

Modeling Students' Units Coordinating Activity

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Dissertation submitted to the faculty of the Virginia Polytechnic Institute and State
University in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

In

Mathematics

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August, 12, 2014

Blacksburg, Virginia

Keywords: Units coordination, Teaching experiment methodology, Fractions,
Multiplicative reasoning

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ABSTRACT

Primarily via constructivist teaching experiment methodology, *units coordination* (Steffe, 1992) has emerged as a useful construct for modeling students' psychological constructions pertaining to several mathematical domains, including counting sequences, whole number multiplicative conceptions, and fractions schemes. I describe how consideration of units coordination as a Piagetian (1970b) structure is useful for modeling units coordination across contexts. In this study, I extend teaching experiment methodology (Steffe & Thompson, 2000) to model the dynamics of students' units coordinating activity across contexts within a teaching experiment, using the construct of *propensity to coordinate units*. Two video-recorded teaching experiments involving pairs of sixth-grade students were analyzed to form a model of the dynamics of students' units coordinating activity. The modeling involved separation of transcriptions into chunks that were coded dichotomously for the units coordinating activity of a single student in each dyad. The two teaching experiments were used to form 5 conjectures about the output of the model that were then tested with a third teaching experiment. The results suggest that modeling units coordination activity via the construct of propensity to coordinate units was useful for describing patterns in the students' perturbations during the teaching sessions. The model was moderately useful for identifying sequences of

interactions that support growth in units coordination. Extensions, modifications, and implications of the modeling approach are discussed.

DEDICATION

I dedicate this work to my wife, Kendra, for her continual love and encouragement these past years of graduate school. I also dedicate this work to my parents, for their unwavering support over the years and for their roles in helping me develop self-confidence, determination and perserverence.

ACKNOWLEDGEMENTS

This work is supported by the National Science Foundation (NSF) under Grant No. DRL-1118571, the Institute for Society, Culture and Environment (ISCE) at Virginia Tech, and the Institute for Creativity, Arts, and Technology (ICAT) at Virginia Tech. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of NSF, ISCE, or ICAT.

It brings me great joy to think of all the individuals who have helped me through the process of this forming this dissertation. In addition to my committee members, I am grateful to Leanna House, Bethany Bray, Dong-Yun Kim, Reza Soltan, and Yasuo Miyazaki for their helpful advice as I began to consider modeling approaches. The efforts of the Learning Transformation Research Group (LTRG) at Virginia Tech was instrumental in data collection, particularly that of computer science doctoral student Serdar Aslan in the software development of the *CandyFactory* and *CandyDepot* games. The support I have received from my fellow mathematics education graduate students has been invaluable. Particularly, I am indebted to Cong ze Xu for her help collecting data for my pilot study and to David Plaxco, George Kuster, Morgan Dominy, and Beth MacDonald for their feedback on my early attempts at explaining my thinking.

Without data and guidance I could not have completed this study. I would like to thank Charles, D. J., Austin, Jane, Justin, Lilly, Michelle, Dylan, and Maddie for participating in the research studies, and I am indebted to Katy Ulrich and Andy Norton for permitting me to analyze their teaching experiments. It is impossible to hyperbolize

how much Andy Norton has impacted my thinking. I feel so fortunate to have had the opportunity to learn from you these past four years. Thank you for everything Andy.

Lastly, I would like to express my gratitude to my parents, in-laws, and extended family for their role in providing me time to write and think. In particular, I am immeasurably fortunate for my wife Kendra, for putting up with me during the more intense writing stages and for providing me with unwavering love and support throughout this process.

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CHAPTER 1: INTRODUCTION

In this chapter, I provide an overview of the study, including a statement of the problem, its importance, my contribution, and my approach. I begin with synopses of my conceptual, theoretical, and methodological frameworks.

Introduction

Simply put, the relationship between the teaching and learning of mathematics is complex. The multitude of decisions that a teacher makes are potentially influenced by an even larger multitude of factors, including those pertaining to the knowledge of the teacher, his aims for teaching, and his teaching philosophies. I consider the goal of teaching quite broadly to be the facilitation of conceptual change in one's students and the art of teaching to be making decisions toward that end. Respectively, meanings for conceptual change and its facilitation are *models* of learning and teaching.

I think of models as purposeful organizations of appropos salience of experiences, which are themselves the product of interaction with and within representations of an unknowable reality. Humans construct models both within and outside of awareness to organize their experiences. *Modeling* is the process of explicitly and purposefully constructing and modifying models (Mooney & Swift, 1999). Modeling involves efforts to improve aspects of models, including validity, reliability, explicability, flexibility, and parsimony. "A model can be more accurate than the data used to build it because it amplifies certain patterns and discards unwanted noise" (Gauch, 1993, p. 468).

In mathematics education research, both the aim of teaching and the aim of modeling directly involve the structuring of abstracted experiences pertaining to student-

teacher-environment interactions. The current study focuses on how deterministic (qualitative) and probabilistic (quantitative) models of these interactions can inform one another. By contributing to our understanding of the complicity between teachers' decisions and students' learning, I hope to reduce the complexity of an aspect of the relationship between the teaching and learning of mathematics.

In this chapter, I discuss my modeling of students' mathematics from a neo-Piagetian perspective in the domain of *units coordination*, the significance and utility of consideration of units coordination in mathematics education research, and my contribution. Currently, it is hypothesized that students' available structures for coordinating units allow for more powerful ways of operating across mathematical domains such as counting, whole number arithmetic, fractions, algebra, and signed number addition (Hackenberg & Lee, under review; Norton & Wilkins, 2012; Steffe & Olive, 2010; Ulrich, 2013). Though researchers have demonstrated some commonalities between units coordination across contexts, how students' units coordination structures are constructed is not well understood in the sense that current understanding of the construction is attributed to a nonlinear process of *reflective abstraction* (Piaget, 1970a; von Glasersfeld, 1995). One aim for the current study is to provide a foundation for modeling the process by which a student eventually (1) constructs a more powerful units coordination structure, in the sense that a units coordination structure that consists of $n + 1$ levels is constructed when only n level structures existed previously or (2) modifies existing n -level structures or $n - 1$ level structures to result in a new n -level structure. By modeling students' units coordination activity, I hope to concurrently shed light on Piaget's process of reflective abstraction that extends existing explications (e.g.,

Dubinsky, 1991; Hershowitz, Schwarz, & Dreyfus, 2001; Norton, 2008; Sfard, 1991; Simon et al, 2010; Simon & Tzur, 2004; Steffe, 1992; Tzur, 2007; von Glasersfeld, 1995; 1998; von Glasersfeld & Steffe, 1991).

Synopsis of Theoretical Constructs

Scheme theory

Building on Piaget's extensive works, von Glasersfeld (1995) describes a scheme as consisting of three parts, building on the notion of a stimulus-response-expected result pattern. The first part is the *recognition template*, the assimilatory structure of a scheme. The recognition template is a *fitting* of an organism's current goals and environment. The *activity* of a scheme is a sequence of mental operations or actions that are performed in order to reach a goal. The *expected result* of a scheme is a necessary result of activity – to the extent that the result satisfactorily *fits* with the goal, the organism is in a state of *equilibrium*, and the recognition template for a particular goal/environment pair is strengthened. With scheme theory, learning is described as modifications in individuals' schemes that serve to satisfactorily restore equilibrium after some *perturbation* of an otherwise closed system.

Units Coordination

I conceive of units coordination as a *structure* that individuals construct, generally outside of awareness, for organizing a system of operative relations between units. Here a *unit* is something that has been *unitized* or set apart for further action, and the development of *composite* units, or units of units, is concurrent with the construction of units coordination structures and operations by which units are related. In the most

general sense, units coordination structures are distinguished by their number of *levels*, the existence of a *unit*, and the existence of *coordination*. Generically, a unit is an object that can be transformed, and coordination refers to a set of transformations that *result* in other units. The structural aspect refers to the idea that the structure completely and totally defines and regulates relationships between units as both possible and necessary.

Consider the following task: You have 3 nickels; how many more nickels would you need to have 35 cents? The mental activity for solving the task begins with *recognition* of the situation; i.e., the fitting of a situational goal to a scheme for which available actions (mental or physical) are expected to give a satisfactory result. The process of fitting a situation to a scheme is called *assimilation*, and a units coordination structure is used in assimilation to quickly and efficiently organize relevant information that might be important for future action.

Upon reading the words “3 nickels,” a student assimilating the situation within a three-level structure would immediately *have available* that 3 nickels is 15 cents, and could anticipate computing the value of seven nickels, simultaneously in terms of nickels and cents, etc., before reading the rest of the task. The units coordination structure is what allows for assimilation and anticipation of the operations that are associated with the units “three,” and “five,” that permit an immediate response of 4 nickels. Furthermore, if asked for explanation, the student might respond, “35 cents is 7 nickels, and 7 nickels is four more than 3 nickels,” even if her original thought process was, “3 nickels is 15 cents, and four more is 20 cents.” The equivalence of the two ways of operating is inherent in the structure.

First, second, and third-Order models

Broadly speaking, my theoretical perspective is that of radical constructivism (von Glasersfeld, 1995). One of the epistemological tenets of radical constructivism is that an individual's schemes are constructed to *fit*, rather than *match*, their experiences in interaction with the world. When confronted with a surprising outcome, or *perturbation*, individuals' schemes are modified. Along the same line of reasoning, my characterizations of students' thinking are merely models that I construct to fit my interpretation of their actions, and to the extent that they don't *fit*, they are modified.

Steffe and Cobb (1982) distinguish between models of another's thinking, which they refer to as "second-order models," and models of one's own thinking, or "first-order" models. A second tenet of radical constructivism guides the construction of second-order models – humans act rationally within their existing conceptual systems. The task of constructing second-order models of a student's thinking involves a conjecturing of the qualities of a student's conceptual system that would explain a teacher-researcher's observations. Like schemes, units coordination structures are elements of second-order models – they are a researcher's constructions, not the student's.

In this study, researchers' established second-order constructs—specific schemes and operations, such as an *iterating* operation, or a partitive unit fractions scheme—are taken as fixed, as my focus is on modeling the dynamics and relationships between students' constructions. I adopt Signe Kastberg's terminology in defining a *model of the dynamics of a second-order model* of students' mathematics a "*third-order model of a student's mathematics*." (personal communication, April 23, 2013). In my theoretical

framework, reflective abstraction, scheme theory, perturbation, abduction, and structuring are each aspects of third-order models of students' mathematics, in that they are used to structure researchers' construction of second-order models and hence contribute to the sequencing of inferences and interventions (cf. Steffe & Thompson, 2000).

The inductive inferences of a student's thinking that a teacher-researcher makes in the course of teaching are based on patterns of behavior-environment interaction.

Because humans *learn*, how they are thinking may be changing *during any sequence of observations* whence inductive inferences of their thinking are made. In my construction of a third-order model of students' mathematics, I thus conceptualize that the *inferences* a teacher-researcher makes are of a students' *propensities*, or probabilistic dispositions, to operate in a conjectured way. Unlike physical propensities, such propensities are not immutable (cf. Peirce, 1935; Popper, 1959). A modeling of a student's propensity for operating in a conjectured way is a third-order modeling of the student's learning.

While the use of a frequentist inference approach has been an invaluable part of research methodologies in mathematics education, I also see value in modeling *changes in inferences* stochastically. In the place of the posing of a hypothetical distribution and fixed data collection methods is the posing of an *a priori* transition formula that is based on the theoretical assumptions of data collection methodology. While each piece of data that is collected is dependent on the previous inferences, the procedures for framing and defining data and data collection are described in detail as in the data collection approach of a frequentist paradigm. The results of the modeling is a plot of the chronological changes in the stability of a student's units coordinating activity across the teaching experiment. Analysis of the plot in conjunction with a qualitative description of the

second-order model inferences is useful for identifying sequences of important interactions, for comparing patterns in inferences across contexts, and for making comparisons across students and teacher-researchers.

Research Questions

1. What are the relationships between a student's units coordinating activities across numerical situations such as counting sequences, fractions, whole number multiplication, and signed integer addition?
2. What sequences of ecological changes facilitate growth in the number of levels of a student's units coordinating structures?
3. How does the modeling of a student's propensity for assimilating an n -level structure contribute to an understanding of how units coordination structures are constructed?

Significance

Connections between mathematical domains

Understanding more about units coordination structures is very important, as units coordination is integral to the construction of number. "The many personalities or subconstructs of rational number that children must conceptually coordinate may all be understood as composition and recompositions of units" (Lamon, 1996, p. 192). Steffe (1992) demonstrated how relationships between students' coordination of counting sequences and unitizing constrained their schemes for multiplication and division. Hackenberg (2007) found that students who did not assimilate a three-level multiplicative conception (i.e., an assimilation like 36 as a unit of 9 fours, each of which is a unit of 4

ones) were unable to conceptualize improper fractions. Ulrich (2012) found that students who could not coordinate three levels of units in activity were unable to count down below zero. Hackenberg and Lee (under review) found that students' multiplicative conceptions also constrained their reasoning with algebraic expressions and equations. Such findings support Steffe's *reorganization hypothesis*, which originally stated that "children's fraction schemes can emerge as *accommodations* in their numerical counting schemes" (Steffe & Olive, 2010, p. 1) and has accordingly been extended to conceptions of linear equations (Hackenberg & Lee, in review) and signed number addition (Ulrich, 2012).

Attending to commonalities and differences

Siegler, Thompson, and Schneider (2011) argue that attending to the commonalities in students' whole number and fractions knowledge as well as their differences might "make possible a more comprehensive theory of numerical development, including detailed analyses of acquisition of understanding of all types of real numbers, acquisitions that are made from infancy through adolescence (p. 289). Modeling students' ways of operating across contexts allows better understanding of how learning takes place outside of an individual's awareness. Modeling the changes in students' *propensity* for stability in units coordination permits identification of multifaceted contingencies that accompany the learning process.

"The structure of consciousness is different from the structure of activity: we are aware of the most relevant properties of objects, but more or less ignorant of the way activity is generated and the way subschemes are activated by superschemes. This hierarchical organization leaves room for improvisation and contingency: schemes and subschemes are often called upon by contingent aspects of situations; it is the reciprocal character of their adaptive function" (Vergnaud, 2009, p. 93).

Methodology

Teaching experiment methodology

Varieties of teaching experiment methodologies focus on particular aspects of the complex relationship between teaching and learning. Researchers of individual students' mental constructs tend to conduct small-group or individual-student experiments in a clinical setting (e.g., Steffe & Thompson, 2000; Siegler & Crowley, 1991; Wagner, 2006). Teacher-researchers whose focus includes sociocultural elements tend to conduct classroom-level teaching experiments in which teachers and researchers collaborate (e.g., Cobb, 2000; Confrey & LaChance, 2000; Streefland, 1993). The former approach generally allows for more fine-grained analysis of an individual and flexibility in instructional decisions that provide insight into *modeling* at the expense of excluding classroom-specific sociocultural factors.

Because of my primary interest in modeling learning, rather than designing instruction, I adapt a radical constructivist paired-student teaching experiment methodology for the construction of second-order models of students' mathematics (Steffe & Thompson, 2000). My approach to building a third-order model, summarized below, involves iterative model formulation and evaluation of agreement with a teacher-researcher's inferences. I constrained measures of model validity and reliability to be applicable to data obtained via this methodology.

Methods

Model building

My approach to answering the research questions above involves building models of the learning that takes place during radical constructivist teaching experiments involving units coordination, with sixth-grade students who were taught by teacher-researchers adopting a neo-Piagetian, radical constructivist epistemology (von Glasersfeld, 1995). While both traditional quantitative and qualitative methodologies were employed, the focus of the study was to explore how mathematics could be used to enrich and build new qualitative models of units coordinating activity. The current study builds off of a short study conducted in the Summer of 2012 with sixth-grade students, in which I was the teacher-researcher and third-order modeler.

Using existing data from two other teaching experiments, I built a model of the students' units coordinating activity that was later tested with the data from a third teaching experiment. To build the model, I began by transcribing, chunking, and coding the teaching sessions from a teaching experiment with Austin, a sixth-grade student who was taught by my adviser, Anderson Norton. I was the witness for this teaching experiment, which meant that I was present for the teaching session and contributed to analysis of the data between teaching sessions but had little to no interaction with the students during the teaching sessions. This teaching experiment actually took place before my the initial third-order modeling, in the Fall of 2011.

Transcription included dialogue and pertinent visual information judged to be in Austin's perceptual field. I then *chunked* the transcription into units of analysis and dichotomously coded the chunks for Austin's *units coordinating activity*. Via the

algorithm generated from my initial study, I used this sequence of codes to generate a plot of Austin's *propensity to coordinate units*. I then interpreted the result of the plot, conjecturing relationships between the values of p , defined as the propensity to *not* coordinate units, and the values of q , defined as the propensity to coordinate units. Chapter 3 describes the methodological constructs of *units coordinating activity* and *propensity to coordinate units*, while Chapter 4 describe the chunking and coding process, as well as the algorithm, in a detailed description of my pilot study.

The next step in model-building was to improve the algorithm by incorporating the *context* of chunks. I had mathematized inferences of units coordinating activity as subjective probabilities for stability in coordinating units using a simple, parsimonious formula based only on the current and previous inference of units coordinating activity. I went back and used the constant comparative method (Glaser, 1965) to construct a manageable set of contextual codes that could be used to improve upon the model. Here the goal was to provide a mechanism for characterizing some inferences of units coordinating activity as stronger than others that would improve the utility of the output.

To build a more useful "improvement" of the model, I needed a measure of model utility. For this, I used the teacher-researcher's qualitative analysis of the teaching experiment, marking explicit indications of Austin's units coordination. The plot of the units coordinating activity would by its consistency with this second-order model. That is, if high values of q and low values of p coincided with inferences of units coordination. In Chapter 5, I outline the steps and approach I used to incorporate context into the model.

I repeated the above activities with the first 10 sessions of a teaching experiment conducted by Catherine Ulrich, focusing on the units coordinating activity of a sixth-grade student named Justin. Ultimately, I found that I was unable to improve upon the approach generated from the initial study and used the initial algorithm in further plots of students' propensity to coordinate units.

I next sought out improvement of my analysis using the output from the context-free algorithm. Here, I explored various methods of objectively characterizing the salient feature of the patterns in the graphs of p and q in conjunction with qualitative descriptions of the contexts of those patterns. My goal was to construct an approach that would be general enough to apply to the third data set, but specific enough to allow for the approach to be predictive as well as explanatory and descriptive. The results of this approach, described in Chapter 6, were used to categorize the Austin and Justin teaching sessions as one of 8 types, firstly quantitatively, by patterns in p and q , and secondly qualitatively, using the second-order model to describe (broadly) the content of the sessions.

Model testing

In the model-building stage, I was continually testing and re-forming my approach in attempt to improve the results of the model, e.g., to make it more useful for explaining, predicting, or describing. This ended in the Spring of 2014, when I formed a final model to test with a third data set. The testing data came from a teaching experiment with a sixth-grade student named Dylan that was conducted purposefully to enable testing of potential conjectures that might arise from the analysis of Austin and Justin. Anderson Norton was the teacher-researcher, and I again served in the role of the

witness. Because of my dual role as third-order model builder, I was careful to not influence task sequencing. The teaching experiment was conducted in the Fall of 2013, as I was analyzing the Austin and Justin data.

I went through the transcribing, chunking, and coding for units coordination activity for the Dylan data prior to completing the final version of the model to be tested. I identified explicit inferences of Dylan's units coordination in the teacher-researcher's descriptive model, and I matched those to chunks via the transcript. To test the validity of the approach for describing units coordination, I explored whether there existed qualification of "high" values of q and "low" values of p that coincided with inferences of units coordinating activity. I formed five conjectures about the patterns in teaching sessions, based on the quantitative and qualitative characterizations, to assess whether the model was explanatory and predictive.

Outline of Methods

1. Transcribed, coded, and chunked nine teaching experiment sessions (each) for Austin attending to existing second-order models of his mathematics as described by the teacher-researcher.
2. Created baseline propensity plots for Austin.
3. Created codes for chunks for Austin's teaching experiment using the constant comparative method.
4. Calculated descriptive statistics of chunks' contexts and considered adjustments to propensity calculation to include weights of particular chunks to fit with descriptions of units coordination as described in the second-order model.

5. Segmented the teaching experiment by session (after first attempting to segment using polynomials of best fit).
6. Qualitatively analyzed the propensity plots of Austin, attending to extended sequences of increases in p , q , or extended sequences of volatility in p and q .
7. Repeated steps 1-6 for Justin
8. Served as witness in a teaching experiment involving Dylan, which included units coordination activity across contexts.
9. Transcribed, chunked and coded all 11 sessions of Dylan's units coordinating activity using the formulation resulting from Step 2.
10. Formed conjectures about the relationships between p , q , *context*, and *perturbation* from the Austin and Justin data.
11. Formed the model of p and q and described the sessions of Dylan's teaching experiment using the approach from Step 10.
12. Evaluated the utility of the model for describing, explaining, and predicting Dylan's trajectory of units coordinating activity.

Results

The results are split into two chapters. In Chapter 6, I describe the results of the modeling process with the Austin and Justin data that led to the formation of a model that could be tested with the Dylan data. Part of the testing was the formation of conjectures to provide indication of validity, reliability, and predictability. In Chapter 7, I describe the results of the testing. This includes evaluation of the use of the model for describing, predicting, and explaining Dylan's units coordinating activity. The appendices contain qualitative and quantitative data used in the development of the third-order models.

Conclusions

In Chapter 8, I interpret the results of Chapter 6 and Chapter 7. I begin by evaluating and describing the limitations and benefits of the modeling approach. I then interpret the results, emphasizing relationships to existing constructs in the mathematics education and educational psychology literature.

CHAPTER 2: THEORETICAL AND CONCEPTUAL CONSTRUCTS

In this chapter, I detail constructs that I use to model students' mathematics. Generally speaking, I adopt a scheme-theoretic perspective, but I include a more general notion of Piagetian *structure*. I describe my conception of reflective abstraction as the process by which conceptual structures and their constituent operations change. I characterize *n*-level units coordination structures and give examples of the structures, schemes, and operations pertaining to units coordination with counting sequences, multiplication, and fractions. I then describe results from prior studies that have informed my hypotheses about the relationships between these conceptual constructs.

Von Glasersfeld and Steffe's Scheme Theory

In radical constructivism, learning is described as change in individuals' schemes. Building on Piaget's extensive works, von Glasersfeld (1995) describes a scheme as consisting of three (not completely distinct) parts (see Figure 2.1). The first part is the *recognition template*, which is the assimilatory part of a scheme. The recognition template is a *fitting* of one's current goals and environment. The *activity* of a scheme is a sequence of mental operations or physical actions that are performed in order to reach a goal. The *expected result* of a scheme is a *necessary* result of activity. Schemes change via interiorization, assimilation, and accommodation.

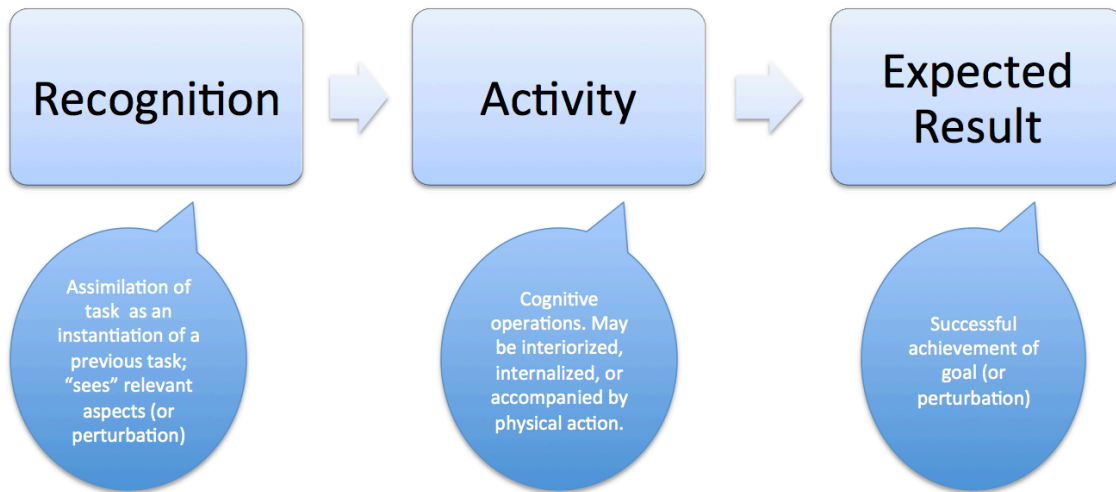


Figure 2.1. The Three Parts of a Scheme

Interiorization

At first, the activity of a scheme might include physical activity as well as cognitive activity. These are called *action* schemes. Schemes for which conscious re-presentations of actions are required, e.g., mental imagery of actions but not the physical actions themselves, are termed *internalized* schemes. Note that re-presentations are distinguished from representations, which are symbols of actions, like pointers. Both action schemes and internalized schemes retain a notion of sequentiality in the three-part structure, as part of the activity portion takes place in conscious awareness. When an internalized scheme becomes *interiorized*, the three parts of a scheme become like a single entity – the recognition template includes the result of the activity.

Only an interiorized scheme can become part of another cognitive scheme, as interiorization is required for a scheme to become a *concept* (Hackenberg, 2007). A *concept* is a “place-holder or variable for some of the properties in the sensory complex

we have abstracted from our experiences of particular things” (von Glasersfeld, 1991, p. 49). The recognition template of a scheme is composed of organizations of concepts.

Assimilation

Assimilation is the experiencing of a situation as fitting the recognition template for an established scheme. If assimilation is successful, meaning that the result of activity fits in an expected way, the organism is in a state of *equilibrium*. As the result of such activity, modification to the scheme is minimal – only the recognition template is affected as the association between a situation and the scheme is strengthened. Repeated assimilation resulting in equilibrium results in the *establishment* of the scheme as both viable and reliable. To the extent that a recognition template for a scheme is broadened, in the sense that situations that were not previously associated with a scheme become assimilated, there is *modification* to the recognition template. If the recognition template is modified substantially, this type of modification is referred to as *generalizing assimilation*.

Accommodation

When an individual does not successfully assimilate a situational goal, there is a state of *perturbation*, which is the impetus to construct a new scheme or modify the scheme. Accommodations are modifications of schemes that occur to satisfy both immediate goals of removing perturbation and meta-goals of improving coherence and efficiency by which to reduce future perturbation. Generalizing assimilation is accommodation that improves efficiency by reducing the need for multiple recognition templates. Like generalizing assimilations, *functional accommodations* are relatively

minor or temporary modifications of schemes that occur during the context of their use. They involve introducing an operation or activity that had previously been external to the scheme (von Glasersfeld & Steffe, 1991). In contrast, *metamorphic accommodations* reflect major changes to the activity portion of a scheme, and they take place after long periods of rest from the related schemes. (Steffe, 1991).

Piagetian Structures

Piaget (1970b) defines a *structure* as comprising “three ideas: the idea of wholeness, the idea of transformation, and the idea of self-regulation” (p. 5). Wholeness refers to the idea that, although structures have individual elements, the relationships between elements are defined by rules that apply to a consolidated totality, rather than particular elements. Transformation refers to the idea that although structures are not static forms, applying a transformation results in a recognizable change. Structures are a framing for “intelligible change that always preserves invariance in certain respects” (p.20). Self-regulation refers to the idea that the “transformations inherent in a structure never lead beyond the system, but always engender elements that belong to it and preserve its laws” (p. 14).

A scheme is an example of a structure. “Piaget (1970a) considered a scheme to be a general structure of a set of actions that conserves itself by repetition, consolidates itself by exercise, and applies itself to situations that vary because of contextual modification” (Hunting, Davis, & Pearn, 1996, p. 356). *Assimilation and accommodation* are meta-operations of the scheme structure. The use of schemes across contexts is what leads to the establishment of invariance in the transformations resulting from the actions of the scheme. Generalizing assimilations, functional accommodations, and metamorphic

accommodations are operations that lead to the establishment or construction of a new scheme. The accompanying meta-operation of *interiorization* leads to consolidation of the actions as a singular transformation. Interiorization permits the development of more complex structures.

Piaget(1970b) describes three parent-structures: algebraic structures, order structures, and topological structures. Unlike the general notion of scheme, the parent-structures are self-regulated by notions of *reversibility*. Algebraic structures are characterized by reversibility taking the form of inversion, as in $5 + (-5) = 0$. Order structures are characterized by reversibility taking the form of permutation of elements, as in $5 > 4$ implies $4 < 5$. Topological structures are characterized by reversibility taking the form of separation and connectedness. The three parent-structures were originally posed by the Bourbaki as irreducible structures for mathematics. They were displaced by category theory, which categorizes types of morphisms that *result* in parent structures (ibid).

Piaget concludes that category theory is a result of reflective abstraction that “derives its substance, not from objects, but from operations performed upon objects, even when the latter are themselves products of reflective abstraction” (1970b, p. 28). When he writes that the substance is “not from objects,” it can be misleading. The operations *are simultaneously* objects *within* category theory and operations *on* parent structures. In my view, all Piagetian structures are the result of what I characterize below as the stronger form of reflective abstraction, in which operations at one level become structured as objects in a higher level.

Reflective Abstraction

Piaget distinguished several types of abstraction, but his characterizations reflected his own changing perspectives throughout his long career (Piaget, 2001; von Glasersfeld, 1995). The first type, *empirical* abstraction, is when abstractions are formed from (sensorimotor) awareness of actions on material objects. The second type, *pseudo-empirical* abstraction, takes place only in the presence of figurative material and involves abstractions on empirical abstractions. For example, if a child picks up four red bars and arranges them in a line above a brown bar, and then says that the brown bar is equal to 4 red bars, it is an example of pseudo-empirical abstraction – the equality of length was abstracted through the child's actions.

In his interpretation of Piaget, von Glasersfeld distinguishes two types of *reflective abstraction* (1995). The first type of reflective abstraction involves a projection and reorganization of a “coordination or pattern of the subjects' own activities or operations on another conceptual level” (p. 105). The structure of operations from a lower level becomes a singular entity that can be operated upon by operations in the higher level. The second type, termed ‘reflected’ abstraction, similarly involves a restructuring on a higher conceptual level, but it also includes awareness of the abstraction. When considering types of structures other than schemes, such as units coordination structures, there is a need to discriminate differently between the types of potential reorganizations that take place as a result of reflective abstraction. Instead of distinguishing between reflective abstractions within or outside of awareness, I distinguish two types of reorganization that might take place as a *weaker* form of reflective abstraction and a *stronger* form of reflective abstraction. In the weaker form of reflective abstraction,

operations of a higher conceptual level *act upon* the structure reflected from the lower level, resulting in (re)structuring of the lower level actions.

Type of Abstraction:	Abstraction upon:	Requires:
Empirical	Things	Sensory-motor material
Pseudo-Empirical	Actions	Sensory-motor material
(type 1) Reflective	Actions	Projection and reorganization
(type 2) Reflected	Actions	Conscious thought, projection, and reorganization

Figure 2.2. Von Glasersfeld's Characterizations of Abstraction

Interiorization is an example of the weaker form of reflective abstraction (Figure 2.3). In the stronger form of reflective abstraction, the operations of the higher level become (re)structured (see Figure 2.4). With the stronger form of reflective abstraction, an interiorized scheme becomes a *concept* within a (higher level) conceptual structure.

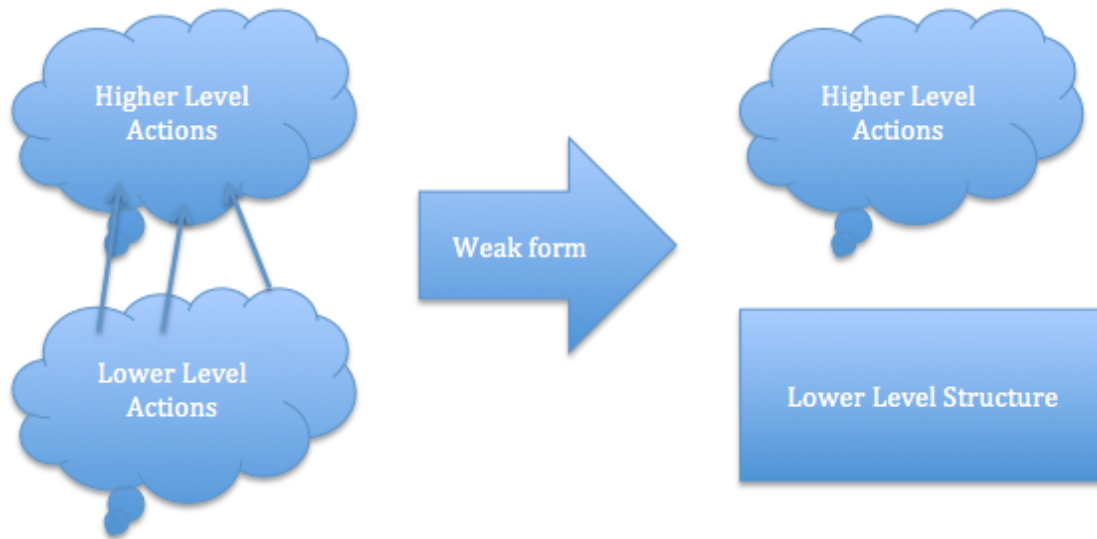


Figure 2.3. Weak Form of Reflective Abstraction

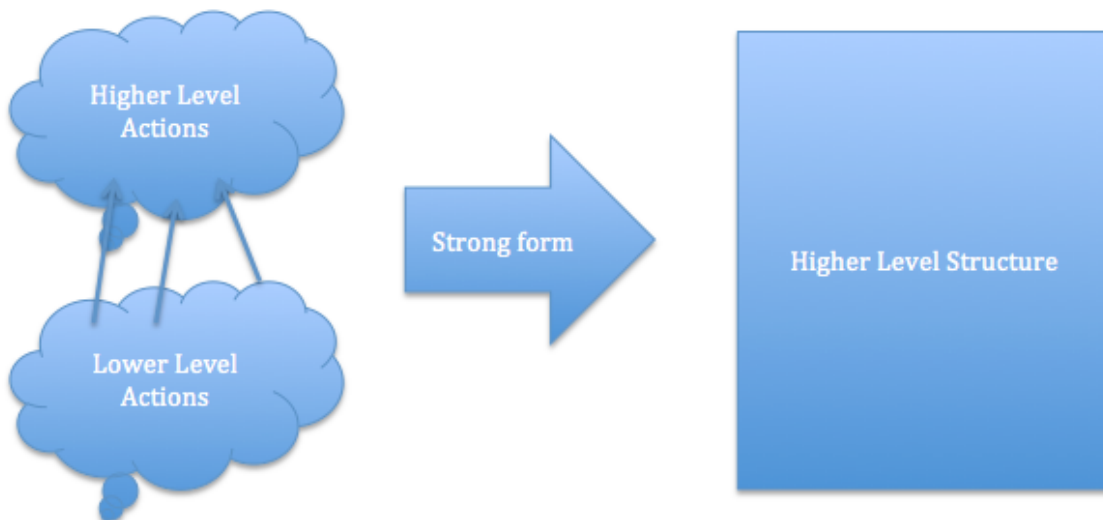


Figure 2.4. Strong Form of Reflective Abstraction

Example of splitting

The following examples may help clarify weak and strong reflective abstraction. The *iterating* operation is the making of connected copies of a unit (Steffe & Olive, 2010). When the iterating operation becomes interiorized as part of a goal-directed scheme, it is the result of the weaker form of reflective abstraction. The weaker form of reflective abstraction results in the concept of *iterable unit*. It does not result in a new way of operating *on* iterating as an object. When the iteration operation can be transformed, i.e., acted upon, rather than merely *used* in a goal-directed situation, it is the result of the stronger form of reflective abstraction. The construction of a splitting operation, i.e., the structuring of iterating and partitioning as inverse operations, is therefore a result of the stronger form of reflective abstraction.

The structuring that results from the stronger form of reflective abstraction implies structuring of the weaker form. For example, the construction of the splitting structure results in iterating and partitioning becoming a single composition, so that splitting replaces extant iterating and partitioning operations. A replacement of sequential iterating and partitioning operations with splitting in one's fraction scheme is a result of the weaker form of reflective abstraction (cf. Norton & Wilkins, 2012). Thus, the structuring *of* the iteration operation as splitting and structuring *with* the splitting operation would seem to happen at the same time.

Note that this simultaneity is partly to blame for disagreements between researchers regarding students mathematical development. The term *splitting* has been used by Confrey with a weaker meaning – as an “action of creating equal parts or copies of an original...that requires only recognition of the type of split and the requirement that

the parts are equal” (Confrey, 1994, p. 300). Confrey models splitting as independent from iterating, leading to a separate system of number based on symmetric or exponential reasoning. However, Confrey characterizes a ‘splitting *structure*’ as the result of *sequential* splitting operations. Confrey claims that the *naming* of a splitting structure is separate and less primitive, and that the counting numbers are used to name the outcome of a split only to be consistent with the result of iteration. For example, thrice doubling the result of a sequence of three 2-splits could just as well be named as three twos if it were not for the inconsistency of three twos meaning 6 ones in a counting world.

Regardless of its naming, the formation of a splitting *structure* as the result of a *sequence* of Confrey splits requires coordination – if not initially of the units resulting from the split, then of the sequence of splitting actions. “Confrey’s splitting analyses seem to point to actions on actions as opposed to number scheme research (e.g., Steffe, 1994) which points to units of units” (Kieren, 1994, p. 392). With a more broad interpretation of “unit,” one may consider Confrey’s splitting structure as the coordination of two types of units: a counting sequence and splitting actions. As discussed above, in order to be a *structure*, in the Piagetian sense, the result of operating must be a *recognizable* transformation. If the operations of a splitting structure are coordinations of a counting sequence with splitting actions, then the result of sequential splitting should also be a coordination of a counting sequence with splitting actions.

Note that the result of any sequence of splits is equivalent to some sequence of 1-splits. For example, successive doubling of a given amount three times is equivalent to successive iterating of a given amount 8 times; successive halving three times is equivalent to equi-partitioning into 8 pieces. Thus, it seems that it cannot be until

Confrey's splitting operation, iterating, and partitioning become simultaneous that Confrey's splitting structure becomes a structure in the Piagetian sense – when Confrey splitting and Steffe splitting are equivalent. This is the result of the stronger form of reflective abstraction, when operations on units and operations on other operations are integrated to form what Norton and Wilkins call the *splitting group* (2012).

In the preceding section, I have described one interpretation of *conceptual change* as being the *result* of reflective abstraction. The example with splitting also illustrates that it is quite difficult to describe the process of reflective abstraction (cf. diSessa & Sherin, 1998).

It is not possible to observe directly any of a subject's [constructs]. We can only infer them from our observations who may or may not bring them to bear on problems. But these very acts of recognizing and solving problems, of asking new questions and creating new problems are the means ... by which a subject constructs new knowledge. This is where reflective abstraction comes in. ... We have little to say about how ... knowledge exists inside a person. It does not seem to reside in memory or in a physiological configuration. All we can say is that a subject will have a propensity for responding to certain kinds of problems in a relatively (but far from totally) consistent way ... which we can (as far as our theory has been developed) describe in terms of [constructs] (Dubinsky, 1991, p. 103).

A model of a *process* of interaction (and intra-action) that results in a determination of a subject's "*having*" a *propensity for stability* in responses to certain problems is a model of a process of reflective abstraction. Such a model necessarily depends on the constructs that are used to describe the inferences. Below, I describe the constructs I use to model students' thinking pertaining to units coordination, in relation to those described in the literature. In Chapter 3, I discuss my interpretation of *propensity for stability* in the context of my methodology.

Units Coordination Schemes, Concepts, and Structures

Recently, radical constructivists have begun incorporating notions of structure to describe units coordination as something other than a scheme. Hackenberg and Tillema (2009) characterized *multiplicative concepts* in terms of the number of levels of units a student could construct, but did not define the term *concept*. Hackenberg (2010b) later defines that "a concept is an interiorized scheme" and "the all-at-once sense of a concept is what [she] interprets Piaget and von Glasersfeld as meaning when they refer to conceptions as structures" (p. 389). Hackenberg (2010b) writes about multiplicative *structures* as the *result* of *interiorization* of multiplicative concepts, which are themselves the result of the *interiorization* of multiplication *schemes*. However, Norton and Wilkins (2012) write about the "organization of three levels of fractional units" (2012, p. 560) and do not refer to levels of units as multiplicative concepts.

Thompson and Saldanha (2003) write about *conceptual schemes* as "stable ways of thinking that entail imagining, connecting, inferring, and understanding situations in particular ways" (p. 12). For Norton, *schemes* are also closely tied to operations (or mental actions). While a scheme can be interiorized and thus become a concept, the reflective abstraction that results in an *organization* of concepts is not considered as an expected result of an organizing meta-scheme. Rather, it is considered the formation of a different type of Piagetian structure (personal communication, February 12, 2013).

Steffe (2001) describes units coordination as both an *operation* and as a *scheme*. Units coordinating is a "mental operation of distributing a composite unit across the elements of another composite unit" (p. 279).

The result of the units coordination permits the child to experience a unit containing five units of two that is not simply a sensory-motor experience. This organized

experience is what I regard as a situation of the scheme. The activity of the scheme in the case of the example is to count by two five times to specify the numerosity of the individual units contained in five units of two, and the results of the scheme is the experience of the immediate past counting activity along with its result. (p. 279)

Steffe (1992) described students' construction of units coordinating *schemes* as reorganizations of their counting sequences, but the relationship between the construction of units coordination schemes and the construction of units coordination structures — the nature of the reorganization — is not well understood. I find the terminology used to describe units coordination lends itself to inconsistent interpretations. The confusion is compounded by the fact that a units coordination structure *for a situation* can be distinguished from the more abstract structure.

With respect to Hackenberg's terminology, I use the term *concept* to refer to an interiorized structure that can be part of a recognition template for a particular scheme, while using also the term *structure* when referring to a generic *n*-level structure. For example, *multiplicative concepts* are particular units coordinating structures in the context of whole number multiplication (Hackenberg, 2010). Recent empirical evidence supports the hypothesis that *n*-level units coordination structures develop in tandem across contexts, necessitating the need to describe units coordination in terms of constructs that transcend context (Boyce & Norton, under review; Hackenberg & Lee, under review).

I use the term *units coordinating activity* to refer to students' (inferred) use of a units coordination structure in either the recognition portion or activity portion of a scheme. I reserve the term (*n*-level) *units coordinating structure* to refer to the number of levels of *interiorized units*. Thus, for someone assimilating a situational goal within an *n*-level units coordinating structure, *units coordinating activity* refers to inferences of

activity that, if interiorized, would be consistent with assimilation with $n + 1$ levels of units.

In the most general sense, units coordination structures are distinguished by their number of *levels*, the existence of at least one *unit*, and the *coordination* across different levels of units. Generically, a unit is an object that can be transformed, and coordination refers to a set of transformations that *result* in other units. The structural aspect refers to the idea that the structure completely and totally defines and regulates relationships between units as both possible and necessary.

An affine space metaphor for units coordination structures

Before I describe my model for the construction of units coordination structures, I offer an affine space metaphor by which to consider their generality and constraints. Take a one-level units coordination structure to be a point, and consider geometric translation. Though a generic translation of a point could be said to determine a line, the result of the action within a one-level structure is another point. Within a 1-level structure, the result of the translation of a point depicted in Figure 2.5 would not include the line segment.



Figure 2.5. Metaphor for a One-level Structure.

A two-level units coordination structure is like a line consisting of points. Though a generic translation of a line could be said to determine a plane, the result of the

translation within a two-level structure is another line. Translations of a line automatically translate its constituent points, but the result is not a plane within a 2-level structure.

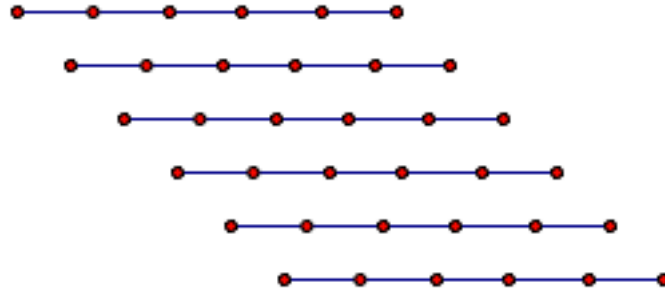


Figure 2.6. Metaphor for a Two-level Structure as a Line Containing Points

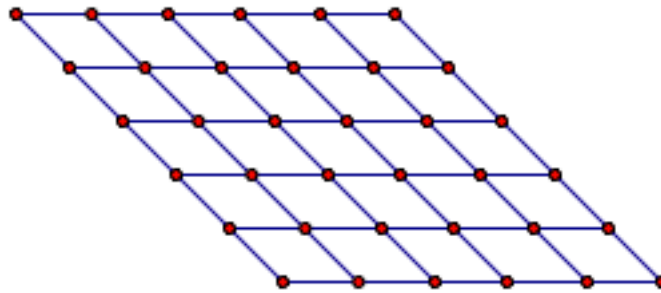


Figure 2.7. Metaphor for a Three-level Structure as a Plane Containing Lines and Points

Similarly, a three-level units coordination structure is like a plane. Just as two generic points determine a line, two generic lines determine a plane, and two generic planes determine a space, the actions on an n -level units coordinating structure eventually form the next level structure. Translations of planes translate both their constitutive lines

and points. Just as no one can *experience* four-dimensional space, no one is theorized to have constructed a four-level units coordinating structure (Ulrich, 2012).

The key aspect of the metaphor is that it distinguishes whether action on a unit transforms or merely translates the unit. Another important use is to illustrate the potential for deficient transformation. Figure 2.6 represents a two-level units coordination structure as translations of line segments, in which the result of each translation is another line segment that can be similarly translated. No new structure is created, but the structure of points within lines is maintained. Figure 2.8 and Figure 2.9 display possible “planes” that correspond with the result of activity on the result of transformations with only two-levels of units structures. Here the new structures for the transformations are deficient, in comparison with the three-level structure of Figure 2.7.

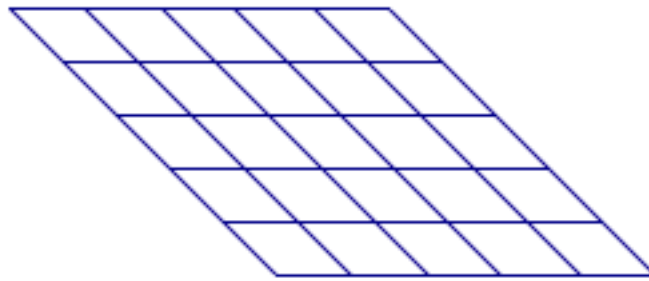


Figure 2.8. A Two-level Structure of a Plane Might Not Include Points

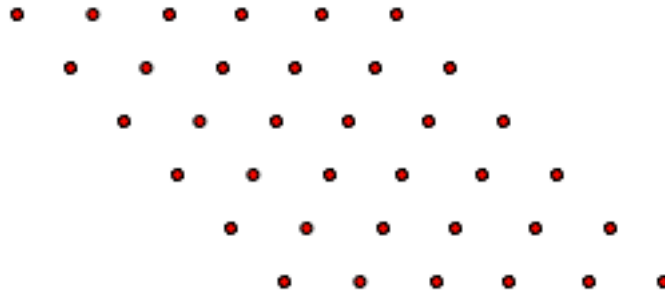


Figure 2.9. A Two-level Structure of a Plane Might Not Include Lines

There are limits to the geometric metaphor of units coordination. While I do not mean to constrain units coordination structures to those for which *containment* is necessary, *composition* of units is required. Units coordination structures could also be for quantitative units rather than purely numerical units. For example, a three-level units coordination structure might coordinate units of distance, time, and speed. I interpret and adopt Steffe's (2001) reorganization hypothesis as positing that students' units coordinating structures *can be* co-constructed with students' construction of counting sequences and unitizing operations.

Von Glasersfeld's model for initial unitizing and counting

Von Glasersfeld's (1981;1995) attentional model for a human's initial conception of an object seems viable to me. Attention is a pulse-like activity that picks out some signals supplied from the nervous system for further processing. Attention can focus both on signals and on re-presentations of (possibly composite) signals. Von Glasersfeld distinguishes between focused and unfocused attention for creating intervals or continuity, and describes the need for memory in order to attend to re-presentations and

reflect on prior foci. Attending is completely internal – it is attention to the signals and re-presentations of signals, and thus is a form of reflection.

Von Glasersfeld describes the forming of a “whole” as bookended by moments of unfocused attention. For example, “O I I I I I I I I O” (unfocused, focused...focused, unfocused). “The concept is the result of a reflective abstraction that separated and retained an operational pattern from the sensorimotor material that provided the occasion for its constitution” (1995, p. 169).

Von Glasersfeld differentiates between noticing a plurality and counting. Counting requires a bounding of some sort, perhaps figurative or temporal. But plurality itself can be made a unitary concept, like ‘books on the table,’ despite there being an unbounded number, through reflective abstraction. Both a plurality and a number are “multitudes composed of units.” To transform plurality into number requires ‘conceptual iteration’ and ‘counting’ (1995, p. 172). Iterating attention is sequential focus on a unitary item, followed by unfocused attention, etc. Counting is the associating of each iteration with a number word that is part of a *fixed sequence*. This coordination of *iterating attention* with utterance of a number word is the initial construction of a 1-level units coordinating structure - it is the child’s construction of an *initial number sequence*.

Number sequences

A number sequence is a “sequence of abstract unit items that contain records of counting acts” (Steffe, 2001, p. 268). The *initial number sequence* is marked by students’ construction of ‘1’ as a unit and of number words, like, ‘five,’ as pointing to the result of counting ‘1’ five times. Here the ‘transformation’ is counting. Each ‘count’

puts the ‘one’ in a new place along a number sequence. The number word ‘five’ refers to a location of a metaphorical point at the end of the sequence, not a *collection* of points.

For students with a *tacitly nested number sequence*, coordination of ‘one’ and ‘five’ is also within a one-level structure. Considering the affine space metaphor, the number word ‘five’ refers to the location of a point along a sequence of line segments connecting 1 and 2, 2 and 3, 3 and 4, and 4 and 5 (see Figure 2.7). Here the transformation is ‘one more’ rather than counting, but the *result* of counting is still only the single location, five. The tacitly nested number sequence enables a student to “count on,” i.e., to know that 3 more than 5 is 8 because 6, 7, 8 is three “movements” from 5. Such double counting is an example of *coordinating* two one-level unit structures *in activity*.

It is the *interiorization* of such activity that results in an *explicitly nested number sequence* (ENS) in which students coordinate numbers like one and five *within* a two-level structure. ‘Five’ then refers to a collection of five one-units, and can properly be called a *composite* unit. With respect to the affine space metaphor, ‘five’ refers to a line. All the ways that ‘one’ could be transformed can now apply to ‘five’ while maintaining the relationship between ‘one’ and ‘five’ and the transformations of ‘one’. While a student with a TNS might be able to reason that there are 3 ‘fives’ in 15 by internally counting by ones and double counting every five internal number utterances, subsequently asking how many threes are in 15 would require a similar activity. For a student who has constructed an ENS, transformation of the ‘fives’ is simultaneous with the transformation of the ‘five ones’, so that the action of counting to 15 would *result* in understanding 15 as ‘5 threes,’ ‘3 fives’ and ‘15 ones.’ The ENS enables students to

anticipate the necessity of the equivalence of transposing double-counting and skip-counting, e.g., the relationship between ‘5, 10, 15’ and ‘3, 6, 9, 12, 15.’

It is with a *generalized number sequence* (GNS) that the above relationships between 15, 5, and 3 are *assimilated* without counting activity, marking the formation of a three-level structure. Within the two-level metaphorical structure, number words like ‘15’ are assimilated as lines containing points or as planes containing lines, but not both simultaneously (see Figures 2.8 and 2.9). When assimilated within a three-level metaphorical structure, the number word ‘15’ now refers to a plane, consisting of lines, consisting of points (see Figure 2.10). The three-level structure of the GNS allows for flexible transformations.

Fraction Schemes

Building off of years of work stemming from Piaget, Inhelder, and Szeminska’s (1960) studies of the development of measurement conceptions, Steffe and Olive (2010) describe a hierarchy of students’ fractions schemes in terms of their associated operations, including operations associated with units coordination. The partitive unit fraction scheme (PUFS) is the first genuine fraction scheme (Steffe, 2001). Like a part-whole scheme (PWS), fractional language is part of the recognition template for a PUFS. Unlike the PWS, the PUFS necessitates sequential operations of partitioning and iterating, and a preliminary notion of measure is inherent in the expected result of the scheme (Norton & Wilkins, 2010). A student with a PUFS understands a *unit* fraction, such as $1/n$, as an *amount* that, when iterated n times, would result in a whole.

A student with a part-whole scheme must have constructed at least 2 levels of units. However, these units are not *fractional units*. The part-whole scheme includes a

disembedding operation, in which a unit within a (composite unit) whole can be removed from the whole and compared to the whole without destroying the whole (Steffe & Olive, 2010). In addition to the operations of the part-whole scheme, the PUFS requires one level of fractional units to be assimilated. With the PUFS, a fraction like $1/n$ is an *iterable fractional unit* (Steffe, 2001, p. 291). In addition to the operations of the part-whole scheme, it also includes an *iterating* operation, in which a whole unit is formed by iterating one of n pieces. The iteration is constrained, however, to forming only the whole – within the PUFS, the fraction $1/n$ is an iterable unit of 1 within n (Steffe & Olive, 2010).

The units coordination structures for coordinating units involving fractional language are as follows. In a PWS, a two-level units coordinating structure is required because of the disembedding operation, but the *units are not the result of iteration* of the partitioned parts. Both the PWS and PUFS require disembedding of a unit from within a composite unit.

Figure 2.10 displays figurative material that could be used to distinguish between a PWS and a PUFS. Suppose a student is presented with the task, “What fraction is the shaded rectangle of the whole rectangle?” When making marks on the figure, a student assimilating the situation within a PWS would likely *segment*, attempting to make equally-sized pieces within the unshaded portion. A student assimilating the situation within a PUFS might also need to make marks, but she would instead *iterate* a measure of the length of the shaded piece to determine the marks’ placement; perhaps by spreading her fingers the width of the shaded piece and repeatedly sliding her fingers across the unshaded piece. A student with a PWS is unlikely to respond $\frac{1}{2}$, but would be unlikely to

be able to give a response without making marks to create a figural representation by which to count. Units coordination structures involving two are special, as the fragmenting of a whole into two ones involves primitive notions of equal sharing that are inherent in the development of unitizing (Piaget & Inhelder, 1967; Steffe & Olive, 2010; cf. Confrey, 1994).



Figure 2.10. A task for assessing the PUFS/PWS

A generalization of the PUFS to *non-unit* proper fractions results in the emergence of a partitive fraction scheme (PFS). A student with a PFS understands a fraction such as m/n as m of an amount that, when iterated n times, would result in the whole. In addition to being constrained to understandings of proper fractions, the PFS is limited by its non-reversibility. Students must construct a *reversible partitive fraction scheme* (RPFS) in order to construct the whole from a given non-unit proper fraction. The RPFS is aptly named because it permits the “reversing” of the operations of the partitive fraction scheme, in that a student conceives that a whole results from iterating, n times, ‘1’ of the m pieces constituting m/n .

With both the PFS and RPFS, non-unit fractions are not iterations of unit fractions. Iterations within the partitive schemes are iterations of ‘1,’ which is part of ‘ n ,’ instead of iterations of ‘ $1/n$.’ A student with an iterative fraction scheme (IFS) understands a fraction m/n as m iterations of the amount $1/n$. This enables the student to iterate to form a size larger than the whole. Thus, the construction of an IFS allows for a conception of improper fractions as “numbers in their own right.”

(Hackenberg, 2007, p. 207)

Multiplicative concepts

Hackenberg and colleagues have written extensively about units coordinating structures for whole numbers, which they term *multiplicative concepts* (Hackenberg, 2007; 2010b; Hackenberg & Lee, under review; Hackenberg & Tillema, 2009). The first multiplicative concept (MC1) involves coordination of two levels of units in activity. This is considered the most basic units coordination for multiplication, and involves “inserting a composite unit into the units of another composite unit” (Hackenberg & Tillema, 2009, p. 3). Students who are yet to construct this coordination are considered MC0. *In activity* means that the student must actually carry out the composition (i.e., by counting or marking with or without visual aids). For example, suppose a student forms a goal of finding out how many threes are in 12. If the student verbally counted by ones to 12, double-counting the first count of three as one, the count from four to six as two, etc., leading to a correct response, this would support an inference that the student had constructed the first multiplicative concept. The result of the activity is a unit, 12, composed of four units (of three).

The second multiplicative concept permits the composition of units described above, but with counting actions interiorized. Students who have constructed the second multiplicative concept also have a greater propensity for strategic reasoning, as they can operate on the units within a composite unit immediately (without having to “make” them). Assimilating two levels of units allows students to coordinate three levels of units in activity by strategically sequencing two-level coordinations. For example, if an MC2 student forms a goal of finding out how many threes are in 24, she might reason that

since there are four threes in 12 and two 12s in 24, then there would be four times two, equals eight 3s in 24.

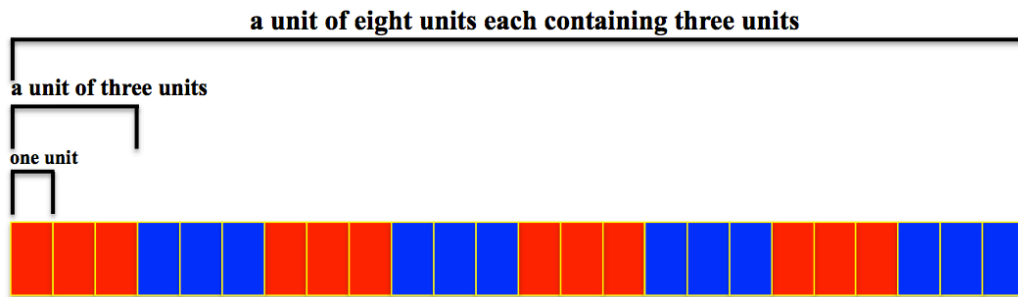


Figure 2.11. A representation of structure for coordinating units within 24

With the third multiplicative concept, a student has interiorized three levels of units. An MC3 student might operate in the same way as an MC2 student in response to the task above, but more quickly and without observable activity. Furthermore, the result of operating would retain a three level structure of 24 as eight units of three units of one unit (see Figure 2.11). Unlike an MC2 student, an MC3 student can strategically and flexibly coordinate three levels of units in support of multiple goals without having to re-compose units in activity. For example, immediately after coordinating 24 as a unit of eight units of three, both an MC2 student and an MC3 student might be able to reason that the sum of five and eight 3s is $24 + 5 = 29$. For an MC2 student, the activity of treating eight 3s as twenty-four 1s when adding is likely to “destroy” the conception of 24 as eight 3s. When faced with a subsequent task of finding the number of 1s in nine 3s, an MC2 student would likely conflate units or re-build a three level structure for 27 in activity. In contrast, an MC3 student would retain the three-level structure of 24 and likely be able to reason immediately that nine 3s is $3 + 24 = 27$.

Initial Hypotheses Regarding Units Coordination Structures

The sequencing of events in a teaching experiment is to some extent shaped by the teacher-researchers' initial hypotheses about *how* changes in students' schemes might take place. For example, there is a general notion that engaging students in tasks involving activity with figurative material and asking students to explain their reasoning during physical activity might encourage their internalization of the activity. Similarly, there is a need for describing hypotheses for how students might come to construct units coordinating structures.

Relationships between units coordinating structures

Hackenberg & Lee (under review) have described the similarities and differences between units coordinating structures with whole numbers, fractions, and algebra; Ulrich (2012) has described units coordinating structures with signed number addition. Though there is evidence that students construct such numerical units coordination structures *simultaneously* across contexts of whole number, fraction, and algebra, relationships between numerical units coordinating structures and quantitative units coordinating structures or of contextual factors regarding such constructions are generally unknown.

Though units coordinating structures for fractions and units coordinating structures for whole numbers share the notion of *n levels* of unit structures, they are not isomorphic (see Table 2.1). In the case of whole number multiplication, a second-level of units assimilation is of a unit, p , composed of p units (ones). In the fractions units coordinating structure, a second level of units assimilation is of a unit, one whole, composed of p fractional units, $1/p^{\text{th}}$. However, a three-level structure for fractions

includes a notion of m/p as m of the amount that, when iterated p times, would make *both* p/p and 1 (for $m > p$). There is no analogous notion for whole number multiplicative concepts. The reorganization hypothesis posits that operations originating with counting are reorganized in the construction of fractions schemes, but the directionality of the relationship between the constructions of units coordinating structures for multiplicative reasoning and units coordinating structures for fractions is not well understood. For example, while both MC2 and a PUFS require an iterating operation, it is unknown whether students construct MC2 before constructing a PUFS.

Table 2.1

Hypothesized Relationships by Number of Levels of Units

Levels	Fractions Schemes	Multiplicative Concepts
1	Pre-fractional (NF)	MC0, MC1
2	PUFS, PFS, RPFS	MC2
3	IFS	MC3

Steffe (2010) argues that iterating operations must precede recursive partitioning operations (partitioning a partition). I hypothesize that n -level units coordinating *structures* might be developed first for units of two and then expanded to other units (cf. Confrey, 1994). This hypothesis is supported by my inferences of a student's ability to coordinate fractional units involving one-half in ways that he couldn't do with one-third in my pilot study.

A hypothesis emerging from my teaching experiences is that students' construction of an n -level units coordinating structure for whole numbers *does* interfere

with their n -level units coordinating structure for fractions, though not in the sense of Streefland's (1993) use of the term *interference*. Streefland (1993) focused on students' reduction of errors involving operations with fraction representations and operations with number representations. I believe that sustained interference may be due to students' assimilating fractional situations within units coordinating structures for whole numbers, rather than constructing fractional units. More specifically, if a student consistently forms situational goals involving fractions for which an existing n -level units coordinating structure for whole numbers is adequate and for which an existing $n-1$ level units coordinating structure for fractional units is inadequate, then the student is unlikely to form an n -level units coordination structure for fractional units.

For example, a student with a part-whole scheme for fractions can successfully assimilate many fractions tasks within a two-level counting structure including a disembedding operation. After a units coordinating structure for unit fractions has been constructed (as part of a PUFS), the student may not experience perturbation when forming a situational goal involving composite fractional units. A part-whole scheme may suffice for a task intended to provoke the construction of a PFS. Instead, a way to facilitate the construction of a PFS is to facilitate the students' formation of a goal that is beyond the limits of a PWS – a task requiring an RPFS (Norton & Wilkins, 2010).

Forming such a goal allows the student to form *operational conjectures* (Norton, 2008), which may lead to functional accommodations. An operational conjecture is an abduction of an operation that, if enacted, would result in an experience of success. In my view, forming an *operational conjecture* is a forming of a *structural conjecture* on a higher level, as an operation must exist as part of some structure in order to reflect upon

it. I hypothesize that the construction of n -level structures for novel contexts (interpreted broadly, and including quantitative units) might follow such a trajectory – the formation of *operational conjectures* is simultaneous with that of *structural conjectures*, which is simultaneous with an (outside of awareness) notion that the existing (*assimilatory*) *units coordinating structure* within a contextual, goal-directed scheme is inadequate.

Facilitating interiorization of units coordination

Students who can coordinate two n -level structures in activity need to develop a necessity for interiorizing the operative relations with the intermediate unit by which to form a $n+1$ level structure. That is, they must face situational goals for which their former way of operating is inadequate. Steffe (1992) provided for such goal formation using tasks involving *missing n -level units*. For example, after a student forms an array of three rows of five chips each and establishes that there were 15 chips total, the teacher might cover the array, say that there were now 25 chips, and ask the student to figure how many rows there would be.

Another way to support interiorization is to provide situations for which a student might form a goal of *retaining* results of prior activity in a sequence of tasks, that is, to take a result of prior activity as given in a subsequent task. Norton and Boyce (2013) provided for such goal formation with a sixth-grade student, Cody. After Cody established relations between four levels of units (chips, cups, boxes, and crates) in activity, the teacher could ask a sequence of tasks in succession that pertain to permutations of the relations. For example, after establishing that there were 2 chips in a cup, 3 cups in a box, and 2 boxes in a crate, the teacher might ask, “How many chips are

in a cup? How many cups are in a box? How many boxes are in a crate? How many cups are in a crate? How many chips are in a crate?” In his response to each task, Cody could refer to the physical situation or a representation of it if he wanted. The hypothesis is that over time, Cody would come to *anticipate* the need for retaining pairwise relationships and hence modify his assimilatory structure so that it would become unnecessary to refer back to figurative material; thus providing a goal of an internalization of the relations between units that could become interiorized.

CHAPTER 3: METHODOLOGICAL CONSTRUCTS

In this chapter, I describe the methodology for the current study. I begin with a description of radical constructivist teaching experiment methodology, including the notions of models of students' mathematics. I then describe why and how I extend the methodology.

Radical Constructivist Teaching Experiment

Over the course of the past five years, I have become familiar with mathematics education research methodologies in which researchers analyze student learning through the activity of teaching. The varieties of teaching experiment methodologies focus on particular aspects of the complex relationship between teaching and learning. Teacher-researchers of individual students' schemes and operations tend to conduct small-group teaching experiments in a clinical setting (Cobb & Steffe, 1983; Steffe & Thompson, 2000), whereas teacher-researchers focused on instruction and discourse tend to conduct classroom-level teaching experiments in a more natural setting (e.g., Cobb, 2000). The former approach generally allows for more fine-grained analysis of individuals and flexibility in instructional decisions at the expense of including fewer sociocultural factors. In either approach, the role of the teacher-researcher and the teacher-researchers' decisions in engendering the learning that is analyzed is paramount.

As my research questions concern students' thinking from a radical constructivist perspective, my data emerges from radical constructivist teaching experiments (Steffe & Thompson, 2000). In Chapter 2, I described some of the conceptual constructs of radical

constructivism: schemes, operations, and reflective abstraction, and their relationship with units coordinating structures. In the present chapter, I describe my interpretation of the constructs pertaining to the modeling inherent in *radical constructivist teaching experiment methodology* (Steffe & Thompson, 2000).

A radical constructivist teaching experiment is a setting in which a teacher-researcher can recursively form and test models about how learning takes place. One of the epistemological tenets of radical constructivism is that individuals' schemes are constructed to *fit*, rather than *match*, their experiences in interaction with the world. When confronted with a surprising outcome, or *perturbation*, individual's schemes are modified. Along the same line of reasoning, characterizations of another's thinking are models constructed to fit one's interpretation of their actions, and to the extent that they do not *fit*, they are modified. As a researcher of students' mathematical thinking, such fitting has multiple aspects, including interpretation of other researchers' models, reflections on one's own thinking, interaction with other teacher-researchers, and, most importantly, interaction with students. The premise motivating modeling efforts is that there are commonalities in human experiences that might enable the models to be useful to others for constructing models that fit their experience, both within and outside schooling settings.

First and Second-order models

Steffe and Cobb (1982) distinguish between models of another's thinking, which they refer to as "second-order models" and models of one's own thinking, or "first-order" models. A second tenet of radical constructivism guides the construction of second-order models – humans act rationally within their existing conceptual systems. The task of

constructing second-order models of a student's thinking involves conjecturing the qualities of the student's conceptual system that would explain a teacher-researcher's observations.

A teacher-researcher's first-order model of his mathematics plays two main roles in the construction of a viable second-order model. First, the teacher-researcher's first-order model of his mathematics guides teaching. In order for a model of a learner to be viable, there should be some indication of ways in which a learner's conceptual structures are superseded by those that seem more useful (von Glasersfeld, 1995). Assuming the teacher-researcher's model of his own mathematics is more powerful than the researcher's model of the student's mathematics, the teacher-researcher's first-order model of the mathematics he is teaching serves in part to qualify what constitutes useful learning on the part of the student. Second, the teacher-researcher's model of his own mathematics influences his modeling. The connections and relationships that describe a conceptual system that explains observations of a student are first-order constructions of the researcher, and include the researchers' mathematics for modeling.

Some examples are in order. In the domain of fractions, Norton (2004; 2008) investigated how students' *conjectural operations* might engender accommodations of their schemes, including the *structuring* of schemes. "Operational conjectures constitute a kind of conjecture in which students use mental actions in novel ways in order to solve problematic situations" (2008, p. 402). Thus, conjectural operations are parts of Norton's second-order models for student's mathematics. As described in his (2008) paper, Norton's notion of conjectural operations is based on his first-order model of Peirce's

(1935) logico-mathematical notion of *abduction*, or posing of a conditional that might explain an otherwise unknown result.

Piaget's (1970a) *INRC* structure is a meta-structure isomorphic to the Klein-4 group. The INRC group is a model of the structural relationships students construct at the level of formal operations and is used to describe the compositions of types of reversibility between algebraic structures and order structures (Inhelder & Piaget, 1958). Here the notion of group is the first-order model, and its application to students' structures is a second-order model. I conceive of similar first and second-order model relationships between researchers' understanding of group theory and their constructions of the schemes and operations of students (see Norton & Wilkins, 2012).

Caution must be used in relating one's first-order knowledge of mathematics and the second-order knowledge of students' mathematics, for it is human nature to perceive structure. Extensive research efforts have been undertaken to construct detailed accounts of the necessary understandings for one to have an understanding of fractions "as operators" (e.g. Behr, Post, Harel, & Lesh, 1992; 1993; Hunting, 1996) or a robust "multiplicative conceptual field" (Vergnaud, 1994) via analyses of first-order models of *adult* thinking of formal mathematical structures. Reliance on such first-order models for interacting with students without consideration of *how* such structures come to be can result in teaching outside of a students' *zone of potential construction* (ZPC). The ZPC consists of the scheme modifications "a student might make in, or as a result of, interactive communication in a mathematical environment" (Steffe, 1991, p. 193). The goal of a constructivist teaching experiment is to engage the student within her ZPC so as to expand it (Norton & D'Ambrosio, 2008; cf. Vygotsky, 1978).

Like the zone of proximal development (ZPD), the zone of potential construction (ZPC) concerns the roles of student-student and student-teacher interactions in student learning. If a task lies within the ZPD of a child, then the child must be necessarily assistable; that is, the child must have a high enough level of development to be guided to the more powerful understanding, but the child must also require assistance (Vygotsky, 1978). In contrast, the ZPC concerns the researchers' *models* of students' mathematics, including constraints and conjectures of how students might potentially reorganize their current ways of operating in accommodation (Norton & D'Ambrosio, 2008). From a radical constructivist perspective, the ZPC is a subset of the ZPD (Norton & D'Ambrosio, 2008).

Third-order Models

Von Glasersfeld and Steffe (1991) describe the utility of second-order models, as well as the critical role of teacher-researchers' decisions in modeling a student's ZPC:

One of the most important aspects of a viable model of the learner would be an indication of at least some of the ways in which a learner's conceptual structures are superseded by structures that seem more adequate from the teacher's point of view. Such modification or increase in an individual's conceptual repertoire is indeed what Piaget called accommodation; yet, in his constructivist theory, change or growth does not take place in response to external information but in response to an internal perturbation. (p. 100)

[W]hat form [these] interactions should take with a particular child at a particular point in that child's conceptual development is not something that can be decided beforehand in any detail. It must depend on the experimenter's experienced intuition at the particular moment. (p. 101)

While I do not disagree with the last statement, because of the lack of qualification of the words "any detail," I believe that consideration of the teacher's inferences while

interacting with a child can be useful for explicating the construction of second-order models. Rather than viewing the decisions of a teacher-researcher as being based on intuition, I maintain that similar processes attributed to students' actions can be considered with respect to the dynamics of student-teacher interaction to enrich the model of the student's mathematics. Although we cannot determine interactions *a priori*, we can study inferences and responses that correspond with "experienced intuition."

In his introduction to students' unit coordination schemes arising from a two-year teaching experiment, Steffe (1992) wrote, "I cannot stress this point enough. I consider most of the schemes [I describe] as much a function of my interventions as they are of the child's constructive power" (p. 263). As interventions are themselves the result of intermediate inferences, I find there is value in modeling not only the change in student thinking upon interventions, but also the changes in the teacher-researcher's thinking *about* the student's thinking both leading up to and following interventions. I believe there is value in analyzing not only the interventions that were productive, in the sense that they resulted in inferences of more powerful schemes, but also those that were not productive in that way.

With the above motivation, I define a model of a second-order model of students' mathematics a "*third-order model of students' mathematics*." The notion of third-order model is not synonymous with third-order derivative, three-level modeling, or the inclusion of cubic terms. Reflective abstraction, scheme theory, perturbation, abduction, and structuring are each aspects of third-order models of students' mathematics, in that they are used to structure researchers' construction of second-order models and hence contribute to the sequencing of inferences and interventions (Steffe & Thompson, 2000).

When a constructivist researcher claims that a student's scheme emerges as the result of reflective abstraction, he is claiming that in his model of a student's mathematics, reflective abstraction on the part of the student is the process by which the scheme emerges. The process that led the researcher to make such an inference is also that of reflective abstraction.

Uses of Stochastic Reasoning in Forming Third-order Models

I consider existing teaching experiment methodologies as allowing for the construction of second-order models that are *deterministic*. Such models are used to explain and conceptualize important variations in a student's thinking, but they do not explicitly model the uncertainty of the inferences within the model of the student's thinking. I consider a model that does model uncertainty to be a *stochastic* model (cf. Saldanha & Liu, 2014). A stochastic model builds on a deterministic model in that in order to conceptualize uncertainty in variation, there must be constructs by which to experience the variation.

Extant use of stochastics to model students' mathematics

Unlike their deterministic counterparts, stochastic third-order models of a student's mathematics are usually not constructed from data obtained directly from a fine-grain longitudinal analysis of inferences of an individual's ways of operating. Rather, researchers conduct clinical interview studies or other "snapshot" measures by which students are compared to one another to make inferences on the variations within a trajectory for a population or sub-population of students (e.g., Gelman, 1972; Hackenberg & Lee, 2013; Pantziara, & Philippou, 2012; Piaget, 1952; Norton & Wilkins, 2009; 2010;

2012; 2013; Wilkins & Norton, 2011). The measurement tools used for assessment are the result of substantial mixed-method and collaborative research efforts that model individuals' development longitudinally.

Wilkins and Norton developed a written assessment of students' fraction schemes and operations in order to provide evidence of the generalizability of the hierarchical nature of some of Steffe & Olive's (2010) fractions schemes (Norton & Wilkins, 2009; 2010; 2012; Wilkins & Norton, 2011). The results from aggregation of their snapshot assessments did not deviate much from hypothesized monotonically increasing patterns. From this, they concluded that students construct the schemes in sequence, with the small number of deviations from the pattern attributed to measurement error. Norton and Wilkins used similar methods to model a student's trajectory for constructing a splitting operation, inferring that the splitting operation is constructed over time with little or no instructional support once a student has constructed a partitive unit fraction scheme (Norton & Wilkins, 2013).

Some researchers (e.g., Fischer, 2009; Siegler & Crowley, 1991; van Geert, 2004) research individual differences *during theorized transition stages* in cognitive development by taking more detailed measurements at particular times. I find such methods analogous to the use of a non-uniform finite element mesh to solve differential equations, in which a finer mesh is used near boundary values. They are used to analyze how individuals whose longitudinal trajectories are similar can have locally dissimilar trajectories. They capture differences in individuals' development over time, allowing for refinement of developmental stages to consider the contextual factors that accompanied individual differences in development.

In contrast, Izsák, Jacobson, Arajao, and Orrill (2012) research individual differences in *cognitive* structures using mixture Rasch modeling. In general, Rasch modeling involves the matching of an item's difficulty and an individual's ability within a single scale. Individuals with high ability are expected to solve all problems of equal or lesser difficulty, while items of lowest difficulty are expected to be solved correctly by all but the lowest able. A mixture Rasch method involves the use of structural equation modeling to propose latent (predictor) variables that are used to group individuals (or items). Izsák et al (2012) found that grouping individuals by their performance on written fractions tasks into two groups explained more of the variance in the data. Upon examination of the tasks (which were not always the most "difficult" ones), Izsák et al. (2012) found that the tasks used to separate groups were those that required three levels of units coordination to be interiorized.

The studies using stochastics described above used the likelihood method of frequentist inference (Cox & Mayo, 2007). Frequentist inference begins not with data, but with a hypothetical sequence of repetitions of a data collection procedure. Competing *a priori* expectations of the data resulting from an infinite sequence of data collection procedures are compared with the data resulting from a finite actualization of such procedures. This is the comparison of a *population distribution* with a *sample distribution*. Characteristics about the sample distribution are compared pairwise with characteristics of the competing hypothesized distributions of the population. Inferences of support of one hypothesized population versus another are the result of a *significantly* better matching of the sample's characteristics to one population's characteristics than

another's. The level of significance (p -value) is determined by a community of practice in order to allow for a characterization of objectivity of such inferences.

A subjective stochastic approach

While the use of a frequentist inference approach has been an invaluable part of research methodologies in mathematics education, I also value modeling *changes in inferences* stochastically. In place of posing a hypothetical distribution and fixed data collection methods of a frequentist approach, a *transition formula* that is independent of observations and data is posed. Just as populations are often assumed to be normally distributed because of the normal distribution's symmetries, simplicities, and asymptotic relationships with the binomial distribution, the particulars of a transition formula are chosen for simplicity, symmetry, and convergence properties. While each piece of data that is collected is dependent on the previous inferences, the procedures for framing and defining data and data collection are described in detail as in the data collection approach of a frequentist paradigm.

Propensities

The inductive inferences of a student's thinking that a teacher-researcher makes in the course of teaching are based on patterns of behavior-environment interaction. They include abductions about why the student might have behaved the way she did — about how the student might have been thinking that would explain the patterns in the teacher-researcher's observations. Conjectures about the ways another thinks do not have truth values of 1 or 0. One can never know with certainty how a human is thinking or not thinking for two reasons. The first is that one must interpret a human's actions in terms

of one's own conceptions. The second is that because humans *learn*, their thinking may be changing *during any sequence of observations* whence inductive inferences of their thinking are made. The first issue is addressed by the consideration of “second-order models” versus “first-order models” of students' mathematics. The second issue is addressed by considering *longitudinal* teaching experiments, during which the *stabilization* of the second-order model permits inferences of learning to be made. To further explore how models become stabilized, I conceptualize that the inferences a teacher-researcher makes during a teaching experiment are of students' *propensities* to think in a conjectured way.

Popper(1959) gives the following characterization of *propensity* -

The propensity interpretation of probability retains the view that probabilities are conjectured or estimated statistical frequencies in long (actual or virtual) sequences...these sequences are defined by the manner in which their elements are generated – that is, by the experimental conditions – [so that conjectured] **probabilities depend on these conditions and may change with them.** ...Probabilities are dispositional properties of these conditions – that is to say, propensities [emphasis added]. (p. 37)

Popper (1959) was referring to properties of the physical world, such as force, in which dispositional properties of conditions were immutable by the action of human observation. The dispositional properties of human subjects are anything but — in the course of a teaching experiment, a central goal is to engender perturbation! A primary condition of the manner in which a sequence of observations is generated during teaching is thus the teacher's inferences of the students' thinking in prior observations. As I describe in the next section, I consider the experimental conditions generating a sequence of observations as consisting of three components: context of physical environment, context of interaction, and a current propensity.

Characterizing the Context of inferences

My approach for modeling transitions in *propensities* began with the development of a framework for collecting data about inferences that took place during a teaching experiment. My approach builds upon an existing extension of teaching experiment methodology – the notion of the hypothetical learning trajectory (Simon, 1995). Steffe’s (2001) *reorganization hypothesis*, that fractions schemes can emerge as accommodations of counting schemes, is described as an example of “mental activity that can lead to the intended concept” (Simon & Tzur, 2004, p. 12). Simon and Tzur suggest that

[I]t is important to note that a more micro-level analysis of [Steffe and colleagues’] work is ... appropriate, an analysis of the activity-effect relationships related to the development of particular fraction schemes. Whereas the particular activity sequences generally involve iteration, it is the differences among these sequences that contribute to an account for how each scheme is built on the foundation of prior schemes. A second challenge derives from using this explanatory mechanism to account for successful learning in the context of particular mathematical tasks. Here the challenge is to identify from among the myriad of activities and effects that can be identified by the observer, a particular coupling of the activities and effects that can account for the learning. (ibid, p. 12).

Rather than focusing solely on propensities for assimilation to a particular scheme with or without cueing from a teacher, I focus on the units coordinating structures that students assimilate across contexts. In my construction of a third-order model of a student’s learning, I consider variations in *experimental conditions* more broadly than the student-teacher interaction. To frame data collection, I thus need to consider types of variations in experimental conditions that may be used to explain differences in students’ assimilations. I entered the next data collection phase with the assumption that whatever inferences I made in sequence *are not* independent, but rather, the observation from which to make a subsequent inference is, to a non-negligible extent, due to a previous

inference. What I could try to make uniform was the method by which I *organized* the data, the method by which I could characterize individual elements in a sequence of observations from which inferences were made.

Simon, Saldanha, McClintock, Akar, Watanabe, and Zembat (2010) offer a methodological framework for conducting analyses of the variation within an individual student's learning trajectory. They use the theoretical framework of *activity-effect relations*, placing attention upon not just students' mathematical structures, but also the *nature of the activities that led to the changes in the structures*. The goal of Simon et al.'s approach is to "explicate the *transition* from the point at which a student did not understand something to where he or she did understand it" (p. 9)." They describe two lenses of analysis: that of the *interaction* between student and teacher and that of a student's cognitive activity. The two are interrelated, in that a teacher's inferences of a student's cognitive activity and conjecture of how that activity might relate to a transition in the student's understanding influence subsequent student-teacher interaction.

Approaches to modeling transfer

In addition to the framework suggested by Simon et al. (2010), I find the characterizations of a student's activities and the interactions between students, teachers, and a student's environment described by Wagner (2006) and Lobato (2006) useful for characterizing contextual changes during the teaching process. As Wagner (2006), Lobato (2006), and Simon et al (2010) each consider contextual change within the interaction of a single student and a teacher-researcher, I expanded their frameworks to include student-student interaction. Below, I justify and detail my choice of frameworks.

Bransford and Schwartz (1999) define two main categories of transfer, that of preparation for future learning (PFL) and that of direct application (DA). In the DA approach, transfer is characterized as “the ability to directly apply one’s previous learning to a new setting or problem” (p. 68). They describe how the research methodologies for the DA approach to transfer usually involve testing performance on problem solving tasks both before and after instruction and practice within the first task’s setting.

Bransford and Schwartz (1999) describe a study by Thorndike and Woodworth (1901) as an exemplar of the traditional DA approach. Individuals were first tested on their ability to estimate areas of squares, given time and instruction with estimating in that setting, and then tested on their ability to estimate areas of triangles and circles. As the performance on the post-test was poor, Thorndike and Wallace concluded that estimating areas was not a general skill. With the DA approach, the relative transferability of knowledge depends on the relationship between the similarity between settings and the ability to solve the latter task; the similarity between the settings is taken as positivist and objectively real.

With the PFL approach, assessment of transfer explores “people’s ability to learn new information and relate their learning to previous experiences” (Bransford & Schwartz, 1999, p. 69). The PFL approach is more amenable to constructivist epistemologies. “Rather than simply viewing transfer as the mapping of old understandings and practices onto a given situation, the PFL perspective emphasizes that people can actively change the given situation into something that is more compatible with their current state and goals” (p. 82). Thus, from a radical constructivist perspective, every given situation involves transfer in the PFL sense, for the perceiving of any

situation is a construction of the mind, based on a fitting of sensori-motor input to one's current mental structures (von Glasersfeld, 1995). While the DA approach to transfer only permits assimilation as positive transfer, the PFL approach also permits accommodation, so that transfer can be said to take place within the individual's cognitive structures. In this way, the PFL approach can be seen as an extension of the DA approach.

In their reconceptualization of transfer in the specific domain of quantitative reasoning, Lobato and Siebert (2002) also critique traditional transfer studies in which researchers distinguish between surface-level or contextual aspects of tasks (which are varied) and structure-level or mathematical aspects of tasks (which are held constant). Lobato (2006) described how quantitative reasoning (Thompson, 1994) was much more difficult for Algebra 1 students than reasoning numerically, exemplifying the notion that simply varying the context can make a task much more complex for students. They contrast the DA approach with Lobato's (2006) *actor-oriented transfer* (AOT), "the personal construction of relations of similarities between activities, or how 'actors' see situations as similar" (Lobato & Siebert, 2002, p. 89). As AOT researchers focus on how individuals *construct similarities* between tasks, AOT can be considered a more focused subset of PFL. Lobato and Siebert (2002) conjecture that students could learn a general process for coping with quantitatively complex situations, and that three aspects of such a process might include isolating attributes, identifying and testing how various quantities affect the attributes, and constructing relationships amongst quantities.

Wagner (2006) also describes a framework for analyzing the process by which an individual comes to see different situations as alike. While Lobato and Siebert (2002)

focus on inferring an individual's process of constructing similarity from observations during relatively few (three, 45 minute) semi-structured interviews with quantitatively complex tasks, Wagner's (2006) "transfer in pieces" (TIP) approach is used to describe "an incremental refinement of knowledge resources that account for – rather than overlook – contextual variation" (p. 1). The main difference in the TIP approach and the AOT approach is that TIP focuses on classifying (relevant) changes in context, from the researchers' perspective, and analyzing *how* changes in the context *introduced by a teacher-researcher* affect an individual's verbal responses. The AOT approach examines in more detail *how* changes in the environment *introduced by the actor* contribute to the actor's construction of similarity. I incorporate both aspects into my characterization of context.

Wagner (2006) argues that the crux to researching transfer is to understand how the mind is able to simultaneously abstract a structure by which to conceive of similarity, yet also abstract a structure by which to conceive of the contextual differences. He adopted a *knowledge-in-pieces* (diSessa, 1993) epistemological view, first developed for adults' understanding of mechanics in physics, in which learning involves "organizing (or reorganizing) and supplementing useful intuitive knowledge in ways that restructure interpretative schemata and more uniformly systemize causal and explanatory understandings" (Wagner, 2006, p. 7). By analyzing a single student's initial perceptions of the importance of problem elements pertaining to the law of large numbers, Wagner was able to distinguish patterns in her sensitivity to problem *type*, *aspect*, or *context*. Type refers to "legitimate mathematical descriptors;" aspect refers to a "detail of a problem that can be a focus of attention;" context refers to "the cover story in which a

problem is embedded” (p. 8). Wagner’s approach to transfer was to analyze the relationships between a context’s characteristics (from the researchers’ perspective), the cueing of an individual’s particular knowledge frame (the researchers’ organization of the individual’s hypothesized organized knowledge), and the associated knowledge (that which is connected to the activated frame) that would subsequently be more readily available in influencing further activity.

To the degree that reflective abstraction results in reorganizations of schemes both within and across contexts, transfer is relevant to the discussion of how units coordination develops. I should note that my discussion of context of transfer has focused on the qualities that seem most relevant to the current study, theories of mathematical learning, and methodologies. Indeed, there is a vast literature based in *situated cognition* that focuses primarily on differences between school mathematics and that of everyday life (e.g., Saxe, 2005) and on transfer within classrooms (e.g. Lave, 1993). I acknowledge that a weakness of my methodology is that it excludes the possibility of capturing such differences. I did consider, in a rather narrow way, students’ *affect*.

Mathematical caring relations

In order to form models of students’ mathematical learning in the context of teaching, a teacher-researcher must have and maintain participants’ willingness to engage in mathematical activity. Hackenberg (2010a) characterizes both cognitive and affective aspects of student-teacher interaction in the service of mathematical learning as *mathematical caring relations*. Mathematical caring is the “orchestration of mathematical learning together with an orientation to monitor and respond to fluctuations

in positive energy available to the self – subjective vitality – that may accompany student-teacher interactions” (p . 237).

Hackenberg (2010a) views student-teacher interaction as a “linked-chain of perturbations” (p. 236). She categorized four types of chains: interactions that seemed to trigger productive mathematical activity, interactions that were not productive, interactions that preserved harmony and increased vitality, and interactions characterized by the teacher’s recalibration of her response. Instead of only monitoring the mathematical context, Hackenberg monitored both her own and her students’ affect. This included consideration of how praise and encouragement was given and received and monitoring of facial expressions, verbal tone, and attentiveness.

My focus is on characterizing only one student’s perspective, and I did not attempt to model the teacher-researcher’s thinking, decision-making, or energy level. I agree with Hackenberg’s claim that the data a teacher-researcher receives from a student in a teaching experiment pertaining to his mathematical ways of operating is necessarily entangled with the student’s energy level. When modeling the variability in units coordination activity, consideration of affect should not be excluded. I included descriptions of students affect that were salient from the video in the transcription. I considered contextual descriptions that included aspects of both student-student and student-teacher interactions that accompanied the teacher-researchers’ (non-explicit) efforts to build and maintain mathematical caring relations (see Figure 6.2).

Data collection and organization

Modeling a student’s transition between discrete points along a trajectory requires a fine-grained modeling of the characteristics of the activities in which a student is

engaged. As described above, I view the characterization of student's activities as having two interrelated components, together constituting an umbrella construct of *contextual changes*. These are student-human interaction and student-ecology interaction. These two categories are more broad but otherwise are similar, respectively, to Tzur and Lambert's (2011) *task pedagogy* and *task features*. Student-human interaction includes teacher-researcher questioning and other observable human actions, including student-student interaction. Student-ecology interaction includes characteristics of the environment, such as task attributes. This seems to be a natural pairing for consideration of the process of abstraction.

“Two separate notions of context are involved: the context of mathematical objects and a set of external factors. The person who abstracts gradually ignores the context of the various mathematical objects. However, the set of external factors may influence this process of abstraction.” (Hershkowitz, Schwarz, & Dreyfus, 2001, p. 197).

To develop sequences of observations in the course of teaching, I partition the entirety of the teaching sessions into a sequence of *chunks*. The motivation in forming the partitioning is to form a coarse partitioning of the teaching experiment into events (chunks) by which to identify units coordination as taking place within a chunk, yet a refined enough partitioning to characterize important qualities of the contextual change between chunks. Chunks are distinguished by qualitative differences in context, along the two dimensions of task context and interaction. Figure 3.1 displays the subcategories for each, along with examples of possible characterizations.

Task Context	Interaction
Aspect: (e.g., coordinate with units of 5 versus units of 8)	Teacher actions: Questioning, Gesturing, Demonstrating, Affirming, Reiterating
Type: (e.g., naming fraction not constructing fractional size)	Peer actions: Responding, Moving figurative materials, Explaining, Gesturing
Setting: (e.g., with fraction rods not verbally)	Student actions: Responding, Gesturing, Explaining, Questioning, Affirming, Manipulating, Affect characteristics

Figure 3.1. Broad Chunking Categories

Judgmental Heuristics and Natural Assessments

Tversky and Kahneman (1993) define *judgmental heuristic* as “a strategy – whether deliberate or not – that relies on a *natural assessment* to produce an estimation or a prediction” (p. 55). A natural assessment is one that is “routinely carried out as part of the perception of events and the comprehension of messages” (p. 54). Natural assessments include “computations of similarity and representativeness, attributions of causality, and evaluations of the availability of associations and exemplars” (p. 54) and, while they are “not tied to the estimation of frequency or probability, they are likely to play a dominant role when such judgments are required” (p. 55). I conceive of a transition propensity model as a model of a *natural assessment* of a development of stability in propensities, not one particularly confined to units coordination. It is a “lifting up” of the 1s and 0s from the chunks and modeling how one might describe the

propensity for a subsequent 1 or 0, given that propensities could never actually be 0 or 1 and that propensities were non-independent, but that the sequence is only *pseudo-random*.

Thus, I formed a goal that the initial calculation of propensity would be independent of a chunk's characterization and parameter-free. Rather, the calculation of propensity at each chunk would be completely determined by (a) an assignment of initial propensity and (b) the sequence of inferences. Though judgmental heuristics have the potential to neglect considerations outside of natural assessments, I anticipated this issue and tried to recognize and remove other considerations *a priori*. Knowing that contextual changes in a chunk would certainly affect propensities, I could model the changes without such considerations, and subsequently consider how they might affect the propensities. I view this construction as a “baseline” model, from my understanding of such techniques from my experience with multi-level linear modeling, with anticipation that I might later add characteristics of the chunks into the calculation. With this motivation, my goal was to model as *simple* a judgmental heuristic as possible.

CHAPTER 4: PILOT STUDY RESULTS

In this chapter, I provide results from a study with a sixth-grade student, Charles, to illustrate the results arising from the methodology described in Chapter 3 that led to the construction of a baseline model. I begin with a description of the data and participant, via a traditional teaching experiment analysis. I then provide the results of the baseline model and suggest an initial approach to using the baseline model and the second-order model together to describe Charles' mathematics. I conclude this chapter with a description of the methodological changes for the current study that were informed by the pilot study.

Teaching Experiment Analysis

Introduction

I collected data as part of a pilot study in which I engaged students in tasks pertaining to fractions and units coordination across a limited number of contexts. I focus on Charles, who was one of four students that I taught in a six-session paired-student constructivist teaching experiment (Steffe & Thompson, 2000) over three weeks during the summer between sixth and seventh grade. Each student's first session consisted of a clinical interview for assessing students' fraction schemes (Wilkins, Norton, & Boyce, 2013). My goals for the remaining five sessions were to (1) build and refine a model for each student's mathematics, to include their ways of operating in multiple task contexts (i.e., with circular and linear fraction models) and sequences, and (2) support (to the extent possible in the short period of time) their construction of the operations theorized as necessary for the emergence of more powerful fraction schemes. For planning

purposes, the videographer/witness, who was familiar with fraction schemes, and I discussed our analysis of the progress of the teaching experiment between sessions.

Charles' proper placement scheme for fractions in the linear context

Beginning with the clinical interview, I noticed that Charles' ways of operating with fractions in a linear context entailed a "proper-placement" scheme, in which he assimilated physical placement (top and bottom) as part of his scheme to achieve a goal of producing a *proper* fraction name. His scheme consisted of the following sequential operations: 1) Make a "top" consist of equivalent marked parts; 2) Count the number of those parts that form the same size as the bottom; 3) Swap the top and bottom if there are more pieces in the top than in the bottom to make the fraction proper; 4) Name the fraction as the number of "top" pieces out of the number of "bottom" pieces. I conjecture that this way of operating had been useful to Charles for making sense of fractions in schooling situations-- i.e., for understanding a fraction as a model of a visual relationship between two whole numbers.

The first accommodation I tried to engender in Charles' ways of operating within the linear fraction context was to include a disembedding operation within his proper-placement scheme. I engaged Charles and his partner in tasks involving figural representations of fractions, using Cuisenaire rods (see Figure 4.1). As seen in line (2) of Protocol 4.1, Charles' proper placement scheme did not explicitly include units – his conception of fraction was to translate a representation of numerator and a representation of denominator to a fractional name. He was disembedding in order to name the 'bottom' fourths – he was treating each single 'red' block as a unit of one, and with it formed a unit of 8 and a unit of 4. But he was not disembedding when he named the fraction – the

formulation of 8 and 4 lost their meaning as eight ones or four ones. In line (24) of Protocol 1, Charles' was able to modify his scheme in activity by 'sweeping away' all but the left-most block from the 'top,' allowing him to maintain equilibrium with his 'proper placement' scheme. I viewed this as units coordination *in activity*, and my hypothesis is/was that he would eventually be able to perform such coordination without having to perform the "sweeping away" activity.

Color	Dimension
Tan (white)	1 cm x 1 cm x 1 cm
Red	2 cm x 1 cm x 1 cm
Lime (green)	3 cm x 1 cm x 1 cm
Purple	4 cm x 1 cm x 1 cm
Yellow	5 cm x 1 cm x 1 cm
Green (dark)	6 cm x 1 cm x 1 cm
Black	7 cm x 1 cm x 1 cm
Brown	8 cm x 1 cm x 1 cm
Blue	9 cm x 1 cm x 1 cm
Orange	10 cm x 1 cm x 1 cm

Figure 4.1. Cuisenaire rods' dimensions by color

Protocol 4.1. Portion of Transcription from Session Two

1. *Teacher*: (Puts out 1 brown bar and 8 red bars). What fraction is 1 red bar out of the brown bar? [Note that the brown bar is the length of four red bars]
2. *Charles*: (Lines 8 red bars above the brown bar). Eight-fourths? An improper

- fraction? Cause you said, 'the red bar of the brown bar.'
3. *Teacher*: What fraction is one (holding up one finger) red bar out of the brown bar?
4. *Charles*: (Touches brown bar). So that would be four-eighths.
5. *Teacher*: Four-eighths, can you say why (interrupted)?
6. *Charles*: That would be the same as one-half.
7. *Teacher*: (To D.J.) What do you think? Do you also think its four-eighths?
8. *D.J.*: Nods.
9. *Teacher*: (Moves 6 red bars and the brown bar to D.J.s desk) What fraction is one red bar of the brown bar now?
10. *D.J.*: (Lines the red bars up, one by one, from left to right above the brown bar)
11. *Charles*: I've got the answer.
12. *D.J.*: One-third. Cause four pieces makes that (the brown) so if there were two it's close, it's like one-half, and if you take that, it's one-third, or something.
13. *Teacher*: Okay, 'cause it's close to one-half?
14. *D.J.*: Yeah, two is close to one-half, so if you take away one it's one-third.
15. *Teacher*: Okay, Charles, what do you think? You said you had the answer, what was your answer?
16. *Charles*: (Confidently) One-fourth.
17. *Teacher*: Can you show me how you got one-fourth? Show it to me, and show it to D.J.
18. *Charles*: (Lines up all 6 red pieces). This brown bar equals up to four pieces. You said to only count one of the pieces, so that would be one-fourth.
19. *Teacher*: Did that make sense to you, D.J.?

20. *D.J.*: It makes more sense than mine.
21. *Teacher*: So you really don't need these other (extra two) pieces do you, to show it?
22. *Charles*: Yeah.
23. *Teacher*: Show it to D.J. so he can see it.
24. *Charles*: See how that equals up to four pieces? So you take these (sweeps away rightmost three) and that's one-fourth.

Through the course of the teaching experiment, Charles engaged in fraction naming and making activities within linear contexts that varied from the proper placement scheme. However, when he was asked to *explain* his reasoning for a response, he consistently presented a representation that was consistent with his proper placement scheme. For example, during the fourth session, Charles was given four red blocks, appended to one another, and asked what one red block was of the whole. His response was $1/3$, which he arrived at after removing the left-most block and placing it above the remaining three. This is consistent with a lack of a disembedding operation component of a part-whole scheme. In the fifth session, he was able to correctly respond $1/4$ to a similar task, without removing the leftmost piece. But when he was asked to explain why, he grabbed a fifth piece from the box and placed it above, to show *why* the piece was $1/4$. When he was asked to name what fraction a single red piece was of a larger piece, he would slide it from left to right, counting by ones, and arrive at a response. But when he was asked to explain *why*, he would grab more reds, line them up to the length of the larger piece, count the number of reds, and then “sweep away” all but the left-most piece.

In the last session, I asked him if he could tell why the red was one-third of the dark green, without using any other red pieces, and he said, “Yeah! One, two, three” while sliding the red across dark green as saying number words. His way of reasoning thus seemed indicative of a partitive unit fraction scheme, but this only occurred when he was unable to make his representation of one-third. While he seemed to be re-presenting the results of figural proper-placement activities, it may be that his inclusion of the disembedding operation within his proper placement scheme might permit his reflection on his necessary operations for naming fractions that could result in a more powerful structure.

Charles’ “Even-fit” conception and whole number unit coordination

A question that that was explored early in the teaching experiment is how Charles might operate in a situation in which the “top” was not initially as large as the “bottom” so that he could not measure the bottom by comparing it directly with the pieces used to form the “top.” In this situation, I inferred that Charles had an “even fit” conception. If he could (in activity) equisegment a given (smaller) piece to form the whole (bottom), then he would name the smaller piece as $1/n$, where n was the number of segments necessary to form the whole. If he couldn’t, then he would name the fraction as $1/1$ (or $1/2$ if it “nearly” fit in twice). However, Charles was inconsistent in his adoption of the even-fit conception, and, as I describe below, his responses varied with whether a task required coordination with whole number or fractional units.

In the third session, he said that a purple rod was one whole of a black, “because the purple bar won’t fit. It’s bigger than [the] space [between a single iteration of the purple and the end of the black bar]. When asked why it would mean “one whole,” he

held up the purple, said “1,” and the black, said “1”, and said, “there’s two pieces, one and one.” Later in the session, after he explained that a tan rod was $\frac{1}{4}$ of a purple rod, by sliding a tan four times from left to right, making marks with his pencil, I asked him why it was not $\frac{1}{1}$ – after all, both are a single pieces, one and one. He explained explicitly that it was because the tan fit in the purple, but the purple didn’t fit into the black.

Later in the third session, I asked Charles and D. J. to give each other tasks. When given a task of finding what fraction a yellow rod was of a blue rod, Charles got a second yellow rod, aligned it with the first yellow, and said the fraction was “Two-ones.” Note that the blue rod is actually slightly shorter than two yellow rods. When I questioned his answer, he said, “Yeah –that would be improper. [The blue] would have to go up there.” I asked him why would it have to go up there, and he said, “ ‘Cause then it would equal one-half.” I then asked D. J. to repeat the question to Charles. After hearing that the question referred to a single yellow rod, Charles grabbed two reds from the box and filled them in the missing gap between a single yellow and the length of the blue. He then aligned the two reds below the yellow, but they didn’t fit within the yellow [too big]. He said, “I guess I’ll try the brown ones.” He then lined up five [tan] rods above the yellow rod and appended four more tan rods to the right of these to fill the gap between the yellow and length of the blue. He then swept away the tan rods and said “Five-ninths.”

Note that in Charles’ response, he treated the yellow rod as being five ones [tans] and the blue rod as being nine ones [tans]. At first, when he attempted to equisegment using two yellows and the result was a bit bigger than the blue, he did *not* respond that the answer was one whole, and instead was inclined to two, and then one-half. When he

reached for the two reds, I believe he was fortuitous that they happened to “fit” in the gap between the length of a single yellow and the blue. When he began to name the yellow in terms of the reds, he again operated differently than before. Under his previous way of operating, he would be expected to say that the red was half of the yellow, but this time he decided to “try the brown ones.” He thus seemed to abduct a recursive segmenting activity. When I asked him to explain his reasoning, he said, “Because five of these [tan] equal up to the yellow, and since [another] yellow doesn’t fit in there [two yellows are longer than the blue] you add four more and that would equal nine. Five plus four is nine, and then take these away [slides away the four] and that would be five-ninths.” Notice the unit coordination – when Charles said “you add four more to the yellow,” he was referring to adding four *tans*, not four yellows. However, he was not referring to four *ninths*. Thus, he was coordinating two levels of whole number units in a linear-context activity with fractions – he was not coordinating two levels of fractional units.

Charles reasoned in a similar way in a reverse setting. When given a piece that he was told was $\frac{1}{5}$ and asked to create the whole, he iterated (marking with pencil) the given piece 5 times to create a marked length. He insisted that without the markings showing that the piece was divided into 5 parts, it would not be the correct whole, because without the markings, and despite being the same length, “it would be one-to-one. It would make it one. ... You need the markings to know it’s one-fifth.”

When given a piece that he was told was $\frac{2}{7}$ and asked to make the whole, Charles physically iterated the given piece 7 times to make a new whole, and seemed to ignore the ‘2.’ When he was explaining his reasoning, he tried iterating a piece that was twice as long as the first given piece (i.e., $\frac{4}{7}$) to attempt to fit the whole that he had just

created by copying the first piece seven times. Thus, he did not seem to understand segmenting and iterating as inverse operations – he tried iterating composed with iteration instead of iteration composed with segmenting. Beyond those that were abstracted empirically (such as pre-partitioned wholes), Charles’ ways of operating were quite limited.

Charles’ Constructions toward a Part-Whole Scheme in the Linear Context

In the final session, I asked Charles a question for which I placed four lime rods appended to one another, and I, while pointing to the third piece from his left, asked him what fraction that piece was of the whole. He removed the indicated piece from the four and placed it above the others, aligned with the left-most piece. He then slid the bottom three lime pieces together, and said, “one-third.” When I asked him to explain his answer, he moved the pieces back to the starting position and said, “You put [the third lime piece above], if it would stay connected...” He repeated the removing of the third piece and sliding to remove the gap, but then said, “Wait! No, one-fourth!” He then re-separated the bottom three, and took a fifth lime out of the box, which he placed into the gap. “There was four pieces, altogether, and that would be one-fourth because 1, 2, 3, 4 [counting the bottom from left to right].” His spontaneous reflection indicated to me that the accommodation of his disembedding operation was becoming interiorized (within his proper placement scheme), hypothetically a necessity for his construction of a part-whole scheme (and eventually a PUFS) in the linear context.

Charles' ways of operating with fractions in the circular area context

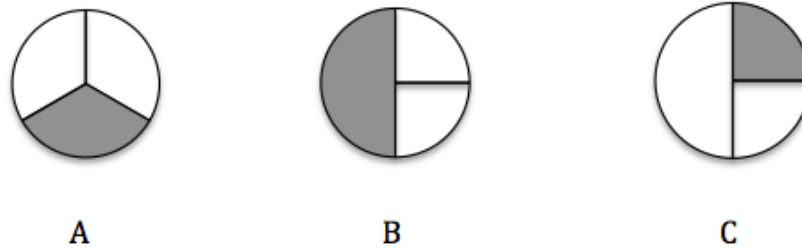


Figure 4.2. Three Circles

In the fourth session, I asked Charles and D.J. to name shaded portions of circles as fractions. I drew three circles, labeled A, B, and C (see Figure 4.2). As described below, Charles' did not seem to have a "proper placement" scheme for this context. Rather, he seemed to operate with a part-whole scheme, although the markings pre-placed upon the circle inconsistently affected his responses to the tasks.

Charles named Circle A " $\frac{1}{3}$ ", Circle B " $\frac{1}{2}$ ", and Circle C " $\frac{1}{3}$;" D.J. said that Circle C was one-fourth instead. When I asked Charles how he got $\frac{1}{3}$ for C, he said, "There's three pieces, and one's covered up, so that would be $\frac{1}{3}$." When I asked D.J. how he got $\frac{1}{4}$ for C, he said that if the radius were extended to become a diameter, there would be four pieces, and thus the shaded part was a fourth. D.J. then drew the segment in, and Charles still said that Circle C was one-third. I asked him, "Why is it still one-third? I see four pieces, not three, now." Charles responded that, "One's covered up, and there's three pieces." I infer that he meant that there were three *more* pieces, for when I asked him what the whole looked like, he said that it had four pieces. Thus, he seemed to be re-interpreting the fraction $\frac{1}{3}$ as a ratio of shaded parts to unshaded parts. When presented with his original C, and asked why it would not be one-half then, he changed

his response. He said that the original C was one-third, and that with the extended diameter, it would be one-fourth.

When I asked Charles why A was one-third, he said that there were three pieces. I then asked him why wasn't B named as one third, as there were three pieces, and one of them was shaded in. His response was that B was one-half "because there's two pieces, and that [the two-fourths] equals up to that [the other half]. There's one whole right there [at the diameter]. There's one, and then there's two pieces, so one-half." I then drew another circle, drew a diameter and shaded in half the circle, which he again said was one-half. I then proceeded to partition the unshaded portion into fifths, and asked him, "Is it still one-half, or did it change?" He responded that it was one-fifth, but D.J. interjected, "It's still a half!" Charles then agreed that it would still be a half, even upon my suggestion that even more lines be drawn in the unshaded part. But when I partitioned the shaded part into two pieces and asked if it was still one-half, he said, "Wait...that would be two-fourths." Asked if that was the same as one-half, he said, "well yeah, because you simplify."

My analysis of Charles' ways of operating with the circular context was limited to only this portion of the fourth session. My inferences are that his "proper placement" scheme and a "part-whole" scheme were both part of his ways of operating with the context. From his initial response to 'C' I inferred that Charles was not attending to the sizes of the pieces, only the number of parts. But his response to 'B' indicated that for one-half, he did include the sizes of the pieces as part of his scheme. He equated the size of one piece and the size of its complement within the whole, despite the latter consisting of two pieces. His conception did not allow for changes to the partitioning marks to be

irrelevant to the naming of a shaded portion of a whole as a fraction. When he suggested two-fourths, there were many more marks on the un-shaded portion of the circle. Thus, while he was unable to simultaneously coordinate the shaded piece as both ‘one’ and ‘two parts, each of which make up half the shaded piece’, he *was* able to project the partitioning onto the unshaded piece, despite its additional markings.

Charles’ “even fit” conception may have played a role with circles as well as in the linear context. In the linear context, when he was able to slide a piece from left to right and “fit” it within a whole, he named the fraction as $1/n$, where n was the number of iterations. Otherwise, he attended to the number of parts, arriving at $1/1$ or $1/2$ with the fraction rods. In the circular context, when he was able to “fit” one part within another, he attended to the size, and otherwise he attended to the number of parts. Thus, it was Charles’ (in)ability to ‘fit’ parts that determined the fractional name for the shaded piece. It seems that he was able to perform such coordination in the circular context with halves, but not thirds, fourths, or fifths.

Charles’ whole number multiplicative reasoning

I gave Charles and D.J. two word problems involving multiplicative reasoning in which they did not have manipulatives or other physical representations. The first problem was during the third session. I asked them, “If we are going to play with the iPad for 7 minutes, and share the minutes equally, how many minutes will each of you get to play?” Charles had the correct answer after only a few seconds and did not use paper or pencil despite their availability. I asked D.J. to explain how he arrived at the answer. D.J. said, “Well, to get equal, we’d both get 6 for equal, we’d get three. Then, there’s 60 seconds in a minute, so if you half that, it’s 30 seconds.” Charles said that he

didn't understand D.J.'s explanation, but after D.J. explained it again, he said that he had done something similar, but would have explained it differently. Unfortunately, I did not ask him to explain how he arrived at the answer of three minutes and thirty seconds. I next asked them, "The last time we played, you played 5 minutes and had 10 correct orders. Do you think you'll have time to make 15 correct orders today?" Charles first response was no, which he explained as the result of thinking that they would have only 3 minutes and 30 seconds total. When I said that they would have seven minutes, he said yes, but he explained that it was because he thought they would *each* get 7 minutes. Thus, Charles did not appear to form a quantity of orders per minute nor use a multiplicative structure for figuring how many minutes it would take to complete 15 orders.

In the fifth session, I asked Charles and D.J. a more direct units coordination question. After clarifying that each of the bars would be the same size, I repeated the following task: "There's ten chocolate bars in a pack. There's eight packs in a box. How many bars are in five boxes?" Charles asked, "There's eight in a box?" I repeated the question, and he stared off in the corner of the room, seemingly concentrating, and then asked if I had the question written down. I suggested that he could write information down as I said the task aloud. After he had the question written down, he said "Ten" and counted on his fingers from one to eight. He then wrote down 80×5 on his paper, and said, "I don't know, 400?" He seemed very unsure and very relieved when D.J. revealed that he had the same response. I asked them how they got 80, and Charles said, "I did the ten times..." and D.J. interjected with the word "eight." When I asked, what does the 80 represent? D.J. said it was "how many [were] in a box" and Charles said, "A pack."

I infer from this limited whole-number unit coordination interaction that Charles was able to coordinate two levels of whole number units in activity. In the first part of the first task, Charles was able to arrive at the correct answer quite quickly. However, I inferred that he was unable to follow D.J.'s explanation, which required understanding of 7 minutes as 6 minutes and 1 minute, with each minute as 60 seconds. D.J.'s explanation thus required two levels of units to be interiorized. Moreover, Charles' inattention to the unit rate or offering of multiplicative reasoning for the second part of the first task is indicative that he did not assimilate the "five minutes and 10 correct orders" as a *rate*, which might be expected for someone who had interiorized two levels of units (I inferred such an understanding was available to D.J.). Additionally, I infer that the way Charles solved the second task was to form two, two-level structures. The first was to form 80 from counting by tens by ones, and the second was to form 400 by algorithmic multiplication of 80 times 5. He did not retain the intermediate result of operating; i.e., that the 80 was 8 tens, which was reflected both by his answer of the 80 representing a pack and his inability to immediately recall how he had arrived at 80. Again, D.J.'s responses served to highlight the difference in the boys' reasoning, as he was able to explain that 80 represented the number of bars per box.

Summary of traditional teaching experiment analysis

In summary, I would characterize Charles as pre-fractional, as I infer that he had yet to construct fractional units (he had not yet constructed a partitive unit fraction scheme). I would also characterize him as coordinating two levels of (whole number) units in activity, what Hackenberg (2007) refers to as the first multiplicative conception. In this section, I have described Charles' schemes for fractions, including a qualitative

description of how Charles' schemes changed over time and in response to particular tasks. In the next section, I will analyze Charles' units coordinating activity during the course of the teaching experiment in more detail. There, my goal is to explore what types of situations allowed for Charles to coordinate units, to explore what types of situations were productive perturbations for Charles, and, conversely, what types of situations did not seem to be productive.

Third-order Model Analysis

Modeling propensities

The transition probability formula I developed was derived from a Markov model proposed by Bush & Mosteller (1953). Bush and Mosteller formulated a model of the behaviorist, conditional approach to learning common at the time, using terms like *alternative states*, *trials*, *probabilities*, and *events*. I was familiar with such Markov properties from previous teaching and coursework, and I realized that for my purposes, inference of units coordination activity (or scheme usage) could be considered alternative states. Events could be situations in which units coordination was inferred.

At first, I considered a general linear transformation that would operate upon *both* the propensities for transitioning from inferring units coordination activity or transitioning from *not* inferring units coordination activity *at each inference*. The four propensities are p : the propensity for not coordinating units given that not coordinating units was the previous state; q : the propensity for coordinating units given that coordinating units was the previous state; $1 - p$, the propensity for coordinating units

given that not coordinating units was the previous state; and $1-q$, the propensity for *not* coordinating units given that coordinating units was the previous state

Propensity computation

Let c_i represent the coding of units coordination for chunk i . If units coordination activity was inferred for chunk i , $c_i = 1$. If units coordination activity was not inferred for chunk i , $c_i = 0$. If I did not make an inference of units coordination activity (e.g., because the activities of the chunk focused on the other student) the chunk would not be included as an event.

Algorithm 4.1

Computing $T_i = \begin{pmatrix} p_i & 1-p_i \\ 1-q_i & q_i \end{pmatrix}, i=1,2,3,\dots$

If $c_{i-1} = 0$:

Define $m = \min(p_{i-1}, 1-p_{i-1})$.

Update $p_i = p_{i-1}(1-m) + m(1-c_i)$.

Leave $p_i = p_{i-1}$ unchanged.

If $c_{i-1} = 1$:

Define $m = \min(q_{i-1}, 1-q_{i-1})$.

Update $q_i = q_{i-1}(1-m) + m(c_i)$.

Leave $q_i = q_{i-1}$ unchanged.

When considering how to determine the four components of T_i , I immediately considered that since decisions made in the teaching experiment influenced the observations from which I made subsequent inferences, I should not consider the propensities formula to be independent of my inferences. I needed to consider that the goals for the teaching experiment are to both model the student's thinking and perturb the student's thinking in order to engender growth. Consecutive "like" inferences have "diminishing returns" in terms of increasing one's conception of stability in likelihood; a

single “unlike” inference drops the conception of such stability by a larger amount than the previous increase. More detail about the origins of Algorithm 4.1 are found in Appendix A.

Qualitative Analyses Across Second and Third-Order Models

The purpose of the baseline model is to model inferences of stability in a student’s propensity for units coordinating activity across chunks over the course of developing a second-order model. There are a multitude of other factors that are involved in a students’ propensity for units coordination; some of these are captured in my characteristics of chunks, and others are not. To describe students’ units coordination activity using the model, I then analyze the characteristics of chunks and the qualitative descriptions of the graph of a plot of the propensities by chunk along with qualitative descriptions of the second-order models of the students’ mathematics.

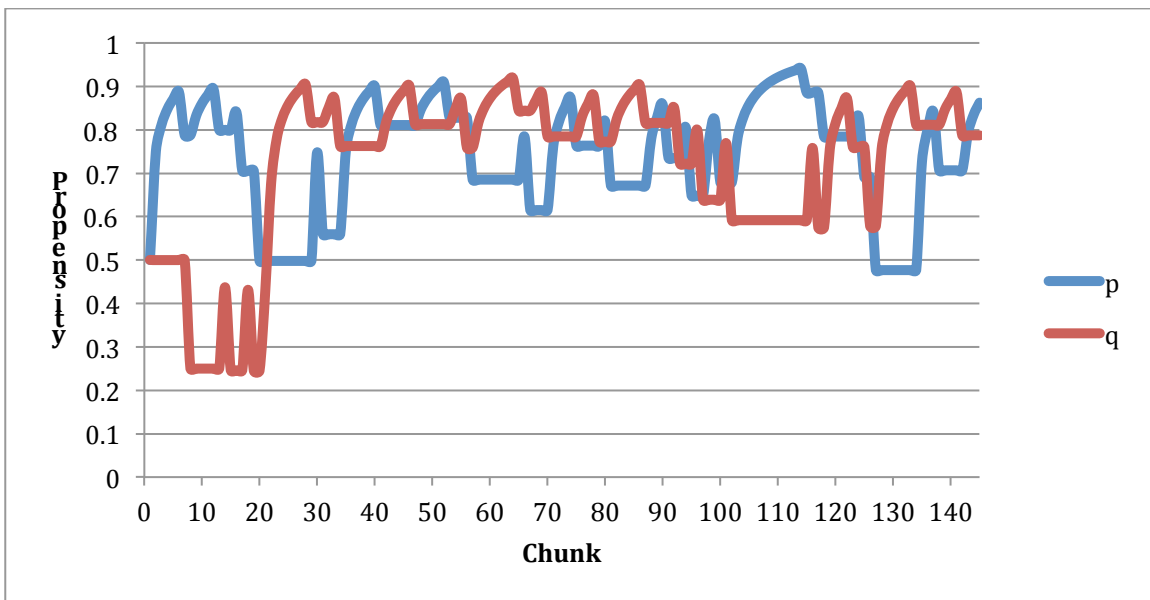


Figure 4.3: Baseline Model for Propensity of Units Coordination by Chunk

As seen in Figure 4.3, there is a great deal of variance in the propensities by chunk, but there is a positive linear trend for both p and q (which indicates increased stability of my inferences of Charles' activity). Another global interpretation is to note the periodicity in the inferences. For example, notice that the pattern of propensities about Chunk 130 have a similar rise and fall pattern to those about Chunk 25. As I will detail below, I find that I can associate increases in p with my attempts at engendering perturbation of Charles' schemes that did not previously include units coordination, and associate increases in q with my attempts at engendering Charles' units coordination within a context that Charles assimilated as "the same." The decreases in p can be associated with a (temporary) resolution to a perturbation to include units coordination, while the decreases in q can be associated with situations in which Charles did not assimilate the "sameness" between a context for which he coordinated units and a subsequent context.

To frame the analysis of the stochastic interpretation in more detail, I begin by describing the general situations of the chunks seen in Figure 4.2. The interactions in the initial Chunks (1 – 11) were in the clinical interview setting, and D.J. was present for each of the subsequent chunks. Tasks in the clinical interview were each in the linear fraction context, using construction paper and pencil/paper. Tasks in Chunks 12-45 were also in the linear fractions context, but they involved the use of Cuisenaire rods. Chunks 46-48 corresponded to the whole number operations pertaining to sharing the iPad previously described in the traditional teaching experiment analysis. Chunks 49-69 were once again in the linear fractions context, with Cuisenaire rods. Chunks 70-81 were with the drawn circles, as described in Figure 4.1. Chunks 82-109 were again in the linear

fractions context with the Cuisenaire rods, but both students also used pencil and paper. Chunks 110 – 118 were the whole number operations pertaining to the chocolate bars described above. Chunks 119 – 125 were again in the linear fractions context, with Cuisenaire rods (but no paper or pencil). Chunks 126-137 involved a fractions word problem, in which students used paper and pencil and gestures to indicate estimate sizes for a foot and an inch. The final eight Chunks once again involved the Cuisenaire rods (Figure 4.1).

It is important to note that inferences of units coordination are rather difficult to assess, and not all inferences are equally reliable or significant. For example, a common cause of an increase in p or q was simply Charles' response to request to explain his reasoning. Thus, it is the sustained patterns of increases (decreases) in p or q , as well as sustained periods of volatility or patterns of periodicity that are immediately relevant. Accordingly, I expound on characteristics of chunks that accompanied what I found to be more noticeable changes in p and q . Specifically, the initial increase in p ; the decrease in p around Chunk 17; the increase in q around Chunk 20; the increase in p around chunk 35; the decrease in p around Chunk 50; the increase in q around Chunk 55; the decrease in both p and q around Chunk 90; the increase in p beginning around Chunk 100; the decrease in p around Chunk 110; and the volatility in p and q between Chunks 115 and 135.

Initial increase in p followed by decrease at Chunk 17

The initial sharp increase in p was during the clinical interview, in which tasks were in the linear fraction context using rectangular pieces cut from construction paper. The tasks were designed to assess whether Charles had an available PUFs, PWS, PFS,

RPFS, and splitting operation. Charles' ways of operating indicated that he did not have these schemes, and as he instead conveyed his proper-placement scheme for fractions, I consistently inferred that he was not coordinating units as part of his scheme for fractions. The decrease in p beginning in Chunk 17 corresponded with Charles' coordination of units by naming a fraction as $1/2$ despite more than one piece in the "top." The initial "sweeping away" idea described in Protocol 1 was first demonstrated to Charles by D.J. in Chunk 17 after D.J. had aligned two red pieces above a purple.

Increase in q at Chunk 22

Immediately after Protocol 4.1, Charles reverted back to a proper-placement conception without disembedding. But in Chunks 22-29, I infer that Charles had constructed a *functional accommodation* of his proper-placement scheme. He consistently needed to "sweep away" all but the left-most piece in the "top" *when naming a fraction while working with the Cuisenaire rods*. Charles even gave some indication that he could coordinate two different two-level structures in activity, as in Chunk 25 (last Chunk of Session 2); he said that since one white piece was one-sixth of a brown piece, then two white pieces would be two-sixths of the brown piece.

Increase in p at Chunk 35

However, the increase in p beginning in Chunk 35 was marked by a task in which I tried to engender further coordination of *two* two-unit structures. Charles named a red rod as half of a purple rod. He also named a red rod as $1/5$ of an orange rod. At this point on Charles' desk there was an orange rod, above which there were two purple rods and a single red rod, above which there were five red rods (See Figure 4.4).

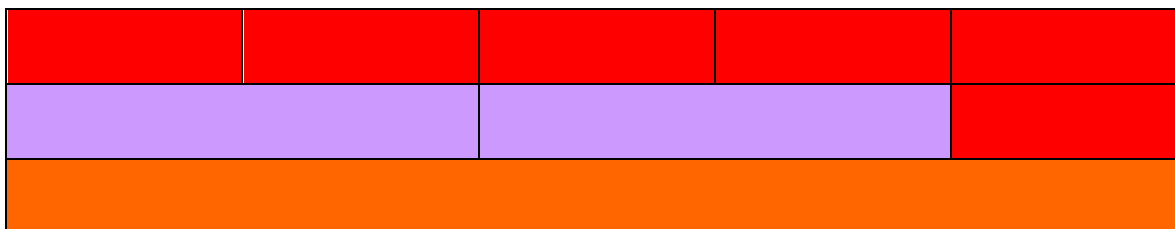


Figure 4.4. Alignment of Fraction Rods in Chunk 35

I then asked him what fraction the purple rod was of the orange rod. At first he said that it would be an improper fraction, but he said that “it couldn’t be, because [the orange] is still more than [the two purple].” I infer that he first considered the orange as 1 and the purple as 2, but upon reflection attended to the size, in that in his conception of an improper fraction, the “top” should also be “bigger” than the bottom. He then explained that an improper fraction would be “the orange of the purple” after moving the orange on top of the purples. I then said, “If this (holding up one red rod) is one-fifth, and two reds make the purple, then how much is the purple?” While D.J. responded two-fifths, and explained his reasoning, Charles said that it didn’t make sense to him. He rearranged the second row of Figure 4.4 so that the red was between the two purples and said, “I’ve got something though. ... One whole, because 1, 2, 3, 4, 5 (tapping finger while sliding across second row) and the [orange] equals five also, five reds. So you’ve got five-fives, then you simplify that.” I asked him to indicate with his fingers what one whole was, and he indicated a single red piece – “because it lines up equally with all the purples.”

Decrease in p and increase in q at Chunk 53

I associate both the decrease in p beginning in Chunk 53 and the subsequent increase in q with Charles’ sliding of a single unit, rather than forming of a whole and

“sweeping away,” when naming a fraction. Charles had observed D.J. iterating a piece from left to right, marking with his pencil at each iteration while counting by ones, and Charles’ acted in a similar manner in a task in which he was asked to name what fraction a tan rod was of a purple rod. He slid the tan rod while counting to four under his breath. When he was finished, D.J. asked for more tan rods, and when I asked Charles if he needed more to explain his answer, he declined. He said, “take this [tan] and put it on [the purple] four times [indicates four jumps from left to right] and that would make [spreads fingers the length of the purple] four pieces. So, you said, what is the tan piece, to the purple piece, so that would be one-fourth.”

Here I infer that Charles had begun to internalize the disembedding operation that had been an accommodation of his proper-placement scheme. I hypothesize that, upon reflection of his own way of operating and his observation of D.J., he realized that he didn’t need to actually form a “whole” tan rod before “sweeping away” all but the left-most piece. Indeed, his way of operating seemed consistent with that of a part-whole scheme, and without further indication, I might have inferred that he “had” a partitive unit fraction scheme. However, he subsequently declined to reject that a purple rod was not $1/1$ of the black rod, as I described in the description of his “even fit” conception. But when asked what fraction the tan was of the black, he thrice slid two tans at a time, counting, “1,2; 3,4; 5,6; 7” before responding, “ $1/7$.” He was then able to name the purple as $4/7$ of the black. “Four of these tan pieces equal up [lines above a purple bar, which is itself above a black bar]. Then you add three more to it [lines to right of other tans]. That would be seven to the black bar and then these four to the purple piece – the purple

piece equals four and the black piece equals seven. Take [all the tans away] and that would be four-sevenths.”

Decrease in both p and q at Chunk 90

The decrease in both p and q beginning in Chunk 90 began with Charles’ insistence on marked pieces for the whole within his proper placement scheme, this time in the reverse context in which he was asked to make the whole when given a piece named as one-fifth. While I inferred that Charles was coordinating units as he formed the whole, he said that red bar of the orange bar would be one-to-one, without marks, but one-fifth, if the same length were marked into five pieces. He definitely indicated perturbation – he twice said that I was confusing him, and he alternated between expressing conceptions of the orange rod as the correct or incorrect whole.

For the next task, I gave D.J. a lime block, which I said was $1/3$. I asked D.J. what $2/3$ would look like. D.J. first formed the whole by appending the three limes and finding the equally-sized block [the blue]. He then found a block that was the length of two limes [the dark green]. When asked to explain his reasoning, D.J. said that he put the three [limes] together and found the piece that was equal, and then took the little piece away. Charles indicated that he did not understand D.J.’s explanations or presentation, despite his intense focus on D.J.’s activities. I then gave them both the task of naming what fraction the dark green was of the blue, and Charles said that he didn’t see how it was two-thirds. It was not until D.J. said, while pointing to the pieces, “There’s the whole (has three limes appended) and that [dark] green piece equals up to the two.” Charles then explained his own thinking as “two of them equal up to [the dark green], and you’re equaling up to three of these pieces, so that would be two-thirds.” My

interpretation of these chunks is that Charles' ways of operating were different when he was presented with a task that required him to assimilate a fraction name versus an otherwise identical task in which he was asked to produce a fraction name.

Increase in p at Chunk 100

The increase in p at Chunk 100 was marked by a task that would seem to require a reversible partitive fraction scheme. I told Charles and D.J. that a red block was $\frac{2}{7}$ of a whole, and asked them to create the whole. Charles' initial response indicated that he anticipated physically iterating the given piece seven times – he asked if the orange bar was the largest available. Thus, his actions indicated to me that he assimilated $\frac{2}{7}$ as one-seventh. He started sliding the red bar across two appended black bars, and subsequently drew a length that was nearly seven times the length of the red bar. He then slid the red bar across this length 7 times, marking with a pencil as he counted by ones, and extended the length to make the seventh copy. He then drew, from this length, a rod of approximate width of the other Cuisenaire rods. He next tried to find a bar in the box that was the size of the bar he had drawn. I suggested that he might need to put two together to form the length. He did not seem to anticipate how the lengths of the bars should be adjusted – when he found that two brown bars were too long, he grabbed an even longer bar. When he tried two black bars and found the result too short (his drawing was slightly off, otherwise this should have been correct) he next tried an even shorter, dark green bar.

After D.J. presented his solution, in which he fit two tan rods to one red rod and made the length of the black bar, Charles reached for a purple bar, and fit two reds below the purple, and said, “Wait –I’ve got it!” but then tried to physically iterate the purple rod

across his drawn bar. When the fourth piece went beyond the length, he said “Never mind.” In summary, it seemed that the context required anticipation that the whole was seven iterations of *half* of the given rod. The task required coordination with fractional units – between $\frac{2}{7}$ and $\frac{1}{7}$. The observations indicate that when Charles was perturbed and unsure of how to make the whole, he did not seem to coordinate units that previously seemed established; i.e., the relationship between the size of a bar and the result of doubling that bar.

Volatility in both p and q in Chunks 115 – 135

The first decreases in p between Chunks 115 and 134 were described in the traditional teaching experiment analysis. Chunk 115 began with Charles’ response to the candy bar and boxes task in the fifth session, in which he eventually coordinated the units in activity, but then did not retain intermediate results of operating. In the sixth session, the decrease in p was marked by his reflection, upon my request for explanation, that he could disembed a piece from *within* a whole and not just from the left. After this, I asked Charles and D.J. to estimate how large $\frac{1}{12}$ of a foot would be. Charles was initially confused by the question, as he asked, “One-twelfth? Isn’t 12 a foot?” But when I asked him, “12 what” he responded that it was 12 inches, and then was able to estimate the size of one inch. Later, he drew his estimate of a foot, partitioned in 12 parts, and shaded in the first piece. I asked him if it mattered that the piece he shaded in was much larger than one of the pieces in the middle. He said, “no, because you’re just making lines into it. The [whole] length is still the same” and repeated this reasoning after T.J.’s suggestion that all the parts should be the same for the shaded part to be one twelfth. I suggested that while they couldn’t be drawn perfectly by hand, Charles would surely notice if I

drew in marks that were quite clearly not uniformly distributed. After I proceeded to mark and shade about $\frac{1}{3}$, and then make 11 additional partitions of amounts ranging between $\frac{1}{20}$ and $\frac{1}{12}$, he agreed that the amount I shaded was not $\frac{1}{12}$ and he tried to re-do his segmenting to make equal parts. Despite the fact that his initial inch was larger than the other pieces, he still tried to segment with that size instead of trying a smaller amount, resulting in his ending up with 11 segments. He then asked, “Can I just draw one more line in [the second piece from the left]?”

Summary

The results of the analysis indicate several types of patterns in Charles’ units coordinating activity across the ecologies of the teaching experiment. Charles’ sustained increase in p corresponded with situations in which he faced tasks that either (a) required at least a partitive unit fraction scheme or (b) required coordination between *two* two-level unit structures. Charles’ sustained increase in q corresponded with situations in which he (a) coordinated with 2 in some way or (b) modified his schemes to include coordinations that were either demonstrated by D.J. or suggested by me. Charles’ sustained decreases in p and q tended to coincide, and they corresponded with situations requiring two-level structures to be anticipated (and hence interiorized). For example, when asked to create a fraction, Charles might first attempt to create a fraction without coordinating units, but then coordinate units to name or explain the name for a fraction that D.J. made. Conversely, there were situations in which Charles made marks that indicated that he was coordinating units, but upon reflection, he reverted to his schemes that did not involve such coordinations.

Conclusions and Implications

The pilot study results demonstrated the utility of the baseline model for capturing a short number of teaching sessions in which the same individual was both the teacher-researcher and third-order modeler. In the current study, these roles were separated, for several reasons. First, this allowed for me to reduce bias in the coding of context and units coordination inferences. When doing both, I would presumably be more likely to re-present the change in context that I had in mind at the time of the session and faced the risk that my coding would be biased by my intent. Second, separating the two roles allowed me to analyze commonalities across teacher-researchers, so that the resulting models would not be as restricted to my own way of thinking. Third, this allowed for me to use existing data sources that would be independent of learning whilst constructing the second and third-order models.

In my forming of the baseline model, I anticipated the need for modification of the formula to include contextual factors that might improve Algorithm 4.1 to more aptly describe the students' propensities' for stability in units coordination. In the pilot study, I considered context when separating chunks, but I did not incorporate context into the third-order model. This is partially because of concerns of bias – the considerations I was using when making units coordination inferences and the considerations I was making when coding context could easily become conflated with my in-the-moment interpretations of the situations. In the next chapter, I describe my methods for extending the baseline model to include context and for analyzing across students and teacher-researchers.

CHAPTER 5: METHODS

In this chapter, I discuss my methods, as motivated and informed by Chapters 3 and 4. I begin by describing my data sources. I then describe the sequential process of transcription, chunking, and coding. I subsequently describe my approach for making models of students' propensity for stability in units coordinating activity become more explanatory and predictive, including a description of how I evaluated the utility of the resulting models with respect to my research goals. I close the chapter with a summary of the procedures.

Data Sources

Including my pilot study with Charles, there were four teaching experiments that were used for constructing models of students' units coordination activity. The second was a teaching experiment with a sixth-grade student, Austin, who was part of a paired teaching experiment in which Anderson Norton, my adviser, was the teacher-researcher and I was the witness and videographer. The third existing data source is a teaching experiment conducted by Catherine Ulrich, also involving a sixth-grade student, Justin. Descriptive and explanatory third-order models of students' units coordination activity were constructed from these three sources, and a fourth data source, a teaching experiment conducted in the Fall semester of 2013 by Anderson Norton (again, with myself as the witness), was used to evaluate their utility. Each of the sessions analyzed was approximately 30 minutes in duration.

The choice of existing data sources provided a variety in units coordination inferences – though each student analyzed was a sixth-grader from the Southeastern

region of the United States, their units coordination structures varied considerably.

Charles assimilated only one level of units, and though he coordinated two levels of units in activity, he was pre-fractional. Austin assimilated two levels of whole number units but not two levels of fractional units, and Justin assimilated three levels of fractional units. The fourth teaching experiment involved Dylan, who assimilated 2 levels of fractional units.

Each student that was analyzed was male. While Charles' partner in the teaching sessions was male, each of the other teaching experiments involved a male-female pairing. The decision to focus on Austin and Justin was due to their general increased willingness to communicate their thinking in comparison with their partner. Dylan and his partner were both quite communicative; I chose to focus on Dylan because I had not analyzed other female students, and I wanted to avoid the possibility of gender differences contributing to a poor model fit.

Student Pseudonym	Levels of Units Assimilated	Teacher-Researcher	Gender of Student	Sessions Transcribed and Analyzed
Charles	1	Boyce	Male	6
Austin	2	Norton	Male	9
Justin	3	Ulrich	Male	10
Dylan	2	Norton	Male	11

Figure 5.1. Data Sources

The teaching experiments most often included the teacher-researcher listed in Figure 5.1, a witness, and a consistent partner, though there were exceptions. In the

teaching experiment with Austin, his classroom teacher observed and at times interjected during four of the sessions. Austin's partner, Jane, was present for all but one of the nine sessions analyzed. In the teaching experiment with Justin, there was a different teacher-researcher during two of the sessions, no partner during the first session, and a different partner during the last two sessions. Unlike the teaching experiments of Austin, Charles, and Dylan, which only involved the use of a single camera, the teaching experiment with Justin included a two-camera configuration, resulting in fewer instances of inaudible or off-camera activity.

For the most part, the students' use of manipulatives was limited to paper and pencil drawings, construction paper cut-outs, and Cuisenaire rods similar to those described in Chapter 4. There were two exceptions. In the teaching experiment with Jane and Austin, there was periodic use of the universal version of *CandyFactory*, an iOS app created by the Learning Transformation Research Group at Virginia Tech (LTRG, 2012). *CandyFactory* is designed to support more powerful conceptions of fractions, by engaging students in partitioning and iterating activities. Gameplay in *CandyFactory* during the teaching sessions was limited to the first two levels, which ensured that improper fractions would not be encountered. In the game, a random number generator produces a "customer order" as a size; the student satisfies the order via sequential actions of partitioning, iterating, measuring, and shipping. In the first level, there are partition marks, which are not present in subsequent levels (see Figure 5.2).



Figure 5.2. Screenshot of CandyFactory Level 1 (LTRG, 2012). Retrieved from <http://www.ltrg.centers.vt.edu>

Another educational game for the iPad, *CandyDepot*, was used in the teaching experiment with Dylan and Maddie. Also created by LTRG, the focus of *CandyDepot* is units coordination (LTRG, 2013). Students first choose unit relationships: the number of bars in a bundle and the number of bundles in a box. The software then produces a sequence of customer orders in terms of bars, bundles, or boxes that the students satisfy by efficiently dragging objects to a truck (see Figure 5.3). Play in *CandyDepot* was restricted to the first level, in which re-unitizing is not required.



Figure 5.3. Screenshot of CandyDepot Level 1 (LTRG, 2013). Retrieved from <http://www.ltrg.centers.vt.edu>

Transcription

I transcribed a subset of observations of each of the first three teaching experiments, chronologically, beginning with the initial session and continuing through a maximum of ten sessions (for the fourth teaching experiment, I also transcribed the final, eleventh session). Most audible student and teacher speech was included verbatim in the transcription, but discourse that seemed irrelevant to the tasks at hand (such as conversation about a volleyball game) was paraphrased. Actions that were visible to the camera were included in the transcription if I inferred that they might have been in the perceptual field of the student of interest. For instance, if there was a potential for the student of interest to see or hear his partner's activity, the partner's activity was included in the transcription. In anticipation of further analysis of both students' units coordinating activity with the final teaching experiment, I transcribed interactions that

were judged in the perceptual field of either student in the transcription of Dylan and Maddie.

I created an Excel spreadsheet for organizing each teaching experiment's transcription. Each teaching session was in a separate sheet, with each sheet initially divided into three columns. The first column lists observations of the teacher/researcher and witness activities. The second column lists observations of the student of interest. The third column is for observations of the student's partner. Figure 5.4 displays an excerpt of the type of descriptions and organization of the transcription.

Oct 12 2011 Teaching Session 4		
Researcher/Teacher/Witness Action	Austin Action	Jane Action
I'm just going to draw something on paper here, and see if y'all can figure out -		
[Draws rectangle about one inch in height and three inches long] Here's a candy bar that I'm going to draw, and I'm going [Shades in leftmost piece] My question is, -to not say it out loud, just think it in your heads - what fraction is shaded? [put Don't say it out loud, just in your heads - what fraction is shaded?		
	[Looking intently]	[Looking intently]
[5 secs] You all got it?		
	Uhmhum	[Nods slightly in affirmation]
Alright, [Jane] you first.		
		[Mouths something] One [Appe:
	[Nods]	
You pretty sure about that?		
		[Nods yes, with eyebrows raised]
If it wasn't one-fifth, what else might it be?		
		[5 seconds pass] One -fourth?
Okay. Austin what do you think?		
	One-fifth [puts arms out, as if to say it's simple]	

Figure 5.4. Transcription Excerpt

Chunking

I continued the approach from my pilot study to chunk the transcriptions. To develop sequences of observations in the course of the teaching experiment, I partitioned the entirety of the transcription into a sequence of *chunks*. There are two main aspects to the chunking process. The first consideration is to distinguish chunks, i.e., to decide where one chunk begins and one chunk ends. The second consideration is a

characterization of chunks. The chunking process begins with the identification of two types of characteristics of context – that of a mathematical task and that of interaction between the environment and the student of interest. Figure 5.5 describes some components of these two types of context. These descriptions are an expansion of my interpretation of context given by Simon et al (2010) and Wagner (2006) for teaching experiments with individuals to also account for student-student interactions.

Task Context	Interaction
Aspect: (e.g., coordinate with units of 5 versus units of 8)	Teacher actions: Questioning, Gesturing, Demonstrating, Affirming, Reiterating
Type: (e.g., naming fraction not constructing fractional size)	Peer actions: Responding, Moving Figurative Materials, Explaining, Gesturing
Setting: (e.g., with fraction rods not verbally)	Student actions: Responding, Gesturing, Explaining, Questioning, Affirming, Manipulating, Affect characteristics

Figure 5.5. Broad Chunking Categories

Separating chunks

Chunks are separated by characterizations of context that, the majority of the time, begins with task presentation and ends with a response by the student of interest that could be used to infer or not infer the coordination of units. If the task context is changed, it must be due to some sort of interaction. If this interaction includes a student response that indicates units coordination activity (or lack thereof), then that response marks the end of the chunk. Other interactions that do not affect the task context that preface such student response are overlaid within the chunk. If the interaction ends

without a response from the student of interest, e.g., if the task was completed by the peer student with the student of interest simply observing, then the chunk would end right before the introduction of a new situation.

Characterizing Chunks – Units Coordination Activity

Once chunks are distinguished, they can be characterized. The first step is the coding of units coordination activity. I use the term *units coordinating activity* to refer to students' (inferred) use of a units coordination structure in *either* the recognition portion or activity portion of a scheme. Engaging students in activity that, if internalized, would support accommodations of their schemes to include a higher-level structure is what I mean when I say a student “coordinated units in activity.” Thus, the term “units coordinating activity” is more broad and includes both interiorized coordination of units and the coordination of units in activity.

If my inference of the activity of the student is that $n + 1$ units were coordinated in the chunk (when either n or $n + 1$ levels were interiorized) then I coded this chunk as ‘1.’ If my inference of the activity of the student is that $n + 1$ units were not coordinated in the chunk (when an opportunity for such coordination to take place was inferred) then I coded this chunk as 0. If my inference of the activity of the student is that no inference about their units coordinating activity could be made, then I coded the chunk as ‘?’.

Examples of inferences leading to the coding of 1s and 0s are listed in Figure 5.6.

Because chunks that were marked with a “?” were not further analyzed, I generally coded ‘1’ or ‘0’ whenever even a weak inference could be made regarding the students’ units coordinating activity.

Inferences Supporting a Code of ‘1’	Inference Supporting a Code of ‘0’
Iterating, partitioning, disembedding	Conflation of units’ numerosity
Attending to size relations in result	Inattention to size relations in result
Affirmation of previous coordinations	Uncertainty of result; empiricism

Figure 5.6. Examples of Inferences Coded with 1 or 0

Characterizing Chunks - Context

As I described in Chapter 3, the separation of chunks was partially motivated to permit subsequent characterization of chunks using my over-arching framing of context as consisting of task context and interaction context. To construct a manageable set of contextual descriptions to incorporate into an adjustment of the baseline models of Austin and Justin, I relied on existing descriptions of the students’ activities from Anderson Norton and Catherine Ulrich, their respective teacher-researchers. To characterize the context of the chunks in the teaching experiment with Austin, I reviewed a manuscript describing Austin’s ways of operating written by Dr. Norton, including references to particular lines of my transcription whence his inferences originated (see Appendix C). Similarly, I reviewed the doctoral dissertation of Dr. Ulrich for inferences of Justin’s mathematics that were explicitly connected to particular lines from her transcription (Ulrich, 2012). I initially only coded the chunks that were identified explicitly in these sources, and I did this coding while blinding myself of my inferences of the students’ units coordination activity by hiding columns in the spreadsheets.

Using open coding, I characterized the chunks that were referred to by Dr. Norton in his description of Austin. I then used the constant comparative method (Glaser, 1965) to refine this list of codes to a manageable number of codes (see Appendix F for analytical memos and codes). I then coded the entirety of the transcript with the resulting codes, either ‘1’ for the presence of the characterization in a particular chunk, or ‘0’. I later repeated this process for characterizing the chunks in the transcription of Justin. To code context in the teaching experiment of Dylan, I combined the codes used for characterizing context from the transcriptions of Austin’s and Justin’s teaching experiments and did not refer to the teacher-researcher’s model for characterizing context.

Improving upon the Baseline Model

Having constructed characteristics of chunks that seem to accompany important changes in Austin’s and Justin’s propensity for stability units coordination across or within chunks, I next considered modification of the baseline model to “fit” inferences described by their teacher-researchers’ second-order models. I first marked, in a separate column in the spreadsheet (see Appendix E), the chunks for which I coded either ‘1’ or ‘0’ for units coordinating activity. I identified the chunks that corresponded to Anderson Norton’s explicit description of Austin’s units coordination, and coded these as ‘1.’ I then adjusted step 2 in Algorithm 5.1 to attempt to fit higher propensities for stability in units coordination activity with Anderson Norton’s inferences. I repeated this process for Justin, using the explicit descriptions of his units coordinating activity from Catherine Ulrich’s dissertation (Ulrich, 2012).

Modifications to the baseline algorithm

Upon the determination of codes to qualify the context of units coordination activity inferences, I considered multiple ways of incorporating these codes to adjust the transition formulae given below in Algorithm 5.1.

Algorithm 5.1.

Computing p_i and q_i , given inference $c_i \in \{0, 1\}$

1. $p_0 = q_0 = 0.5$ 2. For $i = 1, 2, 3, \dots$, - If $c_{i-1} = 0$: a) Define $m = \min(p_{i-1}, 1 - p_{i-1})$. b) Update $p_i = p_{i-1}(1 - m) + m(1 - c_i)$. c) Leave $q_i = q_{i-1}$ unchanged.	- If $c_{i-1} = 1$: a) Define $m = \min(q_{i-1}, 1 - q_{i-1})$. b) Update $q_i = q_{i-1}(1 - m) + m(c_i)$. c) Leave $p_i = p_{i-1}$ unchanged.

Recall that the baseline algorithm includes the following assumptions. The first assumption is that all propensities were equally likely before any observations are made, what I call, *initial equiprobability*. Hence, both p and q are each set initially at 0.5. The second assumption is that the model is symmetric in the sense that updates to p are made in the same manner as updates to q . The third assumption is that a calculated propensity can never be 0 or 1. This follows from the formula above being bounded above by $\sum_1^n \frac{1}{2^n}$.

For simplicity, I considered only linear adjustments to the value of m that would decrease or increase the change in p or q from chunk to chunk, and I required the same adjustment to the calculation of both p and q to maintain symmetry. For each of the j contextual codes identified for the i chunks in which a units coordination inference was

made in a particular teaching experiment, I defined d_{ij} to be 0 if the contextual code was not present for the chunk and 1 if the contextual code was present for the chunk. I made the modification to line 2(a) in Algorithm 5.1, letting $m = m^{b_j \bullet d_j + 1}$, with the vector b_j to be determined via an optimization process. Note that this maintains the assumption that calculated propensities are strictly between 0 and 1, and that the modified version of Algorithm 5.1 is equivalent to the baseline formulation if each $b_j = 0$.

Optimization process

In order to improve upon the baseline model, I constructed a function that would be optimized if the propensity plot were consistent with the teacher-researcher's (second-order modeler's) inferences of units coordination. The assumption underlying such a function is that for chunks for which the second-order modeler explicitly identified inferences of units coordination, the values of p or q should be high (or low) to indicate stability in inferences in units coordination activity. The parameters of u and l were also introduced to qualify "high" and "low" values of q and p , respectively. If a chunk was characterized explicitly in a second-order modeler's description as indicative of units coordination, then the propensity plot would be a good fit if that chunk's value of p was small or if its value of q was high; conversely, if a chunk were characterized explicitly as indicative of a lack of units coordination, then the propensity plot would be a good fit if that chunk's value of p was large or if its value of q was small.

As I discuss further in the Results section, the initial results of the initial approach with Austin's teaching experiment data did not lead to robust values of b , even when the optimization function removed the criteria for inferences of a lack of units coordination

and focused only on fitting with inferences of units coordination. The parameter b was originally planned to be a vector, with parameter values for each context that would allow for the calculations of propensities to differ if a single chunk were coded with multiple contextual codes. Instead, the vector of contextual codes was collapsed into a single dichotomous variable, 1 for the presence of one or more of the identified contexts, and 0 otherwise. As a result, the parameter b is a scalar. The greater the value of b , the more effect the presence of the context has on the propensity calculation. This was appropriate because each of the identified contexts was thought to weaken an inference of units coordination activity and strengthen an inference of not coordinating units.

Analyses of the Plots of Units Coordinating Activity

Segmenting sequences of chunks

In Chapter 4, I interpreted the graphs of p and q in conjunction with the transcript and (my) second-order model, demonstrating how salient aspects of the graphs of p and q together describe the ebbs and flows of units coordinating activity that corresponded with Charles' contexts and periods of perturbations. After (attempting to) improve the algorithm for producing the graphs of p and q for Austin, I similarly examined those graphs in conjunction with both the transcript and the second-order model. The graphs are noisy, and there are many different ways that one can conceptualize the changes in p and q over the course of the chunks. In an effort to improve the predictability and testability of particular plot aspects' correspondence with patterns in activity sequences, I needed an objective way of separating the plots into smaller segments and an objective way of characterizing important aspects of the plots.

In my first attempt to separate the plot into a sequence of segments, I computed and plotted polynomials of best fit for both p and q . I then separated the graph into segments for further analysis by using the extrema and inflection points of the polynomials. Using sixth-degree polynomials was promising. Considering the number of zeros of its first and second derivatives, a sixth-degree polynomial has at most 5 distinct extrema and 4 inflection points. Using two polynomials, one for p and one for q , it was possible to form 18 segment marks at these values. In practice, I found the extrema of one polynomial matched with the inflection points of another, and there were some repeated zeros in the polynomials of best fit.

Using sixth-degree polynomials resulted in a manageable number of segments, but the polynomials were generally not useful for describing how p and q were increasing or decreasing within segments. I instead chose to use teaching sessions to segment the chunks. To characterize sessions, I needed a way to describe the directionality and volatility in both p and q , both within and across sessions. This process involved a sequential posing of formulae, looking at the results for discrimination and fit with my qualitative interpretation of the graphs' meaning.

Characterizing changes in p and q values

In an attempt motivated by a goal of parsimony, I first attempted to use threshold values of u and l to characterize each session. For example, when the values of p remained smaller than l and the values of q remained above u , that would correspond with an increase in stability to coordinate units. Unfortunately, I did not find threshold values that would work well for describing the graphs of both teaching experiments across sessions. I considered using different u and l values for each teaching experiment,

but since the values for the Justin data did not converge, it would require a rather arbitrary choice of values and complicate testing how well the model fit with the Dylan data. I needed an approach that would work well for both teaching experiments.

My approach was to look for a single, objective way to describe the graphs pertaining to both teaching experiments. To account for the values, change, and volatility in p and q , I considered a total of six quantities. First, there was the mean value of p (and q) during the session. Second, there was the range of p (and q) during the session, the maximum value minus the minimum value. Third, there was the difference between the mean value of p (and q) during the session and the value of p (and q) in the chunk entering the session.

Using these six quantities, I created a scale, S , for characterizing a segment of the teaching experiment, using Algorithm 5.2.

Algorithm 5.2

Computing S , given p_i, q_i , for $i \in \{1, 2, 3, \dots, n\}$

1. Compute $mean\ p = \frac{1}{n} \sum_{i=1}^n p_i$ and $mean\ q = \frac{1}{n} \sum_{i=1}^n q_i$.
2. Compute $range\ p = \max(p_i) - \min(p_i)$ and $range\ q = \max(q_i) - \min(q_i)$.
3. Compute $S = \frac{2\ mean\ q - q_0}{range\ q} - \frac{2\ mean\ p - p_0}{range\ p}$.

Note that though the values of p and q are each strictly between 0 and 1, it is possible for the range of q or p to be 0, which would result in undefined S . In practice, there is always variation in both p and q during the teaching sessions. Lower values of S generally correspond with a low propensity for coordinating units (e.g., high p and/or low q) and substantial perturbation (e.g., a large range in either p or q). Conversely, high

values of S correspond with a high propensity for coordinating units and a low degree of perturbation. Next, I detail how I constructed and used S to characterize sessions.

Construction of the Scale Score, S

To form testable conjectures pertaining to the plot of units coordinating activity, I needed to construct a way to objectively characterize the salient aspects of the plots. These were described qualitatively in my analysis of the plot of Charles' units coordinating activity (see Chapter 4). I consider my initial, qualitative interpretations of the plots of p and q as the result of imagining features of curves of best fit. As described in Chapter 4, the salient features of the plots were the increases, decreases and the volatility of p and q , and I used these to help identify two types of interesting segments within teaching experiment sessions – those I associated with Charles' sustained perturbation and those I associated with Charles' growth in units coordination.

To characterize the changes in p and q within each session, I sought an objective way for capturing salient interpretations at the session level. The development of the scale score, S given in Algorithm 5.2 involved successive modifications to a simple formula of looking at the differences in the mean values of p and q within each session. This focused solely on the propensity values and did not incorporate context. I then created a parsimonious set of codes to qualify the context of the underlying the plots' characteristics. These included broad characterizations of the interactions and mathematical domains prevalent in the teaching episodes.

Generally, high values of q and low values of p are associated with successful task completion and satisfactory responses to teacher or student requests for explanations, as this pattern usually corresponds with a greater number of consecutive inferences of units

coordination than consecutive inferences of not coordinating units. Conversely, lower values of q and higher values of p correspond with substantial perturbation. Finding the difference between the mean value of q and the mean value of p during a chunk therefore provides a crude measure of the level of perturbation during a teaching session: higher values of q correspond with lack of perturbation, and higher values of p correspond with greater perturbation. But because only one of p and q changes from chunk to chunk, long strings of consecutive inferences of units coordinating activity (or long strings of consecutive inferences of not coordinating units) result in an increase in only one of q (or p), with the other value remaining nearly constant. Because the values of p and q were not returned to 0.5 at the start of each session, it is possible that the mean value of p could remain high even if there were many consecutive inferences of units coordinating activity.

For example, if there were five consecutive inferences of ‘1,’ the value of q would increase four times, while the value of p would decrease once and then remain constant. If the value of p at the beginning of such a session was high, then merely finding the difference in the mean values of q and p might not be useful for distinguishing a lack of perturbation associated with, for instance, the establishment of a new scheme. To account for such a possibility (observed in Session 9 of the Austin data, see Figure 6.4), I incorporated the starting point of p and q into the characterization of the session’s plot. Instead of finding the difference between $\bar{q}$ and $\bar{p}$, I computed $(2 \bar{q} - q_0) - (2 \bar{p} - p_0)$. Increasing the weight of the mean value in the calculation provided balance, so as to not over-emphasize the initial values in the calculation. Since the range of possible values of p and q was between 0 and 1, doubling the mean value and subtracting the initial value

also resulted in quantities that were generally between 0 and 1. For instance, for $(2 \bar{q} - q_0)$ to be greater than 1, the value of $\bar{q}$ must be greater than 0.5, and for $(2 \bar{q} - q_0)$ to be less than 0, the value of $\bar{q}$ must be less than half of q_0 . Thus, doubling the mean values was an arbitrary but natural choice for increasing their weight in the calculation, and it was this decision that led to the identification of ‘0’ and ‘1’ as cut-off values of S .

Another aspect of the plots that was not still not adequately captured was the volatility within sessions. Although the mean values of p and q accounted for some of the variation, I found it useful to include the range of p and q values in the scale to identify sessions with high volatility in p or q from sessions with a similar mean, initial value, and stable values of p and q . The smaller the range of values of p (and q), the better the change in the mean value captures its trajectory for the duration of the session. Therefore, I weighted the two differences $(2 \bar{q} - q_0)$ and $(2 \bar{p} - p_0)$ by dividing respectively by the range of q and p . Since the range of q and p was between 0 and 1, this increased the absolute value of each quantity. For sessions in which both p and q had a large range, i.e., near 1, the calculation remained (nearly) unchanged. For sessions in which only one of p or q had a large range of values, the larger weight for the other, more stable value effectively dampened the effect of increases in the mean value of q or decreases in the mean value of p that otherwise would be associated with a lack of perturbation.

As with the qualitative interpretation, I sought three gradients – sessions most indicative of establishing of a new scheme, sessions most indicative of perturbation, and the rest. The adjustments to the scale score S permitted association of values of S to each of these

types of situations. I characterized each session of the Austin and Justin teaching sessions as Substantial Perturbation (SP), if the value of S was less than 0; as Equilibrium (E), if the value of S was greater than 1, and as Moderate Perturbation (MP) if the value of S was between 0 and 1 (inclusive).

Conjectured patterns in units coordination activity across and within sessions

Though the S scale provided a way for distinguishing and identifying students' level of perturbation associated with the establishment of a new scheme, other considerations of the types of interactions that might result in similar values of S required further qualitative analysis. Having used the scale score to distinguish the level of perturbation, I returned to the transcription and coding to broadly characterize the dominant aspect of each session and its mathematics domains. I constructed a parsimonious set of codes to further characterize *why* the level of perturbation was low, medium, or high across the Justin and Austin sessions. I used the same approach and codes to characterize teaching sessions with the Dylan data.

As I was considering how to characterize the sessions, I formed conjectures about how the changes in p and q relate to sequences of perturbation-context pairings experienced by Austin and Justin. My approach was to begin with the Austin teaching experiment, and look for salient features of the graphs of the plots of p and q . I then referred back to the second-order model and transcript to form conjectures about the contextual differences that distinguish SP, MP, and E sessions. For example, I distinguished two types of SP sessions – those characterized by conflicting activity sequences and those characterized by demanding, difficult tasks. Repetition of this process with the Justin data resulted in further characterization of MP (total of three

types) and E sessions (total of three types). In my conjecturing below, I assumed that the same 8 characterizations of teaching sessions would apply to the Dylan teaching experiment.

Methods for Testing

Testing validity of baseline model

To test the utility of the approach, I began by transcribing the teaching experiment of Dylan and Maddie. Because both students explained their thinking clearly and I had not identified a student to work with yet, I transcribed inferences that seemed to be in either of the students' awareness. As before, the teacher-researcher formed a second-order model that was independent of my chunking and subsequent coding of units coordinating activity. I identified explicit references to Dylan's units coordination in the second-order model that were matched to lines of the transcription. By hiding columns in the spreadsheet, I blinded myself of my coding of units coordination activity while marking corresponding chunks. I tested the output of Algorithm 5.1 with varying u and l values. The model would be judged as valid if there were high values of u and low values of l that corresponded with optimal fit of the inferences with values of $q > u$ or values of $p < l$ for the chunks identified by the second-order modeler as suggesting that the Dylan was coordinating units.

Testing Conjectures Regarding Session Characterizations

I formed hypotheses about sequences of perturbation-context pairings with Austin and Justin to test with the Dylan data. The first of these conjectures is that the method of computing S will result in separation of E, MP, and SP sessions. The latter conjectures

are related to the perturbation-context pairings. I characterized each type of session using the same typology constructed for the Justin and Austin data. I then attempt to falsify conjectures (formed from the Justin and Austin data) about the sequences of perturbation-context pairings.

Some of these conjectures address inter-session relations. For example, one conjecture is if an Establishing session directly follows an SP session, then the SP session cannot be Demanding. Other conjectures address intra-session relations. For example, an Establishing session beginning with the “easier” of two contexts requiring the same level of units (i.e., from whole number to fraction or from physical manipulative to symbolic manipulative) will result in the student coordinating units in the latter, “harder” context.

Addressing the Research Questions

To conclude this chapter, I restate my research questions with more specific conjectures testable by the methods described above. The value of approach I have taken is only partially determined by reliability, validity, and parsimony considerations. The answer to the research question, “How does the modeling of a students’ propensity for assimilating an n -level structure *contribute* to an understanding of how units coordination structures are constructed?” is also determined by the quality of the conjectures that can be formed and tested. To contribute, the approach must permit the formation and testing of conjectures that are of interest to the mathematics education research community.

My other two research questions are: “What are the relationships between a student’s units coordinating activities across numerical situations such as counting sequences, fractions, whole number multiplication, and signed integer addition?”, and

“How might sequences of ecological changes facilitate growth in the number of levels of a student’s units coordinating structures?” Thus, the other two research questions are about the ability of my approach to form and test two types of potential conjectures. To address my research questions, I thus needed to consider the following:

1. Is the modeling approach descriptive, e.g., is it
 - a. valid?
 - b. reliable?
 - c. parsimonious?
2. Is the modeling approach useful for forming and testing conjectures pertaining to
 - a. relationships between units coordinating activities across numerical situations?
 - b. sequences of ecological changes and growth in units coordination?

Throughout the modeling process, decisions were made with regard to each of these considerations, but they were not treated equally at all times. For example, when choosing data sources, it was extremely important to make sure the teaching experiments would involve multiple numerical situations. Validity, reliability and parsimony considerations were the primary focus of the next stage of the modeling process, as I sought to refine Algorithm 5.1. Consideration of the model utility was heavily emphasized at the end of the process, as I sought to form conjectures pertaining to (2) above that were general enough to hold for the Justin and Austin data and test with the Dylan data.

CHAPTER 6: MODELING RESULTS

In my discussion of my pilot study results (Chapter 4), I described how the plot of p and q could be used in conjunction with second-order model constructs to describe Charles' mathematics. In this chapter, I provide the results of the activities outlined in Chapter 5, which were intended to enhance the model and make it more descriptive, predictive, and explanatory. I begin by describing results pertaining to Austin—specifically, the baseline plot, contextual coding, and optimization results. I next discuss these aspects pertaining to Justin. After discussing the results of modeling the individuals separately, I analyze the patterns of context-perturbation pairings in Justin's and Austin's units coordinating activities and use these patterns to form conjectures about similar patterns in Dylan's units coordinating activity that could be tested.

Third-Order Model of Austin's Units Coordinating Activity

Coding results

The 9 teaching sessions were divided into a total of 358 chunks for which either a '0' or '1' was coded for units coordinating activity (see Figure 6.1). After identifying explicit inferences of units coordinating activity in the second-order modeler's description (given in Appendix C), I matched the line numbers to the containing or adjacent chunks that had been coded '0' or '1' (blinding myself of the coding). Open coding of these chunks led to the formation of 6 contextual categories (see Appendix F for initial contextual codes and memos). I coded the entirety of the 358 chunks with these 6 contextual codes (see Appendix D). The frequency of each code is displayed in Figure 6.2.

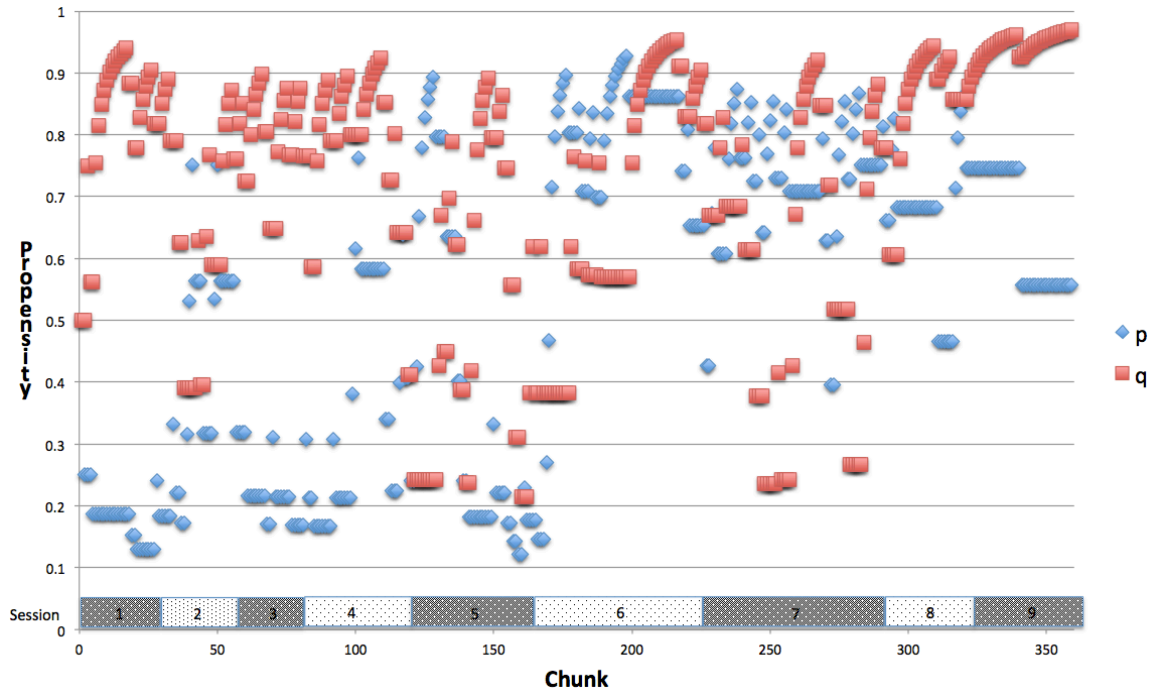


Figure 6.1. Baseline Plot of Austin's Units Coordinating Activity

One important quality of the contextual codes is that the presence of each code would seem to affect the propensities in the same direction, if not in the same magnitude. For each code, its presence strengthens a code of '0' and its absence strengthens a code of '1.' For example, if a chunk was coded as *without* manipulation of figural material, that would make a coding of units coordinating activity more impressive. If a chunk was coded as "reflecting on activity prompt of teacher," that would strengthen a coding of *not* coordinating units. I thus conceptualized that the modification to the baseline model resulting from the contextual coding would be uniform in directionality across the six codes, with the magnitude of each change a parameter to be determined by optimization.

For simplicity, I considered only linear adjustments to the change in p or q from chunk to chunk, and I required the same adjustment to the calculation of both p and q to

maintain the symmetry of Algorithm 5.1. Recall that p is the propensity for stability in *not* coordinating units across chunks, and q is the propensity for stability in coordinating units across chunks. I maintained that only one of p or q would be modified after each observation, and only sought to adjust the increase or decrease so that, potentially, more important inferences would have a greater effect.

Code	Count (358 chunks total)
Reflecting on activity/utterance of partner	33
Reflecting on activity/prompt of teacher	36
Manipulates figurative material	48
Coordination explicitly involves 2	47
Reflecting on previous response	131
Likely rote response	14

Figure 6.2. Contextual Coding – Austin

For each of the j contextual codes identified for the i chunks in which a units coordination inference was made in a particular teaching experiment, I defined d_{ij} to be 0 if the contextual code was not present for the chunk and 1 if the contextual code was present for the chunk. I made the modification to line (a) in Algorithm 5.1, adding the line, $m = m^{b_j \cdot d_{ij} + 1}$, with the vector of parameters, b_j , fixed across the chunks and later determined via an optimization process.

I restricted values of b_j to each be non-negative, in accordance with supposed direction of the 6 contexts affects' on the propensities. The presence of each context should dampen increases in q and amplify increases in p . For example, if an inference of

units coordinating activity was made in which the student was using figurative manipulatives, one would expect no greater an increase in the student's propensity for stability in coordinating units than if no manipulatives were used. Similarly, if a student did not coordinate even units involving the number 'two', then they might be even more likely to continue to not coordinate units that did not involve 'two.'

Note that the modified version of Algorithm 5.1 is equivalent to the baseline formulation if each $b_j = 0$. Recall that m is the minimum of the previous propensity value and its complement. Since m is strictly between 0 and 1, the effect of positive b_j in the algorithm is to further decrease the value of m . The decrease in m results in a smaller increase to q upon successive 1s and a larger increase to p upon successive 0s.

Optimization results

My approach to optimization was to construct 1000 (pseudo) random vectors of b_j values, each between 0 and 3, as seed values in a search for optimal values of b_j . There were 17 chunks associated with instances in which the second-order model of Austin explicitly identified units coordination activity (see Appendix F). Given parameters for u and l , a set of b_j values would be judged an optimal fit if it maximized the number of times (denoted F) that those 17 chunks coincided with q greater than u or p smaller than l . Appendix B gives the code for the function 'usewithfmin' I used to perform the optimization. As its name suggests, the Matlab routine 'fminsearch' was used to step away from the seed value. What I hoped for was convergence toward a particular value or values of b_j regardless of seed value.

What I quickly found is that there was no such convergence, and, furthermore, that the values of b_j were so sensitive to perturbation that `fminsearch` often timed out unsuccessfully. My response was to consider transformations to b_j to reduce the complexity. After initially considering linear transformations to b_j , I reduced the complexity considerably by collapsing d_{ij} to a single variable in i , which was 1 if any of the 6 contexts were inferred and 0 otherwise. This reduced b_j to the one-dimensional b .

Appendix B contains the Matlab code for the function ‘samplemins’ which was used to search for optimal values. After choosing initial values of u and l , I again searched for convergence toward a particular value or values of b . Being confined to a single dimension removed the instability in b , and I searched for robustness in u and l by adjusting those values. Here, I looked for values of u and l that would result in a minimal number of b values that maximize the value of F . I thus looked at the distribution of optimal b values across varying values of u and l . The results, listed in Table 6.1, suggest that $b = 0$, $u = .9$, $l = .45$, meets these criteria.

Table 6.1

Output of samplemins for Austin's Data for Varying Values of u and l Using Collapsed b

u	l	F	# optimal b	mean opt. b	$b = 0$ optimal
.95	.5	12	422	.6966	yes
.9	.5	17	24	.0227	yes
.9	.45	17	14	.0010	yes
.85	.45	17	33	.0425	yes
.8	.45	17	320	.4898	yes
.9	.4	16	19	.0228	yes
.85	.4	17	278	.4512	yes
.85	.35	16	268	.3985	yes

Note: Sample size was 1000

An optimal value of $b = 0$ means that baseline model is as good a fit with the second-order inferences of units coordinating activity as the adjusted model. Figure 6.3 and Figure 6.4 show that for each of the 17 chunks in which inferences of units coordination were made, either values of p are below 0.45 or values of q are above 0.9.

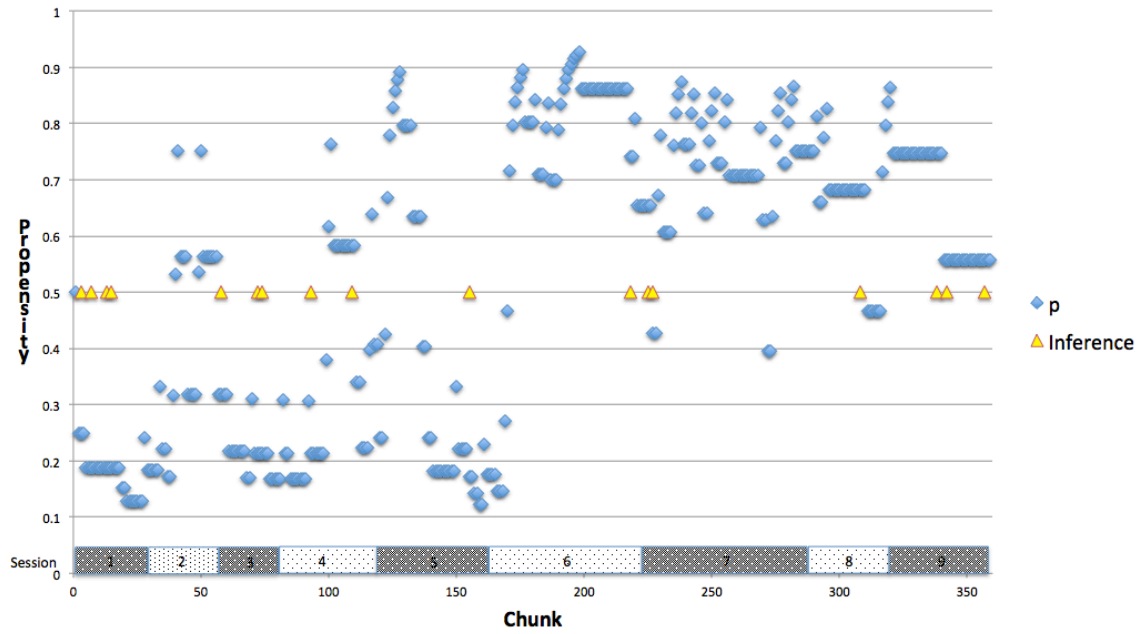


Figure 6.3. Propensity for Austin to Not Coordinate Units

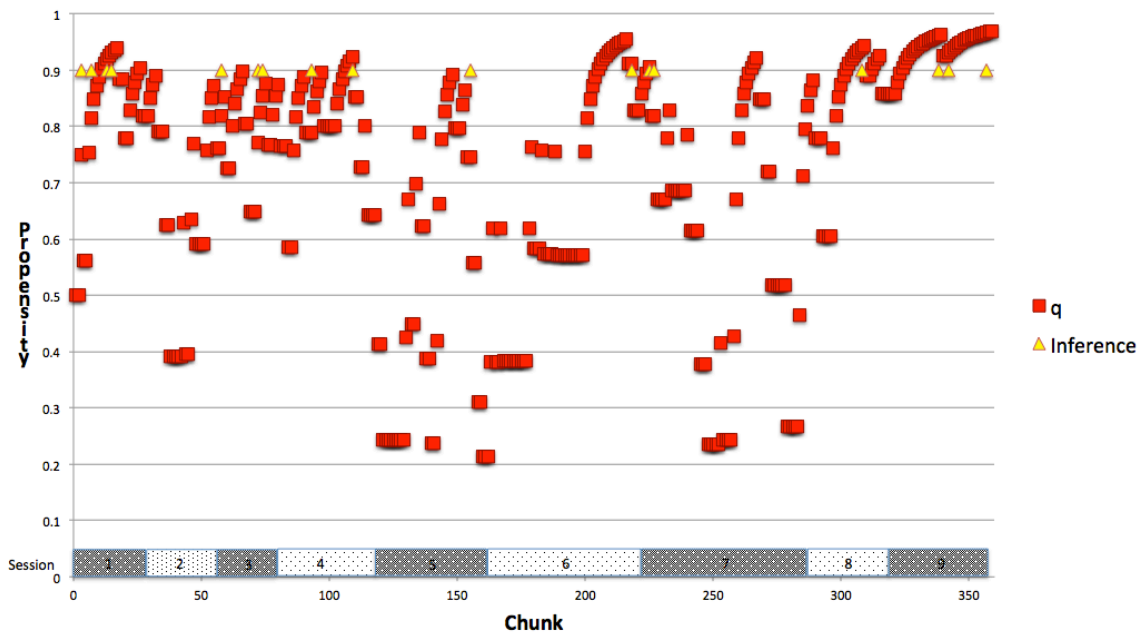


Figure 6.4. Propensity for Austin to Coordinate Units

In further analyses of Austin's units coordination activity, the baseline model was used.

Third-Order Model of Justin's Units Coordinating Activity

Coding results

The 10 teaching sessions were divided into a total of 516 chunks for which either a '0' or '1' was coded for units coordinating activity (see Figure 6.5). After identifying explicit inferences of units coordinating activity in the second-order modeler's description (Ulrich, 2012), I matched the line numbers to the containing or adjacent chunks that had been coded '0' or '1' (hiding the units coordination coding). Open coding of these chunks led to the formation of 4 contextual categories (see Appendix F for initial contextual codes and memos). I coded the entirety of the 516 chunks with these 4 contextual codes; the frequency of each code is displayed in Figure 6.6.

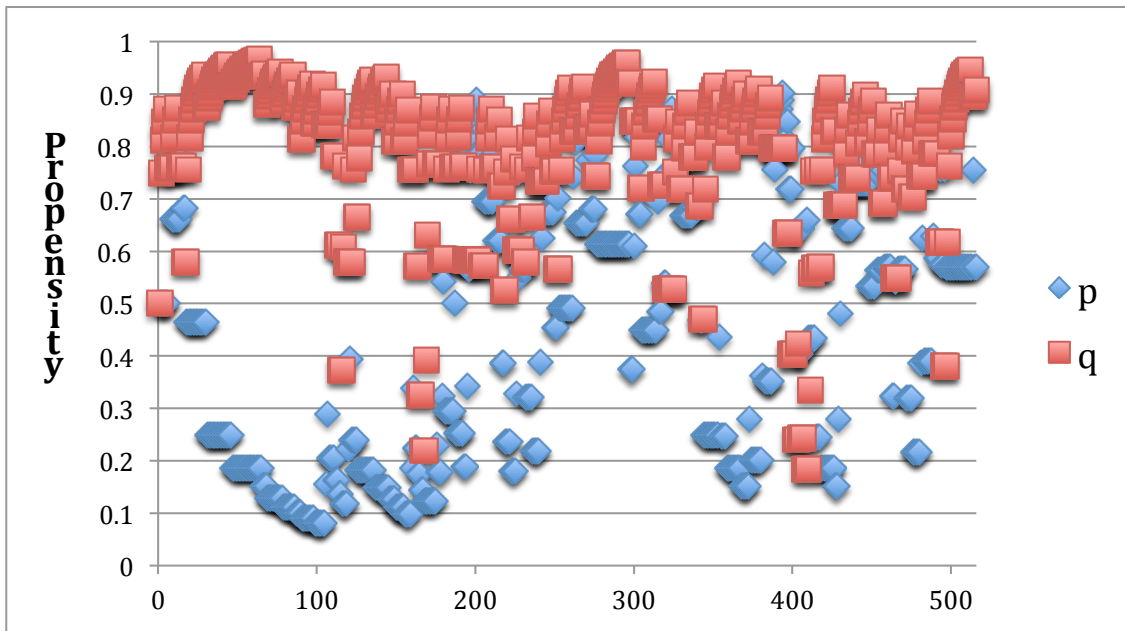


Figure 6.5. Baseline Plot of Justin's Units Coordination Activity

As with the contextual codes arising from the Austin teaching experiment, the presence of each code would seem to affect the propensities in the same direction, if not

by the same magnitude. For each code, its presence strengthens a code of ‘0’ and its absence strengthens a code of ‘1.’ For example, if a chunk was coded as *without* manipulation of figurative material, that would make a coding of units coordinating activity more impressive. If a chunk was coded as “follows teacher request for explanation,” that would strengthen a coding of *not* coordinating units.

Code	Count (516 chunks total)
Follows teacher request for explanation	126
Follows manipulation of figurative material	81
Follows partner response	60
Follows reflection on reflection	39

Figure 6.6. Contextual Codes from Justin’s Data

Optimization results

I pre-emptively collapsed the four variables into a single code that was 1 if any of the contexts were present and 0 otherwise (see Appendix E) in anticipation of using the approach and functions that arose with the Austin data (see Appendix B). This time there were a total of 11 instances of units coordination identified in the second-order model description. The results, listed in Table 6.2, suggest that there is not convergence to a single best value of u , l , and b .

Examination of Figures 6.7 and 6.8 reveals few explicit inferences of units coordinating activity, despite longer teaching sessions and more chunks, and that the units coordination inferences are not distributed throughout the teaching experiment uniformly. I will go into more detail about the issues of improving the model fit in the

Chapter 8, but for now I conclude that I cannot improve upon the baseline model for Justin.

Table 6.2

Output of samplemins for Justin's Data for Varying Values of u and l Using Collapsed b

u	l	F	# optimal b	mean opt. b	$b = 0$ optimal
.75	.35	11	209	.3170	yes
.8	.30	9	66	.0812	yes
.8	.35	10	60	.0836	yes
.85	.35	7	113	.1727	yes
.8	.4	10	61	.0842	yes
.85	.4	7	122	.3259	yes
.8	.45	10	57	.0881	yes
.85	.45	7	183	1.0897	yes
.8	.5	11	136	.1718	yes
.85	.5	10	810	1.7385	no
.9	.5	10	369	1.0592	no
.95	.5	10	366	1.0803	no

Note: Sample size was 1000

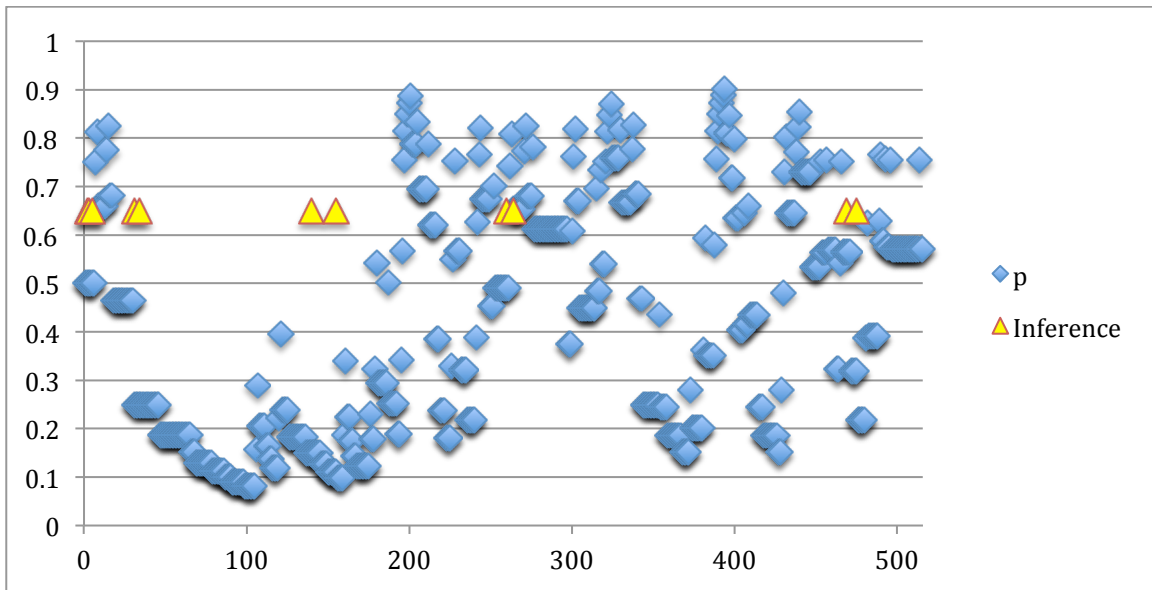


Figure 6.7. Propensity for Justin to Not Coordinate Units

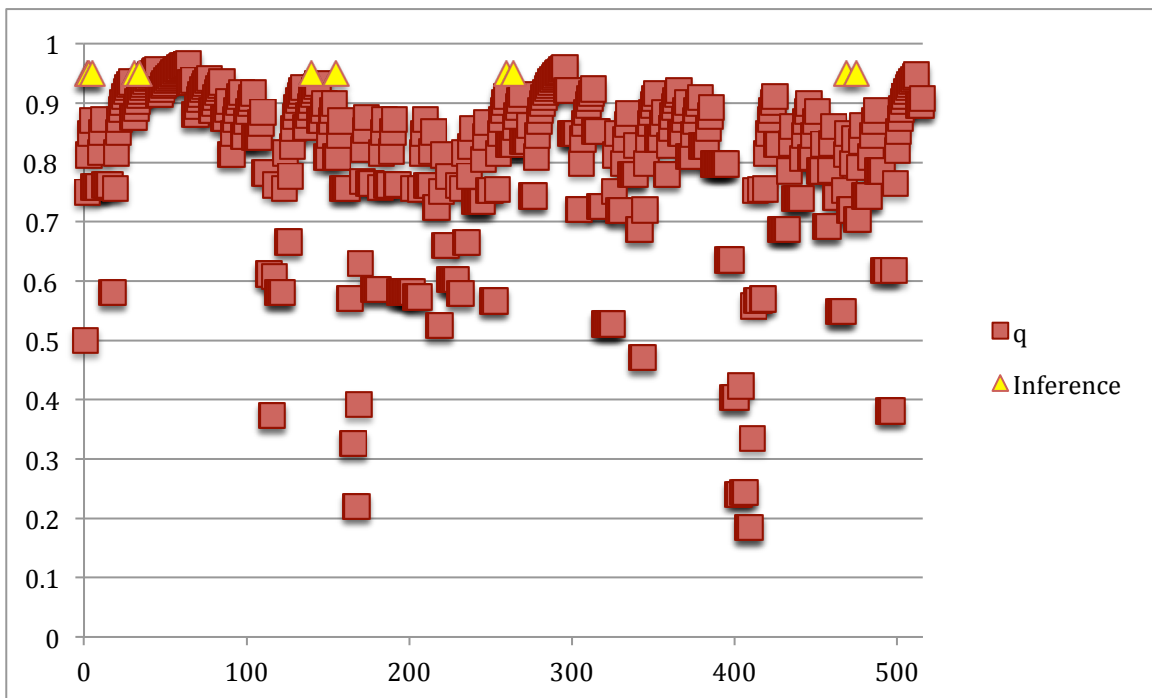


Figure 6.8. Propensity for Justin to Coordinate Units

Analyses of Austin's and Justin's Units Coordinating Activity

Characterizing sessions

To form testable conjectures pertaining to the plot of units coordinating activity, I needed to construct a way to objectively characterize the salient aspects of the plots. These were described qualitatively in my analysis of the plot of Charles' units coordinating activity (see Chapter 4). As with the qualitative interpretation, I sought three gradients – sessions most indicative of establishing of a new scheme, sessions most indicative of perturbation, and the rest. Using the output of Algorithm 5.2, I characterized each session of the Austin and Justin teaching sessions as Substantial Perturbation (SP), if the value of S was less than 0; as Equilibrium (E), if the value of S was greater than 1, and as Moderate Perturbation (MP) if the value of S was between 0 and 1 (inclusive).

Though the S scale provided a way for distinguishing and identifying students' level of perturbation associated with the establishment of a new scheme, other considerations of the types of interactions that might result in similar values of S required further qualitative analysis. Having used the scale score to distinguish the level of perturbation, I returned to the transcription and coding to broadly characterize the dominant aspect of each session and its mathematics domains. My goal was to construct a minimal set of codes to further characterize *why* the level of perturbation was low, medium, or high.

Equilibrium sessions can be described in three ways. There is "Norming," in which the focus is on establishing social norms or expectations for reasoning, interaction and communication. There is "Simplifying," in which the teacher-researcher introduces

tasks that are simpler to reduce the demand on the student after the student experiences prolonged difficulty in a task setting. Finally, there is “Establishing” in which the student operates in a more stable way in a context for which a stable way operating was not previously inferred.

Substantial perturbation can be described in two ways. There is “Demanding,” in which the focus is on tasks for which the student has not constructed the requisite levels of units. Second, there is “Conflicting,” which includes sessions that maintain focus on a single mathematical context for which conflicting activity schemes have been constructed.

Moderate perturbation can be characterized in three ways. First, there is “Interviewing,” in which the teacher-researcher introduces a sequence of seemingly disconnected task contexts, some of which perturb the student. Second, there is “Guiding,” in which a sequence of sub-tasks are presented by a teacher-researcher after presentation of a more difficult task. Third, there is “Moderation,” which is characterized by randomness included in task difficulty, e.g., via games or when tasks are generated by an individual other than the teacher, such as the witness or other student.

Analysis of Austin’s units coordinating activity

Table 6.3 displays the results of the method for characterizing the changes in p and q in the teaching experiment involving Austin. I next describe the contexts of the sessions that accompany the changes in units coordination activity.

Table 6.3

Characterization of the Austin Sessions

Session	Scale Score	Characterization	Type	Task Context(s)
1	2.89	E	Norming	Fractions
2	-0.19	SP	Conflicting	Fractions
3	3.99	E	Establishing	Fractions
4	0.63	MP	Moderating	Fractions
5	0.20	MP	Moderating	Fractions
6	-0.42	SP	Demanding	Fractions
7	-1.20	SP	Demanding	Fractions
8	0.88	MP	Guiding	Fractions
9	10.39	E	Establishing	Whole numbers, fractions

Session 1: Equilibrium

The first session is characterized by high values of q and low values of p . This indicates a lack of perturbation and suggests that the initial activities in which Austin was engaged were not challenging for him. The contexts of these interactions served to introduce norms for the teaching experiment to Austin and Jane, and they were mostly with the first level of CandyFactory. I describe this type of Equilibrium session as “Norming.”

Session 2: Substantial Perturbation

The second session, is characterized by substantial perturbation, with volatility in both p and q . Both p and q drop at the beginning of the segment, and the catalyst for the volatility was the teacher-researcher's questioning of Jane and Austin about what the whole was in Level 1 of CandyFactory. Austin inconsistently conflated the 'whole candy' with either the totality of the candy in the game or the unit fraction. I describe this type of SP session as "Conflicting." The units coordination demands were not too great for Austin – the root of the perturbation was that it was brought to his awareness that he had multiple ways of thinking about fractions that were incompatible. Austin would at times use rule-based strategies that were consistent with coordination of units, without realizing when his other actions contradicted these rules.

Session 3: Equilibrium

The third session consists of stable levels of both p and q . During this Equilibrium session, there was a switch to a new figural context. The students worked with physical manipulatives (Twizzlers). Austin did not completely resolve the issue of naming the whole– he at times "forgot" to name a fraction or to name a whole, but he did not conflate units and was otherwise successful at tasks posed by the teacher-researcher and Jane. I label this session as "Establishing" because Austin seemed to be reconciling his two ways of thinking about fractions, his PUFS and PWS.

Sessions 4 and 5: Moderate Perturbation

The fourth session was marked by moderate perturbation. The session began with the teacher-researcher presenting a rectangle drawn on a piece of paper, partitioned into 5

parts with one shaded. While this task was problematic for Jane, who was unsure whether to name it as one-fourth or one-fifth, it was immediately assimilated within a part-whole scheme by Austin. Austin was perturbed when the teacher-researcher asked what else it might be if not one-fifth. Austin suggested he was trying to think of “one-half of one-fifth” and suggested it might be “three-sevenths.” The teacher-researcher later brought the focus back to the part-whole scheme for the sake of Jane’s understanding, and Austin was successful in coordinating the units. Subsequently, Austin’s classroom teacher was present and interacted considerably as Austin attempted to apply knowledge of rules and procedures for comparing two fractions or producing equivalent fractions.

Both the fourth and fifth sessions were characterized as “Moderation,” as interactions outside of the teacher-researchers’ control moderated the level of perturbation. In the fourth session, the classroom teacher’s interactions increased the level of perturbation. The fifth session began with an RPFS task, to find the whole given $\frac{3}{7}$, which was difficult for Austin. After Austin initially struggled, his classroom teacher interjected by asking Austin how many one-sevenths would be in three-sevenths and how many one-sevenths would be in a whole. The classroom teacher thus lowered the level of perturbation by guiding him to use a part-whole interpretation. The teacher-researcher then asked an easier, PUFS task, to form the whole given $\frac{1}{5}$, which was easier for Austin. The students then chose tasks for each other, and Jane’s task, a request to make a whole given $\frac{2}{4}$, increased the level of demand.

Sessions 6 and 7: Substantial Perturbation

The next two sessions were each characterized by Substantial Perturbation, and I describe both as “Demanding” sessions. In the sixth session, the teacher-researcher

introduced another RPFS task, to find a whole when given a green rod labeled as $\frac{3}{4}$. Austin compared the *complement* of the green rod with the other available Cuisenaire rods to correctly choose the brown rod as the whole. Meanwhile, Jane created a figural representation for $\frac{3}{4}$, aligning 3 reds below the green and 4 reds below the brown to indicate the totality was the fraction. Austin changed his mind about his result, demonstrating uncertainty as he claimed that the red “was the right piece.” Austin was most likely acutely aware that he was perturbed – he often changed his mind and expressed doubt about whether his initial strategy and responses to tasks were appropriate. Throughout the sixth session, Austin was faced with RPFS tasks introduced by the teacher-researcher and by Jane. Unlike in the sixth session, easier (PUFS) tasks were not introduced, and the classroom teacher’s interjections did not moderate the perturbation.

In the seventh session, Jane was absent, and Austin was initially given a commensurate fraction task that was very demanding. He was asked what fraction two tan rods would be of a green rod. After initially naming it as 3 sixths, and eventually two-sixths, Austin said another name could be 4 sixths, or fourths. Subsequent tasks continued to be very demanding for Austin – he seemed tired during the session. When faced with tasks requiring him to create the whole given a figural composite unit fraction, he was more consistently using the complement strategy, and in the process, conflating units. After his mistakes were made apparent, he was faced with the task of finding figural material to represent multiples of a given figural unit fraction, in sequence, e.g., to find $\frac{2}{5}$, then $\frac{3}{5}$, $\frac{4}{5}$, etc. given a piece named as $\frac{1}{5}$. However, a new task of finding $\frac{2}{3}$ given $\frac{1}{3}$ resulted in prolonged sequences of not coordinating units again.

Later, Austin is faced with Customer Order of 9/10 in *CandyFactory*, and he initially attempts 3/11 and later corrects to 3/9 before the classroom teacher interjects with whole number tasks. At the end of the session, Austin seemed fatigued, evidenced by statements like, “I’m pretty sure I’m wrong,” or “Isn’t the denominator supposed to go first?”

Sessions 8: Moderate Perturbation

In the following session, Austin seemed to gain in confidence. I characterize this session as “Guiding.” It begins with the teacher-researcher showing Austin and Jane a clip of them arguing over whether two-fifths of an orange rod should be the yellow rod (since two yellows make an orange, and 5 reds fit into an orange rod) or the purple rod (because 2 reds fit into a purple, and 5 reds fit into a purple). Austin attempts to form the sequence of units $1/5$, $2/5$, $3/5$, $4/5$, $5/5$, but he only attends to the increasing values of the lengths, not the amount of the change. After iterating the red blocks above the purple and orange, he returns to thinking that the purple is $2/5$.

The teacher-researcher goes through sequences of rods named as $1/5$ through $5/5$, then a sequence of rods similarly named as ninths, and then a sequence of rods similarly named as fourths. Austin is successful at naming them, and even correctly names the orange rod as 1 and 1 ninth after initially naming it as 1 and 1 tenth. Austin subsequently adopts the activity of forming a sequence of $1/5$, $2/5$, ... $5/5$ when given the task of finding $1/5$ when given a size named $3/5$, but he treats the given $3/5$ piece as $1/5$, and appends the smallest rod (the tan) to form the larger fractions instead of beginning by partitioning the $3/5$ into 3 parts.

Session 9: Equilibrium

The final session began with a sequence of tasks involving whole number units coordination, in which Austin was very successful with tasks like finding the number of days in three weeks or the amount of nickels needed to make 65 cents. Immediately following these whole number tasks, Austin was given the task of finding a composite fraction, given a unit fraction. In contrast to earlier observations, he was confident in his correct response, despite its inconsistency with Jane's response. Moreover, his activity in *CandyFactory* suggested consistent adjustment in his estimates of customer orders, for example, to correctly attempt $\frac{8}{9}$ when he found that $\frac{7}{8}$ was too small. This session was labeled as "Establishing" as Austin seemed to be establishing a PFS.

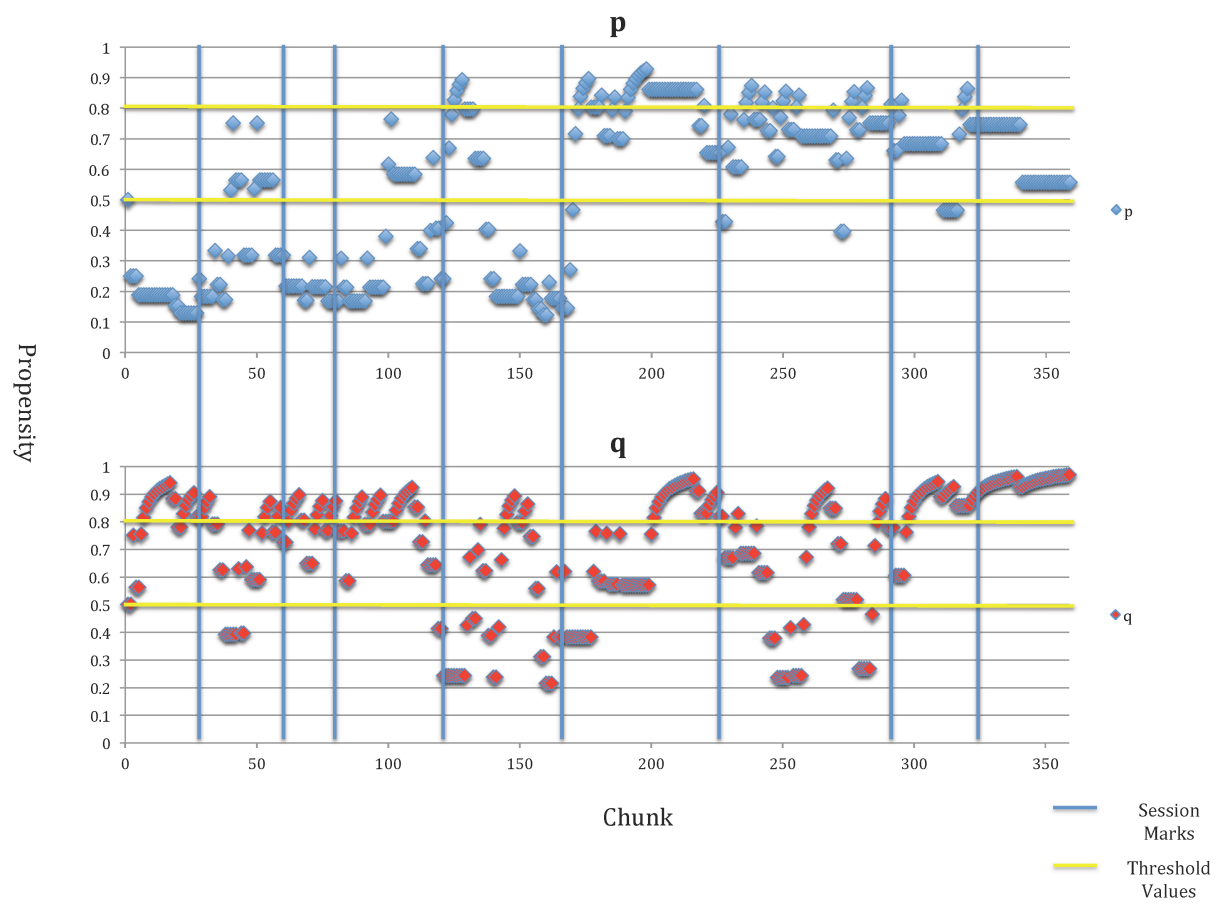


Figure 6.9. Segmenting Austin's Units Coordination Activity

Analysis of Justin's units coordinating activity

Table 6.4 displays the results of the method for characterizing the changes in p and q in the teaching experiment involving Justin. I next describe the contexts of the sessions that accompany the changes in units coordination activity.

Table 6.4

Characterization of the Justin Sessions

Session	Scale Score	Characterization	Type	Task Context(s)
1	0.81	MP	Interviewing	Fractions, whole numbers
2	12.07	E	Norming	Fractions
3	0.88	MP	Guiding	Fractions, whole numbers
4	0.67	MP	Guiding	Fractions
5	0.59	MP	Guiding	Fractions
6	1.30	E	Simplifying	Directed differences: fractions, whole numbers
7	2.86	E	Establishing	Whole numbers, fractions
8	0.58	MP	Guiding	Fractions
9	-0.30	SP	Demanding	Directed differences: whole numbers
10	0.69	MP	Moderating	Directed differences: whole numbers

Session 1: Moderate Perturbation

The first segment of the teaching experiment began with a clinical interview, which included tasks from a variety of contexts, including both fractions and whole numbers. The values of both p and q each rise, which suggest instances of the researcher trying to learn the limits of Justin's mathematics without inducing substantial perturbation. This type of MP is characterized as "Interviewing."

Session 2: Equilibrium

The values of q then continue to rise, while the values of p fall, as the paired teaching experiment begins. This corresponds with the teacher-researcher engaging Justin in fractions tasks in the continuous context – making fractions with strips of paper. This type of session is characterized by "Norming," as the teacher-researcher focuses on establishing norms for communication, and primarily introduces tasks in which both students can experience success.

Sessions 3, 4, and 5: Moderate Perturbation

The third, fourth, and fifth sessions are each characterized as Moderate Perturbation. In the third session, the students begin in a whole number multiplicative context, with tasks requiring coordination of three levels of units, in which both students are successful. The teacher-researcher transitions to fractions tasks requiring coordination of three levels of units, and makes suggestions about sequences of actions the students might do to construct an improper fraction with figural material. For example, when asked to create $9/7$, Justin suggests he would "add two more" to another bar. The teacher-researcher suggests and praises intermediate activities, saying "Maybe you could both divide [a strip of paper] into 7 first? [After they each create a left-most

partition] Yay! You [inaudible] both have great guesses.” I describe this type of Moderate Perturbation as “Guiding.”

The fourth MP session was also characterized as Guiding. In the fourth session, the teacher-researcher begins with a RPFS task, to create a whole given $\frac{4}{9}$. After the students experience success with this, a reversible IFS task is introduced: to create the whole given an amount labeled as $\frac{7}{5}$. When this is difficult, the task is modified: the given amount is labeled as $\frac{4}{3}$ – ostensibly because of a lack of paper materials. Later, a task of producing $\frac{7}{6}$ from a whole partitioned into fifths is introduced. When the students have difficulty, the task is temporarily changed: to divide the fifths into tenths.

I also characterized the fifth session as MP (Guiding). The session began with an easier task – to produce commensurate fractions as multiples of 2. When faced with a task of marking thirds from a whole already partitioned into fourths, Justin formed a goal of making 15 parts – because then 5 of them would be one-third. He is copying Lilly’s activity of partitioning each of the four partitions into three. Instead of correcting Justin’s goal – to form 12 instead of 15 partitions – the teacher stopped him when he finished counting 12 pieces and asked him if he could use that instead. Later, when working with twentieths, Justin named 5 pieces as ‘1 fifth.’ Instead of correcting him immediately, the teacher-researcher stopped him when he counted ‘12’ pieces and asked him if he could use that instead. While the teacher’s interjections serve to guide and encourage Justin’s reflections on the results of his activities throughout the teaching experiment, Guiding sessions are further marked by the introduction of easier sub-tasks in service of a task without abandoning the original task.

Session 6: Equilibrium

Session 6 was labeled as Equilibrium – Simplifying. As with the first Equilibrium session, a primary goal of the session was the introduction of new norms. In this case, a new game was introduced, in which students had to coordinate cumulative differences and weekly differences in a hypothetical weight chart involving both whole numbers and fractions. Justin insists that the increase between 135 and $2/5$ and 148 is 13 and 3 fifths, as he does not coordinate 135 and $2/5$ as a single value. He views two separate calculations, one for finding the complement of 135 (within 148), and one for finding the complement of $2/5$ (which is implicitly within 1). The teacher-researcher poses a new, reverse task, of starting with 135 and $2/5$ and moving to 148. After sequentially adding, $3/5$, 4, and finally 8, resulting in an agreement with Lily's response, Justin seemed to resolve the earlier mistake. While both Lily and Justin made mistakes in both the mixed number setting and the comparison of differences setting, Lily's were more easily rectified. Note each involves coordination of two different *types* of units – whole number and fractional in the first setting, and quantities and differences between quantities in the second setting.

After Justin expressed difficulty with coordinating the differences between mixed numbers and whole numbers, the teacher-researcher changed the context to new games, involving only whole number differences. The session is termed "Simplifying" because new tasks were introduced to maintain equilibrium, in contrast with the introduction of sub-tasks that accompanied "Guiding" MP sessions and the introduction of norms accompanying early E sessions.

Situations in which Justin was tasked with reflecting on the difference between two values of a sequence of positive integers were not problematic. But when he had to compare two differences, i.e., to compare quantities of differences in other quantities, he conflated the units. At the end of the segment, the earlier issue with the mixed numbers resurfaced when the focus returned to similar tasks.

Session 7: Equilibrium

Session 7 was also labeled as Equilibrium, but it was an “Establishing” session. After beginning with multiple whole number units coordinating tasks involving combinatorics, the teacher-researcher returned to a fractions task that had previously been difficult for the students: making $\frac{7}{4}$ from a bar that was partitioned into fifths. After the difficulty persisted, the teacher-researcher introduced an easier task of producing $\frac{1}{4}$ instead. Justin was able to mark the one-fourth by recursively partitioning each of the fifths to form twentieths, without prompting from the teacher. The ease of Justin’s activity suggest that he was establishing a recursive partitioning scheme – this was the first time Justin recursively partitioned without prompting when the partitioning was not halving.

Session 8: Moderate Perturbation

Following the success of Session 7, the task context switched back to fractions in Session 8. In general, the session was characterized as MP, and further classified as Guiding. The initial goal, carried over from the previous session, was for Justin to create 5 fourths from a bar that had been partitioned into 15. The task remained markedly more difficult than the task of finding one-fourth in the previous session, and the teacher-researcher interjected with requests for Justin to reflect on the results of his activities.

Though Justin was able to eventually create the correct strip, his difficulty with creating commensurate (proper) fractions continued during a game that was introduced later in the session. Lilly and Justin each drew a card depicting a fraction as a shaded part of a pre-partitioned bar and the goal was to find the difference in the amounts depicted on the card. When Justin was tasked with partitioning a bar marked into fifths so that thirds would show up, he repeatedly resisted recursively partitioning to form fifteenths. The teacher asked how many pieces there would be if Justin cut each of the pieces in half. Justin realized that would result in 10 pieces. The teacher-researcher suggested that wouldn't work to cut out 3 pieces out of 10 equally, and asked if it might work if he cut each of the 5 pieces into 3. In this manner, the teacher "ask[ed] some questions to see if it would help [Justin] figure it out faster."

Session 9: Substantial Perturbation

The next session is also characterized by substantial perturbation. Justin, now paired with a different student, Michelle, is tasked with coordinating two whole number 'scores,' which are obtained from drawing cards, the directed difference in those scores, and an accumulated directional difference in the scores. Finding the directed differences was not problematic for Justin, generally, as he displayed his ability to flexibly coordinate between 10s and 1s even when encouraged to not use a subtraction algorithm. The accumulated directional differences - the notion of a sum of signed differences - was the focus of the teacher-researcher interventions, as Justin sometimes conflated it with the current difference and other times treated it as two separate (unsigned) quantities. After Justin displayed conflation of units and inconsistency in coordinating the two types of quantities, the teacher-researchers' interventions did not

reduce the perturbation as it had with the MP sessions. I describe it as “Demanding” because of the level of units coordination demanded by the tasks.

Session 10: Moderate Perturbation

The final session analyzed is also characterized by moderate perturbation. One of the differences between it and the previous session was that the teacher-researcher focused more on Justin’s partner, Michelle, who was not able to flexibly work with 10s and 1s to strategically find the differences and was more constrained to working with a subtraction algorithm. The game continued, but the focus was on supporting the students’ reflection on their ways of finding the differences mentally, more so than the quantity of accumulated differences. In contrast to Session 9, Justin did not conflate accumulated differences and current differences in this session. The difficulty that had been so demanding in the previous session was absent, as the teacher attended to the process, rather than result, of Justin’s finding of differences. The witness helped to moderate (increase) the perturbation by requesting Justin to focus on inconsistencies in the results of his activities. I thus characterize this type of MP as “Moderating.”

It seems that Justin had separated the tasks of finding the differences and directed differences and was no longer trying to coordinate the involved quantities as the focus moved to the whole number units coordination. The perturbation in the session involved whole number units coordination. Justin explains that he could go down 3 from 45 to get to 42, and then up 20 to get to 62. Instead of reversing the 3, he added its complement (7) in 10. This is markedly similar to his way of operating with the mixed numbers.

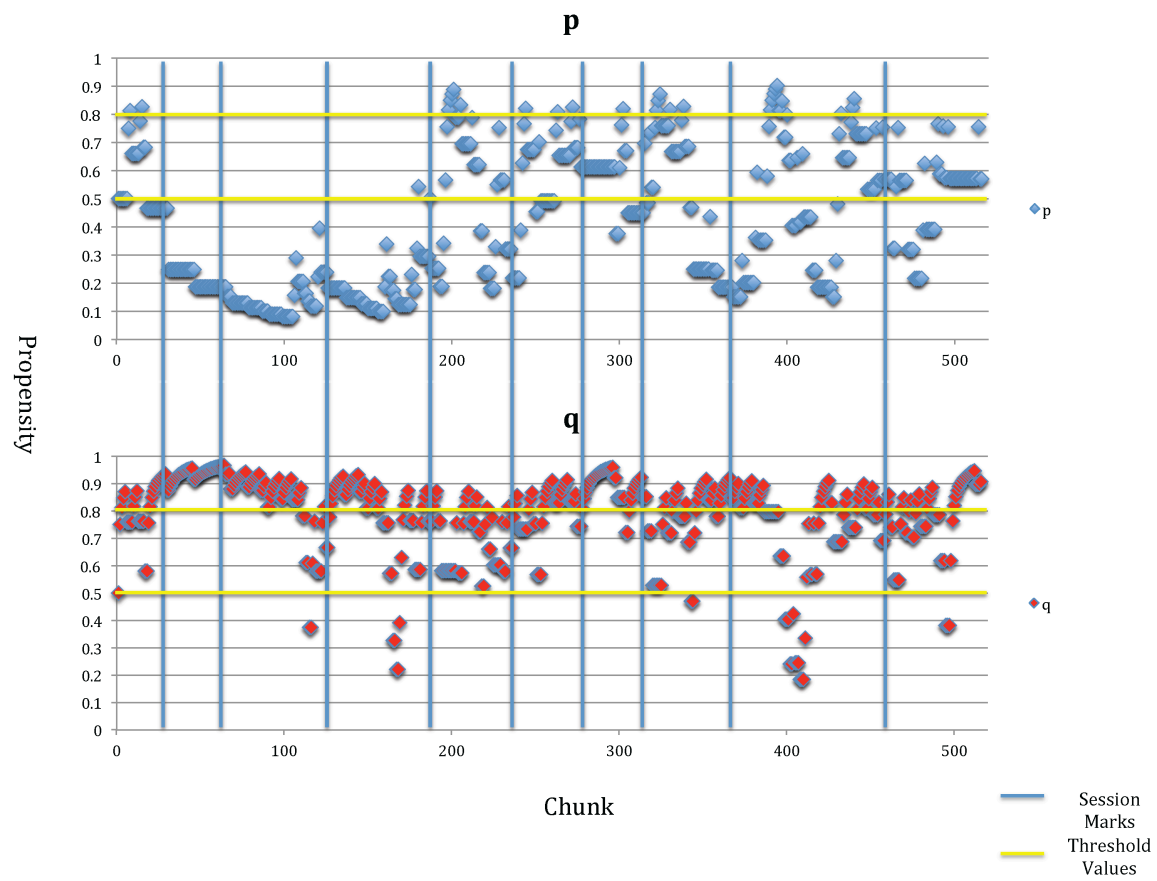


Figure 6.10. Segmenting Justin's Units Coordination Activity

Third-Order Conjectures

For my modeling approach to be useful, it must permit the formation and testing of conjectures that are of interest to the mathematics education research community. I anticipate that characterization of perturbation will be useful for planning and exploring activities across sessions. For example, the third-order model could be used to decide how to introduce and approach new or more difficult tasks. For instance, if sessions characterized by Equilibrium (Establishing) do not follow Substantial Perturbation, a teacher-researcher might expect that seemingly "quick" growth following substantial perturbation is more likely to be capitulation than construction.

Characterization of perturbation might also be useful for reflecting on and exploring task sequencing within sessions. Rather than going into a teaching session with a general idea of a task, scheme, or context, one could plan activity sequences that maintain a high (or low) level of perturbation. For instance, the teacher-researcher might plan to Moderate Perturbation by introducing randomness in the task sequencing.

I believe the model provides an expectation for the variability that transcends student, teacher-researcher, and mathematical context. The approach could be used to compare and contrast the learning of students of all ages, or the approaches of different teacher-researchers. However, the types of conjectures I make now are limited to those that can potentially be falsified via the data obtained from the teaching experiment with Dylan.

The analysis of the third-order models of Austin and Justin's units coordinating activity suggests several conjectures about patterns of interactions that can be tested with the model of Dylan's unit coordinating activity. My knowledge of the content of Dylan's teaching experiment played a large role in the level of detail in the conjectures I chose to

focus on and test. For example, because I knew that Dylan did not use *CandyFactory*, I did not attempt to form conjectures about Austin’s use of that particular app. Because I knew that Dylan did not engage in activities involving directed differences, I did not attempt to form conjectures about Dylan’s units coordination with integers. On the other hand, I only formed conjectures that arose from the modeling results with Justin and Austin. Thus, I limited my characterizations of sessions to those depicted in Figure 6.11.

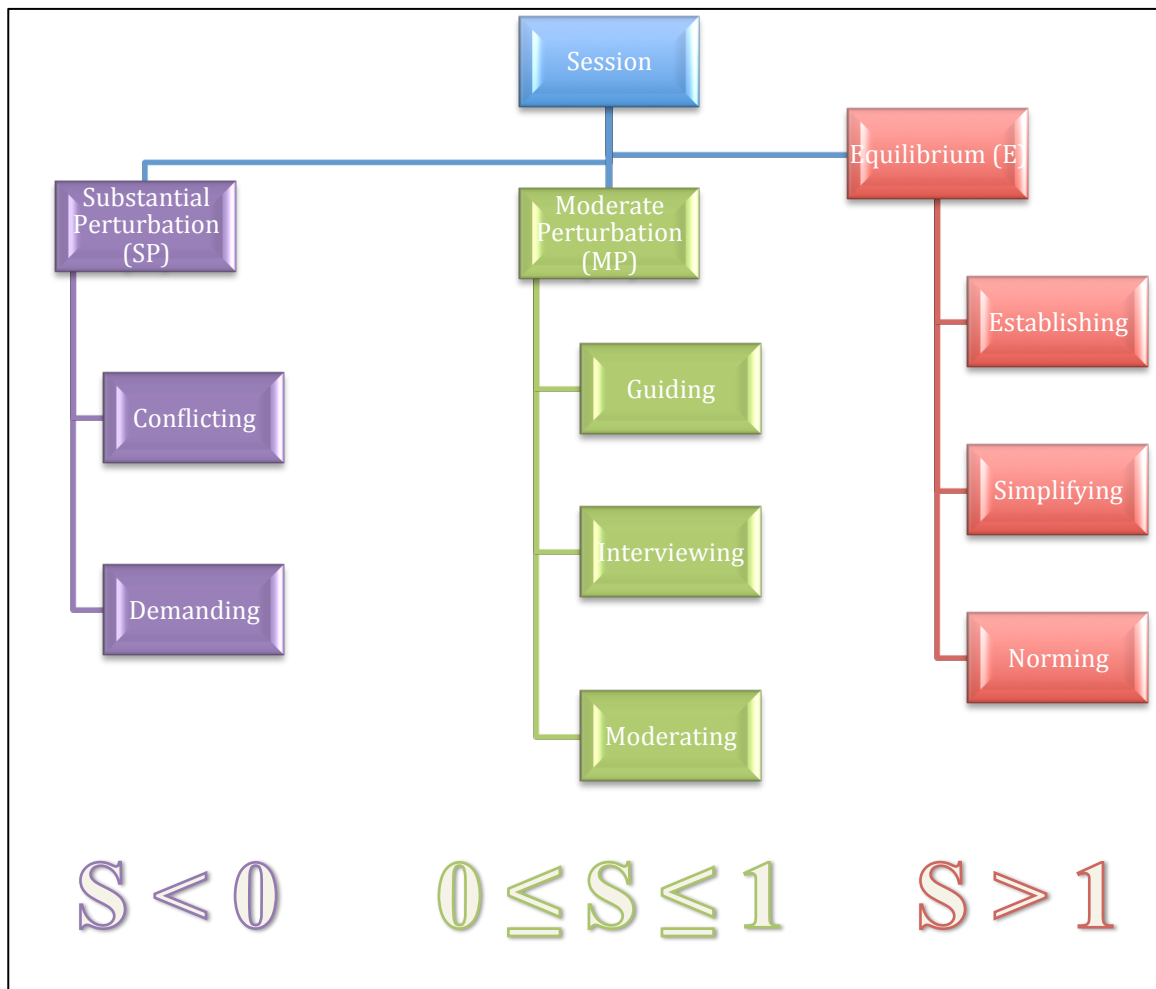


Figure 6.11. Types of Teaching Sessions

Table 6.5

Descriptions of Teaching Session Types

Characterization	Type	Description
E	Establishing	Emergence of scheme or operation
E	Norming	Communication, Interaction Focus
E	Simplifying	Abandoning of harder task
MP	Guiding	Sequential introduction of sub-tasks
MP	Interviewing	Short bursts of task engagement
MP	Moderating	Pseudo-randomness in difficulty
SP	Conflicting	Competing ways of operating
SP	Demanding	Sustained task difficulty

I use the term conjectures, rather than hypotheses, because the teaching experiment and model-building with Dylan was not designed to test them specifically. Finding a lack of support for each conjecture without falsifying it is certainly possible. It is likely that the antecedent will not be true throughout the Dylan teaching experiment for at least one of the following conjectures. Each of the conjectures holds for either the Austin or Justin data, and neither the Justin or Austin data contradicts any of the conjectures. Part of each conjecture is not its “truth,” but its believeability, with and without reliance on the third-order model, and its usefulness. As my approach is partly judged by its utility for predicting and explaining phenomena transcending particular students or researchers, I offer potential implications for the veracity of the conjectures as

well, with the understanding that they would require further research to test generalizability.

My first conjecture is: If an E session directly follows an SP session, then the E session is Establishing and the SP is Conflicting. This pattern will be marked by inversion in p and q , as values of p decrease during the E session while values of q increase during the E session. If this conjecture holds, it would suggest that a teacher inferring Demanding SP in one session should be skeptical of initially inferring E (Equilibrium) directly afterward.

I conjecture that the number of MP sessions will be greater than or equal to the number of SP sessions and greater than or equal to the number of E sessions. I also predict that consecutive SP (Demanding) sessions will not happen if the student's partner is not absent for at least one of the sessions. If these second and third conjectures hold, it would be consistent with the notion that researchers try to avoid a substantial level of perturbation and maintain a moderate level of perturbation throughout the majority of a teaching experiment. If either of these conjectures does not hold, fidelity to the teaching experiment methodology (Steffe & Thompson, 2000) should be questioned.

My fourth conjecture pertains to the utility of MP (Guiding) sessions, in which a teacher-researcher directs a sequence of sub-tasks in service of a goal. I think that if an E (Establishing) session follows an MP (Guiding) session, what becomes established is a sub-activity, not the entire task sequence, and that this activity can be traced back to an earlier session. If MP activities and contexts are identified in which this is found to be true, it could help inform the bounds of the ZPC and ZPD (Norton & D'Ambrosio, 2008) for those contexts. For particular operations or schemes, first introducing an activity,

then introducing a larger activity containing the smaller as a sub-task, and then guiding the student through the larger activity (encouraging reflection between activity sequences) could help the student interiorize the (smaller) activity (von Glasersfeld, 1995).

My fifth conjecture pertains to Establishing sessions. I predict that E (Establishing) sessions begin with the “easier” of numerical contexts requiring the same number of levels of units coordination (Hackenberg & Lee, under review). Values of p remain relatively constant, while values of q rise during the session. If this conjecture holds, it would suggest that a way to help students learn fractions, or integers, etc., is to engage students in activities with whole numbers directly prior to activities with the other context. Figure 6.12 summarizes these 5 conjectures.

Conjecture	Description
1.	If an E session directly follows an SP session, then the E session is Establishing and the SP is Conflicting.
2.	The number of MP sessions will be greater than or equal to the number of SP sessions and greater than or equal to the number of E sessions.
3.	Consecutive SP (Demanding) sessions will not occur unless the student's partner is absent for at least one of the sessions.
4.	If an E (Establishing) session follows an MP (Guiding) session, what becomes established is a sub-activity, not the entire task sequence, and this activity can be traced back to an earlier session.
5.	E (Establishing) sessions begin with the "easier" of numerical contexts requiring the same number of levels of units coordination.

Figure 6.12. Third-order Conjectures

CHAPTER 7: MODEL TESTING RESULTS

In this chapter, I provide the results of testing my approach, including an evaluation of the modeling efforts' utility and an evaluation of the veracity of the previously formed hypotheses. As described in Chapter 5, there were two types of testing. First, I tested the validity of the interpretation of the output of a baseline algorithm (Algorithm 5.1) with the Dylan data, by evaluating how the p and q values matched with the second-order modeler's inferences. Second, I tested the conjectures outlined at the end of Chapter 6 to evaluate the utility of the approach for describing, explaining, and predicting Dylan's units coordinating activity. Lastly, I interpret the results for answering the three research questions.

Testing Agreement With Second-order Model of Dylan

I identified 17 explicit inferences of units coordinating activity in the second-order modeler's descriptions of Dylan's activity (see Appendix G). As depicted in Figure 7.1 and Figure 7.2, the inferences of units coordination were spread throughout the 690 chunks that were marked as 1 or 0 in the teaching experiment. I considered whether each of the associated chunks corresponded with q greater than u or p less than l for varying values of u and l . Table 7.1 lists the values of p and q for these 17 chunks, as computed using the baseline algorithm (Algorithm 5.1).

Since p is the propensity for stability in *not* coordinating units, and q is the propensity for stability in coordinating units, high values of q and low values of p are consistent with a second-order inference of units coordination. Both are not necessary –

for instance, one would expect the value of p could remain quite high if the student repeatedly coordinated units in a new context.

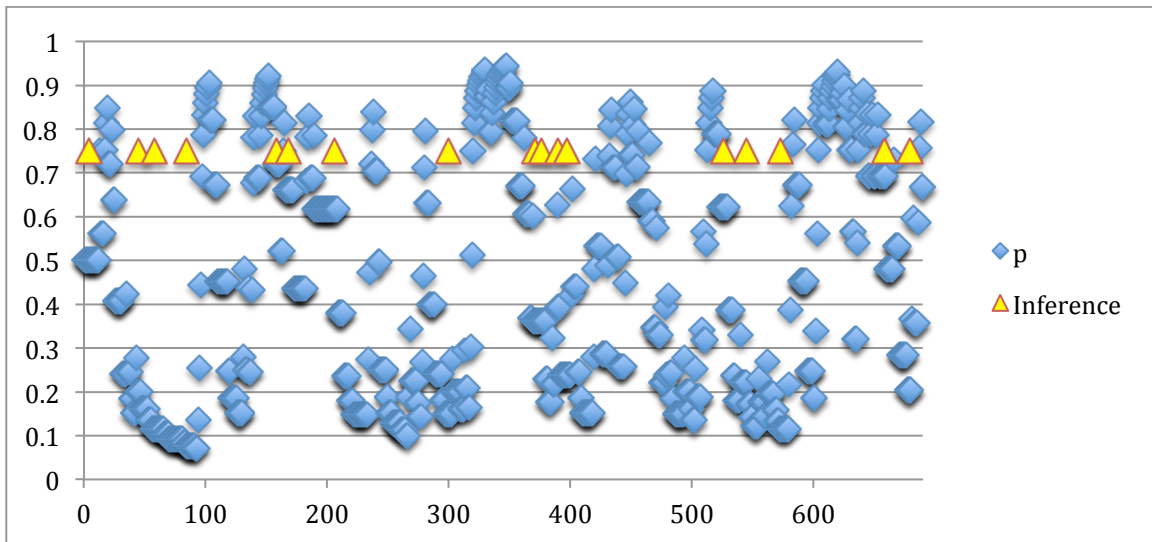


Figure 7.1. Propensity for Dylan to Not Coordinate Units

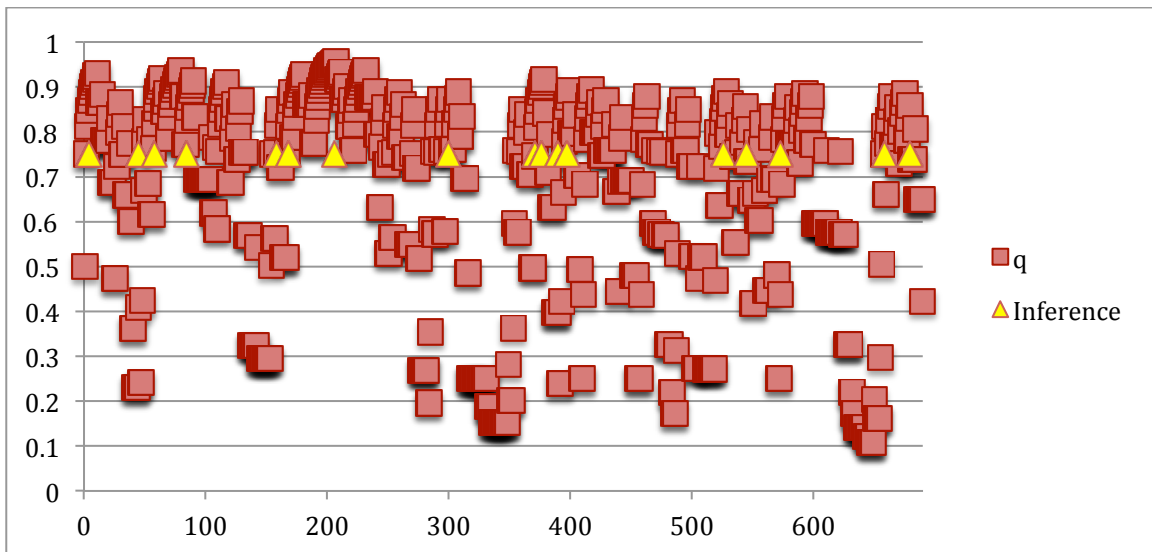


Figure 7.2. Propensity for Dylan to Coordinate Units

Qualitative analyses of Figure 7.1 and Figure 7.2 suggest a great deal of variability in Dylan's units coordinating activity over the course of the teaching experiment. In particular, patterns in the values of p or q , such as minima or maxima of an interpolated polynomial, do not appear to distinguish the inferences of units coordination by the second-order modeler. This is problematic, as it suggests that the context-free third-order model might not agree with the second-order model inferences. However, the lack of identification of a second-order inference is not necessarily a negative indication of units coordination. Table 7.1 lists out the values of p and q for identified chunks. To evaluate the model fit, I considered whether there were threshold values of u and l that would distinguish the identified second-order inferences as units coordination to a better degree than what would be expected by chance.

Table 7.1
Values of p and q Accompanying Second-order Inferences of Units Coordination

Chunk	p	q
4	0.5	0.8477
45	0.2009	0.4083
58	0.1166	0.8193
84	0.0768	0.7723
158	0.7213	0.7539
168	0.6605	0.8127
206	0.6175	0.9549
300	0.1491	0.8496
371	0.3610	0.7466
376	0.3610	0.8998
390	0.3939	0.3977
397	0.2387	0.8569
526	0.6211	0.8537
545	0.2207	0.7337
573	0.1154	0.4370
658	0.6932	0.8125
679	0.2035	0.8587

In the Austin and Justin data, the most restricted values of u and l that permit each of the 28 inferences to fit with the p and q values are $u = .75$ and $l = .45$.

The fit of the output of the baseline algorithm with the second-order modeler's explicit inferences of Dylan's units coordination meets and slightly exceeds this minimal threshold. Values of $u = .75$ and $l = .4$ result in all 17 inferences being successfully matched. That is, for each of the 17 associated chunks, $p < .4$ or $q > .75$.

Letting $u = .9$ and $l = .45$, which was found to be optimal with the Austin data, results in agreement of only 12 of the 17 inferences with the Dylan data. As the value of u is increased or the value of l is decreased, the range of values associated with successful agreement becomes more limited. Accordingly, the likelihood that "successful" model fit may actually be due to chance increases. With $u = .9$ and $l = .45$, 57 percent of chunks lie in the "successful" range. If u is lowered to .75, 72 percent of the chunks meet the threshold (see Table 7.2).

To test whether the agreement between the two models was beyond what would be expected by chance, I compared the ratio of successful fit with units coordination inferences in all 690 chunks with the ratio of successful fit with units coordination inferences in the 17 identified chunks. Fixing l and u results in K of the 690 chunks and k of the 17 identified subset of chunks to be associated with units coordination. I tested whether selecting 17 distinct chunks randomly from the 690 would be likely to achieve as good or better fit.

For fixed u and l , define

$$P(X = k) = \frac{\binom{K}{k} \binom{690 - K}{17 - k}}{\binom{690}{17}}.$$

$P(X = k)$ is the probability that exactly k of 17 chunks chosen at random (without replacement) from the 690 have $p < l$ or $q > u$ (or both). X is thus a discrete random variable with a hypergeometric distribution (Weissstein, 2014). Note that the hypergeometric distribution is appropriate because the total number of chunks is finite and because the output of the test of the fit of a chunk is deterministic once u and l are chosen.

Table 7.2

Statistical Significance of Second-order Agreement using Dylan's Data

u	l	$k_{u,l}$ (percent)	$K_{u,l}$ (percent)	$P(X_{u,l} \geq k_{u,l})$
.9	.45	12 (71)	396 (57)	.1945
.9	.4	12 (71)	374 (54)	.1294
.9	.35	9 (53)	345 (50)	.50
.85	.45	13 (76)	432 (63)	.1738
.85	.4	13 (76)	415 (60)	.1255
.85	.35	11 (65)	391 (57)	.3377
.8	.45	16 (94)	456 (66)	.0079
.8	.4	16 (94)	442 (64)	.0050
.8	.35	14 (82)	423 (61)	.0556
.75	.45	17 (100)	505 (73)	.0046
.75	.4	17 (100)	495 (72)	.0033
.75	.35	15 (88)	479 (69)	.0680

Note. $k_{u,l}$ is the number of the 17 second-order p and q values successfully corresponding with given u and l values. $K_{u,l}$ is the number of the (total) of 690 chunks whose p and q values correspond with units coordination, given u and l values.

The results listed in Table 7.2 suggest that, for the Dylan data, units coordinating inferences by the second-order modeler were not associated with values of q as large as those associated with the Austin data. The chances of a chunk associated with units coordination meeting the thresholds of $u = .9$ and $l = .45$ are not significantly better than what would be expected by chance ($P = .1945$). They were associated with only slightly smaller values of p , as values of u and l that resulted in statistically significant differentiation between identified and unidentified chunks had p values of .4 ($P < .05$).

It seems that the identified inferences of Dylan's units coordination were less likely to accompany long sequences of inferences of units coordinating activity (corresponding with high q) and more likely to accompany short sequences of decreasing propensity to *not* coordinate units (corresponding with lower p). This may be a characteristic of students who coordinate three levels in some contexts but not others. Though there was not a single pair of u and l values that was optimal for the Justin data, the value $u = .75$, $l = .4$ would have fit with those entire 11 inferences, while higher values of u would not (see Table 6.2). It seems that the values of u and l are strongly dependent on differences in the teacher-researchers, as Austin and Dylan had the same teacher-researcher.

Testing Third-order Conjectures

My testing of the conjectures listed at the end of Chapter 6 began with a more fundamental test—whether the calculation of the Scale Score (S) used to characterize teaching segments would discriminate in a manner that would permit the conjectures' potential falsification. The Scale Score is a measure for making inferences from the rise, fall, and volatility of p and q within a segment of the teaching experiment. There are 6 quantities used to calculate the scale score for a segment: the starting value of p , the starting value of q , the mean value of p , the mean value of q , the range of p , and the range of q (see Algorithm 5.2). Recall that negative values of S were associated with substantial perturbation (SP), values of S greater than 1 are associated with equilibrium (E), and values between 0 and 1 (inclusive) are labeled moderate perturbation (MP).

The Dylan teaching experiment was segmented by the 11 teaching sessions. Table 7.3 lists out the values of the 6 quantities used to compute S for each session.

Table 7.4 lists out the value of S , computed using Algorithm 5.2, along with the characterization of the session using one of the 8 descriptions listed in Table 6.5.

Table 7.3
Quantities Used to Calculate S for the Dylan Sessions

Session	Start p	Mean p	Range p	Start q	Mean q	Range q
1	0.5000	0.4614	0.6984	0.5000	0.7065	0.6998
2	0.2009	0.2349	0.8346	0.4083	0.7784	0.6959
3	0.8199	0.5508	0.7699	0.6201	0.6263	0.6158
4	0.6605	0.4890	0.6824	0.8876	0.8779	0.1992
5	0.1478	0.2932	0.7397	0.8501	0.6929	0.7394
6	0.3994	0.5715	0.7961	0.5830	0.4626	0.7371
7	0.6702	0.4205	0.7099	0.7558	0.6989	0.6784
8	0.7142	0.3945	0.6456	0.4795	0.5649	0.7077
9	0.1503	0.3375	0.7656	0.8710	0.6265	0.6195
10	0.1970	0.5084	0.8150	0.6937	0.6824	0.6351
11	0.9304	0.6109	0.6938	0.5765	0.5004	0.7774

Note. Starting values are the values of p and q at the conclusion of the previous chunk except for the initial p and q values of 0.5

Table 7.4
Characterization of the Dylan Sessions

Session	S	Characterization	Task Context(s)
1	0.6991	MP (Interviewing)	Fractions
2	1.328	E (Establishing)	Fractions
3	0.6613	E (Establishing)	Fractions
4	3.893	E (Establishing)	Whole Numbers, Fractions
5	0.1314	MP (Moderating)	Whole Numbers, Fractions
6	- 0.4696	SP (Demanding)	Whole Numbers, Fractions
7	0.7060	MP (Guiding)	Whole Numbers, Fractions
8	0.8029	MP (Guiding)	Whole Numbers, Fractions
9	-0.0688	SP (Conflicting)	Whole Numbers, Fractions
10	0.0505	MP (Guiding)	Whole Numbers, Fractions
11	0.1257	MP (Interviewing)	Whole Numbers, Fractions

Note. S was calculated using Algorithm 5.2. The Characterization and Task Context were determined via analysis of the second-order model and transcription of the teaching sessions.

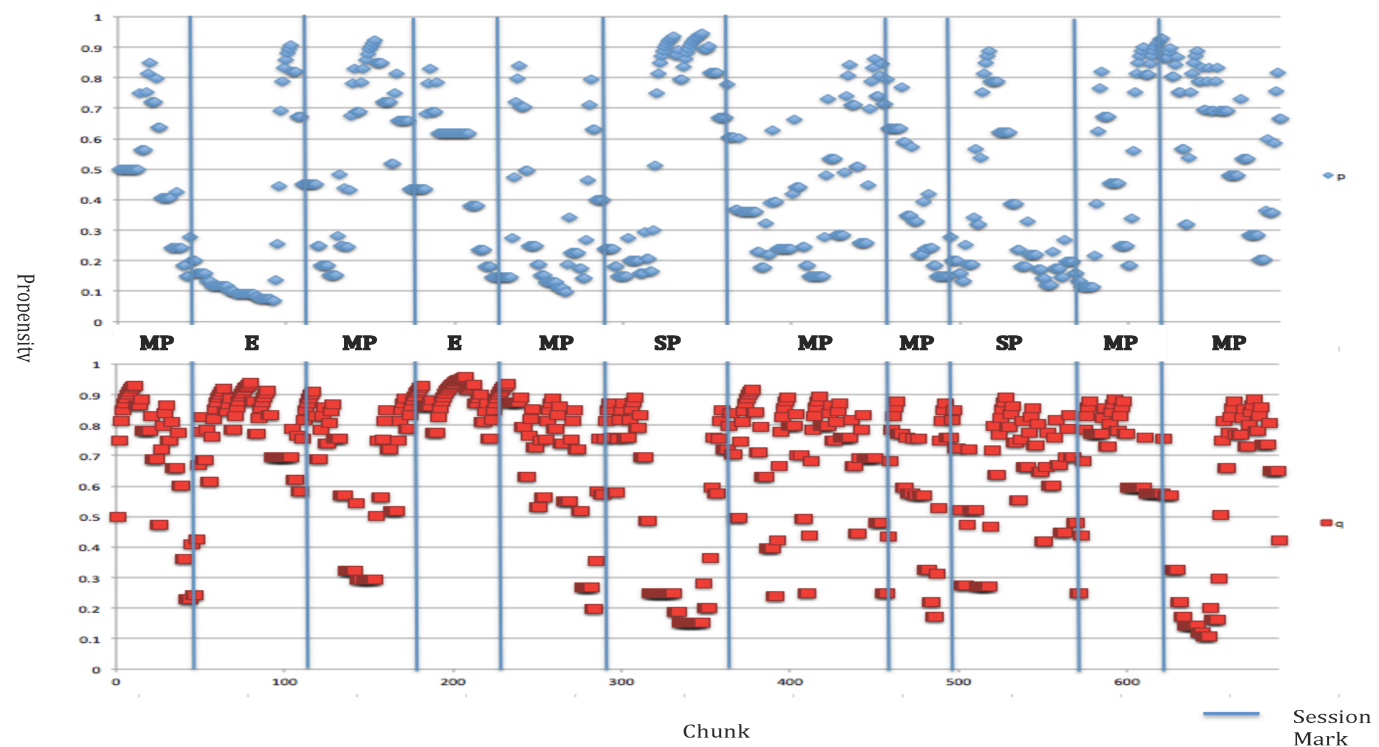


Figure 7.3 Plot of Dylan's Sessions

Characterization of sessions

Having relied on Algorithm 6.2 to distinguish between E, MP, and SP sessions, I went back to the second-order modeler's description and transcription of the Dylan teaching experiment to consider the relationships between the quantities listed in Table 7.3, the shapes of the graphs in Figure 7.3, and the context of the inferences. I constrained myself to characterizing a session as a single type, using the coding developed with the previous sessions.

Session 1: Moderate Perturbation

The first session of the teaching experiment began with free play (Steffe, 1991) intermixed with the introduction of fractions tasks. I characterize the session as "Interviewing." Though the teacher-researcher explicitly worked to establish norms for communication and expectations for student reflection in the very beginning of the session, the rest of the session was marked by a series of exploratory tasks, increasing in difficulty, that are characteristic of interviewing. Dylan was successful on these tasks, but seemed to be attending to only two levels of units. The teacher-researcher introduced doubt about whether a single fraction rod could be a proper fraction. While Dylan was able to find the correctly sized pieces in response to the series of tasks, he also adopted a further requirement that the figural representation of "three-sevenths" must be composed of three rods.

Session 2: Equilibrium

Examination of Figure 7.3 reveals that the majority of Session 2, up until the last few chunks, is characterized by steady increase in q and steady decrease in p that often accompany Equilibrium sessions. I describe the session as "Establishing," as Dylan

reconciles the conflict that had been introduced in the previous session and establishes that proper fractions are sizes that result from iteration of a unit fractional part, regardless of the figural segmentation of the proper fraction representation. At the end of the session, the values of q fall and the values of p rise when Dylan faces a task requiring him to coordinate three levels of units in consideration of commensurate fractions involving halves (two-fourths, four-eighths and one-half).

Session 3: Moderate Perturbation

The third session is characterized as “Moderating.” Maddie created a first task. Dylan was able to find a length corresponding to the whole when given a piece named as $1/12$, coordinating two levels of units. The teacher-researcher increased the demand by posing a reversible IFS task – to find the whole given a fraction named as $5/3$, which perturbed both students. Dylan subsequently chose a task for Maddie in which he had difficulty reconciling his part-whole and measure interpretations of fractions using the figural material from segmented rods. The session is characterized as “Moderating” because of the degree of randomness in the units coordination demands.

Session 4: Equilibrium

The fourth session was labeled as “Establishing.” Unlike the previous session, the teacher-researcher prescribed all the tasks for the session, which involved strips of paper. The first tasks involved whole numbers, such as forming a strip of length seven inches when given a strip of length four inches. These were followed by (reversible) fractions tasks, such as forming the whole and unit when given a strip named $4/9$. Even Maddie commented on the similarity of Dylan’s way of operating across the whole number and fraction tasks. The steady decrease in p during the session was accompanied by a more

parabolic shape in q . This is because in response to the last task of the session, to form a whole and unit given a strip of size $8/5$, Dylan did not iterate a fraction unit to form the whole, naming it “five fives” and not attending to uniform size when partitioning into eight parts. Thus, Dylan established a way of solving RPFS tasks with figural material in which the (two levels of) units were coordinated, and was unable to attend to a third level of units in the reversible IFS task.

Sessions 5: Moderate Perturbation

The fifth session marked the introduction of *CandyDepot* to the teaching experiment sessions. There was randomness in the units coordination requirements of each task. The students took turns solving the tasks, some of which involved whole number orders in terms of bars, and some of which involved (improper) fractional orders of bundles or boxes. This session was characterized as “Moderating.”

When Dylan and Maddie would place an order incorrectly, the teacher-researcher returned to the task to encourage the students to reflect on their reasoning.

Session 6: Substantial Perturbation

The next session involved tasks using physical manipulatives: chips within cups within boxes. Notably, Dylan initially seemed quite confident with the task setup; he noted similarities between the bars tasks and *CandyDepot* in successful whole number and unit fractions tasks. After Maddie reflects on her coordination of three levels of units involving another fractions task, which involved her understanding of one-third as two-sixths, Dylan says he is unable to understand her explanation. When the teacher-researchers subsequently posed an improper fractions task – to find the number of chips in $20/12$ of a box – Dylan conflated the whole and the improper fraction and seemed to

rely on rote calculations until the end of the session. This session was labeled as “Demanding.”

Session 7: Moderate Perturbation

The next session again involved the chips within cups within boxes. The session was characterized as “Guiding.” The tasks of the session consisted primarily of the students individually responding to a sequence of quickly prescribed tasks, building up from lower level units to higher level units, and then back down. The task sequencing generally proceeded unless there was an error in the response, in which case the teacher-researcher would pause and ask for explanation. Dylan used figurative material (looking at the boxes, holding up fingers) to coordinate the units quite successfully when the tasks all involved whole numbers. Both students conflated units when coordinating a third level of fractional units in the session.

Session 8: Moderate Perturbation

The eighth session was characterized as “Guiding.” Dylan and Maddie were given a sequence of tasks similar to those encountered in previous sessions, but with area models instead of linear models. The session began with the task of splitting a square into nine equal pieces, and ended with a task of finding the unit and the whole given a rectangle labeled as $\frac{9}{5}$. In the interim, the students partitioned squares both vertically and horizontally in the service of whole-number and fractions tasks. Maddie coordinated three levels of units in activity, using the figurative material, as she did not seem to struggle with most of the tasks. Dylan often watched or reflected on the results of her activity, adopting it in the subsequent tasks. Unlike Maddie, he struggled on the final, IFS task, initially conflating the given $\frac{9}{5}$ and the whole.

Session 9: Substantial Perturbation

The next session is characterized by substantial perturbation. The students were again playing *CandyDepot*. The randomness of the units coordination required of tasks presented in the game may have increased the perturbation, as Dylan was aware that he “switch[ed] back and forth” between two strategies during the session, which may have contributed to confluences of units. Dylan, at times, demonstrated attention to the referent units, rather than only the formation of the composite units, which often required coordination of a third level of unit. Sometimes he would figure out the number bars needed, and then build that number by treating boxes and bundles as composite units of bars, i.e., first figuring out that 10/3 bundles was 10 bars, and then building up the bars from boxes, bundles and bars. At other times, he would attend to the referent unit, i.e., completing an order of 10/3 boxes by dragging three boxes and one bundle without considering the number of bars. I characterize this type of SP session as “Conflicting,” as Dylan seemed to have two different ways of operating during the session.

Session 10: Moderate Perturbation

The penultimate session was characterized by moderate perturbation, this time, “Guiding.” The context shifted to a continuous setting, as the students worked with relationships between the volume of teaspoons and measuring cups. The setup was similar to the initial chips/cups/boxes tasks, except that the students found relationships between the liquid measures by pouring water and recorded them on paper. Dylan was able to coordinate three levels of units when the units involved whole numbers. Contributing to a moderation of the perturbation was the discrepancy between what Dylan observed when measuring with the water and the mathematical relationships. The

teacher guided Dylan to perturbation when he conflated units by encouraging him to return to physically measuring the relationships with water.

Session 11: Moderate Perturbation

In the final session, Dylan and Maddie go through a series of four activities. I characterize the session as “Interviewing.” The first activity is a return to the water tasks. The teacher explores Dylan’s previous response to a task in which he fortuitously responded correctly, but for which he later wavered between two possible responses. It confirmed that Dylan did not have a structure for coordinating three levels of fractional units. The second task was confirmatory of Dylan’s PFS. He was able to construct $\frac{3}{5}$ from a whole and correctly name the complement of $\frac{3}{5}$ without the whole present. The third activity involved Dylan and Maddie looking at each other’s written assessments from the initial interviews that prefaced the teaching experiment. This again affirmed that Dylan had not constructed a three level structure for fractional units. The last segment of the session returned to *CandyDepot*. Dylan’s actions were impressive in his anticipation of coordination of three levels of (whole number) units, suggesting that he may have begun assimilating three levels of units.

Evaluation of the conjectures

There were five conjectures to be tested; the results are listed in Figure 7.4. Two of the conjectures involved sequences of Equilibrium (E) sessions that did not occur in the teaching experiment and were not falsifiable by the results. The results support the veracity of the other three conjectures, though there is only weak support for the last conjecture.

Conjecture	Description	Result
1.	If an E session directly follows an SP session, then the E session is Establishing and the SP is Conflicting.	Not Applicable
2.	The number of MP sessions will be greater than or equal to the number of SP sessions and greater than or equal to the number of E sessions.	Upheld
3.	Consecutive SP (Demanding) sessions will not occur unless the student's partner is absent for at least one of the sessions.	Upheld
4.	If an E (Establishing) session follows an MP (Guiding) session, what becomes established is a sub-activity, not the entire task sequence, and this activity can be traced back to an earlier session.	Not Applicable
5.	E (Establishing) sessions begin with the "easier" of numerical contexts requiring the same number of levels of units coordination.	Upheld

Figure 7.4. Evaluation of Five Conjectures

In general, the results suggest that though there was no falsification of any of the conjectures, describing the dynamics of the teaching experiment in terms of sessions' units coordination activities may not be fine-grained enough to be useful for making predictions. For the first conjecture, that Equilibrium sessions do not directly follow Substantial Perturbations that are not Conflicting, there is no evidence either way, as both Equilibrium sessions followed MP sessions. There were a total of seven MP sessions, two SP sessions, and two E sessions, so Conjecture 2 was upheld. The two SP sessions were not consecutive, also the latter was not "Demanding," as Conjecture 3 was upheld. Conjecture 4 involved E sessions following MP (Guiding) sessions, of which there were none, and was thus not applicable to the data.

For the last conjecture, that E (Establishing) sessions begin with the "easier" of numerical contexts requiring the same number of levels of units coordination, there were two sessions offering differing levels of support to the conjecture. There was strong supporting qualitative evidence from Session 4 and weaker supporting evidence from Session 2. In Session 4, the units coordination with fractions clearly followed the units coordination with whole numbers. The values of p decrease throughout, and the values of q are consistently high.

In Session 2, there was but a short sequence of tasks involving whole number units coordination prefacing fractions tasks. Also, the entire session was not characterized by a steady increase in q and decrease in p , in conjunction with latter tasks requiring three levels of units. It may be that this session is better characterized by "Norming." As I discuss in Chapter 8, it may also be that would be better to separate the

sessions into sub-segments to better capture within-session contextual changes, especially for difficult to classify sessions.

Results Pertaining to Research Question 1

The approach was somewhat useful for answering my first research question, “What are the relationships between a student’s units coordinating activities across numerical situations such as counting sequences, fractions, whole number multiplication, and signed integer addition? The current study supported and extended the *reorganization hypothesis* (Steffe, 2001), in the findings with Justin and Austin that engaging in units coordination with whole numbers immediately prior to engaging in units coordination with fractions supported units coordination in the latter context. This may or may not have been identifiable without the plots of units coordinating activity, and in the last teaching experiment with Dylan, in which the conjecture was tested, there was moderate support.

I was unable to identify relationships between students’ units coordinating activities regarding counting sequences and signed integer addition. The teaching experiment data I analyzed focused primarily on whole number units coordination and fractions. Though there were instances of counting activity in each of the teaching experiments, engendering growth in a student’s counting sequences was not an explicit goal of any of the teaching experiments. Activities with signed integers were only included with the Justin data. Because I was unable to include the numerical context to improve upon the baseline model, I was unable to form and test more fine-grained conjectures pertaining to the units coordinating activities across the less prevalent contexts.

Results Pertaining to Research Question 2

With regard to my second research question, “What sequences of ecological changes facilitated growth in the number of levels of a student’s units coordinating structures,” I was able to interpret sequences of ecological changes broadly, as gradients in levels of perturbation at the session level and as sequences of engagement involving mathematics content domains (whole numbers, fractions, and integers) within session. The attempt to incorporate context into the algorithm for computing the propensities with the Austin and Justin data was unsuccessful, and this constrained my ability to identify and test conjectures involving more fine-grained sequences of ecological changes. The sequences of ecological changes I was able to model were limited to two dimensions – the characterization of sessions’ perturbations and the mathematical content domains within and across sessions. The approach was successful for identifying and interpreting the task sequencing and interactions utilized by the teacher-researchers to introduce and maintain a productive level of perturbation during and across the teaching experiment sessions.

Results Pertaining to Research Question 3

Regarding the last, over-arching research question, “How does the modeling of a students’ propensity for assimilating an n -level structure contribute to an understanding of how units coordination structures are constructed?,” the coding of units coordinating activity, the plot of the propensity to coordinate units, and the description of the sessions’ perturbations together permitted analysis of the content and structure of radical constructivist teaching experiments focusing on units coordination in multiple domains.

Coding for units coordinating activity across the sessions introduced an approach for analyzing the development of units coordinating structures across the contextual changes of the teaching experiment. Modeling the propensities provided a form for the dynamics of a constructivist teaching experiment that is generally consistent with radical constructivism (von Glasersfeld, 1995) and its teaching experiment methodology (Steffe & Thompson, 2000). Modeling the changes in the propensities using the scale score permitted an objective way to test hypotheses regarding the role of context and interaction in students' development within the flexible structure of the teaching experiment methodology.

CHAPTER 8: CONCLUSIONS

In this chapter, I interpret the results of Chapter 6 and Chapter 7 for answering the research questions:

1. What are the relationships between a student's units coordinating activities across numerical situations such as counting sequences, fractions, or whole number multiplication, and signed integer addition?
2. What sequences of ecological changes facilitate growth in the number of levels of a student's units coordinating structures?
3. How does the modeling of a students' propensity for assimilating an n -level structure contribute to an understanding of how units coordination structures are constructed?

As discussed in Chapter 5, consideration of the first and second research questions guided my approach of modeling students' propensity to coordinate units across contexts. I therefore begin by reflecting on the rationale and evaluating the limitations and benefits of the modeling approach. I then interpret the results, emphasizing contributions relating to existing constructs in the mathematics education and educational psychology literature, with regard to the third question.

Evaluation of the Modeling Approach

Consideration of third-order models

In this study, I have focused on inferences stemming from rather specific sets of data – teaching experiments with sixth-grade students involving experienced teacher-researchers who adopted a radical constructivist epistemological stance (von Glasersfeld,

1995). From this perspective, for every inference involving the thinking of another individual, there is uncertainty, and inferences about changes in the thinking of another individual are absolutely unknowable (Steffe & Cobb, 1983). Furthermore, what is constructed is thought to *fit*, rather than *match*, experienced reality (von Glasersfeld, 1991). I have attempted to apply the principles of radical constructivist epistemology first and foremost to myself in my modeling of mathematical learning. The purpose of my efforts has been to develop an approach that might permit both myself and others to experience, anticipate, and construct models of reality that include intra-student variability in the development of units coordination structures.

The data I analyzed adhered to the teaching experiment methodology (Steffe & Thompson, 2000), as its original purpose was building *second-order models* of students' mathematics.

Second-order models might concern a theorist's or a teacher's knowledge of [a] student's mathematical knowledge, or vice versa, and are necessarily constructed through social interaction. One can legitimately say that they are coconstructed by the observer and the observed. They constitute the primary knowledge of concern in education. (Steffe & Wiegel, 1996, p. 484)

Building second-order models requires both learning how to listen to students and learning how act in ways that allow students to sustain and modify their mathematics (Cobb & Steffe, 1983; Steffe & Wiegel, 1996). Whether focusing on taken-as-shared meanings or individual meanings, "models of learning are unavoidably a function of the teacher's actions" (Steffe & Wiegel, 1996, p. 482).

I describe the models I constructed as *third-order* models for several reasons. Excepting my pilot study, my interactions with teachers and students in the analyzed data did not extend beyond the one-sided role of an observer. My inferences were primarily

formed from analyzing written transcriptions that I constructed and the analysis of the teaching experiments constructed by teacher-researchers. Thus, my observations were a step removed from the interactions between a teacher and a student. In addition to its dependence on the actions of a student and a teacher, a third-order model of learning is dependent on the actions of a *witness*. Though the teaching experiment methodology includes the importance of the witness's role in on-going and retrospective analysis in the teacher-researcher's construction of the second-order model (Steffe & Thompson, 2000), the third-order model I have created extends this role and has resulted in several constructs to consider when planning sessions activities.

Because the silent hypotheses of a witness do not affect the observations made by a teacher about their students, a witness can more easily distinguish and then incorporate the teachers' actions and goals within a model of a student's learning. In forming third-order models of Austin, Justin, and Dylan, I viewed the second-order models of the mathematics of students that were constructed by the teacher-researchers as the *result* of a modeling *process*. Though teaching experiment methodology includes both retrospective and ongoing analysis, the focus is on modeling the product of student learning. In contrast, my model extends our understanding of how a teacher-researcher's understanding of a student's mathematics develops with the students' understanding. My model frames and objectifies the variability of observations involving second-order model constructs.

Consideration of units coordination structures

I conceived of the idea of mathematizing the variability in students' mathematics as I was learning about fractions schemes and watching videos of students engaged in

fractions tasks, beginning in 2010. In trying to understand how students might be thinking, I noticed that I would, at times, adopt less-powerful schemes when engaging with fractions in everyday life. This conflicted with my understanding of a scheme as an established way of operating. I had read the work of Simon and Tzur (2004) and Tzur and Simon (2004) distinguishing between participatory and anticipatory schemes. Early on, I sought to extend the participatory/anticipatory framework to model students' fractions schemes as probabilistic, with many contextual factors, (including but not limited to teacher support) potentially affecting the probability that a student would assimilate a situation within one scheme or another. An original goal was to provide a mechanism for qualifying what it means for a scheme to be “established.”

The consideration of units coordinating *structures*, instead of fractions *schemes*, was motivated by the results of studies suggesting that the researchers' models of the relationships and operations of a PFS and an RPFS were in flux (Hackenberg, 2007; Norton & Wilkins, 2010; Wilkins, Norton, & Boyce, 2013). This phenomenon is not isolated to the partitive fractions schemes, as the meaning of an “iterative” fractions scheme changed over the years as well (Saenz-Ludlow, 1994; Steffe & Olive, 2010). To model probabilistic assimilation of a situation within a set of particular schemes, I needed constructs that were not likely to change in the research community, at least through the course of analysis. At the time of this writing, consideration of units coordination has been advanced in research using a variety of methods, including teaching experiments (Norton & Boyce, 2013b), clinical interviews (Boyce & Norton, submitted; Norton, Boyce, Philips, & Anwyll, in preparation; Norton, Boyce, Hackenberg, Ulrich, & Phillips, in preparation), and written assessments (Norton, Boyce, Philips, & Anwyll, in

preparation; Izsák et al, 2010). Moreover, units coordination transcends mathematics domains of fractions, whole numbers, and integers, which permits analyses across a broad range of contextual settings.

Modeling units coordination as a Piagetian structure (Piaget, 1970b) was not established in the research community at the beginning of this study (cf. Steffe Olive, 2010). At each step of the modeling process, I reflected on whether the model “fit” not only with my inferences, but with the existing theories. Where there was misfit, there was an absence of “objective truth” about what the student was thinking by which to judge whether the theory or the model needed adjustment. As I was modeling the construction of students’ units coordinating structures, I was constructing and reflecting on new constructs that incorporated those that were previously defined, most prevalently, in Steffe & Olive (2010), such as fractions schemes or iterating operations.

Evaluation of the construct of propensity to coordinate units

What is rather unique, and critical to this dissertation thesis, is the creative application of mathematics in modeling the learning of mathematics. I used mathematics to qualify, as well as quantify; I used mathematics to construct, as well as measure. I did not restrict myself to established quantitative methodologies in model formulation. I view my approach to modeling propensities described in Algorithm 5.1 as the result of a process of *mathematical* modeling (Heinz, 2001; See Figure 8.1 and Appendix A).

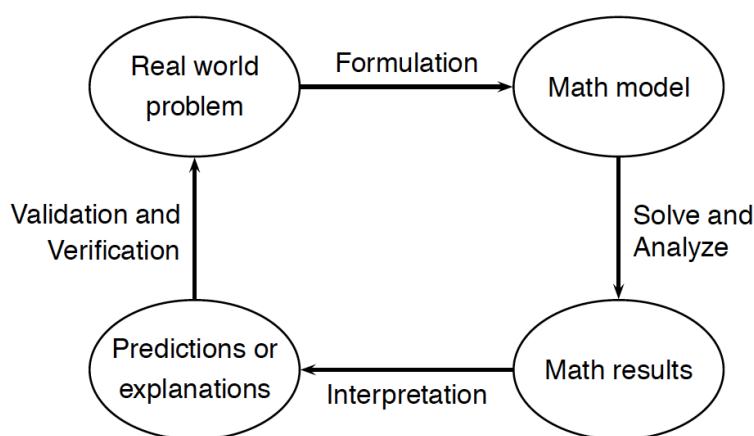


Figure 8.1. The Process of Mathematical Modeling

With reference to Figure 8.1, the “real world problem” was to understand the dynamics of students’ units coordination during teaching experiments. I attempted to incorporate the conjectural activities of both the teacher and the student by forming a model of students’ units coordinating activity during teaching that accounted for both a search for stability and a search for productive perturbation. An early subjective step in the model-building process was perhaps the most critical – the chunking process.

Though even the transcription of teaching experiment data is somewhat subjective, the purpose and result of chunking – to separate the context of observations – is inherently so because of the subjectivity in describing “context.” I qualified my meaning of context, adopting and extending the approaches of Simon et al (2010) and Wagner (2006) as described in Chapter 3. I anticipated and embraced the ambiguity of chunking in the modeling of units coordinating inferences. I purposefully did not model the students’ units coordinating activity using probabilistic methods relying on independent data obtained from a prescribed distribution, and I intentionally sought a

method of describing the units coordination for which minor chunking discrepancies would have a small effect.

Though my inferences of units coordination activity were coded as 0 or 1 (or ?), I was well aware that 0s and 1s were not all “equal.” At this stage in the modeling process, I tried to qualify the context of these inferences – within chunk - to formulate a model that might later become more useful as a stand alone product. Algorithm 5.1 resulted in a noisy model that was simple to apply, but whose results were difficult to interpret. The next step in the modeling process involved increasing the complexity of the algorithm, with the aim of improving its use as a stand alone product. My approach was to build upon, but retain, the ideas for modeling the stability of units coordinating activity across the contexts of the teaching experiment that led to Algorithm 5.1 (see Appendix A). I used the second-order (qualitative) descriptions to validate and improve upon the results.

That I was unable to improve upon Algorithm 5.1 can be thought of both positively and negatively. On the one hand, it suggests that the context-free algorithm was more useful than anticipated. It seems that the types of contexts identified as potentially useful for making adjustments to the propensity calculations were not necessary to achieve good model agreement with the second-order modelers’ inferences of units coordination. This suggests that the teacher-researchers may have taken the context of inferences of units coordinating activity into account when interacting with their students. Certainly this is an area where future research about the generalizability with different teachers and contexts is warranted. Optimistically, the simplicity of the baseline model is good enough to use even more broadly, e.g., with teaching experiments

explicitly focused on topics other than units coordination or those involving teachers with other epistemological perspectives.

On the other hand, it may be that a different method for improving the algorithm or a different approach to assess model validity might prove to be useful. In future work, I plan to try to incorporate model-building approaches other than relying on written second-order descriptions of units coordination. The method of matching the plot of units coordinating activity with the second-order modeler's interpretation did not punish "false positives." I also plan to consider other ways of modifying Algorithm 5.1, perhaps by adjusting the formula to incorporate the passing of time. In particular, I think that an interview with the second-order modeler would be useful to gauge the usefulness of the propensity calculation in Algorithm 5.1 (cf., Rappold, Lavine, & Lozier, 2007).

Evaluation of the characterizations of teaching sessions

Another round of modeling took place after the decision to not adjust Algorithm 5.1. Having constructed propensity plots for Austin and Justin, the "real-world problem" was to characterize the patterns in the plots in a way that would be useful for describing, predicting, or explaining Dylan's units coordinating activity from the plot of his units coordinating activity. The predictions that I was able to make regarding the patterns in Dylan's units coordinating activity were constrained by the lack of contextual distinctions in the plots of p and q . The construction of the scale S in Algorithm 5.2 followed several failed attempts to incorporate mathematics to describe the behavior of the propensity plots in a manner that was both potentially useful and consistent with both the Justin and Austin data.

In forming and analyzing the results of these conjectures, I was interpreting the results of the modeling in terms of radical constructivist notions of perturbation (Piaget, 1970a; Steffe & Wiegel, 1994; von Glasersfeld, 1995), the zone of potential construction (Norton & D'Ambrosio, 2008; Steffe, 1991), and mathematical caring relations (Hackenberg, 2009a). The analysis of the propensity plots of Austin and Justin using Algorithm 5.2 resulted in characterizations of the patterns in students' units coordination activity within each session into 3 gradients of perturbation, labeled "Equilibrium," "Moderate Perturbation," and "Substantial Perturbation." The second step involved analyzing the type of perturbation using the descriptions listed in Table 6.5, and it required qualitative analysis of each teaching session. Though the results were consistent with my predictions, the varied activities within each session were not as cleanly described via the list given in Table 6.5. Exploration of further segmentation of the teaching experiments to characterize the level of perturbation within teaching sessions across the three teaching experiments is warranted.

Contributions

The results of analyzing the third-order models of Austin, Justin, and Dylan suggest that patterns of interactions can be inferred that may be used to inform teaching decisions within teaching experiments. Specifically, to identify the zone of potential construction (ZPC, Steffe, 1991) within the zone of proximal development (ZPD, Vygotsky, 1978), and to attend to mathematical caring relations (MCR, Hackenberg, 2010a). Modeling the variability of observations of a student's units coordinating activity during types of student-teacher interactions also supports and extends the

psychological/neurological notion of a *dynamic structural model* (Fischer & Bidell, 2006).

Zone of proximal development and zone of potential construction

The zone of proximal development of a child is the distance between her actual developmental level as determined by independent problem solving and her level of potential development through problem solving under the guidance or in collaboration with more capable peers (Vygotsky, 1978, p. 86).

Like the zone of proximal development (ZPD), the zone of potential construction (ZPC) concerns the roles of student-student and student-teacher interactions in student learning. If a task lies within the ZPD of a child, then the child must be necessarily assistable; that is, the child must have a high enough level of development to be guided to the more powerful understanding, but the child must also require assistance. In contrast, the ZPC concerns the researchers' *models* of students' mathematics, including constraints and conjectures of how students might potentially reorganize their current ways of operating in accommodation (Norton & D'Ambrosio, 2008). From a radical constructivist perspective, the ZPC is a subset of the ZPD. The complement of the ZPC within the ZPD is the set of tasks for which a student can demonstrate understanding in interpretation of others' actions, but what is constructed are procedural schemes involving current ways of operating rather than new ways of operating (Norton & D'Ambrosio, 2008).

Analysis of the propensity plots of Austin, Justin, and Dylan resulted in characterizations of sessions that may illustrate and identify relationships between the ZPC and ZPD. Engaging a student within his ZPC is the goal of a radical constructivist teaching experiment. If interactions are within the ZPD, but not the ZPC, then the

student may construct procedural schemes to fit with observations of others' activity rather than construct mathematical learning (Norton & D'Ambrosio, 2008).

Consideration of the ZPC and ZPD provides a general way to think about Equilibrium, Moderate Perturbation, and Substantial Perturbation distinctions of teaching sessions. Equilibrium sessions' activities might seem to be outside of the ZPD because the student does not require assistance and is not limited to only participatory and procedural actions – the activities do not present a challenge to the student's existing ways of operating. On the other hand, Substantial Perturbation interactions are generally outside of a student's ZPD for the opposite reason - because the student was unable or unwilling to perform participatory actions. Moderate Perturbation sessions are those within the ZPD. Guiding, Moderating, and Interviewing – the types of MP sessions – describe teacher actions that help to maintain interaction within the student's ZPC.

Sessions described as “Guiding” contain teacher-student interactions suggesting that the teacher introduced a sequence of sub-tasks to assist a student in problem-solving. The type of assistance provided in these sessions may not be intentional scaffolding (Anghileri, 2006) on the part of the teacher. For a radical constructivist teacher, guiding is providing a situation for which a student might conjecture that his current ways of operating are inadequate, by helping the student identify and reflect on inconsistencies in his own way of thinking – guiding is questioning, not showing or telling a student that there are inconsistencies in the results of a teacher's and a student's activity.

In Chapter 6, I formed the conjecture that students may establish a scheme or form an accommodation of an existing way of operating related to a “sub-task” of a Guiding session in a subsequent Establishing session. This is consistent with the idea that

schemes within a child's ZPC are what may potentially become established. The conjecture was formed from the Justin data, and there was neither confirming nor contradictory evidence from the Dylan data. In future work, I plan to construct a model of the units coordinating activity of Dylan's partner, Maddie, to compare her level of perturbation with that of Dylan's. I conjecture that the model will permit distinguishing of "Guiding" to be student-specific, e.g., that the "Guiding" labels would not necessarily apply to the same sessions for multiple students.

Mathematical caring relations

In order to form models of students' mathematical learning in the context of teaching, a teacher-researcher must have and maintain participants' willingness to engage in mathematical activity. Hackenberg (2010a) characterizes both cognitive and affective aspects of student-teacher interaction in the service of mathematical learning as *mathematical caring relations*. Mathematical caring is the "orchestration of mathematical learning together with an orientation to monitor and respond to fluctuations in positive energy available to the self – subjective vitality – that may accompany student-teacher interactions" (p. 237). Hackenberg (2010a) views student-teacher interaction as a "linked-chain of perturbations" (p. 236).

I did not directly model the teacher-researcher's thinking, decision-making, or energy level. The inferences of students' units coordinating activity were dependent upon such constructs though, at least indirectly. For example, in my coding, I gave a student a '0' if he was given the opportunity to coordinate units and, by his response, I inferred that he did not do so. The student's energy level and the teacher-researcher's decision to prompt him with such an opportunity were thus complicit in the modeling of

the student's propensity to coordinate units. Accordingly, the characterizations of sessions as Equilibrium, Moderate Perturbation, or Substantial Perturbation lay the foundation for communicating and describing mathematical caring relations within and across teaching experiments. They provide a new mechanism for reflecting on and planning for the content of sequences of interactions within and across teaching sessions.

A teacher's relinquishment of control over task difficulty, experienced most often in sessions characterizing as Moderating, seems to be a productive way to establish and maintain mathematical caring relations. The introduction of chance, in which some tasks were in the ZPD, and some were not challenging at all for the students, served to moderate the level of perturbation and increase the students' motivation. Part of this moderation was the teacher-researcher's act of encouraging the students to reflect on their reasoning, especially when they encountered a more difficult task. Future research in the development of games like *CandyDepot* or *CandyFactory* (LTRG, 2012; 2013) may consider how algorithms for selecting sequences of units coordinating tasks may effectively moderate perturbation with minimal teacher-student interaction.

Dynamic structural models

In the course of a teaching experiment, we assume that the student is observing and monitoring his own interactions, via reflection, and abducting potential accommodations to both activity and operational structures - both with and outside of awareness. The case of Erlwanger's Benny illustrates how students might form conjectures via patterns in tasks that were intended to be merely incidental (Erlwanger, 1973). The interactions and observations of the teaching experiment and the student's construction of mathematics are contemporaneous, and the construction of the

mathematics of students is the product of on-going reflection on the part of the teacher-researcher. Anticipation of the form of the mathematics of students can be self-fulfilling because, while teachers are suggesting tasks and asking questions to test particular hypotheses, students are constructing mathematics.

Fischer and Biddell (2006) distinguish between the *form* of a Piagetian (1970b) structure – an abstraction used to describe a static characteristic of the structure – and the structure itself – which is necessarily dynamic.

Dynamic structuralism offers an alternative to static conceptions of structure, starting with the recognition of the complexity inherent in human psychological development and the central role of the person in constructing dynamic systems of action and thought. Instead of trying to eliminate or get beyond the complexity of relations among systems, dynamic structuralism uses the tools of contemporary developmental science to analyze patterns in the complexity—how the constructive activity of human agents leads to new relations among systems of action and thought. (Fischer & Biddell, 2006, p. 315)

Dynamic structuralism emphasizes the Piagetian notion (1970b) that structures are not static forms – they permit intelligible change. Because the potential changes within structures are constrained, there are developmental patterns in the variations that can be modeled. Fischer (2009) models developmental sequences by measuring the complexity of *skills*, which are “capacities to act in an organized way in a specific context” (Fischer & Bidell, 2005, p. 321). The level of complexity and the activities of particular skills varies by context and individual. The objects of thought develop in tiers reminiscent of stages of reflective abstraction (Piaget 2001) - they are first limited to actions; later representations; and finally abstractions. The structuring of thought begins with single actions, which become mappings, and progress to systems of mappings that then become objects of the actions of the next tier (Fischer & Bidell, 2006; cf., Sfard, 1991).

Achieving a higher level of complexity does not preclude observations of less-organized systems of action at times, depending on the context of the observations. Furthermore, the trajectory of the development of a higher level of complexity in an individual's skills is dependent on the amount of social-contextual support. Accordingly, Fischer and Biddel model two upper-limits on the complexity of students' skills on particular tasks: functional (without support, everyday level), and optimal (primed, supported) performance. The gap between these two upper limits is "robust, and cannot be removed by simply increasing training, practice, or motivation" (Fischer & Bidell, 2005, p. 354).

The current study, in its focus on modeling the variations in students' units coordinating activity, is generally consistent with the notions of dynamic structuralism. The propensity plots model a developing, changing units coordinating structure that is never fixed and is dependent on interaction. The results extend the theory by providing a mechanism for modeling how specific types of interactions contribute to the development of units coordinating structures beyond levels of support. By modeling the dynamics of students' units coordinating activity as complicit with the level of perturbation, the contextual change, and a teacher's decisions, the current study incorporates notions of *dynamic supports*. This permits a more fine-grained analysis of the variability within the models of students' development beyond functional/optimal upper limits.

Extensions

It remains to be seen whether and how the approaches to modeling students' mathematics introduced in this study may be useful outside of radical constructivist teaching experiments involving units coordination. It may be that only aspects of the

models are useful; for example, the idea of modeling units coordination structures as a propensity, rather than the specific chunking and algorithmic formulation. It may be that the idea of characterizing the level of perturbation by the units coordinating activity is not useful outside of teaching experiments, and that characteristics of tasks' demands (e.g., Stein, Grover, & Henningsen, 1996) are more useful for teaching in classrooms. I do expect the models that were constructed during this study may be useful in teacher education programs to communicate to classroom teachers an expectation of variability in students' mathematical activity, particularly regarding fractions instruction.

Fractions instruction

Several researchers (e.g., Lamon, 2007; Norton & Wilkins, 2011; Steffe, 2010). have documented that unitizing and multiplicative reasoning are requisite for more robust understandings of fractions. The number of levels of whole number units acts as a constraint on students' understanding of fractions, as does the emphasis of school-based instruction solely upon a part-whole interpretation of fractions common to many countries, including the United States (Charalambous, Delaney, Shu, & Mesa, 2010). A part-whole interpretation can be used to assimilate fractional words, such as "three-fifths," to mean three parts from a whole consisting of five parts. If a student (incorrectly) assimilates "one-third" or "one-fifth" from such language within a PUFS (Norton & Boyce, 2013a; Olive & Vomvoridi, 2006), this might be discouraged as "regression" by a teacher focused on students' assimilation of fractional language associated with figural representations, rather than taken as a possible sign of progress in unitizing a fraction. By modeling the variability in students' units coordinating activity, I

hope to lay a foundation for the creation of curricula and assessments that incorporate such variations as part of student's mathematical development.

Numerical and quantitative reasoning

Consideration of units coordination as a structure may also be useful in relating the development of students' *numerical* schemes and their *quantitative* schemes (Thompson, 2011). Thompson (2011) defines *quantification* as a "process of conceptualizing an object and an attribute of it so that the attribute has a unit of measure and the attribute's measure entails a proportional relationship with the unit" (p. 37). With consideration of units coordination as a structure, one can conceive of quantitative reasoning as the utilization or application of a units coordination structure for conceptualizing relationships between quantities.

Quantitative findings from cognitive psychology also support the assertion that units coordination structures may act to constrain future mathematics learning. Siegler, Thompson, and Schneider (2011) found that middle grades' students' understanding of the magnitude of a fraction, as measured by the relative accuracy of representation upon a number line, related significantly more strongly with students' math achievement test scores than did measures of students' proficiency with fractions arithmetic procedures. Similar findings with whole numbers suggest commonalities in students' development. "As with whole numbers, ...[r]elations between fractions magnitude representations and mathematics achievement test scores extend beyond their common relation to arithmetic knowledge" (p. 290).

Commonalities in results stemming from research of students' quantitative reasoning might be explained by consideration of the necessity that students must *unitize*

to form a quantity and apply/construct a units coordination structure to/for *that unit*, rather than *extract* a numerical unit from a situation and apply an existing structure for that numerical unit. Ellis (2007) distinguished between reasoning quantitatively and reasoning with numerical patterns *extracted* from a real-world situation. Ellis' (2007) study focused on comparing the types of generalizations students made. She hypothesized that students' generalizing ability would be differentiated by whether they were reasoning quantitatively or solely numerically, but she instead found that the difference in generalizing was marked by students' construction of an "emergent ratio" between two units.

The distinction affecting students' generalizing can therefore be described not in terms of quantitative reasoning versus number-pattern reasoning, but instead in terms of whether students reasoned with two quantities (or numbers), or whether students constructed and reasoned with a third, emergent-ratio quantity. (Ellis, 2007, p. 471)

The connection between students' units coordinating structures and quantitative reasoning has also been suggested by researchers studying mathematics beyond introductory algebra. Moore (2013) describes how pre-calculus students' construction of arc-length as a quantity coordinating circumference, arc measure, and radius allowed them to form more robust understandings of relationships between degree measure and radian measure. Extending the construct of "emergent-ratio" to include units that vary continuously, students' construction of *covariation* as an *emergent quantity* has also similarly been found as a distinguishing characteristic for students' reasoning with functions (e.g., Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Thompson, 2010). It may be that a similar modeling of a student's propensity to *co-vary* units could be useful for understanding the commonalities between units coordination and quantitative reasoning.

Final Remarks

In this study, I set out to describe what happens during a teaching experiment, using a subjectivist probabilistic approach to model sequences of inferences of units coordinating activity across contexts. The approach was somewhat useful for answering my first research question, “What are the relationships between a student’s units coordinating activities across numerical situations such as counting sequences, fractions, whole number multiplication, and signed integer addition? The current study supported the *reorganization hypothesis* (Steffe, 2001), in the findings with Justin and Austin that engaging in units coordination with whole numbers immediately prior to engaging in units coordination with fractions supported units coordination in the latter context. This may or may not have been identifiable without the plots of units coordinating activity, and in the last teaching experiment, in which the conjecture was tested, there was moderate support. By segmenting teaching sessions more finely, the relationship may become more salient. In a follow-up study, I hope to include the Dylan data in a search for a new segmenting approach.

With regard to my second research question, “What sequences of ecological changes facilitate growth in the number of levels of a student’s units coordinating structures,” I feel that the modeling approach can be improved as well. I was able to interpret sequences of ecological changes very broadly, as gradients in levels of perturbation. The attempt to incorporate context into the algorithm for computing the propensities did not improve the model with the Austin data. With the Austin data, the numerical context was usually fractions. I think that a different approach might be more useful for describing teaching experiments involving multiple numerical contexts. By

constructing a baseline algorithm, I can compare the results with the teacher-researcher inferences in multiple ways. Longer teaching experiments may be necessary to make inferences of the growth in the number of levels of units that are coordinated.

Regarding the last, over-arching research question, “How does the modeling of a students’ propensity for assimilating an n -level structure contribute to an understanding of how units coordination structures are constructed?,” I feel that the answer is that modeling propensities provided a form for the dynamics of a constructivist teaching experiment that is generally consistent with radical constructivism (von Glasersfeld, 1995) and its teaching experiment methodology (Steffe & Thompson, 2000). Developing methods for improving the reliability of the chunking and units coordination inferences will help in the effort to fit the dynamics with others’ observations. As a first step, I hope to collaborate with other researchers using the data from these teaching experiments, to see how the qualitative analysis of other witness-observers matches with the output of the model.

An open question is how consideration of the analysis of the construction of units coordination structures may be useful for understanding students’ mathematics beyond middle-school. It would be helpful to first see if the patterns between the units coordinating activity and levels of perturbation observed with sixth-grade males hold under other conditions. An initial step is the analysis of the units coordinating activity of Maddie. There has been evidence of the constraints students in algebra face writing equations if they have not constructed requisite levels of units (Hackenberg & Lee, under review). Future work could analyze the units coordinating structures of older students, to

see how and whether these constraints and difficulties in algebra (and beyond) may be mitigated by engaging in units coordinating activity in other contexts.

REFERENCES

- Anghileri, J. (2006). Scaffolding practices that enhance mathematics learning. *Journal of Mathematics Teacher Education*, 9, 33-52.
- Behr, M., Harel, G., & Post, T. A., & Lesh, R. (1992). Rational number, ratio, and proportion. In D. A. Grouws (Ed.), *Handbook for research on mathematics teaching and learning*, (pp. 296–333). New York: Macmillan Publishing Company.
- Behr, M., Harel, G., & Post, T. A., & Lesh, R. (1994). Units of quantity: A conceptual basis common to additive and multiplicative structures. In B. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 123-176). Albany, NY: SUNY Press.
- Boyce, S. (2013). A stochastic modeling of a student's units coordinating activity. In M. Martinez, & A. Castro Superfine (Eds.) *Proceedings of the 35th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 1216 – 1223). Chicago: University of Illinois at Chicago.
- Boyce, S., & Norton, A. (2012). Examining the transition from a part-whole to partitive understanding of fractions. In L. R. van Zoest, J.-J. Lo, & J. L. Kratky (Eds.), *Proceedings of the thirty-fourth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (p. 1111-1114).
- Boyce, S., & Norton, A. (in review). *Co-construction of fractions schemes and units coordinating structures*.

- Bransford, J. D., & Schwartz, D. L. (1999). Rethinking transfer: A simple proposal with multiple implications. *Review of Research in Education*, 3(24), 61-100.
- Bush, R. R., & Mosteller, F. (1953). A stochastic model with applications to learning. *The Annals of Mathematical Statistics*, 24(4), 559 – 585.
- Charalambous, C. Y., Delaney, S., Hsu, H., & Mesa, V. (2010). The addition and subtraction of fractions in the textbooks of three countries: A comparative analysis. *Mathematical Thinking and Learning*, 12(2), 117-151.
- Cobb, P. (2000). Conducting teaching experiments in collaboration with teachers. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 307–330). Mahwah, NJ: Lawrence Erlbaum Associates.
- Cobb, P., & Steffe, L. P. (1983). The constructivist researcher as teacher and model-builder. *Journal for Research in Mathematics Education*, 14, 83-94.
- Confrey, J. (1994). Splitting, similarity, and rate of change: A new approach to multiplication and exponential functions. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 293-332). Albany, NY: SUNY Press.
- Confrey, J., & Lachance, A. (2000). A research design model for conjecture-driven teaching experiments. In R. Lesh & E. Kelly (Eds.), *New methodologies in mathematics and science education*. Dordrecht, The Netherlands: Kluwer.
- Confrey, J., & Smith, E. (1995). Splitting, covariation, and their role in the development of exponential functions. *Journal for Research in Mathematics Education*, 66-86.

- Cox, D. R., & Mayo, D. G. (2007). Objectivity and conditionality in frequentist inference. In D. G. Mayo & A. Spanos (Eds.) *Error and inference: recent exchanges on experimental reasoning, reliability, and rationality of science* (pp. 276 – 304). Cambridge : Cambridge University Press.
- diSessa, A. A. (1993). Toward an epistemology of physics. *Cognition and Instruction*, 2/3, 105-225.
- diSessa, A. A., & Sherin, B. L. (1998) What changes in conceptual change? *International Journal of Science Education*, 20, 1155-1191.
- Dubinsky, E. (1991). Reflective abstraction in advanced mathematical thinking. In D. Tall (Ed.), *Advanced Mathematical Thinking*, (pp 95 -123). Dordrecht: Kluwer Academic Publishers.
- Ellis, A. B. (2007). The influence of reasoning with emergent quantities on students' generalizations. *Cognition and Instruction*, 25, 439 – 478.
- Erlwanger, S. H. (1973). Benny's conception of rules and answers in IPI mathematics. *Journal of Children's Mathematical Behavior*, 1(2), 7-26.
- Fischer, K. W. (2009). Mind, brain, and education: Building a scientific groundwork for teaching and learning. *Mind, Brain, and Education*, 3(1), 3-16.
- Fischer, K. W., & Bidell, T. R. (2006). Dynamic development of action, thought, and emotion. In R. M. Lerner (Ed.), *Handbook of child psychology. Vol 1: Theoretical models of human development* (6th ed., pp. 313-399). New York: Wiley.
- Gauch, H. G. (1993). Prediction, parsimony, and noise. *American Scientist*, 81(5), 468-478.

- Gelman, R. (1972). Logical capacity of very young children: number invariance rules. *Child Development*, 43(1), 75-90.
- Glaser, B. G. (1965). The constant comparative method of qualitative analysis. *Social Problems*, 12(4), 436-445.
- Goodson-Espy, T. (1998). The roles of reification and reflective abstraction in the development of algebraic thought: Transitions from arithmetic to algebra. *Educational Studies in Mathematics*, 36, 219-245.
- Gravemeijer, K. (1999). How emergent models may foster the constitution of formal mathematics. *Mathematical Thinking and Learning*, 1(2), 155-177.
- Hackenberg, A. J. (2007). Units coordination and the construction of improper fractions: A revision of the splitting hypothesis. *Journal of Mathematical Behavior*, 26, 27-47.
- Hackenberg, A. J. (2010a). Mathematical caring relations in action. *Journal for Research in Mathematics Education*, 41(3), 236-273.
- Hackenberg, A. J. (2010b). Students' reasoning with reversible multiplicative relationships. *Cognition and Instruction*, 28(4), 383-432.
- Hackenberg, A. J., & Lee, M. Y. (2012). Pre-fractional middle school students' algebraic reasoning. In L.R. Van Zoest, J. -J. Lo, & J. L. Kratky (Eds.). *Proceedings of the 34th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp 943 – 950). Kalamazoo, MI: Western Michigan University.
- Hackenberg, A. J., & Lee, M. (in review). How does students' fractional knowledge influence equation writing? *Journal for Research in Mathematics Education*.

- Hackenberg, A. J., & Tillema, E. S. (2009). Students' whole number multiplicative concepts: A critical constructive resource for fractions composition schemes. *Journal of Mathematical Behavior*, 28, 1-18.
- Hershowitz, R., Schwarz, B. B., & Dreyfus, T. (2001). Abstraction in context: Epistemic actions. *Journal for Research in Mathematics Education*, 32(2), 195-222.
- Heinz, S. (2011). *Mathematical Modeling*. New York: Springer-Verlag
- Hunting, R. P., Davis, G., & Pearn, C. A. (1996). Engaging whole-number knowledge for rational-number learning using a computer-based tool. *Journal for Research in Mathematics Education*, 27(3), 354-379.
- Inhelder, B., & Piaget, J. (1958). The growth of logical thinking from childhood to adolescence. New York: Basic Books.
- Kieren, T. E. (1994) Multiple views of multiplicative structure. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 387-397). Albany, NY: SUNY Press.
- Koichu, B., & Harel, G. (2007). Triadic interaction in clinical task-based interviews with mathematics teachers. *Educational Studies in Mathematics*, 65, 349-365. doi: 10.1007/s10649-006-9054-0
- Izsák, A. (2005). "You have to count the squares": Applying knowledge in pieces to learning rectangular area. *The Journal of the Learning Sciences*, 14(3), 361-403.

- Izsák, A., Jacobson, E., de Araujo, Z., & Orrill, C. H. (2012). Measuring mathematical knowledge for teaching fractions with drawn quantities. *Journal for Research in Mathematics Education*, 43(4), 391-427.
- Lamon, S. J. (1994). Ratio and proportion: Cognitive foundations in unitizing and norming. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 89-122). Albany, NY: SUNY Press.
- Lamon, S. J. (1996). The development of unitizing: Its role in children's partitioning strategies. *Journal for Research in Mathematics Education*, 27 (2), 170-193.
- Lamon, S. J. (2007). Rational numbers and proportional reasoning: Toward a theoretical framework for research. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning*, (pp. 629–667). Charlotte, NC: Information Age Publishing.
- Lave, J. (1993). Situated learning in communities of practice. In L. B. Resnick, J. M. Levine, & S. D. Teasley (Eds.), *Perspectives on socially shared cognition* (pp. 63–82). Washington, DC: American Psychological Association.
- Learning Transformation Research Group (2012). *CandyFactory Educational Game*. Retrieved from <http://ltrg.centers.vt.edu/projects/games/apps/candyFactoryUniversal.htm>
- Learning Transformation Research Group (2013). *CandyDepot Educational Game for iPad*. Retrieved from <http://ltrg.centers.vt.edu/projects/games/apps/candyDepot.htm>
- Lobato, J. (2006). Alternate perspectives on the transfer of learning: History, issues, and challenges for future research. *The Journal of the Learning Sciences* 15(4), 431-449.

- Lobato, J., & Siebert, D. (2002). Quantitative reasoning in a reconceived view of transfer. *The Journal of Mathematical Behavior*, 21(1), 87-116.
- Mack, N. K. (2001). Building on informal knowledge through instruction in a complex content domain: Partitioning, units, and understanding multiplication of fractions. *Journal for research in mathematics education*, 32(3), 267-296.
- Martin, T., & Schwartz, D. L. (2005). Physically distributed learning: Adapting and reinterpreting physical environments in the development of fraction concepts. *Cognitive Science*, 29, 587-625.
- Mooney, D. D., & Swift, R. (1999). *A course in mathematical modeling*. Washington, D. C. : Mathematical Association of America
- Moore, K. C. (2013). Making sense of measuring arcs: a teaching experiment in angle measure. *Educational Studies in Mathematics*, 83, 225 – 245.
- Monaghan, J., & Ozmantar, M. F. (2006). Abstraction and consolidation. *Educational Studies in Mathematics*, 62, 233-258.
- Norton, A. (2004). Students' conjectural operations. (Doctoral dissertation). Retrieved from <http://www.libs.uga.edu/etd>
- Norton, A. (2008). Josh's operational conjectures: Abductions of a splitting operation and the construction of new fractional schemes. *Journal for Research in Mathematics Education*, 39(4), 401-430.
- Norton, A., & Boyce, S. (2013a). A cognitive core for common state standards. *Journal of Mathematical Behavior*, 32, 266-279.

- Norton, A., & Boyce, S. (2013b). Coordinating $n+1$ levels of units. In M. Martinez, & A. Castro Superfine (Eds.) *Proceedings of the 35th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 144 – 147). Chicago: University of Illinois at Chicago.
- Norton, A., & D'Ambrosio, B. S. (2008). ZPC and ZPD: Zones of teaching and learning. *Journal for Research in Mathematics Education*, 39(3), 220-246.
- Norton, A., & Wilkins, J. L. M. (2009). A quantitative analysis of children's splitting operations and fraction schemes. *The Journal of Mathematical Behavior*, 28(2), 150-161.
- Norton, A., & Wilkins, J. L. M. (2010). Students' partitive reasoning. *The Journal of Mathematical Behavior*, 29(4), 181–194.
- Norton, A., & Wilkins, J. L. M. (2012). The splitting group. *Journal for Research in Mathematics Education*, 43(5), 557-583.
- Norton, A., & Wilkins, J. L. M. (2013). Supporting students' construction of the splitting operation. *Cognition and Instruction*, 31(1), 2-28.
- Olive, J., & Steffe, L. P. (2001). The construction of an iterative fractional scheme: The case of Joe. *The Journal of Mathematical Behavior*, 20(4), 413–437.
- Olive, J., & Vomvori, E. (2006). Making sense of instruction on fractions when a student lacks necessary fractional schemes: The case of Tim. *Journal of Mathematical Behavior* 25(1), 18–45.
- Pantziara, M., & Philippou, G. (2012). Levels of students' "conception" of fractions. *Educational Studies in Mathematics*, 79, 61-83.

- Peirce, C. S. (1935). *Collected papers of Charles Sanders Peirce*. C. Hartshorne, P. Weiss, & A. Burkes, (Eds.) Cambridge, MA: Harvard University Press.
- Piaget, J. (1964). *The child's conception of number* (C. Gattegno & F. M. Hodgson, Trans.) London: Routledge & Paul.
- Piaget, J. (1970a). *Genetic epistemology* (E. Duckworth, Trans.). New York: Columbia University Press.
- Piaget, J. (1970b). *Structuralism* (C. Maschler, Trans.). New York, NY: Basic Books. (Original work published 1968)
- Piaget, J. (2001). *Studies in reflecting abstraction* (R. L. Campbell, Trans.). New York: Taylor & Francis. (Original work published 1977)
- Piaget, J., Inhelder, B., & Szeminska, A. (1960). *The child's conception of geometry*. New York: Basic Books. (Original work published 1948)
- Popper, K. R. (1959). The propensity interpretation of probability. *The British Journal for the Philosophy of Science*, 10(37), 25-42.
- Rappold, A. G., Lavine, M., & Lozier, S. (2007). Subjective likelihood for the assessment of trends in the ocean's mixed-layer depth. *Journal of the American Statistical Association*, 102, 771-780.
- Saldanha, L., & Liu, Y. (2014). Challenges of developing coherent probabilistic reasoning: Rethinking randomness and probability from a stochastic perspective. In E. Chernoff & B. Sriraman (Eds.), *Probabilistic Thinking*. Dordrecht: Springer.

- Saxe, G. B. (2004). Practices of quantification from a socio-cultural perspective. In A. Demetriou & A. Raftopoulos (Eds.), *Cognitive Developmental Change* (pp. 241 – 263). New York: Cambridge University Press.
- Seel, N. M. (1999). Educational diagnosis of mental models: Assessment problems and technology-based solutions. *Journal of Structural Learning and Intelligent Systems*, 142(5), 153-185.
- Saenz-Ludlow, A. (1994). Michael's fraction schemes. *Journal for Research in Mathematics Education*, 25(1), 50–85.
- Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as two sides of the same coin. *Educational Studies in Mathematics*, 22, 1-36.
- Sfard, A. (1998). On two metaphors for learning and the dangers of choosing just one. *Educational Researcher*, 27(2), 4-13.
- Siegler, R. S., & Crowley, K. (1991). The microgenetic method: A direct means for studying cognitive development. *American Psychologist*, 46(6), 606-620.
- Siegler, R. S., Thompson, C. A., & Schneider, M. (2011). An integrated theory of whole number and fractions development. *Cognitive Psychology*, 62(4), 273-296.
- Simon, M. A. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics Education*, 26, 114-145.
- Simon, M., Saldanha, L., McClintock, E., Akar, G. K., Watanabe, T., & Zembat, I. (2010). A developing approach to studying students' learning through their mathematical activity. *Cognition and Instruction*, 28 (1), 70 – 112.

- Simon, M. A., & Tzur, R. (2004). Explicating the role of mathematical tasks in conceptual learning: An elaboration of the hypothetical learning trajectory. *Mathematical Thinking and Learning*, 6(2), 91-104.
- Steffe, L. P. (1991). The constructivist teaching experiment: Illustrations and implications. In E. von Glasersfeld (Ed.), *Radical constructivism in mathematics education* (pp. 177-194). New York: Kluwer Academic Publishers.
- Steffe, L. P. (1992). Schemes of action and operation involving composite units. *Learning and Individual Differences*, 4(3), pp. 259-309.
- Steffe, L. P. (1994). Children's multiplying schemes. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 3-39). Albany, NY: SUNY Press.
- Steffe, L. P. (2001). A new hypothesis concerning children's fractional knowledge. *Journal of Mathematical Behavior*, 20, 1-41.
- Steffe, L. P. (2003). Fractional commensurate, composition, and adding schemes: Learning trajectories of Jason and Laura: Grade 5. *Journal of Mathematical Behavior*, 22(3), 237-295.
- Steffe, L. P., & Cobb, P. (1982). The constructivist researcher as teacher and model builder. *Journal for Research in Mathematics Education*, 14(2), 83-94.
- Steffe, L. P., & Olive, J. (2010). *Children's fractional knowledge*. Springer: New York.

- Steffe, L. P., & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of research design in mathematics and science education*, (pp.267–306). Mahwah, NJ: Lawrence Erlbaum Associates.
- Steffe, L. P., & Wiegel, H. G. (1996). On the nature of a model of mathematical learning. In L. P. Steffe, P. Nesher, P. Cobb, G. Goldin, & B. Greer (Eds.) *Theories of Mathematical Learning* (pp. 477 – 498). Mahwah, NJ: Lawrence Erlbaum Associates.
- Stein, M. K., Grover, B. W., & Henningsen, M. (1996). Building student capacity for mathematical thinking and reasoning: An analysis of mathematical tasks used in reform classrooms. *American Educational Research Journal*, 33(2), 455-488.
- Streefland, L. (1991). Fractions in realistic mathematics education. Dordrecht, The Netherlands: Kluwer Academic Publishers.
- Thompson, P. W. (1991). To experience is to conceptualize: A discussion of epistemology and mathematical experience. In L. P. Steffe (Ed.) *Epistemological foundations of mathematical experience* (pp. 260-281). New York: Springer.
- Thompson, P. W. (1993). Quantitative reasoning, complexity, and additive structures. *Educational Studies in Mathematics*, 25(3), 165-208.
- Thompson, P. W. (1994). The development of the concept of speed and its relationship to concepts of rate. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 179-234). Albany, NY: SUNY Press.
- Thompson, P. W. (2011). Quantitative reasoning and mathematical modeling. In L. L. Hatfield, S. Chamberlain & S. Belbase (Eds.), *New perspectives and directions for collaborative research in*

- mathematics education. WISDOMe Mongraphs (Vol. 1, pp. 33-57). Laramie, WY: University of Wyoming.
- Thompson, P. W., & Saldanha, L. A. (2003). Fractions and multiplicative reasoning. In J. Kilpatrick, W. G. Martin, & D. Schifter (Eds.), *A research companion to principles and standards for school mathematics* (pp. 95–113). Reston, VA: National Council of Teachers of Mathematics.
- Thorndike, E. L., & Woodworth, R. S. (1901). The influence of improvement in one mental function upon the efficacy of other functions. *Psychological Review*, 8, 247-261.
- Tversky, A., & Kahneman, D. (1993). Probabilistic reasoning. In A. I. Goldman (Ed.) *Readings in philosophy and cognitive science* (pp. 43-58). Boston: MIT Press.
- Tzur, R. (1999). An integrated study of children's construction of improper fractions and the teacher's role in promoting that learning. *Journal for Research in Mathematics Education*, 30(4), 390-416.
- Tzur, R., & Lambert, M. A. (2011). Intermediate participatory stages as zone of proximal development correlate in constructing counting-on: A plausible conceptual source for childrens' transitory "regress" to counting-all. *Journal for Research in Mathematics Education*, 42(5), 418-450.
- Tzur, R., & Simon, M. (2004). Distinguishing two stages of mathematics conceptual learning. *International Journal of Science and Mathematics Education* 2, 287-304.
- Tzur, R. (2007). Fine-grain assessment of students' mathematical understanding: participatory and anticipatory stages in learning a new mathematical conception. *Educational Studies in Mathematics*, 66, 273-291.

- Ulrich, C. (2012). Additive relationships and signed quantities. (Doctoral dissertation). Retrieved from <http://www.libs.uga.edu/etd>
- Ulrich, C. (2013). Additive versus multiplicative units coordination: An elaboration of existing frameworks and recent findings. In L. P. Steffe, K. C. Moore, & L. L. Hatfield (Eds.) *Epistemic algebraic students: Emerging models of students' algebraic knowing* (pp. 237-265). Laramie, Wyoming: University of Wyoming.
- Vergnaud, G. (1994). Multiplicative conceptual field: What and why? In G. Harel & J. Confrey (Eds.), *The Development of Multiplicative Reasoning in the Learning of Mathematics* (pp. 41-59). Albany, NY: SUNY Press.
- Vergnaud, G. (2009). The theory of conceptual fields. *Human Development*, 59, 83-94.
- van Geert, P. (2004). Dynamic modeling of cognitive development: Time situatedness, and variability. In A. Demetriou & A. Raftopoulos (Eds.) *Cognitive Developmental Change* (pp. 354 – 378).
- von Glasersfeld, E. (1981). An attentional model for the conceptual construction of units and number. *Journal for Research in Mathematics Education*, 12(2), 83-94.
- von Glasersfeld, E. (1991). *An exposition of constructivism: Why some like it radical*. New York: Springer.
- von Glasersfeld, E. (1995). *Radical constructivism: a way of knowing and learning*. Washington, D.C: Falmer Press.
- von Glasersfeld, E. (1998). Why constructivism must be radical. In M. Larochelle, N. Bednarz, & J. Garrison (Eds.), *Constructivism and education* (pp. 23-28). Cambridge: Cambridge University Press.

von Glasersfeld, E., & Steffe, L. P. (1991). Conceptual models in educational research and practice. *The Journal of Educational Thought*, 25(2), 91 – 103.

Vygotsky, L. S. (1978). *Mind in society: The development of higher psychological processes*. Cambridge, MA: Harvard University Press.

Wagner, J. F. (2006). Transfer in pieces. *Cognition and Instruction*, 24(1), 1 – 71.

Wagner, J. F. (2010). A transfer-in-pieces consideration of the perception of structure in the transfer of learning. *Journal of the Learning Sciences*, 19(4), 443-479.

Weisstein, E. W. (2014, May 10). Hypergeometric distribution. *MathWorld – A Wolfram Web Resource*. Retrieved from <http://mathworld.wolfram.com/HypergeometricDistribution.html>

White, P., & Mitchelmore, M. C. (2010). Teaching for abstraction: A model. *Mathematical Thinking and Learning*, 12(3), 205-226.

Wilkins, J. L. M., & Norton, A. (2011). The splitting loope. *Journal for Research in Mathematics Education*, 42(4), 386-416.

Wilkins, J. L. M., Norton, A., & Boyce, S. J. (2013) Validating a written instrument for assessing students' fractions schemes and operations. *The Mathematics Educator*, 22(2), 31-54.

APPENDIX A: Reflection on Modeling Propensities

In this appendix, I reflect on my process for modeling propensities that culminated in the formation of the baseline model discussed in Chapter 4. At first, I considered a general linear transformation that would operate upon *both* the propensities for transitioning from inferring units coordination activity or transitioning from *not* inferring units coordination activity *at each inference*. The four propensities are p : the propensity for not coordinating units given that not coordinating units was the previous state; q : the propensity for coordinating units given that coordinating units was the previous state; $1 - p$, the propensity for coordinating units given that not coordinating units was the previous state; $1 - q$, the propensity for *not* coordinating units given that coordinating units was the previous state.

Before I constructed a formula for updating propensities, I outlined several assumptions that would need to be satisfied. The first was that a calculated propensity should never be 0 or 1. The second was that a formula should be symmetric in the sense that I should make updates to p in the same manner that I make updates to q . The third assumption was that all propensities were equally likely before observation, what I call, *initial equipropensity*. Hence the four entries in the initial T_0 were each 0.5.

My first attempt for calculating the probability change formula involved adding the product of the two rows' entries to the entry and then subtracting the result to determine the other entry in the row. For each inference, only the first row or second row, not both, would change. For example, if the current propensity for coordinating units across chunks was 0.5 and coordinating chunks was the next inference, then q

would increase from 0.5 to $0.5 + 0.5*0.5 = 0.75$. After I implemented this program in Matlab, I found that the model did not discriminate between propensities well. For example, two consecutive initial inferences of *not* coordinating units in activity resulted in propensity of 0.9375. The next inference was also that of *not* coordinating units activity, so the formula resulted in a computed propensity for *not* coordinating units of 0.99609375. This was problematic for me in two ways. The first is that my *subjective* or personal probability conflicted with the propensities I was calculating. Would I really “feel” that there was only a $1 - .99609375$ chance that the next inference would be of units coordination? Perhaps. But if that unlikely inference were to occur, the change in p would barely decrease. It would go from .99609375 to $1 - [(1 - .99609375)*.99609375I + (1-.99609375)] = .99220276$. This most definitely did not fit with my “intuition.”

I realized that increases in p grew exponentially in p , whereas decreases in p grew exponentially in $1-p$. I needed to have a way to dampen the increase in p that better matched my intuition. I needed to have a way to allow propensities to “swing” back and forth with my inferences to better reflect the stability and/or instability in inferences. How could I do that without introducing parameters and maintaining symmetry between calculations of p and q ? My solution was to dampen the growth of increases in p and amplify the decreases in p by making the change in p a function of likelihood of the inference. If I made an inference that should increase p , I would add to p the product of p and the *minimum* of p and $(1-p)$.

Table A.1 displays a comparison of the results both with and without the adjustment. The adjusted method permits the value of p to swing more quickly toward 0.5 and makes it more difficult for the value of p to reach extreme values. Note that the

method of calculation used for q is identical: a sequence of 1110 would result in the values for q listed in Table A.1.

Table A.1

Comparison of Propensity Calculations of p upon inferring 0001

Original Method		Inference	Adjusted Method	
p	$1-p$		p	$1-p$
0.5	0.5	0	0.5	0.5
0.75	0.25	0	0.75	0.25
0.9375	0.0625	0	0.8125	0.1875
0.99609375	0.00390625	0	0.84765625	0.15234375
0.992202759	0.007797241	1	0.718521118	0.281478882

APPENDIX B: Matlab Functions

In this appendix, I provide the code for Matlab functions used for optimizating the baseline model, as outline in Chapter 5. I also provide output from the algorithm for both Austin and Justin.

Function B.1:

```
function [minval, T] = usewithfmin(u, l, coords, vars, inference, Beta)
%This function is used by samplemins in the optimization process to find
%the optimal Beta.
%u is the value that q should be above if inference(i) = 1.
%u should be between 0 and 1
%l is the value that p should be below if inference(i) = 1.
%l should be between 0 and 1
%coords is a nx1-vector of third-order units coordination inferences
with
%entries 0 or 1
%vars is an nxm matrix of contextual codes, with entries 0 or 1
%inference is an nx1 vector of second-order units coordination
inferences
%with values 0 or
%Beta is a vector of length m with real-valued entries
%minval returns the (negative) of the number of times the values of p or
q
%fit with units coordination inference(i)
%T returns the propensity values associated with a particular Beta
seed = [.5 .5 .5 .5];
```

```

    %these are the initial equipropensities, corresponding to p, 1-p, q,
    %1-q
alpha = .5;

    %alpha is the value of m in Algorithm 5.1, initially set to .5
n = length(coords);

    %n is the number of chunks for which units coordination was inferred
tempsum = 0;

    %this is (negative)the number of times the calculated values of
    %p and q fit with inference, given parameters u and l.
for i = 1:n
    T(i,:) = seed;

    %initialization of p, 1-p, q, 1-q
end

for j = 1:n-1
    if coords(j) == 0
        alpha=min(T(j, 1), T(j, 2));
        alpha = alpha^(dot(Beta,vars(j, :)) + 1);
        T(j+1, 1) = T(j, 1) * (1 - alpha) + alpha * (1 - coords(j+1) );
        T(j+1, 2) = 1 - T(j+1, 1);
        T(j+1, 3) = T(j, 3);
        T(j+1, 4) = T(j, 4);
    end

    if coords(j) == 1
        alpha=min(T(j, 3), T(j, 4));
        alpha = alpha^(dot(Beta,vars(j, :)) + 1);
        T(j+1, 1) = T(j, 1);
        T(j+1, 2) = T(j, 2);
    end
end

```

```

        T(j+1, 4) = T(j, 4) * (1 - alpha) + alpha * coords(j+1) ;
        T(j+1, 3) = 1 - T(j+1, 4);
    end

    if inference(j+1) == 1
        if T(j+1, 4) >= u || T(j+1, 2) >= 1
            tempsum = tempsum -1; %negative since minimizing
        end
    end
end

minval = tempsum;

end

```

Function B.2:

```

function [ minBeta, sampleminval, flag, output, averagebetas, minbetas,
maxbetas, numbestfit] = samplemins( u, l, coords, vars, inference,
samplesize )

%samplesize is typically set to 1000

%for other inputs to the function, see the description of usewithfmin.m

%sampleminval returns the smallest value of minval - the optimal number
of

    %third-order /second-order fitting chunks, using usewithfmin.m

%numbestfit returns the number of values of minBeta that achieved the

    %optimal value sampleminval

%minBeta returns the beta values obtained from searching for optimal
value

    %of usewithfmin when searching from a given seed value using

```

```

fminsearch

%averagebetas returns the arithmetic average of values of MinBeta that
    %resulted in an optimal value of sampleminval
%minbetas returns the arithmetic average of values of MinBeta that
    %resulted in an optimal value of sampleminval
%maxbetas returns the arithmetic average of values of MinBeta that
    %resulted in an optimal value of sampleminval
%flag and output are fminsearch variables that can be used to analyze
    %non-convergence of fminsearch


[x, numparametersforbeta] = size(vars);
%the number of parameters to optimize
%X is not used
r = 3.*rand(samplesize, numparametersforbeta);
%these are pseudo-random values between 0 and 3 (used in practice)
    %if search for beta is on [a, b] instead of [0, 3], then
    %r = a + (b - a) .*rand(samplesize,numparametersforbeta);
for i = 1:samplesize
    betaseed = r(i, :);
    [minBeta(i, :), minval(i), flag(i), output(i)] = fminsearch(@(Beta)
usewithfmin(u, l, coords, vars, inference, Beta), betaseed,
optimset('MaxIter', 10000, 'MaxFunEvals', 10000));
    %minval(i) is the optimal value of usewithfmin that is obtained via
    %fminsearch with the seed value betaseed(i, :).
end
%figure
%hist(minval)
sampleminval = min(minval);
goodrange = find(minval == sampleminval);

```

```
%these are the indices that correspond to optimal values
[x, numbestfit] = size(goodrange);
averagebetas = mean(minBeta(goodrange, numparametersforbeta));
maxbetas = -min (-minBeta(goodrange, numparametersforbeta));
minbetas = min (minBeta(goodrange, numparametersforbeta));
end
```

APPENDIX C: Second-order Analyses of Austin

In this appendix, I provide a subset of second-order analyses of the teaching experiment involving Austin and Jane, written by the teacher-researcher, Anderson Norton, that were marked as inferences of units coordination activity. The session and line numbers refer to the transcription I created and provided, which were matched to the nearest chunk.

Session 1: September 21, 2011

[see Lines 6-14; fraction of red rod out of blue rod]

Here, Austin seems to have assimilated the task into a PUFS, iterating the red rod (as a unit of 1) across the length of the blue rod and naming the relationship “sixths

[see Lines 46-56]

When asked what five-eighths means, Austin said it was four more than one eighth, but he did not specify the units of which five-eighths was four more. In line with our model of Austin, we can infer that they were simply units of 1, or even figurative units (“four more red bars”)—not fractional units. This also indicates that Austin was coordinating two levels of units in activity ...

[Lines 260-307]

Austin seems to think that the order could be one-half, if he were allowed to throw away a small part that would be left over after iterating twice. His way of operating on this task up to this point aligns with a PUFs.

[Lines 308-343]

“I partitioned in my head, and it would probably be sevenths.” This explanation suggests that Austin was mentally (and simultaneously) partitioning the whole into various numbers of parts until one of the parts within a partitioning lined up with the customer order.”

Session 3: October 5, 2011

[Lines 20-100]

The students are shown a one-fourth piece and asked to draw the length of the covered whole. ...the accuracy of Austin’s estimate relative to Jane’s fits our models for both

students. Austin may have mentally iterated the one-fourth piece, whereas Jane might have focused on numbers of parts at the expense of attending to iterations of a length.

[Lines 101-169]

After revealing the actual whole, Austin explained, “The reason I chose this as a fourth is I used the whole [lifts up whole] to measure it by fourths.” Once again, Austin’s language aligns with the use of a PUFS (measuring by fourths), while Jane’s seems tied to the specific context of the previous task. It seems that Austin is beginning to more explicitly coordinate fractional units with the whole.

[Lines 265-372]

When Jane gives Austin a “one-fifth” piece, Austin readily makes the whole by iterating the piece five times. Jane does not think he is right, possibly because his cuts were extended due to his placement of the knife blade. This episode provides further evidence that Austin had constructed a PUFS (at least as a participatory scheme).

Session 4: October 12, 2011

[Lines 145- 245]

For the next task Jane hands Austin a “ $\frac{3}{7}$ ” piece. Austin appeared to double the size of the given piece, but upon questioning from his classroom teacher, he claimed that the leftover part, between the given $\frac{3}{7}$ piece and the whole that he made, was $\frac{4}{7}$... Austin had been using a PUFS, or a whole number measurement scheme ...

[Lines 406-478]

Austin initially ... guessed $\frac{3}{5}$, after iterating the length of the customer order twice in the whole and realizing that it would not fit a third time. So, his difficulty in assimilating non-unit fractions as measures continues.

[Lines 479-523]

Janes difficulties also continued, concerning disembedding and the related ambiguity of fractional names. She called a one-fourth piece (one piece disembeeded from four in the whole) “one-fifth” for that reason.

Session 5: October 19, 2011

[

[Lines 329-415]

He did still line up two copies of the given (purple) bar, and referred to the whole as being “2 and 2.” This would fit with a PUFS in which four was being measured in twos. It also fits with his equivalent fraction explanations and his ability to coordinate two levels of units in activity.

Session 6: October 26, 2011

[Lines 396-509]

The teacher-researcher asked them to show one-fifth and then two-fifths of the whole. Austin chose the purple rod because it was commensurate with two red (one-fifth) rods. Jane chose an orange rod aligned with two yellow rods because “that’s five and two” [Line 448]. Seeing Austin’s response, she asks how the purple rod could be five [Line 481]. Austin’s response indicates that he was attempting to coordinate three levels of units, two at a time: “Well, I said this [red] is one-fifth, this [orange] is the whole.”

Session 7: November 2, 2011 (Austin only; Jane absent)

[Lines 5-86]

Austin iterated the two tan rods as if they were a single unit and exclaimed, “Three sets of these and it becomes six.” These actions indicate that Austin assimilated the task within a PUFS but still distinguished the two parts in the iterated unit and even established the whole a three of those composite units. ... He initially confused the units in naming the fraction (“It’s three-sixths”) before the teacher-researcher questioned him and he corrected himself. So, it seems that he was still coordinating two levels of units at a time: the three iterations of the given fraction within the given whole; the six parts created from those three iterations; and then the two units in the part within the six units within the whole.

Session 8: November 9, 2011

[Lines 255-327]

Next, the students built up non-unit fractions from a given $\frac{1}{9}$ piece [tan rod]. Austin had no difficulty with these tasks and even named the orange (10) rod as “one and one ninths”, correcting himself immediately after claiming it was “one and one tenths.”

Session 9: November 16, 2011

[Lines 112-188]

Austin's ability to coordinate multiple levels of unit was further demonstrated when he adjusted his answer for Steve starting with 6 nickels: "That would be 7 ... I subtracted 2."

[Lines 192-249]

Next, the teacher-researcher drew a block, told the students that it was one-sixth of a candy bar, and asked the students to imagine the length of $\frac{5}{6}$. Austin responded by imaginatively iterating the length of the given block five times. This response indicates some progress in his conceptualization of non-unit fractions, as measures of a unit fraction.

[Lines 395-478]

The next customer order was just a little smaller than the whole. Jane guessed $\frac{4}{5}$, and Austin guessed $\frac{3}{4}$. As the students checked their guesses, Austin adjusted them nearer and nearer to the whole: $\frac{5}{6}$, $\frac{6}{7}$, $\frac{7}{8}$, ... It might be premature to claim that Austin had constructed a PFS, but he has clearly made progress in treating non-unit fractions as measures. He is also coordinating three levels of units in activity—though probably still two at a time.

APPENDIX D: Coding of the Austin Data

Following is the data resulting from the coding of units coordination activity, context, and teacher-researcher inference pertaining to Austin. The column “Andy inference” includes ‘1’ where explicit indication of units coordination was inferred, and ‘-1’ where explicit indication of *not* coordinating units was inferred.

Chunk	Units Code	Partner	Teacher	Figurative	2	Reflection	Rote	Andy Inference
1	0			1				
2	1		1					
3	1							1
4	0	1						-1
5	1							
6	1							
7	1							1

8	1			1	1			
9	1	1						
10	1					1		
11	1							
12	1			1				
13	1							1
14	1		1					
15	1					1		1
16	1			1				
17	1							
18	0		1					
19	1					1		
20	0							
21	1						1	-1
22	1					1		

23	1			1				
24	1					1		
25	1			1				
26	1	1						
27	0	1						
28	0					1		
29	1					1		
30	1					1		
31	1		1					
32	1							
33	0	1						
34	0							
35	1							
36	0		1					
37	1			1				

38	0					1		-1
39	0							
40	0		1					
41	0					1		
42	1			1				
43	1	1						
44	0					1		
45	1						1	
46	1						1	-1
47	1	1						
48	0					1		-1
49	0					1		
50	0					1		
51	1			1				
52	1							

53	1			1				
54	1					1		
55	1						1	
56	0			1				
57	1	1		1				
58	1			1				1
59	1	1		1				
60	0			1				
61	1							
62	1			1				
63	1			1				
64	1			1	1			
65	1					1		
66	1			1		1		
67	0					1		

68	1			1				
69	0			1		1		
70	0							
71	1					1		
72	1					1		1
73	1	1			1			
74	1			1				1
75	1					1		
76	0					1		
77	1		1					
78	1		1			1		
79	1	1						
80	1					1		
81	0				1	1		
82	0							

83	1					1		
84	0		1					-1
85	1	1						
86	1					1		
87	1		1					
88	1			1				
89	1			1		1		
90	1	1			1			
91	0					1		
92	0					1		
93	1		1			1		1
94	1			1	1	1		
95	1				1	1		
96	1		1		1	1		
97	1	1	1					

98	0				1	1		
99	0					1		-1
100	0			1		1		
101	0			1		1		
102	1					1		
103	1	1						
104	1					1		
105	1				1		1	
106	1							
107	1			1				
108	1			1				
109	1				1			1
110	0			1		1		
111	1					1		
112	0						1	

113	1					1		
114	1			1			1	
115	0					1		
116	0			1				
117	0						1	
118	1		1					
119	0		1					
120	1		1					
121	0							
122	0					1		
123	0					1		
124	0					1		
125	0		1			1		
126	0			1				
127	0			1				

128	0							
129	1							
130	1							
131	1							
132	0					1		
133	1		1					
134	1		1					
135	1			1				
136	0		1					
137	1		1			1		
138	0							
139	1		1					
140	0							
141	1							
142	1							

143	1							
144	1					1		
145	1							
146	1	1						
147	1					1		
148	1					1		
149	0	1				1		
150	0					1		
151	1	1						
152	1					1		
153	1							
154	0					1		
155	1				1			1
156	0			1	1			
157	1				1			

158	0				1			
159	1				1			
160	0				1	1		
161	0				1			
162	1		1					
163	1							
164	1							
165	0	1						
166	1							
167	1							
168	0							
169	0	1						
170	0					1		
171	0	1						
172	0							

173	0					1		
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APPENDIX E: Coding of the Justin Data

Following is the data resulting from the coding of units coordination activity, context, and teacher-researcher inference pertaining to Justin. The column “Katy Inference” includes ‘1’ where explicit indication of units coordination was inferred. The column “yesorno” is the variable that is ‘1’ if any of the contextual descriptions columns were coded as ‘1’, and ‘0’ otherwise.

Chunk	Units Code	Follows teacher request for exp	Follows manip without marking	Follows partner response	Follows reflection on reflection	Katy Inference	yesorno
1	1						0
2	1		1			1	1
3	1	1				1	1
4	1						0
5	1					1	0

6	0						0
7	0						0
8	0	1			1		1
9	1						0
10	1						0
11	1		1				1
12	1		1				1
13	0						0
14	0						0
15	0						0
16	1	1					1
17	0						0
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19	1		1		1		1
20	1		1				1

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29	1						0
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31	1		1			1	1
32	1		1				1
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42	1						0
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516	1			1		1

APPENDIX F: Coding Memos and Notes

In this appendix, I provide memos and notes created as part of the coding process.

Coding Pertaining to Austin

Austin/Jane Transcription (July 28, 2013)

1. In CandyFactory game, CO = customer order; CB = candy bar; AC = assembled candy
2. In later sessions, coded [R: ...] for actions or speech by teacher ([W: for witness]) In earlier sessions, forgot the [] for Witness
3. Attended to Jane's actions that could have been seen or heard by Austin; left out those that took place while he was clearly not watching what she was doing.
4. Excluded portions of the dialogue that pertained solely to scheduling

Austin/Jane units coordination coding completed on 9/10/2013

One decision to make was whether to code coordination of 2 levels of units when 3 were required. I coded this as '0' if the units were not coordinated correctly, in the case that the teacher (or Austin) subsequently brought this to Austin's awareness.

Hypothesis: Coordinating units in activity in a known context leads to coordinating units in activity in an unknown context. In other words, if a student only coordinates with 2, or with whole numbers, or with manipulatives, etc., the student is more likely to coordinate in activity in a novel setting if that novel setting is prefaced by the known setting. Case in point: last session. Austin can coordinate 2 levels of whole number units – seemingly interiorized. He then follows with flawless units coordination in the remainder of the session. Contrast with session 7, in which Austin is perturbed at the beginning with units coordination questions basically showing that he didn't understand $\frac{2}{6}$ as $\frac{1}{3}$ outside of a figural scheme.

November 7, 2013

Reviewed 1st day's coding and continued adding codes for second day. Focusing on chunks without ? marks first, and making special note to characterize the chunks that Andy had mentioned specifically in his second-order model. One distinction I'm noticing

that may be important is whether Austin anticipates the relation between $1/n$ and $1/(n+1)$ or if he needs to make them (through iteration of figural material) in order for the relation to become salient.

November 23, 2013

Completed adding Andy's descriptions to the spreadsheet and then going back and open coding of the descriptions of the activity near his descriptions. I'm at a cross-roads now. I can either continue to code for Justin's units coordination, then fill in Katy's 2nd order model based on her dissertation, and then code for the descriptions of the activity near those descriptions. Or, I can form a more broad list of codes from just Austin's data first, before looking at Katy's data. The problem will be whether the codes I use for Austin are broad enough to withstand further collapsing. Since my goal is to complete a third-order model of Austin's units coordinating activity before completing a third-order model of Justin's units coordinating activity, I will focus first on identifying codes for Austin's activity that might be used to adjust his model.

One issue that is going to come up is that Andy's descriptions of Austin do not necessarily match with just a single chunk in my coding. There are some places, however, when he points out particular lines and gives strong indication of whether Austin is

coordinating/not. My plan is to focus on just this subset of Andy's second-order model when trying to fit particular variables to adjust the propensity formula to fit maximization of the propensity graph to match with his descriptions.

November 24, 2013

I found that the descriptions I gave were more theoretical, i.e., with regard to a particular scheme or operation. I also found that the description I gave for manipulates figurative material was often negative, i.e., in that it might be more useful in adjusting the model if I consider when he does not manipulate figurative material. But this may change for other students. I also did not distinguish between fractional contexts and whole number contexts, as the multiplicative context was so brief it will serve as a qualitative description. I kept the "coordination explicitly involves 2" because there were several instances where it seemed important to his fortuitous (or otherwise) activity. My next task is to go back and code each chunk for whether each of the codes was present.

Codes from Austin:

Ability to evaluate and reflect on earlier way of operating

Agreement with partners' weaker conception

Ability to evaluate and reflect on earlier ways of operating

Ability to evaluate and reflect on earlier ways of operating

Rote response

explicit verbal indication of iteration

explicit verbal indication of part-whole relation "out of"

characterizes the whole as totality

characterizes the whole as fixed amount (5/5)

Teacher suggests that Austin is confused.

correct response after iterating figural material

Further cited evidence of coordinating units in activity

Awareness of commensurate fractions only after physical material

Rote knowledge of comensurate fractions

Suggests iteration of unit (fraction) after watching partner's physical iteration

Awareness of commensurate fractions

Rote knowledge

Pointed out by Andy as evidence that he did not anticipate this being the same as $9/12$

Anticipates size relation between m/n and x/y after $1/y$ is shown

Incorrectly anticipates size relation between m/n and x/y before $1/y$ is shown; rather, anticipates size relation with $1/2$

Does not anticipate, but is able to reflect on activity to compare size of m/n and $m/(n+1)$

Rote knowledge

Reflects on partner's activity

Physically iterates posited fraction

Does not maintain the whole

Makes whole match the fraction $1/4$; rather than finding $1/4$ that would match the whole

Reflects on ease of his activity with $1/2$

"Measuring by fourths"

Andy suggests this chunk is indicative of Austin's PUFS strengthening, but could be 1-n whole number relation; use of the word "measuring"

Measures with 5ths by iterating a unit piece from left to right, appropriately.

Andy cites this as evidence of Austin's PUFS

using figural material, reflects on comparison of $1/n$ and $1/(n+1)$

Andy suggested this might be him thinking of an equivalent fraction

Conflates consideration of similar size with consideration equivalent name when prompted by teacher

When the whole is not visible, the fraction has no measurement meaning

Andy suggests this is indicative of struggling to coordinate the whole

Reflecting on student response

Reflecting on most recent response

Suggests part-whole interpretation is paramount

Refers to "how big" the piece is not it's name

Anticipates partitioning before iterating to form the whole from $\frac{2}{4}$

Andy marks this as either generalizing assimilation ($\frac{2}{4}$) or whole-number measurement scheme (4 as 2 twos)

Compares size of $\frac{m}{n}$ to $\frac{(m+1)}{n}$ with continuous figural material

Andy says, Either a PUFS or a "whole number measurement scheme"

Unable to anticipate partitioning before iterating with $\frac{3}{7}$ to form the whole

Andy points out that Austin iterated the 3 twice to get toward 7, realizing he couldn't iterate a third time.; he couldn't split to get $\frac{1}{7}$ and then iterate the 1 seven times.

Using continuous figural material, names fraction between $\frac{3}{7}$ and $\frac{4}{7}$ as 3 and $\frac{1}{2}$ 7

Reflecting on his own most recent response.

Andy points out that if they ask what the whole is before activity, it's not confusing. It's only after partitioning or iterating.

Reflects on partner's activity with figurative material

Reverses size comparison of $\frac{3}{5}$ and $\frac{2}{5}$ when iterative figurative material is not available

Andy suggests as evidence that $\frac{1}{5}$ was not iterable for Austin - $\frac{2}{5}$ and $\frac{3}{5}$ are not multiples of

1/5

Considers $2/5$ and $3/5$ the "same thing"

Reflects on partner's response

Andy points out evidence of Austin's awareness that she has exceed the whole

Correctly estimates size of $3/4$ outside of activity

Andy points out that $3/4$ is a reasonable guess, given that he has exhausted fifths and know proper fraction greater than $1/2$

Verbally explains commensurate fractions as figurative equivalence

Andy points out that these are figural and likely rote. He is not seeing $4/6$ as $2/3$, each of which containing $2/6$ most likely.

Does not assimilate $3/5$ as size less than $1/2$

Andy points out that Austin doesn't assimilate $3/5$ here, realizing the order was less than $1/2$

Did not maintain equally-sized pieces in drawing

Demonstrates commensurate fractions is known figuratively

Builds up to whole in activity.

Andy points this out as similar to previous ways of acting with non-unit fractions as measures

2 level units coordination

Prompted by teacher-witness to consider $\frac{3}{7}$ as 3 one-sevenths.

Andy points out this prompt, allowing for sequence of 2 level units coordinations

Coordinates with 2 and 2

Andy points out similarity with Twizzlers task; how less confident, but able to coordinate 2 levels of units at a time (both $\frac{1}{4}$ and $\frac{2}{4}$). Fits with a PUFS in which 4s were being measured with 2s.

Also points out that it matches with his equivalent fractions explanations.

Austin responds to Jane's activity

Andy says Jane initially solves the task and Austin assimilates her explanation

Austin responds to Jane's explanation

Andy says, Whereas Austin was able to assimilate Jane's explanation within a measurement scheme (PUFS) and identify the red rod as one-fourth of the whole

Involves 2; involves forming whole

Andy says Austin is unsure of himself and often corrects himself

Involves 2 ($\frac{2}{3}$)

Andy points out that whereas Austin used to confidently assimilate non-unit fractions as unit fractions, he now seems to be attending to the numerator

Austin is trying to form a whole from a non-unit fraction

Andy describes lines 254-320 as continuing to struggle with units; classroom teacher coaching him to use part-whole strategy

Austin is expressing a task to Jane

Andy points out as interesting that Austin reverts to a unit fraction when task-posing

Austin is explaining his reasoning to Jane

Andy points out as evidence of Austin trying to coordinate 3 levels of units, two at a time

Involves 2

Andy explains Austin interpreted within a PUFS but still distinguished the 2 parts in the iterated unit

Involves 2

Andy explains this as evidence that Austin was coordinating 3 2-level structures:

Involves 2

Task requires going beyond making the whole

Andy says is evidence that Austin still assimilates non-unit fractions within a PUFS

Austin responds to classroom teacher prompt

Andy points out that the classroom teacher is emphasizing a part-whole approach

Austin responds to classroom teacher prompt

Andy explains this is example of teacher-initiated complement strategy

Responding to Jane's activity

Involves 2

Andy explains that this is when Austin realized he was correct, after Jane shows 2 reds = 1 purple

Recalling teacher-initiated prompt from earlier session

Andy says that Austin had no difficulty with these tasks

Makes m/n in rapid succession

Andy points out that this correction was immediate.

Task requires making whole and going beyond whole

Andy points out that when Austin focuss on the measure of a unit fraction, he loses its numeracy, and when he focuses on the numeracy of a unit fraction, he loses its measure

Without figural material, whole number

Andy says that the first few tasks confirmed that Austin was MC2

Without figural material, whole number

Andy points out as particularly strong evidence of MC2

Introduces measurement

Andy points out as progress in Austin's in conceptualization of non-unit fractions as measures of a unit fraction

Andy points out that Austin's activity was very similar to Jane's here.

Adjusts after reflecting on his own response

Andy points out that Austin adjusts size correctly, but that his language is ambiguous.

Adjusts based on his own response

Andy points out that Austin adjusts the sizes correctly, and that he is coordinating 3 levels of units in activity.

I have narrowed down the above descriptions of Austin's environment to 6 codes:

1. Reflecting on activity/utterance of partner

2. Reflecting on activity/prompt of teacher

3. Manipulates figurative material

4. Coordination explicitly involves 2

5. Reflecting on previous response

6. Likely rote response

Coding Pertaining to Justin

January 20, 2014:

Coding of Justin/Lilly transcription

Coded 0 if named fraction as whole number

Coded 1 if added values together mentally; ? if on paper

Coded 0 if expressed need to draw partition marks to distinguish size

Coded 1 if size adjustment was appropriate in fractions tasks; 0 if not or if not sure

Coded 0 if only equipartitioned subset of the whole in the fractions tasks.

Codes from Katy's data

Pencil movements

Explicit counting

Mentions moving 1s around

Whole number addition/subtraction context

Immediacy of response; no prompt for thinking

Use of subtraction algorithm

Response to teacher question of how he was thinking

Response following his own reasoning, explaining his thinking

Fair Sharing

Physical paper manipulation

Iterates without making marks

Manipulates physical material to make size without drawing partitioning marks

Immediate response

Fairsharing

Requires “borrowing; place-value concepts”

Partitioning activity without drawing

Activity with manipulatives without actually drawing anything

Iterates size without making marks

Manipulates physical material

Looks at partner’s work

Direct response to teacher query about his thinking “how would you know.”

Direct reponse to teacher-directed task

Iteration of whole number structures

Request by teacher for him to use particular information.

Involves improper fraction.

Students did not assimilate a figurative material as intended.

Request to “get back to the original bar” teacher-suggested strategy

Describes correct strategy using informal, inconsistent language

Using figurative materials, iterates and adjusts fixed size without making additional marks

Immediacy of response

Response following teacher direct request for reflection

Response following teacher direct request for reflection

Moving numbers other than 1 around in additive situation

Response following teacher request for more exact answer

Response following teacher query about whether partner's response makes sense

Signed quantity tasks

Did not assimilate verbal task description as intended

Looks over at partner's paper

Written description of not assimilating verbal task description as intended

Response following teacher request for clarification of writing

Response following teacher direct request for reflection

Reflection on his own reflection

Response after partner asks him for his result

Erases one response and fills in with another

Uses pencil to count, but not write.

Looks at partner's paper.

Response to encouragement by teacher of correctness.

Looks at partner's paper.

Response following teacher request for evaluation of thinking.

Response to teacher request for evaluation of result.

Response following teacher direct request for reflection.

Uses physical manipulatives (coins)

Response following teacher direct request for reflection.

Response following partner erasing.

March 8, 2014:

I have narrowed down the codes from Justin's data to following 4:

1. Response follows teacher request for reflection/explanation of previous response

Examples: "How did you get that? Does that make sense? Can you explain how...."

2. Response follows manipulation of figurative material, indicating operative activity

Examples: Putting up fingers; iterating size.

3. Response follows perception of partner's response

Examples: Include instances following teacher asking if Justin agrees with Lilly immediately after her verbal response.

Instances in which he agrees/disagrees with Lilly's response.

Instances in which he glances at her paper and immediately writes something similar.

4. Response follows reflection on his own reflection

Example: Changes mind after hearing own explanation. "(initial response) Wait. (second response)".

Example: After iterating and seeing that x iterations don't reach the end, mutters, "it's going to be bigger than this."

APPENDIX G: Second-order Analyses of Dylan

In this appendix, I provide a subset of second-order analyses of the teaching experiment involving Dylan and Maddie, written by the teacher-researcher, Anderson Norton, that were marked as inferences of units coordination activity. The session and line numbers refer to the transcription I created and provided, which were matched to the nearest chunk.

Session 1: October 3, 2013

This episode was designed to engage student in iterating unit fractions, using fractions rods, and to see how they students might coordinate the resulting units (the unit fraction, the whole, and proper fractions of the whole). Free play with the rods led to the first sequence of questions.

[Lines 1-107]

Dylan uses the fact that 5 reds fit into orange, and that two yellows are the same length as five reds, to conclude that two yellows fit into orange. This implies that orange is five yellows for Dylan; he is coordinating two levels of units: orange as a unit of five reds.

[Lines 271-466]

Dylan seemed to assimilate the task as a request for three lime bars, relying on the figurative material to align three lime bars with a single blue bar. He seems to be coordinating two levels of units, at least in activity, but he does not attend to the sevenths at all.

[Lines 496-572]

When the teacher-researcher asked the students to make $\frac{1}{7}$ from a given $\frac{5}{7}$ orange rod, the students responded much as they had with the $\frac{3}{7}$ task. ... Dylan was not conflicted and simply focused on finding a piece that would fit into the orange rod five times. Evidence for Dylan's splitting operation was most apparent when he made $\frac{1}{7}$ from a given $\frac{5}{7}$ rod, presumably treating $\frac{5}{7}$ as a unit of 5.

Session 2: October 8, 2013

[Lines 1-142]

Dylan then asked the teacher to find the complement of the green rod relative to the orange rod. After the teacher responded with a purple rod or two red rods, the teacher and students reviewed what they had learned: that the red rod is one-fifth because it fits into the whole five times, that the green rod was three-fifths because it was three of those rods, and that the whole was two more. This segment established that both students could use figurative material to reason with proper fractions and the whole as a number of iterations of a unit fractional part, though they might be treating that part as a unit of 1 (orange rod as 5 reds, and green rod as 3 reds).

[Lines 346-528]

Dylan identified three units: the purple rod, the red rod, and the tan rod. He named the unit fractions they could make: one-half, one-fourth, one-eighth. He even determined the one-eighth relationship by coordinating units from the other two relationships: “Because 2 of these [tan] go into the red [left aligns below red] and since you do 4 times 2 it equals 8.”

Session 3: October 10, 2013

[Lines 543-]

Dylan seemed focused on producing the complement of $\frac{3}{5}$ in the whole—two fifths. This way of operating fits the part-whole scheme, which involves only two levels of units. It was unclear how Dylan found the two-fifths strip, but both students' solutions were accurate. Later in the episode, as the students were finding the unit, Dylan also seemed to split $\frac{3}{5}$ into three $\frac{1}{5}$ parts).

Session 4: October 15, 2013

[Lines 287-418]

He made adjoined two copies of the $\frac{4}{9}$ strip and then made a half of a half of a third copy and appended it. As such, he seemed to be reasoning with whole numbers: $4+4+(\text{half of half of } 4)$.

Session 5: October 17, 2013

[Lines 3-63]

During this episode, the students played CandyDepot, on Level 1. The first couple of orders involved only whole numbers, which the students easily solved. The tasks demanded no more than two levels of units coordination.

[Lines 113-137]

The next order was for $9/2$ of a box. Still reflecting on the previous task, Maddie did not get involved. Dylan dragged two boxes, then removed one of them and added a bundle. He seemed to be reasoning as he had in the previous task, focusing only on achieving 9 and presuming these units were bars (6 bars in a box and 3 bars in a bundle). Once again, this activity demanded no more than coordinating two levels of units.

[Lines 479-500]

The next order was Dylan's: $20/3$ of a bundle. He immediately began writing out multiples of 3 until he reached 18; he then began circling the numbers in pairs, apparently in reference to the two bundles per box. ... That makes his new approach all the more impressive: He was coordinating pairs of threes as pairs of bundles within boxes! Although his activity may have relied on figurative material (drawing and circling numbers), he was working with three levels of units: a box as a unit of two bundles, each of which was 3 thirds.

Session 6: October 23, 2013

[Lines 146-161]

Chips in $2/3$ of a box. Dylan, who had interpreted the previous question as a request for two thirds (rather than two boxes) answered immediately: 32, "because I knew $1/3$ was 16; 16 plus 16 is 32." His actions were consistent with his previous ways of operating,

especially his reference to adding $2\frac{1}{3}$ s rather than use of a multiplicative relation. He seemed to be coordinating two levels of units at a time: $\frac{2}{3}$ as $\frac{1}{3}$ plus $\frac{1}{3}$; and (previously) 48 as a unit of three 16s.

Session 7: October 29, 2013

[Lines 3-103]

Dylan eventually resolved the issue: “Yeah, because if each chip was 3 cents [raises 3 fingers] and there's 20 chips [staring ahead] 3 times 20 is 60”). He relied on figurative material to coordinate three levels of units and had trouble assimilating abstract units (such as cents, with no visible referent) in that context.

[Lines 104-203]

Rapid fire with Maddie, continued...

Teacher: “What if you had 40 cups; how many chips would there be?”

Surprisingly, Dylan responded first. Furthermore, his response indicated multiple units coordinations. He reasoned that 40 cups was 4 boxes, and each box was 20 chips, so there would be four 20s (80). He did have the benefit of access to figurative material and he did refer to “80 dollars” at first, but he generated his response in about 10 seconds. He explained that he had “counted to 4,” indicating that he counted by 10s four times to determine that there were 4 boxes. He later indicated that he had counted by 20s four times after determining that there would be 4 boxes. So, he seems to have performed two separate units coordinations: 40 as 4 units of 10); and then 80 as 4 units of 20. In each case, he was iterating a composite unit in activity, possibly relying on figurative material. He was also able to transition from one units coordination to the other, again possibly relying on figurative material. We should also note that, like Cody, Dylan worked through boxes to make units coordinations that would have been much simpler if he were able to consider the 40 cups as a unit of 40 units of 2 (the number of chips per cup).

[Lines 204-281]

Even though [Dylan] conflated the 3 80s with 4 80's, his goal to produce the value of 80 chips by iterating 80 is impressive. It suggests that he may have begun coordinating three levels of units in activity.

Session 9: November 6, 2013

[Lines 222-297]

When he later gets 11 bundles, Dylan appropriately adjusts for the new unit, suggesting he may be coordinating three levels of whole number units in activity: the composite units as before but with attention to the referent unit.

[Lines 471-506]

Dylan filled an order for $19\frac{1}{2}$ boxes by shipping 9 boxes and 1 bundle. Although this was incorrect, Dylan appeared to be attending to the referent unit now. As such he may have been coordinating three levels of units in activity: 19 as 9 units of 2 (sic; as whole boxes) with one unit (a half box) left over.

Session 10: November 7, 2013

[Lines 162-199]

As Dylan began pouring water from the half tablespoon into the cup, to determine their relationship, Maddie interrupted him, “You know what you should have done [D]? Figure out how many go into $[1/2 \text{ cup}]$ and then times it by 2. That would be a lot easier.” Dylan seemed confused at first, until Maddie explained further: “The half a cup. See how many of those go into here, instead of having to fill up that whole thing. Then times whatever you get by 2.” Then, Dylan realized that he could just double the previously recorded Tbsp/cup relation (8) to get 16 ... Dylan was able to follow Maddie’s reasoning once he translated it into whole number units and operate from known quantities: the unknown quantity was 2 units of the known quantity of 8. Furthermore, he seemed to understand that if a half spoon fit into the half cup 8 times, then the full unit (spoon) would fit into the half cup half as many times (4); he subsequently reasoned that the full unit would fit into the full cup twice as many times as that (8). This demonstrates reversibility in Dylan’s coordination of whole units, which would require coordinating three levels of whole number, in activity at least. However, the fact that he had to operate sequentially suggests that these three levels were not interiorized. So, he seemed to be coordinating three levels of whole units in activity.

Session 11: November 12, 2013

[Lines 355-418]

Dylan's way of operating is consistent with a PUFS (PFS?). He made three fifths as three iterations of a piece that fit into the whole five times: $\frac{3}{5}$ as three units of a $\frac{1}{5}$ piece. This involves coordinating three levels of units (the unit fraction, the whole, and the proper fraction) in activity.

[Lines 595-822]

In the next task—93 bars--which Maddie was solving, Dylan counted by 4s to 20 and then counted 4 20s to 80, reasoning that each 20 was five 4s. As such, once again he was coordinating at least three levels of units in activity: 80 as 4 units of 20, each of which was 5

units of 4 (boxes). In fact, to coordinate all of these units, at the least, he must have assimilated 20 as a unit of 5 units of 4 (boxes). If he did not rely on boxes as figurative material to collapse the final unit, this would imply he had assimilated three levels of units!