THE DEVELOPMENT OF THE HYDRAULIC ANALOGY

by

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LIST OF SYMBOLS

Symbol	Definition	Unit
$\bar{v}$ (u,v,w)	velocity vector	ft/sec
ρ	mass density of air	$slugs/ft^3$
h	height of water in channel	ft
T	temperature of gas	degrees Rankine
p	pressure	lb/ft^2
γ	ratio of specific heats $\frac{c_p}{c_v}$	- -
g	gravitational acceleration	ft/sec^2
A	area of cross section	ft^2
R	gas constant	ft/° Rankine
z	height above datum line in channel to a point in the flow	ft
E	specific energy	ft
Q	volume rate of flow	ft^3/sec
n	exponent in defining equation for the shape of the channel	- -
c	velocity of sound in a gas or velocity of propagation of a long gravity wave in the water channel	ft/sec
V	specific volume	ft^3/lb
δ	half of body angle	degrees
ϵ	shock angle	degrees
P	pressure ratio across a shock wave	- -
M	ratio of relative velocity of body in fluid to speed of sound in the fluid	- -

Symbols	Definition	Unit
Subscripts		
a	analogous Mach number	- -
c	refers to critical flow	- -
e	experimental value	- -
o	refers to point in undisturbed flow where velocity is negligible	- -
t	theoretical value	- -

MOTTO:

"Analogy as we know, is a good servant, but a bad master: for, when master, it does more to blind than it may previously have done to illuminate". (James Ward, English educator and philosophical writer.).

I

INTRODUCTION

The analogy presented in this paper enables the research worker to find numerical values which are vitally important for the design of high speed airplanes. Using this analogy not only "unmeasurable" values can be changed into easily measurable ones, but a sharp cut in expenses can also be obtained. Wind tunnel research in the field of supersonic aerodynamics is one of the most expensive techniques known today. An average value of one million dollars can be named for the cost of a wind tunnel useful in this field. A very precise experimental unit could be designed and built for less than $2,000. The experimental unit designed by the author cost even less than the above figure.

The analogy between two dimensional gas flow and flow of liquids in an open channel has been known for thirty years. It has been shown that very useful technical data can be obtained regarding aerodynamic forces acting on high speed aircraft by means of this analogy.

Since only the relative velocities between the model and the fluid have importance, two possible experimental solutions are at hand. Either the towing tank or the circulating water channel could be used.

There are several towing tanks in this country at the present time at experimental stations and educational institutions, but there are many difficulties encountered when testing a model. If a model is held stationary in a moving stream of water, routine work proceeds without interruption except for changing models. The time lost in a towing tank in backing up the carriage and model, and waiting for the waves and currents to subside is saved. For the foregoing reasons and for financial ones, as well as considering the time element involved, a circulating water channel has been selected as the experimental tool.

It is to be noted particularly that the circulating water channel to be built is to supplement the wind tunnel and in no way meant to replace it. By the use of this channel, it is possible to obtain much of the basic data about the component parts of an airplane very easily and inexpensively, thereby making available the wind tunnels for more complicated tests.

Three requirements are of primary importance for the channel used in this project. The free surface should be smooth, the velocity across the test section should be uniform, and turbulence should be reduced to a minimum.

This paper intends to extend the validity of the analogy for high Mach numbers by proposing a "modern hydraulic gas" of $\gamma = 1.5$. The aim of the paper is not to improve the experimental techniques, neither to discuss them, but first of all to investigate theoretically the possibilities of extending the use of the analogy for shock wave problems and problems in the transonic region. In fact, from theoretical investigations a general relationship is derived which correlates the shape of the cross section of the channel with the adiabatic gas constant.

All of the previously designed water channels making use of the analogy and testing aerodynamic bodies, by our knowledge, have been of rectangular cross section. In a channel of this type, the ratio of specific heats $\frac{c_p}{c_v} = \gamma$, is 2, while the accepted value for air is 1.4.

The method presented could be used to design a section which would give the desired value of γ equal to 1.4, but a triangular section ($\gamma = 1.5$) will be used because of fabrication difficulties of the other section. In fact, as a simple mathematical analysis shows, a proper cross section can be found for any desired value of $1 < \gamma \leq 2$. Results obtained are in good agreement with the theoretically predicted values.

The apparatus and method described in the paper enables the scientist to do research in the field of supersonic aerodynamics, it helps the teacher to demonstrate phenomena occurring at high speeds and furnishes the designer with a quick method for checking new ideas.

The author therefore believes that schools, factories and research laboratories could be equipped with the inexpensive and simple unit described in this thesis.

The analogy under investigation did not give numerically accurate results in the past and therefore it was known only as an interesting playtoy. The present paper describes a modified form of the analogy with which quantitative results can be obtained. A form of the analogy has been used very satisfactorily at Pennsylvania State College for the investigation of problems related to internal combustion engines.

II

REVIEW OF THE LITERATURE

The following articles and books were used to establish the theory and experimental technique:

Bakhmeteff, B. A.

"Hydraulics of Open Channels"

New York, McGraw-Hill Book Company (1932)

The general flow equations for uniform and varied flow are developed with special application to flow in rectangular channels. Bakhmeteff introduces the concept of the specific energy of the flow, which is the energy head referred to the bottom of the channel cross section. This concept was used to find the wave velocity. By using this term, it is possible to give physical meaning to the critical depth as well as to give an explanation for the hydraulic jump.

The hydraulic jump is a local phenomenon by means of which flow passes in an abrupt manner from shooting water to streaming water. The theory of the hydraulic jump is developed for a channel of general cross section, and in a later chapter, emphasis is placed on the jump in a rectangular channel. Based on Bakhmeteff's analysis, the concept of specific energy proved to be very useful in this thesis.

Comstock, J. P. and Hancock, C. H.

"The Effect of Size of Towing Tank on Model Resistance"

Transactions of the Society of Naval Architects and Marine Engineers

50, 149-198 (1942)

The purpose of this paper was to determine whether wall effect was due to insufficient breadth or insufficient depth of the water in the towing tank. Each dimension was varied separately as well as simultaneously with the other. This also showed the desirable ratio of breadth to depth. These results were used to design the model with a constant obstruction ratio.

A secondary purpose was to obtain information as to the increase of resistance of a wide variety of models in shallow water of unlimited width.

The experiments were conducted in the 8-foot by 4-foot Newport News model tank. The results were obtained by experimentation and presented in graphical form. The results show the increase in resistance due to the effect of testing the models in shallow water.

Courant, R. and Friedricks, K. O.

"Supersonic Flow and Shock Waves".

London, Interscience publisher (1948)

The vector form of the continuity equation and the Bernoulli equation are presented. The vector form of the continuity equation was used in the derivation of the governing equations of the analogy presented in this thesis.

Equations for the motion of waves in shallow water are derived. It is shown that the differential equations governing the motion of shallow water can be replaced by equations which are equivalent to those for a polytropic gas with the exponent $\gamma = 2$

Ferri, Antonio

"Elements of Aerodynamics of Supersonic Flows"

New York, The Macmillan Company (1949)

The general equations of fluid motion are developed for one, two and three dimensional flows. The theory of shock waves is thoroughly developed and both graphical and numerical methods of solution are given. This information was used in the analysis of the flow characteristics of the model used in this experiment. New material on the reflection and interaction of shock and expansion waves and on the measurement of physical quantities is included.

Several charts of the relations between pressure, temperature and the density for a gas when undergoing a shock or expansion wave are tabulated at the end of the book. This information was used to analyze the results obtained in the channel described in this thesis.

Gilmore, R. R., Plesset, M. S., and Crossley, H. E., Jr.

"The Analogy Between Hydraulic Jumps in Liquids and Shock Waves in Gases"

Journal of Applied Physics 21-3, 243-249, March 1950

The theory of the hydraulic jump is presented briefly and the analogy between this phenomenon and the compression shock wave in gases is pointed out. The results of experimental measurements of hydraulic jump intersections on a water table are reported. Considerable disagreement between theory and experiment is found. The discrepancy in the aerodynamic case appears unlike that found in the hydraulic case. Possible reasons for the discrepancy in the hydraulic case are discussed; some sources of error are peculiar to hydraulic jumps and do not apply to shock waves. Such factors limit the water table as an analog device. This paper clearly shows the necessity of the generalization of the hydraulic analogy.

Hancock, C. H.

"The Equipment and Methods Used in Operating the Newport News Hydraulic Laboratory"

Transactions of the Society of Naval Architects and Marine Engineers 56, 39-69 (1948)

In this paper, the small, circulating water channel at the Newport News Hydraulic Laboratory is discussed. The channel is rectangular in cross section and large enough to test small models. Sufficient power is available to operate the channel at supercritical speeds.

Another device of considerable interest is the shallow circulating water channel. The purpose of this channel was to determine its usefulness in producing shallow water waves analogous to shock waves in the supersonic flow of gases. An analogy was developed between the two types of flow specifically for application to the design of steam turbine nozzles. A short description of the channel is given along with a line diagram of the test equipment. Several photographs of cylindrical models are included. This information was very useful in the design of the experimental unit used in the investigation described in this thesis.

Kelland, P.

"On the Theory of Waves"

Proceedings of the Royal Society of Edinburg 14, 497-532 (1840)

In this paper, Kelland derives the equation for the velocity of propagation of a long gravity wave, $c = \sqrt{gh}$ in a rectangular channel. In addition, the height of the wave in a rectangular channel is derived. The results show the tendency of waves in shallow water to become semicircular in shape.

Kelland discusses in Part II of the paper variable wave motion in a triangular channel. The velocity of the propagation of long gravity waves in a triangular channel was found to be $\sqrt{\frac{gh}{2}}$. This equation was verified by experimental tests and the results were presented in tabular form.

Since the velocity of propagation of long gravity waves is analogous to the propagation of sound in air, this equation was found to be very useful in this thesis.

Orlin, W. J., Lindner, N. J. and Bitterly, J. G.

"Application of the Analogy Between Water Flow with a Free Surface and Two-Dimensional Compressible Gas Flow"

National Advisory Committee for Aeronautics Technical Note No. 1185, (1947)

An analogy exists between water flow with a free surface and two-dimensional compressible gas flow if the water flows over a smooth horizontal bottom bounded by vertical walls which is geometrically similar to the corresponding gas flow.

The limitations and conditions of the analogy are discussed. An experiment was conducted using the analogy as applied to the flow about circular cylinders of various diameters at subsonic velocities and extending into the supercritical range.

Reasonably satisfactory agreement of pressure distributions and flow fields exists between water and air flow about corresponding bodies. The development of the measuring apparatus and techniques is presented along with graphs showing the results of this project.

The thesis can be regarded as a continuation and perfection of this report into the supersonic range.

Preiswerk, Ernst

"Application of the Methods of Gas Dynamics to Water Flows with Free Surface"

Part I. Flows with No Energy Dissipation

National Advisory Committee for Aeronautics, T. M. No. 934 (1940)

Part II. Flows with Momentum Discontinuities (Hydraulic Jumps)

National Advisory Committee for Aeronautics, T. M. No. 935 (1940)

In this paper the author discusses how far the analogy exists between the flow of a liquid in a horizontal bottom channel and the two dimensional flow of a compressible gas. A large portion of the paper is devoted to the general theory of the two dimensional gas flows. Although the analogy was developed for a rectangular channel, several of the basic relationships were useful in the theoretical analysis of the problem presented in this thesis.

It is indicated how problems in the field of water flows may be solved directly by the theory of gas dynamics and vice versa. Both graphical and empirical methods of solving two dimensional compressible gas problems are given. A thorough discussion of the method of characteristics is included.

Flows with no energy dissipation are treated in Part I, and flows with hydraulic jumps are treated in Part II. Both conditions are applicable to this thesis.

Saunders, H. E.

"The Circulating Water Channel of the David W. Taylor Model Basin" Transactions of the Society of Naval Architects and Marine Engineers 52, 325-365 (1944)

This paper was chiefly concerned with the design and fabrication of the circulating water channel. Three models of the channel were built before the actual channel was constructed. Many helpful suggestions are included for anyone who is contemplating constructing a similar channel.

A brief history of the previous work done with circulating water channels is included.

The channel is rectangular in cross section and it is possible to investigate flow phenomena in open and closed channels. At the time of publication of this article, no calibration tests of the channel had been made.

III

ABSTRACT

The classical hydraulic analogy between free surface water flow in a rectangular channel and two dimensional gas flow is only _qualitative_ since the analogy assumes "hydraulic gas" with an adiabatic constant $\gamma = \frac{c_p}{c_v} = 2$, which value for air is 1.4. In the thesis, the analogy is generalized and a mathematical derivation is presented resulting in a relation between the adiabatic gas constant and the cross section of the hydraulic channel used. For $\gamma = 2$, rectangular, for $\gamma = 1.5$ triangular and for $\gamma = 1.4$ parabolic cross section is suggested. Using the latter, the analogy becomes exact and _quantitative_ measurements are possible. The corresponding temperature, pressure and density relations for the gas flow are derived in the general and also in the special cases.

Experimental results using a channel of triangular cross section for subsonic and supersonic flow problems are presented, showing very close agreement with theory.

An analysis is presented regarding the shape of the model which is used in the experimental channel and which satisfies the requirements of constant obstruction ratio and boundary conditions. A report on pressure and shock angle measurements is included in the thesis along with a short review of the literature of the hydraulic analogy. It is shown that for investigating compressible flow problems, the proposed channel is superior to the conventional rectangular channel, especially for problems when the effect of the adiabatic gas constant is important.

IV

GENERAL ANALOGY CONSIDERATIONS

Very often in physical sciences, extremely difficult experimental technique and expensive instrumentation are necessary to study certain phenomena important to the research worker. In other cases experimental methods are not known, which immediately excludes experimentation. If an entirely different physical phenomenon can be investigated instead of the original problem, then the above seemingly unsolvable task of measuring an unmeasurable quantity can be accomplished. The above program can be followed only when a relation is established between two physically distinct phenomena.

Analogy research (from the Greek: αναλογια) is the field where different phenomena are compared and connections are established between them. The procedure generally followed consists of 5 steps.

1. The governing equations are developed for the problem under investigation.
2. The analogous governing equations are written down in which different terms represent entirely different physical quantities.
3. The mathematical analogy between the above two sets is established and a dictionary prepared by means of which, terms, variables, coefficients, operations, etc. in the original system are translated into the language of the analogous system.
4. Experimentation performed on the established analogous system.
5. Translation of experimental results, by means of the dictionary of the analogy into the language of the original system.

A special type of analogy research occurs when the analogous system represents basically the same physical phenomena as the original one. We will refer to this type of analogy as similarity research. Although great possibilities are offered by this method, the same experiment is performed with no exceptions.

The literature shows the popularity of analogies in almost every field of scientific research. Workers in the field discover new and surprising analogies every day. An analogy generally is considered good and useful if it is exact (mathematically speaking, i.e. there is a perfect correspondence between the original and the analogous system) and if the analogous system is considerably simpler than the original one (physically speaking, i.e. easily measurable quantities occur in the analogous system).

In the last year several new and known analogies[15,16,17,18*] have been used and it has been shown that from the two criteria mentioned above (mathematical and physical) an unfortunately neglected but important point should be investigated when an analogy research is designed. In setting up the mathematical formulas of the original problem as well as of the analogous system, several assumptions and simplifications are made. These assumptions and simplifications generally mean neglecting terms which do not seem important in the specific problem. When, however, the generally derived formulas are used in an analogy research, it might happen that the neglected terms are the most important ones. For instance, the equation describing the shape of a soapbubble if approximated with a linear equation (squares and products of slopes

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* Numbers refer to references at the end of the thesis.

are neglected) is used as a powerful tool in analogy research in elasticity. The same soapbubble analogy is useful in the study of compressible flow phenomena only if the exact equation of a soapbubble is compared with the equation describing the flow problem.[16]

In this thesis an analogy is worked out and checked by experimental methods for a problem which is vitally important in the nation's war effort. At the present time, supersonic airplane parts are studied in very expensive wind tunnels under rather unfortunate circumstances. Almost unmeasurable quantities have to be measured with great accuracy. The experimental technique in the field of supersonic flight is still struggling with its most basic problems. An analogy is justified undoubtedly in the field from economical as well as from a pure scientific point of view.

The so-called "hydraulic analogy" tried to furnish results in this field but unfortunately the analogy was not exact. It was fairly correct in the low velocity range, where ordinary experimental techniques gave acceptable results at a reasonable cost. Therefore, an improvement and extension of the classical analogy was required which could be established only by carefully following the above outlined procedure.

The author believes that the proposed analogy is another example of systematic research in the field and gives a powerful, relatively inexpensive, accurate, and fast method for obtaining data for high speed flight problems.

V

HISTORY OF THE ANALOGY

Two basically different types of flow of water in a channel have been described in books on hydraulics for more than a hundred years.

The difference between "streaming" and "shooting" flow of water and the so-called "critical" flow separating the two types were observed by many workers in the field of hydro-mechanics. When in the relatively new science, aerodynamics, and especially in gas dynamics, the concept of "subsonic" and "critical" or "transonic" flows of gas were introduced, an analogy was at hand between gas dynamics and hydraulics.

The origin of the so-called "Hydraulic Analogy" was traced back to E. Jouguet's[8] paper published in 1920, which proposed an analogy between two dimensional gas flow and flow of water in an open rectangular channel and gave direct proof by exact mathematical analysis. He introduced the concept of a gas which has an adiabatic gas constant (γ) of 2 and called it "hydraulic gas". It is interesting to note that in the past thirty years several theoretical and experimental papers contributed to this analogy, accepting the initial error of $\gamma = 2$ without trying to introduce a "better" hydraulic gas. The analogy holds for subsonic as well as for supersonic flows. It can be expected that the classical hydraulic gas will show little deviation from air in cases when the value of the adiabatic constant is not one of primary interest. Therefore, even the relatively recent N. A. C. A. reports[10,12] deal with the subsonic applications of the analogy and no experimental investigations were presented with respect to high Mach number flow. The water table experiments of Gilmore, Plesset and Crossley[5] showed very little agreement with the actually observed phenomena in air or with theoretical values.

One of the most important points in the analogy is the fact that the velocity of sound in the gas and the velocity of propagation of long gravity waves in a water channel are analogous. The value of and the mathematical expression for the speed of sound in air is well established. Also, even in elementary textbooks on hydraulics, the formula for the wave velocity is established for a rectangular channel. The discrepancy between the two formulas which results in the unacceptable $\gamma = 2$ value can be improved if the cross section of the channel is changed from rectangular to some other geometric configuration. To find the new shape of the channel cross section, mathematical expressions are necessary for the velocity of water waves in any channel. Therefore, in this outline of the history of the hydraulic analogy, a few words will be said about investigations regarding water wave velocity. As early as 1840, Kelland[9] derived formulas for the wave velocity. Later (1842), Green solved the problem for a special channel. Both men's results were obtained by mathematical analysis which were undoubtedly ingenious but not exact. Modern hydraulics[1], introducing the concept of specific energy, gives an exact method for obtaining the formula for the wave velocity. This method, however, puts certain restrictions on the flow.

VI

THEORETICAL INVESTIGATION

A detailed analysis of certain special channels has been given previously[17, 18]. In this thesis, a general and new mathematical derivation and justification of the analogy will be presented. Different theoretical approaches will be utilized in the derivation of the governing equations.

For a channel of arbitrary but constant cross section, the method of vector analysis will be used to derive the governing equations. For the rectangular and triangular channels, different forms of the continuity equation and the Bernoulli equation will be used to find the relationships between the water flow and two dimensional air flow.

A few simplifying assumptions are required in order to establish the analogy between gas dynamics and open channel flow and are listed below:

1. The gas is a perfect gas (one that obeys $pV = RT$).
2. The flow of gas is steady (variables do not change with time).
3. The flow of gas is irrotational (a velocity potential exists).
4. The gas is an ideal fluid (no viscosity).
5. No body forces are acting on the gas (weight of the gas is neglected).
6. The flow of water is steady (except for wave phenomena).
7. The water is incompressible (compressibility effects are negligible).
8. Water is an ideal fluid (no viscosity).

9. The vertical acceleration of the water is small when compared to the gravitational acceleration[13].

With these assumptions, the analogy can now be developed.

A - Analogy Between Gas Dynamics and Flow of Liquids in An Open Channel of Arbitrary Cross Section

If the analogy between two dimensional gas flow and flow of water in a channel of arbitrary, but constant cross section is established, then the most proper channel shape can be selected. By proper channel is meant the one for which the analogy is perfect. Since the velocity of propogation of long gravity waves is needed, the equation will be derived by utilizing the concept of specific energy and by vector analysis.

The constant channel cross section is represented by the $y = f(z)$ function, which is assumed to be a parabola of n-th order, i.e.

$$y = az^n \qquad \text{- - - - -(1)}$$

We will refer to n as the shape factor of the channel. If the dimensions of the channel are given, then the constant a can be found, which will be called the size factor of the channel.

It can be seen that the $n = 0$ value corresponds to the rectangular, the $n = 1$ to the triangular and the $n = 1/2$ or $n = 2$ values to the parabolic channels.

The cross sectional area of the channel is $A = 2\int_0^h y\,dz = 2\,\frac{ah^{n+1}}{n+1}$ except for the $n = -1$ value which is of no interest since it corresponds to a hyperbolic channel.

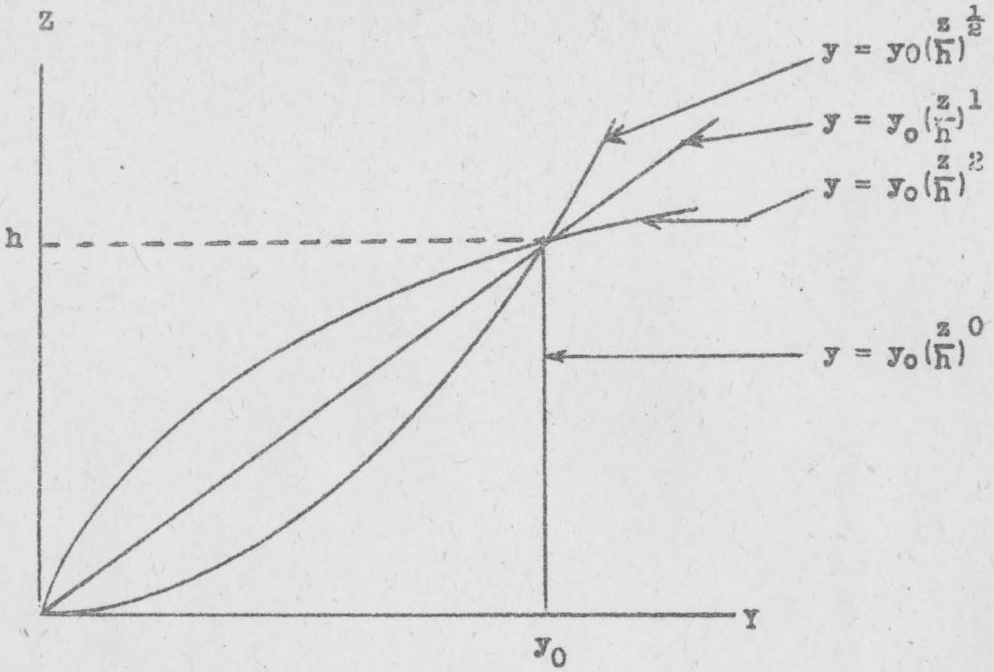


Figure 1

By using Bakhmeteff's[1] definition of specific energy (E) and Figure 1, the critical velocity in an arbitrary shaped channel can be found. The specific energy is defined by:

$$E = h + \frac{v^2}{2g}$$

where h is depth of liquid at certain cross section and v is the average velocity of flow at the same cross section. The volume rate of flow is defined by

$$Q = vA$$

Therefore

$$E = h + \frac{Q^2}{2gA^2}$$

By substituting for the area and recognizing that the critical depth occurs when the specific energy is a minimum for a given rate of flow, the function E(h) may be differentiated and the results equated to zero, which gives:

$$1 - \frac{Q_c^2 (n + 1)^3}{g a^2 h_c h_c^{2(n + 1)}} = 0$$

where h_c denotes the critical depth. If we define Q_c, the critical rate of flow by $Q_c = v_c A_c$ then

$$v_c = \sqrt{\frac{g h_c}{n + 1}}$$

Since the critical velocity in water is the velocity of propagation of long gravity waves,

$$c = v_c = \sqrt{\frac{gh}{n + 1}} \qquad \text{- - - - -(2)}$$

The equation for the velocity of propagation of long gravity waves can be easily derived by the use of vector notation. The continuity equation for the channel flow is

$$\text{div } (\bar{v} h^{n + 1}) = 0 \qquad \text{- - - - -(3)}$$

where $\bar{v}$ (u, v, w) is the velocity vector. The above equation can be expanded and simplified to

$$(n + 1)\, \bar{v} \text{ grad } h + h \text{ div } \bar{v} = 0 \qquad \text{- - - - -(4)}$$

The equation of continuity for the flow of compressible gases is

$$\text{div } (\bar{v} \rho) = 0 \qquad \text{- - - - -(5)}$$

where ρ is the density of the gas.

Comparing equations (3) and (5) it is apparent that the density of the gas and the (n + 1) power of the water depth will be corresponding quantities in the analogy. Equation (5) can be further expanded

$$\bar{v} \text{ grad } \rho + \rho \text{ div } \bar{v} = 0 \qquad \text{- - - - -(6)}$$

The second equation to be considered is the Euler equation for the potential, steady flow of nonviscous fluids,

$$\rho \text{ grad } \frac{\bar{v}^2}{2} = -\text{grad } p + \rho \bar{g} \qquad - - - - -(7)$$

where p is the pressure and $\bar{g}$ is the body force or volume force per unit mass. For the channel flow $p = (h - z)g\rho$ and $\bar{g} = \text{grad } (\rho gz)$, where g is the gravitational acceleration and z the vertical coordinate of any point in the flow. (Figure 1). With the above formula for the pressure, the assumption is made that the vertical acceleration of the water is zero which can be shown by computing the third (z) component of the vectors on the two sides of equation (7). Considering the incompressibility of the water in the channel and using the above expressions for p and $\bar{g}$, the Euler equation is

$$\text{grad } \frac{\bar{v}^2}{2} = -g \text{ grad } h \qquad - - - - -(8)$$

Combining equations (4) and (8)

$$\bar{v} \text{ grad } \frac{\bar{v}^2}{2} = \frac{gh}{n+1} \text{ div } \bar{v} \qquad - - - - -(9)$$

Turning now to the gas flow, the Euler equation of course still has the form of (7). Neglecting the body forces in this case and introducing the velocity of sound by

$$c^2 \text{ grad } \rho = \text{grad } p \qquad - - - - -(10)$$

and combining (6) and (7), one gets

$$\bar{v} \text{ grad } \frac{\bar{v}^2}{2} = c^2 \text{ div } \bar{v} \qquad - - - - -(11)$$

Equation (11) is the well known basic equation of gas dynamics. Comparing equation (9) and (11), c^2 will correspond to $\frac{gh}{n+1}$ which is the square of the velocity of propagation of long gravity waves in a channel defined by equation (1). (The $\sqrt{\frac{gh}{n+1}}$ formula was found by Kelland[4] who in fact showed that the wave velocity is $\sqrt{\frac{Ag}{y_o}}$.) Therefore, in the analogy the velocity of sound in the gas and the wave velocity in the channel are corresponding quantities.

Equations (9) and (11) can be translated into the well known scalar equations for the nonviscous flow of a fluid. Also, the scalar form of equation (9) can be derived by considering the flow of a fluid from an infinitely wide basin and the continuity equation for the flow.

Since the velocity of fluid flowing out of an infinitely wide basin is:

$$V = \sqrt{2 g (h_o - h)} \qquad \text{- - - - -(1 a)}$$

then

$$h = h_o - \frac{V^2}{2g}$$

and

$$\frac{\partial h}{\partial x} = - \frac{1}{2g} \frac{\partial \bar{V}^2}{\partial x}$$

but for two dimensional flow

$$\bar{V}^2 = u^2 + v^2$$

Therefore,

$$\frac{\partial h}{\partial x} = -\frac{1}{g}\left(u \frac{\partial u}{\partial x} + v \frac{\partial v}{\partial x}\right) \qquad \text{- - - - -(2 a)}$$

and

$$\frac{\partial h}{\partial y} = -\frac{1}{g}\left(u \frac{\partial u}{\partial y} + v \frac{\partial v}{\partial y}\right) \qquad \text{- - - - -(3 a)}$$

Writing the continuity equation:

$$\operatorname{div} \bar{v}h^{n+1} = h \operatorname{div} \bar{v} + (n+1)\, \bar{v} \operatorname{grad} h = 0 \quad \text{- - - - -(4 a)}$$

or in scalar form

$$h\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + (n+1)\left[u\frac{\partial h}{\partial x} + v\frac{\partial h}{\partial y}\right] = 0 \quad \text{- - - - -(5 a)}$$

Substituting equations (2 a) and (3 a) in (5 a), we get

$$h\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right] - (n+1)\left[\frac{u^2}{g}\frac{\partial u}{\partial x} + \frac{u v}{g}\frac{\partial v}{\partial x}\right] - (n+1)\left[\frac{u v}{g}\frac{\partial u}{\partial y} + \frac{v^2}{g}\frac{\partial v}{\partial y}\right] = 0$$

$$\text{- - - - -(6 a)}$$

dividing by h and simplifying, we get

$$\frac{\partial u}{\partial x}\left[1 - \frac{u^2}{\frac{gh}{n+1}}\right] + \frac{\partial v}{\partial y}\left[1 - \frac{v^2}{\frac{gh}{n+1}}\right] - \frac{u v}{\frac{gh}{n+1}}\left[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right] = 0 \quad \text{- - - - -(7 a)}$$

This is the differential equation for the velocity of an ideal free surface fluid flowing in a channel of arbitrary, but constant cross section. The corresponding equation for the flow of a two dimensional, nonviscous, compressible gas is:

$$\frac{\partial u}{\partial x}\left[1 - \frac{u^2}{c^2}\right] + \frac{\partial v}{\partial y}\left[1 - \frac{v^2}{c^2}\right] - \frac{u v}{c^2}\left[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right] = 0 \quad \text{- - - - -(8 a)}$$

By comparing equation (7 a) and (8 a), we see they are identical if c^2 is analogous to $\frac{gh}{n+1}$. This relation was shown in equation (2).

In compressible flow phenomena in addition to the continuity and Euler equations, the equation of state and the adiabatic law are used, which are expressed by

$$\frac{p}{\rho} = RT \qquad \text{and} \qquad \frac{p}{p_0} = \left(\frac{\rho}{\rho_0}\right)^{\gamma} \quad \text{- - - - -(12)}$$

where the o subscript refers to reference datum. The above two equations combined give

$$\frac{\rho}{\rho_0} = \left(\frac{T}{T_0}\right)^{\frac{1}{\gamma-1}} \qquad \text{- - - - -(13)}$$

From the continuity equation a relationship was found between the height of water and the density of gas,

$$\frac{\rho}{\rho_0} = \left(\frac{h}{h_0}\right)^{n+1} \qquad \text{- - - - -(14)}$$

Using the above presented equations, we see that

$$\frac{p}{p_0} = \left(\frac{h}{h_0}\right)^{n+2}$$

and - - - - -(15)

$$\frac{T}{T_0} = \frac{h}{h_0}$$

Using equations (13), (14) and (15), it follows that

$$\frac{\rho}{\rho_0} = \left(\frac{h}{h_0}\right)^{n+1} = \left(\frac{h}{h_0}\right)^{\frac{1}{\gamma-1}}$$

therefore, the analogy requires that

$$\gamma = \frac{n+2}{n+1} \qquad \text{- - - - -(16)}$$

Equation (16) can be regarded as the final result of the derivation and it represents a relationship between the adiabatic gas constant and the shape factor of the channel. Equation (16) if solved for n is the formula with which channels can be designed for any required value of the adiabatic gas constant.

The relationship between the adiabatic gas constant (γ) and the shape factor (n) as expressed by equation (16) is shown in Figure 2. From this figure, it is seen that γ can vary between 1 and 2. The shape of the cross section necessary to obtain a value of $\gamma = 1.4$ is shown.

Figure 3 shows the theoretical variation with Mach Number of the ratio of free stream pressure to stagnation pressure for different values of γ. This pressure ratio was calculated by using[4]

$$\frac{p_o}{p_s} = \left(1 + \frac{\gamma - 1}{2} M^2\right)^{\frac{\gamma}{\gamma - 1}}$$

The theoretical variation with Mach Number of the water depth ratio for $\gamma = 1.5$ and $\gamma = 2.0$ is shown. The water depth ratio was calculated by

$$\frac{p}{p_o} = \left(\frac{h}{h_o}\right)^3$$

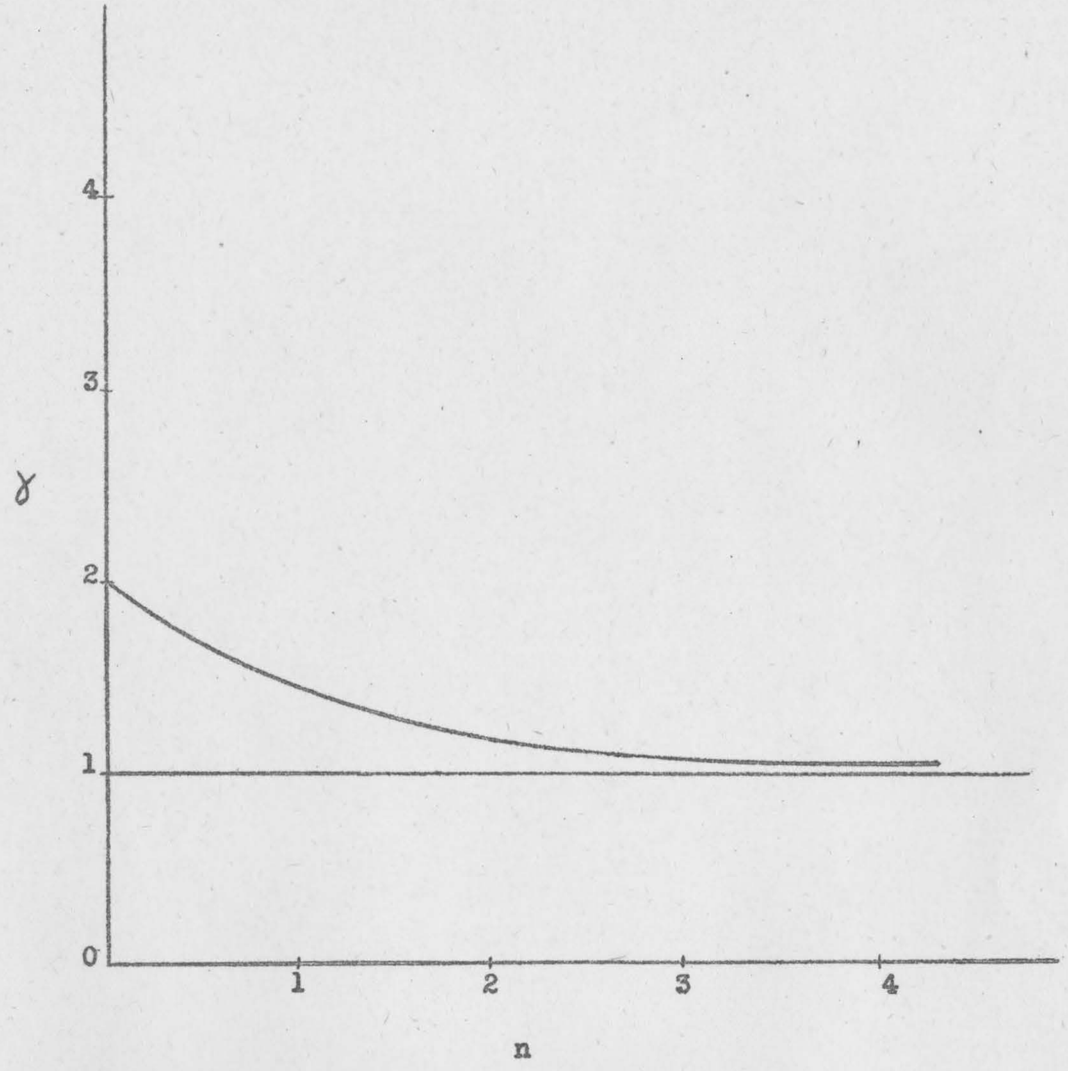
Figure 4 shows the difference between $\gamma = 1.4$ and $\gamma = 1.5$, and $\gamma = 1.4$ and $\gamma = 2.0$ for the ratio of the pressure after a shock wave to the pressure before the shock wave. The pressure ratio across the shock wave was calculated using[4]

$$\frac{p}{p_o} = \frac{2\gamma}{\gamma+1} M_o - \frac{\gamma - 1}{\gamma + 1}$$

The density relations across a normal shock wave are shown in Figure 5 for different values of γ. The density ratio was calculated by using[4]

$$\frac{\rho_o}{\rho} = \frac{2}{\gamma+1} \frac{1}{M_o^2} + \frac{\gamma-1}{\gamma+1}$$

RELATION BETWEEN γ AND N



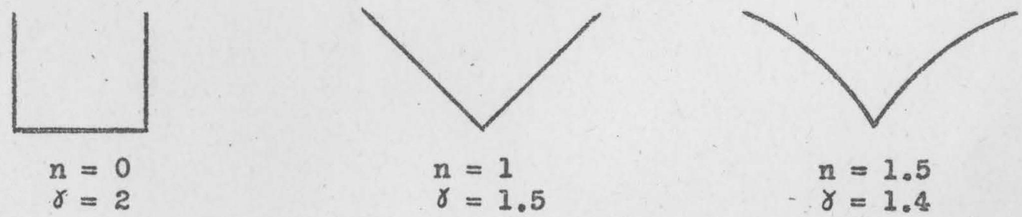


Figure 2

VARIATION OF PRESSURE RATIO AND WATER DEPTH RATIO WITH MACH NUMBER

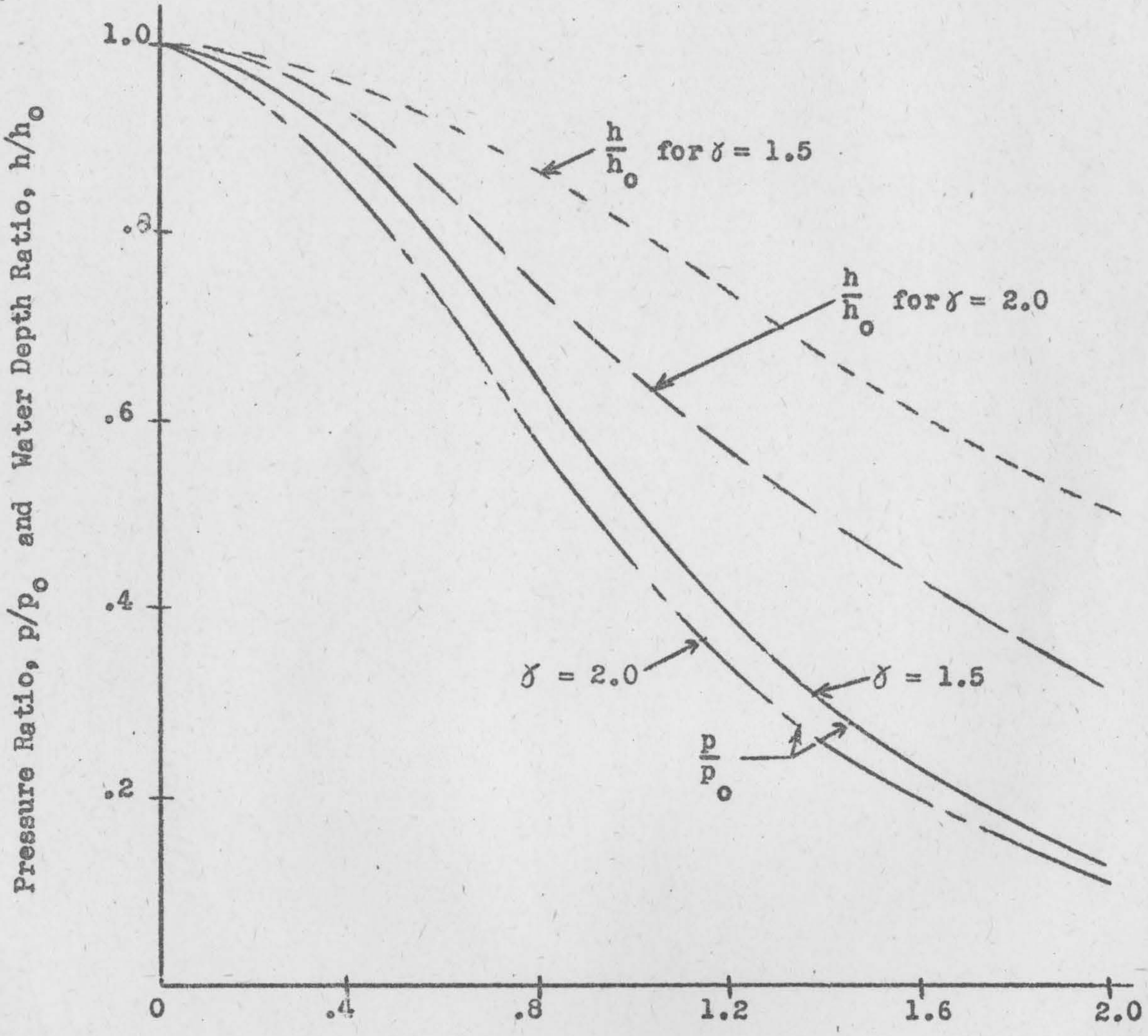


Figure 3

VARIATION OF PRESSURE RATIO ACROSS A NORMAL SHOCK WAVE

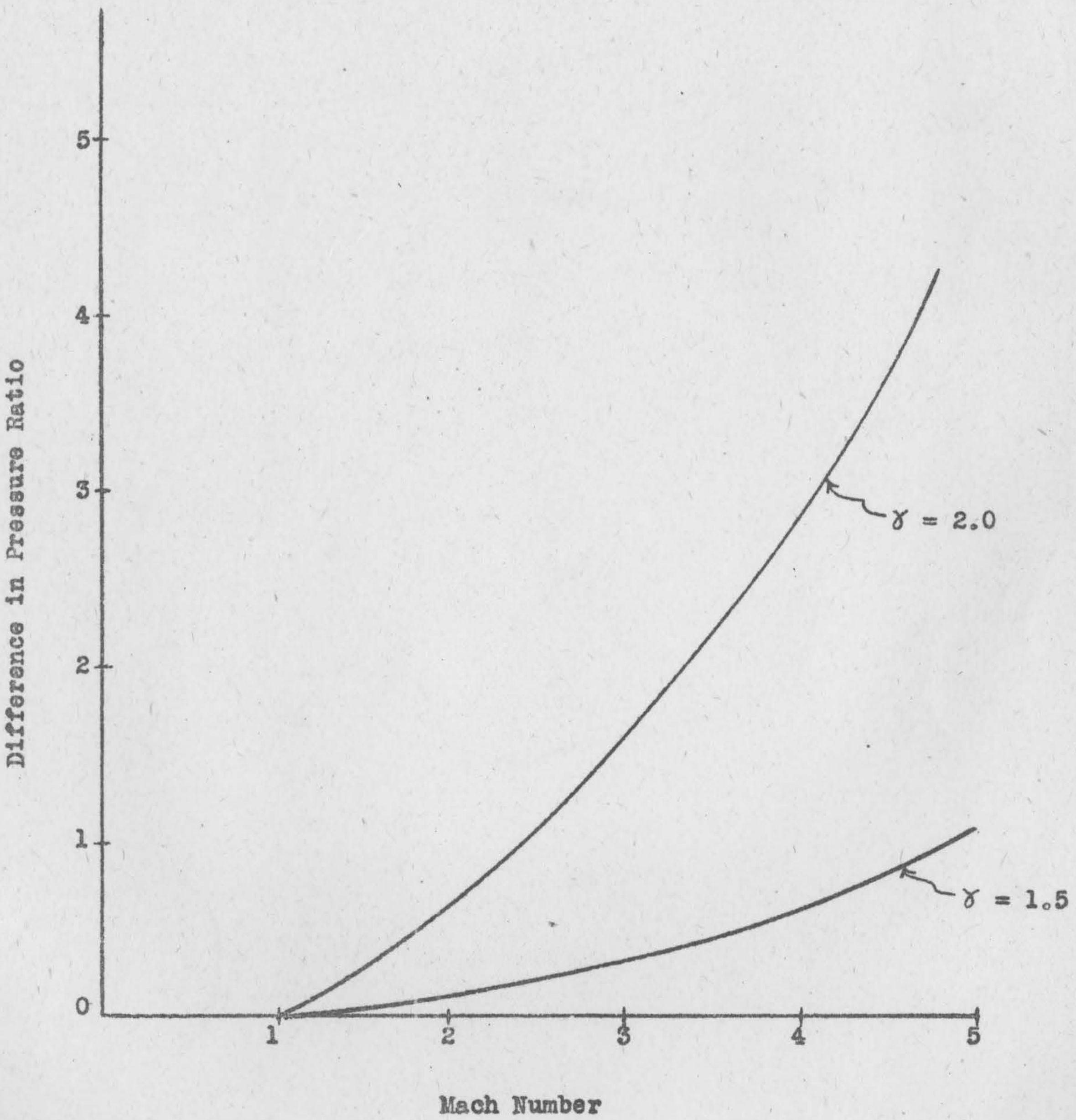


Figure 4

DENSITY RELATIONS ACROSS A NORMAL SHOCK WAVE

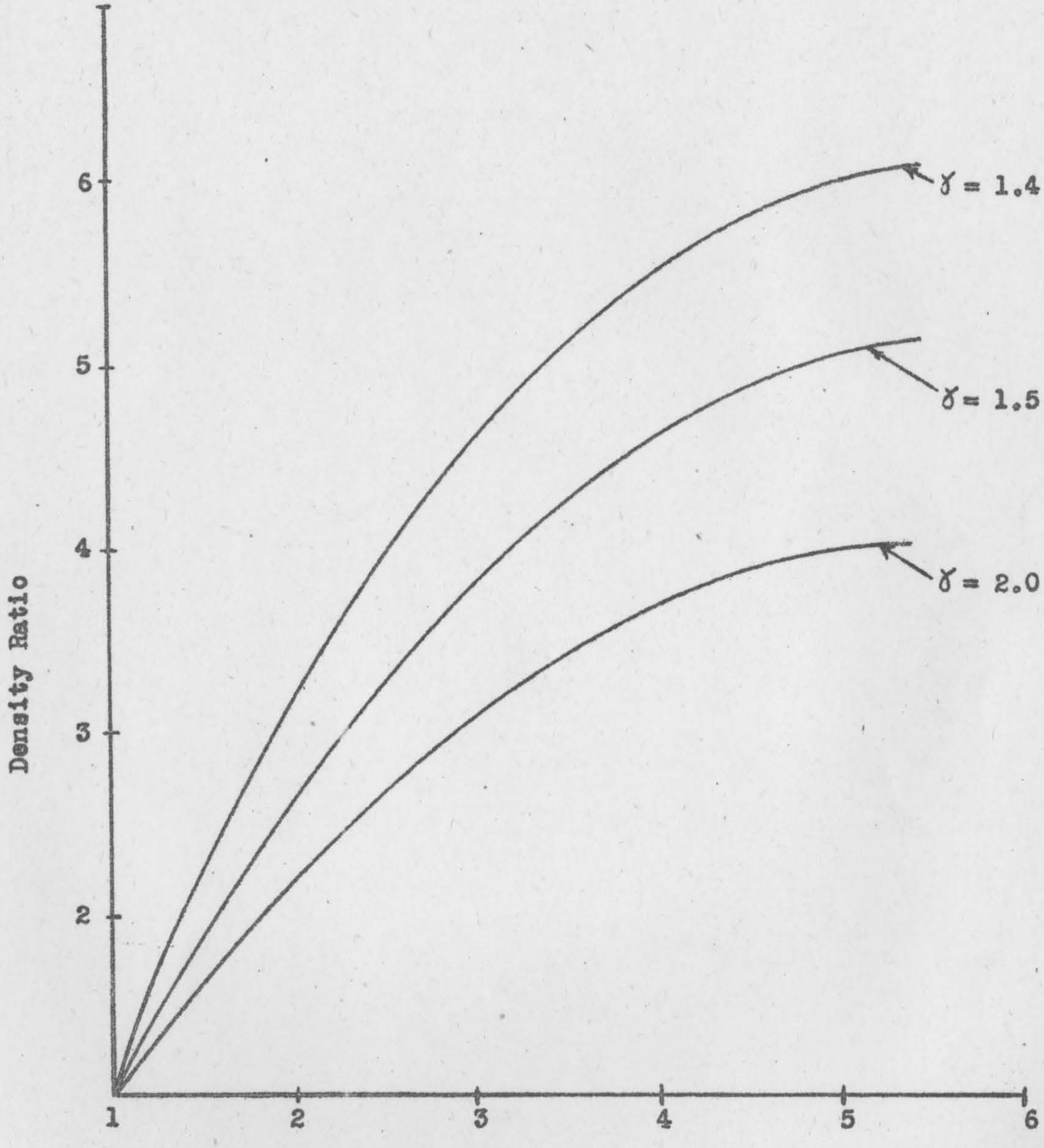


Figure 5

B - Analogy Between Gas Dynamics and Flow of Liquid in an Open Rectangular Channel

This is a special case of the previously developed analogy. To show the relation between gas dynamics and liquid flow in an open rectangular channel, we may write:

<u>Equation of continuity</u>:

$$div(\rho \bar{V}) = 0 \quad \text{or}$$

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0 \quad \text{for gas flow}$$

and $$div(h \bar{V}) = 0 \quad \text{or}$$

$$\frac{\partial(h u)}{\partial x} + \frac{\partial(h v)}{\partial y} + \frac{\partial(h . w)}{\partial z} = 0 \quad \text{for water flow}$$

From these two equations, it follows that:

$$\frac{\rho}{\rho_o} = \frac{h}{h_o} \qquad \text{- - - - -(17)}$$

or from equation (14) when n = 0. From equation (15), we see that the water depth ratio is equal to the gas temperature regardless of the shape of the channel.

$$\frac{h}{h_o} = \frac{T}{T_o} \qquad \text{- - - - -(18)}$$

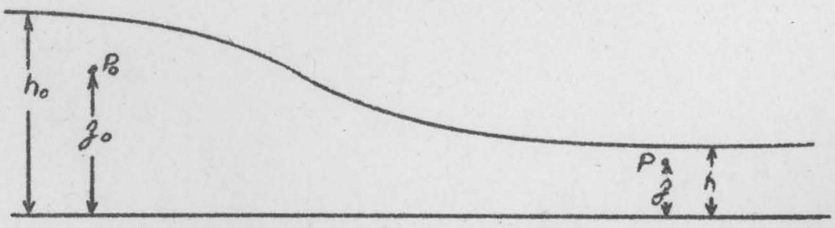


Figure 6

From the conservation of energy and Figure 6, we may write for the water flow, the Bernoulli equation:

$$\frac{\rho v^2}{2} + p + \rho z = C \qquad \text{- - - - -(19)}$$

where

$$p = \rho g (h - z) \qquad \text{- - - - -(20)}$$

From equations (19) and (20) it follows that:

$$\frac{h}{h_0} = 1 - \frac{v^2}{2 c_0^2} \qquad \text{- - - - -(21)}$$

where c_0 is the critical velocity of the propagation of gravity waves,

$$c_0^2 = g h_0$$

For the flow of a compressible gas, we write Bernoulli's compressible gas equation:

$$\frac{V^2}{2} + \int \frac{dp}{\rho} = \text{constant}$$

and

$$\frac{p}{\rho^\gamma} = \text{constant}$$

From these two equations, we get

$$\frac{\rho}{\rho_0} = \left(1 - \frac{\gamma - 1}{2} \frac{V^2}{c_0^2}\right)^{\frac{1}{\gamma-1}} \qquad \text{- - - - -(22)}$$

where c_0 is the velocity of sound in a gas at the stagnation point,

$$c_0^2 = \gamma \frac{p_0}{\rho_0}$$

From equation (17), we see that equations (21) and (22) are identical if:

$$\gamma = 2 \qquad \text{- - - - -(23)}$$

In 1920, Jouguet[8], defined a "hydraulic gas" as a gas having $\gamma = 2$. Since $\gamma = 2$, we see that the water depth ratio corresponding to the gas temperature ratio is:

$$\frac{h}{h_0} = \frac{T}{T_0} = \frac{\rho}{\rho_0} \qquad \text{-----(24)}$$

The same equation could have been obtained from equations (14) and (15) when $n = 0$.

From the equation of state for a perfect gas

$$\frac{p}{p_0} = \frac{\rho}{\rho_0}\frac{T}{T_0} = \left(\frac{h}{h_0}\right)^2 \qquad \text{-----(25)}$$

The same equation could have been obtained from equation (15) with $n = 0$.

From these fundamental relations, it is possible to calculate the temperature, pressure, and density ratios by determining experimentally the water depth ratio only.

The following table is a summary of the relations between the two flows.

Table I

Two-dimensional gas flow ($\gamma = 2$)	Corresponding quantities for flow of water in the rectangular channel
Temperature ratio $\frac{T}{T_o}$	Water depth ratio $\frac{h}{h_o}$
Density ratio $\frac{\rho}{\rho_o}$	Water depth ratio $\frac{h}{h_o}$
Pressure ratio $\frac{p}{p_o}$	Square of water depth ratio $(\frac{h}{h_o})^2$
Velocity of sound $C_o = \sqrt{\frac{\gamma p_o}{\rho_o}}$	Wave velocity $C_o = \sqrt{g h_o}$
Mach number $\frac{v}{c}$	Mach number $\frac{v}{c}$

C - Analogy Between Gas Dynamics and Flow of Liquid in an Open Triangular Channel

This is a special case of the analogy developed in Part A. It can be shown that a liquid flow in a triangular channel more nearly approximates the two dimensional flow of a gas than does liquid flow in a rectangular channel. The ratio of specific heats (γ) is equal to 1.5 for the triangular channel, 2.0 for the rectangular channel, while the corresponding value for air is 1.4. The exact value of γ could be obtained by slightly changing the triangular section. The shape of the channel necessary to attain this value of γ has been derived previously in this chapter.

The equations which govern the analogy between gas dynamics and flow of liquids in an open triangular channel will be derived.

From the continuity equation:

$$\rho\, u = \text{constant} \quad \text{for gas flow}$$

and

$$A\, u = \text{constant} \quad \text{for water flow}$$

From these two equations, it follows that:

$$\frac{\rho}{\rho_0} = \frac{A}{A_0} \qquad \text{- - - - -(26)}$$

where $A = h^2 \tan \alpha$, therefore:

$$\left(\frac{h}{h_0}\right)^2 = \frac{\rho}{\rho_0} \qquad \text{- - - - -(27)}$$

This equation could have been obtained from equation (13) with n = 1.

From the equation of state for a perfect gas:

$$PV = RT$$

and the adiabatic change of state, it follows that:

$$\frac{\rho}{\rho_0} = \left(\frac{T}{T_0}\right)^{\frac{1}{\gamma-1}} \qquad \text{- - - - -(28)}$$

By equating equations (26) and (27), we see that:

$$\left(\frac{h}{h_0}\right)^2 = \left(\frac{T}{T_0}\right)^{\frac{1}{\gamma-1}} \qquad \text{- - - - -(29)}$$

Using equation (29) and solving for γ, we get

$$\gamma = \frac{3}{2} \qquad \text{- - - - -(30)}$$

or by the conservation of energy and Figure 6, we see that:

$$\frac{\rho_0 V_0^2}{2} + P_0 + \rho_0 g z = \frac{\rho V^2}{2} + P + \rho g z$$

If the basin is infinitely wide at p_0, $v_0 = 0$; and if the water is incompressible $\rho = \rho_0$

then

$$P_0 + \rho g z_0 = \frac{\rho V^2}{2} + P + \rho g z$$

Since the vertical acceleration of the water was assumed to be small, we may write:

$$P_0 = (h_0 - z_0)\rho g$$

and

$$P = (h - z)\rho g$$

therefore

$$V^2 = 2g(h_0 - h)$$

where h_0 is the maximum depth of the water in the channel and h is the depth of the water at any point in the channel.

Solving for the water depth ratio

$$\frac{h}{h_0} = 1 - \frac{V^2}{2gh_0}$$

From equation (2) $C_0^2 = \frac{gh_0}{2}$, we get

$$\frac{h}{h_0} = 1 - \frac{V^2}{4C_0^2} \qquad \text{- - - - -(31)}$$

From Bernoulli's compressible gas equation

$$\frac{\rho}{\rho_0} = \left(1 - \frac{\gamma-1}{2}\frac{V^2}{C_0^2}\right)^{\frac{1}{\gamma-1}} \qquad \text{- - - - -(32)}$$

where c_0 is the velocity of sound in a gas, $C_0^2 = \gamma \frac{p_0}{\rho_0}$

From equations (27) and (31)

$$\frac{\rho}{\rho_0} = \left(1 - \frac{V^2}{4C_0^2}\right)^2 \qquad \text{- - - - -(33)}$$

By equating equations (32) and (33) and solving for , we see that must be $\frac{3}{2}$.

$$\gamma = \frac{3}{2} \qquad \text{- - - - -(34)}$$

Therefore, the governing equations of the analogy are:

$$\frac{\rho}{\rho_0} = \left(\frac{h}{h_0}\right)^2 \qquad \text{- - - - -(35)}$$

$$\frac{T}{T_0} = \frac{h}{h_0} \qquad \text{- - - - -(36)}$$

$$\frac{P}{P_o} = \left(\frac{h}{h_o}\right)^3 \qquad - - - - -(37)$$

These last three equations could have been obtained from equations (14) and (15) with n = 1. With these two equations it is possible to calculate the density ratio, pressure ratio and temperature ratio after the water depth ratio for a point in the flow has been determined experimentally.

The following table is a summary of the relationships for the analogy between gas dynamics and liquid flow in a triangular channel.

Table II

Two-dimensional gas flow $\gamma = 1.5$	Corresponding value in the triangular channel
Temperature ratio $\frac{T}{T_o}$	Water depth ratio $\frac{h}{h_o}$
Density ratio $\frac{\rho}{\rho_o}$	Square of water depth ratio $(\frac{h}{h_o})^2$
Pressure ratio $\frac{p}{p_o}$	Cube of water depth ratio $(\frac{h}{h_o})^3$
Velocity of sound $c_o = \sqrt{\gamma \frac{P_o}{\rho_o}}$	Wave velocity $c_o = \sqrt{\frac{gh}{2}}$

Since the adiabatic constant (γ) for air has the value of 1.4, from Figure 2, the n = 1.5 value seems to be the most appropriate. This would indicate a parabolically shaped channel for which the governing equations of the analogy are:

$$\frac{h}{h_o} = \frac{T}{T_o} \qquad - - - - -(38)$$

$$\frac{h}{h_o} = \left(\frac{\rho}{\rho_o}\right)^{\frac{1}{2.4}} \qquad \text{- - - - -(39)}$$

$$\frac{h}{h_o} = \left(\frac{p}{p_o}\right)^{\frac{1}{3.4}} \qquad \text{- - - - -(40)}$$

For fabrication reasons, the n = 1 value which corresponds to a triangular channel will be selected. Attention should be called to the fact that the channel angle does not have any effect on the analogy, i.e., using a triangular channel, any angle can be chosen.

D - Shape of Model

If a rectangular channel is used with parallel walls and horizontal bottom, the shape of the model and that of the prototype are equivalent. If, however, a triangular or parabolic cross section is employed, then the use of a transformed model is necessary. In the following investigation, a detailed analysis will be given regarding the model shape in a triangular channel. The shape of the models in channels of arbitrary cross section will also be outlined.

The obstruction represented by a cylinder in a rectangular channel is independent of the depth of the water and can be represented by the $r = d/\ell$ ratio. (See figure 7).

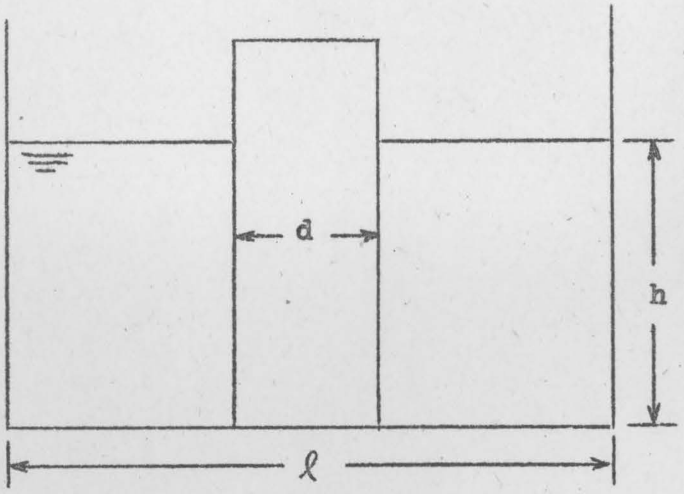


Figure 7

If the same cylinder would be put in a triangular channel, the obstruction ratio (r) would not be constant and an analogy could not be expected between two dimensional gas flow and the flow of water in the triangular channel. (Figure 8).

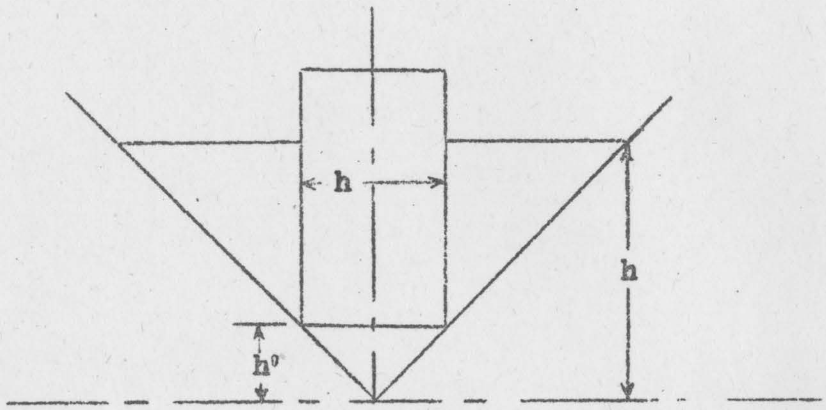


Figure 8

The obstruction ratio for Figure 8 is obtained if the projected cylinder area:

$$(h - h') d$$

is compared to the cross sectional area of the channel:

$$h^2 \tan \alpha$$

Therefore, from Figure 4,

$$r = \frac{(h-h')d}{h^2 \tan \alpha} = \frac{2(h \tan \alpha - d)d}{2 h^2 \tan^2 \alpha} \quad \text{- - - - -(41)}$$

which varies with the variation of h.

As a simple geometric observation, or formula (41) shows, if the cylinder diameter normal to the flow (d) is proportional with the height of the liquid (h), then the obstruction ratio is constant, i.e., independent of h. This indicates the use of a cone in the triangular channel, instead of the same cylinder that was used in the rectangular channel. As it will be shown, this reasoning is almost correct and requires only slight modification in order to obtain the final and proper model shape.

Considering side views of the rectangular and triangular channels, it is obvious that no difference can be observed between them. If, however, a cone is placed in the triangular channel and a cylinder in the rectangular channel, then basically different flow pictures will be obtained and the analogy will not hold. (Figure 9).

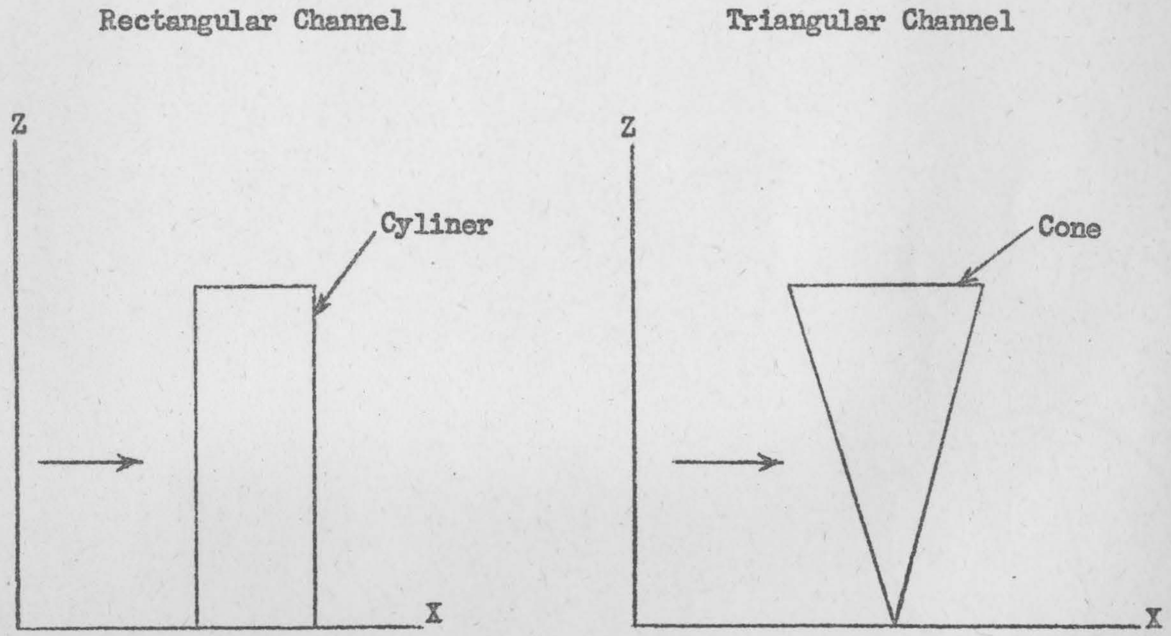


Figure 9

Therefore, one concludes that the model in the triangular channel must show up as a triangle in the front view and as a rectangle in the side view. The construction of this "cylindrical cone" is rather simple and can be described by means of a (X, Y, Z) rectangular coordinate system. (Figure 10).

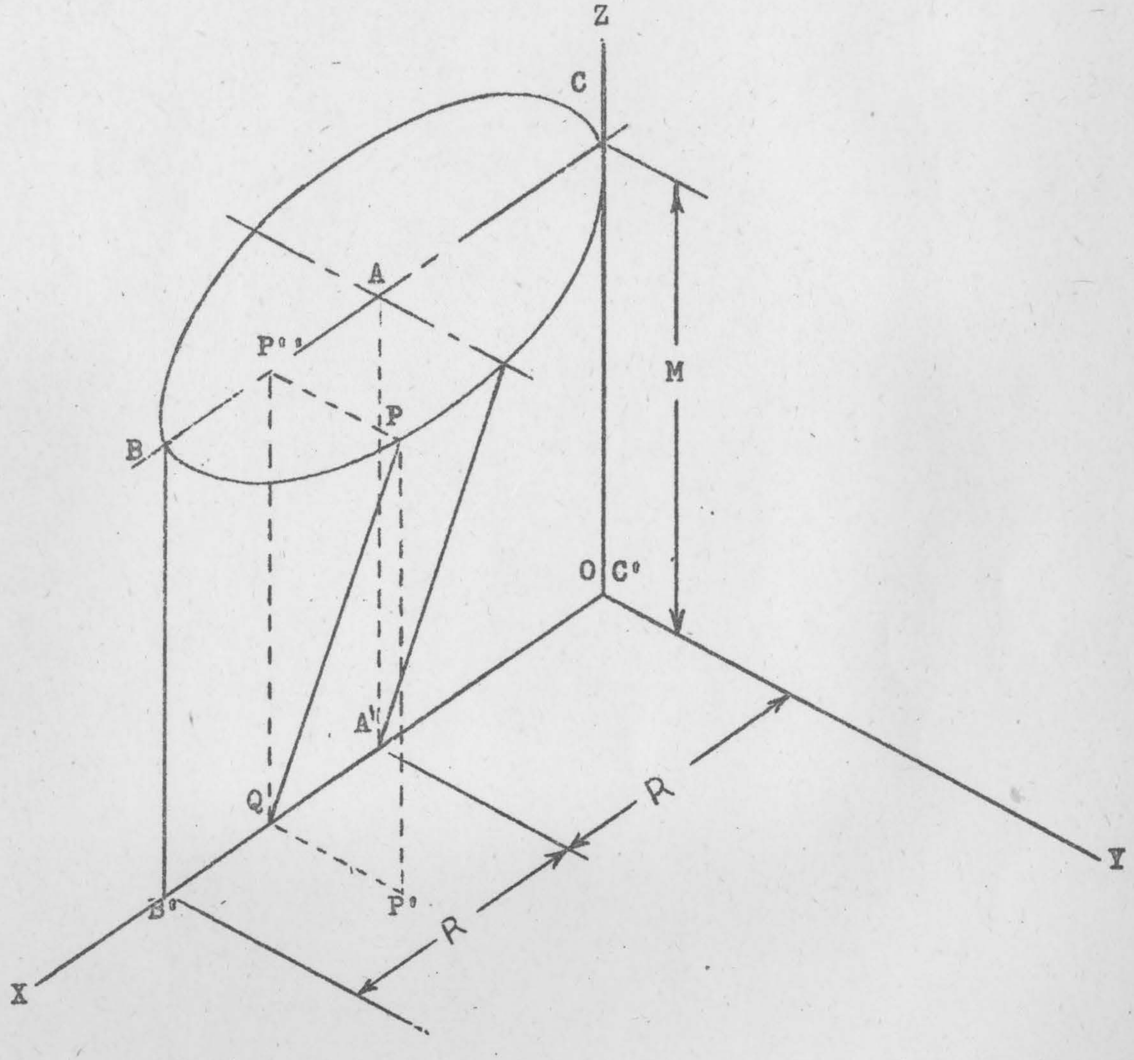


Figure 10

In the Z = M plane (parallel with the (X, Y) coordinate plane), a circle of radius R is located. The center of the circle is at point A, the coordinates of which therefore are X = R, Y = 0, Z = M. The circle now can be drawn. The projection of B on the X axis is B' and the coordinates of B are X = 2R, Y = 0, Z = M. The projection of C on the X axis (C') coincides with the origin of the coordinate system and the coordinates of C are X = 0, Y = 0, Z = M.

Now consider point P on the circle. The projection of P on the (x, y) plane is P' and on the (X, Z) plane P''. The projection of P'' or of P' on the X axis is Q. Connect points P and Q. If the above described construction is repeated and the points of the circle are connected with the points of the B'C' line, a surface is obtained which satisfies the requirements imposed by the constant obstruction ratio and also will be analogous to the cylinder in the rectangular channel.

The equation of the surface just described is

$$Y^2M^2 - 2RZ^2X + Z^2X^2 = 0 \qquad \text{- - - - -(42)}$$

The Z = M section is given circle with radius R:

$$Y^2 + (X-R)^2 = R^2$$

Any other section, parallel with the X, Y plane is an ellipse with the equation:

$$\left(\frac{Y}{Z_o t}\right)^2 + \left(\frac{X-R}{R}\right)^2 = 1 \qquad \text{- - - - -(43)}$$

Where Z_o = Z represents the plane parallel with the (X, Y) plane and

$t = \tan\beta = \frac{R}{M}$ represents the tangent of the half cone angle (β).

The plane sections parallel with the Y, Z) plane are triangles with the equations:

$$Z = \pm Y \frac{M}{(2X_0R - X_0^2)^{\frac{1}{2}}} \qquad - - - - -(44)$$

where $X_0 = X$ represents the plane parallel with the (Y, Z) plane and the $\pm$ sign corresponds to the two sides of the surface.

The obstruction ratio is

$$r = \frac{\tan\beta}{\tan\alpha} = \frac{\text{tangent of half body angle}}{\text{tangent of half channel angle}}$$

and therefore is constant.

A similar analysis can be applied to other bodies. As an example, the important case of a wedge section is mentioned. It is transformed to a hyperbolic paraboloid with the equation:

$$X_0MY + ZX - ZX_0 = 0 \qquad - - - - -(45)$$

where M represents, as before, the Z = M section which is equivalent to the original model and X_0 is seen from Figure 11, where the construction of a surface is shown representing a half wedge.

The fact should be emphasized that in spite of the curved nature of the surfaces obtained by the above analysis, the model can always be made out of triangles since the X = constant plane sections are for any shape triangles. Therefore, the practical application of the analogy is not lessened by the somewhat intricate shape of the model.

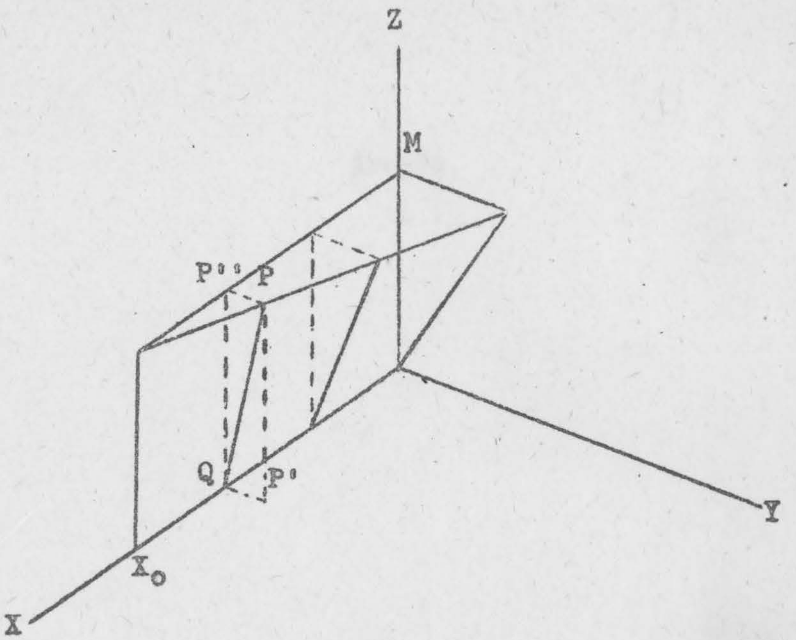


Figure 11

The wedge angle (δ) of the model at the surface of the water was found from the geometry of the model to be:

$$\delta = \arc \tan \frac{h}{6}$$

where h is the water depth measured in inches.

VII

EXPERIMENTAL APPARATUS AND TECHNIQUE

A - Experimental Apparatus

The tests were conducted in a circulating water channel which was designed and constructed at Virginia Polytechnic Institute. The experimental unit was constructed of sheet steel and plexiglas. The unit consists of a settling basin, six foot test channel, return basin, pump, and hydraulic jack. (See Figures 12 and 13).

The settling basin consists of two rectangular tanks, each 30 inches long, 40 inches wide and 30 inches deep. The front side of the forward tank was detachable to facilitate testing channels of different cross sections. This side could be changed by simply removing the holding bolts. The joint between this face and the settling tank was calked with metal sash putty to prevent leaking. In the forward tank a baffle plate was used to insure smooth flow. The baffle plate was constructed of 1/4 inch plywood in which one inch holes were drilled. A movable gate is located between the forward and rear section. By varying the height of this gate, the head in the rear tank can be raised or lowered. The water is pumped into the rear tank and the rate of flow is regulated by a globe valve located directly below the rear tank. A constant head is maintained in this tank by means of an overflow pipe.

The channels tested were six feet long and constructed of clear plexiglas. Two channels, one 90 degrees between sides and one 135 degrees between sides have been constructed and used for tests. The channels were fabricated by placing the plexiglas sheets in a jig and gluing the two sheets together with Duco Cement. The channels were supported at three places on a cantilever beam. One end of the channel was fastened to the front side of the settling basin, a support was placed in the center, and a support was used at the end. A smooth, curving entrance vane was attached to the settling basin where the channel was mounted.

The return basin was a large tank used to collect the water as it flowed out of the channel. Also, it served as a reservoir for the pump.

A centrifugal pump, driven by a 1/6 horsepower electric motor, was connected to the return basin and pumped the water back into the settling basin. A valve was located in the line between the pump and settling tank to facilitate removing the water from the unit.

Since the analogous Mach Number of the flow can be changed by either varying the head in the settling tank or by tilting the channel, a hydraulic jack was placed under the settling tank. By use of the jack, the unit could be pivoted about the forward supports. (See Figure 13).

A depth gage shown in Figure 14 was used to measure the water depths. The gage consisted of a micrometer attached to a fine wire probe. This gage was mounted to a reference board located above the channel.

Both photographs and motion pictures were taken of the flow characteristics of the model. A Speed Graphic Camera and a Keystone 8 mm. Motion Picture Camera were used.

The model, which corresponds to a two dimensional wedge, was constructed of laminated balsa wood. Since every vertical section passed through the model is triangular in shape, the model could be fabricated very readily. The model was sanded until smooth then several coats of wood filler were applied. Since a smooth, colored surface was needed, yellow airplane dope was applied to the model for a finishing coat.

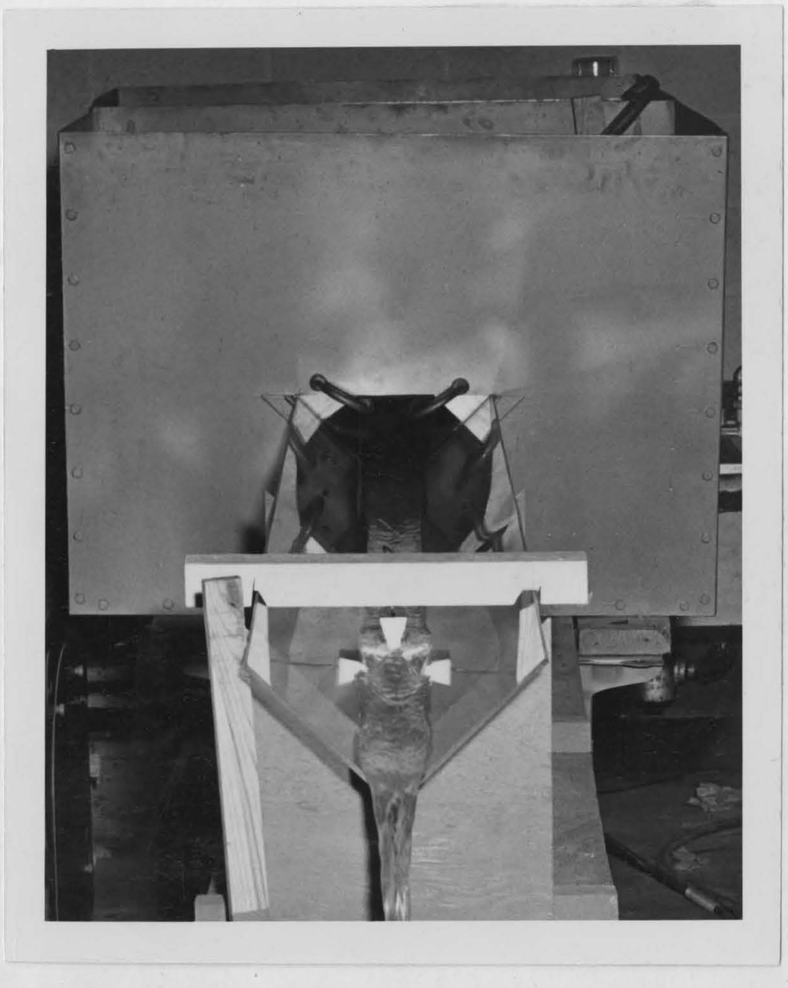


Figure 12

DIAGRAM OF EXPERIMENTAL UNIT

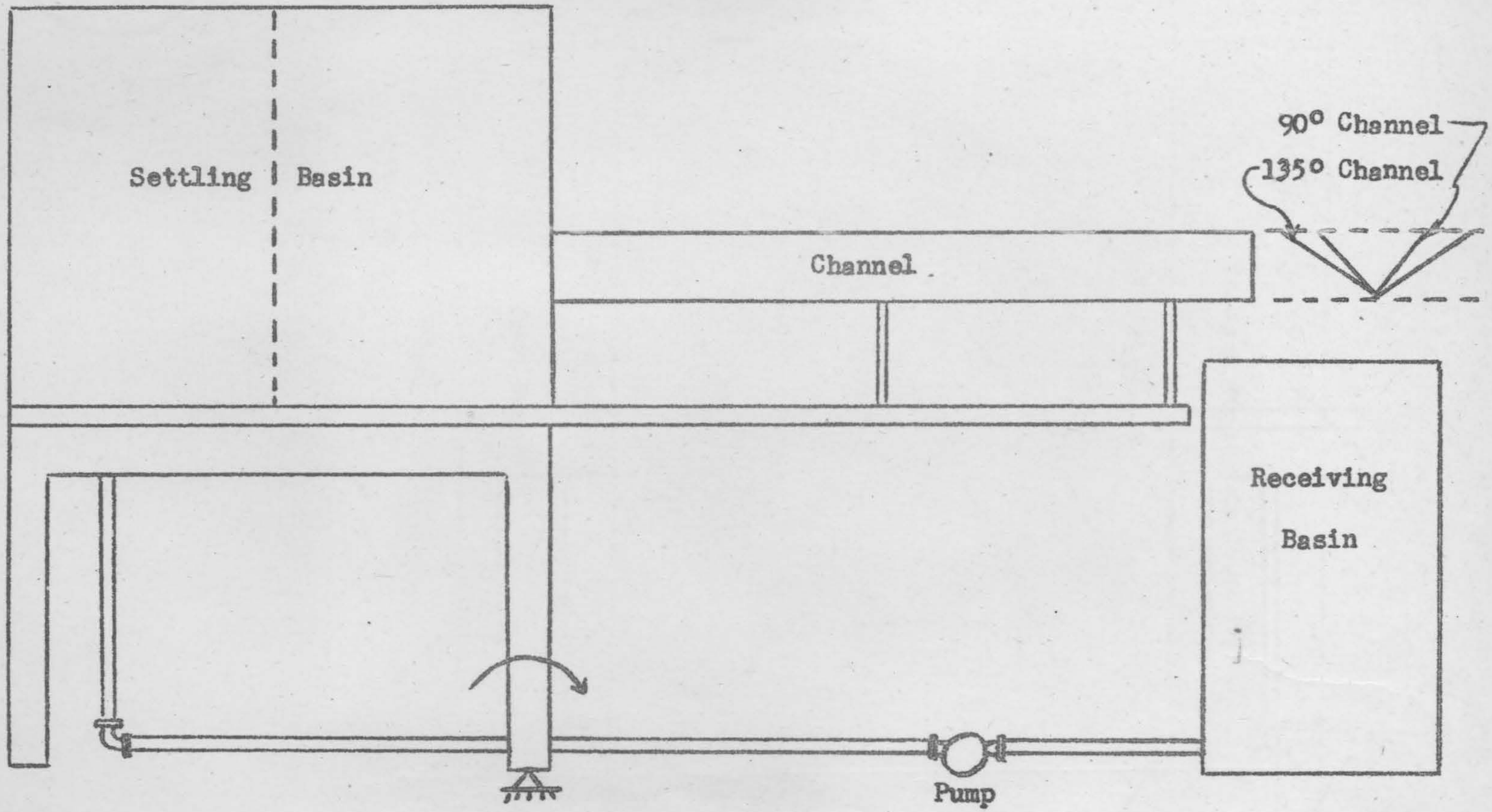


Figure 13

Figure 14

B - Experimental Technique

The desired analogous Mach number was attained by adjusting the height of the water in the settling basin and tilting the channel. The analogous Mach number for the 90 degree channel was found by

$$M = \frac{v}{c} = \frac{Q}{A\left(\frac{gh}{2}\right)^{1/2}} = \frac{1.99\ W}{T\ h^{3/2}} \quad - - - - -(46)$$

For the 135 degree channel, the analogous Mach number can be found by

$$M = \frac{v}{c} = \frac{Q}{A\left(\frac{gh}{2}\right)^{1/2}} = \frac{.823\ W}{T\ h^{3/2}} \quad - - - - -(47)$$

where W is the weight of water, T is the time in seconds and h is the height of water in the channel.

The model was then mounted in test section of the channel and water depth measurements were made. (See Figure 15).

The water depths were measured with a depth gage that could be moved both perpendicular and parallel with the flow. The bottom of the channel was selected as the reference datum for all water depth measurements. Readings were taken both at points near the model and upstream from the model.

Pressure orifices located near the bottom of the model were investigated for determining the water depths. Since the water depth in the channel was less than one inch and the model was tapered, this method was not successful. (See Figure 15).

Surface waves were generated because the water did not wet the sides of the plexiglas channel. In order to eliminate these waves, the surface tension of the water was lowered by adding a detergent (D. F. manufactured by Experiment, Inc.). Since this detergent caused excessive foaming, it was necessary to add an antifoam compound. This combination proved to be very satisfactory.

Photographs were taken of the flow characteristics of the model. From the photographs, it was possible to measure the shock angle. In addition to the photographs, an eight mm. motion picture in color was taken showing subsonic, transonic, supersonic and unsteady flow around the wedge shaped airfoil.

It should be pointed out, that when using a channel with a cross section different from a rectangle, extreme accuracy is required when measuring the water depth. Since this quantity must be raised to a power, a small error in measuring the water depth is greatly magnified.

Figure 15

VIII

RESULTS

A - Theoretical Results

The theoretical results obtained for the analogy between gas dynamics and liquid flow in an open triangular channel are believed to be a closer approximation to the actual flow of air than any of the other present day analogies. By using the governing equations developed in this thesis for the analogy, it is possible to obtain the accepted value of 1.4 for γ. Even the value of 1.5 for γ, obtained by using a triangular channel is a great improvement over the rectangular channel where γ has a value of 2.

A summary of the theoretical investigation is shown in Table 3. This summary lists the relationships between two dimensional gas flow and the flow of liquid in a rectangular channel, triangular channel and a channel of arbitrary cross section. Also, the velocity of sound in a gas and the analogous wave velocity is tabulated. It is interesting to note that the gas temperature ratio is exactly equal to the water depth ratio regardless of the shape of the channel. Another unexpected result was the fact that when using a triangular channel, any angle can be used. The channels used in this investigation were symmetrical with respect to the Z axis.

Table III

Two-Dimensional	Corresponding Values for Liquid Flow in The		
Gas Flow	Rectangular Channel	Triangular Channel	General Channel
ratio of specific heats $\frac{c_p}{c_v} = \gamma = 1.4$	$\gamma = 2$	$\gamma = 1.5$	$\gamma = \frac{n+2}{n+1}$
Temperature ratio $\frac{T}{T_o}$	water depth ratio $\frac{h}{h_o}$	water depth $\frac{h}{h_o}$	water depth ratio $\frac{h}{h_o}$
Density ratio $\frac{\rho}{\rho_o}$	water depth ratio $\frac{h}{h_o}$	square of water depth ratio $(\frac{h}{h_o})^2$	(n+1) power of water depth ratio $(\frac{h}{h_o})^{n+1}$
Pressure ratio $\frac{p}{p_o}$	square of water depth ratio $(\frac{h}{h_o})^2$	cube of water depth ratio $(\frac{h}{h_o})^3$	(n+2) power of water depth ratio $(\frac{h}{h_o})^{n+2}$
Velocity of sound $c_o = \sqrt{\frac{\gamma p_o}{\rho_o}}$	wave velocity $c_o = \sqrt{gh_o}$	wave velocity $c_o = \sqrt{\frac{gh_o}{2}}$	wave velocity $c_o = \sqrt{\frac{gh_o}{n+1}}$
Mach Number $\frac{v}{c}$	Mach Number $\frac{v}{c}$	Mach Number $\frac{v}{c}$	Mach Number $\frac{v}{c}$
Subsonic Flow	streaming water	streaming water	streaming water
Supersonic flow	shooting water	shooting water	shooting water
Shock wave	hydraulic jump	hydraulic jump	hydraulic jump

B - Experimental Results

Results in close agreement with theory have been obtained in the triangular channel for the pressure, density, temperature, and shock angle for a two dimensional, supersonic, wedge-shaped airfoil.

The experimental pressure ratios were obtained by measuring the water depths in the channel and converting these ratios by use of the analogy. As shown in Figures 16, 17 and 18, these results were very close to the theoretically predicted values.

Although the photographs of the flow around the model are not necessary, it is an interesting check on the analogy and experimental technique. Photographs were made and the shock angle was measured directly from the negative. This value for the shock angle compared very closely with the shock angle determined from the water depth measurements.

Several photographs showing the model and shock wave are included. Figure 16 shows the flow around the model when the analogous Mach Number was 2.45. The wedge angle was 6.67 degrees and the experimental shock angle was 29 degrees. The theoretical shock angle was 29.3 degrees[11]. The photograph was taken from the top of the model. The pressure ratio was calculated using the analogy and it agrees very closely with the theoretical value.

Figure 17 is a photograph of the same model at a Mach Number of 1.55. It is readily seen that the shock angle is larger than the shock angle in Figure 16.

Figure 18 is an oblique photograph of the wedge-shaped model at the same Mach Number as Figure 17. The experimental and theoretical values for the shock angle were very close.

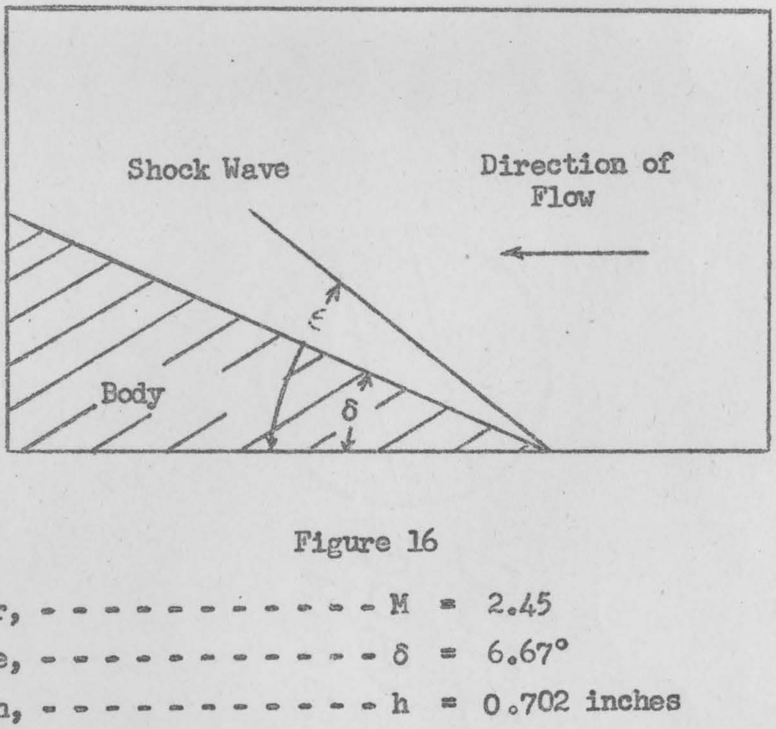


Figure 16

Mach Number, - - - - - - - - - - - - - M = 2.45

Wedge Angle, - - - - - - - - - - - - - δ = 6.67°

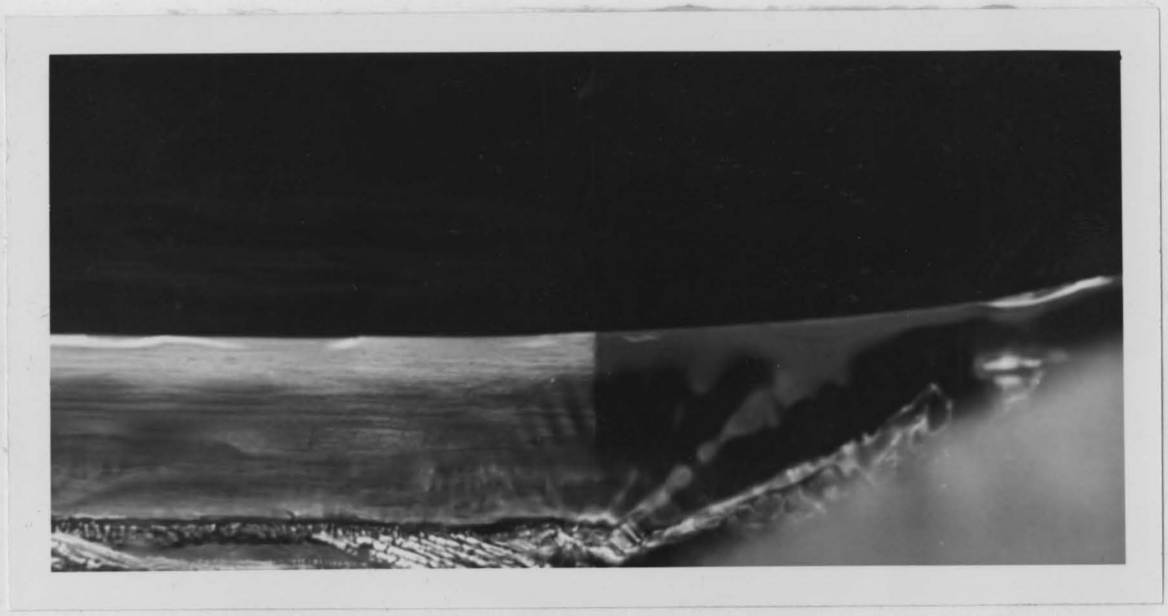
Water Depth, - - - - - - - - - - - - - h = 0.702 inches

Experimental Shock Angle - - - - - - ϵ_e = 29°

Theoretical Shock Angle- - - - - - - ϵ_t = 29.3°

Experimental Pressure Ratio- - - - P_e = 1.512

Theoretical Pressure Ratio - - - - P_t = 1.518



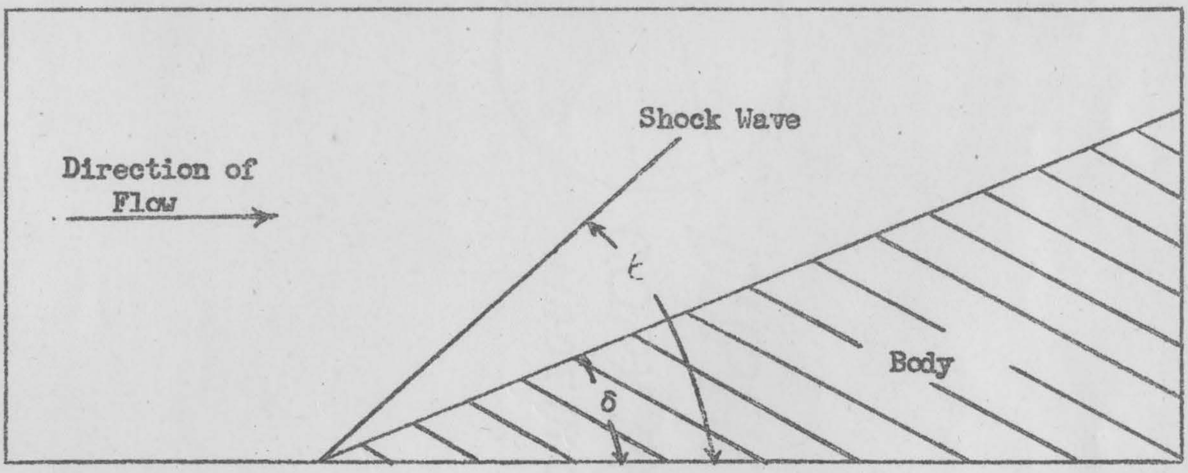


Figure 17

Mach Number, - - - - - - - - - - - - - - - - -	M	=	1.55
Wedge Angle, - - - - - - - - - - - - - - - - -	δ	=	7°
Water Depth, - - - - - - - - - - - - - - - - -	h	=	0.732 inches
Experimental Shock Angle, - - - - - - - - - -	ε_e	=	48°
Theoretical Shock Angle, - - - - - - - - - - -	ε_t	=	49°

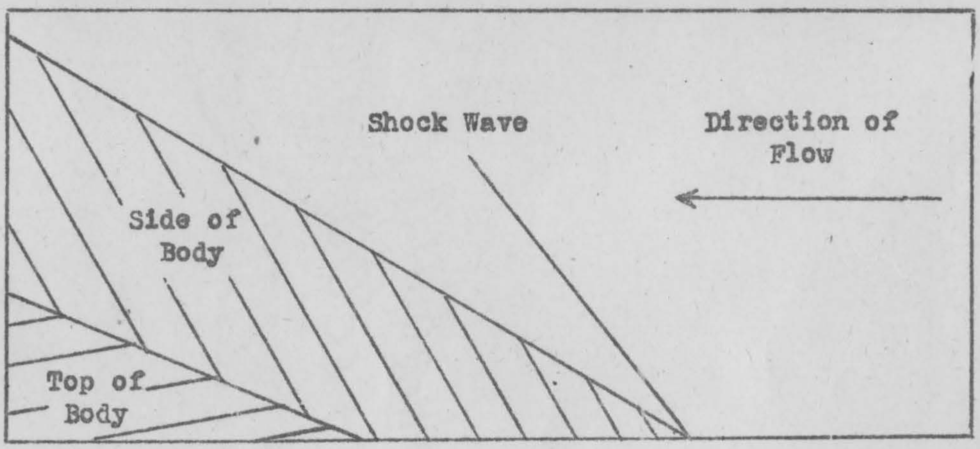


Figure 18

Mach Number, - - - - - - - - - - - - - - - -	M	=	1.55
Wedge Angle, - - - - - - - - - - - - - - - -	δ	=	7°
Water Depth, - - - - - - - - - - - - - - - -	h	=	0.732 inches
Experimental Shock Angle,- - - - - - - - - -	ϵ_e	=	48°
Theoretical Shock Angle, - - - - - - - - - -	ϵ_t	=	49°

The following is a tabulation of experimental tests in the triangular channel on a model that corresponds to a wedge shaped airfoil.

M	δ degrees	h inches	ϵ_e degrees	ϵ_t degrees	P_e	P_t
2.45	6.67	.702	29.0	29.3	1.51	1.51
1.55	7.00	.732	48.0	49.0	----	----
2.28	7.30	.77	----	----	1.48	1.50
2.13	7.30	.77	----	----	1.34	1.33

Table IV

IX

DISCUSSION OF EXPERIMENTAL RESULTS

The photographs in Figures 16, 17, 18 and the results in Table IV show the correspondence of the experimental values with the theoretical values. It is seen that for the same wedge angle, the shock angle decreases as the Mach number of the flow increases. This result is in agreement with the theory of shock waves.

Both the experimental shock angle and the pressure ratio agree very closely with the theoretically predicted values.

X

CONCLUSIONS

This theoretical and experimental investigation has proven that accurate results can be obtained by using the hydraulic analogy. The analogy can be used in either the subsonic, transonic or supersonic range. The theory was intended to be general and showed that an exact analogy exists between free surface water flow and two dimensional gas flow. By theoretical investigation, a new channel was proposed and experiments were conducted. Remarkable correlation between theory and experiment was found and it is believed that the method presented in this thesis represents a closer approximation to the actual flow of air than any of the other present day analogies.

It has been shown that by using the analogy, the pressure, density, and temperature can be easily obtained. In addition, the shock wave phenomenon can be observed. This analogy would be an excellent tool in demonstrating the different types of flow around bodies.

Analogy research in the field of supersonic aerodynamics is not only justified, but also necessary since the present day experimental technique in wind tunnels present economical and scientific difficulties. A large amount of knowledge can be obtained from analogy research in this field. The analogy described in the paper gives accurate results; the proposed apparatus is inexpensive, and experiments can be performed in a very short time.

XI

RECOMMENDATIONS

To facilitate the further use of this analogy, the following suggestions and recommendations are made.

The experimental unit should be constructed so that it will be possible to tilt the channel without tilting the settling basin.

If tests are going to be made in the transonic region, the model will have to be towed (instead of circulating the water) since the water channel will "block" in a manner similar to a wind tunnel at speeds near the velocity of sound.

Since the results obtained from this analogy depend on the water depth measurement, a more precise method is needed. There are several possible solutions to this problem.

The depth at points in the test section can be measured by use of a fine wire probe, which can be moved both vertically and horizontally. A small light mounted in series with the probe would insure positive sign of contact between the probe and the water level. This type of measurement is very valuable at points a short distance away from the model. When the probe touches the surface of the water, it causes a capillary rise of water thereby making this type of measurement undesirable at points very close to the model[12]. (See Figure 14)

A method of measuring the water depth at the surface of the model can be devised by means of small pressure orifices near the bottom of the model. These pressure orifices should be connected to a sensitive manometer[10].

Another method of measuring the water depth employes the use of dye and a special light to photograph the flow around a model. The depth of the water is then determined from the photograph[7].

Probably the best instrument for measuring the water depth would be a depth gage that would seek the water level automatically. A light could be connected to this instrument to signify when the gage had reached the water level.

XII

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XIII

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