# MATCHING TEACHING STRATEGY TO AVAILABLE M-SPACE: A NEO-PIAGETIAN APPROACH TO WORD PROBLEMS

by

Dianna B. Richardson

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APPROVED:

Gerald Brazier, Chairman

Terry Wildman

Lawrence Cross

Jesse Arnold

M. Gerald Cline

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Blacksburg, Virginia

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# INTRODUCTION

## Description of the Problem Situation

Solving mathematics problems, especially those presented as word problems, is difficult for many elementary students. First grade students, many of whom are learning formal mathematics for the first time, may learn addition and subtraction number facts, but may not be able to apply this knowledge when asked to solve word problems. Additionally, basal math texts have not emphasized problem solving activities. One factor in the failure to solve problems may be that the strategies by which children are taught require greater "mental space" than they have available.

The term mental space or M-Space has been coined by a group of Neo-Piagetian researchers who have elaborated upon the theory of cognitive development formulated by Jean Piaget. For Piaget, development involves a series of qualitative changes in thinking due to underlying structural changes occurring in the human organism. Within the last decade, Juan Pascual-Leone and Robbie Case, both Neo-Piagetians, have developed and refined a quantitative index of cognitive developmental competence termed mental space or M-Space. This measure has been postulated to explain the qualitatively different stages of development. Specifically, M-Space describes the number of schemes (psychological entities which account for behavioral regularities) which may be coordinated at one time for action upon stimuli.

Piaget has described developmental changes as occurring in four major stages: sensori-motor, preoperational, concrete operational, and formal operational stages, spanning infancy to adulthood.

The Neo-Piagetian quantitative index defines these stages in terms of a constant quantity of mental space necessary to execute cognitive tasks (termed "a") plus the number of additional schemes which must be coordinated in order to solve the tasks (e.g., "3"). For instance, the conservation problem which exemplifies Piaget's concrete operational stage usually requires an M-Space of a+3 for its successful completion.

According to the theory, to solve problems via a strategy requiring a certain M-Space, the subject needs to possess (1) M-Space at least equal to the amount required by the strategy, (2) the specific necessary schemes in his repertoire, and (3) the tendency to use M-Space to its capacity. To apply the theory to cognitive tasks such as teaching students to solve addition and subtraction word problems, it follows that a teacher would need to have knowledge of the student's available mental space, as well as the mental space required by the teaching strategy which a teacher is using.

Since this theory has been proposed fairly recently and since it is being refined constantly by those who initiated it and by other researchers, some components need further clarification. For example, although the three requirements (stated above) are said to be necessary for problem solving, studies by Pascual-Leone (Pascual-Leone and Smith, 1969) and Case (1975, 1977) in addition to others, have postulated that instruction may play a role in decreasing the amount of M-Space necessary to complete certain cognitive tasks. Essentially, what is postulated is that certain kinds of instruction can facilitate the extinguishing of misleading schemes and/or the chunking of two schemes into one which in turn enables a subject to complete a particular task via instruction which he would not be able to do spontaneously.

Most of the studies which have attempted to lower the strategy requirements to enable a person possessing lower M-Space to successfully complete a task have relied upon task analysis to insure that the strategy requirements are not higher than the M-Space level of the individual. In at least one experiment cited by Pascual-Leone (Pascual-Leone and Smith, 1969) however, 5-6 year olds were taught to perform a certain cognitive task, the class inclusion task, via a didactic teaching procedure. The task had formerly been accessible only to 7-8 year olds.

These studies (which indicate that the M-Space necessary to carry out a particular task can be manipulated) should serve as a ray of hope for teachers; however, there are many instances of instruction which seemingly contradict the notion of M-Space manipulation. To elaborate, this researcher has observed classroom teachers instructing students about particular addition and subtraction strategies until the students could perform the tasks to criterion. After some passage of time, the students were unable to perform the tasks which they were previously trained to accomplish. What is especially striking about the observation is the fact that the tasks would not be expected to far exceed the

M-Space of the students (based upon previous research regarding M-Space levels of 6 and 7 year olds).

Based on the studies cited, one would tend to conclude that instruction would facilitate the previously mentioned "chunking" in addition and subtraction tasks. In fact, Case has stated that "one of the most important functions of instruction should be precisely this: to lower the central processing load (CPL) of assembling and applying more sophisticated executive control structures. While instruction should be able to reduce the CPL of a given structural acquisition; however, it should not be able to eliminate them entirely" (Note 1).

Because of this apparent inconsistency alone, the theory warrants additional study; however, there exist other unexplored aspects of theory appropriateness. Many cognitive tasks exhibit sequential features and the research about addition and subtraction provides strong evidence that these particular tasks develop in a sequence which begins with counting. Children use the counting concepts to acquire other arithmetical knowledge such as counting-all and counting-on addition and counting-back subtraction. (Counting-all is a strategy in which every object in each subset or part of the whole is counted to find the whole or total. Counting-on is a strategy in which a counter realizes that the number which represents one part of the whole is inclusive of all the objects in that part. Knowing this fact allows the counter to start with that representative number and point count the objects in the remaining part to find the whole or total. The reverse technique is used in counting-back.) If students learn the counting-all and counting-on (back) procedures in sequence as certain researchers (Steffe, Richards, and von Glasersfeld, 1979) maintain, the question is whether these two procedures require different levels of M-Space. Task analysis of certain teaching strategies for these two types of addition and subtraction indicate that "counting-all" problems require an a+2 M-Space while "counting-on" (back) problems require an a+3 M-Space.

While other researchers have examined what happens when a single cognitive task is taught to a group of students, they have not determined whether a group of students (in this case, a+2's) who have been taught a sequential task by a strategy which is above their M-Space level (in this case, the counting-on strategy) will "chunk" schemes and execute the task via the a+3 strategy, or whether they will execute the task via an a+2 strategy (in this case the counting-all strategy) which has developed earlier in the sequence.

From a pedagogical point of view, students in the first grade are generally capable of operating at a mental space or M-Space level of a+2 or a+3; therefore, if a teacher is using an a+3 strategy approach, it should be advantageous to know if the a+2's and a+3's are both going to be able to understand and utilize the a+3 strategy or whether other strategies will have to be devised for the a+2's or whether other mathematical subject matter (requiring only an a+2 M-Space) will have to be taught while waiting for the student to develop an M-Space of a+3.

## Purpose

The principal purpose of this study is to determine what effect a student's M-Space level has on the strategy the student uses to solve addition and subtraction word problems when the teaching strategy is matched or mismatched with his M-Space level; specifically, when the M-Space demand of the strategy is above the M-Space level of the subject.

Using the counting-all, counting-on (back), strategies for addition and subtraction, the major research question is the following:

1. Are there differences in a+3 responses for the following groups of subjects: those who have an M-Space of a+2 and are trained to use strategies requiring a+2 M-Space (an M-Space-strategy match) and those who have an M-Space of a+2 who are trained to use strategies requiring a+3 M-Space (an M-Space-strategy mismatch)? The same question is asked for groups having an M-Space of a+2 who are trained to use strategies requiring a+3 M-Space and for those groups having an M-Space of a+3 who are trained to use an a+3 strategy which matches their M-Space level.

For brevity in describing groups, in all subsequent references the abbreviated training group classification will be as follows:

(a) 2-2 group--subjects with an M-Space of a+2 trained via an a+2 strategy.

(b) 2-3 group--subjects with an M-Space of a+2 trained via an a+3 strategy.

(c) 3-2 group--subjects with an M-Space of a+3 trained via an a+2 strategy.

(d) 3-3 group--subjects with an M-Space of a+3 trained via an a+3 strategy.

Although the main purpose of the study has been described, other questions arise based upon the Piagetian and Neo-Piagetian theories and upon past research about the relationship between counting and addition and subtraction. Therefore, each of the additional questions to be answered by this report will also be outlined.

The goal of teaching is not only to produce changes in behavior, but also to promote the transfer of that behavior to new situations which contain unique elements that require the student to use what he has learned in a new way. If instruction does not transfer, it is of little value because one could never be taught every new skill, process, or item of information separately. To understand transfer is to understand how learning to perform a task will affect, in some way, the performance on another task when the materials, context, or procedures vary. In Piagetian literature, the concept of transfer is an extremely important one. The amount and kind of transfer (usually labeled specific and non-specific) is one of the criteria by which one judges whether instruction has been effective.

Each of the remaining research questions can be viewed within the broad framework of transfer. In this study, there are three types of problems specified as "transfer" questions in addition to the original eight posttest questions. These problems are designed to determine transfer when the subject is confronted with different components, materials, or operations. To assess this, the second research question is asked:

2. For each experimental group, is the pattern of responses on transfer problems similar to or different from the pattern which occurs on the eight posttest problems?

The third research question also impinges on the posttest and transfer questions. Steffe et al. (1979) have painstakingly devised a sequence of counting and addition and subtraction development in young children. In one of their findings, they state that although addition and subtraction are inverse operations, certain kinds of counting-on procedures (in addition) develop before certain kinds of counting-back procedures (in subtraction). In view of this, we wish to examine subjects' responses in the two operations separately. Therefore the question:

3. When responses to addition and subtraction problems are viewed separately, do differences exist for (a) posttest problems, (b) transfer problems?

In the fourth research question, incorrect responses are analyzed. Because the responses are not correct, in a broad sense, we can assume that the transfer of learning did not occur. The research question is stated as follows:

4. When responses are examined for each experimental group, are there differences in the proportion of subjects using incorrect strategies?

Two different types of subtraction problems are presented to subjects--straight "take-away" subtraction and implied "comparison" subtraction problems. When analyzing incorrect responses, it may be informative to note whether there are differences between these two different types of subtraction problems. The question is phrased as follows:

5. Are there differences in the incorrect responses for "take-away" subtraction and "comparison" subtraction problems?

The final research question is related to one of the three transfer questions--the student generated problem. In the second research question, the focus was upon the differences in responses between post-test and transfer problems. This last question details the kinds of problems generated. Since it has been shown that there are developmental differences in regard to decoding and encoding abilities (Pascual-Leone and Smith, 1969), it is of interest to determine the kinds of problems students generate when pressed by the examiner. The question asked is the following:

6. What kinds of addition and subtraction problems do subjects generate?

In the following chapters of this paper, research relevant to the six questions will be presented, the design of the study will be described, and the results will be reported and discussed.

## Chapter 2

## REVIEW OF THE LITERATURE

This section contains a review of the literature which is relevant to the study. The two areas of greatest interest are the Neo-Piagetian Theory of M-Space and the sequential development of counting and its relationship to counting-all and counting-on (back) in addition and subtraction problem solving. To emphasize the differences in researcher beliefs about the counting/addition and subtraction sequence and that which is presented in textbooks, we will also explore how basal texts present addition and subtraction. To provide a background and framework for the M-Space theory, theories of development and problem solving are discussed with particular emphasis placed on Piaget's theory since that is the one from which the Neo-Piagetians derive their impetus.

Highlighted in the discussion of M-Space research studies is the role of instruction in facilitating the processing of information when the information to be coordinated is greater than the subject's available M-Space.

The chronological order of discussion of the research is as follows:

--manner in which basal texts present addition and subtraction

--theories of developmental change

--problem solving perspectives and the variables involved

--M-Space theory

--the sequential development of counting and addition and subtraction

Problem Solving in Basal Texts

In its Agenda for Action, the first recommendation of the National Council of Teachers of Mathematics is that problem solving must be the focus of school mathematics in the 1980's. The Council (1980) states the following:

> The development of problem-solving ability should direct the efforts of mathematics educators through the next decade. Performance in problem solving will measure the effectiveness of our personal and national possession of mathematical competence.
>
> Problem solving encompasses a multitude of routine and commonplace as well as nonroutine functions considered to be essential to the day-to-day living of every citizen. But it must also prepare individuals to deal with the special problems they will face in their individual careers.
>
> Problem solving involves applying mathematics to the real world, serving the theory and practice of current and emerging sciences, and resolving issues that extend the frontiers of the mathematical sciences themselves.
>
> This recommendation should not be interpreted to mean that the mathematics to be taught is solely a function of the particular mathematics needed at a given time to solve a given problem. Structural unity and the interrelationships of the whole should not be sacrificed.
>
> True problem-solving power requires a wide repertoire of knowledge, not only of particular skills and concepts but also of the relationships among them and the fundamental principles that unify them. Each problem cannot be treated as an isolated example. This recommendation looks toward the need to solve problems in an uncertain future as well as here and now. As such, mathematics needs to be taught as mathematics, not as an adjunct to its fields of application. This demands a continuing attention to its internal cohesiveness and organizing principles as well as to its uses. (p. 2)

Hopefully, publishers of basal texts will consider this mandate in the future. In fact, differences in how addition and subtraction are taught, and particularly, differences in problem solving methodologies are beginning to be apparent when older series are contrasted with one recently published. Older basal mathematics texts commonly used to teach first graders differ somewhat in terminology and approach, but vary little in their basic objectives for teaching beginning addition and subtraction. Reviewing four widely used basals reveals similar objectives for addition. Nichols, Anderson, Dwight, Flournoy, Kalin, Schleup, and Simon (1974) who authored Holt School Mathematics state the following:

> To show how many in all when the sets are joined. (p. 41)
>
> To relate addition to joining two sets. (p. 43)

Bolsher, Cox, Gibb, Kirkpatrick, Robitaille, Trimble, Vance, Walch, and Wisner (1975), authors of Mathematics Around Us, state this objective:

> Describe an additive situation in terms of joining action. (p. 33)

The authors of Mathematics for Individual Achievement (Denholm, Hankins, Herrick, and Vojtko, 1974) expect that students can accomplish the following task:

> The student can join two sets of objects, one set of which contains one object and determine the number of objects in the resulting set. (p. T124)

Finally, Eicholz, O'Daffer, and Fleenor (1976) who wrote Investigating School Mathematics state the following two objectives for first grade students:

> To reinforce the idea that various combinations can be used to form sets of the same number.
>
> To introduce the idea of the union of two sets. (p. 94)

As the objectives state, to learn addition the student learns to join sets of objects. The following examples of suggested teaching strategies elaborate this methodology. The teaching activities suggested by the authors of Holt School Mathematics (1974) specify that the teacher should:

> Display a set with one member and draw a ring around it. Do the same with a set of two members. Guide the child in naming the numbers in each set and writing the numerals. Elicit the idea that a set with one member joined to one with members forms a set of three members. (p. 41)

A similar activity from Mathematics for Individual Achievement (1974) states that:

> The students should be provided with many opportunities to join sets of objects. Story telling provides an excellent media for joining sets. In the beginning have the students join a set of objects to a set of one object. They may use different colored yarn to ring sets on the display board using a different color to ring the "joined" set. (p. T125)

Mathematics Around Us (1975) requires the student to use the plus sign in his first encounter with addition, but the activity is still one of joining sets:

> Show a set of 3 objects and a set of 2 objects. Ask a child to write on the board how many there are in each set. Move the set of 2 over to join the set of 3. Ask if anyone knows a sign that can be used to show that 2 objects joined the 3 objects. (p. 33)

Early in the books, the student is taught to recognize sets of objects without counting. When he is asked to join sets, the inference seems to be that he will recognize the number of members in a joined set without having to count the total members in it.

The same general procedure is followed for subtraction which is taught as a separating action (i.e., part of a set is separated from the entire set).

Only recently with the publication of new math texts such as Real Math (1980) are the skills associated with addition and subtraction being taught as part of a developmental sequence in which counting skills are viewed as a necessary and integrated part of the sequence. Additionally, many chapters in the Real Math textbook are devoted to forming finger sets, working with concrete objects, and manipulation of objects before addition and subtraction number sentences are presented.

Although the problems presented in the "Introduction to Addition (Subtraction)" sections of basals require that a child use problem solving strategies, there are specific "problem solving" sections designated later in the books. The authors of Holt School Mathematics present different kinds of problem solving ranging from (1) asking a child to observe a dramatization of a number sentence and then writing the sum or difference (p. 48), to (2) choosing an addition sentence that fits a picture problem (p. 63), to (3) presenting a picture depicting the joining of a set (for addition) and the question, "How many _____ in all?". The child must supply either the plus or minus sign for an incomplete number sentence (p. 124). The objectives outlined in Mathematics for Individual Achievement, in contrast, require the child to tell a story involving addition (or subtraction) when given a picture illustrating the situation (p. 78) and Mathematics Around Us presents a combination numeral and picture story and requires that the child write a number sentence for the story (i.e., 4 (picture of owl), 5 more came. How many (picture of owl) came in all?) (p. 203).

Real Math, the only text reviewed which differs in its treatment of problem solving, presents stories from a storybook and the student is expected to work several problems depicting real life situations during each lesson.

The reason for describing how addition and subtraction problem solving is presented in mathematics basals is not to take issue with set theory, but rather to show the positive and negative aspects of how problem solving is presented in these texts.

Steffe, et al. (1979) further explicate the difficulties encountered in using basal texts as the source for teaching beginning addition and subtraction by stating the following:

> Perusal of current commercial primary textbooks in mathematics (and mathematics programs available through other sources) reveal that the program writers, in the main, develop their content sequences through logical analysis ignoring the child's constructions. From the constructivist perspective, the conceptual steps involved in the development of a certain procedure must be integrated with the child's developing conceptual structures. . . . even programs which purportedly are based in cognitive development generally seem to ignore the inherent epistemological problem that is addressed by constructivism. In short, textbook authors produce their instructional analyses in mathematics and instructional technology, largely ignoring the child's own conceptual construction of the mathematics. (p. 41)

Problem solving is an integral part of the mathematics curriculum, but its nature is complex and most educators and mathematicians find it difficult to delineate the attributes of good problem solvers. This may be the reason that math texts deal with addition and subtraction problem solving on a fairly superficial level. If little is known about problem solving processes, it is difficult to present precise strategies for teaching it. According to Lester (1977) "It has been only during the last twenty-five or so years that a major point of view or technique has developed that attempts to relate the important variables that influence problem solving behaviors" (p. 13). This is because of the variety of tasks used in research and the different theoretical positions taken by researchers, ranging from behaviorists to developmental psychologists to gestaltists to information processing theorists. Cohen (1977) reiterates Lester's points stating that:

> Attempts to provide a general characterization of the process of problem solving have met, on the whole, with little success, being either too vague or too incomplete, or both. This is hardly surprising since the kinds of tasks which come under the heading of problem solving are extremely diverse. (p. 46)

While each of the approaches described by Lester and Cohen have revealed interesting insights into the question of the process or processes used in problem solving, until recently none were particularly precise or comprehensive. In the last decade, Juan Pascual-Leone (Pascual-Leone and Smith, 1969; Pascual-Leone, 1970) and Robbie Case (1972a, 1972b, 1978a) have endeavored to integrate a developmentalist point of view with some features of information processing to draft a more comprehensive theory of intellectual development in general and problem solving in particular, since problem solving is a part of intelligence. Their theory, termed a Neo-Piagetian interpretation of development, derives its impetus from the Piagetian theory of learning and development. In the next subsection, developmental change will be viewed from the perspective of Piaget as well as other theorists.

Theories of Developmental Change

Piaget's biologically oriented theory of development recognizes three sources of knowledge in humans: instinctual (innate), developmental, and individually learned (Furth, 1974). Piaget (1971) elaborates on this postulate stating that there are:

> three possible kinds of knowledge: (1) the kind that is linked with hereditary mechanisms (instinct, perception), which may or may not exist in man but which correspond in biological terms to the sphere of characteristics transmitted by the genome; (2) knowledge born of experience, which thus corresponds in biology to phenotypic accommodation; and (3) the logico-mathematical kind of knowledge which is brought about by operational coordinations (functions, etc.) and corresponds, in biology, to regulation systems of any scale, in the hypothesis that elementary logical operations (revisions, dissociations, order, etc.), with their "necessary" characteristic of coherence or non-contradiction, represent the fundamental regulatory organ of intelligence. (p. 100)

The child's logico-mathematical experience (deriving from a child's actions on objects) along with reflective abstraction accounts for development according to Piagetian theory. Reflective abstraction may be defined as "the process by which a learner abstracts what is common among the properties that the members of a certain class of actions between objects possess" (Mick and Brazier, 1979, p. 48). According to Piaget (1970), "reflective abstraction . . . does not derive properties from things but from our ways of acting upon them, the operations we perform on them" (p. 19). Therefore, development is a biological and psychological process, rather than merely an increase in knowledge. This process leads to qualitatively new ways of functioning termed "stages" by Piaget. The stage progression is from the sensorimotor (ties to physical action) to preoperational (characterized by semiotic functioning) to concrete operational (marked by actions on concrete objects and reversibility of thought) to formal operational (characterized by hypothesizing and propositional logic) (Piaget, 1970).

Brainerd's (1978) interpretation of the relationship of Piaget's stage development to cognitive functioning is that it is not possible for someone functioning at one stage to learn concepts from the next higher stage. Supposedly, learning occurs only when the training procedure incorporates that which would develop spontaneously and when the child already partially possesses the concept to be learned. In sum, according to the theory "development constrains learning . . . [and] . . . certain things can only be learned in certain orders. More explicitly, the natural order in which concepts are acquired during spontaneous development cannot be altered in the laboratory" (Brainerd, 1978, p. 96).

While some studies tend to support Piaget's view that training is not feasible (Flavell, 1963; and Mermelstein and Meyer, 1969), other reviews of acceleration studies (Brainerd and Allen, 1971) indicate that training is indeed possible, even when subjects do not possess the to-be-learned concepts (Brainerd, 1978). Additionally, investigators have evidence that younger subjects sometimes outperform older subjects on logical and memorial tasks (Scardamalia, 1977; and Chi, 1978). These findings which run counter to those of Piaget indicate that while Piaget's theory can be a point of origin, some premises may be incorrect or in need of expansion or modification.

Other theorists' ideas and perspectives may also be valuable in designing a more comprehensive theory of development. For example, Odom (1978) states that change in "psychological development is primarily or exclusively based on the perceptual system's changing sensitivity to relations in the environment rather than to changes in cognitive processes" (p. 128). Siegler (1978), while acknowledging Piaget's notions about developmental change, also proposes alternative explanations including those of Case and Brainerd as well as his own approach. He describes Case's M-Space theory which will be discussed in detail later in this paper as emphasizing the memorial capacity of the subject. Brainerd's approach, in comparison, has focused upon the assessment of the various mental actions (i.e., reversibility, addition-subtraction, compensation, and identity) which a subject may use to solve the cognitive task of conservation. Siegler, in yet another possible explanation, has investigated the process of encoding stimuli which is not as complete and/or not as accurate in younger subjects as it is in older ones. Siegler's "incomplete encoding" approach along with others he describes makes it obvious that while much is known about how development occurs, more research about developmental processes is necessary before a comprehensive theory is expounded.

The same lack of conclusiveness is apparent in the investigation of problem solving. It was previously stated that it is difficult to isolate variables which influence problem solving abilities and to reconcile the conflicts surrounding theoretical positions from which it is viewed. Even though this situation exists, much research about problem solving has been conducted; therefore, in the next section, the issue of problem solving is examined more closely.

## Problem Solving--Perspectives and Variables

The complex nature of problem solving was touched upon earlier in this paper. To understand what transpires when a child attempts to solve a problem, one must comprehend the psychological processes which occur in problem solving. Like developmental processes, problem solving is conceptualized in various ways by researchers. This section will spotlight some major perspectives and variables involved in this area of study.

Theoretical perspectives. As discussed earlier, Cohen (1977) views problem solving from the perspective of various theories: stimulus-response, gestaltist, and information processing. According to the stimulus-response view, any problem encountered by an individual represents a "stimulus" and associated with each stimulus is a set of responses. The individual may be conditioned to utilize a certain response by reinforcing that response more strongly than other responses. The conditioning procedure, therefore, accounts for the kind of response which is elicited from the individual. The Gestaltists, in contrast have emphasized the "importance of the perceptual set in problem solving. According to the Gestaltists, proper apprehension of the parts of the problem ensure that the 'force of organization' produce the solution" (Cohen, 1977, pp. 48-49).

The informational approach defines the mental processing of the mind in a manner analogous to the internal workings of a computer. To specify, the informational processing theories possess (1) a sensory intake register through which information flows from the outside to the organism, (2) a processing component, and (3) a storage memory which holds the information on which the human mind works. Greeno's (1973) work with transformation of a situation into a set of sub-processes; Miller, Ganter, and Pribram's (1960) Test-Operate-Test-Exit program; and Newell, Shaw, and Simon's (1960) "state space" (which characterizes a task) and the problem solver's "problem space" (with which he works) are representative of problem-solving models formulated via this approach.

Variables in problem solving. Other investigators have examined the manner in which a child attempts to find the answer to a problem. Cohen (1977) focuses upon three important variables intrinsic to problem solving:

> (1) The nature of the problem (i.e., whether it is well or ill defined, how familiar or unfamiliar it is, and its complexity).
>
> (2) The context of the problem (i.e., the physical location of objects, how questions are asked concerning the problem, and the speed with which an answer is required).
>
> (3) The characteristics of the problem-solvers (i.e., his intelligence, experience with the problem situation or materials, his motivation and his anxiety). (p. 51)

Siegler (1978) concentrates on the third variable, characteristics of the problem-solver, emphasizing the importance of determining whether or not the child is rule governed. He defines rules as the "statements (which) not only summarize data but also have some correspondence to the way in which the data were generated . . . [rules systems] clearly act as both summarizations of data and theories of how data came to be" (p. 119).

For Kintsch (1970) being able to decompose complex problems into subproblems and knowing the goal and remaining goal-directed throughout the process are two important characteristics a problem-solver must possess.

Memory development and problem solving. The characteristic of memory development and its role in problem solving has been investigated widely, especially in connection with information processing. According to Hagan, Jengeward, and Kail (1975), as information flows through the mental system, retrieval of events becomes necessary, and memorial processes play an important role in this. As the child develops, his efforts increase also. According to Brown (1975) "the child's meta-memorial processes (introspective knowledge of the functioning of our memory systems) increases and his 'knowing how to know' (the repertoire of strategies and skills we possess for deliberate memorization activities) also increase" (p. 105). It is essential that the child have this introspective knowledge about his own memorial processes so that he will be able to choose appropriate strategies for tasks and to alter strategies according to the internal and external feedback he receives. Speaking from a Piagetian perspective, Campbell (1976) views memory development as the accumulation of coded information. Memory retention for the purpose of decoding and retrieval is important, but it is equally important to conserve the code, because that is what leads to organizing of remembrances.

Performance models for problem solving. Another perspective is available from Klahr (1978) who states that the development of problem-solving abilities can be viewed according to three performance models: empirical, characteristic, and procedural. In the empirical description, performance measures and their changes over time are assessed; in the global characteristics of the child model, the child is evaluated to see if he is "wholistic or analytic, . . . rule-governed or not, . . . systematic, . . . egocentric" (p. 210); and in the procedural description, a determination is made about "what children know when they exhibit a particular performance level on a task" (p. 210). Although these models may be quite descriptive of what has developed, Klahr warns the researcher to "account for the psychological processes that enable a child to listen to task instruction, assimilate them into his existing general problem-solving processes, and produce something approximating the performance models . . . previously described" (p. 210).

## M-Space Theory

Case's theory, although descriptive in nature, seeks to account for some of the psychological processes occurring in problem-solving. Borrowing heavily from Pascual-Leone and Piaget, Case has derived both a theory of development and a theory of instruction, but it is the former which is of the most interest in this paper. Both Case and Pascual-Leone seek to integrate Piagetian and information processing theories. Pascual-Leone acknowledges Piaget's theory as his point of departure, but he is not primarily concerned with Piaget's logical structures; rather, Pascual-Leone's constructionist theory attempts to explain and predict behavior in terms of models which reflect functional organization. Pascual-Leone's theory of constructive operators contains, as fundamental concepts, schemes (psychological entities which account for behavioral regularities) and scheme boosters (Pascual-Leone, 1976a; Ammon, 1977).

Ammon (1977) states the major tenets of the theory as follows: cognitive functioning is assessed in terms of the functions of schemes; however, all schemes do not function in the same manner. Some schemes, labeled "figurative" schemes, produce mental representations of items of information; other schemes, called "operative" schemes operate as internal representations of rules which act to transform figurative schemes. The theory also contains components which function to increase the activation weight of the schemes upon which they act. Labeled "metaconstructs", the following operators boost the action of schemes: field factors (F), mental attentional energy or working memory (M), affective factors (A), and structural learning (L). These metaconstructs, plus others which may not have been identified yet, have indirect effects; that is, they can cause differential results depending upon the conditions of their activation and the specific schemes to which they apply.

For Pascual-Leone, it is the M-operator which is of primary interest. Central to the theory is the notion of growth of M-Space (working memory) or stated differently, the increase in scheme boosting power of M.

Case's theory is an elaboration of Pascual-Leone's and because of this fact, it contains elements of both Piaget's theory and Pascual-Leone's theory. From Piaget, Case borrows the concept of a stage development. From Pascual-Leone comes the notion that two factors account for stage progression: an increase in the amount of working memory or M-Space and experience with the strategy in question. An overview of Case's (1978a) theory provides the following five postulates:

> (1) Within the major stages of intellectual development, there is a succession of substages . . . and . . . this succession of substages stems from a succession of qualitatively distinct control structures or executive strategies.
>
> (2) Two sorts of factors explain the succession of strategies within any stage. The first is the child's responsiveness to the strategy related experiences he encounters. . . . The second factor is a gradual increase within each stage in the size of the child's working memory. As working memory increases, it becomes easier to acquire and utilize more complex executive strategies.
>
> (3) The gradual increase in working memory does not stem from a structural increase in the attentional capacity of the organism, but rather from an increase in the automaticity of the basic operations it is capable of executing. As these operations become more automatic, their execution requires a smaller proportion of total attentional capacity. The result is that more capacity is available for "storage" or "working". Exactly how

the increase in automaticity occurs is unclear, but it seems that, if experimental input plays a role, it is general rather than specific.

(4) The executive strategies of each major stage involve qualitatively distinct underlying operations and . . . the operation at any given stage must be assembled in working memory from components available at a previous stage. It follows that transition to any given stage depends on the attainment of a certain degree of automaticity during the previous stage.

(5) Implicit in the above postulates is an assumption about the types of learning that cause relatively rapid restructuring of executive strategies. The assumption is that the child's capacity for such learning is present from birth but that it must await a certain size working memory . . . before it may be observed. (p. 64)

The term "working memory" is found throughout Case's five postulates. This would be expected since this term is central to his theory. To understand working memory (referred to in postulate three), it must be viewed in relation to functional operating capacity and functional storage capacity in the human organism. Symbolically, it may be represented as the following: "$O + S = K$ where $O$ = functional operating space; $S$ = functional storage capacity; and $K$ = a constant, equal to the total structural capacity of the system" (Case, 1978a, p. 58). What these symbols mean is that an individual's total working memory functions as either an information storage space or as an information operating space when that person is solving problems.

"Working memory" was termed central computing space or mental space (abbreviated as M-Space) by Pascual-Leone (Pascual-Leone and Smith, 1969; Pascual-Leone, 1970). Other researchers including Case (1972a, 1972b, 1974a), Scardamalia (1977), and Relihan and Restaino (1976) have continued the use of these terms in their writings. Scardamalia (1977) summarizes the definition of M-Space as the maximum number of subjectively discrete schemes that the individual can activate simultaneously through an attentional process (p. 29). This definition corresponds closely to Case's working memory but because the terms are different, the similarity may not be apparent.

According to Scardamalia, in problem solving one must simultaneously activate schemes appropriate to the task. Schemes are the organized sets of reactions indicative of the underlying cognitive structures. Schemes may be classified as either executive, figurative, or operative. Executive schemes are labeled as overall managerial plans that are applied to particular problems or stated differently, they are the internal plans for executing the necessary operations involved. According to Ammon (1977), "Executives . . . direct . . . attention or 'mental energy' toward schemes that are particularly relevant to the goal of behavior" (p. 178). Figurative schemes, according to Case (1974b) are "internal representations of items of information with which a subject is familiar, or of perceptual configurations which he can recognize" (p. 545). Operative schemes are described as internal representations of rules which are used to perform transformations on figurative schemes.

If M-Space is viewed as the maximum number of schemes which may be coordinated at one time, quantification is implicit in this definition. M-Space is a quantitative measure which explains the qualitative psychological changes which mark Piaget's stage transitions.

For each Piagetian substage of development, Pascual-Leone has hypothesized that M-Space occurs as follows:

| Piagetian Substage | Age | Model Value of M (a+k) |
|---|---|---|
| Early preoperational | (3-4) | a+1 |
| Late preoperational | (5-6) | a+2 |
| Early concrete | (7-8) | a+3 |
| Late concrete | (9-10) | a+4 |
| Early formal | (11-12) | a+5 |
| Middle formal | (13-14) | a+6 |
| Late formal | (15-16) | a+7 |

where "a" refers to a constant quantity corresponding to the processing space taken up by activation of the executive scheme, and the "number" refers to the maximum number of figurative and operative schemes which may be coordinated at any one time.

Case, in his refinement of Pascual-Leone's theory, has resliced the major stages into a sequence of four substages per stage (see postulate 1). By using tasks in which the sequencing of executive strategies is apparent such as in Noelting's (1975) orange juice problem, a study is the development of the concept of proportion, Case has noted a sequence of substages. For example, in Noelting's task, "for the simplest strategy, only one item must be considered: the presence or absence of juice. For the second strategy, two items must be considered: the number of orange juice tumblers poured into [container] A and the number of orange juice tumblers poured into [container] B" (Case, 1978a, p. 40). For the third and fourth strategies, three and four items (respectively) must be considered. Only by using tasks like this one involving a sequencing of strategies from simple to complex, can one see the various substages involved.

An additional premise related to the one just discussed is that of substage overlap from one major stage to the next stage. To specify, the structures that Piaget describes as being concrete operational are those structures which have been fully assembled or consolidated and the typical ages at which this occurs are approximately six or seven. Case, in contrast, believes that four year olds are beginning to consolidate the structures described by the concrete operational stage. In the Piagetian theory, "higher order operations build upon and incorporate lower order operations" (Case, 1978a, p. 60). The Neo-Piagetian quantitative description of this statement is that there are a minimum number of schemes which have to be coordinated at the preoperational stage so that beginning concrete operations may be executed. In general, according to Case (Note 2), the fourth substage of each major stage overlaps with the first substage of the next major stage; therefore, the fully consolidated preoperational stage is laying the foundation for the beginning of consolidation at the concrete operational stage even though this may not be apparent unless one is assessing operativity via tasks which have sequencing aspects.

Tasks such as counting and beginning addition and subtraction which have been shown to develop sequentially may be viewed in the context of substage development, a concept which will be explored in greater detail in the next section dealing with counting and addition and subtraction. What follows now, however, is a review of how Case and other researchers have applied the M-Space construct to other logical tasks.

In most of the studies, except for the ones which attempt to validate the M-Space construct, the researchers have examined the effects of instruction on decreasing M-Space demand. Some of the studies involved direct didactic instruction, some involved indirect training, while others involved task analysis of the schemes to be coordinated in order to lower the M-demand; however, in all of these different studies, the researchers claimed to be successful in their training procedures.

Although the results are at odds with the observations of what is occurring in many classroom situations, these researchers might argue that their training techniques exemplify better instructional planning--the kind which facilitates "chunking" of schemes. A counter argument that their training experiments have only trained subjects to criteria and that long term effects are non-existant necessitates a check for delayed posttesting. Because of the design of some studies, delayed posttesting was inappropriate, but it was mentioned in three studies (Case, 1974b; Relihan and Restaino, 1976; and Case, 1977). In all three instances involving delayed testing for periods of several days to two months, the positive results were maintained.

Since Case and other researchers have applied the M-Space construct successfully on a substantial number of experiments, their findings will now be examined in greater detail. Case (1972a), using subjects aged

6, 8, and 10 and the Digit Placement Test in an attempt to validate the M-Space construct, found that subjects could perform the task when it was set at or below their hypothesized M-Space. One important generalization drawn from that study was that M-Space increases with age. In another study (1974a) involving the Digit Placement Task, and 6, 8, and 10 year olds, comparisons were made between subjects who devised their own strategies to solve the tasks and those who were taught a specific strategy. Students who devised a more efficient strategy requiring less M-Space did better than students using a less efficient strategy. When subjects were trained to use a more efficient strategy, there was improvement at all levels.

Two additional studies by Case examined the effects of M-Space demand reduction. In one (1972b), he initiated a study involving kindergarten students in an experimental program. After direct teaching of classification tasks, students of kindergarten age performed as well as fourth grade students on a verbal encoding classification task and as well as second graders on a gestural encoding classification task. The procedure followed was to task analyze the learning sequence regularly used and to determine how to make changes to lower the demands on M-Space. In another study by Case (1974b), a control of variables test (with rods and blocks) was used to compare the structural versus the functional task requirements. From a Piagetian stage concept, the structure of the task is one requiring formal operations; however, by instructing students on a step by step proof checking and proof constructing strategy, the functional level for passage was reduced to the point that 7 and 8 year olds could pass the task.

Using the Piagetian task of conservation and Lefebvre and Pinard's (1972) conflict training procedure to induce conserving responses, Case (1977) attempted to determine whether the individual's subjective certainty had to be disturbed prior to conflict training as Lefebvre and Pinard had hypothesized. Although he found that there was no effect due to preliminary subjective uncertainty, Case did find that pretest to posttest improvement was strongly related to M-Space for which he had tested as a covariable. His results suggest that sensitivity to conflict is a function of the subject's capacity to coordinate cues and to perceive conflict. Accordingly, to resolve the conflict in cues presented in the conservation task, the subject must create or acquire an executive scheme which "constitute[s] a general plan for evaluating the relative quantity of two objects" (Case, 1977, p. 22). The acquisition of the executive scheme (plus the coordination of other figural and operational schemes) accounts for the positive relationship noted in the study between conflict conservation training and M-Space.

Pascual-Leone and Smith (1969), in an article outlining the M-construct, were able to show that 5 and 6 year old students who were taught the logical task of class inclusion via a didactic learning strategy devised by Kohnstamm (1963) performed the task which is usually reserved for 7 and 8 year olds. Analysis of the teaching procedure revealed that the M-Space capacity required of students was within the range of the M-Space which most 5 and 6 year olds normally possess.

In another study of class inclusion problems, Relihan and Restaino

(1976) trained subjects in "an organizing strategy that recodes the interactions between groupings as required of operations in class inclusion" (p. 3). In comparison of control and experimental groups of kindergarteners, all 24 subjects who had received training passed class inclusion posttests while only 3 of 24 non-trained subjects passed.

Scardamalia (1977) applied the M-Space construct to the problem of horizontal décalage which occurs in Piagetian logical tasks. Horizontal décalage is "a term used by Piaget to refer to the asynchronous emergence of various manifestations of the same cognitive structure: for example, the appearance of conservation of weight after conservation of substance" (Scardamalia, 1977, p. 28).

In order to determine whether the information processing demand or M-Space demand of tasks is a factor in décalage, Scardamalia devised combinatorial reasoning tasks in which the number of variables was varied, but the logical characteristics remained constant. To have a framework for comparison of subjects' strategies to the most efficient strategy, a task analysis was conducted (described below).

> [To determine] the M demand for a task . . . the number of schemes that need to be mentally activated in order to execute the most demanding step in a task [is analyzed]. One scheme needs to be activated continually throughout a task sequence. This is what Pascual-Leone calls the executive scheme, and it represents the overall plan of action--in the present case, the plan for making all possible combinations. This scheme is labeled <u>e</u> and is counted separately from other schemes in calculating M demand.
>
> The other schemes involved at any step in task execution will be one or more figurative schemes ($\phi$) representing past or present states of affairs and an operative scheme ($\psi$) representing a transformation. . . . In the present task, the key operative scheme is one representing a rule for deciding which card to trade next. This scheme is labeled "$\psi$trade

rule."

[An example of the schemes a subject must coordinate using this executive or odometer strategy follows:]

ϕthousands column: the thousands column card is the pivotal card.
ϕhundreds column: this hundreds card has not been traded while holding this pivotal card.
ϕtens column: tens cards have been cycled while holding this hundreds card.
ϕtrade rule: trade is the lowest denomination column in which a cycle is incomplete. (Scardamalia, 1977, p. 33)

Results indicate the following: after initially failing tasks set at their maximum M-operator capacity, subjects were able to pass the tasks when the M-demand was lowered sufficiently. She states the following conclusions about this finding:

> Before a statement about lacking logical competence for a given task is warranted, it is essential that the task be presented in a form that has the lowest possible M demand consistent with the logical structure of the task. As the present results demonstrate, the logical capabilities of subjects can be grossly misjudged if this is not done. (p. 36)

In the studies outlined above, two prerequisites in addition to sufficient M-Space are needed for problem solving to occur: having the necessary schemes in one's repertoire of schemes (whether having been taught the overt actions descriptive of the internal schemes or having constructed them oneself) and having the tendency to use M-Space to its capacity. The "field dependence-independence" cognitive style dimension is descriptive of this latter prerequisite. Students who respond or look at problems in the simplest manner or who give unusual weight to salient but misleading cues are assumed to be low M-processors. These field-dependent subjects "assign higher weight to perceptual cues than to cues provided by the task instructions in situations where these two sets of cues suggest conflicting executive schemes" (Case, 1974a, p. 549).

To isolate the effects of the variable M-Space, researchers have controlled for the other two variables. This has been accomplished by: 1) assuring that the schemes being tested are in the learner's repertoire, and 2) testing for field dependency. The measures most often used to test field dependency has been the WISC block design subtest (Case, 1972a, 1974; Scardamalia, 1977) and the Children's Embedded Figure Test (Case and Globerson, 1974).

Case (1975, 1978b) with the aid of his students extended the Neo-Piagetian theory of development to a theory of instruction, making it accessible to those who teach logical tasks in any area of the curriculum. Focusing upon young children's tendency to transfer previously learned strategies to any novel situation where the perceptual field is similar and their inability to consider more than a few items of information at one time (limited M-Space), Case (1978b) postulated three principles for better planning and implementation of instruction: (1) describe the strategy to be taught and the possible strategies which a child might use spontaneously, (2) design the lesson plan so that the child can see the flaws in his reasoning and the reason for using a better strategy, and (3) reduce the M-Space requirements to the lowest possible point while still maintaining the integrity of the task.

In applying these principles, both Case and his students have investigated practical mathematics problems. Lam (Note 4) devised strategies for teaching students to find the sum or addend in subtraction problems of the following types:

$$X - Y = \square \qquad \square - Y = Z \qquad X - \square = Z$$

Steinbach (Note 5) and Stevens (Note 6) examined methods of telling time and teaching addition of fractions, respectively, using the M-Space construct. While the experimenters in the three previous examples used learning tasks which were difficult for students and reorganized them so that the M-Space load was reduced, Case (Note 3) illustrated how his three principles are applied in successfully planning and implementing instruction for missing addend problems of the form $X + \square = Z$. The procedure includes identifying a correct strategy for the task and incorrect strategies used by students, demonstrating the limits of students' spontaneous strategies, reducing the M-Space demands for the task, and identifying necessary prerequisites for acquiring the strategy to be taught.

From the review of the literature, it is apparent that M-Space is a powerful construct which can be used to quantify observed qualitative psychological changes. Examples in this review illustrate how M-Space may be applied to logical tasks including mathematics problems and how it may partially account for horizontal décalages found in logical tasks. Additionally, in formulating a theory of instruction, it is the quantitative variable used to determine or measure the learner's developmental level. There is still much to be learned about M-Space as well as new ways to apply it. In the present study, the mathematical tests of addition and take-away subtraction (which may be conceptualized as either counting-all tasks or as counting-on (back) tasks) are analyzed with respect to M-Space requirements.

## Counting and Addition and Subtraction

In this section, the terminology of "counting-all" and "counting-on" ("back") will be described first and then the relevant research regarding the sequential development and the thinking strategies observed in counting and beginning addition and subtraction will be reviewed.

Counting-on is the procedure whereby the members of one part of a set are counted onto the other part of the set (i.e., in the addition problem, 5 + 3, the child begins with the number 5 and then counts 6, 7, 8). To count-back, the child begins with the sum or total. Each member of one part is counted-back until the entire part has been counted (i.e., in the problem 8 - 3, the child begins with 8 and counts back 7, 6, 5).

In the counting-all procedure, the child begins with the number one and counts every member of both parts of the set. He may physically push all members together to form one set or may point to each member or may only move his eyes, but the counting procedure goes from one to eight.

Attempts to instruct students in basic mathematics, especially to teach them immediate recall of "number facts" has leaned heavily upon the use of drill. Rathmell (1978) explains that drill may be effective in some situations and not others. Rather, students profit from activities involving concrete materials and instruction on new thinking strategies as well as practicing or drilling on basic facts. Concrete materials are important for concept development and, therefore, should definitely be used in initial instruction of a new concept; drill tends to facilitate the kind of thinking that the student is using and thus it is necessary for increased speed and accuracy of responses; instruction in thinking strategies not only helps students know when and how to use the most appropriate and efficient strategies, but also may aid in retrieval of information from a student's memory.

For addition, which Rathmell conceptualizes according to either the "set" model or the "measurement" model, he advocated teaching the following three strategies to young students: 1) Counting-on, 2) recognizing an unknown fact as either one more or one less than a known fact, and 3) compensation or storing to turn an unknown fact into a known fact (9 + 4 is changed to 10 + 3 by thinking "9 and 1 is 10 and 10 plus 3 is 13").

In the past ten years, much has been written about counting and its relation to addition and subtraction. Hendrickson (1979) has inventoried the responses made by first graders to counting and addition and subtraction problems. Although he did not attempt to sequence the counting and arithmetic tasks, he presented such tasks as seeing how far the student could count, whether the student could count objects, whether different arrangements of objects in a set were perceived as having the same number values, whether the student could begin counting at some point other than the beginning of a number sequence, and whether the student could understand and work (with or without manipulatives) simple addition and subtraction orally stated problems.

From his review of responses, Hendrickson concluded that the ability to count orally does not guarantee that a first grader can correctly count objects. Also, the students tended to compare parts with each other or they focused upon the larger part when considering part-whole relationships. The children were able to use manipulative objects to help solve oral problems, but they had their own perception and theories of how to solve tasks; therefore, the problems they attempted to answer sometimes differed from those posed by adult teachers. In one of his more interesting findings, Hendrickson (1979) reported these results about addition and subtraction: "whether an action or a comparison is indicated by the language may make subtraction easier than addition" (p. 21).

Mary Baratta-Lorton in her book <u>Mathematics Their Way</u> (1976) devotes a chapter to counting in which she expresses ideas similar to Hendrickson's. In her assessment, the counting process is not the same for the child as it is for an adult. For the child, knowledge of a counting sequence does not necessarily mean that he understands seriation. As Baratta-Lorton (1976) states "knowing and successfully using the [counting] sequence are two separate, but sequentially related skills" (p. 90).

The sequence of assessments she advocates (and presumably the order in which she believes counting skills emerge) is (1) check for the order of number names, (2) check for one-to-one correspondence, (3) determine whether the child understands invariance or conservation of number, (4) check to see if the student can "count-on," and finally (5) determine whether the child can count backward.

In his book the Piaget Primer - Thinking, Learning, Teaching, Ed Labinowicz (1980) has included a chapter entitled "Children's Levels of Thinking About Numbers" in which he outlines the necessary logical ideas for building a concept of number. He warns that the verbal counting abilities of children may be misleading because they can model adults without understanding their own activities. To understand number, Labinowicz believes that the child must be able to integrate ideas such as one-one-correspondence, conservation, seriation, and class inclusion/ class addition. Furthermore, building a concept of number allows the child to perform number operations such as addition and subtraction.

Some of the most extensive studies about the child's acquisition of counting and addition and subtraction have been conducted by Steffe, Richards, and von Glasersfeld at the University of Georgia. They have formulated models for these acquisitions framed within the theoretical perspective of constructivism, the theory which ascribes the acquisition of knowledge to creative activity of an individual. Steffe et al. (1979) developed their model of counting through intensive teaching experiments with six-, seven-, and eight-year old children. The authors state that "The experiments have yielded results that are in conflict with traditional school practices. During the teaching experiments, it became obvious that (1) children use counting of some type in acquiring a wide variety of knowledge in arithmetic, and (2) children must learn to count" (p. 34). They have delineated counting into six distinct types--(1) rote counting, (2) synchronous counting, (3) point counting, (4) counting with units, (5) cardinal counting, and (6) ordinal counting. Each is briefly defined as follows:

(1) Rote counting--a memorized chain of number names.

(2) Synchronous counting--uttering number names in synchrony with a sequence of external events.

(3) Point counting--uttering number names while pointing to objects in sequence, realizing that each object may be pointed to once and only once.

(4) Counting with units--counting a collection, part of which has been screened from view. (This is a symbolic action.)

(5) Cardinal counting--counting in which the object counted is included in the total collection of objects. The counter can start with any object of the collection and point count-on the others.

(6) Ordinal counting--cardinal counting carried a step further. The counter can count-on while, at the same time, keep a record of those objects counted.

Each of these types of counting can be subdivided into counting forward and counting backward. Steffe et al. show the importance of this subdivision by describing forward and backward counting tasks which an ordinal counter could perform, but a cardinal counter could <u>not</u> perform because of the difference in his ability to use a running tally. Tasks in which the cardinal counter can represent objects by fingers or other objects prior to counting can be adequately performed; however, in a task in which the counter knows the total number of objects and the number of objects in one part of the set which has been screened from view, but does not know how many are in the other screened part, the counter must be able to count backward from the total to the unknown screened part while at the same time keeping in mind the number of objects being counted. Only the ordinal counter can accomplish this. Steffe et al. state that the first three types of counting are qualitative, that is, they are essentially in the province of language. Only with the use of arithmetical units does counting become quantitative and it is this quantitative nature that lays the groundwork for other arithmetic notions such as beginning addition and subtraction.

The authors' model for learning addition and subtraction, specifically, the learning of the inverse relation between addition and subtraction, is closely related to counting. Before the counter can perform addition and subtraction in any other than a rote fashion, he must have reached a level of quantification; that is, he must be counting with units (fourth type of counting).

When the counter can use arithmetical units, he can find addition sums and subtraction differences by using a count-all technique (previously described). By the time that he is able to count-on without a tally, he can use this technique to do any whole number sums in which one addend is small enough to be point counted-on; however, Steffe et al. believe that the counter will still use a count-all technique to do subtraction and will only begin to count-back when he has reached the level of the ordinal counter. When he can count-on and count-back with a running tally, he can begin to relate the reverse operations of addition and subtraction as well as solve missing addend problems.

James Greeno (1979) has also surveyed the research in progress concerning addition and subtraction learned by students in early primary grades and has couched it in a cognitive theory of learning which has three components--(1) a theory of knowledge acquired as a result of learning, (2) a theory of knowledge that students have prior to learning, and (3) a theory of how the transition from the initial state to the acquired-by-learning state occurs. Examining the initial knowledge students have and the knowledge acquired through teaching imparts additional information about counting and addition and subtraction.

In the study of component (2), a theory of students' initial knowledge, Greeno (1979) looks at two issues: (a) the knowledge that preschool children have for counting objects and (b) the ability of preschool children to understand quantitative information in word problems (p. 166). To investigate issue (a), the knowledge children have for counting objects, Greeno refers to a counting model which Gelman and Gallistel (1978) have formulated. In their model, these researchers present five counting principles--three that deal with rules of "how to" count, one that deals with "what to" count, and a final one that involves composite features of the others.

The principles are described as follows:

(1) The One-One Principle--this involves ticking off items in that one and only one tick is used for each item. To do this, the child must partition items into ones counted and ones yet to be counted and tag the items with distinct tags.

(2) The Stable Order Principle--this involves the use of

tags that are produced in a stable or repeatable order. The list of tags should be just as long as the array of items.

(3) The Cardinal Principle--this involves the knowledge that the final tag used represents the cardinal number of the set.

(4) The Abstraction Principle--this involves the determination of what is permissable to be classified together for purposes of counting. Young children do not believe that heterogeneous materials may be grouped together for counting purposes, but this changes with development. By adulthood, individuals will allow that any different events or materials may be classified together for the purpose of counting.

(5) The Order-Invariance Principle--this involves the notion that order is irrelevant in enumerating. If the child knows this, he also knows that the same cardinal number results regardless of the order of tagging.

In retrospect, Gelman describes a sequence in the development of these principles. As early as 2½ years, some children can do the one-one and stable-order tasks, yet these children cannot apply the cardinal principle and even when they begin to apply it, they may not understand it fully. Only later in the developmental sequence, when the order-invariance principle is understood is the cardinal number conserved.

In regard to issue (b), understanding quantitative information, not only do children have some knowledge about how to count, but they also can comprehend quantitative information which allows them to solve some problems before the formal study of arithmetic.

Greeno also describes a four part model for component (1), a theory of students' knowledge acquired through learning; however, for the purpose of this review, only the first part of the model will be discussed in some detail. The part of the model to be studied consists of a description of the procedures used in retrieving basic addition and subtraction facts. Greeno quotes the studies conducted by Resnick and her fellow researchers as examples of the retrieval procedures children use. Groen and Resnick (1977) reported that about half of the 4 and 5 year olds had learned to use a count-on procedure to solve basic addition facts (not word problems) after being given 20-35 practice sessions; however, subtraction appears to be more complicated, at least as far as Woods, Resnick, and Groen (1975) are concerned. They have outlined five procedures which a child may use to answer a subtraction problem (second to fourth graders were included in the investigation).

For single digit problems of the type $m - n = x$ where m and n are integers between 0 and 9 and m is equal to or greater than n:

(1) Start at 0, increment m times, decrement n times. Solution is the value in the counter ($0 + m - n = x$).

(2) Start at m, decrement n times. Solution is the final value in the counter ($m - n = x$).

(3) Start at n, increment ($m - n$) times, that is, until m is reached. Solution is the number of times the counter has been incremented ($n + x = m$).

(4) Start at 0, increment n times, then increment until m is reached. Solution is the number of times that the counter has been incremented after n has been reached ($0 + n + x = m$).

(5) Use either (2) or (3) depending upon which one requires fewer operations.

In summation, it appears that children use a process of incrementing or decrementing (whichever is faster) in solving subtraction problems.

From an examination of the research presented, it is apparent that children must understand and be able to use the counting procedure, that counting is composed of components which develop sequentially, and that beginning addition and subtraction are inextricably tied to counting. Although the research presents some conflicts, especially in the ages at which children can perform certain tasks, the evidence is overwhelming that the tasks discussed are developmental in nature.

In this review, three major items of information are apparent--first, M-Space is a quantitative variable whose increase may explain the differential approaches in solving cognitive tasks; second, M-Space demand may be lowered by certain instructional procedures; and third, counting and addition and subtraction problem solving are sequential cognitive tasks, with the counting-on (back) procedure closely following the counting-all procedure in the developmental sequence.

Given this information, a question arises: when students are at a developmental level (a+2) which allows them to perform the counting-all procedure, does the teaching of the counting-on (back) strategy (normally accessible to students functioning at an a+3 level) facilitate usage of that strategy to solve addition and subtraction word problems? To determine the answer, a+3 strategy responses are checked for different groups to determine if there are differences. A description of this procedure as well as a description of the assignment of subjects to training groups and the teaching methods are discussed in the following section of this paper.

## Chapter 3

## DESIGN OF THE STUDY

The study was designed to determine whether a match between available M-Space and strategy demand is necessary for the usage of a particular strategy to solve a mathematics task, or whether instruction will facilitate the chunking of schemes which allows the task to be solved by a strategy which otherwise would be above the subjects' M-Space. Specifically, the counting-all and counting-on (back) strategies, indicative of two levels in the addition (subtraction) developmental sequence, were taught to subjects possessing either an a+2 M-Space or an a+3 M-Space. The general plan of the experiment was to train groups of a+2 subjects on either the a+2 counting-all strategy or the a+3 counting-on (back) strategy. The same procedure was used for groups of a+3 subjects. The results were analyzed as follows: the a+3 responses were compared for the two groups of subjects who possessed a+2 M-Space (and who were trained on either an a+2 or an a+3 strategy) to determine whether there were differences. The same kind of comparison of a+3 responses was made for the subjects with an M-Space of a+3.

The overall plan of the study involved the following steps: screen subjects for their knowledge of addition and subtraction facts, divide subjects into a+2 and a+3 M-Space groups, teach addition and subtraction word problems to one-half of each group via an a+2 strategy and the other half via an a+3 strategy and compare the a+3 strategy usage on delayed posttests.

## Subjects

The subjects were 139 children ranging in age from 72 months to 96 months in first grades at Independence Elementary School, Manassas Park Elementary School, and Conner Elementary School, located in Manassas Park, Virginia, a blue-collar working community in Northern Virginia. The subjects were both male and female. The reason for selecting this community was that many students from middle class schools have memorized addition and subtraction facts or have learned strategies to complete operations on whole numbers. Children who were identified as gifted or who were assigned to self-contained special education classrooms were not included, modifying the effects of extremes of intelligence.

## Instrumentation

Subjects in first-grade classrooms were administrated four non-commercial, locally devised tests to determine 1) who could count objects to sixteen, 2) who knew numeral-number correspondence to sixteen, 3) who had not memorized addition and subtraction facts to sixteen when presented orally, and 4) who had not memorized addition and subtraction facts to sixteen when presented visually. These tests were administered individually by an adult trainer.

To assess object counting, the subject was asked to match the numerals 1-16 with pictures containing from one to sixteen objects each.

To assess numeral-number correspondence, the subject was required to place in order cards containing the numerals 1-16. Criteria was 100% correct for each task. Knowledge of addition and subtraction facts to 16 was measured both visually and orally using ten addition and subtraction problems. For the orally-presented problems, second and third graders were timed to determine the average length of time required to complete the task if facts were memorized. One half of the visually-presented problems were written in a horizontal format (i.e., 7+4= ___) and one-half were presented in a vertical format (i.e., $\frac{\ \ 7}{+4}$). Criteria for inclusion in the study was 0% correct for both tasks. The criteria was set stringently to help insure that students being trained did not already have a competing knowledge base.

After this initial screening process, the 115 remaining subjects were administered tests that measure M-Space in young subjects--the Cucumber Test and the Backward Digit Span Test. The Backward Digit Span Test, a subtest of the Wechsler Intelligence Scale for Children, has been used by Scardamalia (1977) and, in a slightly revised form, by Case (1977). The Cucumber Test, devised by Avila (1974) and used by Case and Kurland (Note 7) consists of a picture of a figure which looks similar to a clown with different parts of his body shaded in various colors. The subject looks at the picture for five seconds after which he is shown another picture of the clown with no parts colored. His task is to indicate to the experimenter the parts that were colored. The test is devised so that two colored parts measure an M-Space of two, three colored parts measures an M-Space of three, etc. This test was administered twice to each subject while the Backward Digit Span Test was administered once.

Fourteen subjects who either moved or were ill during the testing, one subject whose M-Space equaled a+1, one subject who refused to complete the Cucumber Test, and subjects showing widely varying M-Space test results were eliminated from the study, leaving the 53 subjects to be trained. Subjects in each M-Space level were then randomly assigned to either an a+2 or an a+3 training group. (An a-priori task analysis of the schemes to be coordinated to solve addition and subtraction problems via the strategies taught plus a complete description of addition and subtraction training procedures will be outlined in the following section.)

During both the addition and subtraction training sessions, one problem was read to the subject to determine what spontaneous strategies were used. Next, the experimenter demonstrated the procedure to be used, the subject and the experimenter worked a problem together, and finally, administration of oral problems was begun. If the subject could not complete a problem correctly, the experimenter helped him. Training continued until the subject could correctly complete three problems in sequence using the strategy taught. During the training period, unifix cubes were used as manipulatives by both experimenter and subject.

After four to five weeks, a posttest containing eight problems, four addition and four subtraction, was individually administered to determine the strategy the subject would use. Additionally, three types of transfer problems were administered simultaneously: (1) a three-addend addition problem/two-subtrahend subtraction problem (i.e., A kindergarten class was collecting leaves. They collected 14, but three got crumpled up and then two blew away. How many were left?), (2) an addition and a subtraction problem using Cuisenaire rods, (subjects were trained to use these materials prior to posttesting), and (3) student generated addition and subtraction problems. All addition and subtraction problems were presented in random order.

Subjects who were classified as having an M-Space of a+2 received the M-Space measurements again at posttest time to determine whether their M-Space level had increased from a+2 to a+3.

## Task Analysis and Training Procedure

Various cognitive tasks have been analyzed in terms of the number and kinds of schemes necessary to complete the tasks. Ammon (1977) states that:

> the object of a task analysis is to set down a series of cognitive operations which, if followed step by step, would lead to successful performance of a particular task. In other words, it is like writing a computer program to solve some sort of problem. . . . Each of the steps [may be] analyzed further into the particular schemes that are required and the ways in which they are activated. (pp. 185-186)

An example of task analysis is presented on pages 33-34 of this paper. In a study of the information processing demand of combinatorial reasoning tasks, Scardamalia analyzed the schemes involved in each step of the problem. Other researchers have used similar procedures in analyzing other logical tasks. For example, Pascual-Leone (1976b) has analyzed the conservation task; Robbie Case (Note 3) listed the mental steps involved in executing two missing addend strategies--the subtraction strategy and the estimation-addition-verification strategy; and Relihan and Restaino (1976) investigated the effects of organizational training on the solving of class inclusion problems.

In the present analysis of addition and subtraction tasks, two strategies are advanced for each operation. One requires the coordination of the executive scheme plus three other schemes (a+3) and the other requires coordination of the executive scheme in addition to two other schemes (a+2).

A full task analysis attempts to list each step in the thinking process specifying the information which must be held in consciousness at each step. The point in the sequence which requires the greatest number of schemes to be coordinated represents the maximum amount of M-Space necessary to accomplish the task. The abbreviated analysis presented in this paper is similar to Scardamalia's procedure in that both merely list the greatest number of schemes to be coordinated without describing each separate step in the process.

Addition:

Problem: Bill had 5 frogs in a box and 4 more jumped in. How many frogs does he have now?

Situation: Word problem is read to the student. Unifix cubes are available for student to manipulate. An explanation of the objective of the game is presented.

For a+3 Schemes

Demonstration: Tell subjects that unifix cubes will be used to represent the frogs in the problem. Count out five unifix cubes and make them into a group. Give the conservation demonstration to show subject that "fiveness" is conserved no matter how the cubes are arranged. Place a counter on the group to help the subject keep in mind the number of cubes in the original group; then count the remaining four cubes beginning from five and ending at nine. Count cubes one at a time (i.e., 6, 7, 8, 9). Ask subject if the five frogs in the box have been accounted for (counted). Ask if the four additional frogs have been accounted for. Tell the subject that, after all members in both groups have been combined, the total is another way of naming the two groups.

Schemes: <u>Executive Scheme</u>: directs the listening to and the execution of the problem to find the total number of frogs.

<u>Figurative Scheme</u>: a scheme representing the fact that the numbers in the problem represent the parts (or subsets) of the whole.

<u>Figurative Scheme</u>: a scheme representing the following fact: for the numbers representing parts of the whole, each number representing members of one part may be "counted on" or incremented to the number representing the other part with each incrementation being inclusive of all prior members counted.

<u>Operative Scheme</u>: representing the rule: given parts (or subsets) of the whole, when all members of the parts are accounted for, the total is another name for the parts grouped together.

Abbreviated outlines of procedures and task analyses of a+2 addition, and a+2 and a+3 subtraction follow.

For a+2 Schemes

Demonstration: Count out a group of five cubes. Then count out a group of four cubes. Finally, count all the cubes beginning at one and ending at nine to find the total.

Schemes: <u>Executive Scheme</u>: (outlined previously)

Figurative Scheme: representing the fact that the numbers in the problem represent members of each group.

Operative Scheme: representing the rule: when all members of the groups are counted (from one to the total), the whole (or total) is described.

Subtraction:

Problem: Bill had 9 frogs in a box and 4 jumped out. How many are left?

Situation: Word problem is read to student. Unifix cubes are available to be manipulated.

For a+3 Schemes

Demonstration: Count out and place in a row nine unifix cubes. Separate the group of four which "jumped out". Snap apart the four unifix cubes as you count backward or decrement from the whole (i.e., snap 1--say 8, snap another--say 7, snap another--say 6, another--5). All frogs who jumped out are eliminated or gone. Each time one jumped out, we said the number which is left. Now that all four are gone, the number left is 5.

Schemes: Executive Scheme: directs the listening to and the execution of the problem to find the remaining part.

Figurative Scheme: representing the fact that one of the numbers represents the whole set and the other number represents one of the parts (the ones that jumped out).

Figurative Scheme: representing the fact that given a whole, one of the parts may be decremented from the whole, member by member. The number representing the part (or subset) remaining is inclusive of the remaining members.

Operative Scheme: representing the rule: given a whole, when one part is eliminated, the part remaining is the answer (the number left or the number in the other part).

For a+2 Schemes

Demonstration: After the problem is read, count out nine unifix cubes. Count out four cubes one by one and eliminate. Count the remaining cubes.

Schemes: Executive Scheme: (outlined previously)

Figurative Scheme: representing the fact that one of the numbers represents the whole set and the other number represents one of the parts (the ones that jumped out).

Operative Scheme: representing the rule: given a whole, when one part is eliminated, (scheme for this triggered directly from the field) the part which is left is the answer (the remainder).

Examples of training problems.

Addition:

Billy had a birthday party. He invited 8 boys and 7 girls. How many children were at his party?

The clown at the circus had 7 stars painted on his front and 7 stars painted on his back. How many stars were on the clown?

Subtraction:

Michael had a birthday party and he invited 15 friends. It started to rain so 8 of them left. How many friends stayed at the party?

Twelve Brownie Scouts went to McDonalds. Seven ordered milkshakes and the others ordered cokes. How many ordered cokes?

Examples of posttest questions.

Addition:

Jimmy and Teddy were playing basketball. Jimmy scored 8 points and Teddy scored 6 points. How many points did both boys score?

Karen has 7 crayons in her pencil case and 9 crayons in her desk. How many crayons does she have in all?

Subtraction:

Tim's mother bought 12 eggs at the store, but when she came home, she found that 3 eggs were broken. How many eggs did she have left to use?

Rover had a litter of 14 puppies. Five of the puppies were brown and the rest were black. How many were black?

Examples of transfer posttest questions.

Addend/Subtrahend:

The kindergarten class was collecting leaves for an art project. Jill brought 7 leaves, Bob brought 4 more, and David brought 2. How many leaves did they have in all?

Another kindergarten class did the same art project. They collected 14 leaves in all, but 3 got crumpled up. Then, 2 were lost. How many leaves were left for the children to use in the project?

Cuisenaire:

Jim had 9 cars. He gave 4 away to his friend David. How many cars does he have left?

Heather was helping to plant a flower garden. First she planted 7 daisies and then she planted 3 roses. How many flowers did Heather plant?

Student Generated:

Place unifix cubes and Cuisenaire rods in front of student. Tell student that he can make up an addition problem and use the cubes or rods or fingers or no materials to show how he would solve the problem.

Same procedure for subtraction problem.

To provide information for readers who wish to have additional information about subjects or about certain procedures used, the following section is included.

## Supplementary Information About Subjects and Procedures

Average ages. Table 1 illustrates the age means and standard deviations in months for the four training groups and for the subjects pretested for the study, but not included in it.

## Subjects Eliminated From Study

After administration of screening and pretest measures, 53 subjects remained for training. During the period of time after training and before posttesting (between 28 and 35 days) three subjects were removed from the study: one subject was placed in a self-contained special education class, one subject was taught the "count-on" strategy by a teacher, and one subject was taught the "count-on" strategy, as well as other strategies, by his parents.

## Subjects' M-Space Classifications

As previously stated, subjects with widely-varying responses on the three administrations of the M-Space Measures were eliminated from the study. It was desirable to classify subjects as to M-Space on the basis of consistent responses; however, using the criteria of "same M-Space classification on each test administration" would have eliminated too many subjects from the study. Therefore, subjects exhibiting minor inconsistencies were included.

Of the 26 subjects classified as having an M-Space of 2, twelve were scored as having an a+2 M-Space on all three test administrations; seven were scored as having an a+2 M-Space on two test administrations and an a+1 M-Space on one test administration; and seven were scored as having an a+2 M-Space on two test administrations and an a+3 M-Space on one test administration; of the 24 subjects classified as having an M-Space

Table 1

Age Means and Standard Deviations for
Pretested Subjects and for Training Groups

| Group | N | M | SD |
|---|---|---|---|
| Pretested Only | 89 | 6.65 mo. | .48 mo. |
| 2-2 | 13 | 6.76 mo. | .52 mo. |
| 2-3 | 13 | 6.54 mo. | .52 mo. |
| 3-2 | 11 | 6.74 mo. | .40 mo. |
| 3-3 | 13 | 6.69 mo. | .49 mo. |

of 3, thirteen were scored as having an a+3 M-Space on all three test administrations; and eleven were scored as having an a+3 M-Space on some administrations and an a+4 M-Space on other administrations. Only one subject classified as having an M-Space of 3 scored the following:

| | |
|---|---|
| First administration of Cucumber | 2 M-Space |
| Second administration of Cucumber | 4 M-Space |
| Backward Digit Span | 4 M-Space |

Spontaneous Strategies Used By Subjects

Just prior to training, all subjects were assessed to determine the kinds of strategies they used to solve problems. Strategies were classified into five different types: Count-all (requiring an a+2 M-Space), count-on (back) (requiring an a+3 M-Space), incorrect, not apparent, and other.

The count-all and count-on (back) strategies have been described previously. Examples of "incorrect" strategy usage involve giving no response or adding when subtraction should occur or manipulating numbers in a nonsensical manner; the label "strategy not apparent" is used to describe instances in which the subject gives a correct answer, but cannot detail the strategy used; and "other" strategy usage involves instances in which the subject uses a strategy other than the count-all or count-on (back) strategies to find a correct answer to the question.

The results are presented in Table 2.

Subjects Whose M-Space Increased From a+2 to a+3

At the time of posttesting, M-Space measurements were administered

Table 2

Number of Subjects Using
Each Spontaneous Strategy

| | Kind of Problem | |
|---|---|---|
| Strategy | Addition | Subtraction |
| Count-all | 33 | 43 |
| Count-on (back) | 1 | 0 |
| Incorrect | 9 | 3 |
| Not Apparent | 6 | 2 |
| Other | 1 | 2 |

Note. 53 subjects were trained for the study; however, three subjects were eliminated before posttesting. Data is presented for the 50 remaining subjects who were posttested.

to the 26 subjects who had been classified as having an M-Space of a+2. Three subjects were found to have increased their M-Space levels to a+3. One subject had received a+2 training and two subjects had received a+3 training. For purposes of analysis of results, these subjects were classified as belonging to the a+3 M-Space groups. One subject (KC) was placed in the 3-2 group and the other two subjects (RA and SM) were placed in the 3-3 group.

## Chapter 4

## RESULTS AND DISCUSSION

In this section of the paper, the results and a discussion of each research question will be presented. The first research question is the main focus of the study; however, five additional questions provoked by the M-Space theory and prior research about addition and subtraction are examined.

### M-Space/Strategy Match or Mismatch--Research Question 1

Research question one involves determining whether there would be a difference in subjects' a+3 M-Space responses on the posttest questions for two comparison groups: the 2-2's compared to the 2-3's and the 2-3's compared to the 3-3's.

To compare the 2-2 and 2-3 groups and the 2-3 and 3-3 groups on the eight posttest questions, a one-tailed Mann-Whitney test was utilized. In this test, samples from possibly different populations are tested to determine whether the experimenter can reject the null hypothesis that the two populations are the same. The approach used for execution of this test was to tabulate and rank the number of a+3 responses given by each subject in each group. Subsequently, the rankings were summed for each group (see Table 3 for results). Although the differences in the number of a+3 responses and the differences in rankings are apparent, the Mann-Whitney test clarifies the findings. Statistical analyses of

Table 3

Number of Responses and Sum of the Rankings of a+3 Responses on 8 Posttest Questions for 2-2 & 2-3 Groups and 2-3 & 3-3 Groups

| Group | Number of a+3 Responses for Each Group | Sum of the Rankings of Responses for Each Group |
|---|---|---|
| | 2-2 Group Compared to 2-3 Group | |
| 2-2 | 1 | 162 |
| 2-3 | 4 | 189 |
| | 2-3 Group Compared to 3-3 Group | |
| 2-3 | 4 | 107.5 |
| 3-3 | 66 | 243.5 |

<u>Note</u>. Total number of responses for each group = 104 (8 posttest questions x 13 subjects).

the comparisons of groups illustrate that there were significantly fewer a+3 responses by the 2-3 group than the 3-3 group (Mann-Whitney $\underline{T} = 16.5$; $\underline{w}_p = 27$; $\underline{p} < .001$), but there was no significant difference between the 2-2 group and the 2-3 group ($\underline{T} = 71$; $\underline{w}_p = 59$). This indicates that most subjects in the group having a higher M-Space (a+3) can, under the conditions of this study, perform tasks requiring an a+3 M-Space. Groups with an M-Space of only a+2 cannot perform a+3 tasks even if they are trained to do so.

The 3-2 group was included in the study, but for the purpose of this hypothesis, only the 2-3 group (taught by a strategy requiring more M-Space than subjects have available) was compared to groups in which a match existed, the 2-2 group and the 3-3 group.

Discussion of M-Space/strategy match-mismatch findings. As the reader will recall, the theory states that three prerequisites are necessary for problem solving to occur: (1) sufficient M-Space, (2) exposure to strategy-related experiences, and (3) a tendency to use M-Space to its capacity. As stated earlier, the data in this study suggest that training on a strategy above the subject's M-Space level (the 2-3 training group) does not lead to the coordination of the schemes which allow the a+2 subject to perform on an a+3 task. Prerequisite (1) sufficient M-Space has not been met, but prerequisite (2) exists, namely, exposure to the strategy-related experiences. Recently, however, Case (1979b) has suggested that some a+2 subjects can perform a+3 tasks by chunking two of the required schemes into one, thus reducing the a+3 requirement to a+2. The chunking can occur spontaneously by the individual or the subject may receive instruction which facilitates the coordination of schemes. From the theory and Case's data, it appears that for some subjects, an a+2 M-Space is sufficient (along with prerequisite (2)) to allow a subject to complete a+3 tasks. In fact, in other studies cited in this report, training provided overwhelmingly positive results. In contrast, for the 2-3 group in this study, there were only four a+3 responses out of a total of 104 possible responses to the eight posttest questions while there were 82 a+2 responses for this group. Clearly, the training procedure used in this study (which was an intensive one-to-one teaching experience in which subjects were trained to criterion and then were given a delayed posttest without practice in the interim) was not conducive to coordinating the necessary schemes to complete the counting-on (back) task. What accounts for the differences in these findings? Two possible reasons are the differences in training procedures and the differences in the criteria by which maintenance of the strategy is judged. In regard to the first reason, this training procedure did not attempt to task analyze counting-on (back) in order to lower the M-demand; however, ample opportunity was given in the training to criterion procedure for subjects to spontaneously "chunk" schemes. In reference to the second reason, the criteria for maintenance of the strategy was a four to five week delayed posttest. In some previously cited research, the subjects were posttested immediately after training. In three studies, however, a delayed posttest was included as a part of the procedure and in these cases, positive results were maintained. Therefore, these reasons do not provide unequivocal evidence to account for the differences in findings.

An additional issue must be mentioned. The assumption was made that the task analyses for the counting-all and counting-on (back) strategies were accurate. It is possible that this assumption is not correct, but this is unlikely for two reasons: First, the a+2 and a+3 M-Space levels correspond to the ages and the Piagetian developmental levels of typical first grade subjects; and second, empirical evidence and previous research shows that these particular strategies are learned in sequence and they are utilized by first graders.

Since the third prerequisite, the tendency to use M-Space to its capacity, was mentioned as being important in problem solving, it deserves some discussion. When the testing situation is structured so that certain factors (such as field factors) can interfere with maximum M-Space usage for individuals who are susceptible to them, this can affect efficient strategy usage. The factors can be activated when the individual must execute a structuring of the situation. In the problem solving situations in this experiment, the directions and procedures were structured by the examiner, therefore, the effect of field factors did not greatly affect M-Space utilization.

## Pattern of Responses on Posttest and Transfer Problems--Research Question 2

To investigate the second research question, whether the pattern of responses on transfer problems is similar to or different from the pattern which occurs on the eight posttest problems, we must review the data presented in Table 4. This table gives a comparison of the percentages of responses by strategy for the posttest and three types of transfer questions--addend/subtrahend, Cuisenaire, and student-generated questions.

For the eight posttest questions, percentages of a+3 responses were small and consistent for all groups except the 3-3 group. The percentage of a+3 responses for this group was 63.5% while for the other groups, percentages ranged between 1.0 and 3.8. In contrast, the a+2 responses for the same three groups were 78.8% or higher, while the percentage of a+2 responses for the 3-3 group was only 33.5.

Percentages are given, not only for a+2 and a+3 strategies, but also for five other strategies. Subjects did not always answer questions according to strategies on which they were trained; rather, sometimes they gave correct answers, but no strategy was apparent, and sometimes they used a correct "other" strategy, but it was not one on which they were trained. The reason for including the line-up strategy and memorization of facts strategy, special subcategories of "other" strategies, was because the line-up technique (i.e., a 7 rod and a 3 rod were lined-up together and placed next to the 10 rod forming a match, and thus the answer) was prevalent in solving Cuisenaire rod problems. The memorization of facts strategy was included because it was used often in student-generated problems.

Except for the strategies on which they were trained, subjects used only the incorrect strategy in attempting to solve problems (with the exception of about one percent of subject responses in the 2-3, 3-2, and

Table 4

Percentages of Responses by Strategy
for the Eight Posttest Questions and Transfer Questions[a]

| Group | Eight Posttest Problems | Addend/Subtrahend Problems | Addition & Subtraction Cuisenaire Problems | Addition & Subtraction Student-Generated Problems |
|---|---|---|---|---|
| 2-2 | | | | |
| 3-Strategy | 1.0 | 3.8 | 3.8 | 0.0 |
| 2-Strategy | 85.6 | 84.6 | 65.4 | 5.0 |
| Incorrect Strategy | 13.5 | 11.5 | 23.1 | 19.2 |
| No Strategy Apparent | 0.0 | 0.0 | 7.7 | 3.8 |
| Memorized Facts | 0.0 | 0.0 | 0.0 | 26.9 |
| 2-3 | | | | |
| 3-Strategy | 3.8 | 0.0 | 11.5 | 19.2 |
| 2-Strategy | 78.8 | 92.3 | 61.5 | 30.8 |
| Incorrect Strategy | 16.3 | 7.7 | 11.5 | 11.5 |
| No Strategy Apparent | 0.0 | 0.0 | 0.0 | 3.8 |
| Other Strategy | 1.0 | 0.0 | 15.4 | 0.0 |
| Memorized Facts | 0.0 | 0.0 | 0.0 | 34.6 |
| 3-2 | | | | |
| 3-Strategy | 2.3 | 9.0 | 0.0 | 13.4 |
| 2-Strategy | 87.5 | 90.0 | 86.4 | 72.7 |
| Incorrect Strategy | 9.0 | 0.0 | 4.5 | 0.0 |
| Other Strategy | 1.1 | 0.0 | 0.0 | 0.0 |
| Line-Up Strategy | 0.0 | 0.0 | 9.0 | 0.0 |
| Memorized Facts | 0.0 | 0.0 | 0.0 | 13.6 |
| 3-3 | | | | |
| 3-Strategy | 63.5 | 65.4 | 42.3 | 38.5 |
| 2-Strategy | 33.5 | 34.6 | 34.6 | 11.5 |
| Incorrect Strategy | 2.0 | 0.0 | 3.8 | 7.7 |
| Other Strategy | 1.0 | 0.0 | 3.8 | 19.2 |
| Line-Up Strategy | 0.0 | 0.0 | 15.4 | 0.0 |
| Memorized Facts | 0.0 | 0.0 | 0.0 | 23.0 |

*Note*. The number of subjects in the 2-2, 2-3, and 3-3 groups = 13. The number of subjects in the 3-2 group = 11.

[a]One addition and one subtraction problem for each of three different types of transfer problems were presented to subjects.

3-3 groups which fell into the category of other strategies). Additionally, the usage of incorrect strategies tended to decrease from the 2-2 group to the 3-3 group.

Table 4 also gives percentages for the various transfer problems. The responses to the addend/subtrahend transfer questions (which were included to determine whether adding another part or subset to an addition or subtraction problem affected responses, i.e., 7 + 5 = 12 is a two-addend problem, 7 + 4 + 3 = 14 is a three-addend problem) displayed similarity to the responses for the eight posttest questions with the only major difference being a smaller percentage of incorrect responses.

Cuisenaire rod questions were included to determine whether the subject would transfer the use of strategies from discrete materials to continuous materials. For example, when using unifix cubes, each cube equals one unit. To indicate the number 7, the subject must use seven cubes. When using Cuisenaire rods, the 7 rod (which is seven units long) is used to signify the number 7.

The pattern of responses was fairly consistent with the pattern noted for the posttest questions and the addend/subtrahend questions; however, the following points should be made: for the 2-2 group, there was an increase in the number of incorrect responses in comparison to the eight posttest questions and the addend/subtrahend questions. In the 2-3 group, there was an increase in the a+3 strategy responses and a corresponding decrease in a+2 strategy responses in comparison to the previous sets of questions. Both the 3-2 and 3-3 groups (those with an M-Space of a+3) utilized a "line-up" strategy not used by the a+2 groups.

Nine percent of the 3-2 group and 15.4% of the 3-3 group chose rods which represented the numbers in the problems and lined them up so that they could be compared to a rod which represented the answer.

Student-generated questions required that the subject encode a problem (make up a story problem) and then solve it. This procedure was a reversal of the decoding process which the subject had been using previously to answer the other questions. Many of the problems generated by the students were simple ones for which they had memorized facts. 26.9% of the 2-2 responses, 34.6% of the 2-3 responses, 13.6% of the 3-2 responses, and 23.0% of the 3-3 responses were memorized facts. The use of a+2 strategies for these problems fluctuated widely among groups with the 3-2 group showing the greatest percentage of use (72.7%). The a+3 strategy was used more frequently by the 3-3 group (38.5%) than by other groups, but in comparison to the posttest and the other transfer problems, the 2-3's and the 3-2's exhibited a slight increase in the use of the a+3 strategy.

Addition and Subtraction Sequential Differences--Research Question 3

Research by Steffe et al. (1979) about the developmental sequence of addition and subtraction suggests the need for validation studies; therefore, the posttest and transfer problems have been subdivided into addition and subtraction responses in order to answer the third research question, "When responses to addition and subtraction problems are viewed separately, do differences exist for (a) posttest problems, (b) transfer problems?" Examination of the eight posttest questions in

Table 5 for purposes of comparing addition and subtraction responses revealed that, for the 3-3 group, there were fewer a+3 subtraction responses than addition responses (26 versus 40) and more a+2 subtraction responses than addition responses (23 versus 12). For the other three groups, however, subjects gave more a+2 addition responses than subtraction responses.

Two-sided Sign tests (see Table 6) performed for each group to determine whether there were significant differences in a+2 responses on addition and subtraction posttest questions revealed no significant differences for any of the four groups; however, the 3-3 group response differences were close to the .05 significance level ($t$ = 11.7; $T$ = 12).

There were too few a+3 responses to test for the 2-2, 2-3, and 3-2 groups, but a two-sided Sign test was performed for the 3-3 group with the result that the hypothesis of no difference at the .05 level was not rejected ($n - t$ = 40.96; $T$ = 40), although it was close to the rejection level.

Discussion of the lag between counting-on and counting-back. The dependence of many subjects in the 3-3 group upon the a+2 strategy for subtraction in comparison to their using the a+3 strategy for addition lends credence to the results reported by Steffe et al. (1979). They observed that counting-back lagged behind counting-on and they reasoned that, "If a child has yet to learn counting-back without a tally, finding differences still requires a counting-all technique (when facts are not known automatically)" (p. 39). It seems as if this is what is happening with the 3-3 group. The training procedure in this study was didactic

Table 5

Number of Responses for a+3, a+2, Incorrect, and Other Strategies
Listed as Totals and Listed Separately by Addition and Subtraction Responses

| | Groups | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 2-2 | | | 2-3 | | | 3-2 | | | 3-3 | | |
| | Responses | | | | | | | | | | | |
| Strategies | Total | + | - | Total | + | - | Total | + | - | Total | + | - |
| 8 Posttest Questions | | | | | | | | | | | | |
| a+3 | 1 | 1 | 0 | 4 | 4 | 0 | 2 | 2 | 0 | 66 | 40 | 26 |
| a+2 | 89 | 47 | 42 | 82 | 44 | 38 | 77 | 42 | 35 | 35 | 12 | 23 |
| Incorrect | 14 | 4 | 10 | 18 | 4 | 14 | 8 | 0 | 8 | 2 | 0 | 2 |
| Ot. Strat. | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 1 | 0 | 1 |
| Total | 104 | | | 104 | | | 88 | | | 104 | | |
| Addend/Subtrahend Questions | | | | | | | | | | | | |
| a+3 | 1 | 1 | 0 | 0 | 0 | 0 | 2 | 1 | 1 | 17 | 9 | 8 |
| a+2 | 22 | 11 | 11 | 24 | 12 | 12 | 20 | 10 | 10 | 9 | 4 | 5 |
| Incorrect | 3 | 1 | 2 | 2 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| Ot. Strat. | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Total | 26 | | | 26 | | | 22 | | | 26 | | |
| Cuisenaire Rod Questions | | | | | | | | | | | | |
| a+3 | 1 | 1 | 0 | 3 | 3 | 0 | 0 | 0 | 0 | 11 | 9 | 2 |
| a+2 | 17 | 8 | 9 | 16 | 7 | 9 | 19 | 9 | 10 | 9 | 2 | 7 |
| Incorrect | 6 | 3 | 3 | 3 | 1 | 2 | 1 | 1 | 0 | 1 | 0 | 1 |
| No S. A. | 2 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Ot. Strat. | 0 | 0 | 0 | 4[a] | 2 | 2 | 2[b] | 1 | 1 | 5[c] | 2 | 3 |
| Total | 26 | | | 26 | | | 22 | | | 26 | | |
| Student-Generated Questions | | | | | | | | | | | | |
| a+3 | 0 | 0 | 0 | 5 | 4 | 1 | 3 | 1 | 2 | 10 | 7 | 3 |
| a+2 | 13 | 7 | 6 | 8 | 3 | 5 | 16 | 8 | 8 | 3 | 1 | 2 |
| Incorrect | 5 | 3 | 2 | 3 | 2 | 1 | 0 | 0 | 0 | 2 | 1 | 1 |
| Memo. Ft. | 7 | 2 | 5 | 9 | 4 | 5 | 3 | 2 | 1 | 6 | 2 | 4 |
| No S. A. | 1 | 1 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| Ot. Strat. | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 5 | 2 | 3 |
| Total | 26 | | | 26 | | | 22 | | | 26 | | |

Note. Ot. Strat. = Other Strategies No S. A. = No Strategy Apparent Memo. Ft. = Memorized Facts

[a] Includes one example of memorized facts

[b] Includes two examples of line-up strategy

[c] Includes four examples of line-up strategy

Table 6

Two-Sided Sign Test for Differences in Addition and Subtraction Responses for the 8 Posttest Questions

| Group | Strategy Responses | *n* | *t* | *n-t* | *T* |
|---|---|---|---|---|---|
| 2-2 | a+2 | 89 | 35.26 | 53.74 | 47 |
| 2-3 | a+2 | 82 | 32.13 | 49.87 | 44 |
| 3-2 | a+2 | 77 | 29.90 | 47.10 | 42 |
| 3-3 | a+2 | 35 | 11.70 | 23.30 | 12 |
| 3-3 | a+3 | 66 | 25.04 | 40.96 | 40 |
| 3-3 | a+3 | 66 | 26.34 | 39.66 | 40 |

*Note*. Significance level = .05

[a]One-sided Sign test performed on a+3 strategy responses

in that subjects were told that they should use the procedure taught; therefore, they learned the counting-on and counting-back without a running tally. Counting-on without a tally is much less awkward to apply than the counting-back procedure. To do the latter, a subject must first construct the total (using unifix cubes), then separate one part, and then count-back those members which were separated to find the remainder. For some problems, it seems easier, and in fact, is easier to merely count the remainder. When a subject reaches a level of ordinal counter in which he does not have to represent objects prior to counting, it becomes easier to use the counting-back technique.

Because there is evidence from the theory that more a+3 addition responses than subtraction responses are expected, a one-sided Sign test was also performed for the 3-3 group (see Table 6, $3\text{-}3^{a}$) with the result that the null hypothesis was rejected ($\underline{n} - \underline{t} = 39.66$; $\underline{T} = 40$; $p \leq .05$). There were significantly more a+3 addition responses than subtraction responses.

The addend/subtrahend questions and the Cuisenaire rod questions elicited fairly equal addition and subtraction responses for each strategy with the exception of the 3-3 group. That group exhibited a pattern of addition and subtraction responses on the Cuisenaire rod questions which was similar to the pattern exhibited on the eight post-test problems.

For the student-generated problems, the pattern was similar to the pattern observed in the other problems. There were few a+3 responses except for the 3-3 group, and also, there were fewer a+2 strategy responses in comparison to the other transfer problems. There was greater dependence upon memorized facts instead.

Incorrect Strategy Usage--Research Question 4

Table 5 also contains the data necessary to examine the fourth research question--whether, among the four groups, there are differences in the proportion of subjects using incorrect strategies.

Considering the eight posttest questions, we note that there is a range of incorrect strategy responses from a high of 18 for the 2-3 group to a low of two for the 3-3 group. A Median test was conducted to determine whether the proportion of subjects giving one or more incorrect strategy responses was the same for each group. Table 7 indicates both the number of subjects in each group who exceeded the grand median of subjects giving incorrect responses and the number who did not exceed the grand median. The Median test was chosen because, although the Kruskal-Wallis test is more powerful, "ranks may be assigned to the observations in many ways because of many apparently equal observations . . . [and] . . . the Kruskal-Wallis test may furnish a level of significance much different than the true level of significance" (Conover, 1971, p. 256).

Results indicated a significant difference at the .025 level ($\underline{T}$ = 11.07; 3 df; $\chi^2$ = 9.348). Since the Median test led to rejection of the null hypothesis, subgroups were analyzed to isolate differences. being cognizant that "repeatedly using the same test on subgroups of the original data always distorts the true level of significance of all tests

Table 7

Number of Subjects in the 2-2, 2-3, 3-2, and 3-3 Groups Who Exceed the Grand Median of Subjects Giving Incorrect Responses and the Number Who Are Less Than or Equal to the Grand Median

| | Groups | | | | |
|---|---|---|---|---|---|
| | 2-2 | 2-3 | 3-2 | 3-3 | Totals |
| $>$ Median | 5 | 9 | 8 | 2 | 24 |
| $\leq$ Median | 8 | 4 | 3 | 11 | 26 |
| Totals | 13 | 13 | 11 | 13 | 50 |

but the first . . . [the procedure was used] . . . as an objective 'yardstick' for separating the various populations" (Conover, 1971, p. 170). Differences were found between the 3-3 group and the 3-2 and 2-3 groups, but not between any other groups. Incorrect strategy responses on the eight posttest questions varied according to addition/subtraction operation across all four groups with there being more incorrect strategy usage on subtraction problems than addition problems (see Table 5). For the 2-2 group, 10 of 14 incorrect strategy responses were subtraction responses and for the 2-3 group, 14 of the 18 incorrect strategy responses were subtraction responses. All of the incorrect strategy responses given by the 3-2 and 3-3 groups were subtraction responses.

The three types of transfer questions had relatively few incorrect responses with most occurring in the 2-2 and 2-3 groups. For all groups, the incorrect strategy responses were evenly distributed between addition and subtraction operations. A classification of the kinds of incorrect strategies used by subjects for all posttest and transfer problems is as follows: Added when subtraction was appropriate--15, No strategy apparent, but incorrect answer given--11, Confused by problem, but tried to solve it and gave illogical response--31, and No response--11.

Take-Away Subtraction and Comparison Subtraction--Research Question 5

Steffe et al. (1979) indicate in their research that, when a subject can count-on and count-back with a running tally, he can understand and use the inverse operations of addition and subtraction. Being able to visualize the "whole" and understand how to manipulate the parts in an addition or subtraction problem frees the subject to explore relationships such as comparisons among parts. Steffe et al. believe that this happens only at the highest stage of counting--the "ordinal" level. Not having attained this last level does not prohibit a subject from solving "take-away" subtraction problems, but it is reasonable to assume that a subject who has not attained this level would answer implied "comparison" problems incorrectly more often than an ordinal counter. Since this study contained both types of subtraction problems (take-away and comparison) and since subjects in this study vary in their levels of counting, the question is asked: Are there differences in the incorrect responses for "take-away" and "comparison" subtraction problems?

To answer this question, we continue to analyze the data for incorrect responses. Viewing individual answers classified according to the strategies used allows the reader to determine which posttest subtraction problems were answered incorrectly. Appendix Tables A-H break down the responses by training group and further subdivide them into addition and subtraction problems. Then each of the eight posttest and transfer problems is distributed across strategies.

It is necessary to look at the eight posttest questions in Tables B, D, F, and H (because those are the only ones with "comparison" and "take-away" subtraction problems). For all four groups, one problem (#4), accounted for 24 of the 42 incorrect responses. The problem which states, "Rover had a litter of 14 puppies. Five of them were brown and the rest were black. How many were black?" is an implied comparison problem whereas the other subtraction problems are "take-away" problems. Although this study was originally intended to contain only "take-away" subtraction problems, two comparison subtraction problems were included in the training sessions. Therefore, it was feasible to include an example of this type of problem on the posttest to determine whether it affected responses. Since the one "comparison" subtraction problem accounted for 57% of the incorrect responses in comparison to only 43% for the three "take-away" problems, it is clear that this type of problem affects responses. These results also tend to support the counting sequence devised by Steffe, Richards, and von Glasersfeld and they raise the question of whether different levels of M-Space are required for these two types of problems.

Student-Generated Problems--Research Question 6

The focus of the sixth research question is to determine what kinds of addition and subtraction problems subjects generate. While investigating the second research question, we explored the strategies used; however, the intent of this discussion is to survey the kinds of problems generated.

Since subjects were trained on either a+2 or a+3 strategies, the expectation would be that subjects would use those strategies to solve the problems they generated. In certain cases, subjects did not utilize those strategies; rather, they reacted in a number of different ways. Some failed to dictate problems, one dictated a combination addition/

subtraction problem (5 + 2 - 3 = 4), while another supplied a "comparison" subtraction problem, and two subjects composed nonsensical problems. A number of subjects used number facts which they had memorized to solve the problems they generated. Tables A-H in the appendix allow us to survey the student-generated problems involving memorized facts over 10 (i.e., 8 + 5, 17 - 9) and those involving facts under 10 (i.e., 5 + 3, 9 - 6). Of the 25 responses in which a subject utilized addition and subtraction facts which he had memorized, 20 of those involved facts under 10. Evidently, when given instructions to generate a "story" problem, these students chose the easiest, or at least the problems with which they felt most secure, to answer the questions. By groups, the results for problems involving facts over 10 and under 10 are as follows: 2-2 group--one over, six under; 2-3 group--two over, seven under; 3-2 group--two over, one under; 3-3 group--zero over, six under.

Another interesting fact occurred as students generated their own stories. Although asked to tell an "adding" or addition story, some students reversed the order and communicated a subtraction problem instead. The results were as follows:

| | |
|---|---|
| 2-2 Group | 2 reversals |
| 2-3 Group | 5½ reversals |
| 3-2 Group | 9 reversals |
| 3-3 Group | 5 reversals |

(Note: the ½ in the 2-3 group indicates that the subject told a subtraction problem when asked for an addition problem, but gave no addition response).

The reasons for these reversals are unclear. Perhaps it is because the problem presented prior to this one was a subtraction problem (a Cuisenaire rod subtraction problem), perhaps subtraction problems are easier to envision and solve than addition problems, or perhaps these subjects do not know the difference between addition and subtraction.

## Additional Inquiries

In addition to examining the six research questions, results of three other inquiries are presented: the types of "other" strategies used by subjects, a teacher survey about classroom instruction, and the effects of training sessions.

Other strategies. In regard to the first inquiry, some subjects attempted to solve the problems with diverse strategies which allowed them to answer the problems correctly, but not via the strategy taught. For the eight posttest questions: two subjects used other strategies, one subject stacked and then compared the stacks of unifix cubes to find the answer, and the other subject worked the problem using the cubes and then divided the sum or the remainder of unifix cubes into subsets whose numerosity could be ascertained visually.

For the Cuisenaire rod problems, seven subjects gave the following eleven different "other" strategy responses: counting on fingers--two, using memorized facts--two, lining up rods--six, making small subsets of white Cuisenaire rods--one.

The subjects gave five "other" responses on the student-generated problems: lining up rods strategy--two, counting-on one number--one, using the commutative law in an addition problem--one, generating a comparative subtraction problem and solving it via a memorization strategy--one.

Teacher survey. A teacher survey was conducted for all six first grade math teachers at the end of the study to determine what addition and subtraction facts were taught, what strategies were used to help students memorize facts, and what strategies were used to help students understand the mathematical concepts of addition and subtraction. One teacher had taught addition and subtraction facts to seven, two had taught facts to nine, two had taught facts to 10, and one had taught facts to 18. The latter teacher was in the process of teaching these facts and students had not mastered them. It is possible, however, that her teaching had some effect on the number of "memorization" responses. Strategies used most often to help students memorize facts were drill, use of the number line, counting frames, flash cards, timed quizzes, and counting one number more or one number less. Strategies used most often to help students understand the addition and subtraction concepts were presentation of word problems, use of the number line, use of drill on the addition/subtraction inverse relationship, use of concrete materials, and use of students to dramatize situations.

Before the study began, it became evident that most subjects knew and were using the a+2 count-all strategy. Teachers were advised that the count-on strategy would be taught as the a+3 strategy and were asked to refrain from confounding the results by teaching that strategy in their classrooms. In only one instance did a teacher forget which resulted in the elimination of a subject from the study. The procedure of counting one more or one less to help students memorize facts was sufficiently different from the counting-on strategy so that it probably did not affect the results.

Effects of training sessions. The final inquiry involves determining whether the number of training sessions necessary to reach criterion affects performance on posttest problems. Analysis of the number of training sessions reveals the following:

(1) All subjects requiring more than one training session were either in the 2-3 or 3-3 groups.

(2) Two subjects, one 2-3 and one 3-3, had more than one training session for both addition and subtraction problems.

(3) Ten subjects, six 2-3's and four 3-3's, had more than one training session for subtraction problems only.

(4) Nine subjects had two training sessions; three subjects had three training sessions; and one subject had four training sessions. (Note: one subject had two training sessions for addition and three training sessions for subtraction).

Posttest results for addition problems for subjects receiving more than one training session: the 2-3 subjects answered via an a+2 strategy and the 3-3 subjects answered via an a+3 strategy. Evidently, the additional training did not have the effect of consolidating schemes so that the 2-3 subject could answer via an a+3 strategy.

Posttest results for subtraction problems for subjects receiving more than one training session: for the 2-3 group, all used an a+2 strategy except for one subject who answered all problems incorrectly. For those using the a+2 strategy, some answered problems #4 and #5 incorrectly. For the 3-3 group, one subject used an a+2 strategy, one used an a+3 strategy, and three used an a+2 strategy for some problems and an a+3 strategy for other problems.

A final comment: when subjects were compared group by group, there were few differences in the kinds of strategy responses between those who received only one training session and those who received more than one training session.

Chapter 5

# CONCLUSIONS AND IMPLICATIONS FOR FUTURE RESEARCH

Under the conditions of this study, the following results are enumerated:

(1) Subjects trained according to a strategy which was above their M-Space level did not retain and use the strategy.

(2) Subjects gave similar responses to transfer problems which differed from posttest problems due to the addition of another variable (addend/subtrahend problems) or due to a change in materials (Cuisenaire problems); however, there was somewhat more fluctuation in responses for student-generated problems. The last finding provides evidence that the factors involved in decoding problems may not be the same ones involved in encoding problems.

(3) Viewing a+2 strategy responses, there were no significant differences in addition and subtraction responses for the four groups, but viewing the a+3 strategy responses of the 3-3 group, there were significantly more a+3 addition responses than subtraction responses indicating a possible developmental asynchronism in the counting-on (back) strategies.

(4) Differences existed in incorrect strategy usage among the four groups, specifically, differences were found between the 3-3

group and the 3-2 and 2-3 groups.

(5) Subjects answered comparison subtraction problems incorrectly more often than take-away subtraction problems.

(6) When subjects were allowed to generate their own problems, they tended to devise simple problems and to solve them by using memorized number facts which were less than ten.

The technique utilized in this study involved training subjects to criterion and then posttesting at a later date with no provision for follow-up instruction prior to posttesting, a procedure used in many classrooms, especially for problem solving tasks. Since some teachers have little knowledge about how to teach problem solving strategies and since many mathematics texts currently in use do not devote a great deal of time to the topic, parallels exist between this training methodology and the occurrences in some mathematics classrooms. This is the scenario: students are drilled on a single procedure until they can correctly complete a worksheet. There is no subsequent discussion nor practice provided until the student takes the chapter test.

If it can be assumed that the training procedure presented is analogous to classroom instruction, the implication for teachers is that they will have to become decision-makers about the approach to teaching addition and subtraction skills which develop in sequence. The results indicate that the didactic training approach will work for subjects with an M-Space of a+3, but does not facilitate the use of the counting-on (back) strategy in subjects with an M-Space of a+2. Therefore, other choices must be made--either the teacher must devise teaching strategies which reduce the amount of M-Space necessary to execute the task or the teacher must teach other mathematics subject matter which requires only an M-Space of a+2 to those students who possess that level of M-Space. Since there is evidence that M-Space reduction through task analysis and teaching techniques that foster "chunking" allows the individual to solve problems that would normally be above his M-Space capacity, this information should be kept in mind when planning instruction. Diagnosing the child's M-Space level, task analyzing the requirements of addition and subtraction problems, and varying techniques and materials would appear to be useful in planning better instruction.

It is easy to list these suggestions for applying a developmentally-based theory of instruction, but teachers must understand the demands and feel competent to fulfill them. In regard to diagnosing M-Space, some of the tests available are relatively quick and easy to administer, but the teacher must be trained to administer them and to score the results properly to guarantee accurate information. Additionally, a teacher may not think that it is necessary to assess everyone in the class--M-Space assessment should be only one part of student diagnosis. Task analysis which was also suggested can be laborious and difficult. A teacher does not have to use the formal procedure outlined in this study, however. He can observe the procedure students use, listen to their explanations to determine where difficulties arise, and provide an alternative to the incorrect strategies which the students are utilizing.

This particular procedure is not the only one recommended. Case has also provided three principles for better planning of instruction (described previously on page 35 of this paper). No matter which procedure is used, one must weigh whether the cost in time for diagnosing and planning is worth the results. Additionally, one must consider whether training for specific scheme coordination will promote transfer to other tasks. For mathematics subject matter that follows a sequence and for children who are developing normally, the best policy is to teach the material when the child is developmentally ready to learn it. According to Case (1978b), the real usefulness of this developmental approach to planning instruction is in its application for children who are mentally retarded, handicapped, culturally different, or who need remedial work in particular subject areas.

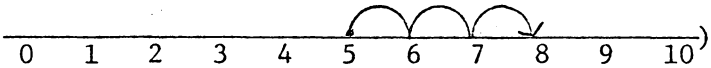
While the intent of this study was to explore and extend knowledge of the M-Space construct by using a didactic "training to criteria" approach applied to two strategies which develop sequentially, the educational implications mentioned above are worthwhile to ponder.

Clearly, this is only one method for studying the problem and because of this research, suggestions for future methodology become more apparent. First, the lengths of training sessions and lengths of time between training sessions can be varied. Second, after training to criterion, subjects can be grouped so that some receive additional instruction between training and posttesting. This will aid in determining the effect of practice upon coordination of schemes. Third, different kinds of training procedures can be utilized. For example, some subjects may model subjects above their M-Space levels, others may be taught via demonstration and rule learning, including how to coordinate and perhaps, chunk the schemes specific to the task. This latter suggestion is not merely focusing upon the items of information to be coordinated, but it also involves facilitating appropriate executive scheme usage.

The role of the executive scheme requires more attention and greater study than it has received in the past. Essentially, executive schemes mobilize mental energy to relevant schemes; therefore, before operations can occur and before schemes can be coordinated, the appropriate executive scheme must be brought to bear upon the situation. Future research should be conducted to determine how the subject acquires the appropriate executive scheme, the role which the executive scheme plays in facilitating coordination of other schemes, and the ways in which other metaconstructs interact with executive schemes.

Various metaconstructs which act as scheme boosters possibly affect the particular executive schemes which a subject uses. Two well known scheme boosters, the F operator (accounts for the effects of field factors) and the Á operator (accounts for the effects of affective factors) have already been described elsewhere in this paper as having a part in determining the activation weights of the schemes which the subject coordinates to solve a problem. It is possible that these metaconstructs not only serve that function, but also affect the executive scheme which determines how the subject approaches a particular task.

Fourth, in conjunction with the suggestion to study the functioning of the executive scheme, task conceptualization is a variable in need of investigation. For example, addition may be viewed as either set combination or as measurement (i.e., number line, 5 + 3




). It would be informative to task analyze the schemes necessary to answer addition problems conceptualized as measurement problems in addition to the task analysis already formulated, train subjects according to the two methods, and then determine if there are differences in subject responses.

The problems themselves warrant additional attention, first in terms of numbers in the problems, and second in terms of the semantics of the problems. In regard to the numbers, more systematic presentation of problems with large numbers to count-on/count-back (i.e., 8 + 7, 16 - 9) versus small numbers (i.e., 9 + 4, 11 - 3) should be made to determine if one strategy tends to be used for large count-on's and another strategy tends to be used for the small count-on's. Focusing upon semantics, Greeno states that "There are three kinds of situations in which students must decide to add: (a) increases in some quantity, as in 'Sue had three marbles, Nancy gave her five more,' (b) combinations of two sets that remain distinct, as in 'Sue has three marbles, Nancy has five, how many do they have altogether?' and (c) comparative specifications, as in 'Sue has three marbles, and Nancy has five more marbles than Sue has. How many marbles does Nancy have?'" (1979, p. 165).

In this study, the difference in one kind of subtraction problem (the implied comparative problem) led to different responses among subjects causing them to answer incorrectly more often than they did for "take-away" subtraction problems. By studying the responses elicited by the different types of addition and subtraction problems described above, investigators will be able to gain information about the semantic processes required to answer the questions and determine whether these kinds of problems fit into the developmental sequence of counting designed by Steffe et al. Greeno has stated that there are three kinds of situations in which students must decide to add and there may well be other situations, also. The point is this: in future studies, these different kinds of problems should be analyzed to determine how they fit the sequence, how much M-Space is required, and what kinds of executive schemes are necessary for their execution.

Another interesting issue for study is how the counting and beginning addition and subtraction data (Steffe et al., 1979) may continue to be combined with Case's theory. In particular, the developmental sequences of counting-all, counting-on (back) <u>without</u> a tally, and counting-on (back) <u>with</u> a tally should be extended. Unfortunately, in this study, there was no "count-on (back) with a running tally" training, only training involving counting-all and counting-on (back) <u>without</u> a tally. It is possible that some subjects were capable of counting-on (back) <u>with</u> a tally, but it cannot be ascertained from the data collected. To determine whether subjects need greater M-Space to be able to count-on (back) <u>with</u> a tally, an additional training component should be included.

A possible limitation of the study is the fact that most subjects already possessed the a+2 strategy (the counting-all strategy) on which two of the groups were trained. Due to this, the effect of training a subject with an a+3 M-Space on a strategy that requires only an a+2 M-Space could not be adequately answered in the present study since it is probable that most subjects acquired the a+2 strategy while they were at an a+2 M-Space level. If replicating this kind of study, a researcher should choose another logical task with which the subject is not familiar, but which is solvable via either an a+2 or an a+3 strategy. Then it may be possible to determine whether the results would differ from the results in the present study.

Other limiting factors are the number of subjects who were eliminated which resulted in a small number of subjects in each group and the fairly homogeneous socio-economic status of subjects. A larger sample with a more diverse SES, perhaps spanning kindergarten through second grade, would provide more data and allow for greater generalization of the conclusions.

One methodological limitation involves the time lapse between screening, pretesting, training, and posttesting. A problem occurred in this study when some subjects changed developmental levels, necessitating a decision about how to classify a subject who was an a+2 when the study began, but developed into an a+3 prior to posttesting.

Another methodological change which, if included in future studies, should elicit additional data involves utilizing a "reminder" component prior to posttesting. In this study, subjects were asked to perform a task which they had not practiced for approximately four weeks. It is probable that the competence level existed for some subjects, but they needed additional exposure or re-exposure to strategy-related experiences. An analogous situation exists when adults are competent to perform complicated mathematical procedures, but must be reminded of, or must research the appropriate formulas before they can be used successfully. This raises the issue of mastery; that is, what performance level do we deem necessary for successful accomplishment of mathematics tasks?

In view of the limitations of this study and the suggestions for future research, it is apparent that additional study of the M-Space construct is warranted. In particular, if the information gleaned from research is beneficial to mathematics classroom teachers, the research will have a practical as well as a theoretical purpose.

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APPENDIX

Table A

2-2 Group Responses Classified According
to Subject and Strategy For Addition Problems

Strategy / Problems

| Subject | a + 2: 1 | 3 | 6 | 8 | A | C | SG | a + 3: 1 | 3 | 6 | 8 | A | C | SG |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| TB | 1 | 1 | 1 | 1 | 1 | | | | | | | | | |
| SG | | 1 | 1 | 1 | | | | 1 | | | | 1 | | |
| LR | 1 | | | | | | | | | | | | | |
| RP | | 1 | 1 | 1 | 1 | | 1 | | | | | | | |
| LT | 1 | 1 | 1 | 1 | 1 | | | | | | | | 1 | |
| MW | 1 | 1 | 1 | 1 | 1 | 1 | | | | | | | | |
| BS* | 1 | 1 | 1 | 1 | 1 | 1 | | | | | | | | |
| TT* | 1 | 1 | 1 | 1 | 1 | | 1 | | | | | | | |
| AT | 1 | 1 | 1 | 1 | 1 | 1 | 1 | | | | | | | |
| MR | 1 | 1 | 1 | 1 | 1 | 1 | 1 | | | | | | | |
| BH | 1 | 1 | 1 | 1 | 1 | 1 | 1 | | | | | | | |
| SC | 1 | 1 | 1 | 1 | 1 | 1 | 1 | | | | | | | |
| KD | 1 | 1 | 1 | 1 | 1 | 1 | 1 | | | | | | | |
| Summary | 11 | 12 | 12 | 12 | 11 | 8 | 7 | 1 | | | | 1 | 1 | |

| Subject | Incorrect: 1 | 3 | 6 | 8 | A | C | SG | Other: 1 | 3 | 6 | 8 | A | C | SG |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| TB | | | | | | | 1 | | | | | | | |
| SG | | | | | | | | | | | | | | |
| LR | | 1 | 1 | 1 | 1 | 1 | 1 | | | | | | | |
| RP | 1 | | | | | 1 | | | | | | | | |
| LT | | | | | | | | | | | | | | |
| MW | | | | | | | 1 | | | | | | | |
| BS* | | | | | | | | | | | | | | |
| TT* | | | | | | 1 | | | | | | | | |
| AT | | | | | | | | | | | | | | |
| MR | | | | | | | | | | | | | | |
| BH | | | | | | | | | | | | | | |
| SC | | | | | | | | | | | | | | |
| KD | | | | | | | | | | | | | | |
| Summary | 1 | 1 | 1 | 1 | 1 | 3 | 3 | | | | | | | |

| Subject | Not Apparent: 1 | 3 | 6 | 8 | A | C | SG | Memorization: 1 | 3 | 6 | 8 | A | C | SG |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| TB | | | | | | 1 | | | | | | | | |
| SG | | | | | | | | | | | | | | $1^{o}$ |
| LR | | | | | | | | | | | | | | |
| RP | | | | | | | | | | | | | | |
| LT | | | | | | | | | | | | | | $1^{u}$ |
| MW | | | | | | | | | | | | | | |
| BS* | | | | | | | | | | | | | | |
| TT* | | | | | | | 1 | | | | | | | |
| AT | | | | | | | | | | | | | | |
| MR | | | | | | | | | | | | | | |
| BH | | | | | | | | | | | | | | |
| SC | | | | | | | | | | | | | | |
| KD | | | | | | | | | | | | | | |
| Summary | | | | | | 1 | 1 | | | | | | | 2 |

Note. The following abbreviations appear in the table:
u-Problem with facts under 10 A-Addend problem SG-Student generated problem
o-Problem with facts over 10 C-Cuisenaire problem

[a] * indicates reversal of addition and subtraction student-generated problems

Table B

2-2 Group Responses Classified According to Subject and Strategy for Subtraction Problems

Strategy

| | a + 2 | | | | | | | a + 3 | | | | | | | Incorrect | | | | | | | Other | | | | | | | Not Apparent | | | | | | | Memorization | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Problems | 2 | 4 | 5 | 7 | S | C | SG | 2 | 4 | 5 | 7 | S | C | SG | 2 | 4 | 5 | 7 | S | C | SG | 2 | 4 | 5 | 7 | S | C | SG | 2 | 4 | 5 | 7 | S | C | SG | 2 | 4 | 5 | 7 | S | C | SG |
| Subject | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| TB | | | | | | | | | | | | | | | 1 | 1 | 1 | 1 | 1 | | | | | | | | | | | | | | | 1 | | | | | | | | $1^u$ |
| SG | 1 | 1 | 1 | 1 | 1 | 1 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | $1^u$ |
| LR | 1 | | | | | | | | | | | | | | | 1 | 1 | 1 | 1 | 1 | 1 | | | | | | | | | | | | | | | | | | | | | |
| RP | 1 | | 1 | 1 | 1 | 1 | | | | | | | | | | 1 | | | | | | | | | | | | | | | | | | | | | | | | | | $1^u$ |
| LT | 1 | 1 | 1 | 1 | 1 | 1 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | $1^u$ |
| MW | 1 | 1 | 1 | 1 | 1 | | | | | | | | | | | | | | | 1 | 1 | | | | | | | | | | | | | | | | | | | | | |
| BS* | 1 | | 1 | 1 | 1 | 1 | | | | | | | | | | 1 | | | | | | | | | | | | | | | | | | | | | | | | | | $1^u$ |
| TT* | 1 | 1 | 1 | 1 | 1 | 1 | 1 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| AT | 1 | 1 | 1 | 1 | 1 | 1 | 1 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| MR | 1 | 1 | 1 | 1 | 1 | 1 | 1 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| BH | 1 | 1 | 1 | 1 | 1 | 1 | 1 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| SC | 1 | 1 | 1 | 1 | 1 | 1 | 1 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| KD | 1 | | 1 | 1 | 1 | | 1 | | | | | | | | | 1 | | | | 1 | | | | | | | | | | | | | | | | | | | | | | |
| Summary | 12 | 8 | 11 | 11 | 11 | 9 | 6 | | | | | | | | 1 | 5 | 2 | 2 | 2 | 3 | 2 | | | | | | | | | | | | | 1 | | | | | | | | 5 |

Note. The following abbreviations appear in the table:
u-Problem with facts under 10
o-Problem with facts over 10
S-Subtrahend problem
C-Cuisenaire problem
SG-Student generated problem

[a] * indicates reversal of addition and subtraction student-generated problems

Table C

2-3 Group Responses Classified According to Subject and Strategy for Addition Problems

| | Strategy: a + 2 | | | | | | | a + 3 | | | | | | | Incorrect | | | | | | | Other | | | | | | | Not Apparent | | | | | | | Memorization | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Problems | 1 | 3 | 6 | 8 | A | C | SG | 1 | 3 | 6 | 8 | A | C | SG | 1 | 3 | 6 | 8 | A | C | SG | 1 | 3 | 6 | 8 | A | C | SG | 1 | 3 | 6 | 8 | A | C | SG | 1 | 3 | 6 | 8 | A | C | SG |
| Subject | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| DM | 1 | 1 | 1 | 1 | 1 | | | | | | | | 1 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | $1^o$ |
| JH | | 1 | 1 | 1 | 1 | | | 1 | | | | | 1 | 1 | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| PC | 1 | 1 | 1 | 1 | 1 | | | | | | | | | | | | | | | 1 | 1 | | | | | | | | | | | | | | | | | | | | | |
| CD | 1 | 1 | 1 | 1 | 1 | 1 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | $1^u$ |
| JM | 1 | | 1 | | 1 | | 1 | | 1 | | 1 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | 1 | |
| GS | 1 | 1 | 1 | 1 | 1 | | | | | | | | 1 | 1 | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| DR | 1 | 1 | 1 | 1 | 1 | 1 | | | | | | | | 1 | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| MP | 1 | 1 | 1 | 1 | 1 | 1 | 1 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| SJ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| TN | 1 | 1 | 1 | 1 | 1 | 1 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | $1^u$ |
| DL | | 1 | 1 | 1 | 1 | 1 | | 1 | | | | | | 1 | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| CS | 1 | 1 | 1 | 1 | 1 | 1 | | | | | | | | | | | | | | | 1 | | | | | | | | | | | | | | | | | | | | | $1^u$ |
| DD | | | | | | | | | | | | | | | 1 | 1 | 1 | 1 | 1 | | | | | | | | 1 | | | | | | | | | | | | | | | |
| Summary | 10 | 11 | 12 | 11 | 12 | 7 | 3 | 2 | 1 | | 1 | | 3 | 4 | 1 | 1 | 1 | 1 | 1 | 1 | 2 | | | | | | 1 | | | | | | | | | | | | | | 1 | 4 |

Note. The following abbreviations appear in the table:
u-Problem with facts under 10 A-Addend problem SG-Student generated problem
o-Problem with facts over 10 C-Cuisenaire problem

[a] * indicates reversal of addition and subtraction student-generated problems

Table D

2-3 Group Responses Classified According to Subject and Strategy for Subtraction Problems

Strategy (Problems), part 1: a + 2, a + 3, Incorrect

| Subject | a + 2: 2 | 4 | 5 | 7 | S | C | SG | a + 3: 2 | 4 | 5 | 7 | S | C | SG | Incorrect: 2 | 4 | 5 | 7 | S | C | SG |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| DM | 1' | 1 | 1 | 1 | 1 | 1 | | | | | | | | 1 | | | | | | | |
| JH | 1 | 1 | 1 | 1 | 1 | 1 | 1 | | | | | | | | | | | | | | |
| PC | 1 | | 1 | 1 | 1 | | | | | | | | | | | 1 | | | | 1 | |
| CD * | 1 | 1 | 1 | 1 | 1 | 1 | 1 | | | | | | | | | | | | | | |
| JM | 1 | | 1 | 1 | 1 | 1 | | | | | | | | | | 1 | | | | | 1 |
| GS | 1 | 1 | 1 | 1 | 1 | 1 | | | | | | | | | | | | | | | |
| DR * | 1 | | 1 | | 1 | | 1 | | | | | | | | | 1 | | 1 | | | |
| MP | 1 | | 1 | 1 | 1 | 1 | 1 | | | | | | | | | 1 | | | | | |
| SJ * | 1 | | | 1 | 1 | 1 | 1 | | | | | | | | | 1 | 1 | | | | |
| TN | 1 | | 1 | 1 | 1 | 1 | | | | | | | | | | 1 | | | | | |
| DL | 1 | | 1 | 1 | 1 | | | | | | | | | | | 1 | | | | 1 | |
| CS * | 1 | | 1 | 1 | 1 | 1 | | | | | | | | | | 1 | | | | | |
| DD | | | | | | | | | | | | | | | 1 | 1 | 1 | 1 | 1 | | |
| Summary | 12 | 4 | 11 | 11 | 12 | 9 | 5 | | | | | | | 1 | 1 | 9 | 2 | 2 | 1 | 2 | 1 |

Strategy (Problems), part 2: Other, Not Apparent, Memorization

| Subject | Other: 2 | 4 | 5 | 7 | S | C | SG | Not Apparent: 2 | 4 | 5 | 7 | S | C | SG | Memorization: 2 | 4 | 5 | 7 | S | C | SG |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| DM | | | | | | | | | | | | | | | | | | | | | |
| JH | | | | | | | | | | | | | | | | | | | | | |
| PC | | | | | | | | | | | | | | | | | | | | | $1^u$ |
| CD * | | | | | | | | | | | | | | | | | | | | | |
| JM | | | | | | | | | | | | | | | | | | | | | |
| GS | | | | | | | | | | | | | | 1 | | | | | | | |
| DR * | | | | | | | 1 | | | | | | | | | | | | | | |
| MP | | | | | | | | | | | | | | | | | | | | | |
| SJ * | | | | | | | | | | | | | | | | | | | | | |
| TN | | | | | | | | | | | | | | | | | | | | | $1^u$ |
| DL | | | | | | | | | | | | | | | | | | | | | $1^u$ |
| CS * | | | | | | | | | | | | | | | | | | | | | $1^o$ |
| DD | | | | | | | 1 | | | | | | | | | | | | | | $1^u$ |
| Summary | | | | | | | 2 | | | | | | | 1 | | | | | | | 5 |

Note. The following abbreviations appear in the table:
u-Problem with facts under 10 S-Subtrahend problem SG-Student generated problem
o-Problem with facts over 10 C-Cuisenaire problem

[a] * indicates reversal of addition and subtraction student-generated problems

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Table E
3-2 Group Responses Classified According
to Subject and Strategy for Addition Problems

| | Strategy | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | a + 2 | | | | | | | a + 3 | | | | | | | Incorrect | | | | | | | Other | | | | | | | Not Apparent | | | | | | | Memorization | | | | | | |
| | Problems | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| | 1 | 3 | 6 | 8 | A | C | SG | 1 | 3 | 6 | 8 | A | C | SG | 1 | 3 | 6 | 8 | A | C | SG | 1 | 3 | 6 | 8 | A | C | SG | 1 | 3 | 6 | 8 | A | C | SG | 1 | 3 | 6 | 8 | A | C | SG |
| Subject | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| TL* | 1 | 1 | 1 | 1 | 1 | 1 | 1 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| OA | 1 | 1 | 1 | 1 | 1 | 1 | 1 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| GS | 1 | 1 | 1 | 1 | 1 | | | | | | | | | | | | | | | | | | | | | | 1 | | | | | | | | | | | | | | | $1^{o}$ |
| CC* | 1 | 1 | 1 | 1 | 1 | 1 | 1 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| SM | 1 | 1 | 1 | 1 | | 1 | 1 | | | | | 1 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| PR | 1 | 1 | 1 | 1 | 1 | 1 | | | | | | | | 1 | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| JW* | | | 1 | 1 | 1 | 1 | | 1 | 1 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | $1^{o}$ |
| NB | 1 | 1 | 1 | 1 | 1 | 1 | 1 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| KC* | 1 | 1 | 1 | 1 | 1 | 1 | 1 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| SF | 1 | 1 | 1 | 1 | 1 | | 1 | | | | | | | | | | | | | 1 | | | | | | | | | | | | | | | | | | | | | | |
| MB | 1 | 1 | 1 | 1 | 1 | 1 | 1 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| Summary | 10 | 10 | 11 | 11 | 10 | 9 | 8 | 1 | 1 | | | 1 | | 1 | | | | | | 1 | | | | | | | 1 | | | | | | | | | | | | | | | 2 |

Note. The following abbreviations appear in the table:
u-Problem with facts under 10
o-Problem with facts over 10
A-Addend problem
C-Cuisenaire problem
SG-Student generated problem

[a] * indicates reversal of addition and subtraction student-generated problems

Table F

3-2 Group Responses Classified according
to Subject and Strategy for Subtraction Problems

| | Strategy | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | a + 2 | | | | | | | a + 3 | | | | | | | Incorrect | | | | | | | Other | | | | | | | Not Apparent | | | | | | | Memorization | | | | | | |
| | Problems | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| Subject | 2 | 4 | 5 | 7 | S | C | SG | 2 | 4 | 5 | 7 | S | C | SG | 2 | 4 | 5 | 7 | S | C | SG | 2 | 4 | 5 | 7 | S | C | SG | 2 | 4 | 5 | 7 | S | C | SG | 2 | 4 | 5 | 7 | S | C | SG |
| TL* | 1 | | 1 | 1 | 1 | 1 | 1 | | | | | | | | | 1 | | | | | | | | | | | | | | | | | | | | | | | | | | |
| OA | 1 | 1 | 1 | 1 | 1 | 1 | 1 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| GS | 1 | 1 | 1 | 1 | 1 | | | | | | | | | | | | | | | | | | | | | | | 1 | | | | | | | | | | | | | | 1[u] |
| CC* | 1 | | 1 | 1 | 1 | 1 | 1 | | | | | | | | | 1 | | | | | | | | | | | | | | | | | | | | | | | | | | |
| SM | 1 | | 1 | 1 | | 1 | 1 | | | | | 1 | | | | 1 | | | | | | | | | | | | | | | | | | | | | | | | | | |
| PR | 1 | 1 | 1 | 1 | 1 | 1 | | | | | | | | 1 | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| JW* | 1 | | 1 | 1 | 1 | 1 | | | | | | | | 1 | | 1 | | | | | | | | | | | | | | | | | | | | | | | | | | |
| NB | | | 1 | 1 | 1 | 1 | 1 | | | | | | | | | 1 | | | | | | 1 | | | | | | | | | | | | | | | | | | | | |
| KC* | 1 | | 1 | 1 | 1 | 1 | 1 | | | | | | | | | 1 | | | | | | | | | | | | | | | | | | | | | | | | | | |
| SF | 1 | | 1 | 1 | 1 | 1 | 1 | | | | | | | | | 1 | | | | | | | | | | | | | | | | | | | | | | | | | | |
| MB | 1 | | 1 | 1 | 1 | 1 | 1 | | | | | | | | | 1 | | | | | | | | | | | | | | | | | | | | | | | | | | |
| Summary | 10 | 3 | 11 | 11 | 10 | 10 | 8 | | | | | 1 | | 2 | | 8 | | | | | | 1 | | | | | | 1 | | | | | | | | | | | | | | 1 |

Note. The following abbreviations appear in the table:
u-Problem with facts under 10
o-Problem with facts over 10
S-Subtrahend problem
C-Cuisenaire problem
SG-Student generated problem

[a] * indicates reversal of addition and subtraction student-generated problems

Table G

3-3 Group Responses Classified According
to Subject and Strategy for Addition Problems

| | Strategy | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | a + 2 | | | | | | | a + 3 | | | | | | | Incorrect | | | | | | | Other | | | | | | | Not Apparent | | | | | | | Memorization | | | | | | |
| | Problems | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| Subject | 1 | 3 | 6 | 8 | A | C | SG | 1 | 3 | 6 | 8 | A | C | SG | 1 | 3 | 6 | 8 | A | C | SG | 1 | 3 | 6 | 8 | A | C | SG | 1 | 3 | 6 | 8 | A | C | SG | 1 | 3 | 6 | 8 | A | C | SG |
| JS | | | | | | | | 1 | 1 | 1 | 1 | 1 | 1 | 1 | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| TD | 1 | 1 | 1 | 1 | 1 | 1 | | | | | | | | | | | | | | | 1 | | | | | | | | | | | | | | | | | | | | | |
| SM | 1 | | | | | | | | 1 | 1 | 1 | 1 | 1 | 1 | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| TW* | 1 | 1 | 1 | 1 | 1 | 1 | | | | | | | | | | | | | | | | | | | | | | 1 | | | | | | | | | | | | | | |
| RA * | | | | | | | | 1 | 1 | 1 | 1 | 1 | 1 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | $1^u$ |
| AG | | | | | | | | 1 | 1 | 1 | 1 | 1 | | | | | | | | | | | | | | | 1 | | | | | | | | | | | | | | | $1^u$ |
| BH * | | 1 | 1 | 1 | 1 | | | 1 | | | | | 1 | 1 | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| RE * | | | | | | | | 1 | 1 | 1 | 1 | 1 | | | | | | | | | | | | | | | 1 | 1 | | | | | | | | | | | | | | |
| JM | | | | | 1 | | 1 | 1 | 1 | 1 | 1 | | 1 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| NW | | | | | | | | 1 | 1 | 1 | 1 | 1 | 1 | 1 | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| BM | | | | | | | | 1 | 1 | 1 | 1 | 1 | 1 | 1 | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| MC * | | | | | | | | 1 | 1 | 1 | 1 | 1 | 1 | 1 | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| LK | | | | | | | | 1 | 1 | 1 | 1 | 1 | 1 | 1 | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| Summary | 3 | 3 | 3 | 3 | 4 | 2 | 1 | 10 | 10 | 10 | 10 | 9 | 9 | 7 | | | | | | | 1 | | | | | | 2 | 2 | | | | | | | | | | | | | | 2 |

Note. The following abbreviations appear in the table:
u-Problem with facts under 10
o-Problem with facts over 10
A-Addend problem
C-Cuisenaire problem
SG-Student generated problem

[a] * indicates reversal of addition and subtraction student-generated problems

Table H

3-3 Group Responses Classified According
to Subject and Strategy for Subtraction Problems

Strategy (Problems)

| Subject | a + 2 | | | | | | | a + 3 | | | | | | | Incorrect | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 2 | 4 | 5 | 7 | S | C | SG | 2 | 4 | 5 | 7 | S | C | SG | 2 | 4 | 5 | 7 | S | C | SG |
| JS | 1 | 1 | | | | | | | | 1 | 1 | 1 | 1 | | | | | | | | |
| TD | 1 | 1 | 1 | 1 | 1 | 1 | | | | | | | | | | | | | | | 1 |
| SM | | | | | | 1 | | 1 | 1 | 1 | 1 | 1 | | | | | | | | | |
| TW * | 1 | | 1 | 1 | 1 | 1 | 1 | | | | | | | | | 1 | | | | | |
| RA * | | | | | | 1 | | 1 | | 1 | 1 | 1 | 1 | | | | | | | | |
| AG | 1 | 1 | | | | | | | | 1 | 1 | 1 | | | | | | | | | |
| BH * | 1 | | | 1 | 1 | | | | | 1 | | | | 1 | | 1 | | | | 1 | |
| RE * | | | | | | | | 1 | 1 | 1 | 1 | 1 | | | | | | | | | |
| JM | 1 | 1 | 1 | 1 | 1 | 1 | 1 | | | | | | | | | | | | | | |
| NW | | 1 | | | | 1 | | 1 | | 1 | 1 | 1 | | 1 | | | | | | | |
| BM | | | | | | | | 1 | 1 | 1 | 1 | 1 | | 1 | | | | | | | |
| MC * | 1 | 1 | | | | | | | 1 | 1 | 1 | 1 | | | | | | | | | |
| LK | | 1 | 1 | 1 | 1 | 1 | | | | | | | | | | | | | | | |
| Summary | 7 | 7 | 4 | 5 | 5 | 7 | 2 | 5 | 4 | 9 | 8 | 8 | 2 | 3 | | 2 | | | | 1 | 1 |

| Subject | Other | | | | | | | Not Apparent | | | | | | | Memorization | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 2 | 4 | 5 | 7 | S | C | SG | 2 | 4 | 5 | 7 | S | C | SG | 2 | 4 | 5 | 7 | S | C | SG |
| JS | | | | | | | 1 | | | | | | | | | | | | | | |
| TD | | | | | | | | | | | | | | | | | | | | | |
| SM | | | | | | | | | | | | | | | | | | | | | 1u |
| TW * | | | | | | | | | | | | | | | | | | | | | |
| RA * | | | | | | | | | | | | | | | | | | | | | 1u |
| AG | | | | | | | | | | | | | | | | | | | | 1 | 1u |
| BH * | | | | | | | | | | | | | | | | | | | | | |
| RE * | | | | | | 1 | 1 | | | | | | | | | | | | | | |
| JM | | | | | | | | | | | | | | | | | | | | | |
| NW | | | | | | | | | | | | | | | | | | | | | |
| BM | | | | | | 1 | | | | | | | | | | | | | | | |
| MC * | | | | | | | 1 | | | | | | | | | | | | | | |
| LK | 1 | | | | | | | | | | | | | | | | | | | | 1u |
| Summary | 1 | | | | | 2 | 3 | | | | | | | | | | | | | 1 | 4 |

Note. The following abbreviations appear in the table:
u-Problem with facts under 10
o-Problem with facts over 10
S-Subtrahend problem
C-Cuisenaire problem
SG-Student generated problem

[a] * indicates reversal of addition and subtraction student-generated problems

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MATCHING TEACHING STRATEGY TO AVAILABLE M-SPACE:
A NEO-PIAGETIAN APPROACH TO WORD PROBLEMS

by

Dianna B. Richardson

(ABSTRACT)

Perspective and Purpose

Recent investigations by Steffe, Richards, and von Glasersfeld (1979) have indicated that addition and subtraction problem solving competencies are developmental in nature and that these competencies build upon counting abilities. They postulate that, in beginning addition and subtraction, a type of problem solving strategy termed counting-all develops prior to another kind of strategy termed counting-on (for addition) and counting-back (for subtraction).

If these tasks are developmental, one may assume that students approach the tasks in qualitatively different ways based upon their developmental levels. Neo-Piagetian researchers have postulated that a quantitative measure of development explains the qualitatively different ways in which children react to the same cognitive task at different stages of development. The measure, termed mental space or M-Space, describes the number of schemes which may be coordinated at one time. First graders, the majority of whom have an M-Space of a+2 or a+3, are

capable of solving addition and subtraction word problems by utilizing the counting-all and/or the counting-on (back) strategies. Given this information, the purpose of this study was to determine what effect M-Space level has on the strategy a subject uses to solve problems when he is trained on a strategy which either matches or mismatches his M-Space level.

## Design

To determine whether a match between M-Space and strategy demand is necessary or whether instruction will facilitate the chunking of schemes which allows the developmental task to be solved by a strategy which would otherwise be above the subject's M-Space level, the following steps occurred: one hundred thirty-nine first graders were pretested to identify those who could count to sixteen, perform numeral/number correspondence to sixteen, but could not solve addition and subtraction number fact problems to sixteen. One hundred fifteen subjects meeting these criteria were given the Cucumber Test and Backward Digit Span Test to assess their M-Space levels. After eliminating subjects before and during training, 50 subjects remained. Twenty-six subjects with an a+2 M-Space were divided into two training groups. Approximately half of the group was trained to use an a+2 strategy (the count-all strategy) to solve addition and subtraction word problems and the other half of the group was trained to use an a+3 strategy (the count-on (back) strategy). The same training procedure was used for the twenty-four subjects with an M-Space of a+3. Four to five weeks later, a delayed posttest consisting of four addition and four subtraction problems and one each of three types of transfer problems was presented.

## Results

Mann-Whitney test results indicated that there were significantly fewer a+3 responses by the subjects with an a+2 M-Space who were trained to use an a+3 strategy than there were for subjects with an a+3 M-Space trained to use an a+3 strategy. However, there was no significant difference between those with an a+2 M-Space trained on an a+2 strategy and those with an a+2 M-Space trained on an a+3 strategy. Results of other research questions indicated that subjects gave similar responses to transfer problems which varied by material or additional variable; for subjects with an a+3 M-Space trained on an a+3 strategy, there were significantly more a+3 addition responses than subtraction responses; the implied comparison subtraction problem was answered incorrectly more often than straight take-away subtraction problems; and students tended to devise simple addition and subtraction problems and solve them by using memorized number facts.

## Discussion

The findings indicate that more study is warranted for the application of the M-Space construct to a theory of how mathematical knowledge develops sequentially, the different ways in which addition and subtraction tasks can be conceptualized, and the instructional implications of applying a developmentally based theory of instruction to mathematics problem solving.