Robust Control for Hybrid, Nonlinear Systems

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(ABSTRACT)

We develop the robust control theories of stopping-time nonlinear systems and switchingcontrol nonlinear systems. We formulate a robust optimal stopping-time control problem for a state-space nonlinear system and give the connection between various notions of lower value function for the associated game (and storage function for the associated dissipative system) with solutions of the appropriate variational inequality (VI). We show that the stopping-time rule can be obtained by solving the VI in the viscosity sense. It also happens that a positive definite supersolution of the VI can be used for stability analysis. We also show how to solve the VI for some prototype examples with one-dimensional state space. For the robust optimal switching-control problem, we establish the Dynamic Programming Principle (DPP) for the lower value function of the associated game and employ it to derive the appropriate system of quasivariational inequalities (SQVI) for the lower value vector function. Moreover we formulate the problem in the L_2 -gain/dissipative system framework. We show that, under appropriate assumptions, continuous switching-storage (vector) functions are characterized as viscosity supersolutions of the SQVI, and that the minimal such storage function is equal to the lower value function for the game. We show that the control strategy achieving the dissipative inequality is obtained by solving the SQVI in the viscosity sense; in fact this solution is also used to address stability analysis of the switching system. In addition we prove the comparison principle between a viscosity subsolution and a viscosity supersolution of the SQVI satisfying a boundary condition and use it to give an alternative derivation of the characterization of the lower value function. Finally we solve the SQVI for a simple one-dimensional example by a direct geometric construction.

To my parents, Tavorn and Somjit Chudoung and in the memory of my grandmother, Yead Khaumpirun

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Chapter 1

Introduction

In this dissertation, we develop the robust control theory for hybrid, nonlinear state-space systems, namely the robust control theory of stopping-time or switching-control problems.

Nonlinear H_{∞} control

Robust (or H_{∞}) control has been studied intensely in the past decade [6, 7, 8, 9, 10, 12, 34, 45, 48, 49, 55]. The notions of *dissipative systems* and *differential games* have been used to solve the nonlinear H_{∞} -control problems. A general nonlinear system used for the formulation of the robust control (or H_{∞} -control) problem is given as

$$\Sigma \begin{cases} \dot{y} = f(y, a, b) \\ w = g(y, a, b) \\ z = h(y, a, b). \end{cases}$$
(1.1)



Figure 1.1: Standard control configuration

Here $y(\cdot) \in \mathbb{R}^n$ denotes the system state, $a(\cdot) \in A \subseteq \mathbb{R}^p$ denotes the control input, $b(\cdot) \in B \subseteq \mathbb{R}^m$ denotes the deterministic unknown disturbance on the system, or exogenous input,



Figure 1.2: Standard closed-loop configuration

 $w(\cdot) \in \mathbb{R}^d$ denotes the observed output, and $z(\cdot) \in \mathbb{R}$ denotes the cost variable. Usually in applications $h(y, a, b) = |\bar{h}(y, a)|^2$ where $\bar{h} : \mathbb{R}^n \times A \to \mathbb{R}^r$.

The H_{∞} -control problem is to find a controller $K : w(\cdot) \to a(\cdot)$ and the smallest number $\gamma^* > 0$ so that the closed-loop system (Σ, K) of Figure 1.2 is γ^* -dissipative and stable. We say that the closed-loop system (Σ, K) is γ -dissipative if there exist an $\gamma > 0$ and a nonnegative real-valued function S with S(0) = 0 such that

$$\begin{cases} \int_0^T h(y_x(s, a, b), a(s), b(s)) ds \le \gamma^2 \int_0^T |b(s)|^2 ds + S(x) \\ \text{for all } x \in \mathbb{R}^n, \text{ all } b \in L^2 \text{ and all } T \ge 0. \end{cases}$$
(1.2)

We have used the notation $y_x(\cdot, a, b)$ for the unique solution of $\dot{y} = f(y, a, b)$ with $y_x(0, a, b) = x$. The inequality (1.2) corresponds to an input-output system having L^2 -gain at most γ where $\int_0^T h(y_x(s, a, b), a(s), b(s)) ds$ replaces the L^2 -norm of the output signal over the time interval [0, T].

In general it is hard to solve the H_{∞} control problem, so we instead solve the suboptimal H_{∞} control problem. The suboptimal H_{∞} control version asks for a preassigned attenuation level $\gamma > \gamma^*$ to find a controller K with some information structure so that the closed-loop system (Σ, K) is γ -dissipative and internally stable, i.e. stable for any initial condition subject to zero disturbance $b \equiv 0$. Notice that the solvability of the suboptimal H_{∞} control problem will depend on the choice of the attenuation level γ . In fact, the main goal of the H_{∞} control theory is to find the admissible controller so that the L^2 -gain of the closed-loop system from disturbances b to outputs z is minimized. That is to compute the real number

 $\gamma^* = \inf\{\gamma > 0: \text{ the suboptimal } H_{\infty} \text{ control problem with index } \gamma \text{ can be solved}\}.$

The optimal disturbance attenuation γ^* is a measure of the robustness of the system and, on the other hand, of the influence that the disturbances have on its behavior. Unfortunately, there is no general effective method to our knowledge, to characterize the value γ^* . In practice, the solution of optimal H_{∞} control problem may be approximated by an iteration of the suboptimal H_{∞} control problem (successively decreasing γ to the optimal disturbance attenuation level γ^*).

Dissipative systems

In an L^2 -gain/dissipative framework, the notion of a storage function for a dissipative system plays a prominent role (see [36, 45]). The dissipative system theory was developed by Willems in the 70's [52]. For a given control $a_*(\cdot) = K(y(\cdot))$, the state feedback closed-loop system

$$\begin{cases} \dot{y} = f(y, a_*, b) \\ \bar{z} = \bar{h}(y, a_*) \end{cases}$$
(1.3)

is said to be *dissipative* with respect to the supply rate $\gamma^2 |b|^2 - |\bar{z}|^2$, for $\gamma > 0$ if there is a nonnegative function $S : \mathbb{R}^n \to \mathbb{R}$, called the *storage function*, such that for all initial condition $y_x(0) = x \in \mathbb{R}^n$, all $T \ge 0$ and all disturbance functions $b \in L^2[0, T]$

$$S(y_x(T)) \le S(x) + \int_0^T [\gamma^2 |b(s)|^2 - |\bar{z}(s)|^2] ds.$$
(1.4)

For brevity we have written $y_x(\cdot)$ instead of $y_x(\cdot, a_*, b)$. The inequality (1.4) is called the *dissipation inequality*. It expresses the fact that the "stored energy" $S(y_x(T))$ at any future time T is at most equal to the sum of the stored energy $S(x) = S(y_x(0))$ at present time 0 and the total externally supplied energy $\int_0^T [\gamma^2 |b(s)|^2 - |\bar{z}(s)|^2] ds$ during the time interval [0, T]. Hence there can be no internal "creation of energy"; only internal dissipation of energy is possible. The dissipative inequality gives

$$S(x) \geq \sup_{b(\cdot), T} \{ \int_0^T [|\bar{z}(s)|^2 - \gamma^2 |b(s)|^2] ds + S(y_x(T)) \}$$

$$\geq \sup_{b(\cdot), T} \{ \int_0^T [|\bar{z}(s)|^2 - \gamma^2 |b(s)|^2] ds \}.$$

We then define the *available storage* as

$$S_a(x) = \sup_{b(\cdot), T} \{ \int_0^T [|\bar{z}(s)|^2 - \gamma^2 |b(s)|^2] ds \}.$$
 (1.5)

The available storage $S_a(x)$ can be interpreted as the maximal "energy" which can be extracted from the system starting in an initial condition y(0) = x. It is well-known that the system is dissipative if and only if the available storage is finite for every initial condition. Moreover if the available storage is finite, then it is again a storage function and is characterized as the minimal storage function.

In general, the supply rate can be any function $s : \mathbb{R}^d \times B \to \mathbb{R}$. Two important choices of the supply rate are

$$\hat{s}(\bar{z}, b) = b^T \bar{z}$$
 and $\bar{s}(\bar{z}, b) = \frac{1}{2} [\gamma^2 |b|^2 - |\bar{z}|^2], \ \gamma > 0.$

The closed-loop system (1.3) is *passive* if it is dissipative with respect to the supply rate $\hat{s}(\bar{z}, b)$ and it has L_2 -gain $\leq \gamma$ if it is dissipative with respect to the supply rate $\bar{s}(\bar{z}, b)$.

If a storage function S is continuously differentiable (C^1) , we see that (1.4) is equivalent to

$$-DS(x) \cdot f(x, a_*, b) - |\bar{h}(x, a_*)|^2 + \gamma^2 |b|^2 \ge 0, \text{ for all } x, b.$$
(1.6)

The inequality (1.6) is called the *differential dissipative inequality*, and it is usually easier to check than (1.4) since we do not have to compute the system trajectories. It is easy to check that the closed-loop system (1.3) is dissipative with the supply rate $\gamma^2 |b|^2 - |\bar{z}|^2$ and with the C^1 storage function S if and only if there is a nonnegative C^1 solution S to the Hamilton-Jacobi-Bellman inequality

$$H^*(x, DS(x)) \ge 0,$$

where the Hamiltonian function H^* is given as

$$H^*(x,p) = \inf_{b} \{-p \cdot f(x,a_*,b) - |\bar{h}(x,a_*)|^2 + \gamma^2 |b|^2\}.$$

It follows from the theory of dynamic programming that if S_a is C^1 , then it is a solution of the Hamilton-Jacobi-Bellman equation (HJBE)

$$H^*(x, DS_a(x)) = 0.$$

In fact, S and S_a can be used as Lyapunov functions in stability analysis (see [45]). However storage functions are usually not everywhere differentiable, nor even continuous. Thus we should want to have a solution of the HJBE in "weak" form. The concept of viscosity solutions comes to play a role.

Viscosity solutions

The theory of viscosity solutions, invented in the early 80's by M.G. Crandall, P.L. Lions and L.C. Evans, provides a PDE framework for dealing with the lack of smoothness of the value functions arising in dynamic optimization problems.

Chapter 1. Introduction

For the definition of a viscosity solution, we consider the first order PDE of the form

$$F(x, Du(x)) = 0 \quad x \in \mathcal{X} \tag{1.7}$$

where \mathcal{X} is an open domain of \mathbb{R}^n and the Hamiltonian F = F(x, r) is a continuous real valued function on $\mathcal{X} \times \mathbb{R}^n$. A function $u \in USC(\mathcal{X})$ is a viscosity subsolution of (1.7) if for any $\varphi \in C^1(\mathcal{X})$

$$F(x_0, DU(x_0)) \le 0$$
 (1.8)

at any local maximum point $x_0 \in \mathcal{X}$ of $u - \varphi$. Similarly, a function $u \in LSC(\mathcal{X})$ is a viscosity supersolution of (1.7) if for any $\varphi \in C^1(\mathcal{X})$

$$F(x_1, DU(x_1)) \ge 0$$
 (1.9)

at any local minimum point $x_1 \in \mathcal{X}$ of $u - \varphi$. A locally bounded function $u : \mathcal{X} \to \mathbb{R}$ is a non-continuous viscosity solution of (1.7) if u^* is a viscosity subsolution of (1.7) and u_* is a viscosity supersolution of (1.7), where u^* and u_* are respectively the upper and lower semicontinuous envelope of u defined as

$$u^*(x) = \limsup_{y \to x} u(y)$$
$$u_*(x) = \liminf_{y \to x} u(y).$$

In the definition of a subsolution we can always assume that x_0 is local strict maximum point for $u - \varphi$ (otherwise replace $\varphi(x)$ by $\varphi(x) + |x - x_0|^2$). Moreover since (1.8) depends on only the value of $D\varphi$ at x_0 , it is not restrictive to assume that $u(x_0) = \varphi(x_0)$. Similar remarks apply to the definition of a supersolution. Geometrically, this means that the validity of the subsolution condition (1.8) for u is tested on the smooth functions "touching from above" the graph of u at x_0 . For more details of viscosity solution theories we refer to [10] and [22].

Differential games

There is a close connection between H_{∞} control and differential games. The theory of (twoperson zero-sum) differential games has a close connection with the worst case analysis of a controlled system with a disturbance. It is well-known that the existence of a solution of the H_{∞} problem can be established by studying the lower value function of the associated game. The theory of differential games started at the beginning of the '60s with the work of Isaacs (see [35]) in the U.S.A. and of Pontryagin (see [41]) in the Soviet Union. The main motivation at that time was the study of the military problems.

For the two-person zero-sum differential games, we consider the system dynamics of the form

$$\dot{y} = f(y, a, b), \quad y(0) = x,$$

where the admissible control $a(\cdot)$ and admissible disturbance $b(\cdot)$ are respectively the control functions of the first and second player. In addition there is a cost functional $J(x, a(\cdot), b(\cdot))$

$$J(x, a, b) = \int_0^\infty l(y_x(s), a(s), b(s))e^{-\lambda s} ds.$$

where l is a running cost and λ is a positive discount factor. J is the cost which the first player wants to minimize and the second player wants to maximize. In other words, -J is the cost the second player has to pay, so the sum of the costs of the two players is zero, which explains the name "zero-sum". We also give an *information pattern* for the two players, namely *nonanticipating strategy* where one player knows the current and past choices of the control made by his opponent (for the other information pattern, see Chapter VIII in [10]). We say that the strategy α (respectively, β) for the first (respectively, second) player is nonanticipating (or causal) if for any t > 0 and $b(s) = \bar{b}(s)$ (respectively, $a(s) = \bar{a}(s)$) for all $s \leq t$ implies $\alpha[\bar{b}](s) = \alpha[\bar{b}](s)$ (respectively, $\beta[\bar{a}](s) = \beta[\bar{a}](s)$) for all $s \leq t$. The *lower value* and *upper value* of a game are then respectively given as

$$v(x) = \inf_{\alpha} \sup_{b} J(x, \alpha[b], b)$$

$$u(x) = \sup_{\beta} \inf_{a} J(x, a, \beta[a]).$$

It is well-known that

$$v(x) \le u(x) \quad \text{for all } x, \tag{1.10}$$

which justifies the terms "lower" and "upper". The inequality (1.10) is not obvious at first glance and requires a proof. Actually, one might first guess the opposite inequality since inf sup \geq sup inf if they are taken over the same sets. We say that the value of the game exists if v(x) = u(x). It is well-known that if the *Isaacs' condition* or *solvability of the small game* holds, i.e.,

$$\sup_{a \in A} \inf_{b \in B} F(x, p, a, b) = \inf_{b \in B} \sup_{a \in A} F(x, p, a, b), \quad \text{for all } x, \ p \in \mathbb{R}^n,$$
(1.11)

where $F(x, p, a, b) = -p \cdot f(x, a, b) - l(x, a, b)$, then the value of the game exists. More generally the condition (1.11) is equivalent to saying that the two-person zero-sum static game over the sets A and B with payoff F has a saddle point. We refer to [10], [12] and references therein for the details of (two-person zero-sum) differential games and for detailed treatment of some advanced topics.

We view the suboptimal H_{∞} control setup as a game with the payoff functional

$$\bar{J}_T(x,a,b) = \int_0^T \bar{\ell}(y_x(t,a,b),a(t),b(t)) dt$$

with

$$\bar{\ell}(y, a, b) = [h(y, a, b) - \gamma^2 |b|^2],$$

where the disturbance player seeks to use b(t) and T to maximize the payoff while the control player seeks to use a(t) to minimize the payoff. The lower-value function \bar{V} for this game is then defined to be

$$\bar{V}(x) = \inf_{\alpha} \sup_{b, T} \bar{J}_T(x, \alpha_x[b], b)$$
(1.12)

where the supremum is over all nonnegative real numbers T and L^2 -disturbance signals $b(\cdot)$, while the infimum is over all nonanticipating control strategies α . The main result concerning this game-theoretic approach to the nonlinear H_{∞} -control problem is: under minimal smoothness and boundedness assumptions on the problem data f and h, the lower value function \bar{V} is the minimal possible storage function, and, if also continuous, is characterized as the minimal viscosity-sense supersolution of the Hamilton-Jacobi-Bellman-Isaacs Equation (HJBIE)

$$\bar{H}(x, D\bar{V}(x)) = 0$$

where we have set

$$\bar{H}(x,p) = \inf_{b} \sup_{a} \{-p \cdot f(x,a,b) - h(x,a,b) + \gamma^{2} |b|^{2} \}.$$

When \bar{V} is smooth and \bar{H} has a saddle point, the optimal control law can be derived from the lower value function $\bar{V}(x)$ via $a_*(x(t)) = a_*(x(t), D\bar{V}(x(t)))$ where

$$a_*(x,p) = \arg \sup_{b} [\inf_{b} \{-p \cdot f(x,a,b) - h(x,a,b) + \gamma^2 |b|^2\}].$$

For details, we refer to [45], [48] and [49].

For practical applications, one must next solve (HJBIE). One approach is via the connection with the stable invariant manifold for the associated Hamiltonian flow (see [45]); it is possible to turn this into a numerical procedure analogous to the method of bicharacteristics for general PDEs (see [23]). Alternatively, one can find an approximation to the true lower value function by computing the lower value function for a discrete-time robust control problem which approximates the true continuous-time problem—see Appendix A in [10] for one illustration of this idea, and [20] and [37] for discrete approximation methods. The approximation methods in [20] and [37] are different. In [20], the author introduced a discrete system and defined the corresponding H_{∞} norm. He characterized the H_{∞} norm of the discrete system in terms of the finite difference inequality (the discrete version of the Hamilton-Jacobi-Bellman inequality in this context) and a property of the value function associated to an ergodic problem, and showed that the ergodic cost index can be used to characterize the discrete H_{∞} norm (similar results for a continuous-time problem were discussed in [27] and [28]). The author then proved the convergence of the discrete norm to the continuous norm and the discrete ergodic cost index to the continuous one. In [37], the authors showed that the available storage was the limit of a corresponding finite-horizon storage function which is the unique solution of the time-dependent version of the Hamilton-Jacobi-Bellman equation. They then introduced the discrete versions of the Hamilton-Jacobi-Bellman equation corresponding to an finite-horizon problem for a controlled Markov chain (this finite-difference scheme was presented in [25] for approximation solution to dynamic programming arising in stochastic optimal control). While there are various methods for improving the efficiency of such numerical approaches to solving HJBIEs, for a state space of high dimension one quickly runs into the "curse of dimensionality". If one's true goal to solve H_{∞} control problem is more qualitative (stability with the satisfaction of some more qualitative performance specs), other approaches which avoid the solution of HJBIEs (such as the Lyapunov-based techniques developed in [38] and [31]) have now been proposed. See also [2] which summarized theories of the modern control and reviewed some recent books in control theory.

Motivation

In this dissertation we develop analogues of all these standard robust control ideas mentioned above for the case (1) of a robust stopping-time problem, and (2) of a robust switching-cost problem.

We formulate a robust optimal control stopping-time problem for a state-space system and give the connection between various notions of lower value function for the associated games (and storage function for the associated dissipative system) with the solution of the appropriate variational inequality (VI) (the analogue of the Hamilton-Jacobi-Bellman-Isaacs equation for this setting). We obtain results of the robust optimal stopping-time problem similar to the standard H_{∞} control problem. We also find the lower-value function W(x)(see(2.12)) explicitly for a prototype problem with one-dimensional state space by a simple, direct, geometric construction (see Section 2.5). The algorithm in Section 2.5 was developed by Professor Ball and the example was given by Professor Day. (Professor Day also provided the example for a robust optimal switching-control problem in Section 3.5.2.) I would like to thank both of them for permitting me to use this.

Our original motivation for study of robust optimal stopping-time problems was as a simpler prototype of the robust control problem with switching costs; the precise connection is explained in Section 3.5.1. However many real-world problems can be formulated as stoppingtime problems. For example an electricity company runs a system of hydroelectric and thermal power stations to provide electricity. This system must be managed to minimize the running cost. For the thermal power stations, at every moment it must be decided whether a shut down power station must be started up, or whether one in operation must be shut down. In stock management, after investors buy stocks, they have to watch those stocks carefully in order to make money. Either stocks go up or go down, they have to decide if they should hold or sell those stocks. Roughly speaking, a stopping-time problem is a management problem. Here *management problems* do not refer to just only business, but also energy, natural resources, information technology problems. For more examples of these problems, see [13]. Engineering and biological complex systems usually have interaction of different types of information which leads to *hybrid* systems, i.e., systems involving both continuous and discrete aspects. The theory of *hybrid control systems* have been developed rapidly (see [17], [18], [19], [39] [43] and references therein). In both engineering and biology systems, the robustness of the systems is essential because it guarantees that the designed systems do not change too dramatically where subjected to exogenous disturbances or modeling errors. In many real-world problems, the control systems have the *hybrid feature*, namely *switching-control*. That is the feature where the system dynamic changes abruptly in respond to a control command, usually with an associated cost. The following examples of switching-control problems are modified from [17]. In satellite control, we consider the system of the form

$$\begin{array}{lll} \dot{y}_1 = & y_2 \\ \dot{y}_2 = & Ca, \end{array}$$

where y_1 and y_2 are respectively angular position and angular velocity of the satellite, $a \in \{-1, 0, 1\}$ depending on whether the reaction jets are full reverse, off, or full on, and C is a given constant. In a manual transmission on a car, we consider

$$\begin{array}{rcl} y_1 &=& y_2 \\ \dot{y}_2 &=& \frac{-g(y_2/a) + b}{1+a}, \end{array}$$

where y_1 is the ground speed, y_2 is the engine RPM, $b \in [0, 1]$ is the throttle position, and $a \in \{1, 2, 3, 4\}$ is the gear shift position. The function g is positive for positive argument.

We formulate a robust optimal control problem for a general nonlinear system with finitely many admissible controls and with positive costs assigned to switching of controls. We give the connection between the lower value (vector) function for an associated game (or the storage function for an associated dissipative system) with the appropriate system of the quasivariational inequalities (the appropriate generalization of the Hamilton-Jacobi-Bellman-Isaacs equation for this context), and the characterization of the continuous lower value vector function as the minimal nonnegative continuous viscosity supersolution of the SQVI.

Our original motivation for robust switching-control problems arose from the problem of designing a real-time feedback control for traffic signals at a highway intersection (see [6], [7]), where the size of the cost imposed on switching can be used as a tuning parameter to lead to more desirable types of traffic-light signalization. Also a positive switching cost eliminates the chattering present in the solution otherwise. We discuss this example more in detail in "future work", Chapter 4.

Related work

Optimal stopping-time problems have a long history in probability theory. There is an introductory exposition in [26] and a more thorough treatment in [13], [30] and [46]. Just

as (deterministic) robust control has many analogies with classical stochastic control, our idea here can be viewed as developing a deterministic robust analogue of optimal stopping. A stochastic stopping-time game is formulated in [13, Section 2.9], but with both players having only the option to stop the system (as opposed to our setup with one player having an input-signal control and the other player having a stopping-time option). A deterministic formulation of an optimal stopping-time problem is discussed in Section III.4.2 of [10], but with a discounted cost rather than dissipation inequality and with no disturbance competing with the control as in the robust approach. Since there is a discount in the running cost, the authors were able to show that the value function is bounded and uniformly continuous, and is the unique viscosity solution of the appropriate variational inequality (VI). Infinitehorizon optimal switching-control problems are discussed in [10, Chapter III, Section 4.4] but with a discount factor in the running cost and no disturbance term. Differential games with switching strategies and switching costs for the case of finite horizon problems are discussed in [53] while the case of infinite-horizon with both control and competing disturbance but with a discount factor in the running cost are discussed in [54]. Again these authors were able to show that the vector value function is continuous and is the unique solution of the appropriate system of quasivariational inequalities (SQVI) because of the positive discount factor in the running cost.

Our derivation of the VI for the stopping-time problem and of the SQVI for the switchingcontrol problem is a direct application of the method of dynamic programming standard in control theory. The technical contribution here to the optimal stopping-time and switchingcost problems can be seen as parallel to that of Soravia in [48] for the nonlinear H_{∞} -control problem: to extend the game-theoretic, dynamic-programming approach to the infinitehorizon setting where, due to a lack of discount factor in the running cost, the running cost is not guaranteed to be integrable over the infinite interval $[0, \infty)$. This forces the introduction of the extra "disturbance player" T in (1.12), (2.12) and (3.11), and complicates many of the proofs (see also [32], [49] and [10, Appendix B] for later, closely related refinements of the nonlinear H_{∞} results). Due to a lack of positive discount factor and the presence of the extra disturbance player T, our lower-value function (see 2.12 for a stopping-time problem and 3.11 for a switching-control problem) probably in general is not continuous, and moreover cannot be characterized simply as the unique solution of the VI (and SQVI) as is the case for finite-horizon problems and problems with a positive discount factor.

We give two derivations of the minimality characterization of the lower value vector function for the switching-control problem. One is the direct argument by using the synthesis method of an optimal control. An alternative derivation of this characterization relies on a general comparison principle for viscosity super- and subsolutions of the SQVI. James ([36]) used this method to obtain a characterization of the storage function for the standard L^2 -gain problem. However for our situation, the comparison principle of the SQVI of the form we want is not in the literature. Thus we prove the comparison principle in Section 3.3.2 and apply it to obtain the characterization of the SQVI with a positive discount $\lambda > 0$ was given in [10, Section II.4.4]. However to the best of our knowledge, it is impossible to adapt the proof there directly to our setting, $\lambda = 0$. So our proof of the comparison principle is from scratch. The basic idea is now a standard technique in the theory of viscosity solutions (see [22]), namely: introduce an auxiliary function depending on some additional parameters of twice the number of variables which is equal to a function of interest on the diagonal, and then use the parameters to penalize the doubling the number of variables. A proof of the comparison principle for the Hamilton-Jacobi-Bellman case is carried out in detail using this technique in [10, Section III3.2].

This dissertation is organized as follows. Following the present chapter, Chapter 2 presents the robust stopping-time control problem, Chapter 3 presents the robust optimal switchingcontrol problem, and finally Chapter 4 presents the conclusions and future work. The material on Chapter 2 overlaps with [3], Chapter 3 except Section 3.3.2 and 3.3.3 with [4] and Section 3.3.2 and 3.3.3 with [5].

Chapter 2

Robust optimal stopping-time problems

In this chapter, we develop the theory of robust stopping-time problem for a general nonlinear state-space system.

2.1 Preliminaries

In this section we give assumptions, definitions and background material needed in subsequent sections. We consider the nonlinear system Σ_{st}

$$\Sigma_{st} \begin{cases} \dot{y} = f(y,b), \quad y(0) = x \in \mathbb{R}^n \\ z = h(y,b), \end{cases}$$
(2.1)

where $y(\cdot) \in \mathbb{R}^n$ denotes the state, $b(\cdot) \in B \subset \mathbb{R}^m$ denotes the deterministic unknown disturbance on the system, and $z(\cdot) \in \mathbb{R}$ denotes the cost function. Usually in the applications $h(y,b) = |\bar{h}(y)|^2$, where $\bar{h} : \mathbb{R}^n \to \mathbb{R}^p$. In addition we assume that we are given a positive stopping cost function $\Phi(y)$.

We make the following assumptions:

(A0) $0 \in B \subseteq \mathbb{R}^m$ and B is closed; $f : \mathbb{R}^n \times B \to \mathbb{R}^n$ and $h : \mathbb{R}^n \times B \to \mathbb{R}$ are continuous;

(A1) f and h are bounded on $B(0, R) \times B$ for all R > 0;

(A2) there are moduli ω_f and ω_h such that

$$|f(x,b) - f(y,b)| \leq \omega_f(|x-y|,R) |h(x,b) - h(y,b)| \leq \omega_h(|x-y|,R),$$

for all $x, y \in B(0, R)$ and R > 0, where a *modulus* is a function $\omega : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$

such that, for all R > 0, $\omega(\cdot, R)$ is continuous, nondecreasing and $\omega(0, R) = 0$;

(A3)
$$(f(x,b) - f(y,b)) \cdot (x-y) \le L|x-y|^2$$
 for all $x, y \in \mathbb{R}^n$ and $b \in B$;

(A4) $\Phi : \mathbb{R}^n \to \mathbb{R}$ is positive and continuous;

(A5) $h(x,0) \ge 0$ for all $x \in \mathbb{R}^n$.

In addition we assume that f(0,0) = 0 and h(0,0) = 0, so that x = 0 is an equilibrium point of the undisturbed system $\dot{y} = f(y,0)$.

Remark 2.1 Note that assumption (A1) eliminates the linear-quadratic case (where f is linear in y and b and h is quadratic in the components of y and b) if B is taken to be an entire Euclidean space \mathbb{R}^m ; if B is restricted to a compact subset, e.g., a large closed ball $B(0, R) \subset \mathbb{R}^m$, then the assumptions (A0)–(A5) do apply in the linear-quadratic case; this is sufficient for many applications. Alternatively one could use the reparametrization technique of Soravia (see [10, Appendix B] and [48]) to reduce the linear-quadratic case to an adapted problem where estimates which are uniform with respect to the disturbance $b \in B$ are satisfied.

For a specified gain parameter $\gamma > 0$ we define the *running cost* function

$$l(y,b) = h(y,b) - \gamma^2 |b|^2,$$
(2.2)

and the Hamiltonian function

$$H(y,p) = \inf_{b \in B} \{-p \cdot f(y,b) - l(y,b)\}.$$
(2.3)

Note that $H(y,p) < +\infty$ for all $y, p \in \mathbb{R}^n$ by (A1). Under assumptions (A0)-(A3), we can show that H is continuous. (The proof is similar to that in [10, page 106].) Let \mathcal{B} denote the set of locally square integrable functions $b : [0, \infty) \to B$. We consider \mathcal{B} to be the set of admissible disturbances. We look at trajectories of the nonlinear dynamical system

$$\dot{y}(s) = f(y(s), b(s)), \quad y(0) = x \in \mathbb{R}^n.$$
 (2.4)

Under the assumptions (A0), (A1) and (A3), for each $b \in \mathcal{B}$ and $x \in \mathbb{R}^n$ the solution of (2.4) exists uniquely for all $s \geq 0$. (The proof of this result is in Section III.5 of [10].) The solution of (2.4) will be denoted by $y_x(s, b)$, or briefly by $y_x(s)$ if there is no confusion. The basic estimates on y_x are the following (for the proofs see Section III.5 of [10]):

$$|y_x(t,b) - y_z(t,b)| \le e^{Lt} |x - z|, \quad t > 0$$
(2.5)

$$|y_x(t,b) - x| \leq M_x t, \quad t \in [0, 1/M_x],$$
(2.6)

$$|y_x(t,b)| \leq (|x| + \sqrt{2Kt})e^{Kt}, \quad t > 0,$$
(2.7)

for all $b \in \mathcal{B}$, where

$$M_x := \sup\{|f(z,b)| : |x-z| \le 1, b \in B\}$$

$$K := L + \sup\{|f(0,b)| : b \in B\}.$$

Let us introduce the notion of upper and lower semicontinuous envelope of a function U: $\mathbb{R}^n \to [-\infty, +\infty]$. These two new functions are, respectively,

$$U^{*}(x) := \limsup_{r \to 0^{+}} \{ U(z) : |z - x| \le r \}$$

$$U_{*}(x) := \liminf_{r \to 0^{+}} \{ U(z) : |z - x| \le r \}.$$

It is well-known that if U is locally bounded, then $U_* \in LSC(\mathbb{R}^n)$ and $U^* \in USC(\mathbb{R}^n)$. Let us now introduce the variational inequality (VI)

$$\max\{H(x, DU(x)), U(x) - \Phi(x)\} = 0, x \in \mathbb{R}^{n}.$$

Definition 2.2 A locally bounded function $U : \mathbb{R}^n \to [-\infty, +\infty]$ is a viscosity subsolution of the VI in \mathbb{R}^n if for any $\Psi \in C^1(\mathbb{R}^n)$

$$\max\{H(x_0, D\Psi(x_0)), \ U(x_0) - \Phi(x_0)\} \le 0$$
(2.8)

at any local maximum point $x_0 \in \mathbb{R}^n$ of $U^* - \Psi$. Similarly, U is a viscosity supersolution of the VI in \mathbb{R}^n if for any $\Psi \in C^1(\mathbb{R}^n)$

$$\max\{H(x_1, D\Psi(x_1)), \ U(x_1) - \Phi(x_1)\} \ge 0$$
(2.9)

at any local minimum point $x_1 \in \mathbb{R}^n$ of $U_* - \Psi$. Finally, U is a viscosity solution of the VI if it is simultaneously a viscosity subsolution and supersolution.

We now describe an alternative way of defining viscosity solutions of the VI by means of the semidifferentials instead of test functions. We define the superdifferential of U^* at x by

$$D^{+}U^{*}(x) := \{ p \in \mathbb{R}^{n} : \limsup_{y \to x} \frac{U^{*}(y) - U^{*}(x) - p \cdot (y - x)}{|x - y|} \le 0 \}$$

and the subdifferential of U_* at x by

$$D^{-}U_{*}(x) := \{ p \in \mathbb{R}^{n} : \liminf_{y \to x} \frac{U_{*}(y) - U_{*}(x) - p \cdot (y - x)}{|x - y|} \ge 0 \}.$$

The following Proposition can be proven by the arguments in [10, page 29 and 294].

Proposition 2.3 (i) A locally bounded function $U : \mathbb{R}^n \to [-\infty, +\infty]$ is a viscosity subsolution of the VI in \mathbb{R}^n if

$$\max\{H(x,p), U(x) - \Phi(x)\} \le 0$$

for all $x \in \mathbb{R}^n$ and all $p \in D^+U^*(x)$.

(ii) A locally bounded function $U : \mathbb{R}^n \to [-\infty, +\infty]$ is a viscosity supersolution of the VI in \mathbb{R}^n if

$$\max\{H(x,p), U(x) - \Phi(x)\} \ge 0$$

for all $x \in \mathbb{R}^n$ and all $p \in D^-U_*(x)$.

We shall need the following theorem in the exposition below. The proof is similar to that in [10, page 92], and hence will be omitted.

Theorem 2.4 Let Ω be an open subset of \mathbb{R}^n and $g : \Omega \times B \to \mathbb{R}$. Assume (A0), (A1), (A3), $g \in C(\Omega \times B)$ and g is bounded on $\Omega \times B$. Then for $U \in C(\Omega)$ the following statements are equivalent:

- (i) $\begin{array}{ll} U(y_x(\nu,b)) U(y_x(t,b)) \geq \int_{\nu}^{t} g(y_x(s),b(s)) \ ds, \\ for \ all \ b \in \mathcal{B}, \ x \in \Omega, \ 0 \leq \nu \leq t < \tau_x[b], \ where \ \tau_x[b] := \inf\{t \geq 0 : y_x(t,b) \notin \Omega\}; \end{array}$
- (ii) $\inf_{b \in B} \{-DU(x) \cdot f(x,b) g(x,b)\} \ge 0$ in Ω in the viscosity sense;
- (iii) $-\inf_{b\in B} \{-DU(x) \cdot f(x,b) g(x,b)\} \le 0$ in Ω in the viscosity sense.

Note that the equivalence of (i) and (ii) is also in [36] and of (i) and (iii) is in [8].

For each $b \in \mathcal{B}$ and $x \in \mathbb{R}^n$, a stopping rule τ associates a single time: $0 \leq \tau_x[b] \leq +\infty$. The essential nonanticipating (or causal) property of stopping rules is that for every $t \geq 0$, whenever two disturbances b and \tilde{b} agree up to t,

$$b(s) = b(s)$$
 for all $s \le t$

then

$$1_{[0,t]}(\tau_x[b]) = 1_{[0,t]}(\tau_x[b])$$
 for all x .

(We have set the notation $1_{[0,T]}$ for the indicator function with $1_{[0,T]}(\tau) = 1$ if $0 \leq \tau \leq T$ and 0 otherwise.) In other words, knowing the history of b(s) for $s \leq t$ is enough to answer the question of whether or not $\tau_x[b] \leq t$. We denote the set of stopping rules which have the nonanticipating property by Γ , i.e.

$$\Gamma := \{\tau : \mathbb{R}^n \times \mathcal{B} \to [0, +\infty] : \tau \text{ is nonanticipating} \}.$$

If $b \in \mathcal{B}$ is a disturbance, $x \in \mathbb{R}^n$ is an initial state and t > 0, then we may consider $y_x(t, b)$ as a new initial state imposed at the time t. If Γ has the additional property

$$\tau_x[b] = t + \tau_{y_x(t,b)}[b_t] \text{ for all } b \in \mathcal{B} \text{ and all } \tau \in \Gamma \text{ with } \tau_x[b] > t$$
(2.10)

(where we have set $b_t(s) = b(t+s)$), we shall refer to Γ as a set of *state-feedback stopping* rules. In this case, given that the system has continued running up to time t, the decision of whether to stop immediately at time t or to continue can be read off from the current value of the state $y_x(t, b)$.

2.2 Formulations

We consider a state-space system Σ_{st} with a positive stopping cost $\Phi(y)$. We set the cost of running the system up to time T with an initial condition x, disturbance b and stopping time τ to be the quantity

$$C_T(x,\tau,b) := \int_0^{T\wedge\tau} h(y_x(s,b),b(s)) \ ds + \mathbb{1}_{[0,T]}(\tau)\Phi(y_x(\tau,b)).$$

We have used the notation $T \wedge \tau$ for $\min\{T, \tau\}$. For a prescribed tolerance level $\gamma > 0$, we seek a function $U(x) \ge 0$ with U(0) = 0 and a stopping-rule $\tau \in [0, \infty]$ so that

$$\begin{cases} C_T(x,\tau,b) \le \gamma^2 \int_0^{T\wedge\tau} |b(s)|^2 \, ds + U(x) \\ \text{for all } b \in \mathcal{B} \text{ and all } T \ge 0. \end{cases}$$
(2.11)

The control decision at each moment of time is then whether to stop and cut one's losses (with penalty $\Phi(y_x(\tau))$) in addition to the accumulated running cost up to time T), or to continue running the system (including the possibility of never stopping the system before the disturbance stops it). In the *open loop* version of the problem, τ is simply a nonnegative extended real number. In the *state feedback* version of the problem, τ is a causal (or nonanticipating) function of the current state in the sense that one decides on whether to stop or continue at a given point in time t as a function of the state vector $y_x(\cdot, b)$ at time t. In the standard game-theoretic formulation of the problem, τ is taken to be a nonanticipating function of the disturbance b, i.e., one decides whether $\tau \leq t$ based solely on the information consisting of the initial state x and the past of the disturbance $b|_{[0,t]}$. The dissipation inequality (2.11) can then be viewed as the analogue of the closed-loop system (with L^2 -norm of output signal being taken to be $C_T(x, \tau, b)$ for each finite-time horizon [0, T]) having L^2 -gain of at most γ . A refinement of the problem then asks for the control τ which gives the best system performance, in the sense that the nonnegative function U(x) is as small as possible.

A closely related formulation is to view the stopping-time system as a game with payoff function

$$J_T(x,\tau,b) = \int_0^{T\wedge\tau} [h(y_x(s,b),b(s)) - \gamma^2 |b|^2] ds + \mathbb{1}_{[0,T]}(\tau) \Phi(y_x(\tau,b))$$

=
$$\int_0^{T\wedge\tau} l(y_x(s,b),b(s)) ds + \mathbb{1}_{[0,T]}(\tau) \Phi(y_x(\tau,b))$$

where the disturbance player tries to use $b(\cdot)$ and T to maximize the payoff, while the control player tries to use the stopping time τ to minimize the payoff. As we shall introduce a variation on this game below, we shall refer to this game as Game I. We define a *lower value* function for Game I as

$$W(x) = \inf_{\tau \in \Gamma} \sup_{b \in \mathcal{B}, \ T \ge 0} \left\{ \mathbb{1}_{[0,T]}(\tau_x[b]) \Phi(y_x(\tau_x[b])) + \int_0^{T \wedge \tau_x[b]} l(y_x(s), b(s)) \ ds \right\}$$
(2.12)

By construction, W(x) gives the smallest possible value which can satisfy (2.11) for some strategy τ .

We now introduce a variation on Game I which we shall call Game II. For the rules of Game II, the maximizing player no longer controls a cutoff time T but rather only the disturbance b, while the minimizing player is constrained to play only nonanticipating stopping-time rules $(x, b) \rightarrow \nu_x[b]$ with finite values $\nu_x[b] < \infty$ for all (x, b), and the payoff function is taken to be $J_{\infty}(x, \nu, b)$. In this formulation, the payoff is guaranteed to be finite due to $\nu_x[b] < \infty$ rather than from $T < \infty$. The lower value function for Game II is then given by

$$V(x) := \inf_{\nu \in \Delta} \sup_{b \in \mathcal{B}} \left\{ \Phi(y_x(\nu_x[b])) + \int_0^{\nu_x[b]} l(y_x(s), b(s)) \ ds \right\},$$
(2.13)

where $\Delta := \{\nu : \mathbb{R}^n \times \mathcal{B} \to [0, \infty) : \nu \text{ nonanticipating} \}$. From the L^2 -gain perspective, V(x) is associated with the desire to optimize the performance bound

$$\int_0^{\nu} h(y_x(s), b(s)) \, ds + \Phi(y_x(\nu)) \le \gamma^2 \int_0^{\nu} |b(s)|^2 \, ds + U(x),$$

(over finite-valued stopping rules ν).

For the purposes of comparison, we also introduce the *available storage function* $S_a(x)$ associated with a disturbance-input to cost-output system (with stopping-time options ignored)

$$S_a(x) := \sup_{b \in \mathcal{B}, \ T \ge 0} \left\{ \int_0^T l(y_x(s), b(s)) \ ds \right\}.$$
 (2.14)

The function $S_a(x)$ is associated with the desire to optimize the standard performance bound associated with L^2 -gain attenuation level γ for an input-output system (with all stopping options ignored)

$$\int_0^T h(y_x(s), b(s)) \, ds \le \gamma^2 \int_0^T |b(s)|^2 \, ds + U(x).$$

Under some technical assumptions, it is well-known that if locally bounded, this available storage function S_a is a viscosity solution in \mathbb{R}^n of the Hamilton-Jacobi-Bellman equation (HJBE)

$$H(x, DS_a(x)) = 0$$

where H is given by (2.3) and if S_a is also continuous, then it is characterized as the minimal, nonnegative, continuous viscosity supersolution of the HJBE.

In addition we introduce the notion of a *stopping-time storage function* S for a stopping-time system (with some particular stopping rule $(x, b) \to \tau_x[b]$ already implemented), namely, a nonnegative function $x \to S(x)$ such that

$$\begin{cases} 1_{(T,\infty)}(\tau_x[b])S(y_x(T,b)) - S(x) \\ \leq \int_0^{T \wedge \tau_x[b]} [\gamma^2|b(s)|^2 - h(y_x(s), b(s))] \ ds - 1_{[0,T]}(\tau_x[b])\Phi(y_x(\tau_x[b], b)) \\ \text{for all } b \in \mathcal{B} \text{ and } T \ge 0. \end{cases}$$
(2.15)

If we set $\tau_x[b] = \infty$ for all $x \in \mathbb{R}^n$ and $b \in \mathcal{B}$, we recover the notion of storage function associated with L^2 -gain supply rate. The control problem then is to find the stoppingtime rule $(x, b) \to \tau_x[b]$ which gives the best performance, as measured by obtaining the minimal possible S(x) as the associated storage function. This suggests that the stoppingtime available storage function $S_{st,a}$ (i.e., the minimal possible stopping-time storage function over all possible stopping rules) should be equal to the lower-value function W for Game I; we shall see that this is indeed the case with appropriate hypotheses imposed.

2.3 Main results

In this section we derive the main results concerning the robust stopping-time problem. We will give the connection between lower value functions W (see (2.12)), V (see (2.13)) and a stopping-time storage function S (see (2.15)) with solutions of the appropriate variational inequality.

2.3.1 The lower value function for Game I

Some simple inequalities are obvious from the definition of W. By using T = 0 (and $\Phi > 0$) in the definition of W, we see that

$$W(x) \ge 0. \tag{2.16}$$

Using $\tau \equiv 0$ gives

$$W(x) \le \Phi(x). \tag{2.17}$$

On the other hand, $\tau \equiv +\infty$ gives

$$W(x) \le S_a(x),\tag{2.18}$$

where S_a is the available storage function given by (2.14).

Proposition 2.5 Assume (A0)-(A4). Then for $x \in \mathbb{R}^n$ and t > 0

$$W(x) \leq \sup_{b \in \mathcal{B}, \ T > 0} \left\{ \int_0^{T \wedge t} l(y_x(s), b(s)) \ ds + \mathbf{1}_{[0,T)}(t) W(y_x(t, b)) \right\}.$$

Proof By the definition of W for each $\epsilon > 0$ there is a stopping-time rule τ^{ϵ} so that

$$W(z) + \epsilon > \int_{0}^{T \wedge \tau_{z}^{\epsilon}[b]} l(y_{z}(s), b(s)) \, ds + 1_{[0,T]}(\tau_{z}^{\epsilon}[b]) \Phi(y_{z}(\tau_{z}^{\epsilon}[b]))$$
(2.19)

for all $z \in \mathbb{R}^n$, T > 0 and $b \in \mathcal{B}$. Fix t > 0. For each $b \in \mathcal{B}$, we define

$$b_t(s) = b(s+t), \ s \ge 0.$$

For $\tau \in \Gamma$ define $\overline{\tau} : \mathbb{R}^n \times \mathcal{B} \to \mathbb{R}^+ \cup \infty$ by

$$\bar{\tau}_x[b] = t + \tau_{y_x(t,b)}[b_t].$$

It is easy to check that $\bar{\tau}$ has the nonanticipating property whenever τ does, and hence $\bar{\tau} \in \Gamma$ for $\tau \in \Gamma$.

Fix $x \in \mathbb{R}^n$. By the definition of W, for each $\epsilon > 0$ and $\tau \in \Gamma$ we may choose $T_{\tau,\epsilon} > 0$ and $b_{\tau,\epsilon} \in \mathcal{B}$ so that

$$W(x) - \epsilon \le \int_0^{T_{\tau,\epsilon} \wedge \tau_x[b_{\tau,\epsilon}]} l(y_x(s), b_{\tau,\epsilon}(s)) ds + \mathbb{1}_{[0,T]}(\tau_x[b]) \Phi(y_x(\tau_x[b_{\tau,\epsilon}])).$$

We may specialize this general inequality to the case where τ is of the form $\overline{\tau}$ for some $\tau \in \Gamma$. In this case $\overline{\tau}_x[b_{\overline{\tau},\epsilon}] \ge t$ and we obtain

$$W(x) - \epsilon \leq \int_{0}^{T_{\bar{\tau},\epsilon} \wedge t} l(y_{x}(s), b_{\bar{\tau},\epsilon}(s)) \, ds + 1_{[0,T_{\bar{\tau},\epsilon})}(t) \left\{ \int_{t}^{T_{\bar{\tau},\epsilon} \wedge \bar{\tau}_{x}[b_{\bar{\tau},\epsilon}]} l(y_{x}(s), b_{\bar{\tau},\epsilon}(s)) \, ds + 1_{[t,T_{\bar{\tau},\epsilon}]}(\bar{\tau}_{x}[b_{\bar{\tau},\epsilon}]) \Phi(y_{x}(\bar{\tau}_{x}[b_{\bar{\tau},\epsilon}])) \right\}.$$

$$(2.20)$$

For $t < T_{\bar{\tau},\epsilon}$, by the change of variable $\nu = s - t$ we have

$$\int_{t}^{T_{\bar{\tau},\epsilon}\wedge\bar{\tau}_{x}[b_{\bar{\tau},\epsilon}]} l(y_{x}(s), b_{\bar{\tau},\epsilon}(s)) \, ds + 1_{[t,T_{\bar{\tau},\epsilon}]}(\bar{\tau}_{x}[b_{\bar{\tau},\epsilon}])\Phi(y_{x}(\bar{\tau}_{x}[b_{\bar{\tau},\epsilon}])) \\
= \int_{0}^{(T_{\bar{\tau},\epsilon}-t)\wedge\tau_{y_{x}(t,b_{\bar{\tau},\epsilon})}[(b_{\bar{\tau},\epsilon})_{t}]} l(y_{y_{x}(t,b_{\bar{\tau},\epsilon})}(\nu), (b_{\bar{\tau},\epsilon})_{t}(\nu)) \, d\nu \\
+ 1_{[0,T_{\bar{\tau},\epsilon}-t]}(\tau_{y_{x}(t,b_{\bar{\tau},\epsilon})}[(b_{\bar{\tau},\epsilon})_{t}])\Phi(y_{y_{x}(t,b_{\bar{\tau},\epsilon})}(\tau_{y_{x}(t,b_{\bar{\tau},\epsilon})}[(b_{\bar{\tau},\epsilon})_{t}]))$$
(2.21)

Apply (2.19) to the case

$$z = y_x(t, b_{\overline{\tau^{\epsilon}}, \epsilon}), \quad b = (b_{\overline{\tau^{\epsilon}}, \epsilon})_t, \quad T = T_{\overline{\tau^{\epsilon}}, \epsilon} - t.$$

Then (2.19) implies that the right hand side of (2.21) (with τ selected to be the τ^{ϵ} as in (2.19)) is bounded above by $W(y_x(t, b_{\overline{\tau^{\epsilon}}, \epsilon})) + \epsilon$. From (2.20) we finally conclude that

$$W(x) - 2\epsilon < \int_0^{T_{\overline{\tau}\overline{\epsilon},\epsilon}\wedge t} \ell(y_x(s), b_{\overline{\tau}\overline{\epsilon},\epsilon}(s)) \, ds + \mathbf{1}_{[0,T_{\overline{\tau}\overline{\epsilon},\epsilon}]}(t) W(y_x(t, b_{\overline{\tau}\overline{\epsilon},\epsilon})) \\ \leq \sup_{b\in\mathcal{B}, T>0} \left\{ \int_0^{T\wedge t} l(y_x(s), b(s)) \, ds + \mathbf{1}_{[0,T)}(t) W(y_x(t, b)) \right\}$$

Since $\epsilon > 0$ is arbitrary, the result follows. \diamondsuit

Proposition 2.6 Assume (A0)-(A4). If $W(x) < \Phi(x)$, then for each $b \in \mathcal{B}$ there exists $\rho = \rho_{x,b} > 0$ such that for all $t \in [0, \rho)$,

$$W(x) \ge \int_0^t l(y_x(s), b(s)) \, ds + W(y_x(t, b)).$$

Proof Fix $x \in \mathbb{R}^n$. By definition of W(x), for each $\epsilon > 0$ there is a choice of strategy $\tau^{\epsilon} \in \Gamma$ so that

$$W(x) + \epsilon > \int_0^{T \wedge \tau_x^{\epsilon}[\widetilde{b}]} \ell(y_x(s), \widetilde{b}(s)) \ ds + \mathbb{1}_{[0,t]}(\tau_x^{\epsilon}[\widetilde{b}]) \Phi(y_x(\tau_x^{\epsilon}[\widetilde{b}]))$$
(2.22)

for all $T \ge 0$ and all $\tilde{b} \in \mathcal{B}$.

We claim that for each $b \in \mathcal{B}$ there is a number $\rho_b > 0$ so that $\tau_x^{\epsilon}[b] > \rho_b$ for all $\epsilon > 0$. If not, then there is a $b \in \mathcal{B}$ and a sequence of positive numbers $\{\epsilon_n\}$ with limit equal to 0 and with $\tau_x^{\epsilon_n}[b]$ tending to 0. Apply (2.22) with ϵ_n replacing ϵ and with b in place of \tilde{b} , use the continuity of Φ and of $y_x(s) = y_x(s, b)$ along with the assumption that b is locally square-integrable to take the limit in (2.22) as $n \to \infty$ to arrive at $W(x) \ge \Phi(x)$, contrary to assumption. We conclude that for each $b \in \mathcal{B}$ there is a $\rho_b > 0$ so that $\tau_x^{\epsilon}[b] \ge \rho_b$ for all $\epsilon > 0$ as asserted.

Fix $b \in \mathcal{B}$ and choose any $t \in [0, \rho_b)$. By definition of $W(y_x(t, b))$, for any $\tau \in \Gamma$ and for any $\epsilon > 0$ there is a choice of $T_{\tau,\epsilon} \ge 0$ and of $b_{\tau,\epsilon} \in \mathcal{B}$ so that

$$W(y_{x}(t,b)) - \epsilon \leq \int_{0}^{T_{\tau,\epsilon} \wedge \tau_{y_{x}(t,b)}[b_{\tau,\epsilon}]} \ell(y_{y_{x}(t,b)}(s), b_{\tau,\epsilon}(s)) \, ds + 1_{[0,T_{\tau,\epsilon}]}(\tau_{y_{x}(t,b)}[b_{\tau,\epsilon}]) \Phi(y_{y_{x}(t,b)}(\tau_{y_{x}(t,b)}[b_{\tau,\epsilon}])).$$
(2.23)

In particular, (2.23) holds for all $\tau \in \Gamma$ for which $\tau_x[b] \ge t$. For any $\hat{b} \in \mathcal{B}$, define $\hat{b}' \in \mathcal{B}$ by

$$\widehat{b}'(s) = \left\{ \begin{array}{ll} b(s), & \text{for } 0 \le s \le t \\ \widehat{b}(s-t), & \text{for } s > t. \end{array} \right.$$

For any $\tau \in \Gamma$ with $\tau_x[b] > t$, we may always find another $\overline{\tau} \in \Gamma$ so that

$$\overline{\tau}_x[\widehat{b}'] = \tau_{y_x(t,b)}[\widehat{b}] + t \text{ for all } \widehat{b} \in \mathcal{B}.$$

From (2.23) we then get that, for any $\tau \in \Gamma$ with $\tau_x[b] \ge t$ and any $\epsilon > 0$,

$$\int_{0}^{t} \ell(y_{x}(s), b(s)) \, ds + W(y_{x}(t, b)) - \epsilon \\
\leq \int_{0}^{t} \ell(y_{x}(s), b(s)) \, ds + \int_{0}^{T_{\tau,\epsilon} \wedge \tau_{y_{x}(t,b)}[b_{\tau,\epsilon}]} \ell(y_{y_{x}(t,b)}(s), b_{\tau,\epsilon}(s)) \, ds \\
+ 1_{[0,T_{\tau,\epsilon}]}(\tau_{y_{x}(t,b)}[b_{\tau,\epsilon}]) \Phi(y_{y_{x}(t,b)}(\tau_{y_{x}(t,b)}[b_{\tau,\epsilon}])) \\
\leq \int_{0}^{t} \ell(y_{x}(s), b(s)) \, ds + \int_{t}^{(T_{\tau,\epsilon}+t) \wedge (\tau_{y_{x}(t,b)}[b_{\tau,\epsilon}]+t)} \ell(y_{x}(s), b'_{\tau,\epsilon}(s)) \, ds \\
+ 1_{[t,T_{\tau,\epsilon}]}(\tau_{y_{x}(t,b)}[b_{\tau,\epsilon}]) \Phi(y_{x}(\tau_{y_{x}(t,b)}[b_{\tau,\epsilon}])) \\
= \int_{0}^{(T_{\tau,\epsilon}+t) \wedge \overline{\tau}_{x}[b'_{\tau,\epsilon}]} \ell(y_{x}(s), b'_{\tau,\epsilon}(s)) \, ds + 1_{[0,T_{\tau,\epsilon}+t]}(\overline{\tau}_{x}[b'_{\tau,\epsilon}]) \Phi(y_{x}(\overline{\tau}_{x}[b'_{\tau,\epsilon}]).$$
(2.24)

It is important to note that for any $\tilde{\tau} \in \Gamma$ with $\tilde{\tau}_x[b] > t$, there is a $\tilde{\tau}'$ in Γ with $\tilde{\tau} = \overline{\tilde{\tau}'}$. To see this for a given $\tilde{\tau} \in \Gamma$ we must find a $\tilde{\tau}' \in \Gamma$ so that

$$\overline{\widetilde{\tau}'_x}[\widehat{b}'] = \begin{cases} \widetilde{\tau}_x[\widehat{b}'] & \text{on the one hand,} \\ \widetilde{\tau}'_{y_x(t,b)}[\widetilde{b}] + t & \text{on the other hand,} \end{cases}$$

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or

$$\widetilde{\tau}_x[\widehat{b}'] = \widetilde{\tau}'_{y_x(t,b)}[\widehat{b}] + t$$

for all $\hat{b} \in \mathcal{B}$. It is always possible to solve for such a $\tilde{\tau}'$ due to the nonanticipating property of $\tilde{\tau}$. We apply this observation in particular to the case $\tilde{\tau} = \tau^{\epsilon}$ where τ^{ϵ} is as in (2.22); thus, for each $\epsilon > 0$ there is a $\tau^{\epsilon'} \in \Gamma$ so that $\overline{\tau^{\epsilon'}} = \tau^{\epsilon}$.

If we now specialize (2.22) to the case where

$$T = T_{\tau^{\epsilon'},\epsilon} + t, \quad \widetilde{b} = b'_{\tau^{\epsilon'},\epsilon},$$

we can continue the estimate in (2.24) (with the general τ replaced with $\tau^{\epsilon'}$) to get

$$\begin{split} \int_0^t \ell(y_x(s), b(s)) \, ds &+ W(y_x(t, b)) - \epsilon \\ &\leq \int_0^{(T_{\tau^{\epsilon'}, \epsilon} + t) \wedge \tau_x^{\epsilon}[b'_{\tau^{\epsilon'}, \epsilon}]} \ell(y_x(s), b'_{\tau^{\epsilon'}, \epsilon}(s)) \, ds + \mathbf{1}_{[0, T_{\tau^{\epsilon'}, \epsilon} + t]}(\tau_x^{\epsilon}[b'_{\tau^{\epsilon'}, \epsilon}]) \Phi(y_x(\tau_x^{\epsilon}[b'_{\tau^{\epsilon'}, \epsilon}])) \\ &< W(x) + \epsilon \end{split}$$

where we used (2.22) for the last step. Since $\epsilon > 0$ is arbitrary, the result follows.

Theorem 2.7 Assume (A0)-(A5). If W is upper semicontinuous, then W is a viscosity subsolution of the VI in \mathbb{R}^n .

Proof Fix $x \in \mathbb{R}^n$. Let $\Psi \in C^1(\mathbb{R}^n)$ be such that x is a local maximum point of $W^* - \Psi$. Since W is upper semicontinuous, we have

$$\Psi(x) - \Psi(z) \le W^*(x) - W^*(z) = W(x) - W(z), \qquad (2.25)$$

for all z in a neighborhood of x. We want to show that

$$\max\{H(x, D\Psi(x)), W(x) - \Phi(x)\} \le 0.$$

Since $W(x) \leq \Phi(x)$, we want to show that $H(x, D\Psi(x)) \leq 0$.

We first consider the case W(x) > 0. Let $\epsilon > 0$ and t > 0. From Proposition 2.5, choose $\hat{b} = \hat{b}_{t,\epsilon} \in \mathcal{B}$ and $\hat{T} = \hat{T}_{t,\epsilon} > 0$ such that

$$W(x) \le \int_0^{\hat{T} \wedge t} [h(y_x(s), \hat{b}(s)) - \gamma^2 |\hat{b}(s)|^2] \, ds + \mathbb{1}_{[0,\hat{T})}(t) W(y_x(t)) + \epsilon \, t.$$
(2.26)

Since $W \ge 0$, we have $1_{[0,\hat{T})}(t)W(y_x(t)) \le W(y_x(\hat{T} \land t))$. Thus

$$W(x) - W(y_x(\hat{T} \wedge t)) \le \int_0^{\hat{T} \wedge t} [h(y_x(x), \hat{b}(s)) - \gamma^2 |\hat{b}(s)|^2] \, ds + \epsilon \, t \tag{2.27}$$

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Under the assumptions on f, for t > 0 small enough, $y_x(s)$ is in the neighborhood of x for which (2.25) holds for all $0 < s \le t$. As a consequence of (2.25) and (2.27), we have

$$\Psi(x) - \Psi(y_x(\hat{T} \wedge t)) \le \int_0^{\hat{T} \wedge t} [h(y_x(s), \hat{b}(s)) - \gamma^2 |\hat{b}(s)|^2] \, ds + \epsilon \, t.$$
(2.28)

Now observe that (2.6) and (A2) imply

$$|f(y_x(s), \hat{b}(s)) - f(x, \hat{b}(s))| \le \omega_f(M_x s, |x| + M_x t_0), \text{ for } 0 < s < t_0$$
(2.29)

and

$$|h(y_x(s), \hat{b}(s)) - h(x, \hat{b}(s))| \le \omega_h(M_x s, |x| + M_x t_0), \text{ for } 0 < s < t_0$$
(2.30)

where t_0 does not depend on ϵ , t and \hat{b} . By (2.30) the integral on the right-hand side of (2.28) can be written as

$$\int_0^{\hat{T} \wedge t} [h(x, \hat{b}(s)) - \gamma^2 |b(s)|^2] ds + o(\hat{T} \wedge t) \quad \text{as } \hat{T} \wedge t \to 0.$$

Thus

$$\Psi(x) - \Psi(y_x(\hat{T} \wedge t)) \le \int_0^{\hat{T} \wedge t} [h(x, \hat{b}(s)) - \gamma^2 |\hat{b}(s)|^2] \, ds + \epsilon \, t + o(\hat{T} \wedge t).$$
(2.31)

Moreover

$$\Psi(x) - \Psi(y_x(\hat{T} \wedge t)) = -\int_0^{T \wedge t} \frac{d}{ds} \Psi(y_x(s)) ds$$

$$= -\int_0^{\hat{T} \wedge t} D\Psi(y_x(s)) \cdot f(y_x(s), \hat{b}(s)) ds$$

$$= -\int_0^{\hat{T} \wedge t} D\Psi(x) \cdot f(x, \hat{b}(s)) ds + o(\hat{T} \wedge t)$$
(2.32)

where we used (2.6), (2.29) and $\Psi \in C^1$ in the last equality to estimate the difference between $D\Psi \cdot f$ computed at $y_x(s)$ and at x, respectively. Plugging (2.32) into (2.31) we get

$$\int_{0}^{\hat{T} \wedge t} - D\Psi(x) \cdot f(x, \hat{b}(s)) ds \le \int_{0}^{\hat{T} \wedge t} [h(x, \hat{b}(s)) - \gamma^{2} |\hat{b}(s)|^{2}] ds + \epsilon t + o(\hat{T} \wedge t).$$

Thus

$$\int_{0}^{\hat{T}\wedge t} \{-D\Psi(x) \cdot f(x,\hat{b}(s)) - h(x,\hat{b}(s)) + \gamma^{2}|\hat{b}(s)|^{2}\} ds \le \epsilon \ t + o(\hat{T}\wedge t).$$
(2.33)

The term in the brackets in the integral is estimated from below by

$$\inf_{b \in B} \{ -D\Psi(x) \cdot f(x,b) - h(x,b) + \gamma^2 |b|^2 \}$$

and hence we have the inequality

$$\inf_{b \in B} \{ -D\Psi(x) \cdot f(x,b) - h(x,b) + \gamma^2 |b|^2 \} \cdot (\hat{T} \wedge t) \le \epsilon \ t + o(\hat{T} \wedge t).$$
(2.34)

At this stage we write $\hat{T}_{t,\epsilon}$ in place of \hat{T} to emphasize the dependence of \hat{T} on t and ϵ . Note that $\frac{t}{\hat{T}_{t,\epsilon} \wedge t} \geq 1$ for all t > 0 and hence $\limsup_{t \to 0} \frac{t}{\hat{T}_{t,\epsilon} \wedge t} \geq 1$. We claim that in fact

$$\limsup_{t \to 0} \frac{t}{t \wedge \hat{T}_{t,\epsilon}} = 1 \text{ (for each } \epsilon > 0).$$

Indeed, if not, then, for each fixed $\epsilon > 0$, there would be a sequence of positive numbers $\{t_n\}$ tending to 0 such that $\hat{T}_{t_n,\epsilon} < t_n$ and $\lim_{n\to\infty} \hat{T}_{t_n,\epsilon}/t_n = \rho_{\epsilon} < 1$. In this case, the inequality (2.26) becomes

$$W(x) \le \int_0^{T_{t_n,\epsilon}} [h(y_x(s),\hat{b}(s)) - \gamma^2 |\hat{b}(s)|^2] ds + \epsilon t_n$$

for all n, from which we get

$$\frac{W(x)}{t_n} \le \frac{1}{t_n} \int_0^{\hat{T}_{t_n,\epsilon}} [h(y_x(s), \hat{b}(s)) - \gamma^2 |\hat{b}(s)|^2] \, ds + \epsilon \tag{2.35}$$

for all n. From (2.4) and (A1) we have an estimate of the form $h(y_x(s), \hat{b}(s)) \leq K_x$ for all s in a sufficiently small interval $[0, \delta)$ (independent of t and ϵ), and hence, for n sufficiently large we have

$$\int_0^{\hat{T}_{t_n,\epsilon}} [h(y_x(s),\hat{b}(s)) - \gamma^2 |\hat{b}(s)|^2] \, ds \le K_x T_{t_n,\epsilon}.$$

Plugging this into (2.35) gives

$$\frac{W(x)}{t_n} \le K_x \frac{\hat{T}_{t_n,\epsilon}}{t_n} + \epsilon.$$

Letting *n* tend to infinity and using the assumption that W(x) > 0 leads to the contradiction $\infty \leq K_x \rho_{\epsilon} + \epsilon \leq K_x + \epsilon < \infty$. Hence $\limsup_{t\to 0} \frac{t}{t \wedge \hat{T}_{t,\epsilon}} = 1$ for each fixed $\epsilon > 0$ as asserted, and we can divide (2.34) by $\hat{T} \wedge t > 0$ and pass to the limit to get

$$\inf_{b\in B} \{-D\Psi(x) \cdot f(x,b) - h(x,b) + \gamma^2 |b|^2\} \le \epsilon.$$

Since ϵ is arbitrary, $H(x, D\Psi(x)) \leq 0$.

It remains to handle the case W(x) = 0. In this case we take $\hat{b} = 0$ and use (A5) and $W \ge 0$ to see that

$$\begin{aligned} W(x) &= 0 &\leq \int_0^t h(y_x(s), 0) \, ds + W(y_x(t)) \\ &= \int_0^t [h(y_x(s), \hat{b}(s)) - \gamma^2 |\hat{b}(s)|^2] \, ds + W(y_x(t)) \end{aligned}$$

for all $t \ge 0$. With this stronger version of (2.26), it is straightforward to follow the procedure in the first part of the proof to arrive at the desired inequality $H(x, D\Psi(x)) \le 0$.

Theorem 2.8 Assume (A0)-(A4). If W is lower semicontinuous, then W is a viscosity supersolution of the VI in \mathbb{R}^n .

Proof Fix $x \in \mathbb{R}^n$. Let $\Psi \in C^1(\mathbb{R}^n)$ be such that x is a local minimum point of $W_* - \Psi$. Since W is lower semicontinuous, we have

$$\Psi(x) - \Psi(z) \ge W_*(x) - W_*(z) = W(x) - W(z), \qquad (2.36)$$

for all z in a neighborhood of x. We want to show that

$$\max\{H(x, D\Psi(x)), W(x) - \Phi(x)\} \ge 0.$$

If $W(x) = \Phi(x)$, the assertion is trivial. Suppose $W(x) < \Phi(x)$. We want to show that $H(x, D\Psi(x)) \ge 0$. By Proposition 2.6, for each $b \in \mathcal{B}$ choose $t_b > 0$ such that

$$W(x) \ge \int_0^t l(y_x(s), b(s))ds + W(y_x(t, b)), \ \forall t \in [0, t_b)$$
(2.37)

Fix an arbitrary $b \in B$ and let $y_x(s)$ be the solution corresponding to the constant disturbance b(s) = b for all s. Under our assumptions on f, there exists $t_1 \in (0, t_b)$ such that $y_x(s)$ is in the neighborhood of x for which (2.36) holds for all $0 < s \le t_1$. From (2.36) and (2.37), we have

$$\frac{1}{t}[\Psi(x) - \Psi(y_x(t))] \ge \frac{1}{t} \int_0^t l(y_x(s), b) ds, \ \forall t \in (0, t_1).$$

Let $t \to 0$, we have

$$-D\Psi(x) \cdot f(x,b) - h(x,b) + \gamma^2 |b| \ge 0.$$

Since $b \in B$ is arbitrary, it follows that

$$H(x, D\Psi(x)) = \inf_{b \in B} \{-D\Psi(x) \cdot f(x, b) - h(x, b) + \gamma^2 |b|\} \ge 0. \diamondsuit$$

Corollary 2.9 Assume (A0)-(A5). If W is continuous, then W is a viscosity solution of the VI in \mathbb{R}^n .

Remark 2.10 Let S_a be the available storage function for the disturbance-to-cost system with stopping options ignored as in (2.14). Since S_a is a viscosity solution of the HJBE in \mathbb{R}^n , we have

$$H(x, D^{-}(S_a)_*(x)) \ge 0$$
 and $H(x, D^{+}(S_a)^*(x)) \le 0, x \in \mathbb{R}^n$.

Thus S_a is a viscosity supersolution of the VI. Moreover if $S_a \leq \Phi$, then S_a is a viscosity solution of the VI.

2.3.2 The lower value function for Game II

Now we will show some inequalities satisfied by the lower value function V(x) for Game II (see (2.13)), and use these inequalities to show that if continuous, V is a viscosity solution of the VI in \mathbb{R}^n . For convenience, set

$$J(x,t,b) = \Phi(y_x(t)) + \int_0^t l(y_x(s),b(s)) \, ds$$

$$\Delta = \{\nu : \mathbb{R}^n \times \mathcal{B} \to [0,\infty) : \nu \text{ is nonanticipating} \}.$$

Thus

$$V(x) := \inf_{\nu \in \Delta} \sup_{b \in \mathcal{B}} J(x, b, \nu[b]).$$

Proposition 2.11 Assume (A0)-(A4).

- (i) Then $V \leq \Phi$. If (A5) also holds, then $0 \leq V \leq \Phi$.
- (ii) Then

$$V(x) \le \sup_{b \in \mathcal{B}} \{ \int_0^t l(y_x(s), b(s)) \, ds + V(y_x(t)) \}, \text{ for all } t \ge 0.$$

(iii) If $V(x) < \Phi(x)$, then for each $b \in \mathcal{B}$ there exists $t_b > 0$ such that

$$V(x) \ge \int_0^t l(y_x(s), b(s)) \, ds + V(y_x(t, b)), \, \forall t \in [0, t_b)$$

Proof (i) Using $\nu \equiv 0$ we have

$$V(x) \le \Phi(x),$$

while using $b \equiv 0$ we have

$$V(x) \geq \inf_{\nu \in \Delta} \{ \int_0^{\nu[0]} h(y_x(s), 0) ds + \Phi(y_x(\nu[0])) \} \\ \geq 0,$$

since h is nonnegative by (A5) and Φ are nonnegative by assumption.

(ii) Fix $x \in \mathbb{R}^n$. For each $\nu \in \Delta$ and $\epsilon > 0$ we may choose $b_{\nu,\epsilon} \in \mathcal{B}$ so that

$$V(x) < J(x, \nu_x[b_{\nu,\epsilon}], b_{\nu,\epsilon}) + \epsilon.$$
(2.38)

Fix t > 0. For each $b \in \mathcal{B}$, define b_t by

$$b_t(s) = b(s+t), \ s \ge 0$$

Notice that $y_x(t,b) = y_{y_x(t,b)}(0,b_t)$.

On the other hand, we may choose $\nu^{\epsilon} \in \Delta$ so that

$$V(z) + \epsilon > J(z, \nu_z^{\epsilon}[b], b)$$
(2.39)

for all $z \in \mathbb{R}^n$ and $b \in \mathcal{B}$. For $\nu \in \Delta$ define $\bar{\nu} : \mathbb{R}^n \times \mathcal{B} \to \mathbb{R}^+$ by

$$\bar{\nu}[b] = t + \nu_{y_x(t,b)}[b_t].$$

One can easily check that $\bar{\nu} \in \Delta$ for each $\nu \in \Delta$. From the definition of J, we have

$$J(x, \bar{\nu}_{x}[b], b) = \Phi(y_{x}(\bar{\nu}[b])) + \int_{0}^{\bar{\nu}[b]} \ell(y_{x}(s), b(s)) \, ds$$

$$= \Phi(y_{x}(\nu_{y_{x}(t,b)}[b_{t}] + t)) + \int_{0}^{t} l(y_{x}(s), b(s)) \, ds$$

$$+ \int_{t}^{\nu_{y_{x}(t,b)}[b_{t}] + t} l(y_{x}(s), b(s)) \, ds$$

$$= \int_{0}^{t} l(y_{x}(s), b(s)) \, ds + \Phi(y_{y_{x}(t,b)}(\nu_{y_{x}(t,b)}[b_{t}]))$$

$$+ \int_{0}^{\nu_{y_{x}(t,b)}[b_{t}]} l(y_{y_{x}(t)}(\alpha), b_{t}(\alpha)) \, d\alpha$$

$$= \int_{0}^{t} l(y_{x}(s), b(s)) \, ds + J(y_{x}(t,b), \nu_{y_{x}(t,b)}[b_{t}], b_{t})$$

(2.40)

If we specialize (2.40) to the case $\nu = \nu^{\epsilon}$ (where ν^{ϵ} is as in (2.39)) and specialize (2.39) to the case $z = y_x(t, b)$ and b of the form b_t , then (2.39) provides the estimate on (2.40)

$$J(x, \bar{\nu}_x[b], b) \le \int_0^t l(y_x(s), b(s)) \, ds + V(y_x(t, b)) + \epsilon \tag{2.41}$$

for all $b \in \mathcal{B}$. If we specialize (2.41) to the case where $b = b_{\bar{\nu},\epsilon}$ and apply (2.38) for the case where ν is of the form $\bar{\nu}$, then (2.38) leads to

$$V(x) < J(x, \bar{\nu}_x[b_{\bar{\nu},\epsilon}], b_{\bar{\nu},\epsilon}) + \epsilon$$

$$\leq \int_0^t l(y_x(s), b_{\bar{\nu},\epsilon}(s)) \, ds + V(y_x(t, b_{\bar{\nu},\epsilon})) + 2\epsilon$$

$$\leq \sup_{b \in \mathcal{B}} \left\{ \int_0^t l(y_x(s), b(s)) \, ds + V(y_x(t, b)) \right\} + 2\epsilon$$

Since ϵ is arbitrary, the result follows.

(iii) The proof of statement (iii) is similar to the proof of Proposition 2.6. \Diamond

Theorem 2.12 Assume (A0)-(A4). If upper semicontinuous, V is a viscosity subsolution of the VI. If lower semicontinuous, V is a viscosity supersolution of the VI. Thus if continuous, V is a viscosity solution of the VI.

Proof First assume that V is upper semicontinuous. We want to show that V is a viscosity subsolution of VI. Since V is upper semicontinuous by assumption, $V^* = V$. Fix $x \in \mathbb{R}^n$. Let $\Psi \in C^1(\mathbb{R}^n)$ and x is a local maximum of $V - \Psi$. We want to show that

$$\max\{H(x, D\Psi(x)), V(x) - \Phi(x)\} \le 0$$
(2.42)

From (i) of Proposition 2.11, $V(x) \leq \Phi(x)$. Thus we want to show that $H(x, D\Psi(x)) \leq 0$. We proceed by contradiction. Suppose that

$$H(x, D\Psi(x)) > \delta > 0.$$

By the definition of H, we therefore have

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$$-D\Psi(x) \cdot f(x,b) - h(x,b) + \gamma^2 |b|^2 > \delta, \ \forall b \in B.$$
(2.43)

Choose R so that $x \in B(0, R)$ and suppose that z is another point in B(0, R) and $b \in B$. We shall need the general estimate

$$\begin{aligned} &|[-D\Psi(z) \cdot f(z,b) - h(z,b)] - [-D\Psi(x) \cdot f(x,b) - h(x,b)]| \\ &\leq |[-D\Psi(z) + D\Psi(x)] \cdot f(z,b)| + |D\Psi(x) \cdot [f(x,b) - f(z,b)]| \\ &+ |[-h(z,b) + h(x,b)]| \\ &\leq \omega_{D\Psi}(|z-x|,R)M_{f,R} + |D\Psi(x)|\omega_{f}(|z-x|,R) + \omega_{h}(|z-x|,R) \end{aligned}$$
(2.44)

where $\omega_{D\Psi}(\cdot, R)$ is a modulus of continuity for $D\Psi(\cdot)$ on B(0, R), where $M_{f,R}$ is a bound on f(z, b) for $(z, b) \in B(0, R) \times B$, and where we use (A1) and (A2). By the continuity of the moduli $\omega_{D\Psi}(\cdot, R)$, $\omega_f(\cdot, R)$ and $\omega_h(\cdot, R)$ at the origin, we deduce that there is a $\delta_R > 0$ so that

$$|z - x| < \delta_R \Longrightarrow |[-D\Psi(z) \cdot f(z, b) - h(z, b)] - [-D\Psi(x) \cdot f(x, b) - h(x, b)]| < \delta/2.$$
(2.45)

Moreover, by (2.7) we know that there is a $t_x > 0$ so that

$$0 \le s \le t_x \Longrightarrow |y_x(s,b) - x| < \delta_R \text{ for all } b \in \mathcal{B}.$$

We conclude that, for $0 \le s \le t_x$ and for all $b \in \mathcal{B}$, from (2.43) and (2.44) combined with (2.45) we have

$$-D\Psi(y_x(s,b) \cdot f(y_x(s,b),b(s)) - h(y_x(s,b),b(s)) + \gamma^2 |b(s)|^2 = [-D\Psi(x) \cdot f(x,b(s)) - h(x,b(s)) + \gamma^2 |b(s)|^2] + \{[-D\Psi(y_x(s,b)) \cdot f(y_x(s),b(s)) - h(y_x(s),b)] - [-D\Psi(x) \cdot f(x,b(s)) - h(x,b(s))]\} \geq \delta - \delta/2 = \delta/2 \text{ for } 0 \le s \le t_x.$$

$$(2.46)$$

Since x is a local maximum of $V - \Psi$, by (2.6) we may assume that $t_x > 0$ also satisfies

$$V(x) - V(y_{x,b}(s)) \ge \Psi(x) - \Psi(y_x(s,b)) \text{ for } 0 < s < t_x \text{ for all } b \in \mathcal{B}.$$
 (2.47)

For any t satisfying $0 < t \le t_x$ we may integrate (2.46) from 0 to t to get

$$\Psi(x) - \Psi(y_{x,b}(t)) > \frac{\delta}{2}t + \int_0^t l(y_x(s), b(s)) \, ds.$$
(2.48)

As a consequence of (2.47) and (2.48), we have

$$V(x) - V(y_x(t,b)) > \frac{\delta}{2}t + \int_0^t l(y_x(s), b(s)) \, ds \text{ for all } b \in \mathcal{B}$$

Thus

$$V(x) \geq \frac{\delta}{2}t + \sup_{b \in \mathcal{B}} \left\{ \int_0^t l(y_x(s), b(s)) \, ds + V(y_x(t, b)) \right\}$$

>
$$\sup_{b \in \mathcal{B}} \left\{ \int_0^t [l(y_x(s), b(s)) \, ds + V(y_x(t, b))] \right\}$$

which contradicts (ii) of Proposition 2.11.

We now assume that V is lower semicontinuous. The proof that V is a viscosity supersolution of VI is similar to the proof of Theorem 2.8. By using (i) and (iii) of Proposition 2.11, one can follow the proof there to show that V is a viscosity supersolution of VI, and the result follows. \diamond

Proposition 2.13 Assume (A0)-(A4). Assume in addition: there is a continuous B-valued function $(x, p) \rightarrow \beta(x, p)$ so that

$$H(x,p) = -p \cdot f(x,\beta(x,p)) - \ell(x,\beta(x,p))$$

Then, if $\widetilde{V} \in C^1(\mathbb{R}^n)$ is a subsolution of VI, then $\widetilde{V} \leq V$. Hence, under these assumptions, if $V \in C^1(\mathbb{R}^n)$, then V is the maximal smooth nonnegative subsolution of VI.

Proof Since $\widetilde{V} \in C^1(\mathbb{R}^n)$ is a subsolution of VI, we have

$$H(x, D\widetilde{V}(x)) \le 0 \text{ and } \widetilde{V}(x) \le \Phi(x), \ \forall x \in \mathbb{R}^n.$$
 (2.49)

Define $\beta_* : \mathbb{R}^n \to B$ by

$$\beta_*(z) = \beta(z, DV(z)) \text{ for } z \in \mathbb{R}^n.$$

Then β_* is continuous in $z \in \mathbb{R}^n$ and from the first part of (2.49) we see that

$$-D\widetilde{V}(z) \cdot f(z,\beta_*(z)) - h(z,\beta_*(z)) + \gamma^2 |\beta_*(z))| = H(z,D\widetilde{V}(z)) \le 0$$
(2.50)

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for all $z \in \mathbb{R}^n$. By assumption β and $D\widetilde{V}$ are continuous; hence $z \to \beta_*(z)$ is continuous on \mathbb{R}^n and the initial-value problem

$$\dot{y}(t) = f(y(t), \beta_*(y(t))), \quad y(0) = x$$

has a solution $y_x^*(t)$. Note that we may regard $t \to b_x^*(t) := \beta_*(y_x^*(t))$ as an element of \mathcal{B} for each $x \in \mathbb{R}^n$, and then $y_x^*(t) = y_x(t, b_x^*(t))$ for all $t \ge 0$. From (2.50) we deduce

$$-D\tilde{V}(y_x(t,b_x^*(t)) \cdot f(y_x(t,b_x^*(t)),b_x^*(t)) - h(y_x(t,b_x^*(t)) + \gamma^2 |b_x^*(t)|^2 \le 0.$$
(2.51)

For $\nu \in \Delta$, integrate (2.51) from 0 to $\nu_x[b_x^*]$ and use the second part of (2.49) to get

$$\begin{split} \widetilde{V}(x) &\leq \int_{0}^{\nu_{x}[b_{x}^{*}]} l(y_{x}(s), b_{x}^{*}(s)) ds + \widetilde{V}(y_{x}(\nu_{x}[b_{x}^{*}], b_{x}^{*})) \\ &\leq \int_{0}^{\nu_{x}[b_{x}^{*}]} l(y_{x}(s), b_{x}^{*}(s)) ds + \Phi(y_{x}(\nu_{x}[b_{x}^{*}], b_{x}^{*})) \\ &\leq \sup_{b \in \mathcal{B}} \{ \int_{0}^{\nu_{x}[b]} l(y_{x}(s), b(s)) \ ds + \Phi(y_{x}(\nu_{x}[b], b)) \} \end{split}$$

Since this holds for each $\nu \in \Delta$, we have

$$\widetilde{V}(x) \le \inf_{\nu \in \Delta} \sup_{b \in \mathcal{B}} \{ \int_0^{\nu_x[b]} l(y_x(s), b(s)) ds + \Phi(y_x(\nu_x[b], b)) \} = V(x)$$

and the result follows. \diamondsuit

2.3.3 The stopping-time storage function

In this subsection, we collect our results concerning stopping-time storage functions defined as in (2.15).

Theorem 2.14 Assume (A0)-(A4). Then a locally bounded stopping-time storage function is a viscosity supersolution of the VI.

Proof Suppose that U is a stopping-time storage function with stopping-time rule τ_U , i.e.

$$\int_{0}^{T \wedge \tau_{U,x}[b]} h(y_{x}(s), b(s)) \, ds + \mathbb{1}_{[0,T]}(\tau_{U,x}[b]) \Phi(y_{x}(\tau_{U,x}[b], b) + \mathbb{1}_{(T,+\infty]}(\tau_{U,x}[b]) U(y_{x}(T, b)) \\
\leq \gamma^{2} \int_{0}^{T \wedge \tau_{U,x}[b]} |b(s)|^{2} \, ds + U(x) \text{ for all } x \in \mathbb{R}^{n}, \ b \in \mathcal{B} \text{ and } T \geq 0.$$
(2.52)

Fix $x \in \mathbb{R}^n$. Let $\Psi \in C^1(\mathbb{R}^n)$ be such that x is a local minimum point of $U - \Psi$. We want to show that

$$H(x, D\Psi(x)) \ge 0$$
 or $U(x) - \Phi(x) \ge 0$.
If $U(x) \ge \Phi(x)$, the result is obvious. It remains to show that $H(x, D\Psi(x)) \ge 0$ when $U(x) < \Phi(x)$.

Fix an arbitrary $b \in B$. Set b(s) = b for all $s \ge 0$. Choose $x_n \in \mathbb{R}^n$ with $\lim_{n\to\infty} x_n = x$ so that $\lim_{n\to\infty} U(x_n) = U_*(x)$. We claim: there is a $\delta > 0$ so that $\tau_{U,x_n}[b] > \delta$ for all nsufficiently large. If the claim were not true, dropping down to a subsequence if necessary, we would have $\lim_{n\to\infty} t_n = 0$ where $t_n := \tau_{U,x_n}[b]$. From (2.52) applied with a fixed Tsufficiently large, we then would have

$$U(x_n) \ge \int_0^{t_n} \ell(y_{x_n}(s), b) \, ds + \Phi(y_{x_n}(t_n, b)).$$
(2.53)

From the estimates (2.5)–(2.7) together with assumption (A3) and the assumed continuity of Φ , we see that $\lim_{n\to\infty} \Phi(y_{x_n}(t_n, b)) = \Phi(x)$ and that $\ell(y_{x_n}(s), b)$ tends uniformly to $\ell(y_x(s), b)$ in s on the interval $[0, \delta]$. Hence we can take limits in (2.53) to get

$$U_*(x) \ge \Phi(x).$$

As $U(x) \ge U_*(x)$ by definition of U_* , this contradicts our assumption that $U(x) < \Phi(x)$, and the claim follows.

Hence, for $0 < t < \delta$, we may apply (2.52), this time with T = t, to get, for each n sufficiently large,

$$U(x_n) \ge \int_0^t \ell(y_{x_n}(s), b) \, ds + U(y_{x_n}(t, b)).$$

Letting n tend to infinity and again using that $\ell(y_{x_n}(s), b)$ tends uniformly in s to $\ell(y_{x_n}(s), b)$ on [0, t] and that $y_{x_n}(t, b)$ tends to $y_x(t, b)$ leads to

$$U_*(x) \ge \int_0^t \ell(y_x(s), b) \, ds + \liminf_{n \to \infty} U(y_{x_n}(t, b)) \tag{2.54}$$

$$\geq \int_0^t \ell(y_x(s), b) \, ds + U_*(y_x(t, b)). \tag{2.55}$$

We now can follow the standard procedure as in the proof of Theorem 2.8 to see that $H(x, D\Psi(x)) \ge 0$ as desired. \diamondsuit

Remark 2.15 The proof of Theorem 2.14 is adapted from the proof of Proposition 3.2 in [32], where it is shown that, under certain conditions, the lower and upper semicontinuous envelopes of a storage function is again a storage function for the classical case (with no stopping options allowed).

Theorem 2.16 Let $U : \mathbb{R}^n \to \mathbb{R}$ be a nonnegative continuous function and $\Omega_U := \{x \in \mathbb{R}^n : U(x) < \Phi(x)\}$. Assume (A0)-(A4) and that B is bounded. If U is viscosity supersolution of VI in \mathbb{R}^n and the stopping rule is given by

$$\tau_{U,x} := \inf\{t : t \ge 0 \text{ and } y_x(t,b) \notin \Omega_U\},$$

$$(2.56)$$

then U is a stopping-time storage function with stopping rule τ_U , and $U \geq W$. Thus if W is continuous and B is bounded, then W is characterized as the minimal nonnegative continuous viscosity supersolution of VI, as well as the minimal possible continuous closed-loop storage function over all possible stopping-time rules $\tau \in \Gamma$.

Proof Since U and Φ are continuous, Ω_U is open. Since $\Phi \geq W$, we have $U(x) \geq W(x)$ and $\tau_{U,x} = 0$ for all $x \in \mathbb{R}^n \setminus \Omega_U$. For $x \in \Omega_U$, we must have $H(x, D^-U(x)) \ge 0$ since U is a viscosity supersolution of VI. Thus

$$\inf_{b \in B} \{ -DU(x) \cdot f(x,b) - h(x,b) + \gamma^2 |b|^2 \} \ge 0,$$

in Ω_U in the viscosity sense. By Theorem 2.4,

$$U(y_x(t_1, b)) \ge \int_{t_1}^{t_2} [h(y_x(s), b(s)) - \gamma^2 |b(s)|^2] ds + U(y_x(t_2, b))$$

for all $b \in \mathcal{B}, x \in \Omega_U, 0 \le t_1 \le t_2 < \tau_{U,x}[b].$

Let $T \ge 0$. Take $t_1 = 0$ and replace t_2 by $T \wedge t_2$ to get

— .

$$U(x) \ge \int_0^{T \wedge t_2} l(y_x(s), b(s)) ds + U(y_x(T \wedge t_2, b)), \ \forall t_2 \in [0, \tau_{U,x}[b]), \ \forall b \in \mathcal{B}.$$

Letting $t_2 \to \tau_{U,x}[b]$, by continuity of U we get

$$U(x) \ge \int_0^{T \wedge \tau_x[b]} l(y_x(s), b(s)) ds + U(y_x(T \wedge \tau_{U,x}[b])), \ \forall b \in \mathcal{B}.$$

Since $y_x(\tau_{U,x}[b]) \in \partial \Omega_U$, we have

$$U(y_{x}(T \wedge \tau_{U,x}[b])) = \begin{cases} U(y_{x}(\tau_{U,x}[b])) & \text{for } 0 \leq \tau_{U,x}[b] \leq T, \\ U(y_{x}(T)) & \text{for } \tau_{U,x}[b] > T \end{cases}$$

$$= 1_{[0,T]}(\tau_{U,x}[b])U(y_{x}(\tau_{U,x}[b])) + 1_{(T,+\infty]}(\tau_{U,x}[b])U(y_{x}(T))$$

$$= 1_{[0,T]}(\tau_{U,x}[b])\Phi(y_{x}(\tau_{U,x}[b])) + 1_{(T,+\infty]}(\tau_{U,x}[b])U(y_{x}(T)).$$

Thus

$$U(x) \geq \int_{0}^{T \wedge \tau_{U,x}[b]} l(y_{x}(s), b(s)) ds + 1_{(T,+\infty]}(\tau_{U,x}[b]) U(y_{x}(T)) + 1_{[0,T]}(\tau_{U,x}[b]) \Phi(y_{x}(\tau_{U,x}[b]).$$

Since this inequality holds for all $b \in \mathcal{B}$ and all $T \ge 0$ and U is nonnegative, we have

$$U(x) \geq \sup_{b \in \mathcal{B}, T \geq 0} \{ \int_{0}^{T \wedge \tau_{U,x}[b]} l(y_{x}(s), b(s)) \, ds + \mathbb{1}_{[0,T]}(\tau_{U,x}[b]) \Phi(y_{x}(\tau_{U,x}[b])) \}$$

$$\geq \inf_{\tau \in \Gamma} \sup_{b \in \mathcal{B}, T \geq 0} \{ \int_{0}^{T \wedge \tau_{U,x}[b]} l(y_{x}(s), b(s)) \, ds + \mathbb{1}_{[0,T]}(\tau_{U,x}[b]) \Phi(y_{x}(\tau_{U,x}[b])) \}$$

$$= W(x).$$

We conclude that W, if continuous, is characterized as the minimal nonnegative continuous viscosity supersolution of the VI, as asserted.

Finally, if W is continuous, from Corollary 2.9 we see that W is a viscosity solution of the VI, and hence in particular a viscosity supersolution. The first part of Theorem 2.16 already proved then implies that W is a stopping-time storage function with stopping rule τ_W . Moreover, if S is any continuous, stopping-time storage function for some stopping-rule τ , from Theorem 2.14 we see that S is a viscosity supersolution of the VI. Again from the first part of Theorem 2.16 already proved, we then see that $S \geq W$, and hence W is also the minimal, continuous stopping-time storage function, as asserted. \diamond

Remark 2.17 The proof of Theorem 2.16 shows that if U is a stopping-time storage function for some stopping-rule τ , then it is also a stopping-time storage function for the stopping-rule τ_U given by (2.56). When the stopping rule τ_U is used, then one can easily check that the function U enjoys the following subordination property with respect to Φ along trajectories of the system:

$$U(x) < \Phi(x) \Longrightarrow U(y_x(t,b)) < \Phi(y_x(t,b)) \text{ for } 0 \le t < \tau_{U,x}[b];$$

$$U(x) \ge \Phi(x) \Longrightarrow \tau_{U,x}[b] = 0 \text{ for all } b \in \mathcal{B}.$$

Remark 2.18 We expect that the hypothesis in Theorem 2.4 that g is bounded can be weakened to g being integrable by use of Aubin viability theory (see [32]); in this case the assumption that B be bounded in Theorem 2.16 can be removed.

2.4 Stability for stopping-time problems

In this section we show how the solution of the VI can be used to prove stability for the closed-loop system with stopping time (Σ_{st}, K) given in Figure 2.1 where $K(y_x(t, b)) = \tau_x[b]$. For a stopping-time system with finite stopping rule, it makes no sense to apply the notion of an asymptotically stable equilibrium point to the stopping-time dissipative system, but the notion of Lyapunov stability still makes sense. If the stopping rule is infinite, then the stopping-time dissipative system becomes a standard dissipative system for which the available storage function can be used to prove stability results. When the stopping-time is finite, the stopping-time storage function can be used to prove Lyapunov stability. In this section we lay these ideas out systematically in detail.

Given a closed-loop stopping-time system (Σ_{st}, K) , we say that the origin is a *stable equilib*rium point of the undisturbed stopping-time system $\dot{y} = f(y, 0)$ if

(a) $y_0(s,0) = 0$ for all $0 \le s \le \tau_0[0]$; and



Figure 2.1: A closed-loop stopping-time system (Σ_{st}, K)

(b) for each $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that

 $\left\{ \begin{array}{l} |x| < \delta \ \text{ implies } |y_x(s,0)| < \epsilon \ \text{ for all } 0 \le s \le \tau_x[0]; \\ \text{ furthermore, } |x| < \delta \ \text{and } \tau_x[0] = \infty \ \text{implies } \lim_{s \to \infty} y_x(s,0) = 0. \end{array} \right.$

We will need the following Lemma in the proof of stability.

Lemma 2.19 If $\phi(\cdot) : \mathbb{R} \to \mathbb{R}$ is nonnegative, uniformly continuous and $\int_0^\infty \phi(s) \, ds < \infty$, then $\lim_{t\to\infty} \phi(t) = 0$.

Proof See [44].

We also assume the following conditions on the system Σ_{st} .

The conditions (A6) and (A7) are modifications of the usual notions of *zero-state observ-ability* and *zero-state detectability*, respectively, for nonlinear controlled systems. For the case of linear systems, it is easy to see that these notions correspond to the usual notions of observability and detectability.

Proposition 2.20 Assume (A0)-(A6) and B is bounded. If U be a nonnegative, continuous viscosity supersolution of the VI in \mathbb{R}^n , then U(x) > 0 for all $x \neq 0$.

Proof Let $x \in \mathbb{R}^n$. Since U is a nonnegative, continuous viscosity supersolution of the VI, we have

$$U(x) \ge \int_0^{T \wedge \tau_x[0]} h(y_x(s,0),0) ds + \mathbf{1}_{[0,T]}(\tau_x[0]) \Phi(y_x(\tau_x[0])) + \mathbf{1}_{(T,+\infty]}(\tau_x[0]) U(y_x(T)), \quad (2.57)$$

for all $T \ge 0$, where $\tau_x[0] = \inf\{t : t \ge 0 \text{ and } U(y_x(t,0)) \ge \Phi(y_x(t,0))\}$. Since U is nonnegative and Φ is positive, we have

$$U(x) \ge \int_0^{T \wedge \tau_x[0]} h(y_x(s,0), 0) ds.$$
(2.58)

Also, from the definition of $\tau_x[0]$ and the assumption that $\Phi(x) > 0$, we see that $\tau_x[0] > 0$ whenever U(x) = 0.

We shall show that if U(x) = 0, then x = 0, from which we get U(x) > 0 for all $x \neq 0$ as wanted. Assume therefore that U(x) = 0. As noted above, this forces $\tau_x[0] > 0$. Furthermore, from (2.58) we get $h(y_x(s,0),0) = 0$ for all $s \in [0, T \land \tau_x[0]]$. By (A6), $y_x(s,0) = 0$ for all $s \in [0, T \land \tau_x[0]]$. Thus $x = y_x(0,0) = 0$ by (A6).

The following is our main result on stability for the case of a stopping-time problem.

Theorem 2.21 Assume (A1)-(A5), (A7) and B is bounded. If U is a nonnegative, continuous, viscosity supersolution of the VI, U(x) > 0 for $x \neq 0$ and U(0) = 0, then x = 0 is the stable equilibrium point of the undisturbed stopping-time system $\dot{y} = f(y, 0)$.

Proof Set $\Omega := \{x \in \mathbb{R}^n : U(x) < \Phi(x)\}$. By the continuity of U and Φ , Ω is open. Since Φ is positive, $U(0) = 0 < \Phi(0)$ and thus $0 \in \Omega$. Choose $\hat{\delta} > 0$ such that $B(0, \hat{\delta}) \subset \Omega$. Since U is a viscosity supersolution of the VI, we have

$$H(x, DU(x)) \ge 0, x \in B(0, \delta)$$
 in a viscosity sense.

By Theorem 2.4, we have

$$\begin{cases} U(x) - U(y_x(t,0)) \ge \int_0^t h(y_x(s,0),0)) \, ds \\ \text{for all } x \in B(0,\hat{\delta}) \text{ and all } 0 \le t \le \tau_x[0], \\ \text{where } \tau_x[0] = \inf\{t \ge 0 : U(y_x(t,0)) = \Phi(y_x(t,0))\}. \end{cases}$$
(2.59)

Since $h(\cdot, 0) \ge 0$, we have

$$0 \le U(y_0(t,0)) \le U(0) = 0$$
, for all $0 \le t \le \tau_0[0]$.

Thus $U(y_0(t,0)) = 0$ for all $0 \le t \le \tau_0[0]$. By the positive definite property of $U, y_0(t,0) = 0$ for all $0 \le t \le \tau_0[0]$.

Next we want to show that for each $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that (1) if $|x| < \delta$ then $|y_x(t,0)| < \epsilon$ for all $0 \le t \le \tau_x[0]$, and (2) if $|x| < \delta$ and $\tau_x[0] = \infty$ then $\lim_{t\to 0} y_x(t,0) = 0$. Let $\epsilon > 0$. Define $r_{\epsilon} = \inf\{U(x) \mid x \in \partial B(0,\epsilon)\}$. Note that $r_{\epsilon} > 0$ because U(x) > 0 for $x \ne 0$, U(0)=0 and $0 \notin \partial B(0,\epsilon)$. By the continuity of U, choose $0 < \delta \le \min\{\epsilon, \hat{\delta}\}$ such that if $|x| < \delta$ then $U(x) < r_{\epsilon}$. Thus for any $x \in B(0, \delta)$, by (2.59) we have

$$U(y_x(t,0)) + \int_0^t h(y_x(s,0),0) \, ds \le U(x) < r_{\epsilon}, \text{ for all } 0 \le t \le \tau_x[0].$$

Since $h(\cdot, 0) \ge 0$, we have

$$U(y_x(t,0)) < r_{\epsilon}$$
, for all $0 \le t \le \tau_x[0]$.

We conclude that there is no $\bar{t} \in [0, \tau_x[0]]$ such that $y_x(\bar{t}, 0) \in \partial B(0, \epsilon)$, and hence by connectedness $y_x(t, 0) \in B(0, \epsilon)$ for $0 \leq t \leq \tau_x[0]$ as required. Moreover if $|x| < \delta$ and $\tau_x[0] = \infty$, then we have

$$U(y_x(t,0)) + \int_0^t h(y_x(s,0),0) ds \le U(x), \text{ for all } t \ge 0.$$

Since U is nonnegative, we have

$$\int_0^t h(y_x(s,0),0)ds \le U(x) \le \infty, \text{ for all } t \ge 0.$$

By the continuity of f and the boundedness of $y_x(\cdot, 0)$, it follows that $y_x(\cdot, 0)$ is uniformly continuous, and so is $h(y_x(\cdot, 0), 0)$. By Lemma 2.19, $\lim_{t\to\infty} h(y_x(t, 0), 0) = 0$. By (A7), we have $\lim_{t\to\infty} y_x(t, 0) = 0$. \diamondsuit

2.5 Computation of optimal stopping-time problem with one-dimensional state space

We now consider the optimal stopping-time problem for the case of one-dimensional state space (n = 1). In addition we assume that the stopping cost function Φ is C^1 on \mathbb{R} .

As a preliminary step, we explicate the structure of solutions U of VI which are piecewise C^1 for the one-dimensional situation. First note that, as a consequence of Proposition 2.3, the VI can be rewritten as three conditions:

$$U(x) \leq \Phi(x) \tag{2.60}$$

$$H(x, D^+U(x)) \leq 0$$
 (2.61)

$$H(x, D^{-}U(x)) \ge 0$$
 for all x with $U(x) < \Phi(x)$ (2.62)

We analyze these conditions under the assumption that the solution U is piecewise C^1 . Thus, each real number x either is a smooth point for U where the derivative U'(x) exists, or x is a nonsmooth point where the one-sided derivatives $U'(x^-)$ and $U'(x^+)$ exist with different values. Then there are three cases:

- (i) x is smooth, then $D^+U(x) = \{U'(x)\} = D^-U(x)$.
- (ii) x is nonsmooth with $U'(x^{-}) < U'(x^{+})$: then $D^{+}U(x) = \emptyset$ and $D^{-}U(x) = [U'(x^{-}), U'(x^{+})]$.
- (iii) x is nonsmooth with $U'(x^-) > U'(x^+)$: then $D^+U(x) = [U'(x^+), U'(x^-)]$ and $D^-U(x) = \emptyset$.

The conditions (2.60)–(2.62) for each of these cases break down as follows:

- (i) U'(x) exists: If $U(x) = \Phi(x)$, then $U'(x) = \Phi'(x)$ since $U \Phi$ has a local maximum at x and $H(x, U'(x)) = H(x, \Phi'(x)) \le 0$; otherwise, $U(x) < \Phi(x)$ and H(x, U'(x)) = 0.
- (ii) $U'(x^-) < U'(x^+)$: Then $U(x) = \Phi(x)$, or $U(x) < \Phi(x)$ and $H(x,p) \ge 0$ for all $p \in [U'(x^-), U'(x^+)]$.
- (iii) $U'(x^{-}) > U'(x^{+})$: Then $H(x, p) \le 0$ for all $p \in [U'(x^{+}), U'(x^{-})]$.

We next analyze in detail the particular case where

$$f(y,b) := g(y) + b; \quad h(y,b) = y^2$$

where both the state vector y and the input-disturbance signal b take values in \mathbb{R} . The terminal cost function Φ is assumed to be a smooth function defined on all of \mathbb{R} with strictly positive values everywhere ($\Phi(x) > 0$ for all $x \in \mathbb{R}$). We assume that g is continuous with continuous, bounded derivative on all of \mathbb{R} . If we take the admissible control set B to be a large but fixed bounded interval $[-M.M] \subset \mathbb{R}$, then the hypotheses (A0)-(A5) are all satisfied. (This assumption is imposed only to guarantee the validity of assumptions (A1)-(A3) required for our general theory; most of the discussion below applies even with B equal to all of \mathbb{R} .) In addition, we assume that

$$g(0) = 0 \text{ and } g'(0) < 0;$$
 (2.63)

this guarantees that 0 is a locally asymptotically stable equilibrium point for the undisturbed system, and that the input-output system $\dot{y} = g(y) + b$, $z = y^2$ (with stopping options ignored) is at least locally γ -dissipative for γ with respect to the equilibrium point 0 (see below). It also then follows from Taylor series approximation that there is a constant $\gamma > 0$ so that

$$|g(x)| < \gamma |x| \text{ for } 0 < |x| < \delta_0 \text{ for some } \delta_0 > 0.$$

$$(2.64)$$

In the sequel we shall assume that γ has been chosen sufficiently large so that (2.64) holds. The Hamiltonian function works out to be

$$H(x,p) = \inf_{b} \{ -(g(x) + b) \cdot p - x^{2} + \gamma^{2} b^{2} \}$$

= $-\frac{1}{4\gamma^{2}} p^{2} - g(x)p - x^{2},$ (2.65)

at least as long as $b_{crit} = \frac{1}{2\gamma^2}p$ is an admissible control, i.e., as long as $|p| \leq 2\gamma^2 M$. (If $|p| > 2\gamma^2 M$, then the constraint $|b| \leq M$ influences the infimum in (2.65) and the formula for H in (2.65) is incorrect.) Provided also that $|x| \leq \gamma |g(x)|$, the equation H(x, p) = 0 then has two distinct real solutions

$$p_{\pm}(x) := 2\gamma^2 \left[-g(x) \pm \sqrt{g(x)^2 - x^2/\gamma^2} \right].$$

In general, $H(x, p) \leq 0$ if and only if $p \leq p_{-}(x)$, $p \geq p_{+}(x)$, or $|x| > \gamma |g(x)|$.

Note that from our assumptions (2.63) and (2.64) on g and γ , we have

$$p_+(0) = p_-(0) = 0,$$

 $p_+(x) > p_-(x) > 0$ for $0 < x < \delta_0$, and $p_-(x) < p_+(x) < 0$ for $-\delta_0 < x < 0.$

We may then solve for the local storage function $S_a(x)$ in the neighborhood $(-\delta_0, \delta_0)$ of the origin:

$$S_a(x) = \begin{cases} \int_0^x p_-(s) \, ds & \text{for } 0 \le x < \delta_0, \\ -\int_x^0 p_+(s) \, ds & \text{for } -\delta_0 < x \le 0. \end{cases}$$
(2.66)

Then $S_a(x) > 0$ for $0 < |x| < \delta_0$ with $S_a(0) = 0$, and is the minimal solution of the HJB equation H(x, S'(x)) = 0 on $(-\delta_0, \delta_0)$.

Note that $S_a(x)$ so defined is the available storage function for the input-output system with all stopping options ignored for the given attenuation level γ satisfying (2.64). From (2.16) and (2.18), we see that in general $0 \leq W(x) \leq S_a(x)$ and hence W(0) = 0. This gives a starting point for our construction of the minimal solution of the VI.

To simplify the description of the algorithm, we assume: (i) $|x| < \gamma |g(x)|$ for 0 < |x|, so $p_{\pm}(x)$ exist and satisfy $p_{-}(x) < p_{+}(x)$ for $x \neq 0$, (ii) $p_{-}(x) < M$ for all x, and (iii) $-M < p_{+}(x)$ for all x. We expect that other degenerate cases can be handled with similar ideas, but with more complicated notation and special considerations.

Our algorithm for constructing the minimal piecewise-smooth solution of the VI for the nice generic case then is as follows:

Step 1. Define $U(x) = S_a(x)$ as in (2.66) for $0 \le x \le x_1$ where x_1 is the first point to the right of 0 where $S_a(x) = \int_0^x p_-(s) \, ds > \Phi(x)$ on some interval $x_1 < x < x_1 + \delta$. If no such x_1 exists, define U(x) by (2.66) for all x > 0.

- Step 2. At the point x_1 , we necessarily have $U(x_1) = \Phi(x_1)$ and $p_-(x_1) \ge \Phi'(x_1)$. Assume that $p_-(x_1) > \Phi'(x_1)$. Then it follows that $H(x, \Phi'(x)) < 0$ for $x_1 \le x \le x + \delta$ for some $\delta > 0$. Let $[x_1, x_2]$ be a maximal interval with left endpoint x_1 on which $H(x, \Phi'(x)) \le 0$. Define $U(x) = \Phi(x)$ for $x_1 \le x \le x_2$.
- **Step 3.** By continuity, necessarily $\Phi'(x) \le p_{-}(x)$ for $x_1 \le x \le x_2$ with equality at $x = x_2$. Assume that $\Phi'(x) > p_{-}(x)$ for x in an interval to the right of x_2 . It follows that

$$\Phi(x_2) + \int_{x_2}^x p_-(s) \, ds < \Phi(x) \tag{2.67}$$

on an interval $(x_2, x + \delta)$ to the right of x_2 . As in Step 1, choose x_3 so that (x_2, x_3) is a maximal interval with left end point equal to x_2 for which (2.67) holds. Define

$$U(x) = \Phi(x_2) + \int_{x_2}^x p_{-}(s) \, ds$$

for $x_2 \leq x \leq x_3$.

- **Step 4.** At x_3 , proceed as in Step 2. Continue this process indefinitely to define U(x) on a maximal interval to the right of 0.
- To construct U(x) at points x to the left of 0, one uses an analogous procedure.
- Step 1-. Define $U(x) = S_a(x)$ as in (2.66) for $x_1^- \le x \le 0$ where x_1^- is the first point to the left of 0 where $-\int_x^0 p_+(s) ds > \Phi(x)$ on some interval $x_1^- \delta < x < x_1^-$. If no such x_1^- exists, use (2.66) to define U(x) for all x < 0.
- Step 2-. At the point x_1^- , we necessarily have $U(x_1^-) = \Phi(x_1^-)$ and $p_+(x_1^-) \leq \Phi'(x_1^-)$. Assume that $p_+(x_1^-) < \Phi'(x_1^-)$. Then it follows that $H(x, \Phi'(x)) < 0$ for $x_1^- - \delta \leq x \leq x_1^-$ for some $\delta > 0$. Let $[x_2^-, x_1^-]$ be a maximal interval with right endpoint x_1^- on which $H(x, \Phi'(x)) \leq 0$. Define $U(x) = \Phi(x)$ for $x_2^- \leq x \leq x_1^-$.
- **Step 3-.** By continuity, necessarily $\Phi'(x) \ge p_+(x)$ for $x_2^- \le x \le x_1^-$ with equality at $x = x_2^-$. Assume that $\Phi'(x) < p_+(x)$ for x in an interval to the left of x_2^- . It follows that

$$\Phi(x_2^-) - \int_x^{x_2^-} p_+(s) \, ds < \Phi(x) \tag{2.68}$$

on an interval $(-\delta + x_2^-, x_2^-)$ to the left of x_2 . As in Step 1-, choose x_3^- so that (x_3^-, x_2^-) is a maximal interval with right end point equal to x_2^- for which (2.68) holds. Define

$$U(x) = \Phi(x_2) - \int_x^{x_2^-} p_+(s) \, ds$$

for $x_3^- \leq x \leq x_2^-$.

Step 4-. At x_3^- , proceed as in Step 2-. Continue this process indefinitely to define U(x) on a maximal interval to the left of 0.

The construction may fail (in particular, Steps 2 and 3) if special tangencies occur, but we expect that it or a minor modification will succeed in most examples.

One can check that this procedure produces a piecewise-smooth solution of VI with the U(0) = 0. We describe how to check this statement for x > 0; the analysis is parallel for x < 0. Indeed, by construction, $U(x) \leq \Phi(x)$ for all x. At a smooth point where $U(x) < \Phi(x)$, we have $U'(x) = p_-(x)$, and hence H(x, DU(x)) = 0 as required at such points. At a smooth point where $U(x) = \Phi(x)$, we have $U'(x) = \Phi'(x) \leq p_-(x)$ from which we get $H(x, DU(x)) \leq 0$ as required at such points. Nonsmooth points occur in the situation arising in Step 2. At such a point $U(x) = \Phi(x)$ and $U'(x^-) = p_-(x) > U'(x^+) = \Phi'(x)$. Hence $H(x, p) \leq 0$ for $p \in [\Phi'(x), p_-(x)] = [U'(x^+), U'(x^-)]$. This verifies the conditions (i)–(iii) for a piecewise- C^1 function U to be a solution of VI.

We next analyze the amount of nonuniqueness in the construction. We assume that our solution U satisfies U(0) = 0 (as is required if U is to be the minimal viscosity subsolution W as noted above). By assumption Φ is strictly positive everywhere, and hence $U(0) < \Phi(0)$. We seek to construct a new solution $\widetilde{U}(x)$ with $\widetilde{U}(0) = U(0)$ but $\widetilde{U}(x) \neq U(x)$ for x in a neighborhood of 0 to the right of 0. By continuity, we still have $\widetilde{U}(x) < \Phi(x)$ for x close to 0 to the right of 0, hence, for such x we must have $H(x, \widetilde{U}'(x)) = 0$. Thus, for smooth points to the right of 0, we have either $\widetilde{U}'(x) = p_{-}(x)$ or $\widetilde{U}'(x) = p_{+}(x)$. If we choose $p_{-}(x)$, we simply recover U(x). Thus, to get a different solution, we must have $\widetilde{U}(x) = U(0) + \int_0^x p_+(s) ds$, leading to $\widetilde{U}(x) > U(x)$ for x > 0 and close to 0. At a point $x_0 > 0$ where $\Phi(x_0) > U(x_0)$, the analysis is similar. Suppose that $\widetilde{U}(x_0) = U(x_0)$ but $\widetilde{U}(x) \neq U(x)$ for x in a neighborhood of x_0 to the right of x_0 . Then by continuity we still have $\widetilde{U}(x) < \Phi(x)$ for x close to x_0 to the right of x_0 , hence, for such x we must have $H(x, \widetilde{U}'(x)) = 0$. Thus, for smooth points to the right of x_0 , we have either $\widetilde{U}'(x) = p_-(x)$ or $\widetilde{U}'(x) = p_+(x)$. If we choose $p_-(x)$, we simply recover U(x). Thus, to get a different solution we must take $\widetilde{U}(x) = U(x_0) + \int_{x_0}^x p_+(s) \, ds$. The point x_0 becomes a nonsmooth point for $\widetilde{U}(x)$ with $\widetilde{U}'(x_0) = p_-(x_0) < \widetilde{U}'(x_0) = p_+(x_0)$ and $\widetilde{U}(x_0) < \Phi(x_0)$. Then the applicable condition for \widetilde{U} to be a viscosity solution of the VI at x_0 is condition (ii); condition (ii) collapses to $H(x,p) \ge 0$ for $p \in [p_-(x_0), p_+(x_0)]$ which is obviously true. In this way we can modify U(x) to a new solution $\widetilde{U}(x)$ but necessarily with $\widetilde{U}(x) > U(x)$ near x_0 . However, once we are following $p_+(x)$ with $\widetilde{U}(x) < \Phi(x)$, it is not possible to switch slopes from $p_+(x)$ back to $p_-(x)$, as in this case the inequality in condition (iii) for a viscosity solution at such a nonsmooth point will be going in the wrong direction. Once $\widetilde{U}(x)$ (following the slope $p_+(x)$) hits a point x_1 where $\widetilde{U}(x_1) = \Phi(x_1)$ we meet an obstruction to continuation of this solution $\widetilde{U}(x)$ to the right of x_1 : at such a point we necessarily have $p_+(x_1) = \widetilde{U}'(x_1) > \Phi'(x_1)$; to guarantee $\widetilde{U}(x) \leq \Phi(x)$ for x to the right of x_1 , we must choose $\widetilde{U}'(x_1^+) \leq \Phi'(x_1)$. But then the applicable condition (iii) for \widetilde{U} to be a viscosity solution of VI at x_1 becomes

$$H(x_1, p) \leq 0 \text{ for } p \in [\widetilde{U}'(x_1^+), \widetilde{U}'(x_1^-)] \supset [\Phi'(x_1), p_+(x_1)]$$

which is necessarily violated. At points x_0 where $U(x_0) = \Phi(x_0)$ and $\Phi'(x_0) < p_-(x_0)$, by a similar argument one can see that it is not possible to arrange for a new solution $\widetilde{U}(x)$ to agree with U at x_0 and depart from U in a neighborhood of x_0 . A parallel analysis as one decreases x away from the initial point x = 0 leads to similar conclusions.

In the analysis above, we see that any solution \widetilde{U} also satisfying the given initial condition $\widetilde{U}(0) = 0$ other than the solution U given by the constructive algorithm given above (a) may have a smaller maximal interval of existence, and (b) satisfies $\widetilde{U}(x) \geq U(x)$ wherever both are defined. Hence we conclude that the construction given above (when applicable) gives the minimal piecewise- C^1 viscosity solution U of the VI satisfying the initial condition U(0) = 0. In this way, we then arrive at the minimal, nonnegative, piecewise- C^1 solution of the VI, and hence at the minimal stopping-time storage function or the lower value function for the game W(x), at least under the assumption that the lower-value function W(x) is piecewise- C^1 .

We now illustrate this construction with the following simple example:

with gain rate

$$\gamma = 2. \tag{2.70}$$

In a linear-quadratic robust control system such as this it would be typical to impose no constraints on the disturbance values $b \in B$. However to satisfy our hypothesis (A1) we will take M = 1, and hence

$$B = [-1, 1]. \tag{2.71}$$

One may check that for $|p| \leq 2\gamma^2 M = 8$ the Hamiltonian (2.3) works out as

$$H(x,p) = \frac{-1}{16}p^2 + px - x^2.$$
 (2.72)

For |p| > 8 the constraint $|b| \le 1$ influences the infimum in (2.3), so that (2.72) is incorrect. However in our example no values of |p| > 8 will occur. We take the stopping cost to be

$$\Phi(x) = \frac{5}{4} + \cos(4x). \tag{2.73}$$

All hypothesis (A0)-(A5) are satisfied.

The solution U(x) of (VI) resulting from the construction is plotted in Figure 2.2 below. We discuss only x > 0 in what follows. (By symmetry we will have U(x) = U(-x) for x < 0.) First, observe that the minimal solution of H(x, p) = 0 is

$$p_{-}(x) = (8 - 4\sqrt{3})x.$$

The minimal nonnegative solution of H(x, S'(x)) = 0 is the available storage function (2.14) for our system (2.69):

$$S_a(x) = \int_0^x p_-(t) \, dt = (4 - 2\sqrt{3})x^2.$$

The initial segment of the solution is



Figure 2.2: Example solution of variational inequality

$$U(x) = S_a(x) \quad \text{for } 0 \le x < x_1.$$

The value $x_1 = .723487$ is determined by solving $S_a(x_1) = \Phi(x_1)$. Next, Step 2 extends the construction of U to

 $U(x) = \Phi(x) \quad \text{for } x_1 \le x < x_2.$

The value of x_2 is the first $x > x_1$ at which $H(x, \Phi'(x)) < 0$ fails. The value $x_2 = .842313$ is located by solving $\Phi'(x) = p_-(x)$.

The construction now proceeds in Step 3, using $U'(x) = p_{-}(x)$ or

$$U(x) = \Phi(x_2) - S_a(x_2) + S_a(x) \quad \text{on an interval } x_2 \le x < x_3,$$

where x_3 is maximal such that $U(x) \leq \Phi(x)$ for $x_2 \leq x < x_3$. This turns out to be $x_3 = 1.84258$. Now we repeat Step 2 to find

$$U(x) = \Phi(x) \quad \text{for } x_3 \le x < x_4,$$

with $x_4 = 2.54367$ the first solution of $\Phi'(x) = p_-(x)$ beyond x_3 . Beyond x_4 we take another section with

$$U(x) = \Phi(x_4) - S_a(x_4) + S_a(x)$$
 on an interval $x_4 \le x < x_5$

with $x_5 = 3.11278$ determined again by $U(x) = \Phi(x)$. Finally, we find that $\Phi'(x) < p_-(x)$ for all $x > x_5$. This means that the remainder of the definition of U is

$$U(x) = \Phi(x)$$
 for $x_5 \le x$.

The optimal stopping rule, as in Theorem 2.16, for this example is to stop at the first instant the state y(t) enters the set

$$[x_1, x_2] \cup [x_3, x_4] \cup [x_5, \infty)$$

on which $U(x) = \Phi(x)$.

In higher dimensions, the initialization U(0) = 0 must be given along a submanifold of codimension 1, as in the method of bicharacteristics, or at an equilibrium point for the Hamiltonian flow, as in the method of stable or antistable invariant manifolds. The analogue of Step 1 is then to use the method of bicharacteristics to produce a solution U(x) of H(x, DU(x)) = 0 for x close to the initial manifold on which $U(x) < \Phi(x)$. This function U(x) will solve the VI as long as the inequality $U(x) \le \Phi(x)$ continues to hold. In higher dimensions, the boundary of the set $\Delta = \{x \colon U(x) = \Phi(x)\}$ no longer consists of isolated points, but rather is a more complicated curve or surface, the free boundary associated with the variational inequality.

Chapter 3

Robust optimal switching-control problems

In this chapter, we formulate a robust optimal control problem for a general nonlinear statespace systems with finitely many admissible controls and with costs assigned to the switching of the controls.

3.1 Preliminaries

In this section we list some assumptions and definitions. We consider a general nonlinear system

$$\Sigma_{sw} \left\{ \begin{array}{ll} \dot{y} = f(y, a, b), \quad y(0) = x \in \mathbb{R}^n \\ z = h(y, a, b) \end{array} \right.$$

where $y(\cdot) \in \mathbb{R}^n$ is the state, $a(\cdot) \in A \subset \mathbb{R}^p$ is the control input, $b(\cdot) \in B \subset \mathbb{R}^m$ is the disturbance and $z(\cdot) \in \mathbb{R}$ is the output of the system. Usually in applications $h(y, a, b) = |\bar{h}(y, a)|^2$, where $\bar{h} : \mathbb{R}^n \times A \to \mathbb{R}^r$. We assume that the set A of admissible controls is a finite set $A = \{a^1, a^2, \cdots, a^r\}$. The control signals $a(\cdot)$ then are necessarily piecewise constant with values in A. We normalize control signals $a(\cdot)$ to be right continuous, and refer to the value a(t) as the current control and $a(t^-)$ as the old current control at time t. We assume that there is a control input index i_0 for which $f(y, a^{i_0}, 0) = 0$ and $h(y, a^{i_0}, 0) = 0$, so that y = 0 is an equilibrium point for the autonomous system induced by setting $a(t) = a^{i_0}$, b(t) = 0 for all $t \ge 0$. In addition we assume that a cost $k(a^i, a^j) \ge 0$ is assigned at each time instant τ_n at which the controller switches from old current control $a(\tau_n^-) = a^i$ to new current control $a(\tau_n) = a^j$. For a given old initial control $a(0^-)$, the associated control decision is to choose switching times

$$0 \le \tau_1 < \tau_2 < \dots, \quad \lim_{n \to \infty} \tau_n = \infty$$

and controls

$$a(\tau_1), a(\tau_2), a(\tau_3), \ldots$$

such that the controller switches from the old current control $a(\tau_n)$ to the (new) current control $a(\tau_n) \neq a(\tau_n)$ at time τ_n , where we set

$$a(t) = \begin{cases} a(0^{-}), & t \in [0, \tau_1), \\ a(\tau_n), & t \in [\tau_n, \tau_{n+1}), n = 1, 2, \dots, \end{cases}$$

if $\tau_1 > 0$ and

$$a(t) = a(\tau_n), \quad t \in [\tau_n, \tau_{n+1}), \ n = 1, 2, \dots,$$

otherwise. We assume that the state $y(\cdot)$ does not jump at the switching time τ_i , i.e., the solution $y(\cdot)$ is assumed to be absolutely continuous. The set of admissible controls for our problem is then given by

$$\mathcal{A} = \{ a(\cdot) = \sum_{i \ge 1} a_{i-1} \mathbb{1}_{[\tau_{i-1}, \tau_i)}(\cdot) : [0, +\infty) \to A | a_i \in A; a_i \neq a_{i-1} \text{ for } i \ge 1, \\ 0 = \tau_0 \le \tau_1 < \tau_2 < \cdots, \tau_i \uparrow \infty \},$$

consisting of piecewise-constant right-continuous functions on $[0, \infty)$ with values in the control set A, where we denote by τ_1, τ_2, \ldots the points at which control switchings occur. We assume that the set B of admissible disturbance values is compact with $0 \in B$ and the set of admissible disturbance functions is given by

$$\mathcal{B} = \{b: [0,\infty) \to B \mid \int_0^T |b(s)|^2 ds < \infty, \text{ for all } T > 0\}.$$

A strategy is a map $\alpha \colon \mathbb{R}^n \times A \times \mathcal{B} \to \mathcal{A}$ with value at (x, a^j) denoted by $\alpha_x^j[\cdot]$. The strategy α assigns control function $a(t) = \alpha_x^j[b](t)$ if the augmented initial condition is (x, a^j) and the disturbance is $b(\cdot)$. Thus, if it happens that $\tau_1 > \tau_0 = 0$, then $a(t) = a_0 = a^j$, for $t \in [\tau_0, \tau_1)$. Otherwise $a(t) = a_1 \neq a^j$, for $t \in [0, \tau_2) = [\tau_1, \tau_2)$ and an instantaneous charge of $k(a^j, a(0))$ is incurred at time 0 in the cost function. A strategy α is said to be *nonanticipating* if, for each $x \in \mathbb{R}^n$ and $j \in \{1, \ldots, r\}$, for any T > 0 and $b, \ \bar{b} \in \mathcal{B}$ with $b(s) = \bar{b}(s)$ for all $s \leq T$, it follows that $\alpha_x^j[b](s) = \alpha_x^j[\bar{b}](s)$ for all $s \leq T$. We denote by Γ the set of all nonanticipating strategies:

$$\Gamma := \{ \alpha \colon \mathbb{R}^n \times A \times \mathcal{B} \to \mathcal{A} \mid \alpha_x^j \text{ is nonanticipating for each } x \in \mathbb{R}^n \text{ and } j = 1, \dots, r \}.$$

We make following assumptions on the problem data f, h, k:

- (A8) $f : \mathbb{R}^n \times A \times B \to \mathbb{R}^n$ and $h : \mathbb{R}^n \times A \times B \to \mathbb{R}$ are continuous;
- (A9) f and h are bounded on $B(0, R) \times A \times B$ for all R > 0;

(A10) there are moduli ω_f and ω_h such that

$$|f(x, a, b) - f(y, a, b)| \leq \omega_f(|x - y|, R)$$

$$|h(x, a, b) - h(y, a, b)| \leq \omega_h(|x - y|, R),$$

for all $x, y \in B(0, R)$, R > 0, $a \in A$ and $b \in B$;

(A11)
$$|f(x,a,b) - f(y,a,b)| \le L|x-y|$$
 for all $x, y \in \mathbb{R}^n$, $a \in A$ and $b \in B$;

(A12) $k: A \times A \to \mathbb{R}$ and

$$\begin{array}{rcl} k(a^{j},a^{i}) &< k(a^{j},a^{d}) + k(a^{d},a^{i}) \\ k(a^{j},a^{i}) &> 0 \\ k(a^{j},a^{j}) &= 0, \end{array}$$

for all a^d , a^i , $a^j \in A$, $d \neq i \neq j$;

(A13) $h(x, a, 0) \ge 0$ for all $x \in \mathbb{R}^n$, $a \in A$.

We look at trajectories of the nonlinear system

$$\begin{cases} \dot{y}(t) = f(y(t), a(t), b(t)) \\ y(0) = x, \end{cases}$$
(3.1)

Under the assumptions (A8), (A9) and (A11), for given $x \in \mathbb{R}^n$, $a \in \mathcal{A}$ and $b \in \mathcal{B}$, the solution of (3.1) exists uniquely for all $t \geq 0$. We denote by $y_x(\cdot, a, b)$ or simply $y_x(\cdot)$ or $y(\cdot)$ the unique solution of (3.1) corresponding to the choice of the initial condition $y(0) = x \in \mathbb{R}^n$, the control $a(\cdot) \in \mathcal{A}$ and the disturbance $b(\cdot) \in \mathcal{B}$. We also have the usual estimates on the trajectories (the proofs are similar to those in Section 2.1):

$$|y_x(t,a,b) - y_z(t,a,b)| \le e^{Lt}|x-z|, \quad t > 0$$
(3.2)

$$|y_x(t,a,b) - x| \leq M_x t, \quad t \in [0, 1/M_x],$$
(3.3)

$$|y_x(t,a,b)| \leq (|x| + \sqrt{2kt})e^{Kt}$$
 (3.4)

for all $a \in \mathcal{A}, b \in \mathcal{B}$, where

$$M_x := \max\{|f(z, a, b)| : |x - z| \le 1, a \in A, b \in B\}$$

$$K := L + \max\{|f(0, a, b)| : a \in A, b \in B\}.$$

For a specified gain parameter $\gamma > 0$, we define the Hamiltonian function $H^j : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ by setting

$$H^{j}(y,p) := \min_{b \in B} \{-p \cdot f(y,a^{j},b) - h(y,a^{j},b) + \gamma^{2}|b|^{2}\}, \ j = 1, \dots, r.$$

Note that $H^j(y,p) < +\infty$ for all $y,p \in \mathbb{R}^n$ by (A9). We now want to show that the Hamiltonian H^j is continuous on $\mathbb{R}^n \times \mathbb{R}^n$.

Lemma 3.1 Assume (A8)-(A10). Then for each $j \in \{1, 2, ..., r\}$, H^j is continuous and it satisfies

$$|H^{j}(x,p) - H^{j}(y,p)| \le L|x-y||p| + \omega_{h}(|x-y|,R)$$

for all $p \in \mathbb{R}^n$, $x, y \in B(0, R)$, R > 0, and

$$|H^{j}(x,p) - H^{j}(x,q)| \le L(|x|+1)|p-q|$$

for all $x, p, q \in \mathbb{R}^n$.

Proof Let $a^j \in A$ and $\epsilon > 0$. Fix $p \in \mathbb{R}^n$ and $x, y \in B(0, R)$, R > 0. Choose $\overline{b} \in B$ such that

$$H^{j}(y,p) \geq -f(y,a^{j},\bar{b}) \cdot p - h(y,a^{j},\bar{b}) + \gamma^{2}|\bar{b}|^{2} - \epsilon.$$

From the definition of H^j , we have

$$H^j(x,p) \le -f(x,a^j,\bar{b}) \cdot p - h(x,a^j,\bar{b}) + \gamma^2 |\bar{b}|^2.$$

Thus

$$\begin{aligned} H^{j}(x,p) - H^{j}(y,p) &\leq f(y,a^{j},\bar{b}) \cdot p - f(x,a^{j},\bar{b}) \cdot p + h(y,a^{j},\bar{b}) - h(x,a^{j},\bar{b}) + \epsilon \\ &\leq |p||f(x,a^{j},\bar{b}) - f(y,a^{j},\bar{b})| + |h(x,a^{j},\bar{b}) - h(y,a^{j},\bar{b})| + \epsilon \\ &\leq L|x-y||p| + \omega_{h}(|x-y|,R) + \epsilon \end{aligned}$$
(3.5)

Next choose $\tilde{b}\in B$ such that

$$H^{j}(x,p) \geq -f(x,a^{j},\tilde{b}) \cdot p - h(x,a^{j},\tilde{b}) + \gamma^{2}|\tilde{b}|^{2} - \epsilon.$$

From the definition of H^j , we have

$$H^{j}(y,p) \leq -f(y,a^{j},\tilde{b}) \cdot p - h(y,a^{j},\tilde{b}) + \gamma^{2}|\tilde{b}|^{2}.$$

Thus

$$\begin{aligned} H^{j}(x,p) - H^{j}(y,p) &\geq -(f(x,a^{j},\tilde{b}) \cdot p - f(y,a^{j},\tilde{b}) \cdot p) - (h(x,a^{j},\tilde{b}) - h(y,a^{j},\tilde{b})) - \epsilon \\ &\geq -|p||f(x,a^{j},\bar{b}) - f(y,a^{j},\bar{b})| - |h(x,a^{j},\bar{b}) - h(y,a^{j},\bar{b})| - \epsilon \\ &\geq -(L|x-y||p| + \omega_{h}(|x-y|,R) + \epsilon) \end{aligned}$$
(3.6)

By (3.5) and (3.6) we have

$$|H^{j}(x,p) - H^{j}(y,p)| \le L|x-y||p| + \omega_{h}(|x-y|,R) + \epsilon.$$

Since ϵ is arbitrary, the result is obtained.

Fix $x, p, q \in \mathbb{R}^n$. Choose $\hat{b} \in B$ such that

$$H^j(x,q) \ge -f(x,a^j,\hat{b}) \cdot q - h(x,a^j,\hat{b}) + \gamma^2 |\hat{b}|^2 - \epsilon.$$

From the definition of H^j , we have

$$H^{j}(x,p) \leq -f(x,a^{j},\hat{b}) \cdot p - h(x,a^{j},\hat{b}) + \gamma^{2}|\hat{b}|^{2}.$$

Thus

$$H^{j}(x,p) - H^{j}(x,q) \leq -f(x,a^{j},\hat{b}) \cdot p + f(x,a^{j},\hat{b}) \cdot q + \epsilon$$

$$\leq |f(x,a^{j},\hat{b})||p-q| + \epsilon$$

$$\leq L(|x|+1)|p-q| + \epsilon \qquad (3.7)$$

Next choose $\hat{b} \in B$ such that

$$H^{j}(x,p) \ge -f(x,a^{j},\widehat{b}) \cdot p - h(x,a^{j},\widehat{b}) + \gamma^{2}|\widehat{b}|^{2} - \epsilon$$

From the definition of H^j , we have

$$H^{j}(x,q) \leq -f(x,a^{j},\widehat{b}) \cdot q - h(x,a^{j},\widehat{b}) + \gamma^{2}|\widehat{b}|^{2}.$$

Thus

$$H^{j}(x,p) - H^{j}(x,q) \geq -(f(x,a^{j},\widehat{b}) \cdot p - f(x,a^{j},\widehat{b}) \cdot q) - \epsilon$$

$$\geq -|f(x,a^{j},\widehat{b})||p-q| - \epsilon$$

$$\geq -(L(|x|+1)|p-q| + \epsilon)$$
(3.8)

By (3.7) and (3.8) we have

$$|H^{j}(x,p) - H^{j}(x,q)| \le L(|x|+1)|p-q| + \epsilon$$

Since ϵ is arbitrary, the result is obtained. \Diamond

We now introduce the system of quasivariational inequalities (SQVI)

$$\max\{H^{j}(x, Du^{j}(x)), \ u^{j}(x) - \min_{i \neq j}\{u^{i}(x) + k(a^{j}, a^{i})\}\} = 0, \ x \in \mathbb{R}^{n}, \ j = 1, 2, \dots, r \quad (3.9)$$

Definition 3.2 A vector function $u = (u^1, u^2, ..., u^r)$, where $u^j \in C(\mathbb{R}^n)$, is a viscosity subsolution of the SQVI (3.9) if for any $\varphi^j \in C^1(\mathbb{R}^n)$

$$\max\{H^{j}(x_{0}, D\varphi^{j}(x_{0})), u^{j}(x_{0}) - \min_{i \neq j}\{u^{i}(x_{0}) + k(a^{j}, a^{i})\}\} \le 0, \ j = 1, 2, \dots, r$$

at any local maximum point $x_0 \in \mathbb{R}^n$ of $u^j - \varphi^j$. Similarly u is a viscosity supersolution of the SQVI (3.9) if for any $\varphi^j \in C^1(\mathbb{R}^n)$

$$\max\{H^{j}(x_{1}, D\varphi^{j}(x_{1})), u^{j}(x_{1}) - \min_{i \neq j}\{u^{i}(x_{1}) + k(a^{j}, a^{i})\}\} \ge 0, \ j = 1, 2, \dots, r$$

at any local minimum point $x_1 \in \mathbb{R}^n$ of $u^j - \varphi^j$. Finally u is a viscosity solution of the SQVI if it is simultaneously a viscosity sub- and supersolution.

3.2 Formulations

We consider a state space system Σ_{sw} with a switching-cost function k. We set the cost of running the system up to time $T \geq 0$ with initial state $y(0) = x \in \mathbb{R}^n$, initial old control setting $a(0^-) = a^j$, control signal $a \in \mathcal{A}$ for $t \geq 0$, and disturbance signal $b \in \mathcal{B}$ to be

$$C_{T^{-}}(x, a^{j}, a, b) = \int_{0}^{T} h(y_{x}(t, a, b), a(t), b(t)) dt + \sum_{\tau: 0 \le \tau < T} k(a(\tau^{-}), a(\tau))$$

As the running $\cot \tilde{\ell}(y(t), a^j, a(t), b(t)) = h(y(t), a(t), b(t)) + k(a(t^-), a(t))$, where $a(t^-) = a^j$ if t = 0, involves not only the value y(t) of the state along with the value of the control a(t)and the value of the disturbance b(t) at time t but also the value of the old current control $a(t^-)$, it makes sense to think of the old current control $a(t^-)$ at time t as part of an augmented state vector $y^{aug}(t) = (y(t), a(t^-))$ at time t. This can be done formally by including $a(t^-)$ as part of the state vector, in which case the switching control problem becomes an *impulse* control problem (see [19], where problems of this sort are set in the general framework of hybrid systems). We shall keep the switching-control formalism here; however, in implementing optimization algorithms, we shall see that it is natural to consider augmented state-feedback controls $(x, a^j) \to a(x, a^j)$ rather than merely state-feedback controls $x \to a(x)$ in order to obtain solutions. We shall refer to such augmented state-feedback controls $(x, a^j) \to a(x, a^j) \in A$ as simply switching state-feedback controllers. Note that while the augmented-state is required to compute the instantaneous running cost at time t, only the (nonaugmented) state vector y(t) is needed to determine the state trajectory past time t for a given input signal $(a(\cdot), b(\cdot))$ past time t.

The precise formulation of our optimal control problem is as follows. For a prescribed attenuation level $\gamma > 0$ and given augmented initial state (x, a^j) , we seek an admissible control signal $a(\cdot) = a_{x,j}(\cdot)$ with $a(0^-) = a^j$ so that

$$C_{T^{-}}(x, a^{j}, a, b) \leq \gamma^{2} \int_{0}^{T} |b(t)|^{2} dt + U_{\gamma}^{j}(x)$$
 (3.10)

for all locally L^2 disturbances b, all positive real numbers T and some nonnegative-valued bias function $U^j_{\gamma}(x)$ with $U^{i_0}_{\gamma}(0) = 0$. Note that this inequality corresponds to an inputoutput system having L^2 -gain at most γ , where C_{T^-} replaces the L^2 -norm of the output signal over the time interval [0, T], and where the equilibrium point is taken to be $(0, a^{i_0})$ in the augmented state space. The dissipation inequality (3.10) then can be viewed as an L^2 -gain inequality, and our problem as the analogue of the nonlinear H^{∞} -control problem for systems with switching costs. In the *open loop* version of the problem, the control signal $a(\cdot)$ is simply a piecewise-constant right-continuous function with values in $A = \{a^1, \ldots, a^r\}$. In the *switching state feedback* version of the problem, $a(\cdot)$ is a function of the current state and current old control, i.e., one decides what control to use at time t based on knowledge of the current augmented state $(y(t), a(t^-))$. In the standard game-theoretic formulation of the problem, $a(\cdot)$ is a nonanticipating function of the disturbance b. A refinement of the problem then asks for the admissible control a with $a(0^-) = a^j$ (with whatever information structure) which gives the best system performance, in the sense that the nonnegative functions $U^j(x)$ is as small as possible. A closely related problem formulation is to view the switching-cost system as a game with payoff function

$$J_{T^{-}}(x, a^{j}, a, b) = \int_{[0,T)} l(y_{x}, a^{j}, a, b), \quad a(0^{-}) = a^{j}, \ j = 1, \dots, r,$$

where we view $l(y_x, a^j, a, b)$ as the measure given by

$$l(y(t), a^{j}, a(t), b(t)) = [h(y(t), a(t), b(t)) - \gamma^{2} |b(t)|^{2}] dt + k(a(t^{-}), a(t))\delta_{t}, \qquad a(0^{-}) = a^{j},$$

where δ_t is the unit point-mass distribution at the point t. In this game setting, the disturbance player seeks to use b(t) and T to maximize the payoff while the control player seeks to use the choice of piecewise-constant right-continuous function a(t) to minimize the payoff. The vector lower-value function $V = (V_{\gamma}^1, \ldots, V_{\gamma}^r)$ of this game is then given by

$$V_{\gamma}^{j}(x) = \inf_{\alpha \in \Gamma} \sup_{b \in \mathcal{B}, \ T \ge 0} J_{T^{-}}(x, a^{j}, \alpha_{x}^{j}[b], b), \ j = 1, \dots, r$$
(3.11)

By letting T tend to 0, we see that each component of the vector-valued lower value function $V_{\gamma}(x) = (V_{\gamma}^{1}(x), \ldots, V_{\gamma}^{r}(x))$ is nonnegative. Then by construction $(V_{\gamma}^{1}, \ldots, V_{\gamma}^{r})$ gives the smallest possible value which can satisfy (3.10) (with V_{γ}^{j} in place of U_{γ}^{j}) for some nonanticipating strategy $(x, a^{j}, b) \to \alpha_{x}^{j}[b](\cdot) = a(\cdot)$.

In the standard theory of nonlinear H^{∞} -control, the notion of *storage function* for a dissipative system plays a prominent role. We say that a nonnegative vector function $S = (S^1, \ldots, S^r)$ on \mathbb{R}^n is a *switching-storage function* for the system Σ_{sw} if for each $j \in \{1, \ldots, r\}$ there is piecewise-constant right-continuous function $a(\cdot)$ with $a(0^-) = a^j$ such that

$$S^{j(t_2)}(y_x(t_2, a, b)) - S^{j(t_1)}(y_x(t_1, a, b)) \leq \int_{t_1}^{t_2} [\gamma^2 |b(s)|^2 - h(y_x(s), a(s), b(s))] \, ds - \sum_{t_1 \leq \tau < t_2} k(a(\tau^-), a(\tau))$$
(3.12)
for all $x \in \mathbb{R}^n, \ b \in \mathcal{B}$ and $0 \leq t_1 < t_2$

(where j(t) is specified by $a(t^{-}) = a^{j(t)}$). The control problem then is to find the switching strategy $\alpha : (x, a^j, b) \to \alpha_x^j[b](\cdot)$ which gives the best performance, as measured by obtaining the minimal possible $S(x) = (S^1(x), \ldots, S^r(x))$ as the associated closed-loop storage function. Note that any vector storage function may serve as the vector bias function $U_{\gamma} = (U_{\gamma}^1, \ldots, U_{\gamma}^r)$ in the L^2 -gain inequality (3.10), if in addition $S^{i_0}(0) = 0$. This suggests that the *switching-cost available storage function* $S_{sc,a}$ (i.e., the minimal possible switchingcost storage function over all possible switching strategies) should equal the lower-value function V_{γ} (3.11) for the game described above. We shall see that this is indeed the case with appropriate hypotheses imposed.

3.3 Main results

In this section we show the connection of the lower value function $V_{\gamma} = (V_{\gamma}^1, \ldots, V_{\gamma}^r)$ (see (3.11)) with the SQVI (3.9). We also address the comparison principle between viscosity subsolution and supersolution of the system of quasivariational inequalities satisfying a boundary comparison.

We show that if continuous, then V_{γ} is the minimal, nonnegative, continuous viscosity supersolution of the SQVI. We give two derivations of this characterization of V_{γ} ; one is a direct argument which parallels the argument given in Chapter 2 for the analogous result for optimal stopping-time problems. The second relies on a general comparison principle for viscosity super- and subsolutions of SQVI.

3.3.1 Dynamic programming

In this subsection we address the Dynamic Programming Principle (DPP) and some properties of a lower value vector function V_{γ} . We then use them to show that if continuous, V_{γ} is a viscosity solution of the SQVI. Throughout this subsection, we assume that V_{γ} is finite.

Proposition 3.3 Assume (A8)-(A12). Then for j = 1, 2, ..., r and $x \in \mathbb{R}^n$

$$V_{\gamma}^{j}(x) \leq \min_{i \neq j} \{ V_{\gamma}^{i}(x) + k(a^{j}, a^{i}) \}$$

Proof Fix a pair of indices $i, j \in \{1, ..., r\}$ with $i \neq j$. For a given $x \in \mathbb{R}^n$, $\alpha \in \Gamma$, $b \in \mathcal{B}$ and T > 0, we have

$$\begin{split} \int_{[0,T)} \ell(y_x(s), a^j, \alpha_x^j[b](x), b(s)) &= k(a^j, \alpha_x^j[b](0)) + \int_{[0,T)} \ell(y_x(s), \alpha_x^j[b](0), \alpha_x^j[b](s), b(s)) \\ &= k(a^j, \alpha_x^j[b](0)) - k(a^i, \alpha_x^j[b](0)) \\ &+ k(a^i, \alpha_x^j[b](0)) + \int_{[0,T)} \ell(y_x(s), \alpha_x^j[b](0), \alpha_x^j[b](s), b(s)) \\ &= k(a^j, \alpha_x^j[b](0)) - k(a^i, \alpha_x^j[b](0)) + \int_{[0,T)} \ell(y_x(s), a^i, \alpha_x^j[b](s), b(s)) \\ &< k(a^j, a^i) + \int_{[0,T)} \ell(y_x(s), a^i, \alpha_x^j[b](s), b(s)) \end{split}$$
(3.13)

where the last inequality follows from (A12). By the definition of $V_{\gamma}^{j}(x)$, we have

$$V_{\gamma}^{j}(x) \leq \sup_{b \in \mathcal{B}, T \geq 0} \int_{[0,T)} \ell(y_{x}(s), a^{j}, \alpha_{x}^{j}[b](s), b(s))$$

for all $\alpha \in \Gamma$. Taking the supremum over $b \in \mathcal{B}$ and $T \ge 0$ on the righthand side of (3.13) therefore gives

$$V_{\gamma}^{j}(x) \le k(a^{j}, a^{i}) + \sup_{b \in \mathcal{B}, T \ge 0} \int_{[0,T)} \ell(y_{x}(s), a^{i}, \alpha_{x}^{j}[b](s), b(s)).$$
(3.14)

Given any strategy $\alpha \in \Gamma$, we can always find another $\tilde{\alpha} \in \Gamma$ with $\tilde{\alpha}_x^i[b] = \alpha_x^j[b]$ for each $b \in \mathcal{B}$, and, conversely, for any $\tilde{\alpha} \in \Gamma$ there is a $\alpha \in \Gamma$ so that $\tilde{\alpha}_x^i$ is determined by α in this way. Hence, taking the infimum over all $\alpha \in \Gamma$ in the last terms on the righthand side of (3.14) leaves us with $V_{\gamma}^i(x)$. Thus

$$V^j_{\gamma}(x) \le k(a^j, a^i) + V^i_{\gamma}(x)$$

Since $i \neq j$ is arbitrary, the result follows. \Diamond

Theorem 3.4 (Dynamic Programming Principle) Assume (A8), (A9) and (A11). Then, for j = 1, 2, ..., r, t > 0 and $x \in \mathbb{R}^n$, we have

$$V_{\gamma}^{j}(x) = \inf_{\alpha \in \Gamma} \sup_{b \in \mathcal{B}, T > 0} \{ \int_{[0, t \wedge T)} l(y_{x}(s, a^{j}, \alpha_{x}^{j}[b], b), \alpha_{x}^{j}[b](s), b(s)) + 1_{[0,T)}(t) V_{\gamma}^{i}(y_{x}(t, \alpha_{x}^{j}[b], b)), \qquad \alpha_{x}^{j}[b](t^{-}) = a^{i} \}.$$
(3.15)

where

$$l(y(s), a^{j}, a(s), b(s)) := [h(y(s), a(s), b(s)) - \gamma^{2} |b(s)|^{2}] ds + k(a(s^{-}), a(s)) \delta_{s}.$$

with $a(0^{-}) = a^{j}$.

Proof Fix $x \in \mathbb{R}^n$, $j \in \{1, 2, ..., r\}$ and t > 0. We denote by $\omega(x)$ the right hand side of (3.15). Let $\epsilon > 0$. For any $z \in \mathbb{R}^n$ and any $a^{\ell} \in A$, we pick $\bar{\alpha} \in \Gamma$ such that

$$V_{\gamma}^{\ell}(z) + \epsilon \ge \int_{[0,T)} l(y_z(s), a^{\ell}, \bar{\alpha}_z^{\ell}[b](s), b(s)), \ \forall b \in \mathcal{B}, \ \forall T > 0$$

$$(3.16)$$

We first want to show that $\omega(x) \geq V_{\gamma}^{j}(x)$. Choose $\hat{\alpha} \in \Gamma$ such that

$$\omega(x) + \epsilon \ge \sup_{b \in \mathcal{B}, \ T \ge 0} \{ \int_{[0, t \wedge T)} l(y_x(s), a^j, \hat{\alpha}_x^j[b](s), b(s)) + \mathbb{1}_{[0, T)}(t) V_{\gamma}^i(y_x(t)), \ \hat{\alpha}_x^j[b](t^-) = a^i \}$$

$$(3.17)$$

For each $b \in \mathcal{B}$, $i \in \{1, \ldots, r\}$ and T > 0, choose $\delta \in \Gamma$ so that

$$\delta_x^j[b](s) = \begin{cases} \hat{\alpha}_x^j[b](s) & s < t \wedge T \\ \bar{\alpha}_z^i[b(\cdot + t \wedge T)](s - (t \wedge T)) & s \ge t \wedge T \end{cases}$$

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with $z := y_x(t \wedge T, \hat{\alpha}_x^j[b], b)$. Clearly, δ_x^j is nonanticipating because $\hat{\alpha}_x^j$ and $\bar{\alpha}_z^i$ are. Note that

$$y_x(s+t\wedge T, \delta_x^j[b], b) = y_z(s, \bar{\alpha}_z^i[b(\cdot+t\wedge T)], b(\cdot+t\wedge T)), \text{ for } s \ge 0$$

Thus by the change of variables $\tau = s + t \wedge T$, we have

$$\int_{[0,T-(t\wedge T))} l(y_z(s), a^i, \bar{\alpha}_z^i[b(\cdot + t \wedge T)](s), b(s+t\wedge T)) = \int_{[t\wedge T,T)} l(y_x(\tau), a^j, \delta_x^j[b](\tau), b(\tau))$$
(3.18)

As a consequence of (3.16), (3.17) and (3.18), we have

$$\begin{split} \omega(x) + 2\epsilon &\geq \sup_{b \in \mathcal{B}, \ T > 0} \{ \int_{[0, t \wedge T)} l(y_x(s), a^j, \hat{\alpha}_x^j[b](s), b(s)) \\ &+ 1_{[0, T)}(t) \int_{[t \wedge T, T)} l(y_z(s), a^i, \bar{\alpha}_z^i[b](s), b(s)) \} \\ &= \sup_{b \in \mathcal{B}, \ T > 0} \{ \int_{[0, T)} l(y_x(s), a^j, \delta_x^j[b](s), b(s)) \} \\ &\geq \inf_{\alpha \in \Gamma} \sup_{b \in \mathcal{B}, \ T > 0} \{ \int_{[0, T)} l(y_x(s), a^j, \alpha_x^j[b](s), b(s)) \} \\ &= V_{\gamma}^j(x) \end{split}$$

Since $\epsilon > 0$ is arbitrary, we conclude that $\omega(x) \ge V_{\gamma}^{j}(x)$.

Next we want to show that $\omega(x) \leq V_{\gamma}^{j}(x)$. From the definition of $\omega(x)$, choose $b_{1} \in \mathcal{B}$ and $T_{1} \geq 0$ such that

$$\omega(x) - \epsilon \le \int_{[0,T_1 \wedge t)} l(y_x(s), a^j, \bar{\alpha}^j_x[b_1](s), b_1(s)) + \mathbb{1}_{[0,T_1)}(t) V^i_{\gamma}(y_x(t))$$
(3.19)

where $\bar{\alpha}_x^j$ is defined as in (3.16) and $\bar{\alpha}_x^j[b_1](t^-) = a^i$ for some $a^i \in A$. If $t \ge T_1$, we have

$$\begin{aligned}
\omega(x) - \epsilon &\leq \int_{[0,T_1)} l(y_x(s), a^j, \bar{\alpha}^j_x[b_1](s), b_1(s)) \\
&\leq \sup_{b \in \mathcal{B}, \ T > 0} \{ \int_{[0,T)} l(y_x(s), a^j, \bar{\alpha}^j_x[b](s), b(s)) \} \\
&\leq V^j_\gamma(x) + \epsilon,
\end{aligned}$$

where the last inequality follows from (3.16). If $t < T_1$, we have

$$\omega(x) - \epsilon \le \int_{[0,t)} l(y_x(s), \bar{\alpha}_x^j[b_1](s), b_1(s)) + V_{\gamma}^i(y_x(t)).$$
(3.20)

Set $z := y_x(t, \bar{\alpha}_x^j[b_1], b_1)$. For each $b \in \mathcal{B}$, define $\tilde{b} \in \mathcal{B}$ by

$$\tilde{b}(s) = \begin{cases} b_1(s) & s < t\\ b(s-t) & s \ge t \end{cases}$$

and choose $\widehat{\alpha} \in \Gamma$ so that

$$\widehat{\alpha}_{z}[b](s) = \overline{\alpha}_{x}^{j}[\widetilde{b}](s+t) \quad \text{for } s \ge 0.$$

By definition of V_{γ}^{i} , choose $b_{2} \in \mathcal{B}$ and $T_{2} > 0$ such that

$$V_{\gamma}^{i}(z) - \epsilon \leq \int_{[0,T_{2})} l(y_{z}(s), a^{i}, \widehat{\alpha}_{z}[b_{2}](s), b_{2}(s)).$$

Then, by change of variable $\tau = s + t$, we have

$$V_{\gamma}^{i}(z) - \epsilon \leq \int_{[t,t+T_{2})} l(y_{x}(\tau), a^{j}, \bar{\alpha}_{x}^{j}[\tilde{b}_{2}](\tau), \tilde{b}_{2}(\tau))$$
(3.21)

As a consequence of (3.20) and (3.21) we have

$$\begin{split} \omega(x) - 2\epsilon &\leq \int_{[0,t)} l(y_x(s), a^j, \bar{\alpha}_x^j[b_1](s), b_1(s)) + \int_{[t,t+T_2)} l(y_x(\tau), a^j, \bar{\alpha}_x^j[\tilde{b}_2](\tau), \tilde{b}_2(\tau)) \\ &= \int_{[0,t+T_2)} l(y_x(\tau), a^j, \bar{\alpha}_x^j[\tilde{b}_2](\tau), \tilde{b}_2(\tau)) \\ &\leq \sup_{b \in \mathcal{B}, T > 0} \{ \int_{[0,T)} l(y_x(\tau), a^j, \bar{\alpha}_x^j[b](\tau), b(\tau)) \} \\ &\leq V_{\gamma}^j(x) + \epsilon, \end{split}$$

where the last inequality follows from (3.16). Since $\epsilon > 0$ is arbitrary, for both cases we have $\omega(x) \leq V_{\gamma}^{j}(x)$ as required. \diamondsuit

Corollary 3.5 Assume (A8)-(A4). Then for each $j \in \{1, \ldots, r\}, x \in \mathbb{R}^n, t > 0$

$$V_{\gamma}^{j}(x) \leq \sup_{b \in \mathcal{B}, T \geq 0} \{ \int_{0}^{t \wedge T} [h(y_{x}(s), a^{j}, b(s)) - \gamma^{2} |b(s)|^{2}] ds + \mathbb{1}_{\{t: t < T\}}(t) V_{\gamma}^{j}(y_{x}(t)) \}.$$

Proof Fix $j \in \{1, \ldots, r\}$, $x \in \mathbb{R}^n$ and t > 0. For each $b \in \mathcal{B}$, define $\alpha_x^j[b](s) = a^j$ for all $s \ge 0$. By Theorem 3.4, we have

$$V_{\gamma}^{j}(x) \leq \sup_{b \in \mathcal{B}, \ T \geq 0} \{ \int_{0}^{t \wedge T} [h(y_{x}(s), a^{j}, b(s)) - \gamma^{2} |b(s)|^{2}] ds + \mathbb{1}_{\{t: t < T\}}(t) V_{\gamma}^{j}(y_{x}(t)) \}. \diamondsuit$$

Proposition 3.6 Assume (A8)-(A12). Suppose that for each $j \in \{1, \ldots, r\}$, V^j is continuous. If $V^j_{\gamma}(x) < \min_{i \neq j} \{V^i_{\gamma}(x) + k(a^j, a^i)\}$, then there exists $\tau = \tau_x > 0$ such that for $0 < t < t_x$

$$V_{\gamma}^{j}(x) = \sup_{b \in \mathcal{B}, \ T \ge 0} \{ \int_{0}^{t \wedge T} [h(y_{x}(s), a^{j}, b(s)) - \gamma^{2} |b(s)|^{2}] ds + \mathbb{1}_{\{t: t < T\}}(t) V_{\gamma}^{j}(y_{x}(t)) \}$$

Proof We assume $V_{\gamma}^{j}(x) < \min_{i \neq j} \{V_{\gamma}^{i}(x) + k(a^{j}, a^{i})\}$. From Corollary 3.5, we know that

$$V_{\gamma}^{j}(x) \leq \sup_{b \in \mathcal{B}, \ T > 0} \{ \int_{0}^{t \wedge T} [h(y_{x}(s), a^{j}, b(s)) - \gamma^{2} |b(s)|^{2}] ds + \mathbb{1}_{[0,T)}(t) V_{\gamma}^{j}(y_{x}(t)) \}, \ \forall t > 0.$$

Suppose there is a sequence $\{t_n\}$ with $0 < t_n < \frac{1}{n}$ for n = 1, 2, ... such that

$$V_{\gamma}^{j}(x) < \sup_{b \in \mathcal{B}, \ T > 0} \{ \int_{0}^{t_{n} \wedge T} [h(y_{x}(s), a^{j}, b(s)) - \gamma^{2} |b(s)|^{2}] ds + \mathbb{1}_{[0,T)}(t_{n}) V_{\gamma}^{j}(y_{x}(t_{n})) \}.$$
(3.22)

Let $w(x, t_n)$ be the right hand side of (3.22). For each t_n , define $\epsilon_n = \frac{1}{3}[w(x, t_n) - V_{\gamma}^j(x)]$. It follows that

$$V_{\gamma}^{j}(x) + \epsilon_{n} < w(x, t_{n}) - \epsilon_{n} \tag{3.23}$$

Choose $b_n \in \mathcal{B}$ and $T_n \ge 0$ such that

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$$w(x,t_n) - \epsilon_n \le \int_0^{t_n \wedge T_n} [h(y_x(s), a^j, b_n(s)) - \gamma^2 |b_n(s)|^2] ds + \mathbb{1}_{[0,T_n)}(t_n) V_{\gamma}^j(y_x(t_n))$$
(3.24)

By Theorem 3.4 choose $\alpha_n \in \Gamma$ such that

$$V_{\gamma}^{j}(x) + \epsilon_{n} \ge \int_{[0,t_{n} \wedge T_{n}]} l(y_{x}(s), a^{j}, (\alpha_{n})_{x}^{j}[b_{n}](s), b_{n}(s)) + 1_{[0,T_{n})}(t_{n})V_{\gamma}^{i_{n}}(y_{x}(t_{n})), \qquad (3.25)$$

where $(\alpha_n)_x^j[b_n](t_n^-) = a^{i_n} \in A$. From (3.23), (3.24) and (3.25), we have

$$\int_{[0,t_n \wedge T_n]} l(y_x(s), a^j, (\alpha_n)_x^j[b_n](s), b_n(s)) + 1_{[0,T_n]}(t_n) V_{\gamma}^{i_n}(y_x(t_n))
< \int_0^{t_n \wedge T_n} [h(y_x(s), a^j, b_n(s)) - \gamma^2 |b_n(s)|^2] ds + 1_{[0,T_n]}(t_n) V_{\gamma}^j(y_x(t_n)).$$
(3.26)

This implies that $(\alpha_n)_x^j[b_n]$ jumps in the interval $[0, t_n \wedge T_n]$. Without loss of generality assume the number of switchings is equal to d_n . If $t_n < T_n$ for infinitely many n, by going

down to a subsequence we may assume $t_n \leq T_n$ for all n. From (3.25) we have

$$\begin{split} V_{\gamma}^{j}(x) &\geq \lim_{n \to \infty} \sup \{ \int_{[0,t_{n} \wedge T_{n})} l(y_{x}(s), a^{j}, \alpha_{x,n}^{j}[b_{n}](s), b_{n}(s)) \\ &+ 1_{[0,T)}(t_{n}) V_{\gamma}^{i_{n}}(y_{x}(t_{n})), \ \alpha_{x,n}^{j}[b_{n}](t_{n}^{-}) = a^{i_{n}} \in A \} \\ &= \limsup_{n \to \infty} \{ \int_{0}^{t_{n}} [h(y_{x}(s), \alpha_{x,n}^{j}[b_{n}](s), b_{n}(s)) - \gamma^{2} |b_{n}(s)|^{2}] ds \\ &+ \sum_{m=1}^{d_{n}} k(a_{m-1}, a_{m}) + V_{\gamma}^{i_{n}}(y_{x}(t_{n})), \ \alpha_{x,n}^{j}[b_{n}](t_{n}) = a^{i_{n}} \in A \} \\ &= \limsup_{n \to \infty} \{ \sum_{m=1}^{d_{n}} k(a_{m-1}, a_{m}) + V_{\gamma}^{i_{n}}(y_{x}(t_{n})), \ \alpha_{x,n}^{j}[b_{n}](t_{n}) = a^{i_{n}} \in A \}. \end{split}$$

By using continuity of $V_{\gamma}^{i_n}$ and $\sum_{m=1}^{d_n} k(a_{m-1}, a_m) > k(a^j, a^{i_n})$, we have

$$V_{\gamma}^{j}(x) \geq \min_{i \neq j} \{ V_{\gamma}^{i}(x) + k(a^{j}, a^{i}) \}$$

which contradicts one of the assumptions. If $t_n \ge T_n$ for infinitely many n, again without loss of generality we may assume $t_n \ge T_n$ for all n. From (3.26) we have

$$\begin{split} &\lim \inf_{n \to \infty} \{ \int_{[0,T_n]} l(y_x(s), a^j \alpha_{x,n}^j [b_n](s), b_n(s)) \} \\ &\leq \lim \sup_{n \to \infty} \{ \int_0^{T_n} [h(y_x(s), a^j, b_n(s)) - \gamma^2 |b_n(s)|^2] ds \}, \end{split}$$

or equivalently,

$$\liminf_{n \to \infty} \{ \int_0^{T_n} [h(y_x(s), \alpha_{x,n}^j[b_n](s), b_n(s)) - \gamma^2 |b_n(s)|^2] ds + \sum_{m=1}^{d_n} k(a_{m-1}, a_m) \} \\ \leq \limsup_{n \to \infty} \{ \int_0^{T_n} [h(y_x(s), a^j, b_n(s)) - \gamma^2 |b_n(s)|^2] ds \}.$$

Thus

$$\liminf_{n \to \infty} \{ \sum_{m=1}^{d_n} k(a_{m-1}, a_m) \} \leq \limsup_{n \to \infty} \{ \int_0^{T_n} [h(y_x(s), a^j, b_n(s))ds \} - \lim_{n \to \infty} \{ \int_0^{T_n} [h(y_x(s), \alpha_{x,n}^j[b_n](s), b_n(s))ds \} \} + \lim_{n \to \infty} \{ \int_0^{T_n} [h(y_x(s), \alpha_{x,n}^j[b_n](s), b_n(s))ds \} \} + \lim_{n \to \infty} \{ \int_0^{T_n} [h(y_x(s), \alpha_{x,n}^j[b_n](s), b_n(s))ds \} \} + \lim_{n \to \infty} \{ \int_0^{T_n} [h(y_x(s), \alpha_{x,n}^j[b_n](s), b_n(s))ds \} \} + \lim_{n \to \infty} \{ \int_0^{T_n} [h(y_x(s), \alpha_{x,n}^j[b_n](s), b_n(s))ds \} \} + \lim_{n \to \infty} \{ \int_0^{T_n} [h(y_x(s), \alpha_{x,n}^j[b_n](s), b_n(s))ds \} \} + \lim_{n \to \infty} \{ \int_0^{T_n} [h(y_x(s), \alpha_{x,n}^j[b_n](s), b_n(s))ds \} \} + \lim_{n \to \infty} \| h(y_x(s), \alpha_{x,n}^j[b_n](s), b_n(s))ds \| + \| h(y_x(s), \alpha_{x,n}$$

and in this case $T_n \to 0$ as $n \to \infty$. Note that the integral terms tend to 0 uniformly with respect to $b_n \in \mathcal{B}$ as $T_n \to 0$ due to the compactness assumption on B, the uniform estimate (3.3), and the continuity assumption (A8) on h. Thus we have

$$\liminf_{n \to \infty} \left\{ \sum_{m=1}^{d_n} k(a_{m-1}, a_m) \right\} \le 0$$

which contradicts (A12). \diamondsuit

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Lemma 3.7 Assume (A8)- (A12) and $V_{\gamma}^{j} \in C(\mathbb{R}^{n}), j = 1, \ldots, r$. If

$$V_{\gamma}^{j}(x) < \min_{i \neq j} \{V_{\gamma}^{i}(x) + k(a^{j}, a^{i}), \}$$

then there exists $\tau = \tau_x > 0$ such that

$$V_{\gamma}^{j}(x) \geq \sup_{b \in \mathcal{B}} \{ \int_{0}^{t} [h(y_{x}(s), a^{j}, b(s)) - \gamma^{2} |b(s)|^{2}] ds + V_{\gamma}^{j}(y_{x}(t)) \}, \ \forall t \in (0, \tau_{x}).$$

Proof From Proposition 3.6, choose $\tau = \tau_x > 0$ such that for all $t \in (0, \tau)$

$$V_{\gamma}^{j}(x) = \sup_{b \in \mathcal{B}, \ T \ge 0} \{ \int_{0}^{t \wedge T} [h(y_{x}(s), a^{j}, b(s)) - \gamma^{2} |b(s)|^{2}] ds + \mathbb{1}_{\{t:t < T\}}(t) V_{\gamma}^{j}(y_{x}(t)) \}.$$

Thus

$$V_{\gamma}^{j}(x) \geq \sup_{b \in \mathcal{B}, T > t} \{ \int_{0}^{t \wedge T} [h(y_{x}(s), a^{j}, b(s)) - \gamma^{2} |b(s)|^{2}] ds + 1_{\{t:t < T\}}(t) V_{\gamma}^{j}(y_{x}(t)) \}$$

$$= \sup_{b \in \mathcal{B}} \{ \int_{0}^{t} [h(y_{x}(s), a^{j}, b(s)) - \gamma^{2} |b(s)|^{2}] ds + V_{\gamma}^{j}(y_{x}(t)) \}. \diamondsuit$$

Theorem 3.8 Assume (A8)-(A13) and $V_{\gamma}^{j} \in C(\mathbb{R}^{n}), j = 1, ..., r$. Then V_{γ} is a viscosity solution of (SQVI)

$$\max\{H^{j}(x, DV^{j}_{\gamma}(x)), V^{j}_{\gamma}(x) - \min_{i \neq j}\{V^{i}_{\gamma}(x) + k(a^{j}, a^{i})\}\} = 0, x \in \mathbb{R}^{n}, \ j = 1, \dots, r.$$

Proof We first show that V_{γ}^{j} is a viscosity supersolution of (SQVI). Fix $x_{0} \in \mathbb{R}^{n}$ and $a^{j} \in A$. Let $\varphi^{j} \in C^{1}(\mathbb{R}^{n})$ and x_{0} is a local minimum of $V_{\gamma}^{j} - \varphi^{j}$. We want to show that

$$\max\{H^{j}(x_{0}, D\varphi^{j}(x_{0})), V^{j}_{\gamma}(x_{0}) - \min_{i \neq j}\{V^{i}_{\gamma}(x_{0}) + k(a^{j}, a^{i})\}\} \ge 0$$
(3.27)

We have two cases to consider

case 1 $V_{\gamma}^{j}(x_{0}) = \min_{i \neq j} \{ V_{\gamma}^{i}(x_{0}) + k(a^{j}, a^{i}) \},$ case 2 $V_{\gamma}^{j}(x_{0}) < \min_{i \neq j} \{ V_{\gamma}^{i}(x_{0}) + k(a^{j}, a^{i}) \}.$

If case 1 occurs, we have

$$\max\{H^{j}(x_{0}, D\varphi^{j}(x_{0})), V^{j}_{\gamma}(x_{0}) - \min_{i \neq j}\{V^{i}_{\gamma}(x_{0}) + k(a^{j}, a^{i})\} \\ \geq V^{j}_{\gamma}(x_{0}) - \min_{i \neq j}\{V^{i}_{\gamma}(x_{0}) + k(a^{j}, a^{i})\} \\ = 0.$$

If case 2 occurs, we want to show that $H^j(x_0, D\varphi^j(x_0)) \ge 0$. Fix $b \in B$ and set b(s) = b for all $s \ge 0$. From Lemma 3.7, choose $\bar{t}_0 > 0$ such that for $t \in (0, \bar{t}_0)$

$$V_{\gamma}^{j}(x_{0}) - V_{\gamma}^{j}(y_{x_{0}}(t)) \ge \int_{0}^{t} [h(y_{x_{0}}(s), a^{j}, b) - \gamma^{2}|b|^{2}] ds.$$
(3.28)

Since x_0 is a local minimum of $V_{\gamma}^j - \varphi^j$, by (3.3) there exists $\hat{t}_0 > 0$ such that

$$\varphi^{j}(x_{0}) - \varphi^{j}(y_{x_{0}}(s, a^{j}, b)) \ge V^{j}_{\gamma}(x_{0}) - V^{j}_{\gamma}(y_{x_{0}}(s, a^{j}, b)), \ 0 < s < \hat{t}_{0}.$$
(3.29)

Set $t_0 = \min{\{\bar{t}_0, \hat{t}_0\}}$. As a consequence of (3.28) and (3.29), we have

$$\varphi^{j}(x_{0}) - \varphi^{j}(y_{x_{0}}(t)) \ge \int_{0}^{t} [h(y_{x_{0}}(s), a^{j}, b) - \gamma^{2} |b|^{2}] ds, \ 0 < t < t_{0}.$$
(3.30)

Divide both sides by t and let $t \to 0$ to get

$$-D\varphi^{j}(x_{0}) \cdot f(x_{0}, a^{j}, b) - h(x_{0}, a^{j}, b) + \gamma^{2}|b|^{2} \ge 0.$$

Since $b \in B$ is arbitrary, we have $H^j(x_0, D\varphi^j(x_0)) \ge 0$.

We next show that V_{γ}^{j} is a viscosity subsolution of (SQVI). Fix $x_{1} \in \mathbb{R}^{n}$ and $a^{j} \in A$. Let $\varphi^{j} \in C^{1}(\mathbb{R}^{n})$ and x_{1} is a local maximum of $V_{\gamma}^{j} - \varphi^{j}$. We want to show that

$$\max\{H^{j}(x_{1}, D\varphi^{j}(x_{1})), V^{j}_{\gamma}(x_{1}) - \min_{i \neq j}\{V^{i}_{\gamma}(x_{1}) + k(a^{j}, a^{i})\}\} \le 0$$
(3.31)

From Proposition 3.3, $V_{\gamma}^{j}(x_{1}) \leq \min_{i \neq j} \{V_{\gamma}^{i}(x_{1}) + k(a^{j}, a^{i})\}$. Thus we want to show that $H^{j}(x_{1}, D\varphi^{j}(x_{1})) \leq 0$.

We first consider the case $V_{\gamma}^{j}(x_{1}) > 0$. Let t > 0 and $\epsilon > 0$. From Corollary 3.5, choose $\hat{b} = \hat{b}_{t,\epsilon} \in \mathcal{B}$ and $\hat{T} = \hat{T}_{t,\epsilon} \ge 0$ such that

$$V_{\gamma}^{j}(x_{1}) \leq \int_{0}^{\hat{T} \wedge t} [h(y_{x_{1}}(s), a^{j}, \hat{b}(s)) - \gamma^{2} |\hat{b}(s)|^{2}] \, ds + \mathbb{1}_{[0,\hat{T})}(t) V_{\gamma}^{j}(y_{x_{1}}(t, \hat{b})) + \epsilon t \tag{3.32}$$

In particular,

$$V_{\gamma}^{j}(x_{1}) \leq \int_{0}^{\hat{T} \wedge t} [h(y_{x_{1}}(s), a^{j}, \hat{b}(s)) - \gamma^{2} |\hat{b}(s)|^{2}] ds + V_{\gamma}^{j}(y_{x_{1}}(\hat{T} \wedge t, \hat{b})) + \epsilon t$$

and hence

$$V_{\gamma}^{j}(x_{1}) - V_{\gamma}^{j}(y_{x_{1}}(\hat{T} \wedge t, \hat{b})) \leq \int_{0}^{\hat{T} \wedge t} [h(y_{x_{1}}(s), a^{j}, \hat{b}(s)) - \gamma^{2} |\hat{b}(s)|^{2}] \, ds + \epsilon t$$
(3.33)

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Since x_1 is a local maximum of $V^j_{\gamma} - \varphi^j$, by (3.3) we may assume that

$$\varphi^{j}(x_{1}) - \varphi^{j}(y_{x_{1}}(s), a^{j}, \hat{b}(s)) \leq V^{j}_{\gamma}(x_{1}) - V^{j}_{\gamma}(y_{x_{1}}(s), a^{j}, \hat{b}(s)), \ 0 < s \leq t$$
(3.34)

Combine (3.33) and (3.34) to get

$$\varphi^{j}(x_{1}) - \varphi^{j}(y_{x_{1}}(\hat{T} \wedge t, a^{j}, \hat{b}(t)) \leq \int_{0}^{T \wedge t} [h(y_{x_{1}}(s), a^{j}, \hat{b}(s)) - \gamma^{2} |\hat{b}(s)|^{2}] \, ds + \epsilon t.$$
(3.35)

Under the assumptions on f and h, it follows that (3.35) is equivalent to

$$\inf_{b \in B} \{ -D\varphi^{j}(x_{1}) \cdot f(x_{1}, a^{j}, b) - h(x_{1}, a^{j}, b) + \gamma^{2} |b|^{2} \} \cdot (\hat{T} \wedge t) \le \epsilon \ t + o(\hat{T} \wedge t)$$
(3.36)

and

$$\limsup_{t \to 0} \frac{t}{t \wedge \hat{T}_{t,\epsilon}} = 1 \text{ (for each } \epsilon > 0\text{)}.$$
(3.37)

For details of the proof, see the proof of Theorem 2.7 in Chapter 2. We now can divide (3.36) by $\hat{T} \wedge t > 0$ and pass to the limit to get

$$\inf_{b \in B} \{ -D\Psi(x) \cdot f(x,b) - h(x,b) + \gamma^2 |b|^2 \} \le \epsilon.$$

Since $\epsilon > 0$ is arbitrary, we conclude that $H(x, D\Psi(x)) \leq 0$.

It remains to handle the case $V_{\gamma}^{j}(x_{1}) = 0$. In this case we take $\hat{b} \equiv 0$ and use (A13) and $V_{\gamma}^{j} \geq 0$ to see that

$$\begin{aligned} V_{\gamma}^{j}(x_{1}) &= 0 &\leq \int_{0}^{t} h(y_{x_{1}}(s), \hat{b}) ds + V_{\gamma}^{j}(y_{x_{1}}(t)) \\ &= \int_{0}^{t} [h(y_{x_{1}}(s), \hat{b}(s)) - \gamma^{2} |\hat{b}(s)|^{2}] ds + V_{\gamma}^{j}(y_{x_{1}}(t)), \end{aligned}$$

for all $t \ge 0$. Then it is straightforward to follow the procedure in the first part of the proof to arrive at the desired inequality $H^j(x_1, D\varphi^j(x_1)) \le 0$.

We next give a connection of a switching storage (vector) function with the SQVI.

Theorem 3.9 Assume (A1)-(A5) and assume that $S = (S^1, \ldots, S^r)$ is a continuous switching storage function for the closed loop system formed by the nonanticipating strategy $\alpha \in \Gamma$. Then S is a viscosity supersolution of SQVI.

Proof. Fix $x \in \mathbb{R}^n$ and $j \in \{1, \ldots, r\}$. Let $\Psi^j \in C^1(\mathbb{R}^n)$ be such that x is a local minimum of $S^j - \Psi^j$. Let $b \in B$. Set b(s) = b for $s \ge 0$. Choose $t_1 > 0$ so that

$$S^{j}(x) - \Psi^{j}(x) \le S^{j}(y_{x}(s, \alpha_{x}^{j}[b], b)) - \Psi^{j}(y_{x}(s, \alpha_{x}^{j}[b], b)), \text{ for all } 0 \le s \le t_{1}.$$
 (3.38)

We have two cases to consider:

case 1 $S^{j}(x) \ge \min_{i \neq j} \{S^{i}(x) + k(a^{j}, a^{i})\},$ case 2 $S^{j}(x) < \min_{i \neq j} \{S^{i}(x) + k(a^{j}, a^{i})\}.$

If case 1 occurs, then

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$$\max \{ H^{j}(x, D\Psi^{j}(x)), S^{j}(x) - \min_{i \neq j} \{ S^{i}(x) + k(a^{j}, a^{i}) \} \}$$

$$\geq S^{j}(x) - \min_{i \neq j} \{ S^{i}(x) + k(a^{j}, a^{i}) \}$$

$$\geq 0.$$

If case 2 occurs, we claim that for each $\hat{b} \in B$ there exists a $t_2 = t_2(\hat{b}) > 0$ such that

$$\alpha_x^j[\widehat{b}](s) = a^j \text{ for } 0 \le s \le t_2.$$

Indeed, if not, then, for each t > 0 there exists a $\bar{b}_t \in \mathcal{B}$ such that

$$\alpha_x^j[\bar{b}_t](t^-) = a^{j(t)} \neq a^j.$$
(3.39)

Since S is a switching storage function, we have

$$S^{j}(x) - S^{j(t)}(y_{x}(t, \alpha_{x}^{j}[\bar{b}_{t}], \bar{b}_{t}))$$

$$\geq \int_{0}^{t} [h(y_{x}(s), \alpha_{x}^{j}[\bar{b}_{t}](s), \bar{b}_{t}(s)) - \gamma^{2} |\bar{b}_{t}(s)|^{2} ds + \sum_{\tau < t} k(a^{j(\tau^{-})}, a^{j(\tau)}).$$

Letting t tend to 0 gives

$$S^{j}(x) - S^{j(0^{+})}(x) \ge k(a^{j}, a^{j(0^{+})}).$$

From (3.39) we see that this implies that $j(0^+) \neq j$. Thus

$$S^{j}(x) \ge \min_{i \neq j} \{S^{i}(x) + k(a^{j}, a^{i})\}$$

which gives a contradiction. Thus the claim is proved.

Since S is a switching storage function, we have

$$S^{j}(x) - S^{j(t)}(y_{x}(t, \alpha_{x}^{j}[b], b))$$

$$\geq \int_{0}^{t} [h(y_{x}(s), \alpha_{x}^{j}[b](s), b(s)) - \gamma^{2} |b(s)|^{2}] ds \text{ for all } 0 < t \le t_{2}.$$
(3.40)

Set $t = \min\{t_1, t_2\}$. Then (3.38) and (3.40) implies that

$$\Psi^{j}(x) - \Psi^{j}(y_{x}(t, \alpha_{x}^{j}[b], b))$$

$$\geq \int_{0}^{t} [h(y_{x}(s), \alpha_{x}^{j}[b](s), b(s)) - \gamma^{2} |b(s)|^{2}] ds \text{ for } 0 < t < t_{3}.$$
(3.41)

Divide (3.41) by t and let t tend to 0 to get

$$-D\Psi^{j}(x) \cdot f(x, a^{j}, b) - h(x, a^{j}, b) + \gamma^{2}|b|^{2} \ge 0.$$

Since this inequality holds for an arbitrary $b \in B$, we have $H^j(x, D\Psi^j(x)) \ge 0$ as required. \diamond

We now proceed to the synthesis of a switching-control strategy achieving the dissipation inequality for a given viscosity supersolution $U = (U^1, \ldots, U^r)$ of SQVI. Given a continuous nonnegative vector function $U = (U^1, \ldots, U^r)$ on \mathbb{R}^n satisfying the condition

$$U^{j}(x) \leq \min_{i \neq j} \{U^{i}(x) + k(a^{j}, a^{i})\} \text{ for all } x \in \mathbb{R}^{n}, \ j = 1, \dots, r,$$

we associate a state-feedback switching strategy $\alpha_U: (y(t), a^j) \to \alpha_U^j(y(t))$ by the rule

$$\alpha^{j}(y(t)) = \begin{cases} a^{j} \text{ if } U^{j}(y(t)) < \min_{i \neq j} \{ U^{i}(y(t)) + k(a^{j}, a^{i}) \}; \\ \text{any } a^{\ell} \neq a^{j} \text{ such that } U^{\ell}(y(t)) + k(a^{j}, a^{\ell}) = \min_{i \neq j} \{ U^{i}(y(t)) + k(a^{j}, a^{i}) \}, \\ \text{otherwise.} \end{cases}$$

$$(3.42)$$

The associated feedback switching strategy is: if the current state is y(t) and the current old control is $a(t^-) = a^j$, then set $a(t) = \alpha^j(y(t))$. Such a strategy can also be expressed as a nonanticipating strategy $\alpha_U : (x, a^j, b) \to \alpha^j_{U,x}[b]$; explicitly for this particular case α_U , we have $\alpha^j_{U,x}[b]$ is given by

$$\alpha_{U,x}^{j}[b](t) = \sum_{n \ge 1} a_{n-1} \mathbf{1}_{[\tau_{n-1},\tau_n)}(t), \text{ for } t \ge 0$$
(3.43)

and $\alpha_{U,x}^{j}[b](0^{-}) = a_0$ where

$$\tau_0 = 0, \ a_0 = a^{j_0} = a^j$$

and for n = 1, 2, 3, ...

$$\tau_n[b] = \begin{cases} \inf\{t > \tau_{n-1} : \ U^{j_{n-1}}(y_{y(\tau_{n-1})}(t - \tau_{n-1}, a^{j_{n-1}}, b(\cdot - \tau_{n-1})) \\ = \min_{i \neq j_{n-1}}\{U^i(y_{y(\tau_{n-1})}(t - \tau_{n-1}, a^{j_{n-1}}, b(\cdot - \tau_{n-1})) + k(a^{j_{n-1}}, a^i)\}, \ y(0) = x\}, \\ +\infty \text{ if the preceding set is empty,} \end{cases}$$

$$a_{n} = a^{j_{n}} = \begin{cases} \sup a^{l} \neq a^{j_{n-1}} \text{ such that} \\ \min_{i \neq j_{n-1}} \{ U^{i}(y_{y(\tau_{n-1})}(\tau_{n} - \tau_{n-1}, a^{j_{n-1}}, b(\cdot - \tau_{n-1})) + k(a^{j_{n-1}}, a^{i}) \} \\ = U^{l}(y_{y(\tau_{n-1})}(\tau_{n} - \tau_{n-1}, a^{j_{n-1}}, b(\cdot - \tau_{n-1})) + k(a^{j_{n-1}}, a^{l}), \ y(0) = x \text{ if } \tau_{n} < \infty; \\ \text{undefined, if } \tau_{n} = \infty. \end{cases}$$

$$(3.44)$$

Note that if $\tau_1 = \tau_0 = 0$, then there is immediate switching from a_0 to a_1 and the n = 1 term in (3.43) is vacuous. Moreover by (A12), $\tau_n > \tau_{n-1}$ for $\tau_{n-1} < \infty$. To see this, we assume that $\tau_n = \tau_{n-1} < \infty$. From the definition of τ_{n-1} and τ_n , we would have

$$U^{j_{n-2}}(y(\tau_{n-1})) = U^{j_{n-1}}(y(\tau_{n-1})) + k(a^{j_{n-2}}, a^{j_{n-1}})$$

= $U^{j_n}(y(\tau_{n-1})) + k(a^{j_{n-1}}, a^{j_n}) + k(a^{j_{n-2}}, a^{j_{n-1}})$ (and hence $j_n \neq j_{n-2}$)
> $U^{j_n}(y(\tau_{n-1})) + k(a^{j_{n-2}}, a^{j_n})$
 $\geq \min_{i \neq j_{n-2}} \{U^i(y(\tau_{n-1})) + k(a^{j_{n-2}}, a^i)\},$

which gives a contradiction.

Theorem 3.10 Assume

(i) (A8)-(A12) hold. (ii) $U = (U^1, \ldots, U^r)$ is a nonnegative continuous viscosity supersolution in \mathbb{R}^n of the SQVI

$$\max\{H^{j}(x, DU^{j}(x)), U^{j}(x) - \min_{i \neq j}\{U^{i}(x) + k(a^{j}, a^{i})\}\} = 0, x \in \mathbb{R}^{n}, j = 1, \dots, r,$$

(*iii*) $U^{j}(x) \leq \min_{i \neq j} \{ U^{i}(x) + k(a^{j}, a^{i}) \}, x \in \mathbb{R}^{n}, j \in \{1, \ldots, r\}.$

Let α_U be the state-feedback strategy defined by (3.42), or equivalently, the nonanticipating disturbance-feedback strategy α_U defined by (3.44). Then $U = (U^1, \ldots, U^r)$ is a storage function for the closed-loop system formed by the strategy α_U . In particular, we have

$$U^{j}(x) \ge \sup_{b \in \mathcal{B}, \ T \ge 0} \{ \int_{[0,T]} l(y_{x}(s), a^{j}, \alpha_{U,x}^{j}[b](s), b(s)) \} \ge V_{\gamma}^{j}(x),$$

for each $x \in \mathbb{R}^n$ and $a^j \in A$. Thus V_{γ} , if continuous, is characterized as the minimal, nonnegative, continuous, viscosity supersolution of the SQVI satisfying condition (iii), as well as the minimal continuous switching storage function satisfying condition (iii) for the closed-loop system associated with some nonanticipating strategy $\alpha_{V_{\gamma}}$.

Proof Let $\alpha_{U,x}^{j}[b](t)$ be the switching strategy defined as in (3.44). We claim that

$$\tau_n \to \infty$$
 as $n \to \infty$.

If $\tau_n = \infty$ for some *n*, then it is trivially true. Otherwise we assume that

$$\lim_{n \to \infty} \tau_n = T < \infty. \tag{3.45}$$

It follows that

$$0 \le \tau_n \le T$$
, for all n ,

since τ_n is nondecreasing. Also from (3.45), we have that a sequence $\{\tau_n\}$ is a Cauchy sequence and hence for all $\nu > 0$ there is some *n* such that $\tau_n < \tau_{n-1} + \nu$. By the definition of τ_n , we choose *t* such that $\tau_n \leq t < \tau_{n-1} + \nu$ and

$$U^{j_{n-1}}(y_x(t)) = U^l(y_x(t)) + k(a^{j_{n-1}}, a^l) \text{ for some } a^l \neq a^{j_{n-1}}$$
(3.46)

(We have written $y_x(t)$ for $y_x(t, \alpha_x^j[b], b)$.) By definition of τ_{n-1} , we have

$$U^{j_{n-2}}(y_x(\tau_{n-1})) = U^{j_{n-1}}(y_x(\tau_{n-1})) + k(a^{j_{n-2}}, a^{j_{n-1}})$$
(3.47)

By (iii), we have

$$U^{j_{n-2}}(y_x(\tau_{n-1})) \leq \min_{i \neq j_{n-2}} \{ U^i(y_x(\tau_{n-1})) + k(a^{j_{n-2}}, a^i) \}$$

$$\leq U^l(y_x(\tau_{n-1})) + k(a^{j_{n-2}}, a^l)$$
(3.48)

From (3.47) and (3.48), we have

$$k(a^{j_{n-2}}, a^{j_{n-1}}) - k(a^{j_{n-2}}, a^l) \le U^l(y_x(\tau_{n-1})) - U^{j_{n-1}}(y_x(\tau_{n-1}))$$
(3.49)

As a consequence of (3.46) and (3.49), we have

$$0 < k(a^{j_{n-2}}, a^{j_{n-1}}) + k(a^{j_{n-1}}, a^{l}) - k(a^{j_{n-2}}, a^{l})$$

$$\leq U^{l}(y_{x}(\tau_{n-1})) - U^{l}(y_{x}(t)) + U^{j_{n-1}}(y_{x}(t)) - U^{j_{n-1}}(y_{x}(\tau_{n-1}))$$

$$\leq \omega_{l}(\nu) + \omega_{j_{n-1}}(\nu)$$

where ω_l and $\omega_{j_{n-1}}$ are moduli of continuity for $U^l(y_x(\cdot))$ and $U^{j_{n-1}}(y_x(\cdot))$ on the interval [0, T] respectively. Letting ν tend to zero now leads to a contradiction, and proves the claim.

Hence $\alpha_x^j[b](t) = \sum a_n \mathbb{1}_{[\tau_{n-1},\tau_n)}(t) \in \Gamma$. Since U is a viscosity solution of the SQVI, we have $H^{j_n}(y_x(s), DU^{j_n}(y_x(s))) \ge 0$, in viscosity solution sense for $\tau_n < s < \tau_{n+1}$. Thus by Theorem 2.4 in page 15, we have

$$U^{j_n}(y_x(s)) - U^{j_n}(y_x(t)) \ge \int_s^t [h(y_x(s), a^{j_n}, b(s)) - \gamma^2 |b(s)|^2] ds,$$

for all $b \in \mathcal{B}$, $\tau_n < s \le t < \tau_{n+1}$. Letting $s \to \tau_n^+$ and $t \to \tau_{n+1}^-$ to get

$$U^{j_n}(y_x(\tau_n)) - U^{j_n}(y_x(\tau_{n+1})) \ge \int_{\tau_n}^{\tau_{n+1}} [h(y_x(s), a^{j_n}, b(s)) - \gamma^2 |b(s)|^2] ds, \ \forall b \in \mathcal{B}.$$
(3.50)

We also have

$$U^{j_n}(y_x(\tau_{n+1})) = U^{j_{n+1}}(y_x(\tau_{n+1})) + k(a^{j_n}, a^{j_{n+1}}), \text{ for } \tau_{n+1} < \infty.$$
(3.51)

Adding (3.50) over $\tau_n \leq T$ and using (3.51), we have

$$\begin{aligned} U^{j_0}(x) &\geq \int_0^T [h(y_x(s), \alpha_x^j[b](s), b(s)) - \gamma^2 |b(s)|^2] ds + \sum_{\tau_n \leq T} k(a_{n-1}, a_n) + U^{j_n}(y_x(T)) \\ &\geq \int_0^T [h(y_x(s), \alpha_x^j[b](s), b(s)) - \gamma^2 |b(s)|^2] ds + \sum_{\tau_n \leq T} k(a_{n-1}, a_n). \end{aligned}$$

Since this inequality holds for arbitrary $b \in \mathcal{B}$ and $T \geq 0$, we have

$$U^{j}(x) \ge \sup_{b \in \mathcal{B}, \ T \ge 0} \left\{ \int_{[0,T]} l(y_{x}(s), a^{j}, \alpha_{x}^{j}[b](s), b(s)) \right\}.$$

Thus $U^j(x) \ge V^j_{\gamma}(x)$. By Theorem 3.8, we know that V_{γ} is a viscosity supersolution of the SQVI if it is continuous. (Note that the proof of viscosity supersolution of V_{γ} in Theorem 3.8 does not use the assumption (A13).) V_{γ} also has the property (iii) by Proposition 3.3. Thus we conclude that if continuous, V_{γ} is the minimal, nonnegative, continuous, viscosity supersolution of the SQVI satisfying the condition (iii).

The first part of this Theorem (Theorem 3.10) already proved then implies that V_{γ} is a switching storage function. Moreover if S is any continuous, switching storage function for some nonanticipating strategy $\alpha_{V_{\gamma}}$, from Theorem 3.9 we see that S is a viscosity supersolution of the SQVI. Again from the first part of this Theorem already proved, we then see that $S \geq V_{\gamma}$ if S has the property (iii), and hence V_{γ} is also the minimal, continuous switching storage function satisfying the condition (iii), as asserted. \diamond

3.3.2 Comparison theorems for solutions of a system of quasivariational inequalities

In this subsection we present comparison theorems for SQVIs. We first give the definition of viscosity sub- and super solution of the system of quasivariational inequalities

$$\max\{u_t^j(x,t) + H^j(x, Du^j(x,t)), u^j(x,t) - \min_{i \neq j}\{u^i(x,t) + k(a^j, a^i)\}\} = 0,$$

for $(x,t) \in \Omega := \mathbb{R}^n \times (0,T), \ T > 0, \ j = 1, 2, \dots, r$

$$(3.52)$$

(where u_t^j denotes the partial derivative with respect to t and $Du^j = D_x u^j$ the gradient with respect to x) for the case of a not necessarily continuous vector function $u = (u^1, \ldots, u^r)$.

Definition 3.11 A vector function $u = (u^1, u^2, ..., u^l)$, where $u^j \in USC(\Omega)$, is a viscosity subsolution of (3.52) if for any $\varphi^j \in C^1(\Omega)$, j = 1, 2, ..., r and any $(x, t) \in \Omega$ such that $u^j - \varphi^j$ has a local maximum (x, t) and

$$\max\{\varphi_t^j(x,t) + H^j(x, D\varphi^j(x,t)), u^j(x,t) - \min_{i \neq j}\{u^i(x,t) + k(a^j, a^i)\} \le 0$$

Similarly, a vector function $u = (u^1, u^2, ..., u^l)$, where $u^j \in LSC(\Omega)$, is a viscosity supersolution of (3.52) if for any $\varphi^j \in C^1(\Omega)$, j = 1, 2, ..., r and any $(x, t) \in \Omega$ such that $u^j - \varphi^j$ has a local minimum (x, t) and

$$\max\{\varphi_t^j(x,t) + H^j(x, D\varphi^j(x,t)), u^j(x,t) - \min_{i \neq j}\{u^i(x,t) + k(a^j, a^i)\} \ge 0.$$

Let $w: \Omega \to \mathbb{R}$ be locally bounded. We define its upper and lower semicontinuous envelopes as, respectively

$$w^*(x,t) := \limsup_{(y,s)\to(x,t)} w(y,s)$$

 $w_*(x,t) := \liminf_{(y,s)\to(x,t)} w(y,s).$

The next Lemma collects some properties of the semicontinuous envelopes which we will use later.

Lemma 3.12 (i) $w_* \le w \le w^*$

- (ii) $w_* = -(-w)^*$
- (iii) $w \text{ is } u.s.c. at (x,t) \text{ if and only if } w(x,t) = w^*(x,t).$ w is l.s.c. at (x,t) if and only if $w(x,t) = w_*(x,t).$
- (iv) $w^*(x,t) = \min\{v(x,t) : v \in USC(\Omega), v \ge w\}$ $w_*(x,t) = \max\{v(x,t) : v \in LSC(\Omega), v \le w\}$

Proof See [10, page 296].

Next we will prove a local comparison result of SQVIs

$$\max\{w_t^j + H^j(x, Dw^j), \ w^j - \min_{i \neq j}\{w^i + k(a^j, a^i)\}\} = 0, \ j = 1, 2, \dots, r$$
(3.53)

in the cone

 $\mathcal{C} := \{ (x,t) : 0 < t < T \text{ and } |x| < C(T-t), \ C > 0 \}.$ (3.54)

Then we apply this result to prove a global comparison principle for a viscosity sub- and supersolution.

Theorem 3.13 Let C > 0 and $H^j : \overline{B}(0, CT) \times \mathbb{R}^n \to \mathbb{R}$ be continuous and satisfy

$$H^{j}(x,p) - H^{j}(x,q)| \leq C|p-q|$$
 (3.55)

$$|H^{j}(x,p) - H^{j}(y,p)| \leq \omega(|x-y|) + \omega(|x-y||p|), \qquad (3.56)$$

for all $x, y \in B(0, CT)$; $p, q \in \mathbb{R}^n$; $j \in \{1, \ldots, r\}$ where ω is a modulus and T > 0. Assume

- (i) (A12) holds;
- (ii) $u^j \in LSC(\overline{C})$ and $v^j \in USC(\overline{C})$ for j = 1, 2, ..., r;
- (iii) $u = (u^1, \ldots, u^r)$ and $v = (v^1, \ldots, v^r)$ are respectively a viscosity super- and subsolution of (3.53) in C.
- (iv) $v(x,0) \le u(x,0)$ for all $x \in B(0,CT)$, i.e. $v^j(x,0) \le u^j(x,0)$ for $j = 1, \ldots, r$,

Then $v \leq u$ in C, i.e. $v^j \leq u^j$ for $j = 1, \ldots, r$.

Proof We prove by contradiction. Assume that there exist $0 < \delta < T$, $\tilde{j} \in \{1, 2, ..., r\}$, $0 < \tilde{t} < T$ and $\tilde{x} \in \mathbb{R}^n$ such that

$$(v^{\tilde{j}} - u^{\tilde{j}})(\tilde{x}, \tilde{t}) = \delta$$
 and $|\tilde{x}| \le C(T - \tilde{t}) - 2\delta$

It follows that

$$0 \le |\tilde{x}| \le C(T - \tilde{t}) - 2\delta < C(T - \tilde{t})$$

Thus $(\tilde{x}, \tilde{t}) \in \mathcal{C}$. Take

$$M > \max_{1 \le j \le l} \sup\{|v^{j}(x,t) - u^{j}(y,s)| : (x,t,y,s) \in \mathcal{C}^{2}\} \ge \delta$$

and also take $g \in C^1(\mathbb{R})$ such that $g' \leq 0$, g(r) = 0 for $r \leq -\delta$, g(r) = -3M for $r \geq 0$. Now define $\langle x \rangle_{\beta} := (|x|^2 + \beta^2)^{\frac{1}{2}}$ and

$$\Phi^{j}(x, y, t, s) := v^{j}(x, t) - u^{j}(y, s) - \frac{|x - y|^{2} + |t - s|^{2}}{2\epsilon} - \eta(t + s) + g(\langle x \rangle_{\beta} - C(T - t)) + g(\langle y \rangle_{\beta} - C(T - s))$$

where ϵ, η, β are positive parameters. Note that Φ^j is upper semicontinuous. Let $(\bar{x}^j_{\epsilon}, \bar{y}^j_{\epsilon}, \bar{t}^j_{\epsilon}, \bar{s}^j_{\epsilon}) \in \bar{C}^2$ be such that

$$\Phi^j(\bar{x}^j_{\epsilon}, \bar{y}^j_{\epsilon}, \bar{t}^j_{\epsilon}, \bar{s}^j_{\epsilon}) = \max_{\bar{\mathcal{C}}^2} \Phi^j$$

Note that $(\bar{x}_{\epsilon}^{j}, \bar{y}_{\epsilon}^{j}, \bar{t}_{\epsilon}^{j}, \bar{s}_{\epsilon}^{j})$ exists because Weierstrass' Theorem on the existence of maxima on compact sets holds for upper semicontinuous functions.

Choose $j_1 \in \{1, 2, ..., r\}$, say $j_1 = 1$, such that

$$\Phi^1(\bar{x}^1_{\epsilon}, \bar{y}^1_{\epsilon}, \bar{t}^1_{\epsilon}, \bar{s}^1_{\epsilon}) = \max_{1 \le j \le l} \Phi^j(\bar{x}^j_{\epsilon}, \bar{y}^j_{\epsilon}, \bar{t}^j_{\epsilon}, \bar{s}^j_{\epsilon})$$
We claim that either $\bar{t}^1_{\epsilon} = 0$ or $\bar{s}^1_{\epsilon} = 0$ or $(\bar{x}^1_{\epsilon}, \bar{y}^1_{\epsilon}, \bar{t}^1_{\epsilon}, \bar{s}^1_{\epsilon}) \in \mathcal{C}^2$ for β and η small enough. If $\bar{t}^1_{\epsilon} = T$, we would have

$$\begin{split} \Phi^{1}(\bar{x}_{\epsilon}^{1},\bar{y}_{\epsilon}^{1},\bar{t}_{\epsilon}^{1},\bar{s}_{\epsilon}^{1}) &= v^{1}(\bar{x}_{\epsilon}^{1},\bar{t}_{\epsilon}^{1}) - u^{1}(\bar{y}_{\epsilon}^{1},\bar{s}_{\epsilon}^{1}) - \frac{|\bar{x}_{\epsilon}^{1} - \bar{y}_{\epsilon}^{1}|^{2} + |\bar{t}_{\epsilon}^{1} - \bar{s}_{\epsilon}^{1}|^{2}}{2\epsilon} - \eta(\bar{t}_{\epsilon}^{1} + \bar{s}_{\epsilon}^{1}) + g(\langle \bar{x}_{\epsilon}^{1} \rangle_{\beta}) + g(\langle \bar{y}_{\epsilon}^{1} \rangle_{\beta} - C(T - \bar{s}_{\epsilon}^{1})) \\ &\leq M - 3M \\ &< 0. \end{split}$$

Similarly if $\bar{s}_{\epsilon}^{1} = T$, we would have $\Phi^{1}(\bar{x}_{\epsilon}^{1}, \bar{y}_{\epsilon}^{1}, \bar{t}_{\epsilon}^{1}, \bar{s}_{\epsilon}^{1}) < 0$. From the definition of $\langle x \rangle_{\beta}$, we have $\langle x \rangle_{\beta} > |x|$ for all x and all $\beta > 0$. If $|\bar{x}_{\epsilon}^{1}| = C(T - \bar{t}_{\epsilon}^{1})$, we have $\langle \bar{x}_{\epsilon}^{1} \rangle_{\beta} - C(T - \bar{t}_{\epsilon}^{1}) > 0$ for any $\beta > 0$. Thus

$$\begin{aligned}
\Phi^{1}(\bar{x}_{\epsilon}^{1}, \bar{y}_{\epsilon}^{1}, \bar{t}_{\epsilon}^{1}, \bar{s}_{\epsilon}^{1}) &\leq v^{1}(\bar{x}_{\epsilon}^{1}, \bar{t}_{\epsilon}^{1}) - u^{1}(\bar{y}_{\epsilon}^{1}, \bar{s}_{\epsilon}^{1}) + g(\angle \bar{x}_{\epsilon}^{1}\rangle_{\beta} - C(T - \bar{t}_{\epsilon}^{1})) \\
&\leq M - 3M \\
&< 0.
\end{aligned}$$

Similarly if $\bar{y}_{\epsilon}^1 = C(T - \bar{s}_{\epsilon}^1)$, we would have $\Phi^1(\bar{x}_{\epsilon}^1, \bar{y}_{\epsilon}^1, \bar{t}_{\epsilon}^1, \bar{s}_{\epsilon}^1) < 0$. However, from the assumption $\tilde{x}| \leq C(T - \tilde{t}) - 2\delta$ and the general inequality $\sqrt{p^2 + q^2} \leq p + q$ with $p = C(T - \tilde{t}) - 2\delta \geq 0$ and $q = \delta > 0$, we see that, for $\beta < \delta$, that

$$\begin{split} \langle \tilde{x} \rangle_{\beta} - C(T - \tilde{t}) &< \langle \tilde{x} \rangle_{\delta} - C(T - \tilde{t}) \\ &\leq \sqrt{[C(T - \tilde{t}) - 2\delta]^2 + \delta^2} - ([C(T - \tilde{t}) - 2\delta] + \delta) - \delta \\ &\leq -\delta \end{split}$$

and hence $g(\langle \tilde{x} \rangle_{\beta} - C(T - \tilde{t})) = 0$ for $\beta < \delta$. Hence

$$\Phi^{1}(\bar{x}_{\epsilon}^{1}, \bar{y}_{\epsilon}^{1}, \bar{t}_{\epsilon}^{1}, \bar{s}_{\epsilon}^{1}) \geq \Phi^{j}(\tilde{x}, \tilde{x}, \tilde{t}, \tilde{t}) \\
= \delta - 2\eta \tilde{t} + 2g(\langle \tilde{x} \rangle_{\beta} - C(T - \tilde{t})) \\
\geq \frac{\delta}{2}$$
(3.57)

for any $\epsilon > 0$, $\beta < \delta$ and $\eta < \frac{\delta}{4t}$. Thus $|\bar{x}_{\epsilon}^1| < C(T - \bar{t}_{\epsilon}^1)$, $|\bar{y}_{\epsilon}^1| < C(T - \bar{s}_{\epsilon}^1)$, $0 \le \bar{t}_{\epsilon}^1 < T$, $0 \le \bar{s}_{\epsilon}^1 < T$ for any $\epsilon > 0$, $\beta < \delta$ and $\eta < \frac{\delta}{4t}$. Thus the claim is proved.

From the inequality

$$\Phi^{1}(\bar{x}_{\epsilon}^{1}, \bar{x}_{\epsilon}^{1}, \bar{t}_{\epsilon}^{1}, \bar{t}_{\epsilon}^{1}) + \Phi^{1}(\bar{y}_{\epsilon}^{1}, \bar{y}_{\epsilon}^{1}, \bar{s}_{\epsilon}^{1}, \bar{s}_{\epsilon}^{1}) \le 2\Phi^{1}(\bar{x}_{\epsilon}^{1}, \bar{y}_{\epsilon}^{1}, \bar{t}_{\epsilon}^{1}, \bar{s}_{\epsilon}^{1})$$

We get

$$\begin{array}{rcl} \frac{|\bar{x}_{\epsilon}^{1}-\bar{y}_{\epsilon}^{1}|^{2}+|\bar{t}_{\epsilon}^{1}-\bar{s}_{\epsilon}^{1}|^{2}}{\epsilon} & \leq & v^{1}(\bar{x}_{\epsilon}^{1},\bar{t}_{\epsilon}^{1})-v^{1}(\bar{y}_{\epsilon}^{1},\bar{s}_{\epsilon}^{1})+u^{1}(\bar{x}_{\epsilon}^{1},\bar{t}_{\epsilon}^{1})-u^{1}(\bar{y}_{\epsilon}^{1},\bar{s}_{\epsilon}^{1})\\ & = & [v^{1}(\bar{x}_{\epsilon}^{1},\bar{t}_{\epsilon}^{1})-u^{1}(\bar{y}_{\epsilon}^{1},\bar{s}_{\epsilon}^{1})]-[v^{1}(\bar{y}_{\epsilon}^{1},\bar{s}_{\epsilon}^{1})-u^{1}(\bar{x}_{\epsilon}^{1},\bar{t}_{\epsilon}^{1})]\\ & \leq & 2M \end{array}$$

It follows that

$$|\bar{x}_{\epsilon}^1 - \bar{y}_{\epsilon}^1|^2 + |\bar{t}_{\epsilon}^1 - \bar{s}_{\epsilon}^1|^2 \le 2M\epsilon \tag{3.58}$$

Thus we have

$$|\bar{x}_{\epsilon}^{1} - \bar{y}_{\epsilon}^{1}|^{2} + |\bar{t}_{\epsilon}^{1} - \bar{s}_{\epsilon}^{1}|^{2} \to 0 \quad \text{as} \quad \epsilon \to 0^{+}$$

$$(3.59)$$

We now want to show that

$$\frac{|\bar{x}^1 - \bar{y}^1|^2 + |\bar{t}^1 - \bar{s}^1|^2}{2\epsilon} \to 0 \text{ as } \epsilon \to 0^+$$
(3.60)

Define

$$S^{1} := \max_{\bar{\mathcal{C}}} \{ v^{1}(x,t) - u^{1}(x,t) - 2\eta t + 2g(\langle x \rangle_{\beta} - C(T-t)) \}$$

Observe that

$$S^{1} \leq \Phi^{1}(\bar{x}_{\epsilon}^{1}, \bar{y}_{\epsilon}^{1}, \bar{t}_{\epsilon}^{1}, \bar{s}_{\epsilon}^{1}) \\ \leq v^{1}(\bar{x}_{\epsilon}^{1}, \bar{t}_{\epsilon}^{1}) - u^{1}(\bar{y}_{\epsilon}^{1}, \bar{s}_{\epsilon}^{1}) - \eta(\bar{t}_{\epsilon}^{1} + \bar{s}_{\epsilon}^{1}) + g(\langle \bar{x}_{\epsilon}^{1} \rangle_{\beta} - C(T - \bar{t}_{\epsilon}^{1})) \\ + g(\angle \bar{y}_{\epsilon}^{1} \rangle_{\beta} - C(T - \bar{s}_{\epsilon}^{1})) \\ := W^{1}(\epsilon)$$

By the definition of Φ^1 , we get (3.60) if we show that

$$W^1(\epsilon) \to S^1 \text{ as } \epsilon \to 0^+$$
 (3.61)

We proceed by contradiction. Assume that (3.61) does not hold. So by the compactness of \overline{C}^2 , there exist $\epsilon_k \to 0^+$, $\overline{x}^1_{\epsilon_k} \to x^*$, $\overline{y}^1_{\epsilon_k} \to y^*$, $\overline{t}^1_{\epsilon_k} \to t^*$, $\overline{s}^1_{\epsilon_k} \to s^*$ such that $\lim_{k\to\infty} W^1(\epsilon_k) > S^1$. But (3.59) implies $x^* = y^*$, $t^* = s^*$. Thus the upper semicontinuity of $v^1 - u^1$ gives

$$\lim_{k \to \infty} W^{1}(\epsilon_{k}) \leq v^{1}(x^{*}, t^{*}) - u^{1}(x^{*}, t^{*}) - 2\eta t^{*} + 2g(\langle x^{*} \rangle_{\beta} - C(T - t^{*}))$$

$$\leq S^{1}$$

which gives a contradiction.

Now we show that neither \bar{t}^1_{ϵ} nor \bar{s}^1_{ϵ} can be zero if ϵ is small enough. In fact if $\bar{t}^1_{\epsilon} = 0$ for an arbitrary ϵ small enough, choose $\epsilon^+_k \to 0^+$ with $t_{\epsilon_k} = 0$ for all k. Then by the compactness of $\bar{\mathcal{C}}^2$ and (3.59) we may assume $\epsilon_k \to 0^+$, $\bar{x}^1_{\epsilon_k} \to x^*$, $\bar{y}^1_{\epsilon_k} \to x^*$, $\bar{t}^1_{\epsilon_k} \to t^*$, $\bar{s}^1_{\epsilon_k} \to t^*$. Observe that $t^* = 0$ because $\bar{t}^1_{\epsilon_k} = 0$ for k. It follows that

$$\begin{split} \limsup_{k \to \infty} \Phi^{1}(\bar{x}_{\epsilon_{k}}^{1}, \bar{y}_{\epsilon_{k}}^{1}, \bar{t}_{\epsilon_{k}}^{1}, \bar{s}_{\epsilon_{k}}^{1}) &\leq \limsup_{k \to \infty} [v^{1}(\bar{x}_{\epsilon_{k}}^{1}, \bar{t}_{\epsilon_{k}}^{1}) - u^{1}(\bar{y}_{\epsilon_{k}}^{1}, \bar{s}_{\epsilon_{k}}^{1})] \\ &\leq v^{1}(x^{*}, t^{*}) - u^{1}(x^{*}, t^{*}) \\ &\leq 0 \end{split}$$

Next we define the test functions on \mathcal{C}

$$\begin{array}{lll} \varphi^{1}(x,t) &:= & u^{1}(\bar{y}_{\epsilon}^{1},\bar{s}_{\epsilon}^{1}) + \frac{|x-\bar{y}_{\epsilon}^{1}|^{2} + |t-\bar{s}_{\epsilon}^{1}|^{2}}{2\epsilon} + \eta(t+\bar{s}_{\epsilon}^{1}) \\ & & -g(\langle x \rangle_{\beta} - C(T-t)) - g(\langle y_{\epsilon}^{1} \rangle_{\beta} - C(T-\bar{s}_{\epsilon}^{1})) \\ \psi^{1}(y,s) &:= & v^{1}(\bar{x}_{\epsilon}^{1},\bar{t}_{\epsilon}^{1}) - \frac{|\bar{x}_{\epsilon}^{1}-y|^{2} + |\bar{t}_{\epsilon}^{1}-s|^{2}}{2\epsilon} - \eta(\bar{t}_{\epsilon}^{1}+s) \\ & + g(\langle \bar{x}_{\epsilon}^{1} \rangle_{\beta} - C(T-\bar{t}_{\epsilon}^{1})) - g(y-C(T-s)) \end{array}$$

Thus $\varphi^1, \psi^1 \in C^1(\mathcal{C}), v^1 - \varphi^1$ has a maximum at $(\bar{x}^1_{\epsilon}, \bar{t}^1_{\epsilon})$ and $u^1 - \psi^1$ has a minimum at $(\bar{y}^1_{\epsilon}, \bar{s}^1_{\epsilon})$. Next we compute the partial derivative of φ^1 and ψ^1 . We have

$$\begin{split} \varphi_t^1(\bar{x}_{\epsilon}^1, \bar{t}_{\epsilon}^1) &= \frac{\bar{t}_{\epsilon}^1 - \bar{s}_{\epsilon}^1}{\epsilon} + \eta - Cg'(\langle \bar{x}_{\epsilon}^1 \rangle_{\beta} - C(T - \bar{t}_{\epsilon}^1)) \\ D\varphi^1(\bar{x}_{\epsilon}^1, \bar{t}_{\epsilon}^1) &= \frac{\bar{x}_{\epsilon}^1 - \bar{y}_{\epsilon}^1}{\epsilon} - g'(\langle \bar{x}_{\epsilon}^1 \rangle_{\beta} - C(T - \bar{t}_{\epsilon}^1)) \frac{\bar{x}_{\epsilon}^1}{\langle \bar{x}_{\epsilon}^1 \rangle_{\beta}} \\ \psi_t^1(\bar{y}_{\epsilon}^1, \bar{s}_{\epsilon}^1) &= \frac{\bar{t}_{\epsilon}^1 - \bar{s}_{\epsilon}^1}{\epsilon} - \eta + Cg'(\langle \bar{y}_{\epsilon}^1 \rangle_{\beta} - C(T - \bar{s}_{\epsilon}^1)) \\ D\psi^1(\bar{y}_{\epsilon}^1, \bar{s}_{\epsilon}^1) &= \frac{\bar{x}_{\epsilon}^1 - \bar{y}_{\epsilon}^1}{\epsilon} + g'(\langle \bar{y}_{\epsilon}^1 \rangle_{\beta} - C(T - \bar{s}_{\epsilon}^1)) \frac{\bar{y}_{\epsilon}^1}{\langle \bar{y}_{\epsilon}^1 \rangle_{\beta}} \end{split}$$

Set $X := \langle \bar{x}_{\epsilon}^1 \rangle_{\beta} - C(T - \bar{t}_{\epsilon}^1)$, $Y := \langle \bar{y}_{\epsilon}^1 \rangle_{\beta} - C(T - \bar{s}_{\epsilon}^1)$. Since u and v are respectively a viscosity supersolution and subsolution of (SQVI), we have

$$\max\{\psi_t^1(\bar{y}_{\epsilon}^1, \bar{s}_{\epsilon}^1) + H^1(\bar{y}_{\epsilon}^1, D\psi^1(\bar{y}_{\epsilon}^1, \bar{s}_{\epsilon}^1)), u^1(\bar{y}_{\epsilon}^1, \bar{s}_{\epsilon}^1) - \min_{i \neq 1}\{u^i(\bar{y}_{\epsilon}^1, \bar{s}_{\epsilon}^1) + k(a^1, a^i)\}\} \ge 0 \quad (3.62)$$

$$\max\{\varphi_t^1(\bar{x}_{\epsilon}^1, \bar{t}_{\epsilon}^1) + H^1(\bar{x}_{\epsilon}^1, D\varphi^1(\bar{x}_{\epsilon}^1, \bar{t}_{\epsilon}^1)), v^1(\bar{x}_{\epsilon}^1, \bar{t}_{\epsilon}^1) - \min_{i \neq 1}\{v^i(\bar{x}_{\epsilon}^1, \bar{t}_{\epsilon}^1) + k(a^1, a^i)\}\} \le 0 \quad (3.63)$$

We have two cases to consider:

 $\begin{aligned} & \textbf{Case 1} \ u^1(\bar{y}_{\epsilon}^1, \bar{s}_{\epsilon}^1) - \min_{i \neq 1} \{ u^i(\bar{y}_{\epsilon}^1, \bar{s}_{\epsilon}^1) + k(a^1, a^i) \} < 0 \\ & \textbf{Case 2} \ u^1(\bar{y}_{\epsilon}^1, \bar{s}_{\epsilon}^1) - \min_{i \neq 1} \{ u^i(\bar{y}_{\epsilon}^1, \bar{s}_{\epsilon}^1) + k(a^1, a^i) \} \ge 0 \end{aligned}$

If Case 1 occurs, we have

$$\psi_t^1(\bar{y}_{\epsilon}^1, \bar{s}_{\epsilon}^1) + H^1(\bar{y}_{\epsilon}^1, D\psi^1(\bar{y}_{\epsilon}^1, \bar{s}_{\epsilon}^1)) \ge 0$$

From (3.63), we have

$$\varphi_t^1(\bar{x}_{\epsilon}^1, \bar{t}_{\epsilon}^1) + H^1(\bar{x}_{\epsilon}^1, D\varphi^1(\bar{x}_{\epsilon}^1, \bar{t}_{\epsilon}^1)) \le 0$$

Thus

$$\psi_t^1(\bar{y}_{\epsilon}^1, \bar{s}_{\epsilon}^1) - \varphi_t^1(\bar{x}_{\epsilon}^1, \bar{t}_{\epsilon}^1) + H^1(\bar{y}_{\epsilon}^1, D\psi^1(\bar{y}_{\epsilon}^1, \bar{s}_{\epsilon}^1)) - H^1(\bar{x}_{\epsilon}^1, D\varphi^1(\bar{x}_{\epsilon}^1, \bar{t}_{\epsilon}^1)) \ge 0$$
(3.64)

By substituting the values of $\psi_t^1(\bar{y}_{\epsilon}^1, \bar{s}_{\epsilon}^1)$, $\varphi_t^1(\bar{x}_{\epsilon}^1, \bar{t}_{\epsilon}^1)$, $D\psi^1(\bar{y}_{\epsilon}^1, \bar{s}_{\epsilon}^1)$, $D\varphi^1(\bar{x}_{\epsilon}^1, \bar{t}_{\epsilon}^1)$ into (3.64), we get

$$C(g'(Y) + g'(X)) + H^{1}(\bar{y}_{\epsilon}^{1}, \frac{\bar{x}_{\epsilon}^{1} - \bar{y}_{\epsilon}^{1}}{\epsilon} + g'(Y) \frac{\bar{y}_{\epsilon}^{1}}{<\bar{y}_{\epsilon}^{1} >_{\beta}}) - H^{1}(\bar{x}_{\epsilon}^{1}, \frac{\bar{x}_{\epsilon}^{1} - \bar{y}_{\epsilon}^{1}}{\epsilon} + g'(X) \frac{\bar{x}_{\epsilon}^{1}}{<\bar{x}_{\epsilon}^{1} >_{\beta}})$$

$$\geq 2\eta$$
(3.65)

Now we add and subtract $H^1(\bar{x}^1_{\epsilon}, \frac{\bar{x}^1_{\epsilon} - \bar{y}^1_{\epsilon}}{\epsilon} + g'(Y) \frac{\bar{y}^1_{\epsilon}}{\langle \bar{y}^1_{\epsilon} \rangle_{\beta}})$ to the left hand side of (3.65), then use (3.55) and (3.56) to get

$$C(g'(Y) + g'(X)) + \omega(|\bar{x}_{\epsilon}^{1} - \bar{y}_{\epsilon}^{1}|) + \omega(\frac{|\bar{x}_{\epsilon}^{1} - \bar{y}_{\epsilon}^{1}|^{2}}{\epsilon} + |\bar{x}_{\epsilon}^{1} - \bar{y}_{\epsilon}^{1}||g'(Y)|\frac{|\bar{y}_{\epsilon}^{1}|}{\langle \bar{y}_{\epsilon}^{1} \rangle_{\beta}}) + C(|g'(Y) - \frac{|\bar{y}_{\epsilon}^{1}|}{\langle \bar{y}_{\epsilon}^{1} \rangle_{\beta}}) - g'(X) - \frac{|\bar{x}_{\epsilon}^{1}|}{\langle \bar{x}_{\epsilon}^{1} \rangle_{\beta}}|) \geq 2\eta.$$

Since $\frac{|\bar{x}_{\epsilon}^1|}{\langle \bar{x}_{\epsilon}^1 \rangle_{\beta}}$, $\frac{|\bar{y}_{\epsilon}^1|}{\langle \bar{y}_{\epsilon}^1 \rangle_{\beta}} < 1$ and ω is nondecreasing, we have

$$C(g'(Y) + g'(X)) + \omega(|\bar{x}_{\epsilon}^{1} - \bar{y}_{\epsilon}^{1}|) + \omega(\frac{|\bar{x}_{\epsilon}^{1} - \bar{y}_{\epsilon}^{1}|^{2}}{\epsilon} + |\bar{x}_{\epsilon}^{1} - \bar{y}_{\epsilon}^{1}||g'(Y)|) + C(|g'(Y)| + |g'(X)|) \ge 2\eta.$$

Since g' = -|g'|, we have

$$\omega(|\bar{x}_{\epsilon}^{1} - \bar{y}_{\epsilon}^{1}|) + \omega(\frac{|\bar{x}_{\epsilon}^{1} - \bar{y}_{\epsilon}^{1}|^{2}}{\epsilon} + |\bar{x}_{\epsilon}^{1} - \bar{y}_{\epsilon}^{1}||g'(Y)|) \ge 2\eta.$$
(3.66)

By using (3.59) and (3.60), if $\epsilon \to 0^+$ then the left hand side of (3.66) goes to zero which gives a contradiction.

If Case 2 occurs, we have

$$u^{1}(\bar{y}_{\epsilon}^{1}, \bar{s}_{\epsilon}^{1}) \geq \min_{i \neq 1} \{ u^{i}(\bar{y}_{\epsilon}^{1}, \bar{s}_{\epsilon}^{1}) + k(a^{1}, a^{i}) \}$$

Choose $j_2 \in \{2, 3, ..., r\}$, say $j_2 = 2$, such that

$$u^{1}(\bar{y}_{\epsilon}^{1}, \bar{s}_{\epsilon}^{1}) \ge u^{2}(\bar{y}_{\epsilon}^{1}, \bar{s}_{\epsilon}^{1}) + k(a^{1}, a^{2})$$
(3.67)

From (3.63), we have

$$v^{1}(\bar{x}_{\epsilon}^{1}, \bar{t}_{\epsilon}^{1}) \leq v^{2}(\bar{x}_{\epsilon}^{1}, \bar{t}_{\epsilon}^{1}) + k(a^{1}, a^{2})$$
(3.68)

From (3.67) and (3.68), we have

$$v^{1}(\bar{x}_{\epsilon}^{1}, \bar{t}_{\epsilon}^{1}) - u^{1}(\bar{y}_{\epsilon}^{1}, \bar{s}_{\epsilon}^{1}) \le v^{2}(\bar{x}_{\epsilon}^{1}, \bar{t}_{\epsilon}^{1}) - u^{2}(\bar{y}_{\epsilon}^{1}, \bar{s}_{\epsilon}^{1})$$
(3.69)

Since $\Phi^1(\bar{x}^1_{\epsilon}, \bar{y}^1_{\epsilon}, \bar{t}^1_{\epsilon}, \bar{s}^1_{\epsilon}) \ge \Phi^2(\bar{x}^1_{\epsilon}, \bar{y}^1_{\epsilon}, \bar{t}^1_{\epsilon}, \bar{s}^1_{\epsilon})$, we have

$$v^{1}(\bar{x}_{\epsilon}^{1}, \bar{t}_{\epsilon}^{1}) - u^{1}(\bar{y}_{\epsilon}^{1}, \bar{s}_{\epsilon}^{1}) \ge v^{2}(\bar{x}_{\epsilon}^{1}, \bar{t}_{\epsilon}^{1}) - u^{2}(\bar{y}_{\epsilon}^{1}, \bar{s}_{\epsilon}^{1})$$
(3.70)

From (3.69) and (3.70), we have

$$v^{1}(\bar{x}_{\epsilon}^{1}, \bar{t}_{\epsilon}^{1}) - u^{1}(\bar{y}_{\epsilon}^{1}, \bar{s}_{\epsilon}^{1}) = v^{2}(\bar{x}_{\epsilon}^{1}, \bar{t}_{\epsilon}^{1}) - u^{2}(\bar{y}_{\epsilon}^{1}, \bar{s}_{\epsilon}^{1})$$

Thus by definition of Φ^j , we have

$$\Phi^2(\bar{x}^1_\epsilon, \bar{y}^1_\epsilon, \bar{t}^1_\epsilon, \bar{s}^1_\epsilon) = \Phi^1(\bar{x}^1_\epsilon, \bar{y}^1_\epsilon, \bar{t}^1_\epsilon, \bar{s}^1_\epsilon)$$

Now repeat the previous consideration with index 2 replacing 1. If Case 1 occurs, we are done. Otherwise there exists $j_3 \in \{1, 3, 4, ..., r\}$ such that

$$v^{2}(\bar{x}_{\epsilon}^{1}, \bar{t}_{\epsilon}^{1}) - u^{2}(\bar{y}_{\epsilon}^{1}, \bar{s}_{\epsilon}^{1}) = v^{j_{3}}(\bar{x}_{\epsilon}^{1}, \bar{t}_{\epsilon}^{1}) - u^{j_{3}}(\bar{y}_{\epsilon}^{1}, \bar{s}_{\epsilon}^{1})$$

Note that $j_3 \neq 1$. If $j_3 = 1$, we would have

$$\begin{aligned} u^{1}(\bar{y}_{\epsilon}^{1}, \bar{s}_{\epsilon}^{1}) &\geq & u^{2}(\bar{y}_{\epsilon}^{1}, \bar{s}_{\epsilon}^{1}) + k(a^{1}, a^{2}) \\ &\geq & u^{1}(\bar{y}_{\epsilon}^{1}, \bar{s}_{\epsilon}^{1}) + k(a^{2}, a^{1}) + k(a^{1}, a^{2}) \end{aligned}$$

This implies that $k(a^2, a^1) + k(a^1, a^2) = 0$ which contradicts (A12). Assume $j_3 = 3$. As before we get

$$\Phi^{3}(\bar{x}_{\epsilon}^{1}, \bar{y}_{\epsilon}^{1}, \bar{t}_{\epsilon}^{1}, \bar{s}_{\epsilon}^{1}) = \Phi^{2}(\bar{x}_{\epsilon}^{1}, \bar{y}_{\epsilon}^{1}, \bar{t}_{\epsilon}^{1}, \bar{s}_{\epsilon}^{1}) = \Phi^{1}(\bar{x}_{\epsilon}^{1}, \bar{y}_{\epsilon}^{1}, \bar{t}_{\epsilon}^{1}, \bar{s}_{\epsilon}^{1})$$

Repeat the process with the index 3 replacing 2 and so on. After finitely many steps, there is an index $j_n \leq r$ for which Case 1 holds. Then the proof is complete. \diamond

Remark 3.14 It is easy to check that Theorem 3.13 holds in the cone $\{(x,t) : 0 < t < T, |x - x_0| < C(T - t)\}$ for any x_0 if (3.55) and (3.56) hold for all $x, y \in B(x_0, CT)$ and $v \le u$ in $B(x_0, CT) \times \{0\}$.

The following is our main comparison theorem for super- and subsolutions of SQVIs.

Theorem 3.15 Let $H^j : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be continuous and satisfy

$$|H^{j}(x,p) - H^{j}(x,q)| \le L(|x|+1)|p-q|$$
(3.71)

for all $x, p, q \in \mathbb{R}^n$, where L > 0 and

$$|H^{j}(x,p) - H^{j}(y,p)| \le \omega(|x-y|,R) + \omega(|x-y||p|,R)$$
(3.72)

for all $p \in \mathbb{R}^n$, $x, y \in B(0, R), R > 0, j \in \{1, 2, \dots, r\}$ where ω is a modulus. Assume

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- (i) (A12) holds;
- (ii) $u^j \in LSC(\mathbb{R}^n \times [0,T])$ and $v^j \in USC(\mathbb{R}^n \times [0,T]), j = 1, \ldots, r;$
- (iii) $u = (u^1, \ldots, u^r)$ and $v = (v^1, \ldots, v^r)$ are respectively a viscosity supersolution and subsolution of (3.53) in $\mathbb{R}^n \times (0, T)$;
- (iv) $v(x,0) \le u(x,0)$ for all $x \in \mathbb{R}^n$.

Then $v \leq u$ in $\mathbb{R}^n \times [0, T]$.

Proof Without loss of generality assume that $T < \frac{1}{L}$, because the proof of the general case is obtained by iterating the following argument on the time intervals of fixed length smaller than $\frac{1}{L}$. Fix $x_0 \in \mathbb{R}^n$ and define

$$r := \frac{LT(|x_0|+1)}{1-LT}, \quad C := L(|x_0|+1+r).$$

By an easy computation, we have

$$r = CT \tag{3.73}$$

We define the cone

$$\mathcal{C}_{x_0} := \{ (x, t) : 0 < t < T, \ |x - x_0| < C(T - t) \}$$

If $x \in B(x_0, CT)$, then $|x| < |x_0| + r$ by (3.73). Thus from (3.71), we have

$$|H(x,p) - H(x,p)| \leq L(|x|+1)|p-q| < L(|x_0|+r+1)|p-q| = C|p-q|$$

Apply Remark 3.14 to get $v \leq u$ in \mathcal{C}_{x_0} . Since $\mathbb{R}^n \times (0,T) = \bigcup_{x_0 \in \mathbb{R}^n} \mathcal{C}_{x_0}$, we have shown that

$$v \le u \quad \text{in } \mathbb{R}^n \times (0, T). \tag{3.74}$$

Next we want to show that $v(x,T) \leq u(x,T)$ for all $x \in \mathbb{R}^n$. From (3.74), we have

$$\lim_{(y,t)\to(x,T)} \sup \{v(y,t) - u(y,t)\} \le 0$$

It follows that

$$v^*(x,T) + (-u)^*(x,T) \le 0$$

Since v is u.s.c at (x, T), we have $v(x, T) = v^*(x, T)$ Thus by using $(-u)^*(x, T) = -u_*(x, T)$, we have $v(x, T) \le u_*(x, T) \le u(x, T)$.

3.3.3 An application of the comparison principle

In this section, we apply a comparison principle to give an alternative derivation of the characterization of a continuous lower value function V_{γ} (see (3.11)) as the minimal, nonnegative, continuous viscosity supersolution of SQVI (3.9).

We set

$$Z^{j}(x,t) = \inf_{\alpha \in \Gamma} \sup_{b \in \mathcal{B}} \int_{[0,t)} l(y_{x}(s), a^{j}, \alpha_{x}^{j}[b](s), b(s)), \ j = 1, \dots, r.$$
(3.75)

Proposition 3.16 Assume (A8)-(A12). Z^j is upper semicontinuous in $\mathbb{R}^n \times [0,T]$ for T > 0.

Proof Fix $x \in \mathbb{R}^n$ and $t \in [0, T]$. Let $\epsilon > 0$. By the definition of $Z^j(x, t)$, choose $\bar{\alpha} \in \Gamma$ such that

$$Z^{j}(x,t) > \sup_{b \in \mathcal{B}} \int_{[0,t)} l(y_{x}(s), a^{j}, \bar{\alpha}^{j}_{x}[b](s), b(s)) - \frac{\epsilon}{4}$$
(3.76)

Let $z \in \mathbb{R}^n$ and $\tau \in [0, T]$. Find $\hat{\alpha} \in \Gamma$ so that, for each $b \in \mathcal{B}$

$$\hat{\alpha}_z^j[b](s) = \begin{cases} \bar{\alpha}_x^j[b](s) & 0 \le s \le t, \\ \bar{\alpha}_x^j[b](t) & s \ge t \end{cases}$$

By the definition of $Z^{j}(z,\tau)$, choose $\bar{b} \in \mathcal{B}$ such that

$$Z^{j}(z,\tau) \leq \sup_{b\in\mathcal{B}} \int_{[0,\tau)} l(y_{z}(s), a^{j}, \bar{\alpha}_{z}^{j}[b](s), b(s)) \\ < \int_{[0,\tau)} l(y_{z}(s), a^{j}, \hat{\alpha}_{z}^{j}[\bar{b}](s), \bar{b}(s)) + \frac{\epsilon}{4}$$
(3.77)

From (3.76) and (3.77), we have

$$Z^{j}(z,\tau) - Z^{j}(x,t) < \int_{[0,\tau)} l(y_{z}(s), a^{j}, \hat{\alpha}_{z}^{j}[\bar{b}](s), \bar{b}(s)) - \int_{[0,t)} l(y_{x}(s), a^{j}, \bar{\alpha}_{x}^{j}[\bar{b}](s), \bar{b}(s)) + \frac{\epsilon}{2}$$
(3.78)

If $\tau < t$, then from (3.78) we have

$$Z^{j}(z,\tau) - Z^{j}(x,t) < \int_{[0,\tau)} [l(y_{z}(s), a^{j}, \bar{\alpha}_{x}^{j}[\bar{b}](s), \bar{b}(s)) - l(y_{x}(s), a^{j}, \bar{\alpha}_{x}^{j}[\bar{b}](s), \bar{b}(s))] - \int_{[\tau,t)} l(y_{x}(s), a^{j}, \bar{\alpha}_{x}^{j}[\bar{b}](s), \bar{b}(s)) + \frac{\epsilon}{2} \leq \int_{0}^{\tau} |h(y_{z}(s), \bar{\alpha}_{x}^{j}[\bar{b}](s), \bar{b}(s)) - h(y_{x}(s), \bar{\alpha}_{x}^{j}[\bar{b}](s), \bar{b}(s))| ds - \int_{\tau}^{t} [h(y_{x}(s), \bar{\alpha}_{x}^{j}[\bar{b}](s), \bar{b}(s)) - \gamma^{2}|\bar{b}|^{2}] ds - \sum_{\tau \leq s < t} k(\bar{\alpha}_{x}^{j}[b](s^{-}), \bar{\alpha}_{x}^{j}[b](s)) + \frac{\epsilon}{2} \leq \int_{0}^{\tau} |h(y_{z}(s), \bar{\alpha}_{x}^{j}[\bar{b}](s), \bar{b}(s)) - h(y_{x}(s), \bar{\alpha}_{x}^{j}[\bar{b}](s), \bar{b}(s))| ds - \int_{\tau}^{t} [h(y_{x}(s), \bar{\alpha}_{x}^{j}[\bar{b}](s), \bar{b}(s)) - \gamma^{2}|\bar{b}|^{2}] ds + \frac{\epsilon}{2} \leq \int_{0}^{\tau} \omega_{h}(|y_{z}(s) - y_{x}(s)|, R_{1}) ds + \int_{\tau}^{t} |h(y_{x}(s), \bar{\alpha}_{x}^{j}[\bar{b}](s), \bar{b}(s)) - \gamma^{2}|\bar{b}|^{2}| ds + \frac{\epsilon}{2} \leq \int_{0}^{t} \omega_{h}(e^{Lt}|z - x|, R_{1}) ds + \int_{\tau}^{t} K_{1} ds + \frac{\epsilon}{2}$$
(3.79)

for some constant K_1 , $R_1 > 0$. If $\tau > t$, then from (3.78) we have

$$Z^{j}(z,\tau) - Z^{j}(x,t) < \int_{[0,t)} [l(y_{z}(s), a^{j}, \hat{\alpha}_{z}^{j}[\bar{b}](s), \bar{b}(s)) - l(y_{x}(s), a^{j}, \bar{\alpha}_{x}^{j}[\bar{b}](s), \bar{b}(s))] + \int_{[t,\tau]} l(y_{z}(s), a^{j}, \hat{\alpha}_{z}^{j}[\bar{b}](s), \bar{b}(s)) + \frac{\epsilon}{2} \leq \int_{0}^{t} \omega_{h}(|y_{z}(s) - y_{x}(s)|, R_{2}) ds + \int_{t}^{\tau} |h(y_{z}(s), \hat{\alpha}_{z}^{j}[\bar{b}](s), \bar{b}(s)) - \gamma^{2} |\bar{b}|^{2} |ds + \frac{\epsilon}{2} \leq \int_{0}^{t} \omega(e^{Lt}|z - x|, R_{2}) ds + \int_{t}^{\tau} K_{2} ds + \frac{\epsilon}{2}$$
(3.80)

for some constant K_2 , $R_2 > 0$. Thus the right hand side of (3.79) and (3.80) can be made less than ϵ if |x - z| and $|t - \tau|$ are small enough. Thus for both cases we have $Z^j(z,\tau) < Z^j(x,t) + \epsilon$ for |x - z|, $|t - \tau|$ small enough. \diamond

Proposition 3.17 Assume (A8)-(A12). Then

$$Z^{j}(x,t) \le \min_{i \ne j} \{ Z^{i}(x,t) + k(a^{j},a^{i}) \}$$

for all $x \in \mathbb{R}^n$, $t > 0, j \in \{1, 2, \dots, r\}$.

Proof It is similar to the proof of Proposition 3.3. \Diamond

Theorem 3.18 Assume (A8)-(A12). Then for all $x \in \mathbb{R}^n$ and $\tau \in (0, t)$,

$$Z^{j}(x,t) = \inf_{\alpha \in \Gamma} \sup_{b \in \mathcal{B}} \{ \int_{[0,\tau)} l(y_{x}(s), a^{j}, \alpha^{j}_{x,t}[b](s), b(s)) + Z^{i}(y_{x}(\tau), t-\tau), \alpha^{j}_{x,t}[b](\tau^{-}) = a^{i} \}$$

Proof It is similar to the proof of Theorem 3.4. \Diamond

Corollary 3.19 Assume (A8)-(A12). Then for all $x \in \mathbb{R}^n$ and $\tau \in (0, t)$,

$$Z^{j}(x,t) \leq \sup_{b \in \mathcal{B}} \{ \int_{0}^{\tau} [h(y_{x}(s), a^{j}, b(s)) - \gamma^{2} |b(s)|^{2}] ds + Z^{j}(y_{x}(\tau), t - \tau) \}$$

Proof Fix $x \in \mathbb{R}^n$, $0 < \tau < t$. For any $b \in \mathcal{B}$, choose $\bar{\alpha} \in \Gamma$ such that $\bar{\alpha}_{x,t}^j[b](s) = a^j$ for all $0 \le s \le \tau$. Thus from Theorem 3.18, we have

$$Z^{j}(x,t) \leq \sup_{b \in \mathcal{B}} \{ \int_{[0,\tau)} l(y_{x}(s), \bar{\alpha}_{x}^{j}[b](s), b(s)) + Z^{j}(y_{x}(\tau), t-\tau) \}$$

$$= \sup_{b \in \mathcal{B}} \{ \int_{0}^{\tau} [h(y_{x}(s), a^{j}, b(s)) - \gamma^{2} |b(s)|^{2}] ds + Z^{j}(y_{x}(\tau), t-\tau) \} \diamond$$

Theorem 3.20 Assume (A8)-(A12). Then $Z = (Z^1, \ldots, Z^r)$ is a viscosity subsolution of (SQVI)

$$\max\{w_t^j + H^j(x, Dw^j), w^j - \min_{i \neq j} \{w^i + k(a^j, a^i)\} = 0 \text{ in } \mathbb{R}^n \times (0, T), j = 1, 2, \dots, r,$$

where $H^j(x, p) = \min_{b \in B} \{-p \cdot f(x, a^j, b) - h(x, a^j, b) + \gamma^2 |b|^2\}.$

Proof Fix $j \in \{1, 2, ..., r\}$, $x \in \mathbb{R}^n$ and $t \in (0, T)$. Let $\varphi^j \in C^1(\mathbb{R}^n \times (0, T))$ such that $Z^j(x, t) = \varphi^j(x, t)$ and $Z^j(z, \nu) \leq \varphi^j(z, \nu)$ for all (z, ν) near (x, t). We want to show that

$$\max\{\varphi_s^j(x,t) + H^j(x, D\varphi^j(x,t)), \ Z^j(x,t) - \min_{i \neq j}\{Z^i(x,t) + k(a^j, a^i)\}\} \le 0.$$
(3.81)

We assume

$$\varphi_s^j(x,t) + H^j(x, D\varphi^j(x,t)) > 0$$

Since φ_s^j and H^j are continuous, choose $\epsilon > 0$ such that

$$\varphi^j_s(z,\nu) + H^j(z,D\varphi^j(z,\nu)) \geq \epsilon$$

for all $z \in B(x, \epsilon)$ and $\nu \in (t - \epsilon t + \epsilon) \cap (0, T)$. Thus by the definition of H^j , we have

$$\varphi_s^j(z,\nu) - D\varphi^j(z,\nu) \cdot f(z,a^j,b) - h(z,a^j,b) + \gamma^2 |b|^2 \ge \epsilon$$
(3.82)

for all $z \in B(x, \epsilon)$, $\nu \in (t - \epsilon, t + \epsilon) \bigcap (0, T)$, $b \in B$.

By assumptions on f and B, choose $\tau > 0$ such that $y_z(s, a^j, b) \in B(x, \epsilon)$ for all $z \in B(x, \frac{\epsilon}{2}), 0 \leq s \leq \tau$ and $\nu - \tau \in (t - \epsilon, t + \epsilon) \cap (0, T)$ for all $\nu \in (t - \epsilon, t + \epsilon) \cap (0, T)$. Let $z \in B(x, \frac{\epsilon}{2}), \nu \in (t - \epsilon, t + \epsilon) \cap (0, T)$. Set $\delta = \frac{\epsilon \tau}{2}$. From Corollary 3.19, choose $\bar{b} \in \mathcal{B}$ such that

$$Z^{j}(z,\nu) \leq \int_{0}^{\tau} [h(y_{z}(s),a^{j},\bar{b}(s)) - \gamma^{2}|\bar{b}(s)|^{2}]ds + Z^{j}(y_{z}(\tau),\nu-\tau) + \delta$$

Since $Z^{j}(y_{z}(\tau), \nu - \tau) \leq \varphi^{j}(y_{z}(\tau), \nu - \tau)$, we have

$$Z^{j}(z,\nu) - \varphi^{j}(z,\nu) \leq \int_{0}^{\tau} [h(y_{z}(s),a^{j},\bar{b}(s)) - \gamma^{2}|\bar{b}(s)|^{2}]ds + \varphi^{j}(y_{z}(\tau),\nu-\tau) - \varphi^{j}(z,\nu) + \delta$$

$$= \int_{0}^{\tau} [h(y_{z}(s),a^{j},\bar{b}(s)) - \gamma^{2}|\bar{b}(s)|^{2} + \frac{d}{ds}\varphi^{j}(y_{z}(s),\nu-s)]ds + \delta$$

Since

$$\frac{d}{ds}\varphi^j(y_z(s),\nu-s) = -\varphi^j_s(y_z(s),\nu-s) + D\varphi^j(y_z(s),\nu-s)\dot{y}_z(s),\nu-s)\dot{y}_z(s),\nu-s$$

It follows that

$$Z^{j}(z,\nu) - \varphi^{j}(z,\nu) \leq \int_{0}^{\tau} [h(y_{z}(s),a^{j},\bar{b}(s)) - \gamma^{2}|\bar{b}(s)|^{2} - \varphi^{j}_{s}(y_{z}(s),\nu-s) + D\varphi^{j}(y_{z}(s),\nu-s) \cdot f(y_{z}(s),a^{j},\bar{b}(s))]ds + \delta$$
$$\leq -\int_{0}^{\tau} \epsilon ds + \delta$$
$$= -\delta$$

which the second inequality follows from (3.82). Thus

$$\limsup_{(z,\nu)\to(x,t)} Z^j(z,\nu) \le \limsup_{(z,\nu)\to(x,t)} \{\varphi^j(z,\nu) - \delta\}$$

Since Z^j is upper semicontinuous, we have

$$Z^{j}(x,t) \le \varphi^{j}(x,t) - \delta < \varphi^{j}(x,t)$$

which gives a contradiction. Thus $\varphi_s^j(x,t) + H^j(x, D\varphi^j(x,t)) \leq 0$. Together with Proposition 3.17, we have (3.81). \diamond

Theorem 3.21 Assume (A8)-(A13). Let $V_{\gamma} = (V_{\gamma}^1, \ldots, V_{\gamma}^r)$ be the lower value vector function where V_{γ}^j is given by (3.11). If $U = (U^1, \ldots, U^r)$, where $U^j \in LSC(\mathbb{R}^n)$ for $j \in \{1, 2, \ldots, r\}$, is a nonnegative viscosity supersolution of (SQVI)

$$\max\{H^{j}(x, DU^{j}(x)), U^{j}(x) - \min_{i \neq j}\{U^{i}(x) + k(a^{j}, a^{i})\}\} = 0, \ x \in \mathbb{R}^{n}, \ j = 1, 2, \dots, r, \quad (3.83)$$

where $H^j(x,p) := \min_{b \in B} \{-p \cdot f(x,a^j,b) - h(x,a^j,b) + \gamma^2 |b|^2\}$, then $U \ge V_{\gamma}$, i.e., $U^j \ge V_{\gamma}^j$ for $j = 1, \ldots, r$. Hence if continuous, V_{γ} is characterized as the minimal nonnegative continuous viscosity supersolution.

Proof Let T > 0. Since U is a viscosity supersolution of (3.83), it is also a viscosity supersolution of

$$\max\{w_t^j + H^j(x, Dw^j(x)), w^j(x) - \min_{i \neq j}\{w^{(x)} + k(a^j, a^i)\}\} = 0$$

in $\mathbb{R}^n \times (0, T), \ j = 1, 2, \dots, r.$ (3.84)

We define

$$Z^{j}(x,t) = \inf_{\alpha \in \Gamma} \sup_{b \in \mathcal{B}} \int_{[0,t]} l(y_{x}(s), a^{j}, \alpha_{x}^{j}[b](s), b(s)), \ \alpha_{x}^{j}[b](0^{-}) = a^{j}, \ j = 1, \dots, r,$$

for all $(x,t) \in \mathbb{R}^n \times [0,T]$. Then $Z = (Z^1, Z^2, \ldots, Z^l)$ is a viscosity subsolution of (3.84) in $\mathbb{R}^n \times (0,T)$ by Theorem 3.20. Fix $(x,t) \in \mathbb{R}^n \times (0,T)$. For any $b \in \mathcal{B}$, choose $\alpha \in \Gamma$ with $\alpha_x^j[b](s) = a^j$ for all $0 \le s \le t$. Thus we have

$$Z^{j}(x,t) \leq \sup_{b \in \mathcal{B}} \int_{0}^{t} [h(y_{x}(s), a^{j}, b(s)) - \gamma^{2} |b(s)|^{2}] ds.$$

This implies that $Z^{j}(x,0) \leq 0$. Since U^{j} is nonnegative, we have $Z^{j}(x,0) \leq 0 \leq U^{j}(x)$. Thus by Theorem 3.15 we have $U^{j}(x) \geq Z^{j}(x,t)$ for all $(x,t) \in \mathbb{R}^{n} \times [0,T]$. Let $\epsilon > 0$. By definition of $Z^{j}(x,T)$ choose $\bar{\alpha} \in \Gamma$ such that

$$Z^{j}(x,T) + \epsilon > \sup_{b \in \mathcal{B}} \int_{[0,T)} l(y_{x}(s), a^{j}, \bar{\alpha}_{x}^{j}[b](s), b(s))$$

Thus

$$U^{j}(x) + \epsilon \ge \sup_{b \in \mathcal{B}} \int_{[0,T)} l(y_{x}(s), a^{j}, \bar{\alpha}_{x}^{j}[b](s), b(s))$$

Since T > 0 is arbitrary, we have

$$U^{j}(x) + \epsilon \geq \sup_{b \in \mathcal{B}, T \ge 0} \int_{[0,T)} l(y_{x}(s), a^{j}, \bar{\alpha}_{x}^{j}[b](s), b(s))$$

$$\geq \inf_{\alpha \in \Gamma} \sup_{b \in \mathcal{B}, T \ge 0} \int_{[0,T)} l(y_{x}(s), a^{j}, \alpha_{x}^{j}[b](s), b(s))$$

$$= V^{j}(x).$$

Since ϵ is arbitrary, $U^j(x) \ge V^j_{\gamma}(x)$. If V_{γ} is continuous, V_{γ} is a continuous viscosity solution of the SQVI (3.83) (see Theorem 3.8. Thus V_{γ} is a viscosity supersolution. The first part of this Theorem already proved then implies that V_{γ} can be characterized as the minimal nonnegative continuous viscosity supersolution, as asserted. \diamond

3.4 Stability for switching-control problems

The usual formulation of the H^{∞} -control problem also involves a stability constraint. In this section we show how the solution of the SQVI can be used for stability analysis. We also prove that, under appropriate conditions, the closed loop system associated with switching strategy α_U corresponding to the supersolution U of the SQVI is stable. The main idea is to use the supersolution U as a Lyapunov function for trajectories of the closed-loop system. Related stability problems for systems with control switching are discussed, e.g., by Branicky in [18].

We consider the system Σ_{sw} with some control strategy α plugged in to get a closed-loop system with disturbance signal as only input

$$\Sigma_{sw} \begin{cases} \dot{y} = f(y, \alpha_x^j[b], b), \ y(0) = x, \ a(0^-) = a^j \\ z = h(y, \alpha_x^j[b], b]. \end{cases}$$

An example of such a strategy α is the canonical strategy α_U (see (3.42) or (3.44)) determined by a continuous supersolution of the SQVI. Moreover, if $V_{\gamma} = (V_{\gamma}^1, \ldots, V_{\gamma}^r)$ is the vector lowervalue function for the associated game as in ((3.11) and we assume that 0 is an equilibrium point for the autonomous system Σ_{sw} by taking $a(s) = a^{i_0}$ and b(s) = 0 (so $f(0, a^{i_0}, 0) = 0$ and $h(0, a^{i_0}, 0) = 0$), then it is easy to check that $V_{\gamma}^{i_0}(0) = 0$. Furthermore, the associated strategy $\alpha = \alpha_{V_{\gamma}}$ has the property that

$$\alpha_0^{i_0}[0] = a^{i_0},\tag{3.85}$$

so 0 is an equilibrium point of the closed-loop system Σ_{sw} with $\alpha = \alpha_{V_{\gamma}}$ and $a(0^{-}) = a^{i_0}$ as well. Our goal is to give conditions which guarantee a sort of converse, starting with any continuous supersolution U of the SQVI.

We say that the closed-loop switching system Σ_{sw} is zero-state observable for initial control setting a^j if, whenever $h(y_x(t), \alpha_x^j[0](t), 0) = 0$ for all $t \ge 0$, then $y_x(t) = y_x(t, \alpha_x^j[0], 0) = 0$ for all $t \ge 0$. We say that the closed-loop system Σ_{sw} is zero-state detectable for initial control setting a^j

if
$$\lim_{t \to \infty} h(y_x(t), \alpha_x^j[0](t), 0) = 0$$
 for some j , then $\lim_{t \to \infty} y_x(t, \alpha_x^j[0], 0) = 0$.

The following proposition gives conditions which guarantee that a particular component U^{j} of a viscosity supersolution $U = (U^{1}, \ldots, U^{r})$ be positive-definite, a conclusion which will be needed as a hypothesis in the stability theorem to follows.

Proposition 3.22 Assume

(i) (A8)-(A13) hold;
(ii) Σ_{sw} is zero-state observable for the initial control setting a^j;
(iii) U = (U¹,...,U^r) is a nonnegative continuous viscosity supersolution of the SQVI

$$\max\{H^{j}(x, DU^{j}(x)), U^{j}(x) - \min_{i \neq j}\{U^{i}(x) + k(a^{j}, a^{i})\}\} = 0, \ x \in \mathbb{R}^{n}, \ j = 1, \dots, r;$$

(iv) $U^{j}(x) \leq \min_{i \neq j} \{ U^{i}(x) + k(a^{j}, a^{i}) \}, x \in \mathbb{R}^{n}, j = 1, \dots, r.$ Then $U^{j}(x) > 0$ for $x \neq 0$.

Proof Let $x \in \mathbb{R}^n$ and $j \in \{1, \ldots, r\}$. By Theorem 3.10, U is a storage function for Σ_{sw} if we use $\alpha = \alpha_U$ given by (3.42) or equivalently, (3.44). Thus

$$U^{j}(x) \geq \int_{[0,T)} l(y_{x}(s), a^{j}, \alpha_{U,x}^{j}[0](s), 0) \, ds + U^{j(T)}(y_{x}(T, \alpha_{U,x}^{j}[0], 0))$$
$$\geq \int_{[0,T)} l(y_{x}(s), a^{j}, \alpha_{U,x}^{j}[0](s), 0) \, ds \text{ for all } T \geq 0.$$

Since k is nonnegative, we have

$$U^{j}(x) \ge \int_{0}^{T} h(y_{x}(s), \alpha_{x}^{j}[0](s), 0) \, ds, \text{ for all } T \ge 0.$$

Thus if $U^j(x) = 0$, then $h(y_x(s, \alpha_x^j[0], 0), \alpha_x^j[0](s), 0) = 0$ for all $s \ge 0$ because h is nonnegative by assumption (A13). Since Σ_{sw} is zero-state observable for initial control setting a^j , it follows that $y_x(s, \alpha_x^j[0], 0) = 0$ for all $s \ge 0$. Thus $x = y_x(0, \alpha_x[0], 0) = 0$. Since U^j is nonnegative, we conclude that if $x \ne 0$ then $U^j(x) > 0$.

Proposition 3.23 Assume

(i) (A8)-(A13) hold;
(ii) U = (U¹,...,U^r) is a nonnegative continuous viscosity supersolution of the SQVI

$$\max\{H^{j}(x, DU^{j}(x)), U^{j}(x) - \min_{i \neq j}\{U^{i}(x) + k(a^{j}, a^{i})\}\} = 0, \ x \in \mathbb{R}^{n}, \ j = 1, \dots, r;$$

(iii) $U^{j}(x) \leq \min_{i \neq j} \{U^{i}(x) + k(a^{j}, a^{i})\}, x \in \mathbb{R}^{n}, j = 1, ..., r;$ (iv) there is an $i_{0} \in \{1, ..., r\}$ such that $U^{i_{0}}(0) = 0$ and $U^{i_{0}}(x) > 0$ for $x \neq 0$. (v) Σ_{sw} is zero-state detectable for all initial control settings $a^{j} \in A$. Then the strategy α_{U} associated with U as in (3.42) or (3.44) is such that $\alpha_{U}^{i_{0}}[0](s) = a^{i_{0}}$ for all s and 0 is a globally asymptotically stable equilibrium point for the system $\dot{y} = f(y, a^{i}, 0)$. Moreover, 0 is a globally asymptotically stable equilibrium point for the system Σ_{sw} , in the sense that the solution $y(t) = y_{x}^{j}(t, \alpha_{U,x}^{j}, 0)$ of

$$\dot{y} = f(y, \alpha_{U,x}^{j}[0], 0), \quad y(0) = x$$

has the property that

$$\lim_{t \to \infty} y_x^j(t, \alpha_{U,x}^j[0], 0) = 0$$

for all $x \in \mathbb{R}^n$ and all $a^j \in A$.

Proof Suppose that $U^{i_0}(0) = 0$ and $U^{i_0}(x) > 0$ for $x \neq 0$. Let $T \geq 0$ and $x \in \mathbb{R}^n$. Since U is a storage function for the closed-loop system formed from Σ_{sw} with $\alpha = \alpha_U$, we have

$$U^{i_0}(x) \ge \int_0^T h(y_x(s), \alpha_x^{i_0}[0](s), 0)) \, ds + \sum_{\tau < T} k(\alpha_{U,x}^{i_0}(\tau^-), \alpha_{U,x}^{i_0}(\tau)) + U^{j(T)}(y_x(T, \alpha_x^{i_0}[0], 0).$$
(3.86)

Since h, k, U are nonnegative and $U^{i_0}(0) = 0$ by our assumptions, we have

$$\sum_{\tau < T} k\left(\alpha_{U,x}^{i_0}[0](\tau^-), \alpha_{U,x}^{i_0}[0](\tau) \right) = 0.$$

This implies that $\alpha_{U,0}^{i_0}[0](t) = a^{i_0}$ for all $0 \le t \le T$. Thus

$$0 \le U^{j(T)}(y_x(T,\alpha_0^{i_0}[0],0) = U^{i_0}(y_x(T,\alpha_0^{i_0}[0],0) \le U^{i_0}(0) = 0.$$

By the positive definite property of U^{i_0} , we have $y_0(T, \alpha_{U,0}^{i_0}[0], 0) = 0$. Since $T \ge 0$ is arbitrary, we conclude that 0 is a equilibrium point of the system $\dot{y} = f(y, a^{i_0}, 0)$.

Next we want to show that 0 is a globally asymptotically stable equilibrium point for the closed-loop switching system Σ_{sw} with $\alpha = \alpha_U$. Again, from the storage-function property of $U = (U^1, \ldots, U^r)$ for the system Σ_{sw} with $\alpha = \alpha_U$, we have

$$\int_0^T h(y_x(s), \alpha_x^j[0](s), 0) \, ds \le U^j(x) < \infty \text{ for all } T > 0.$$

Thus $\lim_{t\to\infty} h(y_x(s, \alpha_{U,x}^j[0], 0) = 0$ by Lemma 2.19. By the detectability assumption (iv), we have $\lim_{t\to\infty} y_x(t, \alpha_{U,x}^j[0], 0) = 0$ as required. \diamondsuit

3.5 Computational issues

The results of Section 3.3 reduce the solution of the robust optimal switching-control problem to the solution of a SQVI. For these results to be useful, of course, one must be able to compute solutions of such an equation, or more precisely for our situation, the minimal viscosity supersolution of such a system of equations. In this section we make a few general observations concerning these issues and give an explicit, direct solution for a simple example with one-dimensional state space.

3.5.1 A connection between solutions of SQVIs and VIs

Suppose that $U = (U^1, \ldots, U^r)$, where $U^1 \in C(\mathbb{R}^n)$, is the minimal viscosity supersolution of SQVI

$$\max\{H^{j}(x, DU^{j}(x)), U^{j}(x) - \min_{i \neq j}\{U^{i}(x) + k(a^{j}, a^{i})\}\} = 0, \ j = 1, \dots, r.$$

Then each U^j can be interpreted as the minimal viscosity supersolution of VI with the Hamiltonian H^j and the stopping cost $\Phi^j = \min_{i \neq j} \{U^i + k(a^j, a^i)\}$.

Given an r-tuple $U = (U^1, \ldots, U^r)$ of nonnegative real-valued functions, define a new r-tuple $F(U) = (F(U)^1, \ldots, F(U)^r)$ of nonnegative real-valued functions by

 $F(U)^j =$ the minimal viscosity supersolution of VI with $H = H^j$ and $\Phi = \Phi^j$.

Note that U is the minimal viscosity supersolution of SQVI if and only if F(U) = U, i.e., if and only if U is a fixed point of F. Formally, one can solve the fixed point problem by guessing a starting point $U_0 = (U_0^1, \ldots, U_0^r)$ and then iterating

$$U_{n+1} = F(U_n), \ n = 0, 1, 2, \dots$$

If $U_n \to U_\infty$ and F is continuous, then from $U_{n+1} = F(U_n)$ one can take the limit to get $U_\infty = F(U_\infty)$ from which we see that U_∞ is a fixed point for F. For finite horizon problems, or problems with a positive discount factor in the running cost, the connection is a little cleaner, as in this situation one has a uniqueness theorem for solutions of the relevant SQVI.

3.5.2 Optimal switching-control problem with one-dimensional state space

In this subsection we consider an optimal switching cost problem with one-dimensional state space. While in principle it should be possible to solve the problem by using the construction in Section 2.5 to perform each iterative step in the procedure outlined in Section 3.5.1, it turns that one can solve explicitly by a direct, geometric, noniterative procedure which we now describe.

We simplify the general problem to the case where there are only two controls $A = \{a^1, a^2\}$, with respective system dynamics given by

$$f(y, a^1, b) = -y + b;$$
 $f(y, a^2, b) = -\mu(y - 1) + b.$

(A value for the parameter $\mu > 1$ will be specified below.) We take the output to be simply the squared state

$$h(y, a, b) = y^2$$

and the switching cost to be given by a parameter $\beta > 0$:

$$k(a^1, a^2) = k(a^2, a^1) = \beta; \quad k(a^1, a^1) = k(a^2, a^2) = 0.$$

All the hypotheses (A8)-(A13) are satisfied. All other assumptions are satisfied with the exception that $B = \mathbb{R}$ is not compact; to alleviate this difficulty, one can restrict B to a large finite interval [-M, M]; to live with this restriction, one may have to adjust the definition of the hamiltonian functions $H^1(x, p)$ and $H^2(x, p)$ in the discussion to follow. We will construct a solution to the (SQVI) for this example via a variation of the algorithm presented in Section 2.5; rather than proving that the solution so constructed is the minimal nonnegative supersolution of SQVI, we verify directly that it is the lower value function $V_{\gamma} = (V_{\gamma}^1, V_{\gamma}^2)$ of the switching-control differential game (3.1).

Because we will take $\mu > 1$, for large |y| the control a = 2 will drive the state toward 0 more strongly than a = 1. However the origin is stable only if a = 1 used when |y| is small. Thus we would expect an optimal strategy to switch to a = 2 for y away from the origin, but then back again to a = 1 near the origin. The details of this will be determined by our solution (V^1, V^2) to (SQVI).

The two Hamiltonian functions work out to be

$$H^{1}(x,p) = px - x^{2} - \frac{1}{4\gamma^{2}}p^{2}$$
$$H^{2}(x,p) = \mu p(1-x) - x^{2} - \frac{1}{4\gamma^{2}}p^{2}.$$

These are both instances of the general formula

$$H(x,p) = \inf_{b} \{-(g(x) + b) \cdot p - x^{2} + \gamma^{2}b^{2}\}$$

= $-pg(x) - x^{2} - \frac{1}{4\gamma^{2}}p^{2}$
= $(\gamma g(x))^{2} - x^{2} - (\frac{1}{2\gamma}p + \gamma g(x))^{2}$

where g(x) = -x for a = 1 and $g(x) = -\mu(x-1)$ for a = 2. Provided $|x| < \gamma |g(x)|$ the equation H(x, p) = 0 has two distinct real solutions:

$$p_{\pm}(x) = -2\gamma^2 g(x) \pm 2\gamma \sqrt{\gamma^2 g(x)^2 - x^2}.$$

We will use $p_{\pm}^{a}(x)$ (a = 1, 2) to refer to these specifically for our two choices of g(x). Observe that $H(x, p) \leq 0$ if and only if $p \leq p_{-}(x)$, $p \geq p_{+}(x)$, or $|x| > \gamma |g(x)|$. This will be important for working with (3.93) below. Note also that the infimum in (3.87) is achieved for $b^* = \frac{1}{2\gamma^2}p$. When $p = p_{\pm}(x)$ in particular we have

$$f(x, a, b^*) = g(x) + \frac{1}{2\gamma^2} p_{\pm}(x)$$

= $\pm \frac{1}{\gamma} \sqrt{\gamma^2 g(x)^2 - x^2},$ (3.87)

which will be positive [negative] in the case of p_+ [p_-] respectively. Moreover, since $H(x, p_{\pm}(x)) = 0$, we will have

$$(g(x) + b^*) \cdot p_{\pm}(x) + x^2 = \gamma^2 (b^*)^2.$$

These observations will be important in confirming the optimality of our switching policy below. The expressions for $p_{\pm}^{1}(x)$ have a simple composite expression: with

$$\rho = \gamma^2 - \gamma \sqrt{\gamma^2 - 1}$$

we have

$$2\rho x = \begin{array}{cc} p_{-}^{1}(x) & \text{if } x \ge 0\\ p_{+}^{1}(x) & \text{if } x \le 0. \end{array}$$
(3.88)

We now exhibit the desired solution of the (SQVI) for the following specific parameter values:

$$\mu = 3, \quad \beta = .4, \quad \gamma = 2.$$
 (3.89)

Let

$$W^{1}(x) = \rho x^{2}$$

$$W^{2}_{-}(x) = \int p^{2}_{-}(x) dx, \text{ for } x \ge 1.2$$

$$W^{2}_{+}(x) = \int p^{2}_{+}(x) dx, \text{ for } x \le \frac{6}{7}.$$

(One may check that for our parameter values $p_{\pm}^2(x)$ is undefined for $\frac{6}{7} < x < 1.2$.) Using values $x_2 \approx -1.31775$, $x_1 = 3/2$, $x_3 \approx 2.55389$ we can present the lower value function(s) for our game:

$$W_{+}^{2}(x) + C_{0} \quad \text{for } x < 0$$

$$V^{2}(x) = \beta + W^{1}(x) \quad \text{for } 0 \le x \le x_{1}$$

$$W_{-}^{2}(x) + C_{1} \quad \text{for } x_{1} < x,$$
(3.90)

where the constants C_0, C_1 are chosen to make V^2 continuous, and

$$V^{1}(x) = \begin{cases} \beta + V^{2}(x) & \text{for } x \leq x_{2} \\ W^{1}(x) & \text{for } x_{2} < x < x_{3} \\ \beta + V^{2}(x) & \text{for } x_{3} \leq x. \end{cases}$$
(3.91)



Figure 3.1: V^1 (solid) and V^2 (dashed)

Graphs are presented in Figure (3.1). Our arguments below depend on a number of inequalities involving DV^a as defined by (3.91), (3.90). For brevity, we will verify several of them graphically rather than algebraically.

The procedure for constructing (3.91), (3.90), and the significance of the particular values x_1, x_2, x_3 , will become apparent as we now work through the verification of (SQVI). Observe that (SQVI) is equivalent to the following three conditions for each of $a \in \{1, 2\}$. (Here a' will generically denote the other value of a: a' = 3 - a.)

$$V^{a}(x) \leq \beta + V^{a'}(x), \text{ for all } x, \qquad (3.92)$$

$$H^{a}(x, D^{+}V^{a}(x)) \leq 0, \text{ for all } x,$$
 (3.93)

$$H^{a}(x, D^{-}V^{a}(x)) \geq 0$$
, for those x with $V^{a}(x) < \beta + V^{a'}(x)$, (3.94)

At points x where both V^1 and V^2 are smooth, these conditions can be expressed more explicitly as: Necessarily $|V^1(x) - V^2(x)| \leq \beta$.

1. If $V^1(x) - V^2(x) = \beta$, then $(V^1)'(x) = (V^2)'(x) =: q(x)$ (since $V^1 - V^2$ has a maximum at x), and $H^1((x) - V^2(x)) \le 0$, $H^2(x) = 0$.

$$H^{2}(x, q(x)) \leq 0, \quad H^{2}(x), q(x)) = 0.$$

2. If $V^1(x) - V^2(x) = -\beta$, then similarly, $(V^1)'(x) = (V^2)'(x) =: q(x)$ and

$$H^{1}(x,q(x)) = 0, \quad H^{2}(x,q(x)) \le 0.$$

3. If $|V^1(x) - V^2(x)| < \beta$, then both

$$H^{1}(x, (V^{1})'(x)) = 0, \quad H^{2}(x, (V^{2})'(x)) = 0.$$

There are a number of other cases, depending on whether x is a smooth point for one or both of V^1 and V^2 and on the relative sizes of the one-sided derivatives of V^a at x is x is a nonsmooth point for V^a . We will work these conditions out as they are needed.

We begin the construction by starting at the point 0. For x close to 0, from the structure of the dynamics and the cost we can see directly that it is optimal to use a^1 . Thus $V^1(x)$ for x close to zero necessarily is equal to the available storage function for the system with disturbance and no control associated with the fixed control $a = a^1$; this available storage function works out to be $W^1(x)$. Similarly, if one starts with the control a^2 and if x > 0, it is optimal to switch immediately to control a^1 and let the dynamics associated with a^1 drive the state to the origin; hence for x > 0 and close to 0, we expect to have $V^2(x) =$ $\beta + W^1(x) = \beta + V^1(x)$. On the other hand, if we start with control a^2 and initial state x < 0and small in magnitude, we do better to use a^2 to drive us to the origin and then switch to a^1 to keep us at the origin; this leads us to conclude that, for small x with x < 0, $V^2(x)$ is the minimal solution of $H^2(x, (V^2)'(x)) = 0$ with initialization $V^2(0) = \beta$. By such direct qualitative reasoning we deduce that the form of $(V^1(x), V^2(x))$ for x in a neighborhood of the origin 0 is as asserted.

For x close to the origin and positive, we are in case (3): we need to check $H^1(x, q(x)) = 0$ while $H^2(x, q(x)) \leq 0$, where the q(x) is the common value of $DV^1(x)$ and $DV^2(x)$, or $p_-^1(x)$. The first equation holds trivially while the second holds as a consequence of $p_-^1(x) < p_-^2(x)$ for $0 \leq x < \delta$ for some $\delta > 0$.

Calculation shows that the first δ for which this latter equality fails is $\delta = x_1 = 3/2$, where $p_-^1(x)$ and $p_-^2(x)$ cross. At this stage, we arrange that $V^2(x)$ follow $p_-^2(x)$ instead of $p_-^1(x)$ (as its derivative) while $V^1(x)$ continues to follow $p_-^1(x)$. Note that the continuation of $V^2(x)$ defined in this way is smooth through x_1 . In this way we have arranged that both Hamilton-Jacobi equations are satisfied $(H^a(x, (V^a)'(x)) = 0 \text{ for } a = 1, 2)$. The only catch is to guarantee that we maintain $|V^1(x) - V^2(x)| \leq \beta$. This condition holds for an interval to the right of x_1 since we have $V^1(x_1) - V^2(x_1) = -\beta$ while $(V^1)'(x) - (V^2)'(x) = p_-^1(x) - p_-^2(x) \geq 0$.

Calculation shows that the first point to the right of x_1 at which $|V^1(x) - V^2(x)| < \beta$ fails is the point x_3 where $V^1(x) - V^2(x) = \beta$; if we continue with the same definitions of $V^1(x)$ and $V^2(x)$ to the right of x_3 , we get $V^1(x) - V^2(x) > \beta$ for x to the immediate right of x_3 . To fix this problem, to the immediate right of x_3 we arrange that $V^2(x)$ still follow $p_-^2(x)$ but now set $V^1(x) = V^2(x) + \beta$. Then points to the immediate right of x_3 are smooth for both V^1 and V^2 and the applicable case for the check of a viscosity solution at such points is case (1). Trivially we still have $H^2(x, (V^2)'(x)) = H^2(x, p_-^2(x)) = 0$ while $H^1(x, (V^1)'(x)) = H^1(x, p_-^2(x)) \leq 0$ since necessarily $V^1(x) - V^2(x)$ is increasing at x_3 from which we get $p_-^1(x) - p_-^2(x) > 0$ on an interval containing x_3 in its interior. At the point x_3 itself, we have $D^+V^2(x_3) = \{p_-^2(x)\} = D^-V^2(x_3)$ while $D^-V^1(x_3) = \emptyset$ and $D^+V^1(x_3) = [p_-^1(x_3), p_-^2(x_3)]$. To check that (V^1, V^2) is a viscosity solution of SQVI at x_3 one simply checks that (i) $H^2(x_3, (V^2)'(x_3)) = H^2(x_3, p_-^2(x_3)) = 0$ and (ii) $H^1(x, p) \leq 0$ for all $p \in [p_-^2(x_3), p_-^1(x_3)]$. The discussion for x < 0 is quite similar to the above. To the immediate left of 0, $V^1(x)$ follows $p_+^1(x)$ rather than $p_-^1(x)$, while $V^2(x)$ follows $p_-^2(x)$. Thus 0 is a smooth point for $V^2(x)$. For points x to the immediate left of 0, we have $H^a(x, (V^a)'(x)) = 0$ for a = 1, 2, so the only remaining issue for (V^1, V^2) to be a viscosity solution at such points is the inequality $|V^1(x) - V^2(x)| \leq \beta$. To verify this, note that $V^1(0) - V^2(0) = -\beta$ and $(V^1)'(x) - (V^2)'(x) = p_+^1(x) - p_+^2(x) < 0$ on an interval $-\delta < x < 0$. These definitions of $V^1(x)$ and $V^2(x)$ work well as x moves to the left away from the origin until we reach the point x_2 where $V^1(x) - V^2(x) = \beta$ and continuation of these definitions for x to the left of x_2 would lead to the unacceptable inequality $V^1(x) - V^2(x) > \beta$. For x to the left of x_2 we tet $V^2(x)$ continue to follow $p_+^2(x)$ while we set $V^1(x) = V^2(x) + \beta$. To the left of x_2 we then have $H^2(x, (V^2)'(x)) = H^2(x, p_+^2(x)) = 0$ while $H^1(x, (V^1)'(x)) = H^1(x, p_+^2(x)) \leq 0$ since we still have $p_+^2(x) > p_+^1(x)$ for x < 0; this verifies that (V^1, V^2) is a viscosity solution of SQVI for $x < x_3$. At $x = x_3$, one checks the viscosity solution conditions by noting that $H^2(x_3, (V^2)'(x_2)) = H^2(x_2, p_+^2(x_2)) = 0$ and $H^1(x_2, p) \leq 0$ for all $p \in D^+V^1(x_2) = [p_+^1(x_2), p_+^2(x_2)]$.

It should be possible to verify that any deviation from this construction which maintains the property that (V^1, V^2) is a viscosity supersolution leads to a larger (V^1, V^2) ; Theorem 3.21 (apart from the technical gap that we have searched only through all piecewise C^1 viscosity supersolutions rather than through all lower semicontinuous viscosity supersolutions—presumably for this simple case, all viscosity supersolutions are in fact piecewise smooth) then implies that (V^1, V^2) constructed as above is the lower-value function for this switching-control game. We shall now give an alternative direct argument that (V^1, V^2) is indeed the lower value function.

The strategy α^* associated with our solution (3.90), (3.91) is easy to describe in state-feedback terms. Define the *switching sets*

$$S_1 = \{x : V^2(x) = \beta + V^1(x)\} = [0, x_1],$$

$$S_2 = \{x : V^1(x) = \beta + V^2(x)\} = (-\infty, x_2] \cup [x_3, \infty).$$

The strategy α^* will instantly switch from a = 1 to a = 2 whenever $y(t) \in S_2$, and instantly switch from a = 2 to a = 1 whenever $y(t) \in S_1$. Otherwise α^* continues using the current control state. Theorem 3.21 would imply that $V_{\gamma}^a \leq V^a$, where V_{γ}^a are the lower values. We will prove directly that in fact $V_{\gamma}^a = V^a$, and that our strategy α^* is optimal. To be precise, we shall show that for any j and any strategy $\alpha \in \Gamma^j$

$$V^{j}(y(0)) \leq \sup_{b \in \mathcal{B}} \sup_{T>0} \left\{ \int_{0}^{T} [h(y_{x}(s), \alpha_{x}^{j}[b](s), b(s)) - \gamma^{2} |b(s)|^{2}] \, ds + \sum_{\tau_{i} \leq T} k(a_{i-i}, a_{i}) \right\}.$$
 (3.95)

Moreover, for our strategy α^* , (3.95) will be an equality for all x, j. The key to this is the existence of a particular "worst case" disturbance, as described in the following proposition.

Proposition 3.24 For any $x \in \mathbb{R}^n$, $j \in \{1, 2\}$ and strategy $\alpha \in \Gamma$, there exists a disturbance $b^* = b^*_{\alpha^j_x} \in \mathcal{B}$ with the property that

$$b^{*}(t) = \frac{1}{2\gamma^{2}} (V^{\alpha[b^{*}])(t)})'(y(t)),$$

holds for all but finitely many t in every interval [0, T].

We emphasize that this proposition is only intended in the context of the particular example and parameter values described above.

Proof Suppose $j, \alpha \in \Gamma^{j}$ and an initial point $x \in \mathbb{R}^{n}$ are given. Begin by considering the solution of

$$\dot{y} = f(y, a^j, \frac{1}{\gamma^2} (V^j)'(y)); \quad y(0) = x.$$
 (3.96)

For j = 2 the right side is C^1 , so the solution is uniquely determined. For j = 1, the right side has discontinuities at x_2 and x_3 , but since $f(x, a^j, \frac{1}{\gamma^2}(V^1)'(x))$ does not change sign across the discontinuities, the solution is again uniquely determined. Graphs of $f(y, a^j, \frac{1}{\gamma^2}(V^j)'(y))$ are provided in Figures 3.2 and 3.3 below. (We comment that although the graphs appear piecewise linear, they are not. Figure 3.2 is linear only for $0 < x < x_1$ and Figure 3.3 is only linear for $x_2 < x < x_3$, as inspection of the formulas shows.) Since $y\dot{y} < 0$ for sufficiently large |y|, it is clear that the solution of (3.96) is defined for all $t \ge 0$. Observe also for j = 1that, for any solution of (3.96), there is at most one value of t for which y(t) is at one of the discontinuities of $(V^1)'$. Thus $(V^j)'(y(t))$ is undefined for at most a single t value.



Figure 3.2: Plot of $f(x, 2, \frac{1}{2\gamma^2}DV^2(x))$.



Figure 3.3: Plot of $f(x, 1, \frac{1}{2\gamma^2}DV^1(x))$.

Now consider the disturbance $b(t) = \frac{1}{\gamma^2} (V^j)'(y(t))$. The control $\alpha_x[b](t)$ produced for this disturbance will only take the value j on the initial interval: $0 = \tau_0 \le t \le \tau_1$. We define $b^*(t) = b(t) = \frac{1}{\gamma^2} (V^j)'(y(t))$ for these t. At $t = \tau_1$ the control $\alpha_x[b]$ will switch from j to j'. We therefore redefine y(t) for $t > \tau_1$ as the solution of

$$\dot{y} = f(y, j, \frac{1}{\gamma^2} (V^{j'})'(y))$$

with initial value $y(\tau_1)$ as already determined. Likewise, redefine $b(t) = \frac{1}{\gamma^2} (V^{j'})'(y(t))$ for $t > \tau_1$. Because we have not changed b on $[0, \tau_1]$, the nonanticipating property of strategies insures that $\alpha_x[b](t)$ for $t \le \tau_1$ and τ_1 remain the same for this revised b. Using the new b, the control $\alpha_x[b](t)$ determines the next switching time τ_2 . We know that $\tau_1 < \tau_2 \le \infty$ and $\alpha_x[b](t) = j'$ for $\tau_1 < t \le \tau_2$. We now extend our definition of b^* with $b^*(t) = b(t)$ for $\tau_1 < t \le \tau_2$. At τ_2 the control switches again, back to j. So we now redefine y(t) and b(t) for $t > \tau_2$ by taking $y(\tau_1)$ as already determined, solving

$$\dot{y} = f(y, j, \frac{1}{\gamma^2} (V^j)'(y))$$

and redefining $b(t) = \frac{1}{\gamma^2} (V^j)'(y(t))$ for $t > \tau_2$. For $t \le \tau_2$ the values of b(t), $\alpha_x[b](t)$, and y(t) remain unchanged, again by the nonanticipating hypothesis. We now identify the switching time τ_3 associated with $\alpha_x[b](t)$, and extend our definition for $\tau_2 < t \le \tau_3$ using $b^*(t) = b(t)$. At τ_3 the control will switch again to j', so continue our redefinition process again for $t > \tau_3$.

Continuing this redefinition and extension process, we produce the desired disturbance $b^*(t)$ and state trajectory y(t) associated with the control $\alpha_x[b^*](t)$ satisfying the requirements of the proposition. The only conceivable failure of this construction would be if the switching times τ_i that are generated in the construction were to have a finite limit: $\lim \tau_i = s < \infty$. Our hypotheses on the strategy α disallow this however, for the following reason. If it were the case that $\lim \tau_i = s < \infty$, then extend our definition of b^* in any way to $t \ge s$, say $b^*(t) = 0$. By hypothesis, $\alpha[b^*]$ is an admissible control in \mathcal{A}^j , which means in particular that its switching times τ_i no not have a finite accumulation point. But extension of b^* for t > s does not alter the switching times $\tau_i < s$, by the nonanticipating property again. This would mean that $\alpha[b^*]$ does have an infinite number of $\tau_i < s$, which is a contradiction. Finally, by our comments above, on each interval $[\tau_i, \tau_{i+1}]$ there is at most a single t value at which $(V^{\alpha[b^*])'(t)}(t)$ is undefined. Thus there are at most a finite number of such t in any [0, T]. \diamond

Consider now any strategy $\alpha \in \Gamma^{j}$, x = y(0) and b^{*} be as in the proposition. On any time interval $[\tau_{i}, \tau_{i+1}]$ between consecutive switching times, (3.93) and the fact that $b^{*}(t)$ achieves the infimum in (3.87) for x = y(t) and $p = (V^{a_{i}})'(x)$ implies that (for all but finitely many t)

$$\frac{d}{dt}V^{a_i}(y(t)) \ge (\gamma b^*(t))^2 - h(y(t), a_i, b^*(t)).$$

Thus for any $\tau_i < t \leq \tau_{i+1}$ we have

$$V^{a_i}(y(t)) - V^{a_i}(y(\tau_i)) \ge \int_{\tau_i}^t \gamma^2 |b^*|^2 - h \, ds.$$

Across a switching time τ_i we have from (3.92)

$$V^{a_i} - V^{a_{i-1}} \ge -\beta = -k(a_{i-1}, a_i).$$

Adding these inequalities over $\tau_i \leq T$ we see that

$$V^{\alpha[b^*](T)}(y(T)) - V^{\alpha[b^*](0)}(y(0)) \ge -\left\{\int_0^T [h - \gamma^2 |b^*|^2] \, ds + \sum_{\tau_i \le T} k(a_{i-1}, a_i)\right\}.$$

Rearranging, this says that

$$V^{\alpha[b^*](T)}(y(T)) + \left\{ \int_0^T [h - \gamma^2 |b^*|^2] \, ds + \sum_{\tau_i \le T} k(a_{i-1}, a_i) \right\} \ge V^{\alpha[b^*](0)}(y(0)). \tag{3.97}$$

When we consider α^* specifically, we recognize that

$$H^{a_i}(y(t), (V^{a_i})'(y(t))) = 0$$

(where we set in general $H^{a^i} = H^i$ and $V^{a^i} = V^i$ for i = 1, 2) for t between the τ_i , and at τ_i

$$V^{a_{i+1}} - V^{a_i} = -\beta = -k(a_{i+1}, a_i).$$

This means that (3.97) is an equality for α^* specifically.

To finish our optimality argument we will show that for α in general above, as $T \to \infty$ we must have either $y(T) \to 0$ and $\alpha[b^*](T) \to 1$, or else

$$\int_{0}^{T} [h - \gamma^{2} |b^{*}|^{2}] ds + \sum_{\tau_{i} \leq T} k(a_{i-1}, a_{i}) \to +\infty.$$
(3.98)

In the case of α^* specifically, we will have the former possibility. Since $V^1(0) = 0$ and is continuous, these facts imply (3.95) as claimed. The verification of these asserted limiting properties depends on some particular inequalities for $(V^a)'(x)$ as determined by (3.91), (3.90). First, we assert that, for both a = 1 and a = 2,

$$h - \gamma^2 |b^*|^2 = |x|^2 - \frac{1}{4\gamma^2} [(V^a)'(x)]^2 > 0, \text{ for } x \neq 0.$$
 (3.99)

Moreover $|x|^2 - \frac{1}{4\gamma^2}[(V^a)'(x)^2)$ has a positive lower bound on $\{x : |x| \ge \epsilon\}$ for each $\epsilon > 0$. Instead of what would be a very tedious algebraic demonstration of this, we simply offer the graphical demonstration in Figure 3.4. For the parameter values (3.89) we have plotted $b^* = \frac{1}{2\gamma}(V^a)'(x)$ and q = x (dashed lines) as functions of x. The validity of (3.99) is apparent.



Figure 3.4: Graphical verification of (3.99) for DV^1 (left) and DV^2 (right)

The other fact we need is that for a = 2 and the corresponding disturbance $b^*(t)$, the statedynamics does not have an equilibrium at 0. This is easy to see, because at x = 0 we have $b^* = \frac{1}{2\gamma^2}(V^2)'(0) = 0$, but $f(0, a^2, b^*) = -\mu + b^*$. A graph of $f(x, a^2, b^*) = -\mu(x-1) + \frac{1}{2\gamma^2}(V^2)'(x)$ is provided in Figure 3.2, where we see a unique equilibrium just beyond x = 1.

In the case of a = 1 however, $\dot{x} = f(x, a^1, \frac{1}{2\gamma^2}(V^1)'(x))$ has a unique globally asymptotically stable equilibrium at x = 0, as is evident in Figure 3.3.

We turn then to the verification of the assertion of (3.98) or its alternative: assuming (3.98) to be false we claim that $y(T) \to 0$ and $\alpha[b^*](T) \to 1$. By the nonnegativity from (3.99) we must have both

$$\sum_{\tau_i < \infty} k(a_{i-1}, a_i) < \infty, \text{ and } \int_0^\infty [h - \gamma^2 |b^*|^2] \, dx < \infty.$$
(3.100)

The first of these implies that there are only a finite number of switches; $\alpha[b^*](t) = i^*$ is constant from some time on. It is not possible that $i^* = 2$ because in that case y(t) would be converging to the positive equilibrium of Figure 3.2, which implies by (3.99) that, as $t \to \infty$,

$$h(y(t), a_{i^*}, b^*(t)) - \gamma^* |b^*(t)|^2 \to C > 0.$$

This contradicts the second part of (3.100). Therefore $i^* = 1$, which shows that $\alpha[b^*](T) \to 1$. But since $\alpha[b^*](t) = 1$ from some point on, the stability illustrated in Figure 3.3 means that $y(t) \to 0$ as claimed. This completes our verification of the optimality of the strategy α^* .

Chapter 4

Conclusions and future work

This chapter gives conclusions of the dissertation and a view of future research.

4.1 Conclusions

We have developed the theories of robust stopping-time control and switching-cost control problems. We formulated the problems in both an L^2 -gain/dissipative framework and a game-theoretic framework.

Our main results concerning the robust stopping-time problems are as follows: under minimal smoothness and boundedness assumptions on the problem data

1. If the lower value function W (see (2.12)) for Game I is upper semicontinuous, then W is a viscosity subsolution in \mathbb{R}^n of the variational inequality (VI) given by

$$\max\{H(x, DV(x)), V(x) - \Phi(x)\} = 0, \ x \in \mathbb{R}^n,$$
(4.1)

where $H(x, p) = \inf_{b} \{-p \cdot f(x, b) - h(x, b) + \gamma |b|^2\}$. If W is lower semicontinuous, then W is a viscosity supersolution of the VI (4.1). Thus if W is continuous, W is a viscosity solution of the VI (4.1). In fact, if W is continuous, then W can be characterized as the minimal, nonnegative, continuous viscosity supersolution of the VI (4.1).

- 2. If continuous, the lower value function V (see (2.13)) for Game II is a viscosity solution of the VI. Moreover in certain cases V is characterized as the maximal viscosity subsolution of the VI (4.1).
- 3. Any locally bounded stopping-time storage function for some stopping-time strategy τ is a viscosity supersolution of the VI (4.1) (for the definition of the stopping-time storage function see (2.15)); conversely, if U is any nonnegative continuous viscosity

supersolution of the VI (4.1), then U is a stopping-time storage function with stoppingtime rule of state-feedback form given by $\tau_{U,x}[b] = \inf\{t \ge 0 : U(y_x(t,b) \ge \Phi(y_x(t,b))\},$ and $U \ge W$.

It also happens that a positive definite supersolution U of the VI can be used to prove stability of the equilibrium point 0 for the system with zero disturbance $\dot{y} = f(y, 0)$. We also obtain the lower-value function W(x) explicitly for a prototype problem with one-dimensional state space by a simple, direct, geometric construction.

Our main results concerning the robust optimal switching-cost problem are as follows: under minimal smoothness assumptions on the problem data and compactness of the set B,

- (i) $V_{\gamma}^{j}(x) \leq \min_{i \neq j} \{V_{\gamma}^{i}(x) + k(a^{j}, a^{i})\}, x \in \mathbb{R}^{n}, j = 1, \dots r$ (for the definition of $V_{\gamma} = (V_{\gamma}^{1}, \dots, V_{\gamma}^{r}),$ see (3.11)).
- (ii) If continuous, V_{γ} is a viscosity solution in \mathbb{R}^n of the system of quasivariational inequalities (SQVI)

$$\max\{H^{j}(x, DV_{\gamma}^{j}(x)), V^{j}(x) - \min_{i \neq j}\{V_{\gamma}^{i}(x) + k(a^{j}, a^{i})\}\} = 0, \ x \in \mathbb{R}^{n}, \ j = 1, \dots, r,$$

$$(4.2)$$

where
$$H^{j}(x,p) = \min_{b} \{ -p \cdot f(x,a^{j},b) - h(x,a^{j},b) + \gamma^{2}|b|^{2} \}.$$

- (iii) If $S = (S^1, \ldots, S^r)$ is a continuous switching storage function for some strategy α (for the definition of the switching storage function see (3.12)), then S is a nonnegative viscosity supersolution of the SQVI (4.2).
- (iv) If $U_{\gamma} = (U_{\gamma}^1, \ldots, U_{\gamma}^r)$ is a nonnegative, continuous viscosity supersolution (4.2) and U_{γ} has the property (i), then there is a canonical choice of switching state-feedback control strategy $\alpha_{U_{\gamma}} : (x, a^j, b) \to \alpha_{U_{\gamma},x}^j[b]$ such that U_{γ} is a switching storage function for the closed-loop system formed by using the strategy $\alpha_{U_{\gamma}}$; thus,

$$U_{\gamma}^{j}(x) \geq \sup_{b, T} \{ \int_{[0,T)} l(y_{x}(s), a^{j}, \alpha_{U_{\gamma}, x}^{j}[b](s), b(s)) \} \geq V_{\gamma}^{j}(x).$$

and the lower-value (vector) function V_{γ} , if continuous, is characterized as the minimal, nonnegative, continuous viscosity supersolution of (4.2) having property (i) above, as well as the minimal continuous function satisfying property (i) which is a switching storage function for the closed-loop system associated with some nonanticipating strategy $\alpha_{V_{\gamma}}$.

We gave two derivation of this characterization of V_{γ} . The first method is a direct argument which parallels the argument for optimal stopping-time problems and the second method relies on a general comparison principle for viscosity super- and subsolutions of SQVI which is proved in Section 3.3.2. The usual formulation of the H_{∞} -control problem also involves a stability constraint. We also prove that, under appropriate conditions, the closed loop system associated with switching strategy α_U corresponding to the supersolution U of the SQVI is stable. The main idea is to use the supersolution U as a Lyapunov function for trajectories of the closed-loop system.

4.2 Future work

Studying robust stopping-time control problems and robust switching-control problems has opened the door to further new research.

Methods for solving the VI/SQVI

We have seen that a robust stopping-time control problem (respectively, a robust switchingcontrol problem) is solvable if we can find a solution of the variational inequality (VI) (respectively, the system of quasivariational inequalities (SQVI)). To make these theories useful, the methods for solving the VI/SQVI should be developed. The VI/SQVI is the generalization of the Hamilton-Jacobi-Bellman-Isaacs equation(HJBIE) in this setting, so we hope that the already existing methods for solving the HJBIE should be adapted to solve the VI/SQVI. There are two possible ways to solve the VI/SQVI: (i) via adaptation of the method of bicharacteristics; or (ii) via solving an approximating discrete-time control problem, or more generally an approximating discrete-time game problem. We now discuss the ideas we have in mind for these two methods.

We have seen that the first part of the VI/SQVI is the Hamilton-Jacobi-Bellman equation (HJBE) which can be solved by the method of bicharacteristics via the connection with the stable invariant manifold for the associated Hamiltonian flow (see [45, Chapter 7]). Thus it is possible to apply this method to solve the VI/SQVI. We did apply this method for solving the VI/SQVI of a simple one-dimensional example. The examples in Section 2.5 and 3.5.2 can be seen as adaptations of this method for simple cases with one-dimensional state space. We also tried to apply this method for the simple two-dimensional example of the switching-problem. The author together with Professor Day wrote MATLAB/MATHEMATICA programs to simulate the problem, but the result was inconclusive. We hope to do more analysis for adaption of the method of bicharacteristics to solve the VI/SQVI.

Alternatively, we might compute the lower value function for a discrete-time problem which approximates the continuous-time problem: by introducing the discrete-time system of stopping-time/switching-control problem, defining the discrete lower value function, applying dynamical programming method to derive the discrete version of the VI/SQVI, and finally showing the convergence of the discrete lower value function to the continuous one.

For the stopping-time problem, the discrete-time system we have in mind is in the form: for a fixed $\delta > 0$

$$\Sigma_{st,\delta} \begin{cases} y(k+1) = y(k) + \delta f(y(k), b_k), \ y(0) = x \\ z(k) = h(y(k), b_k). \end{cases}$$

We denote by B^k the set of sequences taking values in B and $\sum_{k=0}^{N} |b_k|^2 < \infty$ for all N. For given sequence $\beta = \{b_k\} \in B^k$ and initial condition $y(0) = x \in \mathbb{R}^n$, the trajectory of the system $\Sigma_{st,\delta}$ will be determined and we denote it by $y_k = y_x(k,\beta)$. We define the cost functional to be in the form

$$J_{\delta}(x,\beta,\tau,T) = \sum_{k=0}^{(\tau \wedge T)-1} \delta(h(y_k,b_k) - \gamma |b_k|^2) + \mathbb{1}_{[0,T)}(\tau) \Phi(y_{\tau}),$$

where summation over the empty set is equal to zero. The discrete lower value function W_{δ} is defined as

$$W_{\delta}(x) = \inf_{\tau} \sup_{\beta, T} J_{\delta}(x, \beta, \tau[\beta], T),$$

where the supremum is over all nonnegative natural numbers T and $\beta \in B^k$, while the infimum is over all nonanticipating stopping-time strategies τ with values in $\mathbb{N} \bigcup \{\infty\}$. The following inequalities about W_{δ} are obvious. Setting T = 0 ($\Phi > 0$) implies that

$$W_{\delta}(x) \ge 0$$

while setting $\tau \equiv 0$ implies

$$W_{\delta}(x) \leq \Phi(x).$$

By applying the dynamical programming method, it is possible to show that

$$W_{\delta}(x) = \min\left\{\sup_{b} \left\{W_{\delta}(x+\delta f(x,b)) + \delta(h(x,b)-\gamma^2|b|^2\right\}, \Phi(x)\right\}.$$

Then this equation is the discrete version of the VI. It should be possible to show that $W_{\delta}(x)$ converses to W(x) (given by (2.12)) as δ goes to 0. The discrete approximation of the stopping-time problem was discussed in [33] and [50], but in the case of the differential games which both players stop the system in order to minimize or maximize the cost functional.

It is possible that one can use this discrete approximation of the stopping-time to solve the switching-control problem since there is an interesting connection between the solutions of the VI and SQVI (see Section 3.5.1). The solution of an SQVI is a fixed point of a map which assigns to a given vector function the collection of solutions of a decoupled system of VIs, or, at the level of value functions, the lower value vector function for a switching-control problem is a fixed point of a map which assigns to an n-tuple of nonnegative-real valued functions the set of lower value functions for a decoupled collection of stopping-time problems (with different terminal cost functions determined by the input vector function). In principle, it should therefore be possible (at least in some cases) to find the lower value function for a

switching-control problem by iteratively solving for the value functions of a decoupled system of stopping-time problems, and thereby reduce solution of a switching-cost problem to the iterative solution of decoupled systems of VIs. This idea is discussed in [13] in the context of the stochastic, diffusion problems, and a similar remark giving a connection between the impulsive control problem and the stopping time problem is given in [10, Chapter III Section 4.3], where some convergence results are also given. Thus one can view stopping-time problems as having pedagogical value as stepping stones to the more complicated impulsivecontrol and switching-control problems. We wish to prove the convergence results in case of the VI/SQVI as in [10].

Problems with boundary projection dynamics

In many real-world applications, there is a restriction on the state-spaces. Thus if we want to apply theories developed in this dissertation, the theories with restricted state-space should be developed. We hope to develop the analogous theories to those developed in this dissertation for this case. We wish to formulate the robust optimal switching-cost control problem for a state-space system with projection dynamics in the state evolution. The projected dynamical system (PDS) was proposed in [24] and it was applied for the standard robust control problem of the restricted state-space system in [6] and [7], see also [1] for the connection to the viscosity notion.

We define the PDS associated with a dynamical system $\dot{y} = f(y, a, b)$ and a closed, convex set Ω as

$$\dot{y} = \pi_{\Omega}(y, f(y, a, b)), \quad y(0) = x \in \Omega, \tag{4.3}$$

where a map π has the following property: if y is in the interior of Ω , then $\pi_{\Omega}(y, f(y, a, b)) = f(x, a, b)$; if y is on the boundary of Ω and f(y, a, b) "point out of Ω ", then we use the projection of f(y, a, b) onto the tangent space of Ω (see Figure 4.1). This ensures that the trajectories of the PDS will always remain in the set Ω once the initial condition y(0) = x starts in Ω (see [6] and [24]).

We now assume that the admissible control set is $A = \{a^1, \ldots, a^r\}$. The general problem is to find the state-feedback control $y \to a(y)$ (or nonanticipating strategy $(x, a^j, b) \to \alpha_x^j[b]$) which guarantees the dissipative inequality

$$\int_0^T h(y_x(s), \alpha_x^j[b](s), b(s)) ds + \sum_{\tau} k(\alpha_x^j[b](\tau^-), \alpha_x^j[b](\tau)) \leq \gamma^2 \int_0^T |b(s)|^2 ds + U^j(x), \quad \alpha_x^j[b](0^-) = a^j$$

(where y_x now solves (4.3) with $a = \alpha[b]$) for all disturbances $b \in \mathcal{B}$, initial conditions $y(0) = x \in \Omega$ and initial control $a(0^-) = a^j \in A$. The related game formulation is the lower



Figure 4.1: Projected dynamics of the state space

value function $V_{\gamma} = (V_{\gamma}^1, \dots, V_{\gamma}^r)$ is defined by

$$V_{\gamma}^{j}(x) = \inf_{\alpha \in \Gamma} \sup_{b \in \mathcal{B}, \ T \ge 0} \int_{[0,T)} l(x, a^{j}, \alpha_{x}^{j}[b], b), \quad j = 1, \dots, r$$
(4.4)

where

$$l(y(t), a^{j}, a(t), b(t)) = [h(y(t), a(t), b(t)) - \gamma^{2} |b(t)|^{2}] dt + k(a(t^{-}), a(t))\delta_{t}, \qquad a(0^{-}) = a^{j},$$

and δ_t is the unit point-mass distribution at the point t. The problem is to work out the theory of Chapter 3 and the various approaches to computational implementation given in the methods for solving the VI/SQVI subsection for more complicated situation where discontinuities enter the system dynamics on the boundary.

As we mentioned in the introduction, our original motivation for the study of the robust switching-cost control problem came from the problem of designing a real-time feedback control for traffic signals at a highway intersection. This problem was discussed in [6] and [7], but the switching costs were not considered. Thus the control law produced the "chattering" phenomenon. We expect that the positive switching costs will prevent fast switching of the control.

The simple model for traffic flow at an isolated intersection of the two one-way streets of Figure 4.2 (modified from [6]) is defined as

$$\dot{y} = b + Sa$$
, $S = \begin{bmatrix} -s_1 & 0\\ 0 & -s_2 \end{bmatrix}$, $z = y$,

where $y = (y_1, y_2) \in \Omega = \{(y_1, y_2) \in \mathbb{R}^2 : y_i \ge 0, i = 1, 2\}, b = (b_1, b_2) \text{ and } a = (a_1, a_2) \in A$ where the set A of admissible controls is $\{(1, 0), (0, 1)\}$. We set $a^1 = (1, 0)$ and $a^2 = (0, 1)$.

The variables and parameters appearing in the model are as follows:



Figure 4.2: A simple two-way intersection

State variables:

 y_1 is the queue length of the traffic stream in approach X; y_2 is the queue length of the traffic stream in approach Y;

Exogenous inputs:

 b_1 is the arrival rate of the vehicles at approach X; b_2 is the arrival rate of the vehicles at approach Y;

Parameters:

 s_1 is the saturation flow rate of approach X (vehicles/lane/s); s_2 is the saturation flow rate of approach Y (vehicles/lane/s).

We allow b_i to be nonnegative, corresponding to departure of the vehicles from the two approaches. We assume both s_i to be strictly positive. The interpretation is that $a = (a_1, a_2) = (1, 0)$ corresponds to a green light for approach X (and red light for Y). Likewise $a = (a_1, a_2) = (0, 1)$ corresponds to a red light for X and a green light for Y.

We put the following assumption on the state equation: the difference between the arrival density and the flow served represents exactly the dynamic rate of the queue length at that approach, as long as the queue length to be served is positive. That is

$$\dot{y}_i = \delta(y_i, b_i - s_i a_i), \quad i = 1, 2,$$
(4.5)

where the function δ is given by

$$\delta(x, v) = \begin{cases} v & \text{if } x > 0 \text{ or if } x = 0 \text{ and } v > 0, \\ 0 & \text{otherwise.} \end{cases}$$

The effect of δ is to produce the projected dynamical system, as in (4.3). (Such models can be applied to more general queueing problems as well). The problem is then to design the statefeedback controller $y \to a(y)$ (or nonanticipating strategy $(x, a^j) \to \alpha_x^j[\cdot], x \in \Omega, a^j \in A$) which guarantees

$$\begin{split} &\int_0^T |y(s)|^2 ds + \sum_{\tau} k(\alpha_x^j[b](\tau^-), \alpha_x^j[b](\tau)) \\ &\leq \gamma^2 \int_0^T |b(s)|^2 ds + U^j(x), \quad \alpha_x^j[b](0^-) = a^j \end{split}$$

for all $T \ge 0$ and locally L_2 disturbance $b(\cdot)$ where $y(\cdot)$ solves (4.5) with an initial condition y(0) = x and control $a(\cdot) = \alpha_x^j[\cdot]$ with $a(0) = a^j$. This problem was solved in [6] (and more generally in [7] for an *n*-phase traffic signals) with the switching cost function $k \equiv 0$. The lower value function (or minimal storage function) there was shown to solve a Hamilton-Jacobi-Bellman equation with piecewise-defined nonsmooth Hamiltonian. A closely related problem with boundary dynamics but no switching costs was formulated in [1], where the value function is shown to satisfy a Hamilton-Jacobi-Bellman equation with smooth Hamiltonian but with certain additional boundary conditions taken in the viscosity sense.

Formulations using maximum principle

Pontraygin's maximum principle (MP) and Bellman's dynamic programming (DP) are the main tools in the optimal control theory. The MP, which involves the adjoint (or costate) vector and the Hamiltonian system, is the most classical and useful necessary condition of optimality. While the fundamental idea of the DP is that the value function, if smooth enough, satisfies the Hamilton-Jacobi-Bellman equation, it is well-known that there is a connection between the MP and the DP for the classical control theory. That is, if the value function is smooth, then its gradient is equal to the costate vector (e.g., [29] and also see [10] and [56] for the case of nonsmooth value function).

It was shown in [19] that the switching control system could be converted to be the impulsive control system. Deterministic impulsive control problem was discussed in [10, Section III.4.3] (see also [11]), where the authors applied the dynamic programming principle to show that the value function satisfies the *quasivariational inequality*. The maximum principle for the impulse control was developed by Blaquiere ([14], [15] and [16]), see also [42] for a different proof and [51] for a different setting. Our question is if there is any connection between the MP and the DP for impulsive control problem.

For the sake of completeness we include the minimum principle of the impulse control problem invented by Blaquiere here. We consider a dynamical system under the control of an agent J_0 who influences the evolution of the state $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$ in some planning period through his choice of a feedback control s in a prescribed control set S_0 . The evolution of state exhibits both sudden jumps and continuous changes.

We assume that the state lies in some open subset Ω of \mathbb{R}^n , and one of its components, say y_n , is the time t. Let $A \subseteq \mathbb{R}^{d_1}$ and $M \subseteq \mathbb{R}^{d_2}$ be prescribed nonempty, open sets of point a and α , respectively. Let K_a and K_α be prescribed nonempty subsets of A and M, respectively. Let P and Π be the sets of all functions defined on Ω with range in K_a and K_α respectively. Let Δ be the collection of all closed subsets of Ω , in the topology of \mathbb{R}^n .

We define the control set $S_0 = \Delta \times P \times \Pi$. In other words, J_0 will influence the evolution of the state through his choice of a closed subset of Ω , say Ψ , and a pair of functions defined on Ω , say $(p(\cdot), \pi(\cdot)) \in P \times \Pi$.

Let $l(\cdot) : \Omega \times A \to \mathbb{R}, \ k(\cdot) : \Omega \times M \to \mathbb{R}, \ f(\cdot) : \Omega \times A \to \mathbb{R}^n$ and $g(\cdot) : \Omega \times M \to \mathbb{R}^n$ be prescribed C^1 functions where

$$f = (f_1, \dots, f_n), \quad g = (g_1, \dots, g_n)$$

$$f_n(y, a) \equiv 1, \qquad g_n(y, \alpha) \equiv 0.$$

We say that a feedback control $s = (\Psi, p(\cdot), \pi(\cdot)) \in S_0$ is admissible if and only if

$$y \in \Psi \Rightarrow y + g(y, \pi(y)) \in \Omega - \Psi.$$

Let S be the set of all admissible feedback controls.

We say that a function $y(\cdot) : I = [0, \infty) \to \overline{\Omega}$ is a *path* generated by $s = (\Psi, p(\cdot), \pi(\cdot)) \in S$ from the initial condition $x \in \Omega$ if and only if

- (i) y(0) = x;
- (ii) $y(\cdot)$ is piecewise continuous on I; let T(I) denote the set of its discontinuity points;
- (iii) $y(t) = y(t^{-})$ for $t \in I, t \neq 0$;
- (iv) $t \in T(I) \Rightarrow y(t) \in \Psi$ and $y(t^+) = y(t) + g(y(t), \pi(y(t)));$
- (v) for all $t \in I T(I), y(t) \in \Omega \Psi$;
- (vi) $y(\cdot)$ is differentiable and y'(t) = f(y(t), p(y(t))) a.e. $t \in I$.

For each initial condition y(0) = x and $s = (\Psi, p(\cdot), \pi(\cdot)) \in S$ we define the cost functional

$$J(x,s) = \int_{I} f_0(y_x(s), p(y_x(s))ds + \sum_{s \in T(I)} g_0(y_x(s), \pi(y_x(s))),$$

and the value function

$$V(x) = \inf_{s \in S} J(x, s).$$
 (4.6)

Let $y^*(\cdot) : I \to \overline{\Omega}$ be a path generated by the optimal feedback control $s^* = (\Psi^*, p^*(\cdot), \pi^*(\cdot)))$ on the interval I and let $\lambda(\cdot) : I \to \mathbb{R}^{n+1}$ be a piecewise continuous function, with $\lambda(t) = \lambda(t^-)$ for $t \in (0, \infty)$. Let

$$H(\lambda, y, a) = \sum_{i=0}^{n-1} \lambda_i f_i(y, a),$$

$$\mathcal{H}(y, \alpha) = \sum_{i=0}^{n-1} \lambda_i(t_c^+) g_i(y, \alpha)$$

with

$$\lambda = (\lambda_0, \dots, \lambda_n), \quad t \in I, \ t_c \in T(I)$$

We say that $\lambda(\cdot)$ corresponds to s^* and $y^*(\cdot)$ if and only if on any subinterval $[t_i, t_{i+1}] \subset I$ on which $y^*(\cdot)$ is continuous, $\lambda(\cdot)$ is a solution of the set of differential equations

$$\dot{\lambda}_i = -\frac{\partial H}{\partial y_i}(\lambda, y, a), \quad y = y^*(t), \ a = p^*(y),$$

and, at any point of discontinuity of $x^*(\cdot)$, say t_c ,

$$\lambda_i(t_c) = \lambda_i(t_c^+) + \frac{\partial \mathcal{H}}{\partial y_i}(y, \alpha), \quad y = y^*(t_c), \ \alpha = \pi^*(y).$$

Theorem 4.1 If $y^*(\cdot) : I = [0, \infty) \to \overline{\Omega}$ is a path generated by the optimal feedback control $s^* = (\Psi^*, p^*(\cdot), \pi^*(\cdot))$, satisfying some assumptions (see [16]), then there exists a nonzero piecewise continuous vector function $\lambda(\cdot) : I \to \mathbb{R}^{n+1}$, corresponding to $s^* = (\Psi^*, p^*(\cdot), \pi^*(\cdot))$ and $y^*(\cdot)$ so that

(i) on any subinterval $[t_i, t_{i+1}] \subset I$ on which $y^*(\cdot)$ is continuous,

$$\min_{a \in K_{a}} H(\lambda(t), y^{*}(t), a) = H(\lambda(t), y^{*}(t), p^{*}(y^{*}(t)));$$

(ii) at any discontinuity point, say t_c of $y^*(\cdot)$,

$$\min_{\alpha \in K_{\alpha}} \mathcal{H}(y^*(t_c), \alpha) = \mathcal{H}(y^*(t_c), \pi^*(y^*(t_c)));$$

(iii)
$$\min_{a \in K_a} H(\lambda(t_c^+), y^*(t_c^+), a) - \min_{a \in K_a} H(\lambda(t_c), y^*(t_c), a) \\ = \frac{\partial \mathcal{H}}{\partial y_n}(y, \alpha) \quad y = y^*(t_c), \alpha = \pi^*(y);$$

(iv)
$$\lambda_0(t) = 1$$
, for all $t \in I$.

We then have the following open question of the impulsive control: Is there is any connection between the value function defined in (4.6) and a costate function λ ?. The game version of the minimum principle for the impulsive control problem with finite time horizon was discussed in [14] (see also [15] for necessary and sufficient conditions for optimal impulsive control). It would also be of interest to formulate a maximum (or minimum) principle tailored specifically for control problems with switching costs.

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