda_A).$$

Before the proof of our main result we will establish a relation between the core entropy of a finite subdivision rule and the Perron eigenvalue of its edge recursion matrix despite not having mesh approaching 0 for the finite subdivision rules to which this result applies. We now move on to describing some useful tools and concepts from complex dynamics. In particular, from the theory of systems defined by the iteration of post-critically finite quadratic polynomials.

## 2.3 Complex Quadratic Dynamics

The objective of this section is two-fold: 1) define Misiurewicz polynomials and describe Hubbard trees and 2) discuss the dynamical features of Misiurewicz polynomials and Hubbard trees pertinent to this work. We wish to meet these objectives without lengthy recapitulation of theoretical results whose generality is far greater than the setting that the majority of this work focuses on. We thus content ourselves by simply stating the results we require and rely on the many oustanding references that exist today for the details behind these results. We first establish our dynamical terminology.

Let $X$ be a topological space and let $f : X \to X$ be a continuous map. For each $x \in X$, the **orbit** of $x$ is the set $\mathcal{O}(x) = \{f^n(x)\}_{n\in\mathbb{N}_0}$ of iterates of $x$ under $f$ including $x$. For each $x \in X$, the set $\mathcal{O}^+(x) = \{f^n(x)\}_{n\in\mathbb{N}}$ is the **forward orbit** of $x$ and is the sequence of iterates of $x$ starting with the first iterate $f(x)$. A point $x \in X$ is **(pre)periodic** if the orbit $\mathcal{O}(x)$ is finite. A point is **periodic** if $f^k(x) = x$ for some $k \in \mathbb{N}$. The minimal such $k$ is called the **period** of $x$ and $x$ is **fixed** if its period is 1. A point $x \in X$ is **strictly preperiodic** if there exists an $m \in \mathbb{N}$ such that $f^m(x)$ is periodic but $x, f(x), f^2(x), \ldots, f^{m-1}(x)$ are not periodic.

**Definition.** Let $U$ be an open subset of $\widehat{\mathbb{C}}$ and $R : U \to U$ be a rational function with fixed point at $z_0$. Suppose there is an analytic homeomorphism $\phi$ that maps $U$ to some neighborhood $V$ of 0 with $\phi(z_0) = 0$. Moreover, suppose there is a map $f : V \to V$ such that $R = \phi^{-1} \circ f \circ \phi$. Then we say that $R$ is **locally, analytically conjugate** to $f$.

### 2.3.1 Misiurewicz Polynomials

Let $\widehat{\mathbb{C}}$ be the Riemann sphere and let $c \in \mathbb{C}$. In complex quadratic dynamics, one studies iteration of maps $p : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ given by

$$p(z) = z^2 + c.$$

Such quadratics will be said to be in **standard form with parameter** $c$. In this setting, and for polynomials in general, a natural way to dichotomize the phase space $\widehat{\mathbb{C}}$ is into the *filled Julia set* and the *basin of attraction* of $\infty$. These sets can be defined as follows. The **basin of attraction** of $\infty$ is the set

$$W^s(\infty) = \{z \in \widehat{\mathbb{C}} : p^n(z) \to \infty \text{ as } n \to \infty\}.$$

The **filled Julia set** $K(p)$ is the complement of the basin of attraction of $\infty$, $K(p) = \widehat{\mathbb{C}} \setminus W^s(\infty)$. In other words, the filled Julia set of a polynomial is the set of points in $\widehat{\mathbb{C}}$ whose orbits remain bounded under iteration. The rest of the points in $\widehat{\mathbb{C}}$ converge to $\infty$ under iteration. With this dichotomy, we define the **Julia set** to be the topological boundary of the filled Julia set. Defining the Julia set in this way is in contrast to the standard theoretical development in complex dynamics which takes as its starting point the theory of normal

families of functions but is well-suited for our objectives in this section.

In the particular setting of complex polynomial dynamics, a theorem proved by Fatou in [13] but originally attributed to Böttcher asserts the existence of local conjugacies near $\infty$ to power maps near 0.

**Proposition 2.3.1.** *A polynomial of degree $d$ is locally conjugate to $z \mapsto z^d$ on some neighborhood of $\infty$.*

The conjugacies whose existence is asserted in this proposition are commonly referred to as **Böttcher maps** or **Böttcher coordinates**. Given the local nature of Böttcher maps, a natural question arises: What is the natural domain for a Böttcher map? That is, if one were to try and define Böttcher coordinates on the largest or most inclusive neighborhood of $\infty$ in $\widehat{\mathbb{C}}$, what would this neighborhood be? The answer to this question, in the setting of quadratic polynomials, is determined by the topology of the Julia set and underlies another natural dichotomy— that of the space of parameters $c \in \mathbb{C}$. Namely, the Julia sets of quadratic polynomials given by $p(z) = z^2 + c$ turn out to be either connected or totally disconnected. The following proposition determines the connectedness of the Julia set of a quadratic polynomial in standard form solely through the orbit of its critical point.

**Proposition 2.3.2.** *Let $p_c$ be a polynomial in standard form with parameter $c$. Let $J(p_c)$ be the Julia set of $p_c$ and let $\mathcal{O}^+(0) = \{c_k = p_c^k(c_0) : k \in \mathbb{N}\}$ be the forward orbit of the critical point $c_0 = 0$. Then $J(p_c)$ is disconnected if and only if $c_k \to \infty$ as $k \to \infty$. If so, $J(p_c)$ is a Cantor set. In particular, $J(p_c)$ is totally disconnected.*

This proposition is one way to see the motivation behind the definition of the **Mandelbrot set** $\mathcal{M}$ given by

$$\mathcal{M} = \{c \in \mathbb{C} : c_0 \in K(p_c)\}$$

where $p_c(z) = z^2 + c$, $c_0 = 0$ is the sole finite critical point of $p_c$, and $K(p_c)$ is the filled Julia set of $p_c$.

Given a quadratic polynomial $p_c$ in standard form, work done by Douady and Hubbard in [8], [9] demonstrates a natural or preferred choice of Böttcher map. The next proposition states the preferred domain of the Böttcher maps for quadratic polynomials whose parameter $c$ is in the Mandelbrot set. We will define *external rays* and *external angles* using these preferred Böttcher coordinates and then define Misiurewicz polynomials. We conclude this subsection with an example of a Misiurewicz polynomial and note its salient dynamical features.

**Proposition 2.3.3.** *Let $c \in \mathcal{M}$, let $p_c$ be the quadratic polynomial in standard form with parameter $c$, let $K(p_c)$ be the filled Julia set of $p_c$, and let $\overline{\mathbb{D}}$ be the closed unit disk in $\mathbb{C}$. Then there exists a Böttcher map $\phi_c : \widehat{\mathbb{C}} \setminus K(p_c) \to \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ such that the diagram*

$$\begin{tikzcd}
\widehat{\mathbb{C}} \setminus K(p_c) \arrow[r, "p_c"] \arrow[d, "\phi_c"'] & \widehat{\mathbb{C}} \setminus K(p_c) \arrow[d, "\phi_c"] \\
\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}} \arrow[r, "z \mapsto z^2"] & \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}
\end{tikzcd}$$

*commutes.*

By stipulating that the Böttcher map $\phi_c$ satisfy $\phi_c(z)/z \to 1$ as $|z| \to \infty$, one obtains a uniqueness for the Böttcher coordinates given in this proposition up to the multiplication by a square root of unity. In other words, for maps on $\widehat{\mathbb{C}}$ defined by quadratic polynomials in standard form with parameters in the Mandelbrot set, there is a preferred choice of Böttcher coordinates such that the natural domain for the Böttcher map is the basin of attraction of $\infty$ (see [19]). Therefore, Böttcher maps, in this setting, are often thought of as providing polar coordinates to the basin of attraction of $\infty$. Let $\psi_c = \phi_c^{-1}$ and let us parametrize polar angle in $\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ by the number of turns using $t \in \mathbb{R}/\mathbb{Z}$. An **external ray** of $K(p_c)$ is a set

$$R(t) = \psi_c\Big(\{\rho e^{2\pi i t}\}_{\rho > 1}\Big).$$

We call

$$\gamma(t) = \lim_{\rho \to 1} \psi_c\Big(\rho e^{2\pi i t}\Big)$$

the **landing point** of the external ray of argument $t$, if the limit exists.

A dense subset of parameters in the topological boundary of the Mandelbrot set correspond to the particular quadratic polynomials of interest, the *Misiurewicz polynomials*. These polynomials can be defined as follows:

**Definition.** A **Misiurewicz polynomial** map is a map $p : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ of the form $p(z) = z^2 + c$ where the orbit of the critical point $c_0 = 0$ is strictly preperiodic.

The Julia sets of Misiurewicz polynomials are dendrites and are therefore locally connected, as we show in an archetypal example below. Therefore, the landing points of external rays exist for all $t \in \mathbb{R}/\mathbb{Z}$ by the following theorem of Carathéodory.

**Theorem 2.3.4** (Carathéodory)**.** *A univalent map $\phi : \mathbb{D} \to \widehat{\mathbb{C}}$ has a continuous extension to $\partial\mathbb{D}$ if and only if $\partial\phi(\mathbb{D}) \subset \widehat{\mathbb{C}}$ is locally connected.*

In other words, this theorem by Carathéodory tells us when the preferred Böttcher maps further extend to the boundary, which, in the current setting, means to the Julia set $J(p_c)$ giving us a semi-conjugacy. Letting $\overline{\phi_c} : \widehat{\mathbb{C}} \setminus \mathring{K}(p_c) \to \widehat{\mathbb{C}} \setminus \mathbb{D}$ be the extension of $\phi_c$ to the

boundary circle $\partial\mathbb{D}$ and $\varphi = \overline{\phi_c}|_{\partial\mathbb{D}}$, we have the following commutative diagram often referred to as the **Carathéodory semi-conjugacy**

$$\begin{tikzcd} J(p_c) \arrow[r, "p_c|_{J(p_c)}"] \arrow[d, "\varphi"'] & J(p_c) \arrow[d, "\varphi"] \\ \partial\mathbb{D} \arrow[r, "z\mapsto z^2"] & \partial\mathbb{D} \end{tikzcd}$$

We now look at an example to showcase the salient features pertient to this work that all Misiurewicz polynomials possess.

**Example 2.3.5.** Let $p : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be given by $p(z) = z^2 + i$.

The critical point $c_0 = 0$, is strictly preperiodic and has orbit diagrammed by the order of iteration below

$$\begin{tikzcd} 0 \arrow[r] & i \arrow[r] & -1+i \arrow[r] & -i \arrow[l, bend left] \end{tikzcd}$$

The filled Julia set $K(p)$ is equal to the Julia set for this polynomial and is depicted below.

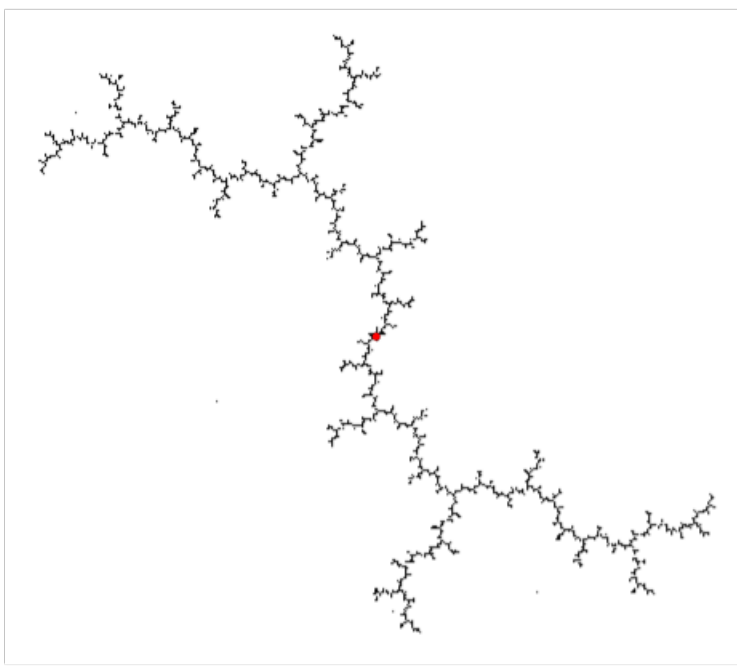

Figure 2.9: The filled Julia set of $p$; figure produced using the software package Mandel. The critical point $c_0 = 0$ is marked in the center of this figure with a red dot in the Julia set.

The Julia set is, topologically, a **dendrite**, a locally connected continuum that contains no simple closed curves. Of particular note is the invariant nature of the Julia set. As mentioned

above, an archetypal construction process for the finite subdivision rules under consideration in this work is to determine a *viable* 1-skeleton for the model subdivision complex and pull this structure back under the map one wishes to realize as a subdivision map. A natural starting point for viable 1-skeleta are set-wise invariant pieces of classical invariant subsets like the Julia set and its complement.

Before moving on to elucidating the finite subdivision rule realization of Example 1.2.2, we must describe another very useful and intruiging relation in the setting of complex quadratic dynamics. One may view Böttcher's result on the existence of Böttcher coordinates as a manifestation of Riemann's mapping theorem. Douady and Hubbard articulate this perspective in [8], [9]. The definition we've made above for external rays is only in the setting of filled Julia sets of quadratic polynomials in standard form. That is, the external rays we've defined are only ever external to the filled Julia set of some quadratic polynomial in standard form. The Riemann mapping theorem, however, may be applied to any subset $K \subset \mathbb{C}$ that is compact, connected, and full (meaning the complement of $K$ in $\mathbb{C}$ is connected). One has, in this more general setting, the existence of unique pairs $(r, \phi_K)$ where $r \in \mathbb{R}$ and $\phi_K : \mathbb{C} \setminus K \to \mathbb{C} \setminus \overline{\mathbb{D}}_r$ is an analytic homeomorphism such that $\phi_K(z)/z \to 1$ as $|z| \to \infty$ and $\overline{\mathbb{D}}_r$ is the closed disk of radius $r$ centered about the origin. One can then similarly define *external rays* and their *landing points* for any compact, connected, and full subset $K \subset \mathbb{C}$.

One particular compact, connected, and full subset is the Mandelbrot set $\mathcal{M}$. It is not known whether the external ray of every angle $t \in \mathbb{R}/\mathbb{Z}$ lands on the Mandelbrot set. The assertion that every external ray lands is a form of the well-known MLC Conjecture which asserts that the Mandelbrot set is locally connected. This conjecture remains unproven today. The very useful and intriguing relation we wish to describe, however, only cares about whether the external rays of *rational* arguments $t \in \mathbb{R}/\mathbb{Z}$ land. It is clear that by referring to elements of $\mathbb{R}/\mathbb{Z}$ as *rational* we mean that they have rational representatives. This turns out to be true.

Let $t \in [0, 1)$ be a rational representative of a point in $\mathbb{R}/\mathbb{Z}$. Let $R(t)$ be the external ray of $\mathcal{M}$ and let $\gamma(t)$ be its landing point. There is no a priori reason to suspect that the angle $t$ would contain any information relating to the dynamics of $p(z) = z^2 + \gamma(t)$ and yet it does:

**Remark 2.3.6.** *The orbit of $t/2$ or $(t+1)/2$ under the doubling map $x \mapsto 2x \mod 1$ has the same structure as the orbit of the critical point $c_0 = 0$ under the map $p(z) = z^2 + \gamma(t)$.*

For example, we saw that the orbit of 0 under $p(z) = z^2 + i$ was

$$\begin{tikzcd} 0 \arrow[r] & i \arrow[r] & -1+i \arrow[r] & -i \arrow[l, bend left] \end{tikzcd} .$$

The external ray of angle 1/6 turns out to land at $i$ and we note

$$\begin{tikzcd} 1/12 \arrow[r] & 1/6 \arrow[r] & 1/3 \arrow[r] & 2/3 \arrow[l, bend left] \end{tikzcd} .$$

See [8], [9], [12] for details. We will sometimes refer to points in either the filled Julia set of a quadratic polynomial in standard form or the Mandelbrot set using angles of external rays which land at those points. This is the primary manner in which we will use the Böttcher coordinates which conjugate the dynamics outside of $K(p)$ to $z \mapsto z^2$ outside of $\overline{\mathbb{D}}$.

We are now able to elucidate Example 2.1.2. Namely, we can identify specific subsets of $\widehat{\mathbb{C}}$ which correspond to the elements of the model subdivision complex of Example 2.1.2 along with its subdivided complex. We will see that Example 2.1.2 realizes the quadratic polynomial map given by $p(z) = z^2 + i$.

### 2.3.2 Finite Subdivision Rule Realization

In this subsection, we detail the finite subdivision rule realization of the Misiurewicz polynomial map given by $p(z) = z^2 + i$.

**Example 2.3.7.** Let $\phi$ be the Böttcher coordinates for $p$. We have the conjugacy provided by the Böttcher map depicted in the following commuting diagram.

$$\begin{tikzcd}
\widehat{\mathbb{C}} \setminus K(p) \arrow[r, "p"] \arrow[d, "\phi"'] & \widehat{\mathbb{C}} \setminus K(p) \arrow[d, "\phi"] \\
\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}} \arrow[r, "z \mapsto z^2"] & \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}
\end{tikzcd}$$

We start by identifying the vertices of the model subdivision complex. The vertices labeled $c_1$, $c_2$, and $c_3$ in the model subdivision complex correspond to the points $i$, $-1+i$, and $-i$, the finite post-critical points of the map $p$. The point $b$ corresponds to the landing point of three external rays of $K(p)$ with angles $1/7$, $2/7$, and $4/7$, and $\infty$ is $\infty$. The edges which connect vertices to $\infty$ are the external rays emanating from their landing points and the rest are the unique arcs contained in the Julia set. We depict the model subdivision complex in the standard local chart of $\widehat{\mathbb{C}}$ that does not contain the point $\infty$ in the figure below.

The edges which connect the points in the post-critical set of $p$ to the point $b$ comprise a special invariant subset of the dynamical plane of $p$. This set is called the *Hubbard tree* of $p$. We define and describe relevant properties of these trees in the next subsection.

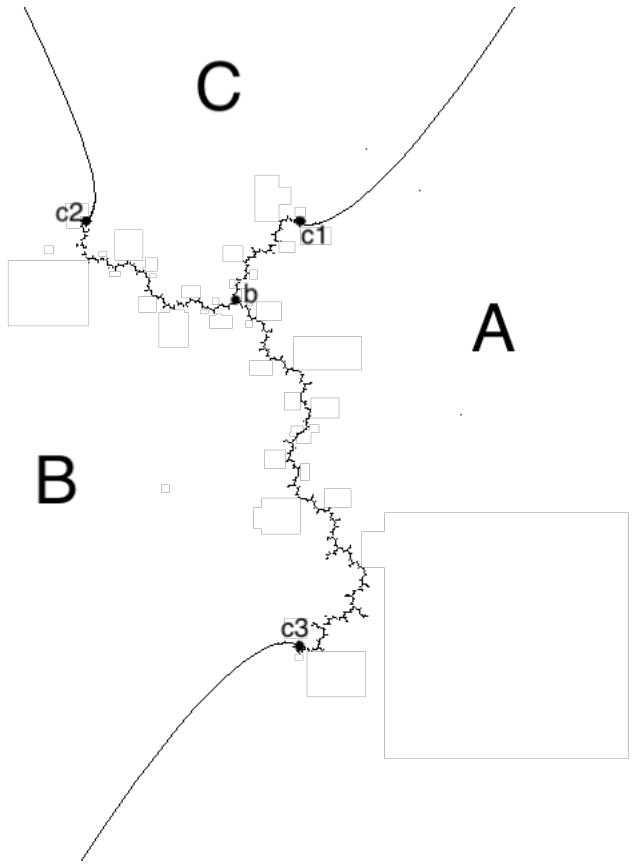


Figure 2.10: The model subdivision complex $S_{\mathcal{R}}$ of Example 2.1.2. The edges of the Hubbard tree are depicted using the software package Mandel. The external rays landing at the post-critical points with angles 1/6, 1/3, 2/3.

### 2.3.3 Hubbard Trees

In this subsection, we define Hubbard trees of Misiurewicz polynomials and state properties of the dynamics of a Misiurewicz polynomial's restriction to its Hubbard tree. We wish to provide information to convey general context and motivation for these objects while still avoiding lengthy recapitulation of the relevant work. We start by pointing to several outstanding references and then give the definitions and descriptions of particular relevance for this work.

There are several references that one can point to for detailed developments of Hubbard trees, each of which are distinguished by context and motivation. First and foremost is the work of Douady and Hubbard in [8], [9] in which they introduce Hubbard trees associated to any post-critically finite polynomial. Bruin and Schleicher in [5] begin with abstract topological dynamical systems given by iteration of maps on trees where the maps satisfy certain fundamental properties that are exhibited by the restrictions of post-critically finite polynomial

maps to their Hubbard trees. They investigate these models of Hubbard trees as dynamical systems and determine, among other things, the existence and uniqueness of these models by constructively realizing them with symbolic systems, describe an algorithmic way to obtain the topology of the tree from a symbolic description of the orbit of its critical point, and determine when such trees have embeddings in the complex plane thereby determining when there exists a quadratic polynomial whose restriction to its Hubbard tree realizes the dynamics of their abstract systems. In a similar vein, Poirier in [23] applies results of [22] to determine the realizability of abstract Hubbard trees. It would be useful to consult any one of the above references for general information about Hubbard trees and informative abstract models of the dynamics of polynomial maps restricted to their Hubbard trees. Since we pay particular attention to Misiurewicz polynomials in this work, we will define Hubbard trees as Alseda and Fagella do in [3] as doing so aligns very well with our goal in this subsection.

Alseda and Fagella in [3] examine the dynamical properties of post-critically finite polynomials that are preserved or reflected in the polynomial's restriction to its Hubbard tree; in particular if a property known as the *renormalizability* of a polynomial can be deduced from the dynamics of its restriction to its Hubbard tree. We follow their definition of Hubbard trees for Misiurewicz polynomials.

**Definition.** Let $f$ be a Misiurewicz polynomial and let $J(f)$ be its Julia set (recall that $K(f) = J(f)$ for Misiurewicz polynomials). Given a subset $A$ of $J(f)$, the **convex hull** of $A$, denoted $[A]$, is the smallest (by the partial order of containment) closed connected subset of $J(f)$ that contains $A$. Let $P_f \subset J(p)$ be the post-critical set of $f$. The **Hubbard tree** of $f$ is the set

$$H(f) = [P_f].$$

Some of the properties of a Misiurewicz polynomial's restriction to its Hubbard tree are summarized in the following proposition. Some of these properties are stated in [3]. These properties are part of the many properties that are stated and proven in Chapter 1 of [23]. These properties are also used to define the abstract Hubbard trees of [5]. We denote by $[x, y]$ the convex hull of two points $x, y \in J(f)$; such hulls correspond to the arcs in $J(f)$ connecting $x$ and $y$. We call the set $[x, y] \setminus \{x, y\}$ the **interior** of the arc $[x, y]$. We also designate a standard graph structure on $H(f)$. Any point $x \in H(f)$ such that $H(f) \setminus \{x\}$ has at least three components is a **branch point** of $H(f)$. Similarly, we refer to points $x \in J(f)$ such that $J(f) \setminus \{x\}$ has at least three components as branch points. We define the vertex set $V(f) = P_f \cup \{v \in H(f) : v \text{ is a branch point of } H(f)\}$. Edges of the standard graph structure on $H(f)$ are the arcs $[u, v]$ where $u, v \in V(f)$ and $[u, v]$ does not contain any vertices in its interior.

**Proposition 2.3.8.** *Let $f$ be a Misiurewicz polynomial, let $H$ be its Hubbard tree, and let $g = f|_H$.*

*(a) $g : H \to H$ is continuous and surjective;*

*(b) every point in $H$ has at most two preimages under $g$;*

*(c) at every point other than the critical point, the map $g$ is a local homeomorphism onto its image;*

*(d) all endpoints of $H$ are on the critical orbit;*

*(e) if $x$ and $y$ with $x \neq y$ are branch points or points on the critical orbit, then there is an $n \geq 0$ such that $g^n([x, y])$ contains the critical point.*

One way to deduce part (e) of the above proposition is through the following proposition. This is Proposition 2.15 of [23] and applies to any post-critically finite polynomial $f$, so, in particular, applies to Misiurewicz polynomials.

**Proposition 2.3.9.** *Let $\Omega(f) \subset J(f)$ be the singleton consisting of the sole finite critical point 0, let $M$ be a finite invariant set containing $\Omega(f)$, and let $T(M) = [M]$. Let $T^*(M)$ be the family whose elements are the closures of components of $T(M) \setminus \Omega(f)$. Let $v, v' \in J(f) \cap T(M)$ be two (pre)periodic points. If, for all $n \geq 0$, $f^n(v)$ and $f^n(v')$ belong to the same element of $T^*(M)$, then $v = v'$.*

We will use this proposition in the proof of Lemma 3.1.3. The definition given above does not suffice to specifically designate a Hubbard tree for polynomials that are not Misiurewicz polynomials. For post-critically finite polynomials in general, one must designate a notion of "regulated arc" in order to define Hubbard trees uniquely. This is because the filled Julia set for postcritically finite polynomials does not always have empty interior and one must therefore designate a specific path within the bounded components of the Fatou set of the polynomial when choosing arcs within the filled Julia set connecting elements of the post-critical set. For Misiurewicz polynomials, of particular interest in this work, the definition made above will suffice as the filled Julia sets for such polynomials are dendrites.

Having introduced convex hulls in filled Julia sets of Misiurewicz polynomials, we can now consider an example which demonstrates the existence of finite subdivision rules realizing Misiurewicz polynomials for which the 1-skeleton of the model subdivision complex is not comprised solely of edges in the filled Julia set and its complement.

**Example 2.3.10.** Let $f$ be the Misiurewicz polynomial with filled Julia $K(f)$ given by $f(z) = z^2 + i$. We first observe, by 2.3.6, the landing point of angle $t = 0$ has an orbit which consists of a single fixed point

$$\gamma_f(0) \mapsto \gamma_f(0).$$

We will consider a curve in the complement of $K(f)$ which converges to the landing point $\gamma_f(0)$ that is not an external ray. We will then note this curve's set-wise invariance and use it to build a finite subdivision rule realizing $f$. Let $\varphi : \widehat{\mathbb{C}} \setminus K(f) \to \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ be the preferred Böttcher map conjugating the dynamics of $f$ outside $K(f)$ to $z \mapsto z^2$ outside $\overline{\mathbb{D}}$. Consider the spiral curve $\alpha : (0, \infty) \to \widehat{\mathbb{C}}$ in $\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ given by the parametrization

$$\alpha(t) = (t\cos(2\pi\log(t)), t\sin(2\pi\log(t))).$$

We will abuse notation and use $\alpha$ to refer both to the curve as a function and to its image as a subset of $\widehat{\mathbb{C}}$. We will pay particular attention to the portion of this curve outside $\mathbb{D}$. We consider the image of $\alpha$ under $z \mapsto z^2$. We obtain the curve given by the parametrization

$$(\alpha(t))^2 = (t^2 \cos(4\pi \log(t)), t^2 \sin(4\pi \log(t))).$$

We see from the parametrization that we obtain the same curve. We have therefore obtained a subset of $\widehat{\mathbb{C}} \setminus \mathbb{D}$ that is set-wise invariant under $z \mapsto z^2$ and intersect the preimage of the landing point $\gamma_f(t)$ at $z = 1$. Let $\sigma = \varphi^{-1}(\alpha)$ be the preimage of $\alpha$ under the Böttcher coordinates. By the conjugacy provided by the Böttcher coordinates, we see that the preimage spiral is set-wise invariant under $f$.

Let $P_f$ be the post-critical set of $f$ and let $l_\sigma = \lim_{t \to 1} \varphi^{-1}(\alpha(t))$, the landing point of the spiral on the Julia set $K(f)$. Let $T = [P_f \cup \{l_\sigma\}]$ be the convex hull of the post-critical set along with $l_\sigma$ in $K(f)$. We can define the model subdivision complex realizing $f$ by setting the vertices to be $P_f \cup \{l_\sigma\}$ along with any branch points in $K(f)$ and $\infty$. We let the edges of the model subdivision complex be the minimal arcs connecting vertices in $K(f)$ and the spiral arc connecting $\infty$ and its landing point.

Taking the preimage of the model subdivision complex to obtain a subdivided complex $\mathcal{R}(S_\mathcal{R})$, we've obtained a finite subdivision rule realizing $f$ whose model subdivision complex does not consist solely of the convex hull of a finite invariant subset of $K(p)$ and external rays. This illustrates a constructive procedure for determining finite subdivisions rules that should work to realize any Misiurewicz polynomial.

We can now define Thurston's original core entropy in the setting of Misiurewicz polynomials. We remark that this quantity is defined for all post-critically finite polynomial maps on the Riemann sphere.

**Definition.** Let $f$ be a Misiurewicz polynomial map and let $H(f)$ be its Hubbard tree. Let $g = f|_{H(f)}$ be the restriction of $f$ to $H(f)$. The **core entropy** of $f$ (or **Thurston's core entropy**) is the topological entropy of $g$.

Part of what our main result demonstrates is that there is scarcely a distinction between the two definitions of core entropy in the original setting in which this quantity was considered. However, before demonstrating the main result, we take care to distinguish the two this way.

## 2.4 Matings of Quadratic Misiurewicz Polynomials

In this section, we define and describe a few ways to construct, from a given pair of quadratic Misiurewicz polynomials, a dynamical system given by a map known as a *mating*. We are ultimately interested in utilizing the work of Wilkerson in [27]. In this work, Wilkerson constructs finite subdivision rule structures realizing various families of matings of Misiurewicz polynomials. A consequence of the main result of this work is that, for certain finite subdivision rule construction's of Wilkerson's realizing these systems, the core entropy is the max of the core entropies of the polynomials.

### 2.4.1 Matings

We begin with a mating construction known as the *formal mating*. Although the construction described below applies equally well to any complex polynomial, we instead remain in the setting of interest and describe matings of Misiurewicz polynomials. The domain of the mating is built from a combination of the domains of two Misiurewicz polynomials. However, instead of the natural domain $\widehat{\mathbb{C}}$, we use a different compactification of the plane so as to make the domain a topological closed disk. We start by describing this closed disk domain and a Misiurewicz polynomial's extension to the boundary circle of this disk.

**Formal Mating Domain.**
$\widehat{\mathbb{C}}$ is obtained as the one-point compactification of $\mathbb{C}$ by adding the point $\infty$. We may similarly obtain a compactification by adding a circle's worth of points as follows. Let the set of symbols $\mathcal{I} = \{\infty \cdot e^{2\pi i t} : t \in \mathbb{R}/\mathbb{Z}\}$ be called the **circle at infinity**. We add the circle at infinity to $\mathbb{C}$ to form the set $\widetilde{\mathbb{C}} = \mathbb{C} \cup \mathcal{I}$. Endow $\widetilde{\mathbb{C}}$ with a natural topology making it homeomorphic to a disk. We can now exploit the content of Section 1.3.1 to define a continuous extension of a Misiurewicz polynomial $f : \mathbb{C} \to \mathbb{C}$ to the circle at infinity. In particular, we know that in $\widehat{\mathbb{C}}$, at $\infty$, any Misiurewicz polynomial is locally analytically conjugate to the squaring map. It is this behavior that enables one to continuously extend $f$ to the circle at infinity by defining $f(\infty \cdot e^{2\pi i t}) = \infty \cdot e^{2\pi i (2t)}$. We will abuse notation and not refer to the continuously extended map with a distinct symbol. We can now describe the domain of the formal mating.

Let $f : \widetilde{\mathbb{C}}_f \to \widetilde{\mathbb{C}}_f$ and $g : \widetilde{\mathbb{C}}_g \to \widetilde{\mathbb{C}}_g$ be two Misiurewicz polynomials that have been continuously extended to their domains compactified by circles at infinity. We have given subscripts to the domains of $f$ and $g$ to explicitly distinguish them as two separate copies of the same space. Let $\widetilde{\mathbb{C}}_f \sqcup \widetilde{\mathbb{C}}_g$ be the disjoint union. We let $(x, f)$, $x \in \mathbb{C}$, denote elements in $\widetilde{\mathbb{C}}_f \sqcup \widetilde{\mathbb{C}}_g$ in the hemisphere corresponding to the domain of $f$ and we let $(x, g)$ denote elements in $\widetilde{\mathbb{C}}_f \sqcup \widetilde{\mathbb{C}}_g$ that are in the hemisphere corresponding to the domain of $g$. We define an equivalence relation $\sim$ on $\widetilde{\mathbb{C}}_f \sqcup \widetilde{\mathbb{C}}_g$ given by $x \sim x$ for all $x \in \widetilde{\mathbb{C}}_f \setminus \mathcal{I}_f \cup \widetilde{\mathbb{C}}_g \setminus \mathcal{I}_g$ and $(\infty \cdot e^{2\pi i t}, f) \sim (\infty \cdot e^{-2\pi i t}, g)$ for all $t \in \mathbb{R}/\mathbb{Z}$. That is, glue the circles at infinity for the domains of $f$ and $g$ along opposite directions according to the order on $\mathbb{R}/\mathbb{Z}$. Let $S^2_{f,g} = \widetilde{\mathbb{C}}_f \sqcup \widetilde{\mathbb{C}}_g / \sim$ endowed with the quotient topology. The space $S^2_{f,g}$ is a topological 2-sphere comprised of open hemispheres identified

with the original planar domains of $f$ and $g$ and a single equator identified with both of the circles at infinity. This 2-sphere is the domain of the *formal mating*. We now define the formal mating.

**Definition.** Let $f$ and $g$ be Misiurewicz polynomial maps as above. The **formal mating** $F = f \perp\!\!\!\perp g : S^2_{f,g} \to S^2_{f,g}$ of $f$ and $g$ is given by

$$F = f \perp\!\!\!\perp g = \begin{cases} f, & \text{on } \widetilde{\mathbb{C}}_f, \\ g, & \text{on } \widetilde{\mathbb{C}}_g \end{cases}.$$

That is, the mating $F$ acts like $f$ and $g$ on their respective open hemispheres and acts like the doubling map along the equator. Points such as the **post-critical points** or **critical points** of the formal mating are those points for $f$ and $g$ identified in $S^2_{f,g}$. Now we want to define two more mating constructions built from the formal mating. In particular, one defines equivalence relations on the 2-sphere $S^2_{f,g}$ and determines naturally induced maps on the quotient space. We start by defining *external rays* for the formal mating of two Misiurewicz polynomials.

Recall that any Misiurewicz polynomial $f_c(z) = z^2 + c$ has a Böttcher map $\phi_c$ conjugating the dynamics on the complement of the filled Julia set to the squaring map on the complement of the unit disk. The external rays were defined to be the inverse images of the rays emanating from the origin outside the unit disk under the Böttcher map. Changing our domains from $\widehat{\mathbb{C}}$ to $\widetilde{\mathbb{C}}$ does not change the definition of external rays of a Misiurewicz polynomial. However, the inverse of the Böttcher map is no longer "centered" about a single point at $\infty$. That is, when our domain was $\widehat{\mathbb{C}}$, the end of each external ray opposite the landing point was a single point $\infty$. In the domain $\widetilde{\mathbb{C}}$, for all $t \in \mathbb{R}/\mathbb{Z}$, the end of an external ray $R(t)$ opposite to its landing point becomes the point $\infty \cdot e^{2\pi i t}$.

Given two Misiurewicz polynomials $f$ and $g$, for all $t \in \mathbb{R}/\mathbb{Z}$, we denote by $R_f(t)$ and $R_g(t)$ the external rays of argument $t$ of the filled Julia sets $K(f)$ and $K(g)$. We define the **external ray** of the formal mating $F = f \perp\!\!\!\perp g$ with argument $t \in \mathbb{R}/\mathbb{Z}$ to be the set given by the equivalence classes

$$R_F(t) = (R_f(t), f) \sqcup (R_g(-t), g) \cup \{[(\infty \cdot e^{2\pi i t}, f)]\}.$$

That is, the external ray of $F$ of argument $t$ is comprised of the external rays of $f$ of argument $t$ along with the external ray of $g$ of argument $-t$ and the point at which these external rays connect along the equator. We let the **landing points** $\gamma_F(t)$ of $R_F(t)$ be the set of landing points $\{\gamma_f(t), \gamma_g(-t)\}$.

The formal mating of two Misiurewicz polynomials is a branched covering of a topological 2-sphere. In [11], Douady and Hubbard prove a result of Thurston's for when orientation-preserving post-critically finite branched covering maps of the 2-sphere are *equivalent* to

a rational map. We define the particular notion of equivalence, known in the literature as *Thurston equivalence*. We do this for completeness and to designate a notion of *matability*.

**Thurston Maps and Thurston Equivalence.**

Let $f : S^2 \to S^2$ be an orientation-preserving branched covering map. Let $\deg_x(f)$ be the local degree of $f$ at $x$. We define two sets of points in $S^2$. The **critical set** of $f$ is given by

$$\Omega_f = \{x : \deg_f(x) > 1\}.$$

The **post-critical set** is the set given by

$$P_f = \bigcup_{n \in \mathbb{N}} f^n(\Omega_f)$$

A map $f$ is called **post-critically finite** if $P_f$ is a finite set. Post-critically finite orientation-preserving branched covering maps of degree at least two are known in the literature as **Thurston maps**.

The matings defined in this work are Thurston maps. Thurston's topological characterization theorem was motivated by the desire to determine when matings were dynamically *equivalent* to rational maps. Topological conjugacy as the notion of equivalence turns out to be too strong a notion of equivalence. For example, the formal mating is never topologically conjugate to a rational map [21]. *Thurston equivalence*, defined below, achieves a suitable notion of equivalence that is weaker than topological conjugacy.

**Definition. (Thurston Equivalence)** Two branched mappings $f, g : S^2 \to S^2$ are **Thurston equivalent** if and only if there exist homeomorphisms $\theta, \theta' : S^2 \to S^2$ such that the diagram

$$\begin{tikzcd}
(S^2, P_f) \arrow[r, "f"] \arrow[d, "\theta"'] & (S^2, P_f) \arrow[d, "\theta'"] \\
(S^2, P_g) \arrow[r, "g"] & (S^2, P_g)
\end{tikzcd}$$

commutes, and $\theta$ is isotopic to $\theta'$ relative to $P_f$.

We will say that a map $f$ is **obstructed** if it is *not* Thurston equivalent to a rational map. We say that two Misiurewicz polynomials are **strongly mateable** if the formal mating $F = f \amalg g$ is Thurston equivalent to a rational map. We now define two equivalence relations on the domain of the formal mating $S^2_{f,g}$ of two Misiurewicz polynomials. These equivalence relations, in some circumstances, induce two additional types of matings. One is called the *topological mating* the other is called the *degenerate mating*. We describe relations among

the three conceptions of mating after finishing our definitions.

**Ray Equivalence.**
Here we formalize the notion of gluing the filled Julia sets of Misiurewicz polynomials together. Given Misiurewicz polynomials $f$ and $g$, their filled Julia sets are identified as subsets in the domain of the formal mating $S^2_{f,g}$. For each $x$ in the filled Julia set of $f$ there exists at least one $t \in \mathbb{R}/\mathbb{Z}$ such that the external ray $R_f(t)$ lands at $x$. $R_f(t)$ meets the equatorial circle at infinity in $S^2_{f,g}$ at $\infty \cdot e^{2\pi i t}$ which is identified with $\infty \cdot e^{-2\pi i t}$ in $\widetilde{\mathbb{C}}_g$. Since the filled Julia sets of Misiurewicz polynomials are locally connected, each ray lands at some point in the filled Julia set and so, in particular, $R_g(-t)$ lands at some point on the filled Julia set of $g$. Taking these components together, we see we have $R_F(t) \cup \gamma_F(t)$. The *ray equivalence relation* identifies all points in $S^2_{f,g}$ contained in such rays.

More precisely, we define the **ray equivalence relation** on $S^2_{f,g}$ to be the relation $\sim_{\text{ray}}$ generated by

$$x \sim_{\text{ray}} y \text{ iff } x, y \in R_F(t) \cup \gamma_F(t) \text{ for some } t \in \mathbb{R}/\mathbb{Z}.$$

Note, we identify points that share *some* external ray and landing point of the formal mating. There may be multiple rays landing at the same points in $J(f)$ or $J(g)$ and so the equivalence classes can be complicated.

**The Topological Mating.**
The domain of the topological mating is the quotient space $S^2_{f,g}/\sim_{\text{ray}}$. The **topological mating** is the map $F^* : S^2_{f,g}/\sim_{\text{ray}} \to S^2_{f,g}/\sim_{\text{ray}}$ which $F$ induces on the quotient space.

There are a few key theoretical considerations to be made in considering topological matings [21].

1. The quotient space $S^2_{f,g}/\sim_{\text{ray}}$ may not be a Hausdorff space. Equivalently, the associated subset of $S^2_{f,g} \times S^2_{f,g}$ determined by $\sim_{\text{ray}}$ may not be a closed subset.

2. The quotient space $S^2_{f,g}/\sim_{\text{ray}}$ may not be homeomorphic to a 2-sphere.

3. The induced map may not be Thurston equivalent to a rational map.

Defining topological matings as quotients of formal matings, in a sense, reflects the motivation for defining our last type of matings called *degenerate matings* (see below). Topological matings are very interesting, but difficult to work with directly. Understanding theoretically that it is possible for a topological mating to be Thurston equivalent to a rational map leads naturally to the question of when this happens, that is: for which polynomial pairs? Thurston's topological characterization of rational maps determines when a Thurston map is Thurston equivalent to a rational map but it does not designate a general algorithmic way to check for when obsructions exist or how to find them and is certainly not particular to maps that arise as matings. In the particular case of Thurston maps that do arise as matings,

however, work has been done to apply Thurston's theorem to understand when such matings are obstructed.

This work, in part, gave rise to this approach to understanding topological matings as quotients of formal matings. With this approach, it is natural to ask whether *Thurston equivalence to a rational map* is a property which is preserved under ray-equivalence. That is, is it true that a topological mating is Thurston equivalent to a rational map if and only if the formal mating which gives rise to it under ray equivalence is also Thurston equivalent to a rational map? It turns out that there exists instances in which a formal mating is obstructed, yet the topological mating is not. This leads to the notion of a *removable obstruction*. Works of Tan Lei, M. Rees, M. Shishikura ([25], [24]) gave rise to a notion of mating that is intermediate between the formal mating and the topological mating that is obtained by *removing the obstruction*. This intermediate mating is what's known as the **degenerate mating**. We define this notion of mating below.

**The Degenerate Mating.**
Let $F : S^2_{f,g} \to S^2_{f,g}$ be the formal mating of two Misiurewicz polynomials $f$ and $g$. We first define the domain of the degenerate mating which we will denote $S^2_{\text{degen}}$. We denote the degenerate mating by $G : S^2_{\text{degen}} \to S^2_{\text{degen}}$. We obtain this domain as a quotient of $S^2_{f,g}$ under a relation $\sim_{\text{degen}}$ whose only non-trivial classes consist of a particular collection of ray equivalence classes (that is, the rest of the points are identified to themselves). We define the non-trivial classes of $\sim_{\text{degen}}$ as follows. Let

$$\begin{aligned} Y' &= \{\text{ray equivalence classes containing at least two points of } \Omega_F \cup P_F\}, \\ Y &= \{\zeta | \zeta \text{ is a ray equivalence class intersecting } \Omega_F \cup P_F \text{ and there exist} \\ &\qquad m \geq 0,\ \eta \in Y' \text{ such that } F^m(\zeta) = \eta\}. \end{aligned}$$

The selection critera for the ray equivalence classes to be identified under $\sim_{\text{degen}}$ can be described as follows. Start with ray equivalence classes that contain at least two post-critical or critical points of $F$; these are elements of $Y'$. For Misiurewicz polynomials, these points are located on the filled Julia sets of the mated polynomials. Add to these classes any ray equivalence class that contains a critical or post-critical point for which some iterated image (possibly itself) is equal to the iterated image of an element of $Y'$. You can find such classes by taking preimages of the members of $Y'$ and choosing the ray equivalence classes that contain at least one point in $\Omega_F \cup P_F$.

If $Y = \varnothing$, we define $S^2_{\text{degen}} = S^2_{f,g}$. If $Y \neq \varnothing$ and all $\zeta \in Y$ are topological trees (in particular, each element is simply connected), we set the non-trivial classes of $\sim_{\text{degen}}$ to be the elements of $Y$ and define $S^2_{\text{degen}} = S^2_{f,g}/\sim_{\text{degen}}$. If $Y \neq \varnothing$ and there is an element $\zeta \in Y$ that is not simply connected, then we do not define the degenerate mating. This defines the domain of the degenerate mating in all cases in which it is to be defined.

The map itself is defined as follows. If $Y = \varnothing$, then we let $G = F$. If $Y \neq \varnothing$ and all $\zeta \in Y$ are simply connected, then we define $G : S^2_{\text{degen}} \to S^2_{\text{degen}}$ as follows (following [24]): Let $\pi : S^2_{f,g} \to S^2_{\text{degen}}$ be the canonical projection map. There exists a continuous map $\overline{G} : S^2_{\text{degen}} \to S^2_{\text{degen}}$ such that $\overline{G} \circ \pi = \pi \circ F$. This map $\overline{G}$ is a branched covering that coincides with $F/\sim_{\text{degen}}$ everywhere except in a neighborhood of $\pi(F^{-1}(Y) \setminus Y)$. Each component of $F^{-1}(Y) \setminus Y$ is a tree and its image under $F$ is a point. We modify the map $\overline{G}$ in a small neighborhood of $\pi(F^{-1}(Y) \setminus Y)$ to obtain a branched covering map $G : S^2_{\text{degen}} \to S^2_{\text{degen}}$ as follows. Take pairwise disjoint open disk neighborhoods of each element in $F^{-1}(Y) \setminus Y$ and choose homeomorphisms between these neighborhoods and their images. This defines $G : S^2_{\text{degen}} \to S^2_{\text{degen}}$ up to Thurston equivalence.

We say that a pair of Misiurewicz polynomials is **matable** if the degenerate mating exists and is Thurston equivalent to a rational map. Shishikura in [24] shows that if the degenerate mating exists and is Thurston equivalent to a rational map, there exists a continuous semi-conjugacy between $S^2_{f,g}$ and $\widehat{\mathbb{C}}$ conjugating the formal mating $F$ with a rational map and, futher, shows that this semi-conjugacy induces a topological conjugacy between the topological mating and the rational map. See [25], [27], [18], for additional details on the definitions of these maps. We will provide archetypal examples that illustrate the properties of these maps that are of most importance to us in this work.

### 2.4.2 Finite Subdivision Rules Realizing Matings

The matings defined in the last section, although fairly straightforward to conceptually describe, are not particularly easy to understand. To start, the notion of gluing two dendrites to form a topological 2-sphere is difficult to accept, let alone visualize. In [27], Wilkerson constructs finite subdivision rules which realize both formal and degenerate matings toward the end of elucidating matings and their dynamics. In this subsection we state the families of matings realized by her constructions whose core entropy we consider in Section 3.2. We illustrate the constructions with a few examples. We begin with a construction which realizes formal matings of Misiurewicz polynomials by finite subdivision rules.

Let $f$ and $g$ be two quadratic Misiurewicz polynomials. Let $P_f$ and $P_g$ be the post-critical sets of $f$ and $g$ and let $J(f)$ and $J(g)$ be the Julia sets of $f$ and $g$. We construct a finite subdivision rule that realizes a mating in a way similar to the archetypal construction for finite subdivision rules realizing Misiurewicz polynomials. That is, we designate a viable candidate for a 1-skeleton in the domain of the mating, pull the resulting CW structure back under the mating so as to yield a subdivided complex, and check the cellular nature of the mating by construction.

**Definition. (Construction 0)** Given a formal mating $F = f \perp\!\!\!\perp g$, select a finite set of external rays of the mating $\{R_F(t_k)\}_{k=1}^n$ such that all of the forward images under $F$ of each ray $R_F(t_k)$ is contained in the collection. In the underlying space $S^2_{f,g}$, we designate a cell

decomposition starting by specifying the 1-skeleton. Let $S^1_{\mathcal{R}}$ be the union of the convex hull of $P_f \cup \{\gamma_f(t_k)\}_{k=1}^n$ in $J(f)$, the convex hull of $P_g \cup \{\gamma_g(-t_k)\}_{k=1}^n$ in $J(g)$, and the collection of external rays of the mating $\{R_F(t_k)\}_{k=1}^n$. We further specify the cell decomposition on $S^2_{f,g}$ by designating the 0-cells to be the points in $P_f \cup \{\gamma_f(t_k)\}_{k=1}^n \cup P_g \cup \{\gamma_g(-t_k)\}_{k=1}^n$ along with any branch points in either of the convex hulls taken in $J(f)$ and $J(g)$. Letting the complement of the 1-skeleton specified above naturally comprise the open 2-cells, we have a CW structure on $S^2_{f,g}$ which we take to be our model subdivision complex, $S_{\mathcal{R}}$. Let $\mathcal{R}(S_{\mathcal{R}})$ be the cell decomposition determing by taking the preimage of $S_{\mathcal{R}}$ under $F$. The mating $F : \mathcal{R}(S_{\mathcal{R}}) \to S_{\mathcal{R}}$ is then cellular with respect to $\mathcal{R}(S_{\mathcal{R}})$ and $S_{\mathcal{R}}$ by construction. The finite subdivision rule $\mathcal{R}$ defined as above is a finite subdivision rule of **Construction 0**.

The definition above differs only superficially from what is found in [27]. The idea behind the construction is as follows: a standard way to build a finite subdivision rule starting with the model subdivision complex for a single Misiurewicz polynomial is to take a finite invariant collection of external rays that includes at least the rays which land at the post-critical set, take the convex hull in the Julia set of the landing points of this finite collection of rays, and mark post-critical points, branch points, and landing points as vertices and let the open 1-cells and 2-cells follow naturally from the choice of vertices. This idea is naturally extended to matings by taking a finite invariant collection of external rays of the mating and similarly building the model subdivision complex by following this standard construction in each hemisphere. We elucidate this construction in the following example.

**Example 2.4.1.** Let us start by describing a different way to denote a quadratic Misiurewicz polynomial. We recall Remark 2.3.6

*The orbit of $t/2$ or $(t+1)/2$ under the doubling map $x \mapsto 2x \mod 1$ has the same structure as the orbit of the critical point $c_0 = 0$ under the map $p(z) = z^2 + \gamma(t)$.*

That is, using the angle of landing points of external rays of the Mandelbrot set, a parameter $c = \gamma(t)$ corresponding to a rational value of $t \in \mathbb{R}/\mathbb{Z}$ defines a quadratic polynomial in standard form $f_c(z) = z^2 + c$ whose critical orbit $\{f_c^k(0)\}_{k\in\mathbb{N}_0}$ is in direct correspondence to, or has the exact same structure as, the orbit $\{2^k(\frac{t}{2})\}_{k\in\mathbb{N}_0} = \{2^k(\frac{t+1}{2})\}_{k\in\mathbb{N}_0}$ of the angle $(t/2) \in \mathbb{R}/\mathbb{Z}$ under doubling. It is for this reason that we sometimes denote $f_c$ by $f_t$. This notation is cleaner, because often $c$ is the solution to some high degree polynomial in $c$ and hence cumbersome if not impossible to represent in some neat form. This notation also offers information more readily because the orbit $\{2^k(\frac{t}{2})\}_{k\in\mathbb{N}_0} = \{2^k(\frac{t+1}{2})\}_{k\in\mathbb{N}_0}$ is more readily computable than the orbit $\{f_c^k(0)\}_{k\in\mathbb{N}_0}$ even if one was given $c$ in some neat form.

We take as our starting point the quadratic Misiurewicz polynomial $f_{1/4}$ and its filled Julia set.

Given the Carathéodory semiconjugacy, the choice of a finite collection of external rays where the forward orbit of each ray is contained in the collection amounts to choosing a finite number of rational arguments in $\mathbb{R}/\mathbb{Z}$ and taking the forward orbits of each. For example,

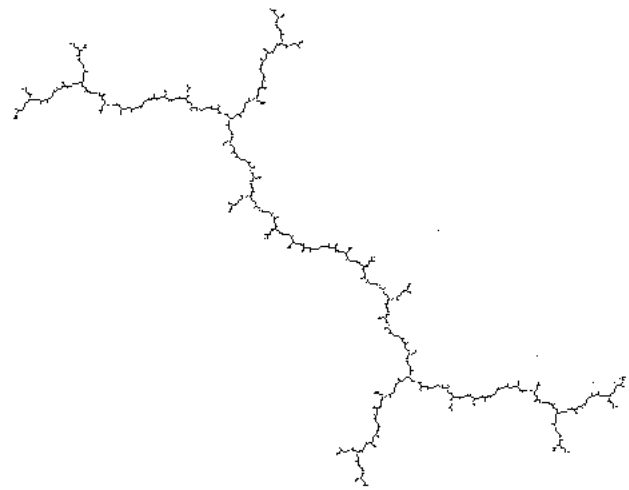

Figure 2.11: The filled Julia set of $f_{1/4}$; figure produced using the software package Mandel.

one could simply take the single angle $t = 1/4$. The orbit under doubling of $t/2 = 1/8$ is given by

$$1/8 \longrightarrow 1/4 \longrightarrow 1/2 \longrightarrow 1 \circlearrowleft .$$

Thus, taking the convex hull of the landing points $\gamma_{f_{1/4}}(1/8), \gamma_{f_{1/4}}(1/4), \gamma_{f_{1/4}}(1/2), \gamma_{f_{1/4}}(1)$, and adding the external rays of these argument, one obtains something which looks like the depiction in Figure 2.12.

Designating the post-critical points, which coincide with the landing points of the external rays we've selected, and branch points as vertices, we have a model subdivision complex for the Misiurewicz polynomial $f_{1/4}$. We depict the other hemisphere of $F$ in Figure 2.13. This hemisphere has a cell decomposition whose vertices correspond to the landing points $\gamma_{f_{1/4}}(-1/8), \gamma_{f_{1/4}}(-1/4), \gamma_{f_{1/4}}(-1/2), \gamma_{f_{1/4}}(-1)$ along with branch points. The rest of the 1-skeleton is comprised of the external rays with these angles and the convex hull of the landing points in the other copy of $J(f_{1/4})$.

To build off of this construction and create a model subdivision complex for the formal mating $F = f_{1/4} \perp\!\!\!\perp f_{1/4}$ of $f_{1/4}$ with itself, we simply duplicate this construction in $\widetilde{\mathbb{C}}$ and glue the two copies of $\widetilde{\mathbb{C}}$ together along their circles at infinity. We depict a local view of the domain of $F$ in Figure 2.14.

The domain of the degenerate mating of two quadratic Misiurewicz polynomials starts with an equivalence relation on the domain of the formal mating. The identifications in the domain of the formal mating may not be immediately clear. We illustrate the identifications made by this relation using an example.

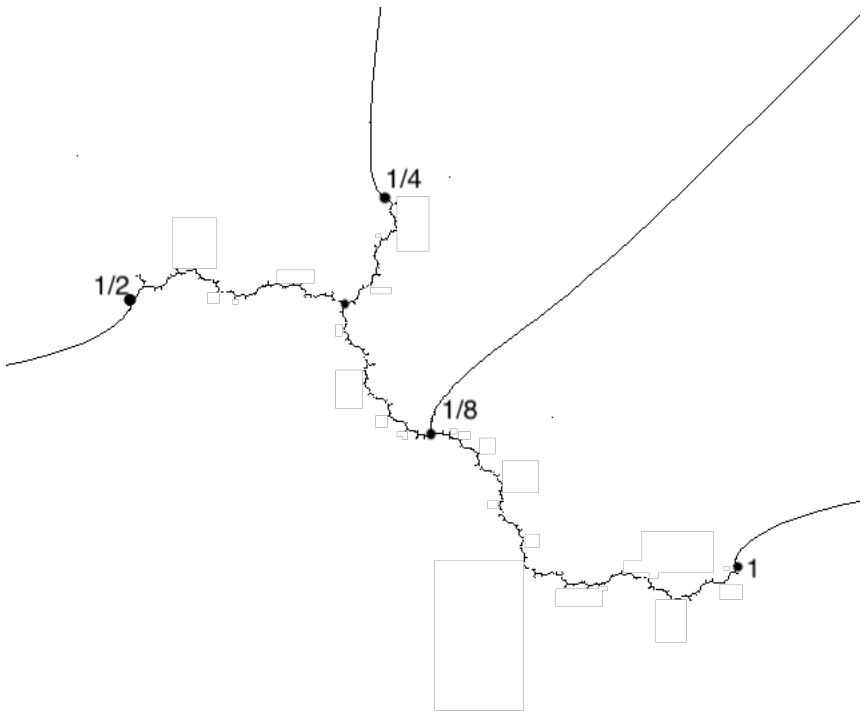


Figure 2.12: The convex hull of the landing points $\gamma_{f_{1/4}}(1/8), \gamma_{f_{1/4}}(1/4), \gamma_{f_{1/4}}(1/2), \gamma_{f_{1/4}}(1)$ in $J(f_{1/4})$ along with the external rays with arguments 1/8, 1/4, 1/2, and 1. Note, in this depiction, we are looking at a local view of one hemisphere for the domain of $F$.

**Example 2.4.2.** We saw at the end of Section 1.3.1 that $f_i = f_{1/6}$. We illustrate the identifications made by $\sim_{\text{degen}}$ in the domain of the formal mating $F = f_{1/6} \perp\!\!\!\perp f_{1/6}$. We start with the equivalence classes of $\sim_{\text{ray}}$ that contain at least two post-critical points of $F$. Again, by the Carathéodory semiconjugacy, the arguments of the external rays which land at post-critical points are the arguments present in the orbit of $(1/6)/2$ under doubling. Therefore, the external rays corresponding to these arguments correspond to classes of $\sim_{\text{degen}}$. These are $[x_1] = R_F(1/12) \cup \gamma_F(1/12)$, $[x_2] = R_F(7/12) \cup \gamma_F(7/12)$, $[x_3] = R_F(1/6) \cup \gamma_F(1/6)$, $[x_4] = R_F(1/3) \cup \gamma_F(1/3)$, $[x_5] = R_F(2/3) \cup \gamma_F(2/3)$. We depict these in Figure 2.15.

The rays of argument $t/2$ and $(t+1)/2$ both land at the critical point of $f_t$ for any rational valued $t \in \mathbb{R}/\mathbb{Z}$ which accounts for the addition of the external ray $R_F(7/12)$ above. We now consider iterated preimages of these classes under $F$ which contain at least one point on the critical orbit of $F$. Preimages of $R_F(t)$ are $R_F(t/2)$ and $R_F((t+1)/2)$. There is one and only one ray landing at each post-critical point and exactly two rays landing at the critical point. Therefore, taking preimages of the angles of the $[x_i]$, we see that there are no other non-trivial classes of $\sim_{\text{degen}}$ to be identified.

We now give Wilkerson's construction for finite subdivision rule structures which realize degenerate matings. We are still in the setting of Construction 0, namely, we let $f$ and $g$ be

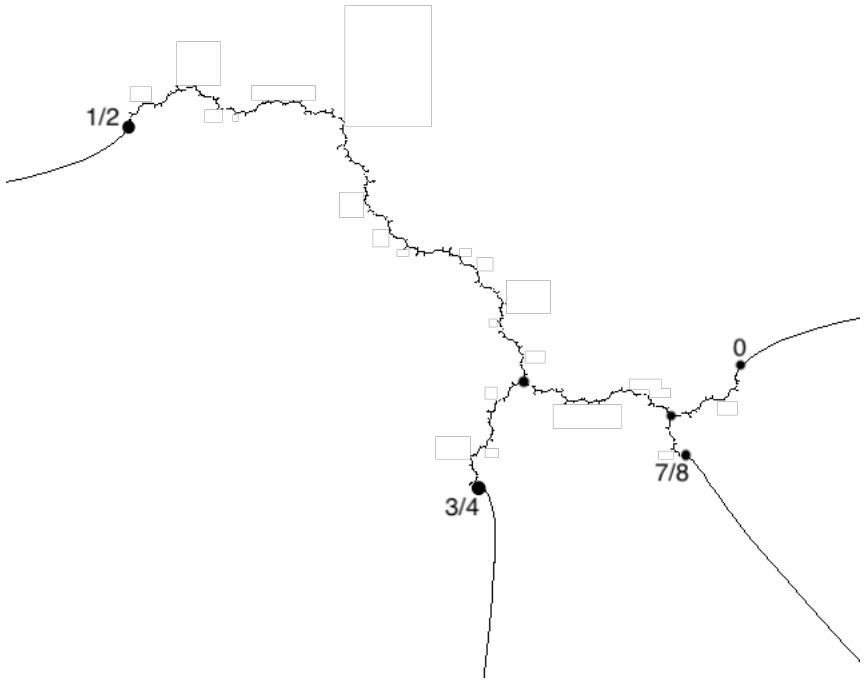


Figure 2.13: The convex hull of the landing points $\gamma_{f_{1/4}}(1/8), \gamma_{f_{1/4}}(1/4), \gamma_{f_{1/4}}(1/2), \gamma_{f_{1/4}}(1)$ in $J(f_{1/4})$ along with the external rays with arguments 1/8, 1/4, 1/2, and 1. Note, in this depiction, we are looking at a local view of one hemisphere for the domain of $F$.

two quadratic Misiurewicz polynomials.

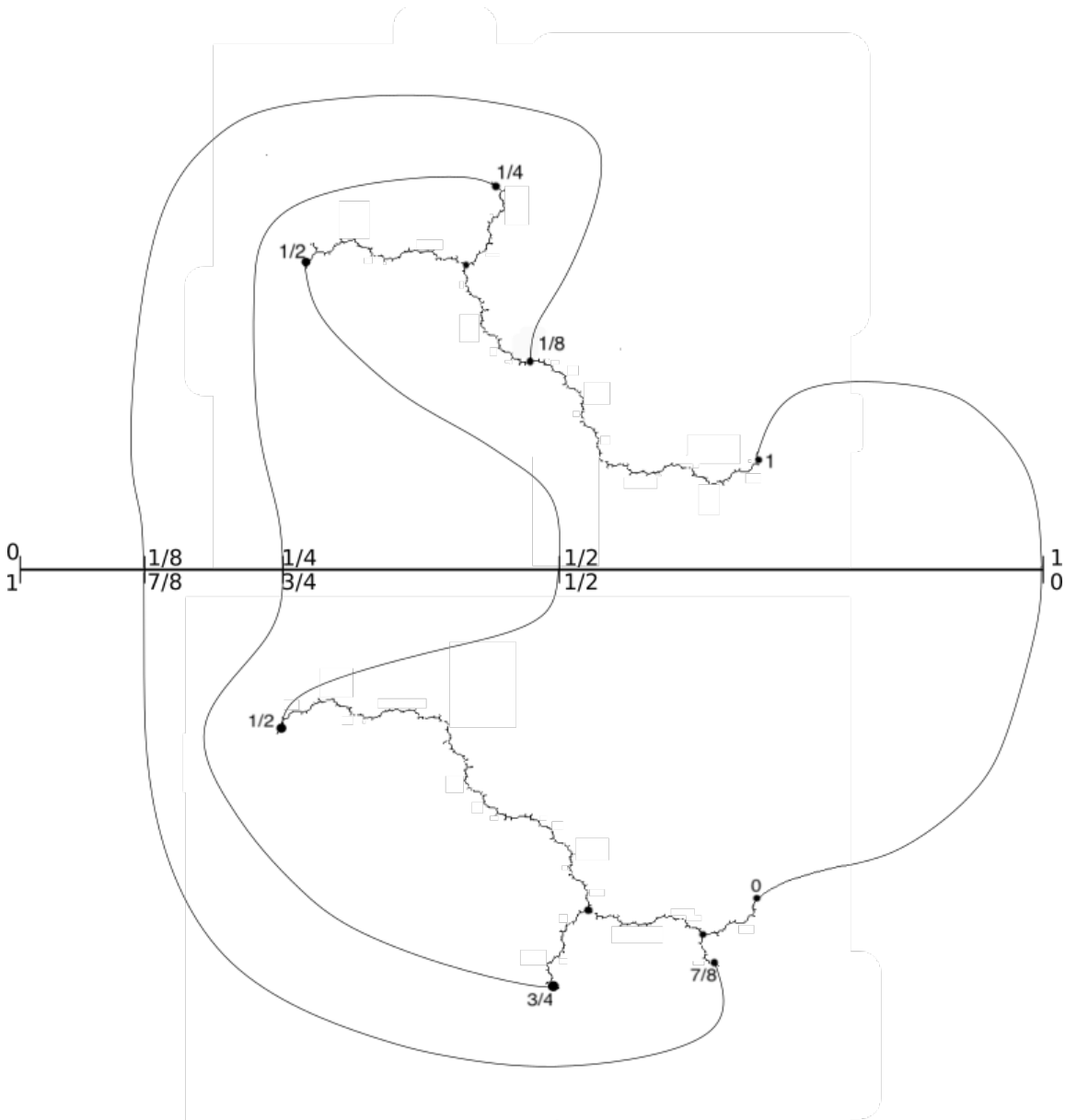


Figure 2.14: The domain of the formal mating $F = f_{1/4} \perp\!\!\!\perp f_{1/4}$ realized by a finite subdivision rule. That is, the model subdivision complex $S_{\mathcal{R}}$ of $\mathcal{R}$. We show the equator. The arguments above the equator correspond to the arguments for the top hemisphere's circle at infinity and the arguments below the equator correspond to the arguments for the bottom hemisphere's circle at infinity.

**Definition. (Construction 1)** Given a degenerate mating $G = f \perp\!\!\!\perp_d g$, we designate the cell decomposition on the domain $S^2_{f,g}$ by identifying a subcollection of classes under $\sim_{\text{degen}}$. Identify the classes of $\sim_{\text{degen}}$ that contain at least two points on the Hubbard trees of $f$

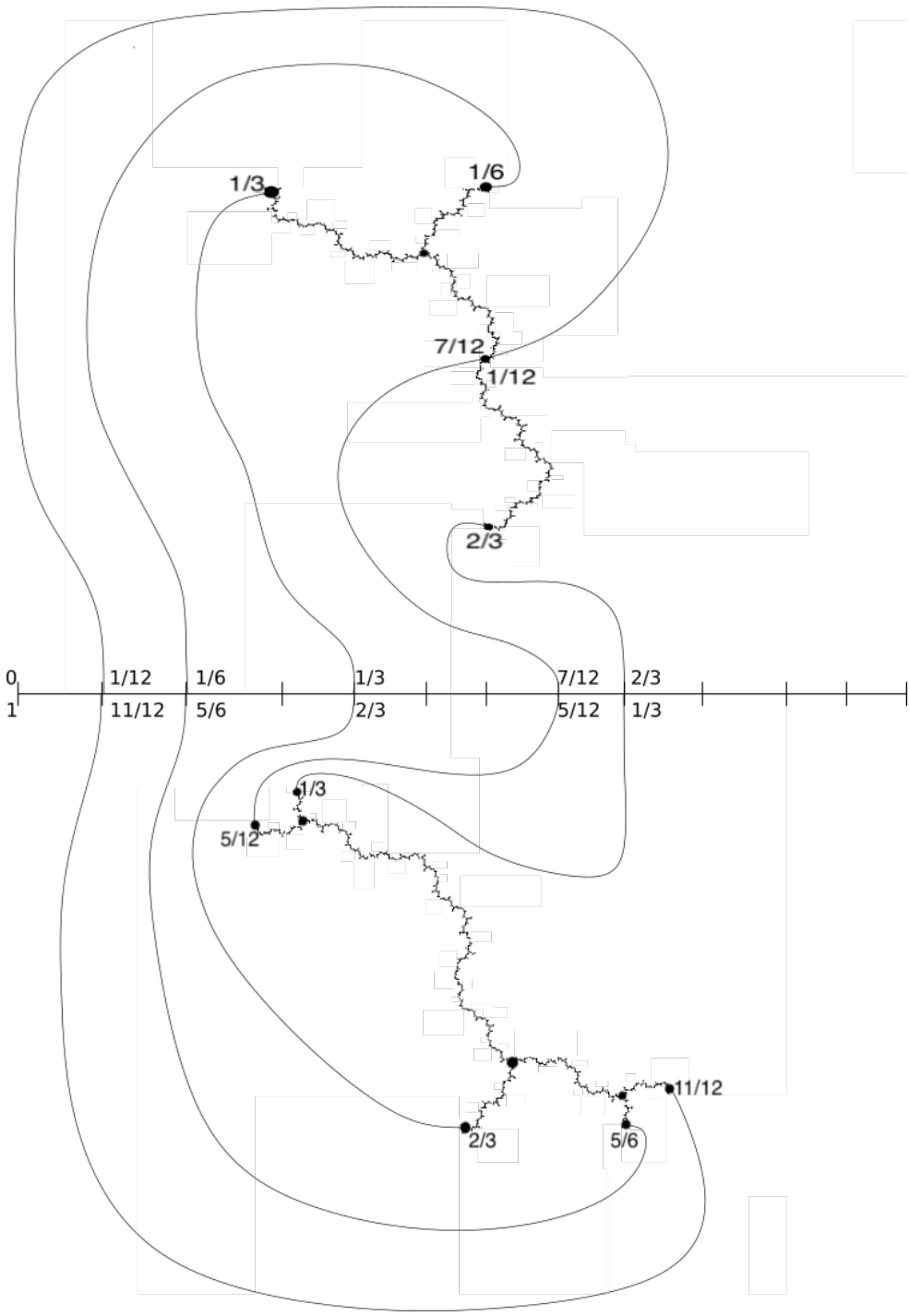


Figure 2.15: A depiction of the equivalence classes of $\sim_{\text{degen}}$ seen in the domain of the formal mating $F = f_{1/6} \perp\!\!\!\perp f_{1/6}$.

and $g$. The vertices are the classes of post-critical points and branch points of the formal mating $F = f \perp\!\!\!\perp g$. The edges are naturally determined using the standard graph structure of the Hubbard trees of $f$ and $g$ (refer to Section 1.3.3). The 1-skeleton is then the union of the Hubbard trees of $f$ and $g$ in $S^2_{f,g}$ identified under the above specification. The CW complex determined in this way is the model subdivision complex $S_{\mathcal{R}}$. If the complex determined by taking the preimage of $S_{\mathcal{R}}$ under $G$ yields a subdivided complex $\mathcal{R}(S_{\mathcal{R}})$ and if $G : \mathcal{R}(S_{\mathcal{R}}) \to S_{\mathcal{R}}$ is a subdivision map, we call the finite subdivision rule $\mathcal{R}$ a finite subdivision rule of **Construction 1**.

Wilkerson provides necessary and sufficient conditions for when this construction yields a finite subdivision rule which realizes the degenerate mating.

**Theorem 2.4.3.** *Let $K = \{y_1, y_2, \ldots, y_n\} \subset H(f) \sqcup H(g)$, $n > 1$, with $y_i \sim_{degen} y_j$ for $i, j \in 1, \ldots, m\}$ be a collection of points. We call such collections of points* ***point groupings****. We call a collection of points $x_1, x_2, \ldots, x_m \in (f \perp\!\!\!\perp g)^{-1}(K)$ such that $x_i \sim_{ray} x_j$ for all $i, j \in \{1, \ldots, m\}$ a* ***preimage point grouping*** *of $K$. We say that a preimage point grouping $\{x_1, x_2, \ldots, x_m\}$* ***satisfies condition (∗)*** *if and only if $\{x_1, x_2, \ldots, x_m\}/\sim_{degen}$ is a single point. Then, Construction 1 admits a finite subdivision rule with subdivision map $G = f \perp\!\!\!\perp_d g$ if and only if $\sim_{degen}$ identifies some point of $H(f)$ with some point of $H(g)$ and each preimage point grouping of all identified point groupings on $H(f) \sqcup H(g)$ satisfies condition $(*)$.*

This theorem establishes that, in order to obtain a finite subdivision rule under Construction 1, one must have that the preimages of points on the Hubbard trees of $f$ and $g$ are identified under $\sim_{\text{degen}}$. That is, for points on the Hubbard tree, one needs the relation $\sim_{\text{degen}}$ to be preserved under preimages. We illustrate an instance in which this construction successfully yields a finite subdivision rule realizing a mating using the degenerate mating of $f_{1/6}$ and itself. We will show that Construction 1 for this example satifies the conditions of Theorem 22.

**Example 2.4.4.** Let $F = f_{1/6} \perp\!\!\!\perp f_{1/6}$ and let $G = f_{1/6} \perp\!\!\!\perp_d f_{1/6}$. We saw in Example 21 the equivalence classes of $\sim_{\text{degen}}$. The subset of these classes which identify points on the two copies of the Hubbard tree of $f_{1/6}$ are $R_F(1/3)$ and $R_F(2/3)$. Taking preimages, we obtain the preimage point groupings corresponding to the landing points $\gamma_F(1/6)$ and $\gamma_F(1/3)$. Both of these landing points of $F$ are identified under $\sim_{\text{degen}}$ and, therefore, each of the preimage point groupings of all of the point groupings satisfy condition $(*)$ and we thus obtain a finite subdivision rule realizing $G$ under Construction 1.

We depict the model subdivision complex obtained from this construction below. Compare the identifications being made by the degenerate mating and note that we are only depicting the pertinent identifications and elements of Construction 1. Namely, we don't bother to depict the identifications that are made by $\sim_{\text{degen}}$ taking place outside the two copies of the Hubbard tree of $f_{1/6}$.

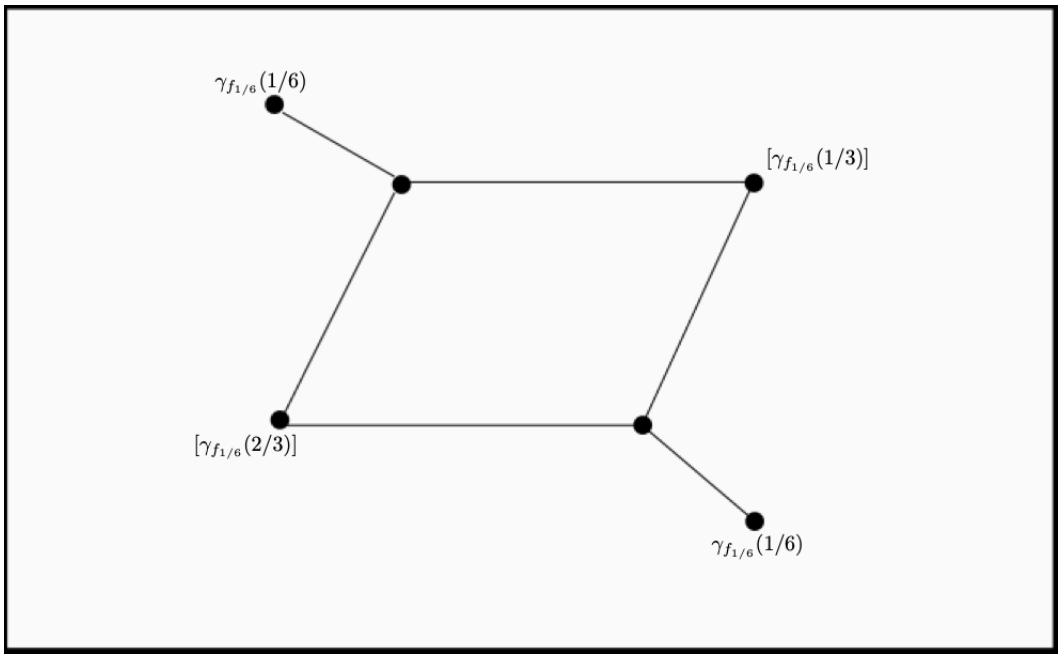


Figure 2.16: The domain of the degenerate self mating $G = f_{1/6} \perp\!\!\!\perp f_{1/6}$

# Chapter 3

# Main Results

## 3.1 Core Entropy of Finite Subdivision Rules: Misiurewicz Polynomials

In this section, we prove the following result.

**Theorem 3.1.1.** *Let $p$ be a Misiurewicz polynomial map with Julia set $J(p)$, Fatou set $F(p)$, and Hubbard tree $H(p)$. Let $\mathcal{R}$ be a finite subdivision rule realizing $p$, let $S^1_{\mathcal{R}}$ be the 1-skeleton of the model subdivision complex $S_{\mathcal{R}}$, and let $\varphi = p|_{S^1_{\mathcal{R}}}$ and let $\tau = p|_{H(p)}$. If for each open edge $e$ of $S_{\mathcal{R}}$, either $e \subset J(p)$ or $e \subset F(p)$, and the union of the edges in $J(p)$ is connected, then $h(\varphi) = h(\tau)$ or $h(\varphi) = 0$. In particular, if all open edges $e$ of $S_{\mathcal{R}}$ are such that $e \subset F(p)$, then $h(\varphi) = 0$. If there is an open edge $e$ of $S_{\mathcal{R}}$ such that $e \subset J(p)$ and the union of the edges contained in $J(p)$ is connected, then $h(\varphi) = h(\tau)$.*

Upon considering Thurston's core entropy, it is natural to wonder why the core chosen is the Hubbard tree. That is, it is natural to consider whether there might exist other conceivable 1-dimensional subsystems, or cores, associated to the systems defined by iteration of Misiurewicz polynomials such that the topological entropy of such restrictions yields something more interesting than $\log(2)$.

The 1-skeleta of model subdivision complexes of finite subdivision rules which realize Misiurewicz polynomials comprise a general class of 1-dimensional subsystems. This result tells us that for a large class of viable *cores* associated to a Misiurewicz polynomial, one will not obtain new possibilities for the entropy. As discussed in the preliminaries, one particular way to construct a finite subdivision rule realizing a Misiurewicz polynomial is by adding to its Hubbard tree, equipped with a standard graph structure, external rays so as to construct a model subdivision complex which recursively subdivides under preimages of the Misiurewicz polynomial map. Obtaining Thurston's core entropy for such constructions makes core entropy of finite subdivision rules a generalization of Thurston's core entropy.

We now establish the relation between the core entropy of a finite subdivision rule and the Perron eigenvalue of its edge recursion matrix. Let $\mathcal{R}$ be a finite subdivision rule realizing a Misiurewicz polynomial. We start by considering the dynamics on the edges in the Fatou set.

For Misiurewicz polynomials, the Fatou set coincides with the attracting basin of $\infty$. Therefore, all points along the edges contained in the Fatou set converge to $\infty$ under iteration. By Proposition 2.2.4, we see that the topological entropy of such a system must equal the topological entropy of the system $\infty \mapsto \infty$ which is 0. We will see in the proof of the main result, the edge recursion matrices for such systems have a Perron eigenvalue of 1. Hence, the logarithm of the Perron eigenvalue of the edge recursion matrix for a system of edges in the model subdivision complex contained in the Fatou set is also 0. Thus, in this setting, the core entropy coincides with the logarithm of the Perron eigenvalue of the edge recursion matrix. We now consider systems of edges of a model subdivision complex contained in the filled Julia set. We will assume that the union of the edges comprising such systems is connected. Since the Julia sets of Misiurewicz polynomials are dendrites, this gives us a dynamical system on a tree in the Julia set.

A theorem of [3] (Theorem 2.9) gives us that the topological entropy of such a system equals the logarithm of the Perron eigenvalue of the edge recursion matrix provided we mark the critical point as a vertex. Due to the forward-invariance of the Fatou and Julia sets, the edges of the model subdivision complex contained in the Fatou set will never subdivide into subedges whose types are identified with edges in the Julia set and vice versa. This manifests as a decomposition of the edge recursion matrix into blocks corresponding to edges in the Julia set and edges in the Fatou set. We can therefore equate the core entropy of a finite subdivision rule with the logarithm of the Perron eigenvalue of the edge recursion matrix and we will exploit this equivalence in the proof of the main result by using both quantities as distinct forms of core entropy.

We begin by proving the following lemma that will be used in proving Theorem 3.1.1 when (b) is assumed to hold.

**Lemma 3.1.2.** *Let $p$ be a Misiurewicz polynomial map realized by a finite subdivision rule $\mathcal{R}$ with model subdivision complex $S_{\mathcal{R}}$, subdivided complex $\mathcal{R}(S_{\mathcal{R}})$, and Julia set $J(p)$. If $e$ is an open edge of $S_{\mathcal{R}}$ and $e \subset \hat{\mathbb{C}} \setminus J(p)$, then $e$ does not properly subdivide.*

**Proof.** We first note that it is clear, because of the Carathéodory seminconjugacy, for any $x \in \hat{\mathbb{C}} \setminus J(p)$ with $x \neq \infty$, the orbit $\{p^n(x)\}_{n\in\mathbb{N}}$ is an infinite collection of points in $\hat{\mathbb{C}} \setminus J(p)$. Let $e$ be an open edge of $S_R$. We proceed by contradiction and suppose that $e$ properly subdivides. By definition, this means there exists an open 0-cell, that is, a vertex $v$ of $\mathcal{R}(S_{\mathcal{R}})$ such that $v \in e$, hence $v \in \hat{\mathbb{C}} \setminus J(p)$. Since $p$ is cellular, this means $p(v)$ is a vertex of $S_{\mathcal{R}}$. We first observe that this implies that $v \neq \infty$. This is because $p(\infty) = \infty$ and therefore, $\infty$ would be a vertex of $S_{\mathcal{R}}$ and therefore cannot be in an open edge of $S_{\mathcal{R}}$. Since every vertex of $S_{\mathcal{R}}$ is also a vertex of $\mathcal{R}(S_{\mathcal{R}})$ and $p$ is cellular with respect to $\mathcal{R}(S_{\mathcal{R}})$ and $S_{\mathcal{R}}$, it follows

that $p^n(v)$ for all $n \in \mathbb{N}$ is a vertex of $S_R$. Therefore, since $S_{\mathcal{R}}$ is a finite complex, this implies $\{p^n(v)\}_{n\in\mathbb{N}}$ is a finite set. However, because $v \in \hat{\mathbb{C}} \setminus J(p)$ and $v \neq \infty$, we have from the above note, that $\{p^n(v)\}_{n\in\mathbb{N}}$ must be an infinite set. Thus, we have a contradiction, and so open edges in $\hat{\mathbb{C}} \setminus J(p)$ do not properly subdivide. ■

Lemma 3.1.2 will be invoked when we are assessing a 1-skeleton whose open edges are all contained in the Fatou set of a Misiurewicz polynomial. We will now show that, in the setting of Theorem 3.1.1, if there exists an edge of the model subdivision complex contained in the Julia set of the polynomial and if the union of the edges contained in the Julia set is connected, then the Hubbard tree of that polynomial must necessarily be contained in the 1-skeleton of $S_{\mathcal{R}}$. The proof of this lemma will make use of Proposition 2.3.8.

**Lemma 3.1.3.** *Let $p$ be a Misiurewicz polynomial realized by a finite subdivision rule $\mathcal{R}$ with subdivision complex $S_{\mathcal{R}}$, Julia set $J(p)$, and Hubbard tree $H(f)$. If an open edge $e$ in $S_{\mathcal{R}}$ is contained in $J(p)$ and the union of the edges of $S_{\mathcal{R}}$ contained in $J(p)$ is connected, then $H(f) \subset S^1_{\mathcal{R}}$.*

**Proof.** Let $e$ be an open edge of $S_{\mathcal{R}}$ such that $e \subset J(p)$. Let $c_0$ be the critical point of $p$. The Hubbard tree is the smallest closed connected subset of $J(p)$ that contains the post-critical set $\mathcal{P} = \{p(c_0), p^2(c_0), \dots\}$. We will show that some iterated image of $e$ must contain $\mathcal{P}$ thereby showing, by the minimality of $H(p)$, that $H(p) \subseteq S^1_{\mathcal{R}}$. Let $v, v'$ be the vertices of $e$. Note that $v$ and $v'$ must be (pre)periodic under $p$ because there are only finitely many vertices of $S_{\mathcal{R}}$. We start by showing that the iterates of $v$ and $v'$ under $p$ must, at some point, be on opposite sides of $c_0$. Removing $c_0$ from $J(p)$ yields exactly two components. Let $T_0$ and $T_1$ be the closures of the connected components of $J(p)$ obtained by removing $c_0$. Suppose, for the sake of contradiction, that for all $n \in \mathbb{N}_0$ either

$$(p^n(v) \in T_0 \text{ and } p^n(v') \in T_0) \text{ or } (p^n(v) \in T_1 \text{ and } p^n(v') \in T_1).$$

That is, assume that for each iterate, the vertices of $e$ are in the same component, either $T_0$ or $T_1$. By Proposition 2.3.8, this implies that $v = v'$. Therefore, the closure of $e$ is not simply connected in $J(p)$, i.e. there is some subset of $J(p)$ homeomorphic to a circle. Since $p$ is a Misiurewicz polynomial, $J(p)$ is a dendrite, and hence, we have a contradicition as we cannot have a loop in a dendrite. Therefore, there must exist an $n \in \mathbb{N}_0$ such that $p^n(v)$ is in the closure of one component of $J(p) \setminus \{c_0\}$, say, $T_0$, and $p^n(v')$ is in the other. Since all iterates of $e$ must be contained in $S^1_{\mathcal{R}}$ and $S^1_{\mathcal{R}}$ is path connected, we see that there must be a path $\pi$ in $S^1_{\mathcal{R}}$ connecting the iterates $p^n(v)$ and $p^n(v')$. This path $\pi$ must also be contained in some iterated image of $e$, $p^m(e)$. The path $\pi$ must contain $c_0$ and therefore $S^1_{\mathcal{R}}$ must contain $c_0$ and, by forward-invariance, some iterates of $e$ must contain $\mathcal{P}$. Therefore, the union of all edges contained in $J(p)$ is a closed connected subset of $J(p)$ which contains the post-critical set. Therefore, we have, by the minimality of $H(p)$, $H(p) \subset S^1_{\mathcal{R}}$. ■

We now prove Theorem 3.1.1. We briefly describe the structure of the proof. Let $X_1 = S^1_{\mathcal{R}} \cap J(p)$ and let $X_2 = \overline{S^1_{\mathcal{R}} \cap F(p)}$. $X_1$ and $X_2$ are closed subsets of $S^1_{\mathcal{R}}$ invariant under $\varphi$.

Let $\varphi_1 = \varphi|_{X_1}$ and let $\varphi_2 = \varphi|_{X_2}$. By Proposition 2.2.3, we then have that

$$h(\varphi) = \max\{h(\varphi_1), h(\varphi_2)\}.$$

We first dispense with the case in which all open edges of $S^1_{\mathcal{R}}$ are contained in $F(p)$. In this case, we show that $h(\varphi) = 0$. Therefore, if there exists an open edge of $S_{\mathcal{R}}$ contained in $J(p)$, by the equality above, we will have $h(\varphi) = h(\varphi_1)$. We then move onto assessing $h(\varphi_1)$. We know in this case that $H(p) \subseteq X_1$. We will let $V = \{v \in S_{\mathcal{R}} : v \text{ is a vertex}, \varphi^n(v) \notin H(p) \text{ for all } n \in \mathbb{N}_0\}$, $Y = H(p) \cup V$, and let $\tau = \varphi|_Y$. We will use Proposition 2.2.4 to show that $h(\varphi_1) = h(\tau)$. We will then show that $h(\tau)$ is Thurston's core entropy by an application of Proposition 2.2.3.

**Proof.** (of Theorem 3.1.1) We start by showing that if all of the open edges of $S_{\mathcal{R}}$ are contained in $F(p)$, then $h(\varphi) = 0$.

Suppose all of the open edges of $S_{\mathcal{R}}$ are contained in $F(p)$. By Lemma 3.1.2, none of the edges of $S_{\mathcal{R}}$ can properly subdivide. We label the edges of $S_{\mathcal{R}}$ $1, \ldots, m$ and let $A$ be the edge recursion matrix of $\mathcal{R}$. Recall that the edge recursion matrix $A = (a_{ij})$ is defined by $a_{ij}$ is the number of subedges, that is, edges of the subdivided complex $\mathcal{R}(S_{\mathcal{R}})$, of type $i$ that are in edge $j$, where $i, j \in \{1, \ldots, m\}$. Since no edge properly subdivides, for each $j \in \{1, \ldots, m\}$, there exists a unique $i \in \{1, \ldots, m\}$ such that $a_{ij} = 1$. The rest of the entries of $A$ are 0. We consider what this means for the graph of the edge recursion matrix $G_A$. Since each column of $A$ has one and only one nonzero entry of 1, each vertex of $G_A$ has exactly 1 incoming edge. Further, there can only be $m$ edges in $G_A$. We now consider what this implies about the communication classes of $G_A$.

First, for a communication class consisting of a single vertex, there is either an edge from that vertex to itself, or there is not. The irreducible component corresponding to such a class would then be a $1 \times 1$ matrix with entry 1 or 0. Now suppose $C$ is a communication class of $G$ whose vertex set consists of at least 2 vertices. Let $V_C$ and $E_C$ be the vertex and edge sets of $C$, respectively. We will show that every vertex of $C$ must be on exactly one cycle. Let $v_1, v_2 \in V_C$. Since $v_1$ and $v_2$ are in the same communication class, there must exist a path starting from $v_1$ ending at $v_2$ and a path starting at $v_2$ ending at $v_1$. Concatenating one path after the other yields a loop in $G$. Let $\pi = e_1 e_2 \cdots e_k$ be one such loop, where $i(e_1) = v_1$ and $t(e_k) = v_1$. We note that since there can only be exactly one incoming edge for each vertex in $G$, we have $i(e_j) \neq t(e(j))$ for all $j \in \{1, \ldots, k\}$, for if there was a self-loop at any vertex along the loop $\pi$, the number of incoming edges at that vertex would be greater than 1. The loop $\pi$ cannot exit the communication class $C$. Since the loop must return, if the loop were to exit the communication class, the communication class would be enlarged by whichever vertices it encountered outside this class. Thus, so far, we have that $\pi$ is a loop without any self-looping edges such that each vertex encountered along this loop is in the communication class $C$. All that is left to show is that every vertex of $C$ is on this loop. For the sake of contradiction, suppose there was a vertex $u \in C$ such that $u$ is neither the terminal nor initial

vertex for any of the edges in $\pi$. Let $v_j$ be any vertex along the path $\pi$. There must exist a path starting from $u$ ending at $v_j$ and a path starting from $v_j$ ending at $u$. The vertex $v_j$ already has an incoming edge from $\pi$ and since $u$ is not along $\pi$, any path from $u$ to $v_j$ must then increase the number of incoming edges at $v_j$. We have a contradiction, and therefore, every vertex of $C$ must be on a cycle. The uniqueness of this cycle is clear again from the assumption that there is exactly one incoming edge for each vertex. Thus, every vertex of a communication class exists on a unique cycle.

It is clear that the adjacency matrices for such communication classes are permutation matrices. That is, there must exist exactly one nonzero entry of 1 in each row and in each column. The eigenvalues of a permutation matrix are roots of unity, and, in particular, include 1. The Perron eigenvalue of $A$ is therefore 1 and the core entropy is therefore 0.

We have now established that, if all open edges of $S_{\mathcal{R}}$ are contained in $F(p)$, then $h(\varphi) = 0$. In general, this may not be the case. Let $X_1 = S^1_{\mathcal{R}} \cap J(p)$ and let $X_2 = \overline{S^1_{\mathcal{R}} \cap F(p)}$. Both $X_1$ and $X_2$ are closed subsets of $S^1_{\mathcal{R}}$. By the invariance of $J(p)$ and $F(p)$ under $p$, we have that $\varphi(X_1) \subseteq X_1$ and $\varphi(X_2) \subseteq X_2$. Let $\varphi_1 = \varphi|_{X_1}$ and let $\varphi_2 = \varphi_{X_2}$. By Proposition 2.2.3, we have that

$$h(\varphi) = \max\{h(\varphi_1), h(\varphi_2)\}.$$

We see that the system $(X_2, \varphi_2)$ is just like the system we considered in the first part of this proof and so $h(\varphi_2) = 0$. Therefore, if there exists an open edge of $S_{\mathcal{R}}$ contained in $J(p)$, the above equation becomes $h(\varphi) = h(\varphi_1)$. We now analyze the topological entropy of $\varphi_1$. We will show that $h(\varphi_1)$ is equal to Thurston's core entropy. We start by defining a closed invariant subset $Y \subseteq X_1$ such that all of the points $x \in X_1$ tend, under iteration, to $Y$.

Let $V = \{v \in S_{\mathcal{R}} : v \text{ is a vertex of } X_1 \text{ and } \varphi^n(v) \notin H(p) \text{ for all } n \in \mathbb{N}_0\}$. That is, $V$ is the set of vertices of the model subdivision complex which do not end up in $H(p)$ after some number of iterations of $p$. Let $Y = H(p) \cup V$. $Y$ is a closed subset of $X_1$ such that $\varphi_1(Y) \subset Y$. We now show how another application of Proposition 2.2.3 tells us that $h(\varphi_1|_Y) = h(p|_{H(p)})$.

Let $\tau = \varphi|_Y$, $Y_1 = H(p)$, and $Y_2 = V$. Let $\tau_1 = \tau|_{Y_1}$ and let $\tau_2 = \tau|_{Y_2}$. Both $Y_1$ and $Y_2$ are closed subsets of $Y$ and $\tau(Y_1) \subseteq Y_1$ and $\tau(Y_2) \subseteq Y_2$. By Proposition 2.2.3, we have that $h(\tau) = \max\{h(\tau_1, h(\tau_2)\}$. The system $(Y_2, \tau_2)$ is clearly such that $h(\tau_2) = 0$. Indeed, it is a discrete set of points comprised of a finite collection of (pre)periodic orbits and so an analysis similar to the one made in the beginning of this proof applies (i.e. take a Markov partition comprised of each individual orbit). Therefore, $h(\tau) = h(\tau_1) = h(p|_{H(p)})$. Now we finish the proof by showing how Proposition 2.2.4 implies that $h(\varphi_1) = h(\tau)$.

$Y$ is a closed subset of $X_1$ and $\varphi_1(Y) \subseteq Y$. The proof of Lemma 2.3 actually demonstrates that the images of an open edge of $S_{\mathcal{R}}$ contained in $J(p)$ eventually intersects the Hubbard tree. Due to the expansive nature of the dynamics of $p|_{H(p)}$, we see that for each $x \in X_1$ the iterates $\varphi_1^n(x)$ must tend to $Y$ uniformly on compact subsets of $X_1 \setminus Y$. Therefore, Proposi-

tion 2.2.4 and the preceding paragraph give us $h(\varphi_1) = h(\tau) = h(p|_{H(p)})$. ■.

This theorem shows, in particular, that a finite subdivision rule realizing a Misiurewicz polynomial, whose edges satisfy reasonable hypotheses, possibly vacuously, has core entropy value that is either Thurston's core entropy or 0. We note that we could replace the assumption that the union of the edges of the model subdivision complex contained in the Julia set is connected with the weaker assumption of $H(p) \subseteq S_{\mathcal{R}}$. In the following section, we examine the core entropy of maps that are not among the family of maps for which core entropy was originally defined. Recall, Thurston's core entropy of a post-critically finite polynomial is the topological entropy of the polynomial map's restriction to its Hubbard tree. We consider the core entropy of matings of Misiurewicz polynomials.

## 3.2 The Core Entropies of Quadratic Matings

In this section, we apply the ideas in the proof of our main result to prove the following result. Let $f$ and $g$ be quadratic Misiurewicz polynomials.

**Theorem 3.2.1.** *Let $F = f \perp\!\!\!\perp g$ be the formal mating of $f$ and $g$ and let $G = f \perp\!\!\!\perp_d g$ be the degenerate mating of $f$ and $g$. Let $\mathcal{R}$ be a finite subdivision rule realizing $F$ of Construction 0 and let $\mathcal{R}'$ be a finite subdivision rule of Construction 1 realizing $G$. Let $h(f)$ and $h(g)$ be Thurston's core entropy of $f$ and $g$. Then*

$$h(\mathcal{R}) = max\{h(f), h(g)\} = h(\mathcal{R}').$$

**Proof.** We start with assessing the core entropy of $\mathcal{R}$. Let $\varphi = F|S^1_{\mathcal{R}}$. The 1-skeleton $S^1_{\mathcal{R}}$ is comprised of a finite set of external rays $\{\mathcal{R}_F(t_k)\}_{k=1}^n$ containing all of the forward images of each of its elements, the convex hull of the set of points $P_f \cup \{\gamma_f(t_k)\}_{k=1}^n$, and the convex hull of the set of points $P_g \cup \{\gamma_g(-t_k)\}_{k=1}^n$. The latter two components contain the Hubbard trees of the polynomials $f$ and $g$. As in the proof of Theorem 3.1.1, we are poised to use Propositions 2.2.3 and 2.2.4 to demonstrate the desired result. We first designate a decomposition of $S^1_{\mathcal{R}}$ into invariant subsets.

Let $V = \{v \in S_{\mathcal{R}} : v \text{ is a vertex and } F^k(v) \notin H(f) \cup H(g) \text{ for all } k \in \mathbb{N}_0\}$. Let $Y_1$ be the convex hull of $P_f \cup \{\gamma_f(t_k)\}_{k=1}^n$ in $J(f)$ and let $Y_2$ be the convex hull of $P_g \cup \{\gamma_g(-t_k)\}_{k=1}^n$ in $J(g)$. Let $X_1 = Y_1 \cup Y_2 \cup V$ and let $X_2 = \overline{S^1_{\mathcal{R}} \setminus X_1}$. $X_1$ is the closed set comprised of the Hubbard trees of $f$ and $g$ and the vertices of $S_{\mathcal{R}}$ which do not end up in the Hubbard trees under iteration. $X_2$ is the closed set comprised of the external rays in the 1-skeleton and their landing points. As such, both of these sets are forward invariant under $F$. Let $\varphi_1 = F|_{X_1}$ and let $\varphi_2 = F|_{X_2}$. By Proposition 2.2.3, we have that

$$h(\varphi) = \max\{h(\varphi_1), h(\varphi_2)\}.$$

The dynamics of $(X_2, \varphi_2)$ is a system consisting of the permutation of external rays much like the first part of the proof of Theorem 20. As such, we know its entropy to be 0 and hence we have $h(\varphi) = h(\varphi_1)$.

The dynamics of the formal mating in each hemisphere of its domain is exactly that of a Misiurewicz polynomial. Therefore, as in the proof of Theorem 3.1.1, we can further decompose $X_1$ into $H = H(f) \cup H(g) \cup V$ and its complement $H'$. Since $H$ is a closed invariant subset of $X_1$ and for each point $x \in X_1$, $\varphi_1^k(x) \to H$ as $k \to \infty$, by Proposition 2.2.4, we have $h(\varphi_1) = h(\varphi_1|_H)$. We are in the same position to apply Proposition 2.2.3 once again by decomposing $H$ into $H(f) \cup H(g)$ and $V$. Since the dynamical system of $(V, F|_V)$ permutes a collection of vertices, we see that its entropy is 0. Therefore, we have

$$h(\varphi) = h(\varphi_1) = h(\varphi_1|_H) = h(\varphi_1|_{H(f)\cup H(g)}).$$

Since $H(f)$ and $H(g)$ are disjoint, this amounts to determinining the maximum between $h(F|_{H(f)}) = h(f|_{H(f)})$ and $h(F|_{H(g)}) = h(g|_{H(f)})$ and this is the desired result.

We now assess the core entropy of $\mathcal{R}'$. If there is no distinction between the formal mating and the degenerate mating, then $\mathcal{R} = \mathcal{R}'$. Therefore, in this instance, the above argument applies equally well to demonstrate the result for $\mathcal{R}'$. Assume $R \neq \mathcal{R}'$. We note that the distinction between the model subdivision complexes of $S_{\mathcal{R}}$ and $S_{\mathcal{R}'}$ is that a finite number of external rays connecting points on the Hubbard trees of $f$ and $g$ have been identified. These finitely many identifications do not affect the essence of the argument made above.

In this case, the 1-skeleton is comprised of a pair of Hubbard trees that have been identified in finitely many points. Thus, one may let $X_1 = H(f)$ and $X_2 = H(g)$, let $\varphi_1 = \varphi|_{X_1}$, $\varphi_2 = \varphi|_{X_2}$ and by Proposition 6, we have $h(\varphi) = \max\{h(\varphi_1), h(\varphi_2)\}$. We see, however, that $\varphi_1 = f|_{H(f)}$ and $\varphi_2 = g|_{H(g)}$ and so, without having to employ Proposition 2.2.4, we've obtained the desired result. ■.

## 3.3 Distinct Phenomena for Core Entropy of FSRs

We've shown above, that in the original setting of quadratic Misiurewicz polynomials, core entropy by way of a finite subdivision rule realizing the polynomial coincides with Thurston's core entropy or is 0. Interesting phenomena arise when considering the core entropy of finite subdivision rules which realize maps that are outside this original setting. For example, core entropy of finite subdivision rules is not an invariant of Thurston equivalence. We demonstrate this phenomenon through an example.

**Example 3.3.1.** In this example, we take the formal self-mating $F = f_{1/6} \perp\!\!\!\perp f_{1/6}$. We start by realizing $F$ with a finite subdivision rule $\mathcal{R}$ using Construction 0.

We must choose a finite collection of external rays closed under forward iteration. We choose the rays $\{R_F(1/2), R_F(1)\}$. Marking the landing points $\{\gamma_F(1/2), \gamma_F(1)\}$, branch points, and post-critical points in both copies of $K(f_{1/6})$, we obtain the model subdivision complex of $\mathcal{R}$. We depict the model subdivision complex with its edges numbered in Figure 3.1.

The model subdivision complex has 16 edges. We can determine their subdivisions by noting the forward images of each edge taking care to note that edges 2 and 11 contain critical points of $F$. In Figure 3.1 we have labeled the vertices and we provide a diagram of the orbits of these points below.

$$\begin{tikzcd}
a_0 \arrow[r] & a_1 \arrow[loop right] \\
b_0 \arrow[r] & b_1 \arrow[r] & b_2 \arrow[loop right] \\
c_1 \arrow[r] & c_2 \arrow[r, bend right] & c_3 \arrow[l, bend right]
\end{tikzcd}.$$

We depict the subdivisions for each edge in Figure 3.2. The edge recursion matrix we obtain from the edge subdivision rules above is given by the $16 \times 16$ matrix shown below. We note the block diagonal structure obtained. From top left to bottom right, the blocks correspond to the edge recursion pattern on the Hubbard tree of $f_{1/6}$ in one hemisphere, the external rays, and the Hubbard tree of $f_{1/6}$ in the other hemisphere, respectively.

$$E_1 = \begin{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} & 0 & 0 \\ 0 & \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} & 0 \\ 0 & 0 & \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{bmatrix}$$

The largest real eigenvalue of $E_1$ is approximately 1.52137971. We thus calculate the core entropy to be $h(\mathcal{R}) \approx \ln(1.52137971)$.

We now modify the finite subdivision rule to obtain a clearly Thurston equivalent, in fact, equivalent, map realized by a distinct rule $\mathcal{R}'$. We add the intersection points between the external rays and the equatorial circle at infinity as vertices. This splits each external ray into two pieces adding two more edges. We also include two new edges contained in the equator between these added vertices. We then have the model subdivision complex which is depicted in Figure 3.3.

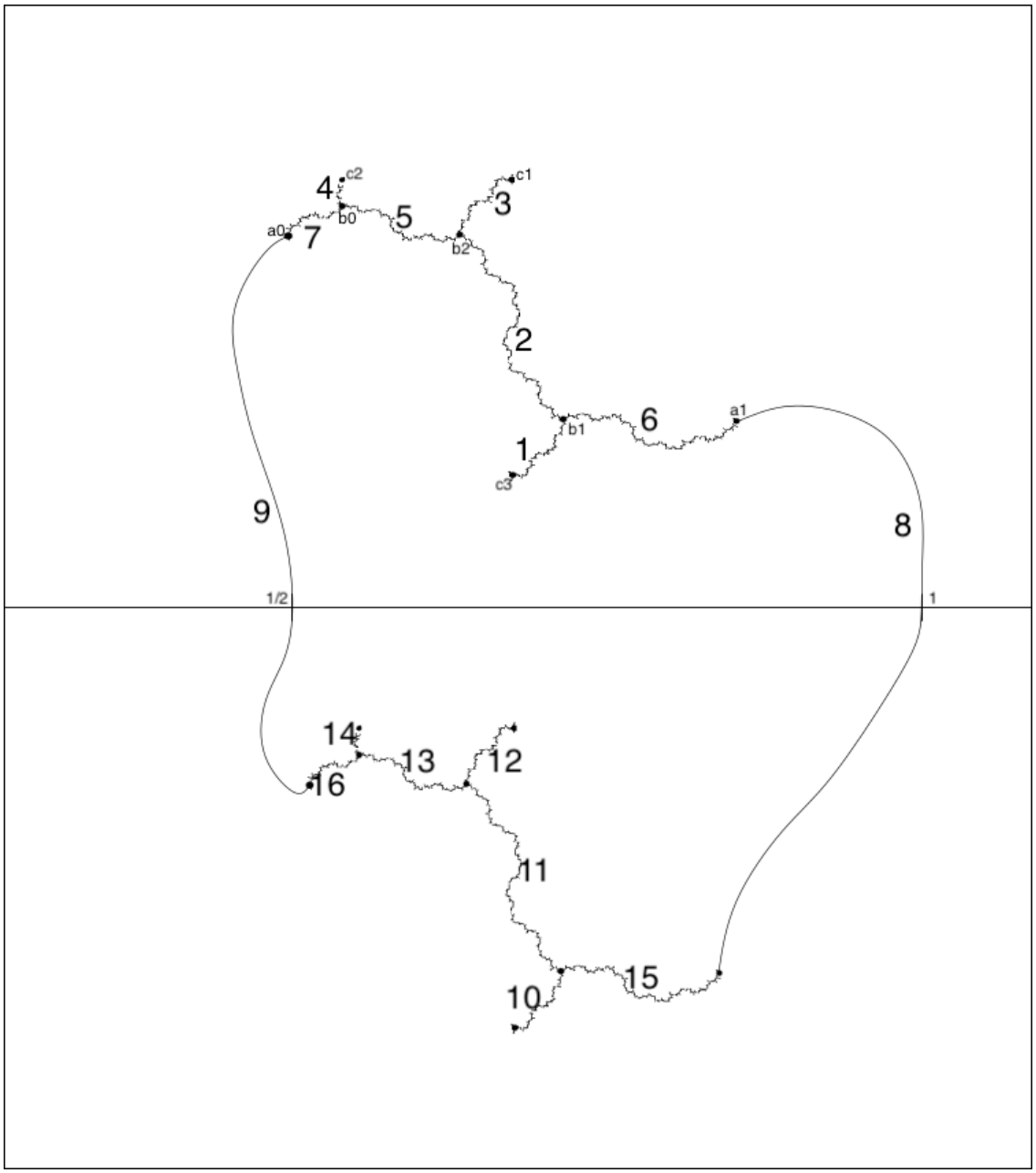


Figure 3.1: The model subdivision complex for $\mathcal{R}$. We show the equator along with the angles of the external angles we've chosen but we do not include it as part of the 1-skeleton of $S_{\mathcal{R}}$.

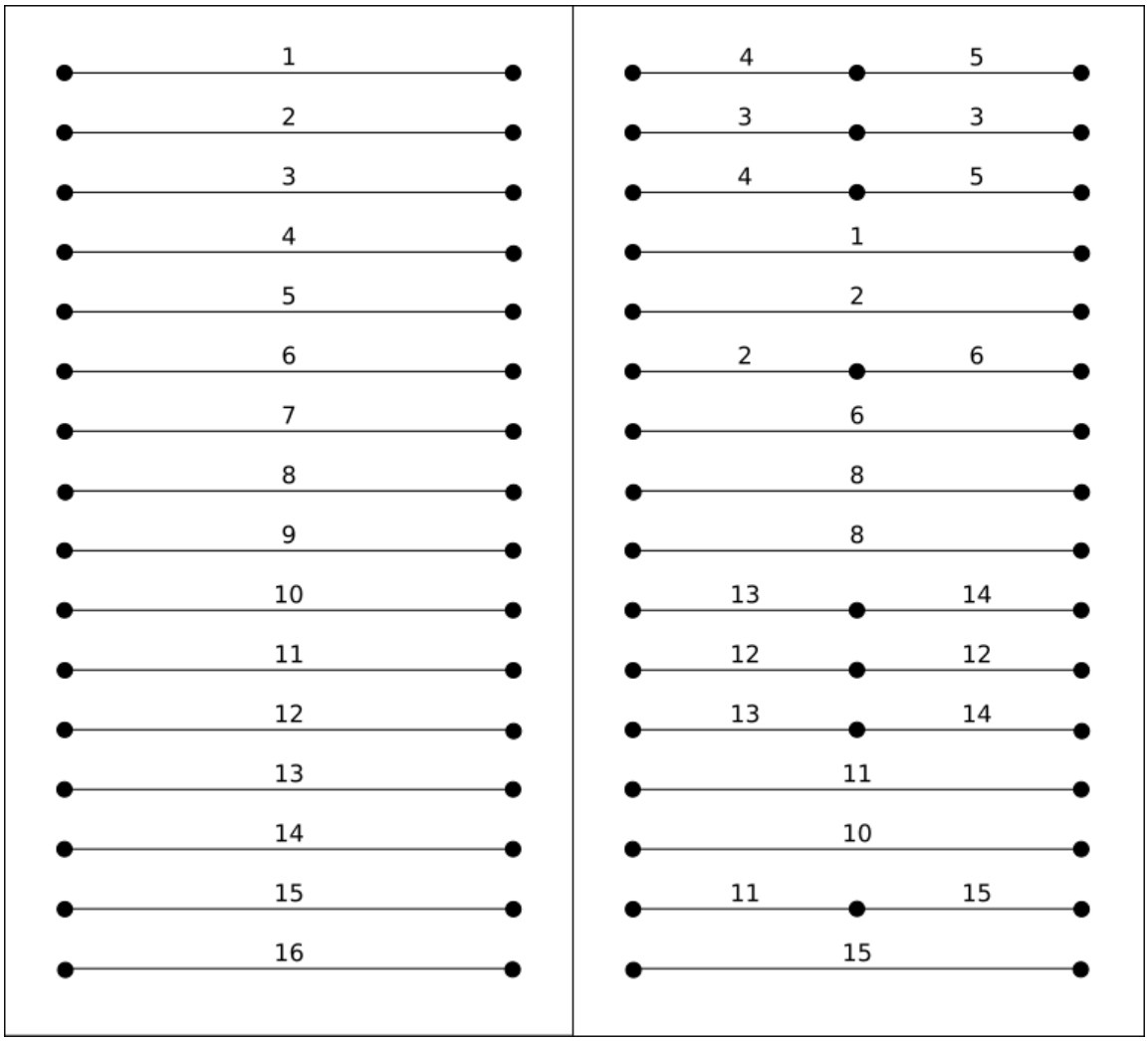


Figure 3.2: The edge subdivisions for each edge type of $\mathcal{R}$. The left panel depicts edge types while the right panel depicts their subdivisions.

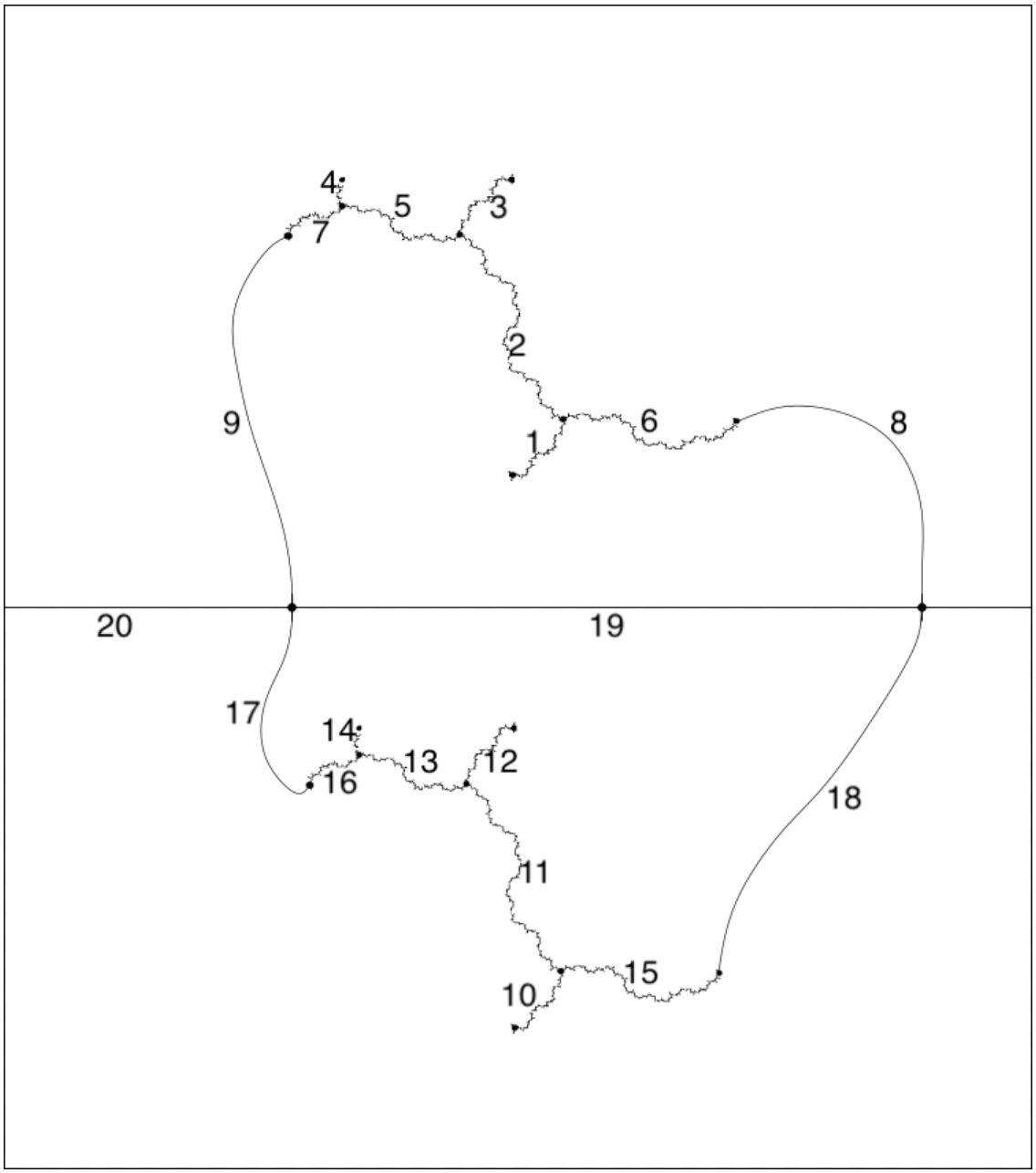


Figure 3.3: The model subdivision complex for $\mathcal{R}'$. We show the equator dashed along with the angles of the external angles we've chosen. We mark the intersection points between equator and external rays as vertices and include the equatorial arcs as edges in $S^1_{\mathcal{R}}$

This model subdivision complex has 20 edges and has edge subdivision rules depicted in Figure 3.4 with its edge recursion matrix following.

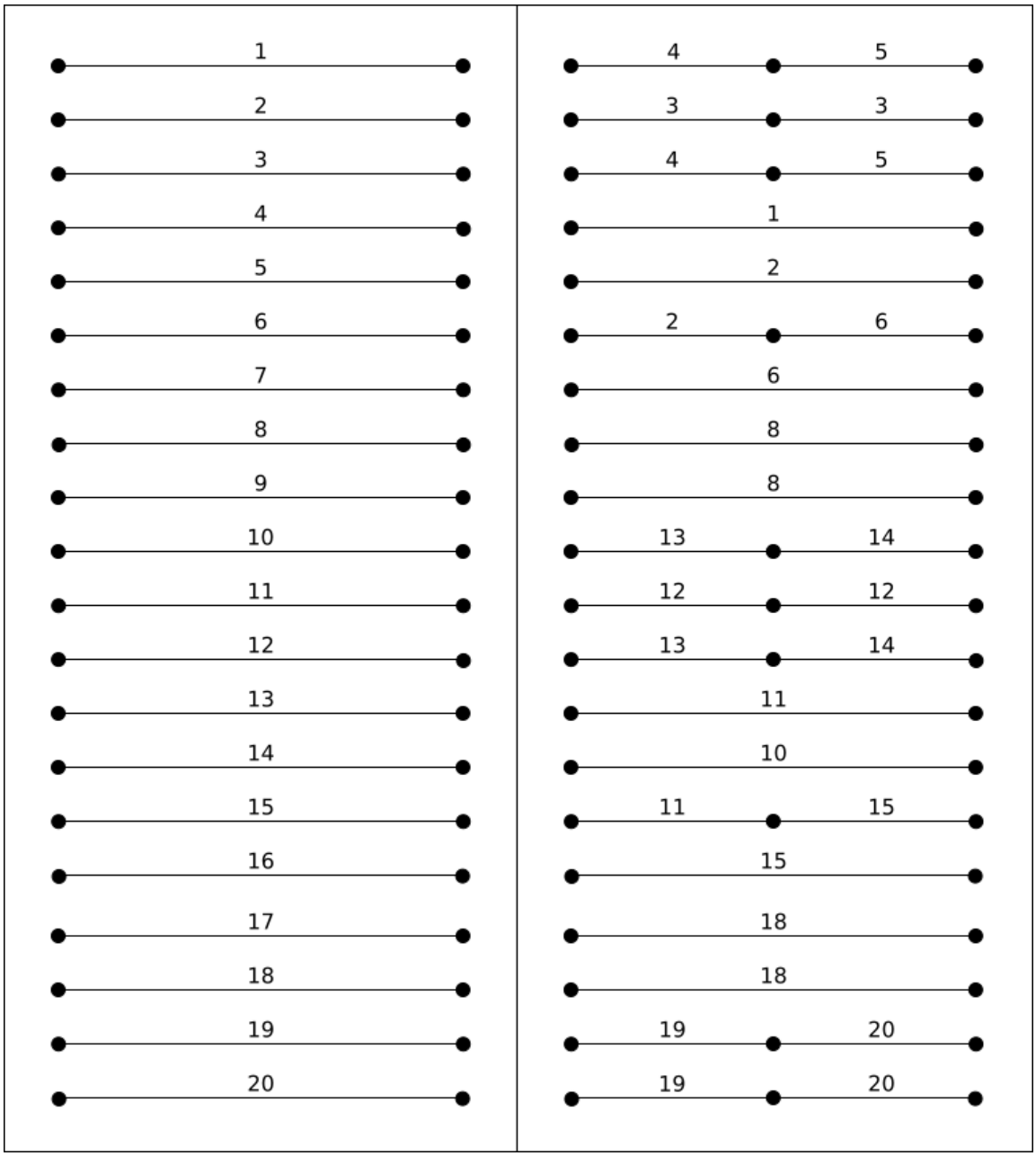


Figure 3.4: The edge subdivisions for each edge type of $\mathcal{R}'$. The left panel depicts edge types while the right panel depicts their subdivisions.

$$E_2 = \begin{bmatrix} \begin{bmatrix} 0&0&0&1&0&0&0\\ 0&0&0&0&1&1&0\\ 0&2&0&0&0&0&0\\ 1&0&1&0&0&0&0\\ 1&0&1&0&0&0&0\\ 0&0&0&0&0&1&1\\ 0&0&0&0&0&0&0 \end{bmatrix} & 0 & 0 & 0 & 0 \\ 0 & \begin{bmatrix} 1&1\\ 0&0 \end{bmatrix} & 0 & 0 & 0 \\ 0 & 0 & \begin{bmatrix} 0&0&0&0&1&0&0\\ 0&0&0&1&0&1&0\\ 0&2&0&0&0&0&0\\ 1&0&1&0&0&0&0\\ 1&0&1&0&0&0&0\\ 0&0&0&0&0&1&1\\ 0&0&0&0&0&0&0 \end{bmatrix} & 0 & 0 \\ 0 & 0 & 0 & \begin{bmatrix} 1&1\\ 0&0 \end{bmatrix} & 0 \\ 0 & 0 & 0 & 0 & \begin{bmatrix} 1&1\\ 1&1 \end{bmatrix} \end{bmatrix}$$

Note, the preimage of $S_{\mathcal{R}'}$ under $F$ yields vertices along the equator corresponding to the external rays of argument $1/4$ and $3/4$. The largest real eigenvalue of $E_2$ is $2 = \deg(F)$. We thus calculate the core entropy of this finite subdivision rule to be $h(\mathcal{R}') = \log(2) \neq h(\mathcal{R})$.

This alteration in particular leaves the Hubbard trees cores alone and yet one obtains distinct core entropy values. Constructing a Thurston equivalence from a Misiurewicz polynomial which leaves the Hubbard tree unchanged yields equivalent Thurston core entropy whereas the same cannot be guaranteed for core entropy of the finite subdivision rules which realize the Misiurewicz polynomial.

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